PROOF-THEORETIC SEMANTICS
Assessment and Future Perspectives

Proceedings of the
Third Tübingen Conference on Proof-Theoretic Semantics
Tübingen, 27–30 March 2019

Edited by

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Presentations

Preface

The third Tübingen conference on proof-theoretic semantics took place in the *Alte Aula* of the University of Tübingen from the 27th to the 30th of March 2019. These proceedings contain all abstracts, the slides of most presentations and also a number of full papers, including a transcript of the talk by Per Martin-Löf.

This conference stands in the tradition of conferences and workshops in Tübingen on proof-theoretic semantics and general proof theory: notably the conference on substructural logics in 1990, the first conference on proof-theoretic semantics in 1999, the second conference on proof-theoretic semantics in 2013, and the conference on general proof theory celebrating 50 years of Dag Prawitz's "Natural Deduction" in 2017 (online proceedings). It also marks the end of a 10 year ANR-DFG funded collaboration between Paris and Tübingen on the topic of "Hypothetical Reasoning".

Two researchers, who would have given an invited talk at this conference, have sadly passed away: Kosta Došen in October 2017 and Roy Dyckhoff in August 2018. We therefore included in the programme special memorial sessions dedicated to them.

We gratefully acknowledge financial support from DFG (Deutsche Forschungsgemeinschaft) and DLMPST (Division of Logic, Methodology and Philosophy of Science and Technology).

We would like to thank Marine Gaudefroy-Bergmann for the perfect organisation of the conference, as well as Katrin Graß and Natalie Clarius, who assisted her. We also thank Natalie for her help in the preparation of these proceedings.

Thomas Piecha
Peter Schroeder-Heister

The organisers would like to thank all participants for making the conference such an enjoyable event. (*From left to right:* Thomas Piecha, Natalie Clarius, Peter Schroeder-Heister, Marine Gaudefroy-Bergmann, Katrin Graß.)

Conference Photo

Photo: Katrin Graß

From left to right, standing:
Eugenio Orlandelli, Nissim Francez, Bartosz Więckowski, Peter Schuster, Heinrich Wansing, Kai Tanter, Hermógenes Oliveira, Andrzej Indrzejczak, Gerhard Jäger, Hermann Haeusler, Dale Miller, René Gazzari, Andreas Röhler, Zhaohui Luo, Lutz Straßburger, Greg Restall, Preston Stovall, Philip Scott, Matteo Acclavio, Dag Prawitz, Simon Docherty, David Pym, Hosea von Hauff, Sara Negri, Giulio Guerrieri, Patrik Sestic, Bogdan Dicher, Dominik Peteler, Marie Duží, Hidenori Kurokawa, David Binder, Clément Lion, Tor Sandqvist, Sara Ayhan, Ansten Klev, Paolo Maffezioli, Per Martin-Löf, Paolo Pistone, Luca Tranchini, Valeria de Paiva, Michele Abrusci, Alberto Naibo, Luiz Carlos Pereira, Tim Richter, Thomas Piecha, Paulo Santos, Göran Sundholm, Dominik Kirst, Elio La Rosa, Natalie Clarius, Ulrich Felgner, Constantin Brîncuş, Ingo Skupin, Henk Zeevat, Marianna Girlando, Wilfried Keller, Reinhard Kahle, Michael Arndt, Christian Ittner, Simon Hosemann, Mattia Petrolo, Will Stafford, Joachim Klappenecker, Peter Schroeder-Heister, Nuria Brede, Geiza Hamazaki

From left to right, sitting:
Pablo Cobreros, Andrei Rodin, Dilectiss Liu, Wagner de Campos Sanz, Ernst Zimmermann, Marine Gaudefroy-Bergmann

Plenary Speakers

Vito Michele Abrusci (Roma Tre)

Valeria de Paiva (Cupertino, CA)

Marie Duží (VSB – Technical University of Ostrava)

Nissim Francez (The Technion, Haifa)

Andrzej Indrzejczak (University of Łódź)

Gerhard Jäger (University of Bern)

Reinhard Kahle (Universidade Nova de Lisboa & University of Tübingen)

Per Martin-Löf (Stockholm University)

Dale Miller (Inria Saclay & LIX, Palaiseau)

Sara Negri (University of Helsinki)

Luiz Carlos Pereira (PUC Rio de Janeiro)

Francesca Poggiolesi (CNRS, IHPST Paris, UMR 8590)

Dag Prawitz (Stockholm University)

David Pym (UCL London)

Greg Restall (University of Melbourne)

Peter Schroeder-Heister (University of Tübingen)

Philip Scott (University of Ottawa)

Göran Sundholm (Leiden University)

Heinrich Wansing (Ruhr-University Bochum)

Programme

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On Synonymy in Proof-Theoretic Semantics

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The Calculus of Natural Calculation

14:15-14:45 Sara Ayhan
The Meaning of Proofs in Different Proof Systems

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Nested Sequent Calculi for Lewis' Counterfactual Logics

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Abstracts

Paradoxes, Intuitionism, and Proof-Theoretic Semantics

Reinhard Kahle

Universidade Nova de Lisboa & University of Tübingen

We review paradoxes like Russell's, the Liar, and Curry's in the context of intuitionistic logic. One may observe that one cannot blame the underlying logic for the paradoxes, but has to take into account the particular concept formations. For proof-theoretic semantics, however, this comes with the challenge to block some forms of direct axiomatization of the Liar.

(Joint work with Paulo Guilherme Santos, Universidade Nova de Lisboa.)

On Synonymy in Proof-Theoretic Semantics

Heinrich Wansing

Ruhr-University Bochum

The topic of identity of proofs was put on the agenda of general (or structural) proof theory at an early stage. The relevant question is: When are the differences between two distinct proofs (understood as linguistic entities, proof figures) of one and the same formula so inessential that it is justified to identify the two proofs? It seems that a certain question that is in a sense dual to the one concerning identifying syntactically distinct proofs of one and the same formula has never been raised so far: When are the differences between two distinct formulas so inessential that these formulas admit of identical proofs? One reason why this question has, as it seems, not been asked could be that one may expect structurally different formulas to have distinct proofs that reflect the differing structure of the formulas under consideration. The question appears to be less unnatural if the idea of working with more than one kind of derivations is taken seriously. If a distinction is drawn between proofs and disproofs (or refutations) as primitive entities, it is quite conceivable that a proof of one formula amounts to a disproof of another formula, and vice versa. In particular, if a proof of a formula A is identified with (amounts to) a disproof of the negation $\neg A$ of A, and a disproof of A amounts to a proof of $\neg A$, then, obviously, the syntactically distinct formulas A and $\neg\neg A$ have one and the same proof or one and the same disproof if they are provable or disprovable at all.

Once the possibility of identifying proofs and disproofs of distinct formulas is recognized, it may be used to define a notion of synonymy between formulas: Two formulas A and B are synonymous just in case the following two conditions hold, for any finite multi-sets of assumptions (formulas taken to be true), Δ, and counter-assumptions (formulas taken to be false), Γ, (i) for every proof of A from Γ and Δ, there exists an identical proof of B from Γ and Δ, and vice versa, and (ii) for every disproof of A from Γ and Δ, there exists an identical disproof of B from Γ and Δ, and vice versa. The present paper develops this idea.

Moving the First Steps towards the Study of Proofs-Why

Francesca Poggiolesi

CNRS, IHPST Paris, UMR 8590

Starting from the dichotomy proofs-that and proofs-why and exploiting the ressources of proof theory, our aim is to move the first steps towards the formalization of proofs-why.

The Calculus of Natural Calculation

René Gazzari

University of Tübingen

Gentzen's calculus of Natural Deduction is, indeed, a natural representation of informal (mathematical) reasoning. But mathematicians do not only reason in their proofs, they also have to calculate with mathematical objects represented by the terms of a formal language; the traditional formalisations of such calculations seem to be artificial.

The calculus of Natural Calculation is an extension of Gentzen's calculus by suitable term rules allowing for a more natural treatment of equalities and, therefore, an adequate representation of the calculations found in informal proofs. We present this extension, and discuss some of its central properties. We conclude our talk with a sketch of some other extensions of Natural Deduction motivated by the introduction of term rules.

The Meaning of Proofs in Different Proof Systems

Sara Ayhan

Ruhr-University Bochum

In this talk I want to address the question of how the meaning of proofs can be determined from the viewpoint of proof-theoretic semantics (PTS). The origins of PTS lie in the question of what constitutes the meaning of the logical connectives and its response: the rules of inference that govern the use of the connective. However, what if we go a step further and ask about the meaning of a proof as a whole? From a PTS point of view it seems a sensible approach to base the meaning of a proof on the rules it contains. This can be extended by two questions: It can firstly be asked what the sense of a proof actually consists of. Asking this question also means to dive into the question whether similar distinctions as Frege's in the case of singular terms or sentences are possible for proofs, e.g. if we can distinguish a mere syntactic divergence from a divergence in meaning. If we have two (syntactically) different derivations, does this always lead to a difference, firstly, in sense and, secondly, in denotation? The second question would then be whether there are differences in determining the sense of proofs for different calculi (in this case: natural deduction vs. sequent calculi).

Hypo: A Simple Constructive Semantics for Intuitionistic Sentential Logic; Soundness and Completeness

Wagner de Campos Sanz

Universidade Federal de Goiás

Hypo is a constructive semantics for Intuitionistic Sentential Logic (ISL). Constructions and proofs are not primitives in it. It is built over one *primitive relation*: the relation of *semantical consequence* ($\Vdash$). The meaning of logical constants is made explicit in terms of this relation. Clauses establish sufficient and necessary conditions for the occurrence of a logical constant either as a *hypothesis* among others or as a *consequence* of that set of hypotheses. A *hypothesis* is a sentential content supposed to hold, i.e. taken *as if it were true* (which is different to being taken as asserted to be true). We can read $\Gamma \Vdash E$ as "E [has been] recognized to follow from hypotheses Γ" and $\Gamma \nVdash E$ as "E [has not been] recognized to follow from hypotheses Γ". The semantics contains Leftist and Rightist clauses giving sufficient and necessary conditions for each logical constant. It is completed by Constructive Structural Clauses. ISL can be proved complete and sound with respect to Hypo.

The Original Sin of Proof-Theoretic Semantics

Bogdan Dicher & Francesco Paoli

University of Lisbon & University of Cagliari

Proof-theoretic semantics is an alternative to model-theoretic semantics. It aims to explain the meaning of the logical constants in terms of the inference rules that govern their behaviour in proofs. We argue that this must be construed as the task of explaining these meanings relative to a logic, i.e., to a consequence relation. Alas, there is no agreed set of properties that a relation must have in order to qualify as a consequence relation. Moreover, the association of a consequence relation to a logical calculus is not as straightforward as it may seem. We show that these facts are problematic for the proof-theoretic project but the problems can be solved. Our thesis is that the consequence relation relevant for proof-theoretic semantics is the one given be the sequent-to-sequent derivability relation in Gentzen systems.

Adequacy for a Proof-Theoretic Semantics Based on Elimination Rules

Hermógenes Oliveira

University of Tübingen

I will prove completeness and soundness of propositional intuitionistic logic with respect to an adaptation of Dummett's proof-theoretic notion of validity based on elimination rules.

On Prawitz' Ecumenical System

Luiz Carlos Pereira

PUC Rio de Janeiro

A classical mathematician can be eclectic, the intuitionist cannot. A classical mathematician may consider the intuitionistic position quite interesting, since constructive proofs, although usually longer, are more informative than indirect classical proofs, since they have an algorithmic nature and satisfy interesting informative properties. To the intuitionistic mathematician however, there seems to be no alternative but to revise and revoke the universal validity of certain classical principles of reasoning. In 2015 Dag Prawitz (see [3]) proposed an ecumenical system, a codification where classical logic and the intuitionistic logic could coexist "in peace". The main idea behind this codification is that the classical mathematician and the intuitionist share the constants for conjunction, negation and the universal quantifier, but each has her/his own disjunction, implication and existential quantifier. Similar ideas are present in Dowek [1] and Krauss [2], but without Prawitz' philosophical motivations. The aims of the present paper are: (i) to investigate the proof theory and the semantics for Prawitz' ecumenical system, (ii) to compare Prawitz' system with other ecumenical approaches, and (iii) to propose new ecumenical systems.

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Validity of Inferences Reconsidered

Dag Prawitz

Stockholm University

My aim is to explicate a concept of valid inference that is congruent with the use of valid inferences to support our assertions and thereby extend our knowledge. The most common definition of the validity of inference in contemporary logic and philosophy is obviously not relevant when one has this aim in mind. Other proposed definitions within proof-theoretic semantics or with a constructivist origin turn out to have serious failings in this respect. In this talk I shall reject some principles that those definitions were built on and consider some other principles that I think are essential for determining the wanted concept of valid inference. Some of them are based on Gentzen's ideas behind his system of introduction and elimination rules, now developed in a different way than in the rejected definitions.

Logic and Ethics

Per Martin-Löf

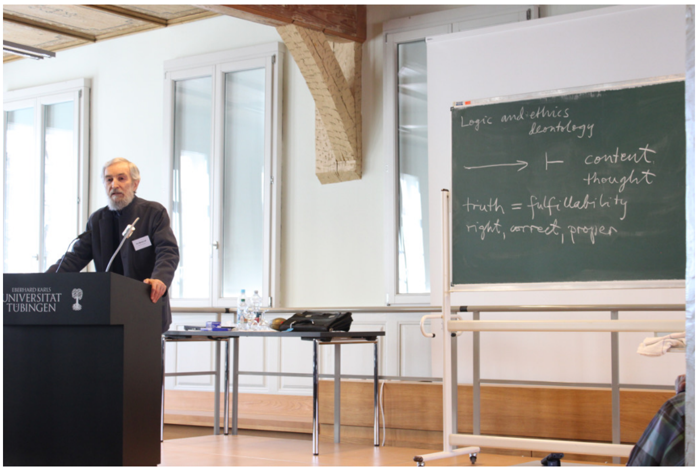
Stockholm University

The condition under which it is correct (proper) to make an assertion is that the assertor knows how (is able) to perform the task which constitutes the content of the assertion (correctness condition for assertions).

To make an assertion is to commit (obligate) yourself to performing the task which constitutes the content of the assertion (commitment account of assertion).

The condition under which it is correct (proper) to undertake an obligation (make a commitment) is that the obligor knows how (is able) to fulfil it (ought implies can).

The relation between the preceding three principles is simple: the correctness condition for assertions follows from the commitment account of assertion taken together with the ought-implies-can principle. Both the commitment account of assertion and the ought-implies-can principle bring in the notion of duty (obligation) and hence implicitly, by the correlativity of rights and duties, the notion of right. On the other hand, the notions of right and duty are the key notions of deontological ethics. Thus, all in all, logic has not only an ontological layer and an epistemological layer, but also a deontological layer underlying the epistemological one. It can be avoided only by treating the notion of knowledge how (can) as a primitive notion, thereby abstaining from relating it to the notions of right and duty (may and must).



Validity

Göran Sundholm

Leiden University

The talk will distinguish and consider interrelations (if any) between five different notions of validity.

1. Validity of a proof (demonstration);
2. Validity of an inference;
3. (Logical) Validity of a wff (proposition);
4. (Logical) Validity as a consequence among wff's (proposition);
5. Vality of derivations in the sense of Prawitz.

If time permits, the significance of 1:st order predicate calculus for logic, when considered as the study of valid inference, will be discussed.

A New Approach to Proof-Theoretic Semantics for Classical Logic

Mario Piazza & Gabriele Pulcini

SNS, Pisa & Universidade Nova de Lisboa

In this talk we introduce a notion of partial truth within the confines of classical logic: truth-values become discrete elements of the set of rational numbers $\mathbb{Q}$ within the interval [0, 1]. The main feature of our approach is its purely syntactical dimension: truth-values exhaust their function in decorating the axioms and the logical rules of *classical* sequent calculus. Such a decoration is tailored to keep record, along the proofs, of the number of occurrences of *identity axioms*, encoding the primitive logical fact that anything implies itself. The lesson, then, is that the art of proving in classical logic amounts to surveying the multiset of identities in the final claim of the proof. We demonstrate the effectiveness of our formalism by showing that any two proofs of the *same* theorem A confer to A the *same* interpretation, namely the interpretation given *by the ratio between the number of identity axioms and the total number of axioms in the proof of* A. This ratio gives direct expression to the notion of partial truth [2].

The new semantical picture can be summarized as follows: on the one hand, all classical tautologies are preserved, not because they are true for any assignment but because all their components are identities; on the other hand, contingencies and contradictions vanish into a stratified universe. This situation, it turns out, engenders an infinite series of suprasystems of classical logic which are paraconsistent in spirit. Unlike Makinson's supraclassical systems [1, 3], these logics are non-Tarskian but they happily satisfy structurality, i.e., theoremhood is preserved under uniform substitution.

Finally, we show that our semantical framework may offer a new way to approach the problem of providing a proof-theoretic semantics for classical logic. The idea is that a semantics in terms of proofs needs not rely a priori on some sort of epistemic content transmitted along proofs via logical rules, like what happens in the proof-theoretic semantics for intuitionistic logic [4].

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[3] M. Piazza and G. Pulcini. Uniqueness of axiomatic extensions of cut-free classical propositional logic. *Logic Journal ofthe IGPL*, 24(5):708–718, 2016.

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Formal Semantics in Modern Type Theories: An Overview

Zhaohui Luo

Royal Holloway, University of London

I'll give an overview, and report some recent developments, of Formal Semantics in Modern Type Theories (MTT-semantics) by focussing on:

1. The rich structures in MTTs, together with subtyping, make MTTs a nice and powerful framework for formal semantics of natural language;
2. MTT-semantics is both model-theoretic and proof-theoretic and hence a very attractive semantic framework.

By explaining the first point, we'll introduce MTT-semantics and, at the same time, show that the use and development of coercive subtyping play a crucial role in making MTT-semantics viable. The second point shows that MTTs provide a unique and nice semantic framework that was not available before for linguistic semantics. Being model-theoretic, MTT-semantics provides a wide coverage of various linguistic features and, being proof-theoretic, its foundational languages have proof-theoretic meaning theory based on inferential uses (appealing philosophically and theoretically) and it establishes a solid foundation for practical reasoning in natural languages on proof assistants such as Coq (appealing practically). Altogether, this strengthens the argument that MTT-semantics is a promising framework for formal semantics, both theoretically and practically.

Abstract Semantic Conditions and the Incompleteness of Intuitionistic Propositional Logic with respect to Proof-Theoretic Semantics

Thomas Piecha

University of Tübingen

In [1] it was shown that intuitionistic propositional logic (IPC) is semantically incomplete with respect to certain notions of proof-theoretic validity. This questioned a claim by Prawitz, who was the first to propose a proof-theoretic notion of validity, and claimed completeness for it [3, 4]. In this talk we put these and related results into a more general context [2]. We formulate five abstract semantic conditions for proof-theoretic validity, which every proof-theoretic semantics is supposed to satisfy. We then show that if in addition certain more special conditions are assumed, IPC fails to be complete. Here a crucial role is played by the generalized disjunction principle. Several concrete notions of proof-theoretic validity are considered, and it is shown which of the conditions rendering IPC incomplete they meet.

From the point of view of proof-theoretic semantics, intuitionistic logic has always been considered the main alternative to classical logic. However, in view of the results to be discussed in this talk, intuitionistic logic does not capture basic ideas of proof-theoretic semantics. Given the fact that a semantics should be primary over a syntactic specification of a logic, we observe that intuitionistic logic falls short of what is valid according to proof-theoretic semantics. The incompleteness of intuitionistic logic with respect to such a semantics therefore raises the question of whether there is an intermediate logic between intuitionistic and classical logic which is complete with respect to it.

(Joint work with Peter Schroeder-Heister, University of Tübingen.)

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The Jacobson Radical and Glivenko's Theorem

Peter Schuster

University of Verona

Alongside the analogy between maximal ideals and complete theories, the Jacobson radical carries over from ideals of commutative rings to theories of propositional calculi. This prompts a variant of Lindenbaum's Lemma that relates classical validity and intuitionistic provability, and the syntactical counterpart of which is Glivenko's Theorem. Apart from shedding some more light on intermediate logics, this eventually prompts a non-trivial interpretation in logic of Rinaldi, Schuster and Wessel's conservation criterion for Scott-style entailment relations (BSL 2017 & Indag. Math. 2018).

(Joint work with Giulio Fellin & Daniel Wessel, both University of Verona).

Inquisitive Proof-Theoretic Semantics

Will Stafford

UC Irvine

When looking at the logic captured by proof-theoretic validity one must pay special attention to the sets of atomic rules considered. This talk will explore how the sets of atomic rules that are allowed can be modified to give a notion of proof-theoretic validity which is equivalent to inquisitive semantics. This offers a new perspective both on inquisitive semantics and the flexibility of proof-theoretic validity.

Nested Sequent Calculi for Lewis' Counterfactual Logics

Marianna Girlando & Nicola Olivetti

Aix-Marseille University and University of Helsinki & Aix-Marseille University

Counterfactual logics are a family of modal logics introduced by Lewis to formalise conditional sentences of the form "if A was the case, then B would have been the case". These logics extend the language of classical propositional logics with a two-place modal operator standing for comparative plausibility, by means of which the counterfactual conditional can be defined. The semantics of counterfactual logics is given in terms of neighbourhood models with the property that the family of neighbourhoods associated to each world is nested. In a previous paper, we defined sequent calculi for some logics in Lewis' family by enriching the structure of sequents by means of conditional blocks, which are syntactic objects standing for disjunctions of comparative plausibility formulas. In the present work we define nested sequents for counterfactual logics. The main advantages of these proof systems are that they are fully invertible (differently from the sequent calculi), and allow one to obtain a direct completeness proof with respect to neighbourhood models. Finally, the nested structure of a sequent can be represented by a tree of labels, therefore obtaining a natural translation with labelled proof systems.

Proof-Theoretic Semantics for Natural Language

Nissim Francez

The Technion, Haifa

The purpose of this talk is to highlight Proof-Theoretic Semantics (PTS) as a viable alternative to the traditional model-theoretic semantics (MTS). PTS is well-established within logic. I will show how it successfully extends from logic to natural language (NL).

The talk has two parts:

1. A brief exposition of PTS, not necessarily in connection to NL. I will present the notion of a *meaning-conferring proof-system* (reflecting "use"), and the induced notions of *grounds for assertion* and *proof-theoretic consequence*, both underlying the proof-theoretic rendering of meaning.
2. A contrastive review of some of the applications of PTS to NL and advantages of PTS as a theory of meaning for NL.

   The following applications of PTS to NL will be presented.

   (a) The meaning of *determiners*, with an emphasis of *conservativity* being a non-issue.
   (b) The meaning of *intensional transitive verbs (ITVs)* and their monotonicity.
   (c) The meaning of adjective-noun combination.
   (d) The meaning of transitive verbs with an implicit object, with an emphasis of the source of equality of meaning.

   In all these applications, it will be shown how ontological commitments as to what kind of objects (sometimes dubious!) populating models are replaced by an appeal to syntactic entities within a meaning-conferring proof-system.

Geometric Rules in Infinitary Logic

Sara Negri

University of Helsinki

Geometric logic studies what are called geometric or coherent theories, with contributions and application areas from such diverse fields as structural proof theory, category theory, constructive mathematics, modal and non-classical logics, and automated deduction; only recently, geometric logic has entered the focus of a systematic proof-theoretic investigation.

Gentzen's systems of deduction, sequent calculus and natural deduction, have been recognised as an answer to Hilbert's 24th problem by providing the basis for a general theory of proof methods in mathematics that overcomes the limitations of axiomatic systems. Natural deduction and sequent calculus, however, give a transparent analysis of the structure of proofs that works to perfection for pure logic, but when they are augmented with axioms for mathematical theories, much of their strong properties are lost.

The method of axioms-as-rules studies the transformation of certain classes of axioms into rules of inference of a suitable form that maintain the good structural properties of the extended systems. Coherent theories are very well placed into this programme, in fact, they can be translated into structure-preserving inference rules in a natural fashion.

The methodology is here extended to geometric theories by augmenting infinitary G3-style sequent calculi with a rule scheme for infinitary geometric implications. As an application, it is shown that by bringing the classical and intuitionistic calculi close together, the infinitary Barr's conservation theorem for geometric theories becomes an immediate result.

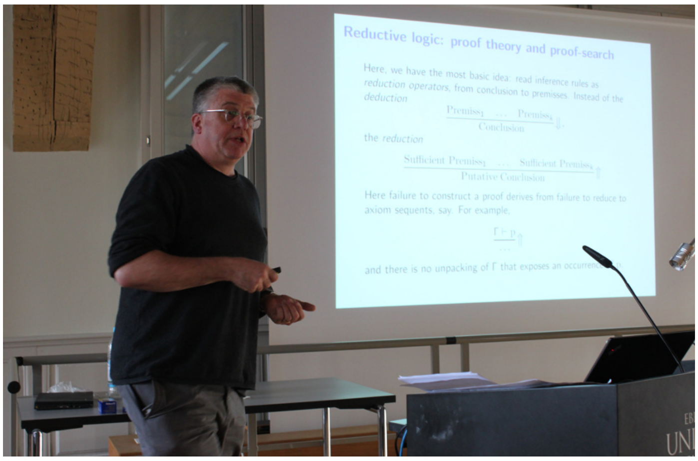
(Based on joint work with the late Roy Dyckhoff.)

Reductive Logic and Proof-Theoretic Semantics

David Pym

UCL London

While the deductive approach to logic begins with premisses and in step-by-step fashion applies proof rules to derive conclusions, the complementary reductive approach instead begins with a putative conclusion and searches for premisses sufficient for a legitimate derivation to exist by systematically reducing the space of possible proofs. Not only does this picture perhaps more closely resemble the way in which mathematicians actually prove theorems and, more generally, the way in which people solve problems using formal representations, it also encapsulates diverse applications of logic in computer science such as the programming paradigm known as logic programming, the proof-search problem at the heart of symbolic AI and automated theorem proving, precondition generation in program verification and more. It is also reflected at the level of truth-functional semantics – the perspective on logic utilized for the purpose of model-checking and thus verifying the correctness of industrial systems – wherein the truth value of a formula is calculated according to the truth values of its constituent parts. Developing the mathematical theory of reductive logic – proof theory, semantics, and control – for wide classes of substructural, modal, intuitionistic, and classical systems is a substantial ongoing project which is drawing upon not only the traditional tools of proof theory, model theory, and category theory but also upon the more recently developed theory of coalgebra and its applications in modelling stateful systems. Underlying this mathematical work is the basic observation that the spaces of objects that must be explored in reductive logic, either in the setting of proof-search or in the setting of model-checking, are bigger than the corresponding spaces of proofs or semantics realizers. This situation closely resembles that which gives rise to the structures used to set up concepts of validity in proof-theoretic semantics, and is used to support additional structure that is of use computationally.



Proof-Theoretic Semantics: Some Open Problems

Peter Schroeder-Heister

University of Tübingen

I discuss three problems that are particularly interesting for the future direction of proof-theoretic semantics:

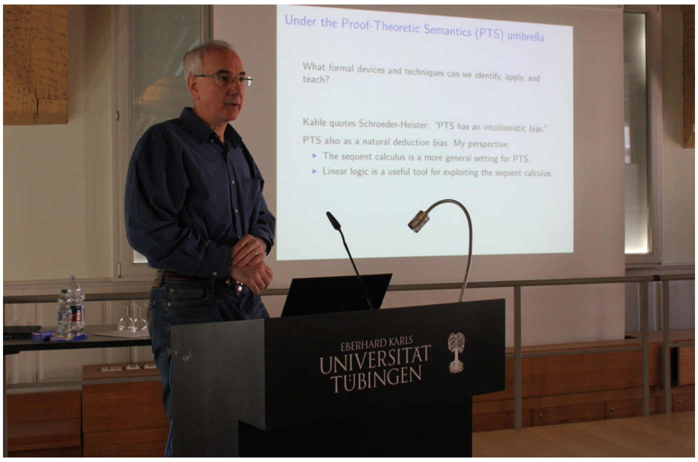
1. The problem of the general adequacy of validity-based semantics, given the incompleteness results recently obtained by Piecha and Schroeder-Heister.
2. The idea of an intensional approach to proof-theoretics semantics, here applied to the problem of harmony of introduction and elimination rules.
3. The proof-theoretic semantics of atomic statements, with particular emphasis on the complementary notions of generality and substitution.

Applying a Linear Logic Perspective to Arithmetic

Dale Miller

Inria Saclay & LIX, Palaiseau

[without abstract]



Going Without: A Linear Modality and its Role

Valeria de Paiva

Cupertino, CA

Linear type theories have been with us for more than 25 years, from the beginning of Linear Logic, and they still have much to teach us. I want to discuss and compare four linear type theories and show how an overlooked system has much potential to help with open issues in modal type theory. We discuss the Linear Lambda Calculus (Benton et al.), Plotkin and Barber's DILL (Dual Intuitionistic Linear Logic) and Benton's Linear-non-Linear (LNL) type theories. Then we recall how DILL can be transformed into ILT (Intuitionistic and Linear Type Theory), as described by Maietti et al. (2000), a type theory without a modality, but with two functions spaces, and what we gain with this transformation. Finally, we speculate on how this transformation might be helpful for other modal type theories, their models and envisaged applications.

Proof-Theoretic Semantics and Paradoxical Languages

Mattia Petrolo

Federal University of ABC

In proof-theoretic semantics there are two main approaches to the definition of validity for derivations, one based on introduction rules and the other on elimination rules. In the context of usual systems for intuitionistic logic, these approaches enjoy an important symmetry: both allow to show that all derivations are valid. However, this symmetry breaks in the case of proof systems dealing with paradoxical derivations. To substantiate this claim we exploit two different analyses of paradoxes in natural deduction. We show that in one case elimination rules cannot be justified starting from introduction rules and, in the other, introduction rules cannot be justified starting from elimination rules. As a consequence, the proof-theoretic semantics of paradoxical languages arising from the introduction-based and the elimination-based approach might be very different. Indeed, we argue that these two semantic frameworks open the way to two different notions of paradoxality in proof-theoretic semantics: either as failure of some compositional principle (e.g. MP or cut), as it would result from an introduction-based approach, or as some notion of partial function, as it would result from an elimination-based approach.

The Role of Structural Reasoning in the Genesis of Graph Theory

Michael Arndt

University of Tübingen

The seminal book on graph theory by Dénes Kőnig, published in the year 1936, collected notions and results from precursory works from the mid to late 19th century by Hamilton, Cayley, Sylvester and others. More importantly, Kőnig himself contributed many of his own results that he had obtained in the more than 20 years that he had been working on this subject matter.

What is noteworthy from a logician's perspective is the fact that the fundamentals of what he calls directed graphs are taken almost exhaustively from Paul Hertz' 1922 article on structural reasoning about sentences of the form $a \to b$. This is not a fact that is well known, neither in logical nor in graph theoretical circles, even though Kőnig fully acknowledges it in his book. In view of the numerous trends in the recent decades to describe and explicate logical matters by means of graphs, the fact that it was Hertz' foundation of structural reasoning that informed basic notions of graph theory in the first place is highly significant.

The main goal of this talk is to summarize Hertz' article and demonstrate how Kőnig integrated the notions and results presented therein in his book. This is followed by an exposition of how and when Hertz' results were reinvented in terms of graph theory. A critical assessment of the opinion expressed by both Hertz and Kőnig that the more general sentences of the form $(a_1, \ldots, a_n) \to b$, introduced by Hertz in a companion article in 1923, cannot be interpreted by graphs concludes this talk.

Preservation of Structural Properties in Intuitionistic Extensions of an Inference Relation

Tor Sandqvist

KTH Stockholm

The paper approaches cut elimination from a new angle. On the basis of an arbitrary inference relation among logically atomic sentences, an inference relation on a language possessing logical operators is defined by means of inductive clauses similar to the operator-introducing rules of a cut-free intuitionistic sequent calculus. The logical terminology of the richer language is not uniquely specified, but assumed to satisfy certain conditions of a general nature, allowing for, but not requiring, the existence of infinite conjunctions and disjunctions. We investigate to what extent structural properties of the given atomic relation are preserved through the extension to the full language. While closure under the Cut rule narrowly construed is not in general thus preserved, two properties jointly amounting to closure under the ordinary structural rules, including Cut, are.

In the interest of conceptual economy, deducibility relations are specified purely inductively, rather than in terms of the existence of formal proofs of any kind.

Are the Open-Ended Rules for Negation Categorical?

Constantin C. Brîncuş

University of Bucharest

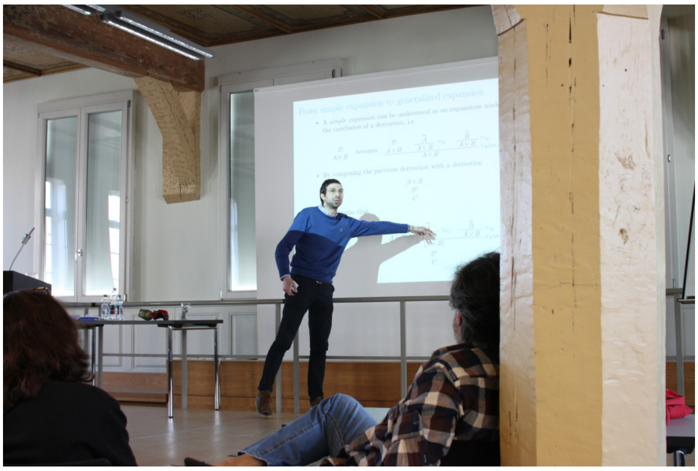
Van McGee has recently argued that Belnap's criteria constrain the formal rules of classical natural deduction to uniquely determine the semantic values of the logical connectives and quantifiers if the rules are taken to be open-ended, i.e., if they are truth-preserving within any mathematically possible extension of the original language. An assumption of his argument is that for any class of models there is a mathematically possible language in which there is a sentence true in just those models. This assumption, however, is problematic for the class of models of classical propositional logic. I argue that the existence of non-normal models for the classical propositional connectives, and in particular for negation, i.e., models for which the calculi remain sound and complete, but in which the logical constants have different meanings than the standard ones, undermines McGee's argument.

Stability in Sequent Calculus

Hidenori Kurokawa & Alberto Naibo

Kanazawa University & University Paris 1, IHPST

In this paper we argue that, from the perspective of proof-theoretic semantics, the proof transformation associated with Dummett's notion of stability becomes more perspicuous when looked through the lens of sequent calculus. From such a perspective, our aim is threefold. First, we analyze a generalized form of proof-expansion recently discussed in the literature (esp. by Luca Tranchini) as a formal way to characterize stability. We show that this expansion can be understood in terms of a specific proof transformation in sequent calculus, i.e. the possibility of permuting upward a cut with respect to a derivation of an identity sequent. Secondly, we use the sequent calculus framework in order to consider a dual version of stability (where the duality consists in changing the position of the identity derivation with respect to the cut). We show that when no structural rules are involved, stability and its dual allow one to separate the style of the connectives. Finally, we claim that our approach based on sequent calculus allows one to reconsider the nature of harmony and stability, as both take part in the process of cut elimination. In the case of the most standard connectives, harmony uniformly corresponds to reduction of the complexity of a cut, while stability uniformly corresponds to reduction of the height of a cut. We explore how far this idea can be extended.



From Syntactic Proofs to Combinatorial Proofs

Matteo Acclavio & Lutz Straßburger

Inria Saclay & LIX, École Polytechnique

Proof theory plays an important role in many areas of computer science. As the name suggests, proof theory aims to study proofs as mathematical objects and their interactions. However, unlike many other mathematical fields, proof theory is not able to identify its objects: we do not have a clear understanding of when two proofs are the same.

In this talk we investigate Hughes' combinatorial proofs as a notion of proof identity for classical logic.

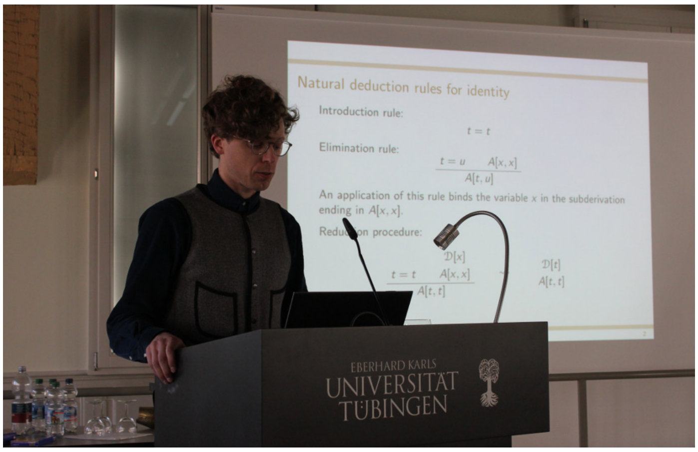
We show how to represent proofs from various syntactical formalisms, including sequent calculus, analytic tableaux, and resolution, by means of combinatorial proofs. For each of these formalisms, a natural notion of proof identity is given by certain rule permutations. We compare these notions with the ones combinatorial proofs enforce. This allows to compare proofs given in different formalisms.

The Logic and Harmony of Identity

Ansten Mørch Klev

Czech Academy of Sciences, Prague

The standard natural deduction rules for the identity predicate have seemed to some not to be harmonious. Stephen Read has suggested an alternative introduction rule that restores harmony but requires second-order logic for its proper functioning. I shall argue that the standard rules are in fact harmonious. To this end, natural deduction will be enriched with a theory of definitional identity. This leads to a novel conception of what counts as a canonical derivation, on the basis of which the identity elimination rule may be justified.

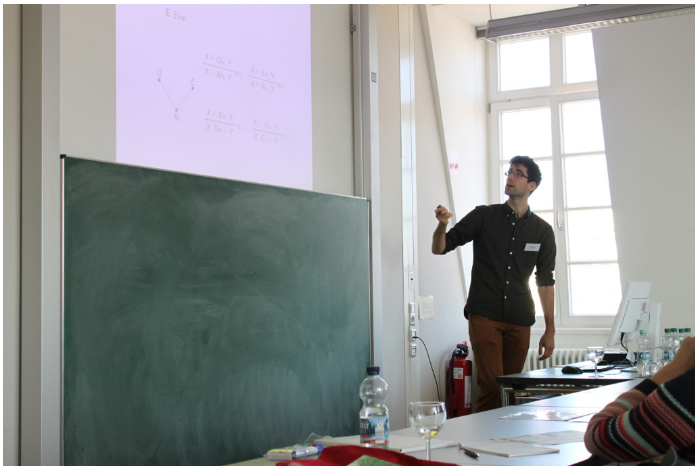


An Inferentialist Semantics for Atomics, Predicates, and Names

Kai Tanter

Monash University

Inferentialism is a theory in the philosophy of language which claims that the meanings of expressions are constituted by inferential roles or relations, rather than truth and reference. Instead of a traditional model-theoretic semantics, inferentialism naturally lends itself to a proof-theoretic semantics, where meaning is understood in terms of inference rules with a proof system. Most work in proof-theoretic semantics has focused on logical constants, with comparatively little work on the semantics of non-logical vocabulary. Drawing on Robert Brandom's notion of material inference and Greg Restall's bilateralist interpretation of the multiple conclusion sequent calculus, I present a proof-theoretic semantics for atomic sentences and their component names and predicates. The resulting system has several interesting features: (i) the rules for atomic sentences are determined by those for their component predicates and names; (ii) cut elimination for the system can be proved; (iii) model theoretic extensions can be interpreted as idealisations derived from the more fundamental inference rules; (iv) in contrast to other proof-theoretic approaches to non-logical vocabulary, the inference rules are of a similar kind to those for logical vocabulary and priority is given to neither the introduction (right) nor elimination (left) rules.



Harmony, Higher-Order Rules, and the Curry-Howard-Lambek Correspondence

Yoshihiro Maruyama

Kyoto University

Different instances of Curry-Howard-Lambek correspondence have been developed; yet a general mechanism underpinning them is still unclear. Building upon the theory of higher-order rules, we explicate a uniform mechanism to derive from logical constants the corresponding categorical constructions, and thereby establish the logic-category correspondence at a general level on the basis of general-elimination harmony (GE harmony for short). This means that general-elimination harmony yields the Curry-Howard-Lambek correspondence: any logical constants harmoniously definable within the system of higher-order rules can in principle be given categorical semantics. We also have a look at several examples of this general mechanism, one of which is a contradictory constant as in the liar paradox, corresponding to certain "liar categories." Since not all logical constants defined via GE harmony come from adjoint situations, we also compare GE harmony with Lawverian categorical harmony. To this end, we revisit the idea of structural functors by Kosta Došen, which allows us to revise and sharpen Lawverian harmony. We then consider how many of GE-harmonious logical constants come from categorical adjunctions, or how many of them are categorically harmonious as well.

Modes of Assumptions and Moods of Implications

Bartosz Więckowski

Goethe University Frankfurt

We define a natural deduction system which permits, depending on the canonical derivability of a formula in a given reference proof system, a factual or a counterfactual mode of assuming it. The system contains I/E-rules for factual and for counterfactual implication which make use of these modes. Derivations in this system normalize and normal derivations have the subformula property. In contrast to proof systems for counterfactual logics which result from internalizing possible worlds truth conditions, the system admits a foundationally autarkical proof-theoretic semantics for counterfactuals.

The Calculus of Higher-Level Rules in Modern Dress

Paolo Pistone & Luca Tranchini

University of Tübingen

The "categorification" of lambda-calculi and natural deduction systems by means of higher-order categories is a recurrent theme in last years' research. While most type systems can be presented as categories of finite order (e.g. 2-operads), it is well-known that Martin-Löf Type Theory generates, under certain hypotheses, a weak omega-category.

In this talk we develop the suggestion that Peter Schroeder-Heister's 1984 extension of natural deduction with higher-level rules (HLR) provides a second example of a category of infinite order. We propose a formulation of this system within the language of opetopes, one of the existing approaches to weak omega-categories. We show that this formulation of HLR allows for a cleaner description of higher-level substitution, and leads to a better understanding of normalization and identity of proof.

Proof-Theoretic Semantics of Natural Deduction Based on Inversion

Ernst Zimmermann

University of Tübingen

The talk presents a full Proof-Theoretic Semantics of Natural Deduction based on a slightly modified inversion principle of Prawitz (D. Prawitz, *Natural Deduction*, 1965), namely the following extension of Prawitz' inversion definition: The elimination rule for a connective q may invert the introduction rule for q, but also vice versa, the introduction rule for a connective q may invert the elimination rule for q. Such an extension of inversion, to be defined in detail, gives the following inversion theorem:

> Inversion for two rules of connective q (intro rule, elim rule) exists iff a conversion of a maximum formula for q exists. For short: inversion equals conversion.

The inversion theorem is specified to two fragments of logics: Lambek Calculus, LC and Intuitionistic Linear Logic, ILL with four propositional connectives: two multiplicatives, implication and conjunction, two additives, conjunction and disjunction. LC is defined by using elimination rules by composition. ILL is defined by using general elimination rules.

The considerations show that in LC and ILL for the four connectives exactly one of the pair intro rule, elim rule is inverting. Climbing up in the substructural hierarchy more structural rules are available, connective differences collapse, and inversion gets arbitrary. For instance for intuitionistic conjunction and disjunction intro rule and elim rule are inverting – due to structural rules.

Equality of Proofs

Philip Scott

University of Ottawa

Since the early work of F. W. Lawvere and especially the later work of J. Lambek, categorical proof theorists have examined the question “what are natural equations between proofs?”. Thus, the so-called Curry-Howard-Lambek correspondences, fundamental in categorical logics, use appropriate term calculi to annotate proof-trees. We ask: what are appropriate equations? And what criteria should we use to justify them?

We discuss such questions from several viewpoints and applications from the literature: coherence theorems in categories and multicategories (the original motivation of Lambek), the equations and constructions of various free categories and their model theory, and recent advances in categorical recursion theory. Time permitting, we discuss some more recent structures in categorical proof theory arising in theoretical computer science, mathematics, and physics.

Isomorphisms in a Category of Proofs

Greg Restall

University of Melbourne

In this talk, I will show how a category of formulas and classical proofs can give rise to three different hyperintensional notions of sameness of content.

One of these notions is very fine-grained, going so far as to distinguish p and $p \wedge p$, while identifying other distinct pairs of formulas, such as $p \wedge q$ and $a \wedge p$; p and $\neg\neg p$; or $\neg(p \wedge q)$ and $\neg p \vee \neg q$.

Another relation is more coarsely grained, and gives the same account of identity of content as equivalence in Angell's logic of analytic containment. A third notion of sameness of content is defined, which is intermediate between Angell's and Parry's logics of analytic containment. Along the way we show how purely classical proof theory gives resources to define hyperintensional distinctions thought to be the domain of properly non-classical logics.

Natural Language Processing by Natural Deduction

Marie Duží

VSB – Technical University of Ostrava

I introduce the system for deriving inferable, or implicit knowledge from the explicit textual data by means of natural deduction adjusted to my background theory of Transparent Intensional Logic (TIL). TIL is the system with procedural semantics that assigns abstract procedures to terms of natural language as their meanings. TIL operates with a fundamental dichotomy between procedures and their products, i.e. functions. This dichotomy corresponds to two basic ways in which a procedure can occur within another procedure, namely displayed or executed. If the procedure is displayed, then the procedure itself is an object of predication; we say that it occurs hyperintensionally. If the procedure is executed, then it is a constituent of another procedure. Applying the rules of natural deduction to constituents is unproblematic. However, when quantifying into a hyperintensional context, we are confronted with technical complications that arise from the fact that a displayed procedure does not produce an object to operate on. Rather, the procedure itself is an object to operate on. As a way out, I introduce the substitution method that operates on procedures and is broadly applied not only to quantifying into, but also to β-reduction by value and anaphora resolution.

Proof-Theoretic Approach to Definite Descriptions

Andrzej Indrzejczak

University of Łódź

The talk is concerned with two topics which usually do not come together. Definite descriptions are more often discussed in the framework of the philosophy of language. In logic very often a reductionist perspective of Russell is taken as the last word in this question. Moreover, even serious logical research on definite descriptions (as developed for example in the area of free logics) usually is carried out by means of semantical methods. Modern proof-theoretic apparatus, in particular sequent calculus, was rarely applied in this field so far. We focus on proof-theoretic features and problems with their application to the description-operator as an additional constant. It seems that the application of techniques taken from structural proof theory may shed a new light on the good and bad sides of different approaches to definite descriptions. The application of sequent calculus to definite descriptions will be examined on some theories due to Hilbert and Bernays (and developed by Stenlund), Frege (developed by Kalish and Montague) and Lambert (as developed by Garson).

Huge Proofs, Redundant Proofs and some Reasons in favor of NP = PSPACE

Edward Hermann Haeusler & Lew Gordeev

PUC Rio de Janeiro & University of Tübingen

We discuss an argument in favour of NP = PSPACE. Any exponentially sized linearly bounded height proof P of A in implicational minimal logic is highly redundant. This is expressed by the fact that there is at least one derivation $*P$ that occurs exponentially many times as sub-derivation of P. This is a consequence of the fact that any tree-like proof is labeled with linearly many formulas (subformulas from A) and the proof is linearly height-bounded. There may exist more than one (different) derivation that occurs exponentially many times as sub-derivations of P. They and the way that they glue in each other to form the proof itself raises a kind of spectral analysis of proofs; components are the occurring derivations and the analysis is the way they combine by means of repetitions to the whole proof. We decompose exponentially linearly height-bounded proofs into, somehow, combinations of polynomially sized derivations. This combination resembles the horizontal compression method we presented previously. We show a new horizontal compression, based on rewriting rules, that obtains a polynomially sized dag-like (compressed) proof of A. We provide a polynomial algorithm for the verification of the validity of dag-like proofs.

A Dialogical Reconstruction of Brouwer's Creating Subject?

Clément Lion

University of Lille

The problem of the creating subject has been widely rejected amongst mathematicians due to its vagueness. Even Kreisel's attempt to reconstruct it through an axiomatic theory, using a new connective, has been recently confronted with the impossibility of supplying it with genuine meaning explanations without postulating a non-constructive proof. Our proposal is to investigate how a dialogical treatment of the problem could be elaborated out of the peircian distinction between iconic, indexical and symbolical functions of a sign. The semantic problems underlying Kreisel's axiomatic reconstruction would concern only an objectified (and symbolic) form of subjectivity, which is only at stake at a "strategic level".

Extra-Logical Proof-Theoretic Semantics in HoTT

Andrei Rodin

Russian Academy of Sciences

Kant famously argued that elementary geometrical statements such as Euclid's Triangle Angle Sum Theorem cannot be deduced from the first principles by purely logical means because their proofs require extra-logical geometrical constructions. The discovery of non-Euclidean geometries in the 19th century made Kant's analysis of geometrical reasoning untenable in its original form, and throughout the following 20th century it was generally viewed as fundamentally mistaken or at least wholly outdated. However, the recently emerged Homotopy Type theory (HoTT) and the related program of building new "univalent" foundations of mathematics provide a formal and conceptual basis for revising, once again, the epistemic role and logical function of extra-logical constructions in mathematical (and other) proofs.

The key feature of HoTT, which modifies the intended logical semantics of Martin-Löf Type theory (MLTT), is the homotopical cumulative hierarchy of types, which distinguishes between types of different homotopy levels. This hierarchy suggests a simple (albeit not uncontroversial) criterion of logicality according to which only types with at most one term qualify as propositions, and only applications of MLTT rules to propositions and their terms (proofs), that is, to judgements in the traditional sense of the term, qualify as logical inferences stricto sensu. According to the same criterion, applications of the same schematic rules to types and terms of higher homotopy levels are extra-logical constructions. Such extra-logical applications of deductive rules also have a proof-theoretic impact because the obtained non-propositional constructions serve as witnesses for associated propositions (formally obtained via the propositional truncation of higher types) in a manner similar to which elementary geometrical constructions support the traditional geometrical proofs. For this reason it is justified, in our view, to qualify the standard semantics of HoTT just outlined as proof-theoretic. Using HoTT as a motivating example I would like to discuss further the role of logical and extra-logical elements in formal proofs.

Interpolation in Extensions of First-Order Logic

Guido Gherardi, Paolo Maffezioli & Eugenio Orlandelli

Università di Bologna, Universidad de Barcelona & Università di Bologna

We provide a constructive proof of Craig's interpolation theorem for extensions of classical and intuitionistic first-order logic with a special type of geometric axioms, called singular geometric axioms. As a corollary, we obtain a direct proof of interpolation for (classical and intuitionistic) first-order logic with identity, as well as interpolation for several mathematical theories including strict partial orders, apartness, positive partial orders and positive linear orders.

Proof-Nets: Tools for Studying Equivalence between Proofs and Proof-Theoretic Semantics

Vito Michele Abrusci

Roma Tre

I will discuss how proof-nets (geometrical representations of linear logic proofs) may be considered as related to the themes investigated in the domain of proof-theoretic semantics and may be tools for future works in this field.

Predicative Hierarchies

Gerhard Jäger

University of Bern

Starting off from the usual (often informal) definitions of predicativity and the Feferman-Schütte characterization of the limits of predicativity, we turn to a few theories of second order arithmetic and set theory that "spoil" the general picture. We use these irritations as an opportunity to reconsider some of the familiar ingredients of predicativity from a different perspective, leading to what I call metapredicativity.

Presentations

Proof-Theoretic Semantics

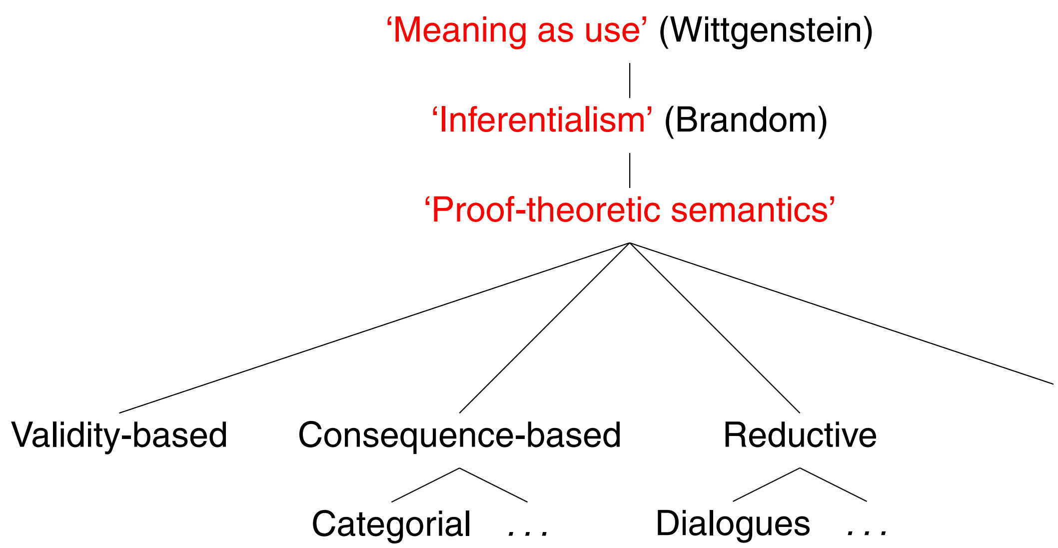
Assessment and Future Perspectives

Tübingen, 27-30 March 2018

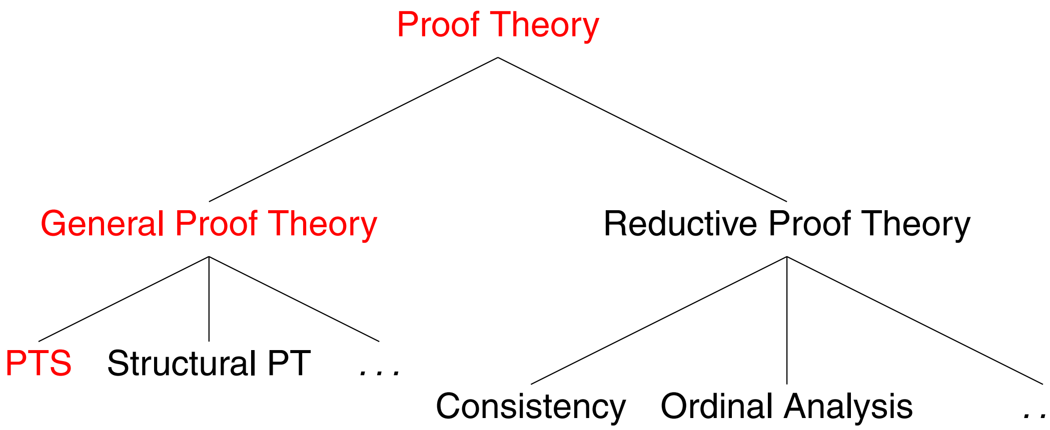
Opening

Peter Schroeder-Heister

The philosophical taxonomy



The proof-theoretic taxonomy



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The Tübingen conferences

- 1997 “Proof-Theoretic Semantics”
- 2013 “Advances in Proof-Theoretic Semantics”
- 2015 “General Proof-Theory”
- 2019 “Proof-Theoretic Semantics — Assessment and Future Perspectives”

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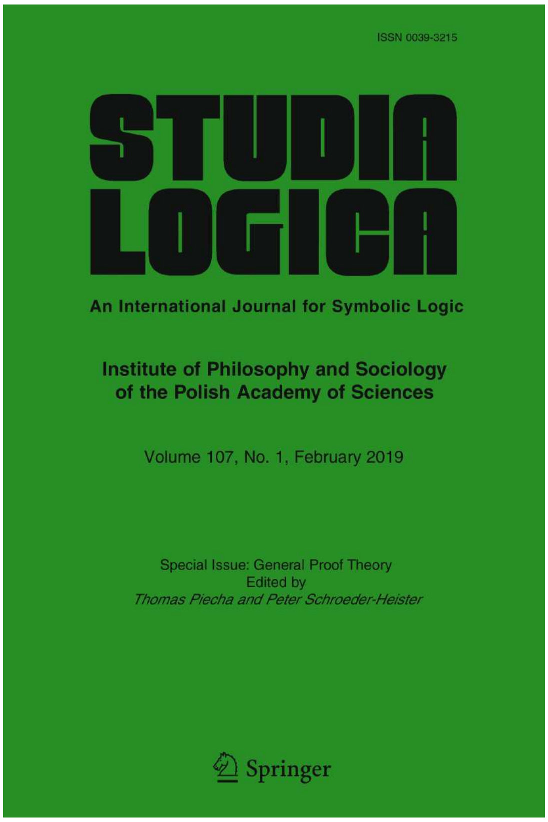

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The development of the community in proof-theoretic semantics

Proof-Theoretic Semantics 1997:

Had still the flavour of a *contradictio in adiecto*, both in mathematical and in philosophical logic

Proof-theoretic semantics 20 years after:

Has grown into a mature and well-recognised subject, in particular in philosophy.

“Proof-theoretic semantics” has become a common term which is considered natural.

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"Proof-theoretic semantics"

I am happy that "proof-theoretic semantics" caught.

I proposed it in 1985 to not leave the term "semantics" to the denotationalists alone — so not just Carnap, Tarski, ...

This is our second big terminological achievement in Tübingen, next to "substructural logic".
It is actually a 'natural' term, whereas 'substructural' is more artificial.

Terms are important.

"Substructural logic" even made it to the AMS Mathematical Subject Classification list (03B47). "Proof-theoretic semantics" not yet (but we haven't done anything to achieve this goal so far).

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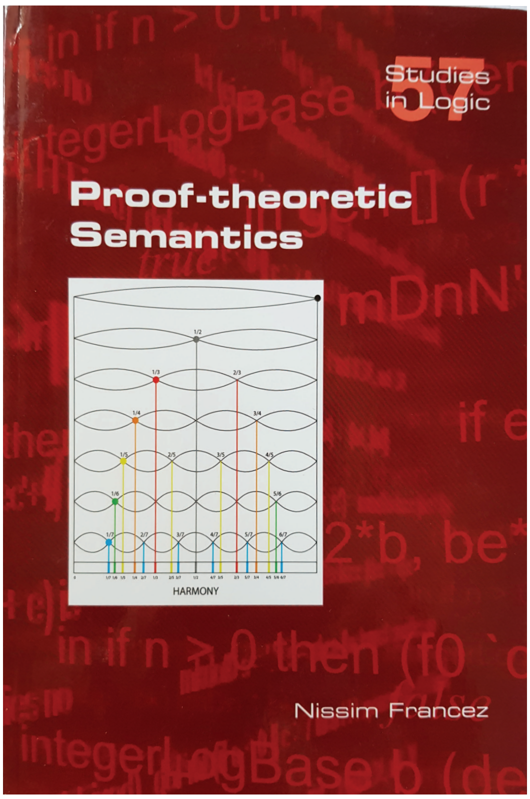

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Continuity

Four speakers have been present at all proof-theoretic semantics and general proof theory conferences since 20 years ago:

- Reinhard Kahle (co-organiser of the first conference)
- Per Martin-Löf
- Dag Prawitz
- Göran Sundholm

Unfortunately,

- Kosta Došen
- Roy Dyckhoff
- Lars Hallnäs
- Bill Tait

who attended the past conferences, were not able to come.

in memoriam

Of those who were present already at the first conference, four have died:

- Michael Dummett †2011
- Grigori Mints †2014
- Kosta Došen †2017
- Roy Dyckhoff †2018

To Kosta and Roy, who died recently, who had very close ties to our group and who would definitely be present, we will dedicate sessions of talks, given by speakers who had worked with him or worked in their specific field:

- To Roy on Thursday, with Sara Negri and Nissim Francez speaking
- To Kosta on Friday, with Phil Scott and Greg Restall speaking

The Tübingen group

is now represented at the meeting with contributed papers:

- Michael Arndt
- René Gazzari
- Lev Gordeev
- Hermógenes Oliveira
- Thomas Piecha
- Paolo Pistone
- Peter Schroeder-Heister
- Luca Tranchini
- Ernst Zimmermann

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The programme

is large both in terms of invited speakers and contributed papers.

That we received so many submissions, speaks for the attractiveness of the topic.

This means that we have to run parallel sessions, as we did not want to reject too many good contributions.

For the plenary speakers, we have 45 min per talk including discussion.
For the papers in parallel sessions, we have 30 min per talk.

Which means that the programme is pretty tight.

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Thanks: Financial support

DFG Deutsche Forschungsgemeinschaft
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I wish you

a good conference

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Paradoxes, Intuitionism, and Proof-Theoretic Semantics

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Proof-Theoretic Semantics III

Tübingen

27.3.2019

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Cantor

Cantor 1898

Unter einer *fertigen Menge* verstehe man jede Vielheit, bei welcher alle Elemente *ohne Widerspruch* als *zusammenseiend* und daher als *ein Ding für sich* gedacht werden können.

As a *finished set* one may consider each multiplicity for which all elements can be thought of as *being together without contradiction*, and therefore *as a thing in itself*.

- In this view, there are no paradoxes for Cantor: a resulting contradiction just shows that the mulitplicity is not a finished one.
- Thus, a "paradox" is a just *reductio-ad-absurdum* argument to show that the set in question doesn't exist.
- But such arguments are performed *a posteriori*.

Hilbert

- Hilbert did not agree with Cantor's perspective.

Hilbert 1917

Wenn wir einen mathematischen Beweis erst am Resultate auf seine Zulässigkeit prüfen können, so brauchen wir überhaupt keinen Beweis.

If we can verify the admissibility of a mathematical proof only at the result, we do not need any proof at all.

- The consistency of a set should be ensured "a priori".

Weyl on the Grelling–Nelson Paradox

Weyl 1918

But anyone who forgets that a proposition with such a structure can be meaningless is in danger of becoming trapped in absurdity—as a famous "paradox," essentially due to Russell, shows.

Let a word which signifies a property be called *autological* if this word itself possesses the property which it signifies; if it does not possess that property, let it be called *heterological*. . . .

Now what about the word "heterological" itself? If it is autological, then it has the property which it expresses and, so, is heterological. If, on the other hand, it is heterological, then it does not have this property and, so, is autological. Formalism regards this as an insoluble contradiction;

but in reality this is a matter of scholasticism of the worst sort: for the slightest consideration shows that absolutely no sense can be attached to the question of whether the word "heterological" is itself auto- or heterological.

Weyl on the Grelling–Nelson Paradox

- Weyl's warning, that propositions can be meaningless, can be taken as an indication that one would have to renounce the *tertium-non-datur* for such propositions.
- The natural choice seems to be intuitionistic logic, which is distinguished for leaving out the *tertium-non-datur* from classical logic.
- We will see, however, that intuitionistic logic is not of much help regarding the paradoxes.

Paradoxes in an intuitionistic setting

- Russell's Paradox: $R = \{X | X \notin X\}$.
- Interestingly, a simple inspection of the proof of contradiction from R shows that it does not depend on the *tertium-non-datur*: the proof can be carried out by logical reasoning which is intuitionistically valid.
- Thus, we cannot resolve Russell's Paradox by just replacing the underlying classical logic by intuitionistic logic.

- What about the Liar?
- For the Liar Paradox, the standard argument uses indeed classical logic, arguing by a case distinction on assuming that it is true or that it is false, both leading to contradictions.

Curry's Paradox

- In 1942, Curry presented a paradox to simplify the *Kleene-Rosser Paradox*.
- In essence, Curry's paradox is based in the definability of sentences expressing

This sentence implies φ.

for any sentence φ.

- Only requiring some very simple rules for implication one can obtain, from the defined sentence, φ.
- As, in this way, every formula of the system is derivable, the system is inconsistent.

Curry's Paradox

- Apparently, this paradox does not even involve negation.
- In particular, the Curry's reasoning is intuitionistically valid.
- Now, replace φ in Curry's sentence by $\bot$ (*falsum*):

This sentence implies $\bot$.

- But intuitionistic negation $\neg\varphi$ can be defined as $\varphi \to \bot$.
- Thus, Curry's paradox (using $\bot$) is nothing else than the Liar in intuitionistic terms.
- As for Russell, revoking the bivalence is not enough to ban the paradox.
- Thus, the paradoxes are independent of the underlying logic.

Concept formation (rather than logic)

- As changing from classical to intuitionistic logic does not resolve the paradoxes, one cannot held the logic responsible for them.
- In contrast, one may look to the **concept formation** involved in the paradoxes.
- In a classical setting, one takes care of concept formation by careful choices of interpretations in standard semantics.
  - This, however, involves either a staunch platonistic insight in the interpretation or, at least, a firm confidence in set-theoretic constructions for them.
- Also Brouwer can cope with the problem, if one puts the concept formation ahead of the logic.
  - This is in line with his idea that mathematics goes ahead of logic: "Mathematics is independent of logic" and "Logic depends upon mathematics".

Concept formation (Brouwer)

- Russell's paradox:
  - Brouwer clearly rejected Cantorian set theory as such and abstract set formation principles are plainly anti-intuitionistic.
- Liar and Curry's Paradox:
  - They depend on self-referential features of formal languages (Gödelization).
  - But such formal features are not the subject of intuitionism.
- Thus, the paradoxes may even support Brouwer's anti-logicistic convictions.
- This perspective also vindicates Weyl:
  - His criticism of (the scholasticism around) the Grelling–Nelson Paradox was intended, not to advocate many-valued logic, but rather to demand a careful delimitation of the "categories" (Weyl) to which a meaningful proposition is affiliated.

Historical note

- Brouwer's conception of intuitionism had nothing to do with the paradoxes.
  - Brouwer doesn't even mention the paradoxes, except for two instances to reject Cantorian or axiomatic set theory.
- In contrast, Hilbert's foundation research was indeed motivated by the paradoxes.
- Hilbert's search for consistency proofs was intended to block the paradoxes: "Consistency implies Existence."
- Brouwer, once, even granted him the possibility to carry out successfully consistency proofs.
  But they would be of no use in Brouwer's eyes, as:
  "Consistency does not provide Meaning."
- In essence, Hilbert and Brouwer talked at cross purposes on the foundations of mathematics.

Proof-Theoretic Semantics

Schroeder-Heister

- Proof-theoretic semantics "is based on the fundamental assumption that the central notion in terms of which meanings are assigned to certain expressions of our language, in particular to logical constants, is that of *proof* rather than *truth*. In this sense proof-theoretic semantics is *semantics in terms of proofs*"
- "Proof-theoretic semantics is intuitionistically biased"
- "Most forms of proof-theoretic semantics are intuitionistic in spirit, which means in particular that principles of classical logic such as the law of excluded middle or the double negation law are rejected or at least considered problematic."

- Proof-theoretic semantics is, thus, confronted with the paradoxes in the same way as intuitionism:
  - the underlying logic does not help.

Proof-Theoretic Semantics

Following the discussion of intuition, proof-theoretic semantics may deal with the paradoxes along the same lines:

Taming the concept formations.

- For Russell's Paradox, we have to provide a proof-theoretic semantics for the set formation principles, expecting that such a semantics blocks the possibility to introduce the "Russell set".

  $\rightarrow$ Such an approach was initiated by Hallnäs.
- The Liar requires, in its usual form, a truth predicate and, a forteriori, some form of *Gödelization*.

  $\rightarrow$ There are plenty of truth theories around and giving them a proof-theoretic semantics can be subsumed under the "open problem" of *Proof-Theoretic Semantics Beyond Logic*.

But there is another challange.

Paradoxical rules

- One may axiomatize the Liar directly, by introducing a self-contradicting atom R with $R \leftrightarrow \neg R$.
- **Paradoxical rules**

$$\frac{\Gamma, \neg R \ \vdash \ C}{\Gamma, R \ \vdash \ C} (R \vdash) \qquad\qquad \frac{\Gamma \ \vdash \ \neg R}{\Gamma \ \vdash \ R} (\vdash R)$$

- We are facing here a "*tonk*-like phenomenon" ($\rightarrow$ Tranchini): the definition of $(R \vdash)$ and $(\vdash R)$ would spoil the calculus.
- To deal with *tonk*, *proof-theoretic harmony* was conceived as a possible solution ($\rightarrow$ Dummett).
- But the two paradoxical rules appear to be in perfect harmony ($\rightarrow$ Read).

Normalizability

- Next to harmony, we may consider *normalizability of proofs*.
- Tennant used normalizability to block the paradoxical rules.
- A combination of harmony and normalizability might provide a common treatment of *tonk* and the paradoxical rules (→ Tranchini).
- Note, however, that normalizability is not any longer a *local* property, but a *global* one.
  - We cannot assign a proof-theoretic meaning to the connectives by solely inspect the given rules, but we have to prove properties of derivability in general.

Normalizability

- In principle, normalizability could be proven, for a given set of axioms, before performing the single proofs. But there are two problems:
  - Normalizability will be, in general, an undecidable property.
  - Proof-theoretic semantics would depend on such a (meta-)proof of normalizability, which we consider delicate with respect to the philosophical aims of proof-theoretic semantics.
- Using normalizability as a proof-theoretic principle, is there superordinate philosophical argument that this blocks *all* potential contradictions?
  - Are we better off than in set theory?
- Global proof-theoretic conditions, designed to ensure consistency, already failed! (→ Frege and others)

Definitional Freedom

- With reference to Hallnäs, Schroeder-Heister proposes a possible solution:

Definitional Freedom

- Under this freedom, one does not forbid any rules, but has to single out the "well-behaved" ones by (*a posteriori*) mathematical arguments.
- The situation would be analogous to recursion theory, where one does not forbid the definition of partial functions, but rather singles out, *a posteriori*, the functions which are total (or their domain).
- For recursion theory, there are formal systems to reason about partiality, namely *free logics* or the *logic of partial terms*.
- For logical calculi, there exist a largely forgotten attempt by Behmann, but a modern, worked out formalism is still a desideratum.

Definitional Freedom

- Incorporating the reasoning about the "well-behaviour" of definitions would, in fact, vindicate Weyl and, in part, Hilbert:
  - for Weyl, the slightest (or not so slight) consideration about the sense of a definition would turn explicit;
  - for Hilbert, the admissibility of (the concepts used in) a proof would not be checked at the result, but build in in the reasoning.
- One may note that paradoxes work with *locally correct reasoning*, reasoning which should be admitted.

Schroeder-Heister

This connects the proof theory of clausal definitions with theories of paradoxes, which conceive paradoxes as based on locally correct reasoning.

Conclusion

- Paradoxes are not phenomena of the underlying logic, but of the concept formations.
- In proof-theoretic semantics, reasoning about the concept formations has to be part of the game:

Schroeder-Heister

- We strongly propose definitional freedom in the sense that there should be one or several formats for definitions, but within this format one should be free.
- Whether a certain definition is well-behaved is a matter of (mathematical) 'observation', and not something to be guaranteed from the very beginning.

Cantor

Das Wesen der Mathematik liegt in ihrer Freiheit.

The essence of Mathematics is based in its freedom.

Moving the first steps towards the study of proofs-why

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March 27th 2019, Proof-Theoretic Semantics 2019, Tübingen, Germany

Introduction/Aim of the talk
From proofs-that to proofs-why
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The notion of proof, together with some other notions like truth or knowledge, is one of the milestone concepts of occidental logic and philosophy of science

Poggiolesi, F. Proof-why and Proof Theory

When it comes to proofs, (at least) one question seems to naturally arise

What do proofs serve for ?

Since the Antiquity, two different answers have been identified

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Proofs that

ἀπόδειξις ᾽ότι
Proofs *quia*

The goal of proofs is to show *that* something is true

Proofs why

ἀπόδειξις διότι
Proofs *propter quid*

The goal of proofs is to show *why* something is true

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The distinction seems easy to grasp when considered in the real word

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Proofs that

Proofs why

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Proofs that

Proofs why

these oranges are similar

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Proofs that

Proofs why

Are you sure ?

these oranges are similar

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Proofs that

Proofs why

These oranges have the same shape and the same size

these oranges are similar

Proofs that

Proofs why

These oranges have the same shape and the same size

It follows that

these oranges are similar

Proofs that

Proofs why

These oranges have the same shape and the same size

we can infer that

these oranges are similar

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Proofs that

Proofs why

These oranges have the same shape and the same size

we can infer that

these oranges are similar

these oranges are similar

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Proofs that

These oranges have the same shape and the same size

we can infer that

these oranges are similar

Proofs why

Why ?

these oranges are similar

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Proofs that

These oranges have the same shape and the same size

we can infer that

these oranges are similar

Proofs why

These oranges have the same shape and the same size

these oranges are similar

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Proofs that	Proofs why
These oranges have the same shape and the same size	These oranges have the same shape and the same size
we can infer that	are the reasons why
these oranges are similar	these oranges are similar

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Proofs that	Proofs why
These oranges have the same shape and the same size	These oranges have the same shape and the same size
we can infer that	explain why
these oranges are similar	these oranges are similar

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Proofs that	Proofs why
These oranges have the same shape and the same size	These oranges have the same shape and the same size
we can infer that	are the grounds for
these oranges are similar	these oranges are similar

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Proofs that		Proofs why
These oranges have the same shape and the same size		These oranges have the same shape and the same size
we can infer that	←	are the grounds for
these oranges are similar		these oranges are similar

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Proofs that	Proofs why
These oranges are similar	These oranges have the same shape and the same size
we can infer that	
theses oranges have the same size	these oranges are similar

Proofs that	Proofs why
These oranges are similar	These oranges have the same shape and the same size
$\updownarrow$	
theses oranges have the same size	these oranges are similar

Proofs that	Proofs why
These oranges are similar	These oranges have the same shape and the same size
↕	the converse does not hold
theses oranges have the same size	these oranges are similar

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▻ *The problem is that of understanding whether this contraposition which seems natural in an informal context, also makes sense or has a value in a formal - mathematical/logical - context*

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In the philosophical and mathematical tradition the attitude has been quite negative

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An exception to this attitude has been represented by the great mathematician and philosopher Bernard Bolzano who shaped his whole logic, mathematics and theory of science around this dichotomy

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Logic

$A \wedge B$ is inferable from A, B, as well as A and B are both inferable from $A \wedge B$. [B. Bolzano, WS, §221]

No one will ever think that the reason of the truth of A or the reason of the truth of B lies in the truth $A \wedge B$, while, when considering the inverse, one realizes that the reason of the truth $A \wedge B$ cannot but consist in A, B. [B. Bolzano, WS, §221]

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Despite his brilliant intuitions, the work of Bolzano remains incomplete

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General Project[1]

Our general project is to develop a neo-Bolzanian theory of proofs-why (or grounding) using contemporary conceptual and formal tools (in the sciences and not metaphysical !)

1. ANR JCJC Project *Insights from Bolzano*, 2019-2021.

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Our aim today

We will present some steps towards the formalization of proofs-why (or grounding) along the lines of what has already been done with proofs-that

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Proofs-that have been protagonist in the history of logic and mathematics

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Gentzen introduced the calculus of *natural deduction*, $\mathcal{ND}$, which can be thought of as our best formalization of the logical aspects of proofs-that

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Thus we have...

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(Logical aspects of) Proof-that	$\longrightarrow$	Meta-linguistic relation called *derivation*, $\vdash$
Each inference step	$\longrightarrow$	Introduction/elimination rules $\dfrac{A_1, ..., A_m}{B}$
Linguistic counterpart of proof-that (via deduction theorem)	$\longrightarrow$	Material implication $A_1 \wedge ... \wedge A_m \rightarrow B$

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Given the strict link between proofs-that and proofs-why and the great developments of the research on proofs-that

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▻ The general insight is to do the same with proof-why

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(Logical aspects of) Proof-why → Meta-linguistic relation called *formal explanation*, $|\!\sim$

Each grounding step → Grounding rules

$$\frac{A_1, \ldots A_m}{B}$$

Linguistic counterpart of proof-why (via deduction theorem) → Connective to be read in terms of *because*

$$A_1 \wedge \ldots \wedge A_m \rhd B$$

Please see *F. Poggiolesi, On constructing a logic for the notion of complete and immediate formal grounding, Synthese, 195 : 1231 - 1254, 2018*

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(Logical aspects of) Proof-why → Meta-linguistic relation called *formal explanation*, $|\!\sim$

Each grounding step → Grounding rules

$$\frac{A_1, \ldots A_m}{B}$$

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$$\frac{A_1, ..., A_m}{B}$$

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$$\frac{A_1, ...A_m}{\overbrace{}B}$$

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But how do we pass from $\dfrac{A_1, ..., A_m}{B}$ to $\underset{B}{\overset{A_1, ...A_m}{\smile}}$?

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$\underset{B}{\overset{A_1, ...A_m}{\smile}}$

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$$\frac{A_1, ...A_m}{B}$$

1. Positive Derivability

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$$\frac{A_1, ...A_m}{B}$$

1. $A_1, ..., A_m \vdash B$

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$$\underbrace{A_1, \ldots A_m}_{B}$$

1. Linked to the dichotomy with proofs-that

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$$\underbrace{A_1, \ldots A_m}_{B}$$

2. Negative Derivability

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$$\underbrace{A_1, ...A_m}_{B}$$

2. $\neg A_1, ..., \neg A_m \vdash \neg B$

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$$\underbrace{A_1, ...A_m}_{B}$$

2. Linked to the counterfactual dependence proper of explanatory step (see Lewis, Woodward, Reutlinger, Saatsi)

$$\underbrace{A_1, ...A_m}_{B}$$

2. It can be weakened, as shown in Poggiolesi and Fancez (2019)[2]

2. F. Poggiolesi and N. Francez, Towards a general definition of logic dependent truths-grounding, Erkentnnis, submitted.

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$$\underbrace{A_1, ...A_m}_{B}$$

3. Complexity

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$$\frac{A_1, \ldots A_m}{B}$$

3. Refinement of notion of logical complexity, but it is a long and laborious definition[2]

2. F. Poggiolesi, On defining the notion of complete and immediate grounding, Synthese, 193 : 3147 - 3167, 2016

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$$\frac{A_1, \ldots A_m}{B}$$

3. Linked to the old idea that in explanation complexity always increases, to be found in Descartes, Leibniz, Arnauld, Pascal, Bolzano, ...

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Proofs-why (or grounding) = dependence [given by positive and negative derivability] + directionality [given by complexity]

In order to formulate grounding rules, our starting point are the introduction rules of the classical natural deduction calculus (because of positive derivability and complexity)

$$\dfrac{\begin{matrix}\vdots & \vdots \\ A & B\end{matrix}}{A \wedge B}\ \wedge I$$

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B}\ \vee I \qquad\qquad \dfrac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B}\ \vee I$$

$$\dfrac{\begin{matrix}\not{A} \\ \vdots \\ \bot\end{matrix}}{\neg A}\ \neg I$$

Conjunction

$$\widetilde{A \wedge B}$$

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Conjunction

$$\widetilde{A \wedge B}$$

positive derivability

Conjunction

$$\widetilde{A \wedge B}$$

positive derivability
negative derivability

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Conjunction

$$\widetilde{A \wedge B}$$

positive derivability
negative derivability
complexity

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Conjunction

$$\frac{A, B}{A \wedge B}$$

positive derivability
negative derivability
complexity

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Conjunction

$$\frac{A, B}{A \wedge B}$$

positive derivability $\lightning$
negative derivability
complexity

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Conjunction

$$\dfrac{A, B}{A \wedge B}$$

positive derivability $\lightning$
negative derivability $\lightning$
complexity

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Conjunction

$$\dfrac{A, B}{A \wedge B}$$

positive derivability $\lightning$
negative derivability $\lightning$
complexity $\lightning$

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Disjunction

$$\widetilde{A \vee B}$$

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Disjunction

$$\widetilde{A \vee B}$$

positive derivability

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Disjunction

$$\widetilde{A \vee B}$$

positive derivability
negative derivability

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Disjunction

$$\widetilde{A \vee B}$$

positive derivability
negative derivability
complexity

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Disjunction

$$\frac{A}{A \vee B}$$

positive derivability
negative derivability
complexity

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Disjunction

$$\frac{A}{A \vee B}$$

positive derivability ↯
negative derivability
complexity

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Disjunction

$$\frac{A}{A \vee B}$$

positive derivability ↯
negative derivability
complexity

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Disjunction

$$\frac{A}{A \vee B}$$

positive derivability ↯
negative derivability
complexity

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How can we change the rule so to get the grounding relation ?

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What do we add to the premise ?

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Consider a disjunctive truth like $p \vee q$ and suppose that p is true
p then formally explains $p \vee q$. But is there anything else that explains $p \vee q$?

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The answer would seem to depend on the truth of q

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If q is true, then p and q together are the *complete* explanation of $p \vee q$

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And if q is false ? In this case, p would seem to constitute the complete explanation of $p \vee q$: but this is only because q is false

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In the case where q is false, it still has a role to play in determining the grounds of $p \vee q$: its falsity *ensures that* or is *a condition for* p to be the complete explanation for $p \vee q$

To capture this role, we shall say that p is the complete explanation for $p \vee q$ under the robust condition[2] that q is false

2. As remarked by Francez, robust conditions can be seen as *contexts*.

Disjunction

$$\widetilde{A \vee B}$$

positive derivability
negative derivability
complexity

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Disjunction

$$\frac{A, B}{\widetilde{A \vee B}}$$

positive derivability
negative derivability
complexity

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Disjunction

$$\frac{A, B}{A \vee B}$$

positive derivability ↯
negative derivability
complexity

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Disjunction

$$\frac{A, B}{A \vee B}$$

positive derivability ↯
negative derivability ↯
complexity

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Disjunction

$$\frac{A, B}{\widetilde{A \vee B}}$$

positive derivability ↯
negative derivability ↯
complexity ↯

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Disjunction

$$\widetilde{A \vee B}$$

positive derivability
negative derivability
complexity

Poggiolesi, F. Proof-why and Proof Theory

Disjunction

$$\frac{[\neg B]\ A}{A \vee B}$$

positive derivability
negative derivability
complexity

Introduction/Aim of the talk
From proofs-that to proofs-why
Grounding rules
Conclusion

Disjunction

$$\frac{[\neg B]\ A}{A \vee B}$$

positive derivability $\lightning$
negative derivability
complexity

Poggiolesi, F. Proof-why and Proof Theory

Disjunction

$$\frac{[\neg B]\ A}{A \vee B}$$

positive derivability ↯
negative derivability ↯
complexity

Disjunction

$$\frac{[\neg B]\ A}{A \vee B}$$

positive derivability ↯
negative derivability ↯
complexity ↯

Disjunction

$$\frac{[\neg B]A}{A \vee B} \qquad \frac{[\neg A]B}{A \vee B} \qquad \frac{A, B}{A \vee B}$$

positive derivability ↯
negative derivability ↯
complexity ↯

Introduction/Aim of the talk
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Negation

An adequate formulation for the grounding rule of negation is an open problem in the current literature on metaphysical grounding but also in some passages of the WS of Bolzano

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Negation

$\widetilde{\neg A}$

positive derivability
negative derivability
complexity

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Negation

$\widetilde{\neg A}$

positive derivability
negative derivability
complexity

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Negation

$$\frac{\not A}{\neg A}$$

positive derivability
negative derivability
complexity

INTRODUCTION/AIM OF THE TALK
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Negation

$$\frac{B}{\neg A}$$

positive derivability
negative derivability
complexity

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Negation

$$\frac{B?}{\neg A}$$

positive derivability
negative derivability
complexity

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Negation

$$\begin{array}{c} A,\ B \\ \vdots \\ \bot \end{array}$$

positive derivability
negative derivability
complexity

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Negation

$$\begin{array}{c} A,\ B \\ \vdots\vdots \\ \bot \end{array}$$

positive derivability ↯
negative derivability ↯
complexity ↯

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Negation

$$\begin{array}{c} \not{A}, B \\ \vdots\vdots \\ \bot \\ \hline B \\ \widetilde{\quad} \\ \neg A \end{array}$$

positive derivability ↯
negative derivability ↯
complexity ↯

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Negation[3]

$$\begin{array}{c} \not{A}, \not{B} \\ \vdots \\ \bot \\ \hline B \\ \hline \neg A \end{array}$$

positive derivability ↯
negative derivability ↯
complexity ↯

3. As in P. Schroeder-Heister (1984)

Introduction/Aim of the talk
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What is (are) the reason(s) for "it is not raining" to be true ?
The reason is that it is sunny, if it is sunny and it is raining lead to a contradiction
What is the reason for "the wall is not black" to be true ?
The reason is that the wall is white, if the wall is black and the wall is white lead to a contradiction

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Grounding Rules

$$\begin{array}{c} A, B \\ \sim \\ A \wedge B \end{array}$$

$$\begin{array}{c} [\neg B] A \\ \sim \\ A \vee B \end{array} \qquad \begin{array}{c} [\neg A] B \\ \sim \\ A \vee B \end{array} \qquad \begin{array}{c} A, B \\ \sim \\ A \vee B \end{array}$$

$$\begin{array}{c} [\neg D]\ A, B, \not{C} \\ \vdots \\ \bot \\ \hline [D]\ A, B \\ \sim \\ \neg C \end{array}$$

Introduction/Aim of the talk
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Grounding rules
Conclusion

Grounding Rules

$$\begin{array}{c} A, B \\ \sim \\ A \wedge B \end{array}$$

$$\begin{array}{c} [\neg B] A \\ \sim \\ A \vee B \end{array} \qquad \begin{array}{c} [\neg A] B \\ \sim \\ A \vee B \end{array} \qquad \begin{array}{c} A, B \\ \sim \\ A \vee B \end{array}$$

$$\begin{array}{c} [\neg D]\ A, B, \not{C} \\ \vdots \\ \bot \\ \hline [D]\ A, B \\ \sim \\ \neg C \end{array}$$

Where in each rule all formulas that occur above the tilde form a consistent multiset

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Future Research

- Extension to quantifiers
- Generalization of the approach so to cover connectives which are not classical (e.g. relevant implications !) or the standard ones
- Adequate semantics for the grounding rules
- Links with Bolzano's work
- Links with causality and mathematical explanatory proofs

Introduction/Aim of the talk
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THANK YOU

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The Calculus of Natural Calculation

René Gazzari, University of Tübingen

Proof-Theoretic Semantics
27. March 2019

Motivation: A Natural Calculus

1. *Natural Deduction:* Gentzen's intention is "*to set up a formal system which comes as close as possible to actual reasoning*" (Collected Papers, p. 68) $\rightsquigarrow$ Natural Deduction.
2. *argumentations:* The representation of argumentations in Natural Deduction is natural.
3. *reality:* Mathematicians do not only argue in their proofs, but they also calculate.
4. *calculations:* The representation of calculations in Natural Deduction via the traditional identity rules is not natural. $\rightsquigarrow$ Calculus of Natural Calculation with a natural representation of calculations.

Analysis: Representation of Calculations

Informal Mathematical Reasoning with Terms

1. *calculate:* $t_1 \overset{\text{(Justification)}}{=} t_2 \overset{\text{(Justification)}}{=} t_3 \overset{\text{(Justification)}}{=} t_4$
2. *evaluate:* By the transitivity of equality, we infer $t_1 = t_4$.

Traditional Formalisation I (Natural Deduction)

$$\dfrac{\dfrac{\dfrac{\text{Reason}}{t_1 = t_2} \quad \dfrac{\text{Reason}}{t_2 = t_3}}{t_1 = t_3} \quad \dfrac{\text{Reason}}{t_3 = t_4}}{t_1 = t_4}$$

Analysis: Representation of Calculations

Informal Mathematical Reasoning with Terms

1. *calculate:* $t_1 \overset{\text{(Justification)}}{=} t_2 \overset{\text{(Justification)}}{=} t_3 \overset{\text{(Justification)}}{=} t_4$
2. *evaluate:* By the transitivity of equality, we infer $t_1 = t_4$.

Traditional Formalisation II (Natural Deduction)

$$\dfrac{\dfrac{\text{Reason}}{t_1 = t_2} \quad \dfrac{\dfrac{\text{Reason}}{t_2 = t_3} \quad \dfrac{\text{Reason}}{t_3 = t_4}}{t_2 = t_4}}{t_1 = t_4}$$

Analysis: Representation of Calculations

Informal Mathematical Reasoning with Terms

1. *calculate:* $t_1 \overset{\text{(Justification)}}{=} t_2 \overset{\text{(Justification)}}{=} t_3 \overset{\text{(Justification)}}{=} t_4$
2. *evaluate:* By the transitivity of equality, we infer $t_1 = t_4$.

Intended Formalisation - Calculation

$$\dfrac{\dfrac{\dfrac{t_1 \qquad \text{Justification}}{t_2}\,(=) \qquad \text{Justification}}{t_3}\,(=) \qquad \text{Justification}}{t_4}\,(=)$$

Introduction
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A Theory of Occurrences

Analysis: Representation of Calculations

Informal Mathematical Reasoning with Terms

1. *calculate:* $t_1 \overset{\text{(Justification)}}{=} t_2 \overset{\text{(Justification)}}{=} t_3 \overset{\text{(Justification)}}{=} t_4$
2. *evaluate:* By the transitivity of equality, we infer $t_1 = t_4$.

Intended Formalisation - Evaluation

$$\dfrac{\dfrac{\dfrac{\dfrac{[t_1]^1 \qquad \text{Justification}}{t_2}\,(=) \qquad \text{Justification}}{t_3}\,(=) \qquad \text{Justification}}{t_4}\,(=)}{t_1 = t_4}\,(1)$$

Analysis: Intended Representation

1. *calculations:* Representation of a calculation by a sequence of calculation steps (linear, extendable, corresponding to the informal calculation).
2. *evaluation:* Representation of the evaluation step, in which the result of a calculation is inferred.
3. *dealing with terms:* Avoiding the artificial intermediate equations by direct transformations of terms.

Method: Marking Positions

1. *marking positions:* We need a good method to mark positions (of occurrences of subterms in terms). $\rightsquigarrow$ nominal forms as defined by Schütte.
2. *principle idea:* the position of a subterm in a term is represented by a nominal term in which the intended subterm is replaced by a nominal symbol $*$.
3. *examples:*
   1. $(0+0)+0 \rightsquigarrow (0+*)+0$
   2. $(0+0)+0 \rightsquigarrow (*+0)+*$
   3. $(0+0)+0 \rightsquigarrow *+0$

Nominal Forms

1. *nominal terms:* *Nominal terms* are defined as standard terms, but with an additional clause: $*$ is an atomic nominal term.
2. *general substitution function:* If $\mathtt{t}, \mathtt{s}$ are nominal terms, then $\mathtt{t}[\mathtt{s}]$ is the result of replacing all $*$ in $\mathtt{t}$ by $\mathtt{s}$. (recursive definition)
3. *examples:*

$$*[* + 0] \;\doteq\; * + 0 \quad ; \quad * + 0[0 + 0] \;\doteq\; (0 + 0) + 0$$

4. *position:* A nominal term $\mathtt{t}$ is the *position* of a term s in a standard term t, if $\mathtt{t}[s] \doteq t$.

Remarks (Nominal Forms)

1. *nominal formulae:* the introduced concepts are easily carried over to nominal formulae $\mathtt{A}$.
2. *theory of occurrences:* a generalisation of these concepts is suitable for a theory of occurrences
3. *advantage:* with these concepts, a precise formulation of inference rules is possible

New Rules – Basic Term Rules

1. *new atomic derivation:* Every term t is a derivation.

2. *calculation step (positive ; negative):*

$$(E{=})\ \frac{\mathtt{r}[t] \qquad t = s}{\mathtt{r}[s]} \quad ; \quad (E{=})\ \frac{s = t \qquad \mathtt{r}[t]}{\mathtt{r}[s]}$$

   Substitution of some occurrences of t in $r \doteq \mathtt{r}[t]$.

3. *evaluation step:*

$$\cfrac{\begin{matrix}[t] \\ \vdots \\ s\end{matrix}}{t = s}\ (I{=})$$

Introduction
The Calculus of Natural Calculation
Smaller-Than Relation
Introduction of the Basic Term Rules
Auxiliary Calculations
Important Results

Remarks (Term Rules)

1. *discharge of terms:* The discharge of terms is mandatory.
2. *immediate introduction:* Having assumed a term t, an immediate introduction of the equation $t = t$ is possible. This corresponds with reflexivity of the identity relation.
3. *positive/negative rules:* The positive and negative rules correspond with the symmetry of the identity relation.
4. *extendability:* We can extend a calculation by arbitrary calculation step. This corresponds to transitivity of the identity relation.

Arithmetical Example

1. In the next example, we provide a formal proof in the new calculus for:

$$\mathrm{PA} \Vdash \forall x.\, x + 0 = 0 + x$$

2. We use the following axioms of PA:
   1. $(A_1) \;\; \risingdotseq \;\; \forall x.\, x + 0 = x$
   2. $(A_2) \;\; \risingdotseq \;\; \forall xy.\, x + S(y) = S(x + y)$
3. We only provide the induction base and the inductive step; the rest follows by induction schema.

Commutativity of Addition with 0

Statement: $\mathrm{PA} \Vdash \forall x : x + 0 = 0 + x$

- Infer $A(0) \risingdotseq 0 + 0 = 0 + 0$:

$$\frac{[0 + 0]}{0 + 0 = 0 + 0}\,(I{=})$$

Commutativity of Addition with 0

Statement: $\mathrm{PA} \Vdash \forall x : x + 0 = 0 + x$

- Assume $A(x) \doteq x + 0 = 0 + x$.
- Infer $A(S(x)) \doteq S(x) + 0 = 0 + S(x)$:

$$\dfrac{\dfrac{\dfrac{A_2}{0+S(x)=S(0+x)} \qquad \dfrac{\dfrac{\dfrac{A_1}{x+0=x} \qquad \dfrac{[S(x)+0] \qquad \dfrac{A_1}{S(x)+0=S(x)}}{S(x)}}{S(x+0)} \qquad x+0=0+x}{S(0+x)}}{0+S(x)}}{S(x)+0=0+S(x)}\ (I{=})$$

More Rules

- We introduce some more rules reflecting that the identity relation is a congruence relation.

New Rules – Auxiliary Calculations

4. *auxiliary calculations - term (positive ; negative):*

$$\dfrac{\mathtt{r}[t] \qquad \begin{matrix}[t]\\ \vdots \\ s\end{matrix}}{\mathtt{r}[s]}\,(\mathrm{A}_{\mathrm{trm}}) \quad ; \quad \dfrac{\begin{matrix}[s]\\ \vdots \\ t\end{matrix} \qquad \mathtt{r}[t]}{\mathtt{r}[s]}\,(\mathrm{A}_{\mathrm{trm}})$$

Substitution of some occurrences of t in $r \simeq \mathtt{r}[t]$.

5. *auxiliary calculation - formula (positive ; negative):*

$$\dfrac{\mathtt{A}[t] \qquad \begin{matrix}[t]\\ \vdots \\ s\end{matrix}}{\mathtt{A}[s]}\,(\mathrm{A}_{\mathrm{fml}}) \quad ; \quad \dfrac{\begin{matrix}[s]\\ \vdots \\ t\end{matrix} \qquad \mathtt{A}[t]}{\mathtt{A}[s]}\,(\mathrm{A}_{\mathrm{fml}})$$

Substitution of some occurrences of t in $A \simeq \mathtt{A}[t]$.

Basic Proof-Theoretic Results

Some Propositions

1. Every calculation can be transformed into a dual calculation (bottom up) with the same justifications (but having alternated positions).
2. Every calculation can be transformed into a linear calculation by integrating auxiliary calculations.
3. The calculus is complete and sound (with respect to Natural Deduction with traditional identity rules).

Motivation: Smaller-Than Calculations

1. *other relations:* We are motivated to consider term rules for other relation symbols representing other relations.
2. *correspondence:* We want to see how the features of the calculus correspond with properties of the formalised relation.

$\rightsquigarrow$ Investigation of smaller-than calculations in the additive fragment of the ring of integers.

Language of the Ring of Integers

Non-Logical Symbols

1. *constant symbols:* 0, 1
2. *function symbols:* $+$ (binary) and $-$ (unary)
3. *relations symbol:* $<$

Positive and Negative Positions

1. *intention:* A p-form p marks a positive position in a term (increasing the respective subterm increases the full term), an n-form n marks a negative position.

Definition

1. $*$ is a p-form.
2. If t is a p-form (an n-form), then also $\mathtt{t}[* + s]$, $\mathtt{t}[s + *]$.
3. If t is a p-form (an n-form), then $\mathtt{t}[-*]$ is an n-form (a p-form).

1. *Schütte:* The definition is motivated by Schütte (p-form and n-form with respect to implications).
2. *syntactical definition:* The definition is purely syntactical, and depends neither on provability nor on validity.

More Term Rules

1. *elimination of $<$:*

$$(<)\ \frac{s < r \qquad \mathtt{n}[r]}{\mathtt{n}[s]} \quad ; \quad (<)\ \frac{\mathtt{p}[s] \qquad s < r}{\mathtt{p}[r]}$$

2. *introduction of $<$:*

$$\frac{\begin{array}{c}[t]\\ s\end{array}}{t < s}$$

Side conditions:

- The discharge of t is mandatory.
- We discuss later the consequences of some restrictions on the length of calculations.

Example

$\Vdash \forall x.\ 0 < x \to -x < 0$

$$\dfrac{\dfrac{\dfrac{\dfrac{[0 < x]^2 \qquad [-x]^1}{0}}{-x < 0}\,(1)}{0 < x \to -x < 0}\,(2)}{\forall x.\ 0 < x \to -x < 0}$$

1. Observe: $-x \doteqdot - * [x]$, and $-* \doteqdot *[-*]$ is an n-form.

First Results

Compatibility Axioms

T_C is the set of the following axioms:

1. *addition:* $\forall xyz.\ x < y \to x + z < y + z$
   $\forall xyz.\ x < y \to z + x < z + x$
2. *negation:* $\forall xy.\ x < y \to -y < -x$

One-step calculations are complete and sound with respect to T_C.

1. *completeness:* $\Vdash_1 T_C$
2. *soundness:* For all one-step calculations $\mathfrak{C}$: $T_C, \mathbf{hyp}(\mathfrak{C}) \vdash \mathbf{res}(\mathfrak{C})$

Transitivity

Transitivity

Presupposing T_C-completeness, two-step calculations are equivalent to transitivity.

1. *completeness:* $\Vdash_2 A_{\text{tr}}$

$$\dfrac{\dfrac{\dfrac{[x]^1 \quad \dfrac{x < y \wedge y < z}{x < y}}{y} \quad \dfrac{x < y \wedge y < z}{y < z}}{z}}{x < z}\,(1)$$

2. *soundness:* For all two-step calculations $\mathfrak{C}$:

$$T_C + A_{\text{tr}},\ \mathbf{hyp}(\mathfrak{C}) \vdash \mathbf{Res}(\mathfrak{C})$$

Reflexivity

Reflexivity

Immediate introductions (zero-step calculations) are equivalent to reflexivity.

1. *completeness:* $\Vdash_0 A_{\text{ref}}$

$$\dfrac{\dfrac{[x]}{x < x}}{\forall x.\, x < x}$$

2. *soundness:* $A_{\text{ref}} \vdash t < t$ for all terms t.

Remarks

1. Subsequently, we discuss the strict version of smaller-than.

Anti-Symmetry

Rule of Anti-Symmetry

$$\dfrac{\begin{matrix}[t] \\ s\end{matrix} \qquad \begin{matrix}[s] \\ t\end{matrix}}{\bot}\ (<)$$

1. The rule of anti-symmetry is complete.
2. Presupposing T_C-completeness, the rule of anti-symmetry is sound.

Example: Anti-Reflexivity ($\Vdash \forall x.\ \neg x < x$)

$$\dfrac{\dfrac{\dfrac{\dfrac{[x]^1 \qquad [x < x]^2}{x} \qquad \dfrac{[x]^1 \qquad [x < x]^2}{x}}{\bot}\ (1)}{\neg x < x}\ (2)}{\forall x.\ \neg x < x}$$

Linearity and Orientation

Rules of Linearity and Orientation

$$\dfrac{\begin{matrix}[t < s] \\ A\end{matrix} \qquad \begin{matrix}[t = s] \\ A\end{matrix} \qquad \begin{matrix}[s < t] \\ A\end{matrix}}{A} \quad ; \quad \dfrac{}{0 < 1}$$

Remarks:

1. Both rules are standard (formulae) rules.
2. Completeness and soundness of the rule of linearity due to the inference rules of disjunction.
3. Completeness and soundness of the rule of orientation is immediate.

Example: Calculation in the Ring of Integers

$T_R \Vdash \forall xyz.\ x+z<y+z \ \rightarrow\ x<y$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x]^1}{(x+z)-z}\ (T_R{:=}) \qquad [x+z<y+z]^2}{(y+z)-z}\ (\dagger{:}<)}{y}\ (T_R{:=})}{x<y}\ (1)}{\dfrac{x+z<y+z \rightarrow x<y}{\forall xyz.\ x+z<y+z\ \rightarrow\ x<y}}\ (2)$$

($\dagger$) Observe that $* - z \doteq * + (-z) \doteq *[* + (-z)]$ is a p-form.

Conclusion

1. *Natural Calculations:* We have introduced the calculus of Natural Calculations for a natural representation of equality calculations.
2. *Smaller-Than Relation:* We extended the calculus by incorporating smaller-than calculations.
3. *Correspondence:* We investigated the correspondence between some properties of the smaller-than relation and the features of the calculus.

The Meaning of Proofs in Different Proof Systems

Sara Ayhan (Ruhr University Bochum)

Proof-Theoretic Semantics:
Assessment and Future Perspectives
Third Tübingen Conference on Proof-Theoretic Semantics

27 March, 2019

Main Questions

What is the meaning of proofs in a proof-theoretic semantics account?

- How do we get from meaning of logical constants to meaning of proofs as a whole?

→ Meaning of proof must be in some way based on rules of inference it contains

→ Approach proposed by Tranchini: distinguishes for a derivation to have a denotation (a proof object it refers to) and to have sense: "being constituted of applications of correct inferences rules" (Tranchini 2016: 508)

Main Questions

- What exactly does the sense of a derivation consist of?
- → Can we make a distinction between sense and denotation of proofs analogous to Frege's distinction for singular terms or sentences?
- What is the relation of different kinds of proof systems with respect to such a distinction?
- → Do two derivations with the same denotation in different proof systems always differ in sense or can sense be shared over two proof systems?

The basic approach

- There can be different ways to go from the same premises to the same conclusion, not only in different (kinds of) proof systems but also within one proof system (cf. Restall 2017 for an approach in classical logic)
- Focus so far:
  - Normal vs. non-normal proofs in Natural Deduction (ND)
  - Proofs containing an application of the cut rule vs. cut-free proofs in Sequent Calculus (SC)
- However, this can also happen due to changing the order of rule applications
- → Does this change the denotation of the derivation, i.e. the proof it refers to, or only the sense, i.e. the way the inference is built up?

The basic approach

- Encoding the proof systems with λ-terms

$\rightarrow$ makes sense and denotation transparent:

  - **Denotation** is referred to by the normal form of the term that denotes the sequent or formula to be proven $\rightarrow$ henceforth: the 'end-term'

  $\rightarrow$ Two derivations with β-equivalent end-terms denote the same proof

  - Concerning **sense**, usually the difference between normal and non-normal terms is mentioned (Girard 1990, Tranchini 2016, Restall 2017)

Different senses: normal vs. non-normal terms

$\mathrm{ND}_{p \supset p}$

$$\dfrac{[x : p]}{\lambda x.x : p \supset p}\supset\mathrm{I}$$

$\mathrm{ND}_{\text{non-normal}\, p \supset p}$

$$\dfrac{\dfrac{\dfrac{[x : p]}{\lambda x.x : p \supset p}\supset\mathrm{I} \quad \dfrac{[y : q]}{\lambda y.y : q \supset q}\supset\mathrm{I}}{\langle \lambda x.x, \lambda y.y \rangle : (p \supset p) \wedge (q \supset q)}\wedge\mathrm{I}}{fst(\langle \lambda x.x, \lambda y.y \rangle) : p \supset p}\wedge\mathrm{E}$$

$fst(\langle \lambda x.x, \lambda y.y \rangle) \rightsquigarrow \lambda x.x$

Different senses: normal vs. non-normal terms

SC$\vdash (p \wedge p) \supset (p \vee p)$

$$\dfrac{\dfrac{\dfrac{\overline{z : p \vdash z : p}}{y : p \wedge p \vdash fst(y) : p}\,\wedge\mathrm{L}}{y : p \wedge p \vdash \mathtt{inl}\, fst(y) : p \vee p}\,\vee\mathrm{R}}{\vdash \lambda y.\mathtt{inl}\, fst(y) : (p \wedge p) \supset (p \vee p)}\,\supset\mathrm{R}$$

SC$_{\text{cut}}\vdash (p \wedge p) \supset (p \vee p)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{z : p \vdash z : p}}{y : p \wedge p \vdash fst(y) : p}\,\wedge\mathrm{L} \quad \dfrac{\overline{z : p \vdash z : p}}{y : p \wedge p \vdash snd(y) : p}\,\wedge\mathrm{L}}{y : p \wedge p, y : p \wedge p \vdash \langle fst(y), snd(y)\rangle : p \wedge p}\,\wedge\mathrm{R}}{y : p \wedge p \vdash \langle fst(y), snd(y)\rangle : p \wedge p}\,\mathrm{C} \quad \dfrac{\overline{z : p \vdash z : p}}{y : p \wedge p \vdash fst(y) : p}\,\wedge\mathrm{L}}{y : p \wedge p \vdash fst\,\langle fst(y), snd(y)\rangle : p}\,\mathrm{cut}}{y : p \wedge p \vdash \mathtt{inl}\, fst\,\langle fst(y), snd(y)\rangle : p \vee p}\,\vee\mathrm{R}}{\vdash \lambda y.\mathtt{inl}\, fst\,\langle fst(y), snd(y)\rangle : (p \wedge p) \supset (p \vee p)}\,\supset\mathrm{R}$$

$\lambda y.\mathtt{inl}\, fst\,\langle fst(y), snd(y)\rangle \rightsquigarrow \lambda y.\mathtt{inl}\, fst(y)$

The basic approach

- Encoding the proof systems with λ-terms

$\rightarrow$ makes connection between change of order of rule applications and sense-denotation-distinction transparent:

  - In SC it is often possible to change the order of rule application
  - However, this does not necessarily lead to a different proof (in ND it does)
  - Sometimes it does, sometimes it doesn't: we need terms to distinguish the cases

Distinguishing sense and denotation of proofs

Are these different proofs?

$SC_1 \vdash ((q \land r) \lor p) \supset ((p \lor q) \land (p \lor r))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,\mathrm{Rf}}{q \vdash p \lor q}\,\lor\mathrm{R}}{q \land r \vdash p \lor q}\,\land\mathrm{L} \quad \dfrac{\dfrac{\dfrac{}{r \vdash r}\,\mathrm{Rf}}{r \vdash p \lor r}\,\lor\mathrm{R}}{q \land r \vdash p \lor r}\,\land\mathrm{L}}{q \land r, q \land r \vdash (p \lor q) \land (p \lor r)}\,\land\mathrm{R}}{q \land r \vdash (p \lor q) \land (p \lor r)}\,\mathrm{C} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\mathrm{Rf}}{p \vdash p \lor q}\,\lor\mathrm{R} \quad \dfrac{\dfrac{}{p \vdash p}\,\mathrm{Rf}}{p \vdash p \lor r}\,\lor\mathrm{R}}{p, p \vdash (p \lor q) \land (p \lor r)}\,\land\mathrm{R}}{p \vdash (p \lor q) \land (p \lor r)}\,\mathrm{C}}{(q \land r) \lor p \vdash (p \lor q) \land (p \lor r)}\,\lor\mathrm{L}}{\vdash ((q \land r) \lor p) \supset ((p \lor q) \land (p \lor r))}\,\supset\mathrm{R}$$

$SC_2 \vdash ((q \land r) \lor p) \supset ((p \lor q) \land (p \lor r))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,\mathrm{Rf}}{q \land r \vdash q}\,\land\mathrm{L}}{q \land r \vdash p \lor q}\,\lor\mathrm{R} \quad \dfrac{\dfrac{\dfrac{}{r \vdash r}\,\mathrm{Rf}}{q \land r \vdash r}\,\land\mathrm{L}}{q \land r \vdash p \lor r}\,\lor\mathrm{R}}{q \land r, q \land r \vdash (p \lor q) \land (p \lor r)}\,\land\mathrm{R}}{q \land r \vdash (p \lor q) \land (p \lor r)}\,\mathrm{C} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\mathrm{Rf}}{p \vdash p \lor q}\,\lor\mathrm{R} \quad \dfrac{\dfrac{}{p \vdash p}\,\mathrm{Rf}}{p \vdash p \lor r}\,\lor\mathrm{R}}{p, p \vdash (p \lor q) \land (p \lor r)}\,\land\mathrm{R}}{p \vdash (p \lor q) \land (p \lor r)}\,\mathrm{C}}{(q \land r) \lor p \vdash (p \lor q) \land (p \lor r)}\,\lor\mathrm{L}}{\vdash ((q \land r) \lor p) \supset ((p \lor q) \land (p \lor r))}\,\supset\mathrm{R}$$

Distinguishing sense and denotation of proofs

Are these different proofs?

$SC_1 \vdash ((q \land r) \lor p) \supset ((p \lor q) \land (p \lor r))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,\mathrm{Rf}}{q \vdash p \lor q}\,\lor\mathrm{R}}{q \land r \vdash p \lor q}\,\land\mathrm{L} \quad \dfrac{\dfrac{\dfrac{}{r \vdash r}\,\mathrm{Rf}}{r \vdash p \lor r}\,\lor\mathrm{R}}{q \land r \vdash p \lor r}\,\land\mathrm{L}}{q \land r, q \land r \vdash (p \lor q) \land (p \lor r)}\,\land\mathrm{R}}{q \land r \vdash (p \lor q) \land (p \lor r)}\,\mathrm{C} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\mathrm{Rf}}{p \vdash p \lor q}\,\lor\mathrm{R} \quad \dfrac{\dfrac{}{p \vdash p}\,\mathrm{Rf}}{p \vdash p \lor r}\,\lor\mathrm{R}}{p, p \vdash (p \lor q) \land (p \lor r)}\,\land\mathrm{R}}{p \vdash (p \lor q) \land (p \lor r)}\,\mathrm{C}}{(q \land r) \lor p \vdash (p \lor q) \land (p \lor r)}\,\lor\mathrm{L}}{\vdash ((q \land r) \lor p) \supset ((p \lor q) \land (p \lor r))}\,\supset\mathrm{R}$$

$SC_3 \vdash ((q \land r) \lor p) \supset ((p \lor q) \land (p \lor r))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,\mathrm{Rf}}{q \vdash p \lor q}\,\lor\mathrm{R}}{q \land r \vdash p \lor q}\,\land\mathrm{L} \quad \dfrac{\dfrac{}{p \vdash p}\,\mathrm{Rf}}{p \vdash p \lor q}\,\lor\mathrm{R}}{(q \land r) \lor p \vdash p \lor q}\,\lor\mathrm{L} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{r \vdash r}\,\mathrm{Rf}}{r \vdash p \lor r}\,\lor\mathrm{R}}{q \land r \vdash p \lor r}\,\land\mathrm{L} \quad \dfrac{\dfrac{}{p \vdash p}\,\mathrm{Rf}}{p \vdash p \lor r}\,\lor\mathrm{R}}{(q \land r) \lor p \vdash p \lor r}\,\lor\mathrm{L}}{(q \land r) \lor p, (q \land r) \lor p \vdash (p \lor q) \land (p \lor r)}\,\land\mathrm{R}}{(q \land r) \lor p \vdash (p \lor q) \land (p \lor r)}\,\mathrm{C}}{\vdash ((q \land r) \lor p) \supset ((p \lor q) \land (p \lor r))}\,\supset\mathrm{R}$$

Change of order does not lead to different proofs

$SC_1 \vdash ((q \wedge r) \vee p) \supset ((p \vee q) \wedge (p \vee r))$

$y : q \vdash y : q$ (Rf) $\quad z : r \vdash z : r$ (Rf)

$y : q \vdash \mathtt{inry} : p \vee q$ (∨R) $\quad z : r \vdash \mathtt{inrz} : p \vee r$ (∨R) $\quad x : p \vdash x : p$ (Rf) $\quad x : p \vdash x : p$ (Rf)

$v : q \wedge r \vdash \mathtt{inr}\mathit{fst}(v) : p \vee q$ (∧L) $\quad v : q \wedge r \vdash \mathtt{inr}\mathit{snd}(v) : p \vee r$ (∧L) $\quad x : p \vdash \mathtt{inl}x : p \vee q$ (∨R) $\quad x : p \vdash \mathtt{inl}x : p \vee r$ (∨R)

$v : q \wedge r, v : q \wedge r \vdash \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle : (p \vee q) \wedge (p \vee r)$ (∧R) $\quad x : p, x : p \vdash \langle \mathtt{inl}x, \mathtt{inl}x\rangle : (p \vee q) \wedge (p \vee r)$ (∧R)

$v : q \wedge r \vdash \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle : (p \vee q) \wedge (p \vee r)$ (C) $\quad x : p \vdash \langle \mathtt{inl}x, \mathtt{inl}x\rangle : (p \vee q) \wedge (p \vee r)$ (C)

$u : (q \wedge r) \vee p \vdash \mathtt{case\ u}\ \{v.\ \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle \mid x.\ \langle \mathtt{inl}x, \mathtt{inl}x\rangle\} : (p \vee q) \wedge (p \vee r)$ (∨L)

$\vdash \lambda u.\mathtt{case\ u}\ \{v.\ \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle \mid x.\ \langle \mathtt{inl}x, \mathtt{inl}x\rangle\} : ((q \wedge r) \vee p) \supset ((p \vee q) \wedge (p \vee r))$ (⊃R)

$SC_2 \vdash ((q \wedge r) \vee p) \supset ((p \vee q) \wedge (p \vee r))$

$y : q \vdash y : q$ (Rf) $\quad z : r \vdash z : r$ (Rf)

$v : q \wedge r \vdash \mathit{fst}(v) : q$ (∧L) $\quad v : q \wedge r \vdash \mathit{snd}(v) : r$ (∧L) $\quad x : p \vdash x : p$ (Rf) $\quad x : p \vdash x : p$ (Rf)

$v : q \wedge r \vdash \mathtt{inr}\mathit{fst}(v) : p \vee q$ (∨R) $\quad v : q \wedge r \vdash \mathtt{inr}\mathit{snd}(v) : p \vee r$ (∨R) $\quad x : p \vdash \mathtt{inl}x : p \vee q$ (∨R) $\quad x : p \vdash \mathtt{inl}x : p \vee r$ (∨)

$v : q \wedge r, v : q \wedge r \vdash \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle : (p \vee q) \wedge (p \vee r)$ (∧R) $\quad x : p, x : p \vdash \langle \mathtt{inl}x, \mathtt{inl}x\rangle : (p \vee q) \wedge (p \vee r)$ (∧)

$v : q \wedge r \vdash \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle : (p \vee q) \wedge (p \vee r)$ (C) $\quad x : p \vdash \langle \mathtt{inl}x, \mathtt{inl}x\rangle : (p \vee q) \wedge (p \vee r)$ (C)

$u : (q \wedge r) \vee p \vdash \mathtt{case\ u}\ \{v.\ \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle \mid x.\ \langle \mathtt{inl}x, \mathtt{inl}x\rangle\} : (p \vee q) \wedge (p \vee r)$ (∨L)

$\vdash \lambda u.\mathtt{case\ u}\ \{v.\ \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle \mid x.\ \langle \mathtt{inl}x, \mathtt{inl}x\rangle\} : ((q \wedge r) \vee p) \supset ((p \vee q) \wedge (p \vee r))$ (⊃R)

Change of order *does* lead to different proofs

$SC_1 \vdash ((q \wedge r) \vee p) \supset ((p \vee q) \wedge (p \vee r))$

$y : q \vdash y : q$ (Rf) $\quad z : r \vdash z : r$ (Rf)

$y : q \vdash \mathtt{inry} : p \vee q$ (∨R) $\quad z : r \vdash \mathtt{inrz} : p \vee r$ (∨R) $\quad x : p \vdash x : p$ (Rf) $\quad x : p \vdash x : p$ (Rf)

$v : q \wedge r \vdash \mathtt{inr}\mathit{fst}(v) : p \vee q$ (∧L) $\quad v : q \wedge r \vdash \mathtt{inr}\mathit{snd}(v) : p \vee r$ (∧L) $\quad x : p \vdash \mathtt{inl}x : p \vee q$ (∨R) $\quad x : p \vdash \mathtt{inl}x : p \vee r$ (∨)

$v : q \wedge r, v : q \wedge r \vdash \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle : (p \vee q) \wedge (p \vee r)$ (∧R) $\quad x : p, x : p \vdash \langle \mathtt{inl}x, \mathtt{inl}x\rangle : (p \vee q) \wedge (p \vee r)$ (∧)

$v : q \wedge r \vdash \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle : (p \vee q) \wedge (p \vee r)$ (C) $\quad x : p \vdash \langle \mathtt{inl}x, \mathtt{inl}x\rangle : (p \vee q) \wedge (p \vee r)$ (C)

$u : (q \wedge r) \vee p \vdash \mathtt{case\ u}\ \{v.\ \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle \mid x.\ \langle \mathtt{inl}x, \mathtt{inl}x\rangle\} : (p \vee q) \wedge (p \vee r)$ (∨L)

$\vdash \lambda u.\mathtt{case\ u}\ \{v.\ \langle \mathtt{inr}\mathit{fst}(v), \mathtt{inr}\mathit{snd}(v)\rangle \mid x.\ \langle \mathtt{inl}x, \mathtt{inl}x\rangle\} : ((q \wedge r) \vee p) \supset ((p \vee q) \wedge (p \vee r))$ (⊃R)

$SC_3 \vdash ((q \wedge r) \vee p) \supset ((p \vee q) \wedge (p \vee r))$

$y : q \vdash y : q$ (Rf) $\quad z : r \vdash z : r$ (Rf)

$y : q \vdash \mathtt{inry} : p \vee q$ (∨R) $\quad x : p \vdash x : p$ (Rf) $\quad z : r \vdash \mathtt{inrz} : p \vee r$ (∨R) $\quad x : p \vdash x : p$ (Rf)

$v : q \wedge r \vdash \mathtt{inr}\mathit{fst}(v) : p \vee q$ (∧L) $\quad x : p \vdash \mathtt{inl}x : p \vee q$ (∨R) $\quad v : q \wedge r \vdash \mathtt{inrsnd}(v) : p \vee r$ (∧L) $\quad x : p \vdash \mathtt{inl}x : p \vee r$ (∨I)

$u : (q \wedge r) \vee p \vdash \mathtt{case\ u}\ \{v.\mathtt{inr}\mathit{fst}(v) \mid x.\mathtt{inl}x\} : p \vee q$ (∨L) $\quad u : (q \wedge r) \vee p \vdash \mathtt{case\ u}\ \{v.\mathtt{inrsnd}(v) \mid x.\mathtt{inl}x\} : p \vee r$ (∨I)

$u : (q \wedge r) \vee p, u : (q \wedge r) \vee p \vdash \langle \mathtt{case\ u}\ \{v.\mathtt{inr}\mathit{fst}(v) \mid x.\mathtt{inl}x\}, \mathtt{case\ u}\ \{v.\mathtt{inrsnd}(v) \mid x.\mathtt{inl}x\}\rangle : (p \vee q) \wedge (p \vee r)$ (∧R)

$u : (q \wedge r) \vee p \vdash \langle \mathtt{case\ u}\ \{v.\mathtt{inr}\mathit{fst}(v) \mid x.\mathtt{inl}x\}, \mathtt{case\ u}\ \{v.\mathtt{inrsnd}(v) \mid x.\mathtt{inl}x\}\rangle : (p \vee q) \wedge (p \vee r)$ (C)

$\vdash \lambda u.\ \langle \mathtt{case\ u}\ \{v.\mathtt{inr}\mathit{fst}(v) \mid x.\mathtt{inl}x\}, \mathtt{case\ u}\ \{v.\mathtt{inrsnd}(v) \mid x.\mathtt{inl}x\}\rangle : ((q \wedge r) \vee p) \supset ((p \vee q) \wedge (p \vee r))$ (⊃R)

Distinguishing sense and denotation in proofs

Encoding the rules with λ-terms makes transparent why some changes in the order of rule applications leads to different terms, while others do not:

- In SC the $\wedge$L rule as well as the $\supset$L rule are substitution operations
- → they can change their place in the order because only the inner structure of the term is affected (for $\supset$L only if the right premise is not an axiom)
- → no completely new term is created
- In ND there are no substitution operations as rules, i.e. there's always a new kind of term produced

Distinguishing sense and denotation in proofs

- As can be seen in the case above a change of order of rule applications can lead to different proofs (SC_1 and SC_3)
- In the other case (SC_1 and SC_2) there is only a difference in sense, but not in denotation

Sense of a derivation

- Consists of the *set of terms that occur within the derivation*
- If a sense-denoting set can be obtained from another by replacing any occurrence of a variable (bound or free) by another variable of the same type, they express the same sense

Philosophical motivation

Fregean sense is

- a procedure to determine its denotation (Dummett 1973)
- a sequence of instructions; terms represent programs; purpose of a program is to calculate its denotation (Girard 1990)

→ There can be the same underlying proof in different proof systems, like in ND and SC but also within the same proof system: there can be difference in sense but not in denotation

→ The proof is essentially the same but the way it is given to us, the way of inference, differs, i.e. the sense, differs

→ This can be read off the set of terms occurring in a derivation: they end up building the same term but the way it is built differs

Example for different senses

Example: SC_1 vs. SC_2 (same denotation)

- Sense of SC_1:

$$\{x, y, z, u, v, \mathtt{inl}x, \mathtt{inr}y, \mathtt{inr}z, \langle \mathtt{inl}x, \mathtt{inl}x\rangle, \mathtt{inr}fst(v), \mathtt{inr}snd(v), \langle \mathtt{inr}fst(v), \mathtt{inr}snd(v)\rangle, \mathtt{case}\ \mathtt{u}\ \{v.\langle \mathtt{inr}fst(v), \mathtt{inr}snd(v)\rangle \mid x.\langle \mathtt{inl}x, \mathtt{inl}x\rangle\}, \lambda u.\mathtt{case}\ \mathtt{u}\ \{v.\langle \mathtt{inr}fst(v), \mathtt{inr}snd(v)\rangle \mid x.\langle \mathtt{inl}x, \mathtt{inl}x\rangle\}\}$$

- Sense of SC_2:

$$\{x, y, z, u, v, \mathtt{inl}x, \langle \mathtt{inl}x, \mathtt{inl}x\rangle, fst(v), snd(v), \mathtt{inr}fst(v), \mathtt{inr}snd(v), \langle \mathtt{inr}fst(v), \mathtt{inr}snd(v)\rangle, \mathtt{case}\ \mathtt{u}\ \{v.\langle \mathtt{inr}fst(v), \mathtt{inr}snd(v)\rangle \mid x.\langle \mathtt{inl}x, \mathtt{inl}x\rangle\}, \lambda u.\mathtt{case}\ \mathtt{u}\ \{v.\langle \mathtt{inr}fst(v), \mathtt{inr}snd(v)\rangle \mid x.\langle \mathtt{inl}x, \mathtt{inl}x\rangle\}\}$$

Frege's cases

1. Different signs, same sense, same denotation; examples for:
   - Singular Terms: "Gottlob's brother", "the brother of Gottlob"
   - Sentences: "*M* gave document *A* to *N*", "Document *A* was given to *N* by *M*" (Frege 1979: 141)
2. Different signs, difference in sense, same denotation; examples for:
   - Singular Terms: "morning star", "evening star"
   - Sentences: "The morning star is the planet Venus", "The evening star is the planet Venus"

- The following cases should NOT happen according to Frege:
  - One sign, different senses: ambiguous terms (happen in natural languages but shouldn't occur in formal languages)
  - One sense, different denotations: the sense should determine the reference

Transferring Frege's cases onto the context of proofs

1. Different signs, same sense (same set of terms within derivation), same denotation (same term for the endsequent), this case is possible between the two proof systems:

ND$_{\vdash (p \vee p) \supset (p \wedge p)}$

$$\dfrac{\dfrac{\dfrac{[y : p \vee p] \quad [x : p] \quad [x : p]}{\texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\} : p}\vee\mathrm{E} \qquad \dfrac{[y : p \vee p] \quad [x : p] \quad [x : p]}{\texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\} : p}\vee\mathrm{E}}{\langle \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}, \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}\rangle : p \wedge p}\wedge\mathrm{I}}{\lambda y.\langle \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}, \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}\rangle : (p \vee p) \supset (p \wedge p)}\supset\mathrm{I}$$

SC$_{\vdash (p \vee p) \supset (p \wedge p)}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : p \vdash x : p} \quad \dfrac{}{x : p \vdash x : p}}{y : p \vee p \vdash \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\} : p}\vee\mathrm{L} \qquad \dfrac{\dfrac{}{x : p \vdash x : p} \quad \dfrac{}{x : p \vdash x : p}}{y : p \vee p \vdash \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\} : p}\vee\mathrm{L}}{y : p \vee p, y : p \vee p \vdash \langle \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}, \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}\rangle : p \wedge p}\wedge\mathrm{R}}{y : p \vee p \vdash \langle \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}, \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}\rangle : p \wedge p}\mathrm{C}}{\vdash \lambda y.\langle \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}, \texttt{case}\ \ \texttt{y}\ \{x.x \mid x.x\}\rangle : (p \vee p) \supset (p \wedge p)}\supset\mathrm{R}$$

Transferring Frege's cases onto the context of proofs

2. Different signs, difference in sense, same denotation: see example above between SC_1 and SC_2 but also consistent with distinguishing between SC-derivations containing cut vs. cut-free ones:

$SC_{\vdash (p \wedge p) \supset (p \vee p)}$, Sense: $\{z, y, fst(y), \mathtt{inl}\, fst(y), \lambda y.\mathtt{inl}\, fst(y)\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z : p \vdash z : p}}{y : p \wedge p \vdash fst(y) : p}\wedge L}{y : p \wedge p \vdash \mathtt{inl}\, fst(y) : p \vee p}\vee R}{\vdash \lambda y.\mathtt{inl}\, fst(y) : (p \wedge p) \supset (p \vee p)}\supset R$$

SC-cut$_{\vdash (p \wedge p) \supset (p \vee p)}$, Sense: $\{z, y, fst(y), \mathtt{inl}\, z, \mathtt{inl}\, fst(y), \lambda y.\mathtt{inl}\, fst(y)\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z : p \vdash z : p}}{y : p \wedge p \vdash fst(y) : p}\wedge L \quad \dfrac{\dfrac{}{z : p \vdash z : p}}{z : p \vdash \mathtt{inl}\, z : p \vee p}\vee R}{y : p \wedge p \vdash \mathtt{inl}\, fst(y) : p \vee p}\text{cut}}{\vdash \lambda y.\mathtt{inl}\, fst(y) : (p \wedge p) \supset (p \vee p)}\supset R$$

Conclusion

- Frege was interested in identity: we can also use these results to say something about proof identity
- Identity over different calculi or within the same calculus consists in having end-terms reducible to the same normal form:
  - With syntactically different derivations in ND this can only happen when we have one derivation in normal and the other in non-normal form (reducible to the former)
  - In SC this can happen when
    1. we have one derivation in cut-free form and one corresponding derivation containing cut, or
    2. when there's a change in the order of rule applications, either of the:
    - $\wedge$L rule, or of the
    - $\supset$L rule, though iff its right premise is not an axiom
- If they are supposed to be *identical in meaning* then this means that the way the inference is given is essentially the same, so the set of terms building up the end-term must be the same

Thanks for your attention!

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Term-annotated natural deduction

$$\frac{\begin{matrix}\Gamma & & \Delta\\ \vdots & & \vdots\\ s:A & & t:B\end{matrix}}{\langle s,t\rangle : A\wedge B}\wedge\mathrm{I} \qquad \frac{\begin{matrix}\Gamma\\ \vdots\\ t:A\wedge B\end{matrix}}{\mathit{fst}(t):A}\wedge\mathrm{E}_1 \qquad \frac{\begin{matrix}\Gamma\\ \vdots\\ t:A\wedge B\end{matrix}}{\mathit{snd}(t):B}\wedge\mathrm{E}_2$$

$$\frac{\begin{matrix}\Gamma\\ \vdots\\ s:A\end{matrix}}{\mathtt{inls}:A\vee B}\vee\mathrm{I}_1 \qquad \frac{\begin{matrix}\Gamma\\ \vdots\\ t:B\end{matrix}}{\mathtt{inrt}:A\vee B}\vee\mathrm{I}_2 \qquad \frac{\begin{matrix}\Gamma & & \Delta,[x:A] & & \Theta,[y:B]\\ \vdots & & \vdots & & \vdots\\ r:A\vee B & & s:C & & t:C\end{matrix}}{\mathtt{case\ r}\ \{x.s \mid y.t\}:C}\vee\mathrm{E}$$

$$\frac{\begin{matrix}\Gamma,[x:A]\\ \vdots\\ t:B\end{matrix}}{\lambda x.t : A\supset B}\supset\mathrm{I} \qquad \frac{\begin{matrix}\Delta & & \Gamma\\ \vdots & & \vdots\\ s:A\supset B & & t:A\end{matrix}}{\mathit{App}(s,t):B}\supset\mathrm{E} \qquad \frac{\begin{matrix}\Gamma\\ \vdots\\ t:\bot\end{matrix}}{\mathit{abort}(t):A}\bot\mathrm{E}$$

Term-annotated G0ip

Logical axiom:

$$\frac{}{x:A\vdash x:A}\mathrm{Rf}$$

Logical rules:

$$\frac{\Gamma\vdash s:A \quad \Delta\vdash t:B}{\Gamma,\Delta\vdash\langle s,t\rangle : A\wedge B}\wedge\mathrm{R}$$

$$\frac{\Gamma,x:A\vdash s:C}{\Gamma,y:A\wedge B\vdash s[\mathit{fst}(y)/x]:C}\wedge\mathrm{L}_1 \qquad \frac{\Gamma,x:B\vdash s:C}{\Gamma,y:A\wedge B\vdash s[\mathit{snd}(y)/x]:C}\wedge\mathrm{L}_2$$

$$\frac{\Gamma\vdash s:A}{\Gamma\vdash \mathtt{inls}:A\vee B}\vee\mathrm{R}_1 \qquad \frac{\Gamma\vdash t:B}{\Gamma\vdash \mathtt{inrt}:A\vee B}\vee\mathrm{R}_2$$

$$\frac{\Gamma,x:A\vdash s:C \quad \Delta,y:B\vdash t:C}{\Gamma,z:A\vee B\vdash \mathtt{case\ z}\ \{x.s\mid y.t\}:C}\vee\mathrm{L}$$

$$\frac{\Gamma,x:A\vdash t:B}{\Gamma\vdash\lambda x.t:A\supset B}\supset\mathrm{R} \qquad \frac{\Gamma\vdash t:A \quad \Delta,y:B\vdash s:C}{\Gamma,\Delta,x:A\supset B\vdash s[\mathit{App}(x,t)/y]:C}\supset\mathrm{L}$$

$$\frac{}{x:\bot\vdash \mathit{abort}(x):C}\bot\mathrm{L}$$

Appendix

Term-annotated G0ip

Structural rules:
Weakening:

$$\frac{\Gamma \vdash t : C}{\Gamma, x : A \vdash t : C}\,\mathrm{W}$$

Contraction:

$$\frac{\Gamma, x : A, y : A \vdash t : C}{\Gamma, x : A \vdash t[x/y] : C}$$

Cut:

$$\frac{\Gamma \vdash t : D \qquad \Delta, x : D \vdash s : C}{\Gamma, \Delta \vdash s[t/x] : C}$$

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Hypo: A Simple Constructive Semantics for Intuitionistic Sentential Logic; Soundness and Completeness

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Abstract

This paper examines semantics for Intuitionistic Sentential Logic. A somewhat new semantics is going to be proposed: we call it Hypo. Its basic relation is neither that of truth in a model, or a world, nor that of proof or construction. It is the relation of consequence from hypotheses. The meaning of each logical constant is made explicit by clauses stated in terms of this relation. Also, Hypo can be extended to cover Classical Sentential Logic. Inferential validity in both cases, intuitionist and classical sentential logic, is easily derivable.

Keywords: intuitionistic semantics, classical semantics, hypotheses, sentential logic, logical consequence

1 Introduction

Tarskian semantics employ truth clauses in order to make explicit the meaning of logical constants. The informal BHK semantics presentation in Heyting [Hey56] employs assertion clauses for the same purpose. In the second, constructions are intended as the basic elements of the semantics, and an assertion involves the possession of such a construction.

Heyting's Intuitionist Sentential Logic (ISL) is the focus here. This logic was developed through distinct and temporally separated steps. And the step of stating a formal semantics was the not the first one.

At the beginning of the 20's in the XXth century, Johansson proposed what is now known as minimal logic. After, in 1925, Kolmogorov proposed an

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elucidation of what should be intuitionistic logic from his standpoint. It turned out that his proposal was identical to Johansson's minimal logic. This logic does not contain the *ex contradictione quodlibet* principle. Only later, in the beginning of the 30's, Heyting presented the now most accepted formalization of intuitionistic logic. Differently of Johansson's and Kolmogorov's, Heyting's formalization contains the *ex contradictione* principle.

Kripke formal semantics[1] for intuitionistic logic appeared in the 60's [Kri65].[2] It does not count with agreement of orthodox intuitionists, because, among other things, in knowledge states either a sentence is forced or it is not. Heyting characterizes logical constants based on the concept of mental construction. In the 1970's the intuitionist interpretation was reformulated and called Proof Interpretation, eliminating the subjective concept of mental construction in favor of the concept of proof, according to van Atten [Att17].

The principal *locus* containing a non-formal exposition for the meaning of logical constants is Heyting's *Intuitionism: An Introduction* [Hey56]. The interpretation there given lies under the heading BHK interpretation.[3]

Other slightly distinct semantical analyses called Proof-theoretic Semantics appeared from the 70's on, among them Dummett's pragmatist and verificationist definitions [Dum81] and Prawitz's analysis of Proof Theory [Pra71]. A common characteristic of all semantics mentioned above is what Schroeder-Heister [Sch18], as also Kosta Došen [Dos15], have called the categorical transmission view. In classical semantics truth is transmitted from premises to conclusion, while in proof-theoretical semantics the concept of proof or construction takes that role. Prawitz [Pra72] even conjectured that the logic semantically supported by proof-theoretical analyses would be exactly intuitionistic logic. But this has been recently shown not to be the case by Piecha & Schroeder-Heister [Pie19].

A non-categorical semantics is one where the transmission view is not assumed when explaining meaning of logical constants. This is exactly the case of the proposal below.

It is not our objective to debate what is the faithful interpretation of intuitionistic logical constants. We want rather to point out that Heyting's Sentential Logic can be interpreted in a straightforward and simple way without using the concept of proof or construction. The interpretation to be offered does not appeal to any non-constructivist principles. It is a basis for defining the derived concepts of closed proof and of inference. Therefore, Intuitionistic Sentential Logic has, at least, another very natural interpretation.

The concept we are envisaging here is the primitive concept of semantical consequence which comes entrenched with another primitive concept: that of hypothesis. One dogma of traditional semantics, according to Kosta Došen

[1] Orthodox intuitionists have some difficulties in accepting Kripke semantics as a faithful account of the intuitionist original position.

[2] For an exposition see [Fit69].

[3] Brouwer-Heyting-Kolmogorov.

([Dos15], p. 150) is that “the correctness of the hypothetical notions reduces to the preservation of the correctness of the categorical ones”. We are going to defy that dogma, although not in the same way as this author envisaged. One question that has to be settled is how to understand the notion of hypothesis.

2 ISL Sequent Calculus

Our considerations will be based on a handy formulation of the ISL Sequent Calculus.[4] Its inference rules will be presented linearly. We suppose given a recursively defined sentential language $\mathfrak{L}$ containing the atomic absurd sentence $\bot$ and no sentential parameters. Capital Latin letters C, D, etc. (with or without subindexes) represent sentences of language $\mathfrak{L}$. Small Latin letters c, d, etc. (with or without subindexes) represent sentences belonging to $\mathfrak{L}_{\mathrm{At}}$, the subset of atomic sentences of $\mathfrak{L}$. Capital Greek letters Γ and Δ represent finite subsets of $\mathfrak{L}$ (including the empty set). Θ and Λ represent either finite or unbounded subsets of $\mathfrak{L}$ (including the empty set). Small Greek letters γ and δ (with or without subindexes) represent finite subsets of $\mathfrak{L}_{\mathrm{At}}$ (including the empty set). Capital Greek letters Ψ and Ω represent finite multisets[5] of sentences of $\mathfrak{L}$ (including the empty multiset). Small Greek letters λ and μ (with or without subindexes) represent finite multisets of $\mathfrak{L}_{\mathrm{At}}$ (including the empty multiset). The sentential intuitionist sequent logic LJ^{s}, is defined by the following rules:

Structural rules

Basic sequents: $C \vdash C$
Thinning on the left: $\Omega \vdash C \implies \Omega, D \vdash C$
Contraction on the left: $\Omega, E, E, \Psi \vdash C \implies \Omega, E, \Psi \vdash C$
Cut: $(\Omega \vdash C \text{ and } C, \Psi \vdash D) \implies \Omega, \Psi \vdash D$

Operational rules

Implication introduction on the right: $\Omega, C \vdash D \implies \Omega \vdash C \to D$
Implication introduction on the left:
$\Omega \vdash C \text{ and } \Psi, D \vdash E \implies \Omega, \Psi, C \to D \vdash E$
Disjunction introduction on the right: $\Omega \vdash C \implies \Omega \vdash C \vee D$
$\Omega \vdash D \implies \Omega \vdash C \vee D$
Disjunction introduction on the left:
$\Omega, C \vdash E \text{ and } \Omega, D \vdash E \implies \Omega, C \vee D \vdash E$
Conjunction introduction on the right: $\Omega \vdash C \text{ and } \Omega \vdash D \implies \Omega \vdash C \wedge D$
Conjunction introduction on the left: $\Omega, C, D \vdash E \implies \Omega, C \wedge D \vdash E$

[4] Not because we consider it a privileged calculus *vis-à-vis* Natural Deduction, but because it makes comparison with semantic clauses easier.
[5] That is, sets admitting multiple occurrences of a same element.

Absurd "rule" (as a basic sequent): $\bot \vdash e$
Negation is a defined constant: $\neg C \equiv_{\text{df}} C \to \bot$.

A syntactical proof in LJ^s is a structure (either a sequence or a tree, reader's choice) of sequents governed by the above rules including axiom sequents belonging to a theory T. When all sentences in an axiom sequent are atomic we say it is an atomic axiom. In order to obtain the classical sequent calculus LK^s from LJ^s, an extra structural rule is added, Peirce's rule[6]:

Discharge of implication on the left: $\Omega, E \to F, \Psi \vdash E \implies \Omega, \Psi \vdash E$

Theorem 1. *For any proof over* LJ^s *there is a corresponding proof with the same conclusion in which there is no instance of Cut.*

Proof. Roughly similar to the traditional one, see [Gen35]. QED

3 Semantics

Although in contemporary logic the word "semantics" is many times assumed to be synonymous with model theory, here we stress its original sense as a theory focusing the meaning of logical constants.

From a philosophical point of view, semantical clauses for logical constants are supposed to contain an explicitation[7] of their sense in terms of the meaning given to the constituent(s) part(s) via a basic primitive relation. Usually, a model theory is completed by a adding a stipulation of how the basic relation applies in the case of atomic sentences.

In classical Tarskian semantics, clause

$$\vDash^M C \land D \iff \vDash^M C \text{ and } \vDash^M D$$

for example, can naturally be read as: *$C \land D$ is true, in model M, if and only if both C is true, in model M, and D is true, in model M*. In case C (and/or D) is atomic, the model has to stipulate either that C (and/or D) *is true* or that C (and/or D) *is not true* (*tertium non datur*), which means (according to a convention rarely made explicit) one of two: it is true or *it is false* (via bivalence). A model enters as a parametric basic mathematical entity, which among other things turns the definition clauses into a way of formulating countermodels.

Next, in classical semantics, comes the definition of logical consequence: $\Gamma \Vdash C$ *if and only if C is true in all models in which Γ is true*. $\Gamma \Vdash C$ is read accordingly as: C follows from Γ. $\Gamma \nVdash C$ is read as: C does not follow from Γ.

[6]It is a hypothesis discharging rule.

[7]We use as a verb the expression "to explicitate". It means to unfold, to make it clear, to make it explicit. All derived forms are to be acquainted too.

These readings carry an *ontological commitment* since it is assumed that the case is determined no matter if we know it or not.

One interesting problem consists in explaining what means "all models".[8] The problem is more acute in first order logic. Constructivists object that the validity of *tertium non datur* principle for sentences, and in particular the atomic ones, cannot be independent of the activity of a subject which should be in possession of a procedure for deciding what is the case.

When we come to intuitionistic logic the simple strategy of Tarskian semantics is not available, since bivalence is not regarded as a valid constructive principle. In Kripke semantics an evolved substitute has been proposed. Kripke builds his semantics upon a somewhat different relation.

For comparison, Kripke's semantical clause for conjunction is like:

$$\alpha \vDash^M C \wedge D \iff \alpha \vDash^M C \text{ and } \alpha \vDash^M D$$

The clause is read as: *$C \wedge D$ is forced [is true] in the state of knowledge [possible world] α, of model M, if and only if both C is forced [true] in α, of model M, and D is forced [true] in α, of model M.*[9] This semantics is based on a relation of being forced in a state of knowledge of a model – which can be assimilated to being true in a possible world of the model.[10] Possible worlds are representatives of states of knowledge. From a formal point of view it is clear how to define a state of knowledge, but the concept associated is not exempt of philosophical difficulties as, for example, the problem of understanding what is an *actual* infinite state of knowledge.

Complementing Kripke's clauses, there comes together a total stipulation of which atomic sentences are forced or not forced in each state of knowledge, *tertium non datur*. Either a state of knowledge forces an atomic sentence (to be true) or it does not force it (to be true). Next, it can be proved that any molecular sentence is also either forced or not forced in each state of knowledge, due to the way clauses were stated. Consistency is assumed to be a basic attribute of states of knowledge or possible worlds.

A priori there is no major difference between the two clauses for conjunction in Tarskian semantics and in Kripkean semantics. In fact, at the sentential level, those two semantics really differ only over implication. We should consider that the real divergence between classical sentential logic and intuitionistic sentential logic is somehow related to implication.

On close inspection, states of knowledge are defined by sets of atomic

[8]We reserve the word "model" specifically to that step in which is established a relation of consequence for atomic sentences.

[9]Or, what should be taken as the same, that $C \wedge D$ is forced in the possible world α if and only if C is forced in α and D is forced in α.

[10]Not being true in a state of knowledge does not mean to be false in this state of knowledge, although being false in a state of knowledge does mean not being true in this state of knowledge. This is the deep reason why Kripke has assumed *tertium non datur* for the relation of being forced in a state of knowledge.

sentences, those that are forced in the state. Thus, a Kripke model is a structured set of states of knowledge ordered by inclusion. It is natural to look at Kripke semantics as consisting of clauses for establishing a relation between a set of atomic sentences, *or atomic hypotheses*, and another sentence, *the consequent*. The final step in defining validity uses quantification over *all Kripke models*.

4 Hypo Semantics

4.1 The question

Our aim here is to answer a simple question. Considering the notion of hypothesis as semantically primitive, could one obtain a characterization of logical constants using the relation of semantical consequence – a relation involving a finite set of hypotheses and a consequent? We claim that the answer to this question is positive. We are going to call the semantics to be formulated: *Hypo Semantics*. Actually, the answer is so that neither the concept of truth neither the concept of proof are taken as primitives in this semantics. Instead, the notion of semantical consequence (for sentential logical constants), with its immanent notion of hypothesis, is going to be elucidated for each sentential constant.

4.2 A constructive semantical consequence relation and its negation

The relation of semantical consequence is represented as: $\Gamma \Vdash_A C$. Read as an assertion it means: *A recognized C to follow from the set of hypotheses* Γ. "A" stands for an agent and the act of recognizing has a legal sense too.

There is only one way of negating the relation described above. This negation is what Heyting calls *factual negation* ([Hey56] pp. 18–19), and it is going to be represented by $\text{Not}(\Gamma \Vdash_A C)$. Its assertion means: *A did not recognize C to follow from the set of hypotheses* Γ. As recognition is itself an act it has to be realized in time. Non-recognition means here a failure in achieving recognition in the past, failure that can occur by different reasons.

The relation and its negations can be used for making assertions about me and others. "A" can be replaced by "I", "you", "he", "she" etc. The explicit reference to a cognitive agent precludes any ontological commitment in assertions.

However, when someone wants to make a semantical statement involving the relation of semantical consequence things change. The index "A" represents an idiosyncrasy irrelevant for the point. Semantical clauses are intended to make explicit the rules governing recognition independently of any particular agents. As we are far from suggesting that the acts of an agent are not relevant for establishing recognizability, we will develop a second degree concept, that of *recognition of recognizability*. The expression $\Gamma \Vdash C$ without an indexing agent here means: *it is recognizable that C follows from the set of hypotheses* Γ

or *it can be recognized that C follows from the set of hypotheses* Γ. That is, it expresses recognizability. Therefore, establishing $\Gamma \Vdash C$ means to recognize recognizability. For that, a series of acts are required to be performed by an agent.

The negation of the semantical statement is $\Gamma \nVdash C$ and it means: the supposition that C would be recognizable as following from the set of hypotheses Γ is absurd (or implies an absurd).

We claim that the canonical way for making explicit the meaning of a logical constant is by means of necessary and sufficient conditions for that constant in the context of recognizability of semantical consequence.

4.3 Cognition and acts

The previous condition for the assertion of recognition, and also of recognition of recognizability, is a *cognition act by a subject*. Here lies the constructive perspective and its *epistemological nature*. *The cognition acts* we are referring to are not restricted to acts of categorical proof. An act of verification that a consequent follows from a set of hypotheses requires a previous supposition act. What follows the supposition act is therefore a series of deduction steps from hypotheses. The final legal act of stating recognition, even of recognizability, can be enacted when the necessary and sufficient conditions are shown to hold. And, constructively, this is to be considered so when the acts involved in the necessary and sufficient conditions have been performed.

Acts are developed in time. No human is logically omniscient, since he cannot effect all possible acts of logical cognition in a finite lapse of time.

4.4 Failure

With factual negation, Heyting wants to express a failure in action by a special negation. When an act was not effected, different causes could be a reason for it. A failure in recognition occurs when the attempt to verify a necessary condition fails. It can be of two distinct natures: (i) by showing that at least one of the necessary conditions implies an absurd; (ii) the attempts to carry a verification fail, ending without success and without any refutation. Failure of kind (i) is *semantical*. Failure (ii) is *pragmatical*. Other two extreme sides of pragmatical failures are: (iii) when there is no attempt to verify the conditions; (iv) although the conditions were already verified, the last act of recognition is missing.

Pragmatical failures and the extremes of it are beyond the scope of a semantical investigation.

Semantical failures obtains when: (i) a deduction showing that the supposition of recognizability results an absurd ($\Gamma \nVdash C$); (ii) if the consistency of the semantics and the consistency of Γ were previously granted, a deduction showing $\Gamma, C \Vdash \bot$. Item (i) means that: $\Gamma \nVdash C \implies$ it should be the case that, for any A, Not($\Gamma \Vdash_A C$).[11] Additionally, item (ii) means that: $\Gamma, C \Vdash \bot \implies \Gamma \nVdash C$.

4.5 Hypothesis, truth and proof

The general concepts of truth and of categorical proof are not considered primitives here. We assume to be plainly meaningful to talk about consequence from hypotheses, and even so of consequence from hypotheses in metahypothetical terms. As we see it, the explicitation of meaning involves it. An exemplification is to be found in the proof of Theorem 4 below.

Finite sets of hypotheses are strong enough for making explicit the meaning of sentential logical constants.

Hypotheses are introduced by an act of supposition which lies on every element of the set of hypotheses. Thus, all members of the set must be somehow accessible. Hypotheses are a certain sentential content of which we are invited, or are inviting – it depends on our dialogical position in argumentation –, to accept as a starting point for reasoning. This invitation is more specifically an invitation to take the information expressed by a sentence "*as if* it were true", without any definitive commitment of the participants in dialogue. The general case is that the truth value is attached provisionally to the content, without presupposing possession or existence of a proof of the hypothesized sentence, not even an open proof.

To make a hypothesis[12] is not the same as to suppose the possession of a proof. Under the supposition that someone is in possession of a proof of the P=NP statement, it would be concluded that this person has the right to win a prize[13], while a similar conclusion does not follow from the supposition of P=NP being true.

No invitation for hypothesizing can be considered effective without guaranteeing a way of determining if a given sentence is being taken as a hypothesis or not. An *unbounded*, i.e. potentially infinite, list of hypotheses is in principle acceptable, but only in the measure that an effective method for determining when a sentence belongs or not to this list is given.

4.6 Semantical clauses

Clauses for logical constants are divided into two groups: those in which the logical constant occurs in the *left side of the primitive relation*, as a *hypothesis* among others, and those in which it occurs in the *right side*, as the *consequent*. The symbol "$\cong$" in the clauses is used to express the fact that the clause makes

[11] This implication is of mixed nature joining recognizability and the impossibility of effecting an act.

[12] One example of a hypothesis being proposed to an audience of non-mathematicians, let's say, is: Suppose that the squared root of two were rational. This is not an assertion, but an invitation, and yet an act of speech.

[13] See https://www.claymath.org/millennium-problems/millennium-prize-problems.

explicit the meaning of semantical consequence statement (at the left side of "$\cong$") through necessary and sufficient conditions (at the right side of "$\cong$"). Clauses are asymmetrical. The left side contains the "explicitandum", and the right side the "expliciens".[14] The explicitation clauses must (i) accurately explicitate the meaning of the logical constants and, at the same time, (ii) be so formulated as to be formally correct.

4.6.1 Clauses for the sentential logical constants

The semantical clauses are:

Leftist clauses:

($\wedge^{l}$) $\Gamma, C \wedge D \Vdash E \cong \Gamma, C, D \Vdash E$

($\vee^{l}$) $\Gamma, C \vee D \Vdash E \cong \Gamma, C \Vdash E$ and $\Gamma, D \Vdash E$

($\rightarrow^{l}$) $\Gamma, C \rightarrow D \Vdash E \cong$ for any Δ: $(\Delta, C \Vdash D \implies \Gamma, \Delta \Vdash E)$

($\bot^{l}$) $\Gamma, \bot \Vdash c \cong$ under any condition (for atomic c).

Rightist clauses:

($\wedge^{r}$) $\Gamma \Vdash C \wedge D \cong \Gamma \Vdash C$ and $\Gamma \Vdash D$

($\vee^{r}$) $\Gamma \Vdash C \vee D \cong$ for any E: $((\Gamma, C \Vdash E$ and $\Gamma, D \Vdash E) \implies \Gamma \Vdash E)$

($\rightarrow^{r}$) $\Gamma \Vdash C \rightarrow D \cong \Gamma, C \Vdash D$

($\bot^{r}$) $\Gamma \Vdash \bot \cong$ for any atomic e: $\Gamma \Vdash e$.

Theorem 2. *(i) For any E: $\Gamma, \bot \Vdash E$; (ii) $\Gamma \Vdash \bot \iff$ for any E: $\Gamma \Vdash E$.*

Proof. Straightforward induction. QED

The spellings of both clauses for conjunction are:

> For E being recognized to follow from the set of hypotheses $\Gamma, C \wedge D$ *it is necessary and sufficient* to be recognizable that E follows from the set of hypotheses Γ, C, D.[15]

> For $C \wedge D$ being recognized to follow from the set of hypotheses Γ *it is necessary and sufficient* to be recognizable that C follows from the set of hypotheses Γ and to be recognizable that D follows from the set Γ.

[14] The words *explicandum* and *explicans* are good alternatives, they are already taken in the literature. However, these terms seem to suppose an assumption: that the fact of making meaning explicit would do as an explanation of meaning, thesis with which we are inclined to agree, but which we are not going to argue for.

[15] The expression "for . . . being recognized to follow . . ." is to be understood as a description of a potential act of recognition. The conditions presented as sufficient and necessary in the right hand side do not involve ontological determination. The expression "to be recognizable" indicates that it is a second order relation that must be taken as a condition. It indicates what constitutes the grounds for accomplishing a further act of recognition (by an agent), the one implied in the *explicitandum*.

The spelling for right implication is:

> For $C \to D$ being recognized to follow from the set of hypotheses Γ *it is necessary and sufficient* to be recognizable that D follows from the set of hypotheses Γ, C.

The reading of other clauses is a variation of the above examples, quantified conditions will receive special attention below.

4.6.2 Semantical principles for hypotheses

As hypotheses are a primitive element in the above clauses, there must be principles laying out how to semantically operate with them:

(Idempotence)	$C \Vdash C$
(Loading hypothesis) / (Load)	$\Gamma \Vdash D \implies \Gamma, \Delta \Vdash D$
(Dropping hypothesis) / (Drop)	$(\Gamma, C \Vdash D$ and $\Gamma \Vdash C) \implies \Gamma \Vdash D$

This group constitutes the *structural consequence* (SC) *principles*.

(Idempotence), as any other semantical principle, is a general semantical rule, it is schematic. An assertive act of recognizability can be issued for a given sentence C: It is recognizable that C follows from hypothesis C. This act states the rule.[16] Also, an indexed assertive act of recognition can also be issued for a given sentence C: I recognize that C follows from hypothesis C.

(Load) establishes when the set of hypotheses can be extended. It is the semantical version of the weakening rule of sequent calculus. The recognizability of a semantic consequence relation with an extended set of hypotheses can be effected once a former recognizability with a lesser set of hypotheses has already been done. We claim that this principle, as stated, does not make sense if the members of the set in the left hand side of the relation were considered (intuitionistic) assertions.[17]

The (Drop) principle is roughly *complementary* to (Load). It states the conditions under which a hypothesis can be dropped still keeping recognition of consequence.[18]

Theorem 3 (Transitivity)**.** $(\Gamma \Vdash C$ *and* $C, \Delta \Vdash E) \implies \Gamma, \Delta \Vdash E$.

Proof. Suppose $\Gamma \Vdash C$ and $C, \Delta \Vdash E$. By using (Load) we obtain $\Gamma, \Delta \Vdash C$ and also $C, \Gamma, \Delta \Vdash E$. By (Drop) it results $\Gamma, \Delta \Vdash E$. QED

(Transitivity) is the semantical version of the cut rule for sequent calculus.

[16] (Idempotence) can be obtained by induction from an atomic version of (Idempotence). The proof makes use of leftist and rightist logical constant clauses and requires the (Load) principle.

[17] Intuitionists tend to conceive premises as assertions taking part in an inference. They also equate assertion with the possession of a proof for the sentence. Notice that no counterfactual hypothesis is formulated as an assertion, since the mood used in them is not the indicative.

[18] Classical logic is the result of adding a principle allowing a new way of dropping hypotheses. See below.

4.6.3 *Predicativity* versus *impredicativity*

Conditions of clauses ($\to^{l}$), ($\vee^{r}$) and ($\bot^{r}$) contain a quantification. A reading "by range" might suggest impredicativity.

For avoiding impredicativity, quantifiers are to be read in the same way that Heyting [Hey56] did for quantifiers in the BHK interpretation. The rightist absurd clause is read as:

($\bot^{r}$) For $\bot$ being recognized to follow from the set of hypotheses Γ *it is necessary and sufficient*, for any given atomic sentence e, to be recognizable that e WOULD follow from the set of hypotheses Γ.

What does this condition mean? It means that in order to establish recognizability, if one has shown a schematic semantical deduction of a sentential metaparameter c from hypotheses Γ, such that c does not occur in Γ, one has accomplished the task. It is clear that by substituting an atomic sentence s in place of this metaparameter c in the entire deduction we obtain a semantical deduction of s from hypotheses Γ. But there is a proviso: no *explicandum* of greater or equal complexity can be either introduced by sufficient conditions or eliminated by necessary conditions in the course of the schematic deduction. This makes explicit how to understand the auxiliary "WOULD".

A similar attitude applies to the other cases:

($\to^{l}$) For E being recognized to follow from the set of hypotheses $\Gamma, C \to D$ *it is necessary and sufficient*, for any given set of hypotheses Δ, to be recognizable that E WOULD follow from Γ, Δ on the supposition that D be recognizable to follow from the set of hypotheses Γ, Δ, C.[19]

($\vee^{r}$) For $C \vee D$ being recognized to follow from the set of hypotheses Γ *it is necessary and sufficient*, for any given sentence E, to be recognizable that E WOULD follow from the set of hypotheses Γ on the supposition that: (i) E be recognizable to follow from hypothesis C; and (ii) E be recognizable to follow from hypothesis D.

We claim that these clauses can be used[20] without violating predicativity if the proviso is respected. An example follows in the proof below, in particular lines (1) to (3):

Theorem 4. $C \Vdash C \vee D$.

Proof.

(1) $C, C \Vdash E$ and $D, C \Vdash E$	suppose
(2) $C, C \Vdash E$	from 1

[19] The metaparameter is here for finite sets of sentences.

[20] With some precaution.

(3)	for any E: $(C, C \Vdash E$ and $D, C \Vdash E \implies C \Vdash E)$	(1), (2), $(C, C = C)$, discharging (1)
(4)	$C \Vdash C \vee D$	$(\vee^{r})$ over (3)

QED[21]

Parameter E in the above schema is used freely in the steps (1) and (2), and bounded in (3). The schema giving rise to the universal quantifier in line (3) is from (1) to (2). C and D are parameters that can be freely substituted by sentences in the schema. An ideal agent can recognize that $C \vee D$ follows from hypothesis C once he is either able to build the schema or to understand it. His behavior *cum* understanding entitles him to perform the recognition of recognizability.

4.6.4 Equivalences

Some notable equivalences between clauses are worth of attention.

Theorem 5. *All rightist clauses are equivalent to the respective leftist clauses* under SC principles.

Proof. We examine here the case of left implication, the other cases are to be treated similarly. Assuming $(\rightarrow^{r})$ let's show $(\rightarrow^{l})$. First, the necessary condition direction. Suppose $\Gamma, C \rightarrow D \Vdash E$. By (Load) $\Gamma, \Delta, C \rightarrow D \Vdash E$. Suppose $\Delta, C \Vdash D$. By $(\rightarrow^{r})$ $\Delta \Vdash C \rightarrow D$. By (Load) $\Gamma, \Delta \Vdash C \rightarrow D$. By (Drop) $\Gamma, \Delta \Vdash E$. Second, the sufficient condition direction. Suppose for any Δ: $(\Delta, C \Vdash D \implies \Gamma, \Delta \Vdash E)$. By instantiation, $(C \rightarrow D, C \Vdash D) \implies (\Gamma, C \rightarrow D \Vdash E)$. We obtain $C \rightarrow D, C \Vdash D$ as follows. By (Idempotence), $C \rightarrow D \Vdash C \rightarrow D$. By $(\rightarrow^{r})$, $C \rightarrow D, C \Vdash D$.[22] QED

Some clauses can also be reformulated in different ways.

Theorem 6. *The following are alternatives explicitations that could be used for expressing necessary and sufficient conditions, instead of the respective original clauses,* under SC principles*:*

$(\wedge^{l^*})$ $\Gamma, C \wedge D \Vdash E \cong$ *For any* Δ*:* $((\Gamma, \Delta \Vdash C$ *and* $\Gamma, \Delta \Vdash D) \implies \Gamma, \Delta \Vdash E)$

$(\wedge^{r^*})$ $\Gamma \Vdash C \wedge D \cong$ *For any* E*:* $(C, D \Vdash E \implies \Gamma \Vdash E)$

$(\vee^{l^*})$ $\Gamma, C \vee D \Vdash E \cong$ *For any* Δ*:* $((\Delta \Vdash C$ *or* $\Delta \Vdash D) \implies \Gamma, \Delta \Vdash E)$

$(\rightarrow^{r^*})$ $\Gamma \Vdash C \rightarrow D \cong$ *For any* Δ*:* $(\Delta \Vdash C \implies \Gamma, \Delta \Vdash D)$ *Right 1*

$(\rightarrow^{r^{**}})$ $\Gamma \Vdash C \rightarrow D \cong$ *For any* E*:* $(D \Vdash E \implies \Gamma, C \Vdash E)$ *Right 2*

[21] Another proof of this same consequence relation can be given, a proof more in the style of an axiomatic proof. We present it below.

[22] See Appendix A for a full proof concerning implication.

Proof. Straightforward. QED

Theorem 7. *The following equivalence is implied by the leftist implication clause of Hypo, where $\overline{\top}$ represents any constructive tautological condition, but not vice versa:* (†) $C, C \to D \Vdash D \iff \overline{\top}$.

Proof. Suppose $C, C \to D \Vdash D \iff \overline{\top}$. (i) Suppose that for any $\Delta : (\Delta, C \Vdash D \implies \Gamma, \Delta \Vdash E)$. By instantiation, $C \to D, C \Vdash D \implies \Gamma, C \to D \Vdash E$. Since $\overline{\top}$, then $C, C \to D \Vdash D$. Therefore, $\Gamma, C \to D \Vdash E$. Now comes the reason why $(\to^{l})$ does not follow from (†). Suppose $\Gamma, C \to D \Vdash E$. Suppose $\Delta, C \Vdash D$. If the rightist clause $(\to^{r})$ were used, then we would obtain that for any $\Delta : (\Delta, C \Vdash D \implies \Gamma, \Delta \Vdash E)$ by (Transitivity). However, $(\to^{r})$ is equivalent to $(\to^{l})$, according to Theorem 5, what would then make us go into a vicious circle. From (†) it is impossible to prove $\Gamma, C \Vdash D \implies \Gamma \Vdash C \to D$. An easy instance where this can be visualized is $C \Vdash C \implies \Vdash C \to C$. Although D can be substituted by $C \to C$, we could not get rid of the hypothesis C in (†). QED

The statement of necessary and sufficient conditions for a logical constant does not imply that any deduction of the *explicitandum* has to be equal to a deduction of the *expliciens*, and for a good reason. The *expliciens* is formulated in metalanguage and uses quantification in some clauses. No *explicitandum* contains a metalinguistic logical constant. The *explicitandum* represents an object language deduction. Actually, this fact and Theorem 5 provide a way of obtaining a calculus whose rules have premises and conclusion which belong only to the sentential language, as it is the case with LJ^{s}. We have more to say about this point below.

4.7 Pragmatics *versus* semantics

It is pragmatically the case, for any given agent A, that either $\Gamma \Vdash_A C$ or that $\text{Not}(\Gamma \Vdash_A C)$. Since factual negation involves a judgement about matters of fact, the verb is used in the past.

A principle stating that either the relation of consequence can be recognized or it cannot be recognized corresponds to decidability:

(Dec) $\Gamma \Vdash C$ or $\Gamma \nVdash C$.

The pragmatical ground is not a sufficient support for taking (Dec) as a semantical principle. For the time being, we have no grounds to support (Dec). But, as we are going to see, this principle is correct for sentential languages.

Theorem 8. *The following equivalence is an alternative to the rightist implication clause only by assuming (Dec):*

$(\to^{r\#})$ $\Gamma \Vdash C \to D \cong$ *For any* Δ*:* $(\Delta \nVdash C$ *or* $\Gamma, \Delta \Vdash D)$.

Proof. It suffices to show that

$$\text{For any } \Delta\text{: } (\Delta \Vdash C \implies \Gamma, \Delta \Vdash D) \iff \text{For any } \Delta\text{: } (\Delta \nVdash C \text{ or } \Gamma, \Delta \Vdash D)$$

QED

4.8 Structural Hypo

Independent of the impredicativity question, Hypo clauses with quantification represent a problem in the measure that they are not amenable to a usual syntactical treatment, and in full form they would require a more complex calculus.

Let's say that *a clause is structural* if its condition is a finite metaexpression containing only the punctuation marks "(" and ")" for disambiguating expressions; the relation "$\Vdash$"; the metasentential logical constants "and" and "tautology"; finally, free sentential parameters and free parameters for finite sets of sentences. A *structural clause* is directly *amenable* to a *syntactical treatment*. They can easily be converted into rules of a finitary calculus – with one or more rules for the necessary conditions and one or more for the sufficient conditions. The formulation of syntactical rules for representing leftist implication and rightist disjunction clauses would require an extension of the sentential language with some sort of quantification. This is the same as adding forms into the recursively defined language, making it a kind of second-order sentential language.

There are different ways of choosing structural suited characterization of the semantics, such that each logical constant clause is structural. A particular wise choice is the following, we call it

Structural Hypo:

($\wedge^{l}$) $\quad \Gamma, C \wedge D \Vdash E \cong \Gamma, C, D \Vdash E$

($\vee^{l}$) $\quad \Gamma, C \vee D \Vdash E \cong \Gamma, C \Vdash E$ and $\Gamma, D \Vdash E$

($\to^{r}$) $\quad \Gamma \Vdash C \to D \cong \Gamma, C \Vdash D$

($\bot^{l}$) $\quad \Gamma, \bot \Vdash e \cong$ tautology (for e atomic).

SC principles – containing a simpler version of the (Load)[23]:

(Idempotence) $\quad C \Vdash C$

(Load) $\quad \Gamma \Vdash C \implies \Gamma, D \Vdash C$

(Drop) $\quad (\Gamma \Vdash C$ and $\Gamma, C \Vdash E) \implies \Gamma \Vdash E$

When considering a judgment of recognizability like $\Gamma \Vdash C$ as holding in Structural Hypo this will be the same as to say that there is a *Hypo*

[23] Observe that since we are considering the sets of hypotheses to be finite, any instance of the original (Load) principle – $\Gamma \Vdash C \implies \Gamma, \Delta \Vdash C$ – can always be obtained by a finite number of applications of the simpler form.

semantical proof of this statement which proceeds exactly like an *axiomatic proof*, since every occurrence of a sentence in this proof is either an instance of (Idempotence); or of $\Gamma, \bot \Vdash e$; or is the conclusion of a deduction according to one of the clauses, deduction whose premisses already belong to the deduction. As an example we restate Theorem 4 with a semantical "axiomatic proof":

Theorem 9 (second version of Theorem 4)**.** $C \Vdash C \vee D$.

Proof.
(1) $C \vee D \Vdash C \vee D$ (Idempotence)
(2) $C \Vdash C \vee D$ by one of the necessary condition readings of ($\vee^{I}$).
QED

5 Logical consequence

The concept of *logical consequence* for the sentential environment is now defined as follows for a set Λ of hypotheses maybe unbounded: $\Lambda \vDash C$ if and only if there is a finite $\Gamma \subseteq \Lambda$ such that $\Gamma \Vdash C$.[24]

6 Considerations

We claim that *the above characterization of logical consequence formally captures the constructive meaning of intuitionist sentential constants*.

This is so, *first of all*, because a logical constant meaning is made explicit by means of two semantical clauses presenting necessary and sufficient conditions for the use of the constant, one as hypothesis the other as consequent. Some thought that this would be better done through inferences or deduction rules. But formal rules are not completely transparent, they presuppose some understanding of what is being done and how it is being done. For example, constructivists tend to assimilate the occurrence of any formula C as an assertion, but this interpretation is far from obvious. Thus, it is simply better to state all sufficient and necessary conditions, making explicit all the metalogical constants employed once and for all. Additionally, according to the dogma we have pointed above, the use of rules has sometimes been made with the objective of reducing the hypothetical to the categorical, and we intend to show here that the reverse endeavor is feasible, simple and offers an alternative semantics for ISL.

Second, there is only one structure over which the semantics is build, a structure that is more complex than a simple Kripke model but at the same time much simpler than the set of all Kripke models. This structure is the set of all finite sets of hypotheses, i.e., the semilattice of all finite subsets of the sentential language. Such a structure is going to be examined below.

[24]This definition is good even for the case of classical logic to be examined below.

Third, the meaning of each logical constant is made explicit by using the relation of semantical consequence, taken as a primitive relation. This relation involves another primitive notion: that of hypothesis, which then has also to be made semantically explicit. This was done by the statement of (Idempotence), (Load) and (Drop). From now on, any definition of simple consequence (in particular of simple atomic consequence) can be closed by the relation of logical consequence thus generating what is called a Theory.[25]

At this point is seems fair to make an observation about Došen's ([Dos15], p. 153) remarks concerning the consequence relation. There are more points of agreement with him than disagreement. We also see as a dogma the conception according to which categorical notions should be primitive with respect of hypothetical notions. But, according to this author:

> Since B is a consequence of A whenever the implication $A \to B$ is true or correct, there would be no essential difference between the theory of inference and the theory of implication. An inference is often written vertically, with the premise above the conclusion, A/B, and an implication is written horizontally $A \to B$, but besides that, and purely grammatical matters, there would not be much difference.

We do not agree with that. The relation of consequence used for making explicit the meaning of logical constants is the strict relation of *semantical consequence*. However, other concepts of consequence not entirely based on semantical notions are a reality. Think about factual laws, for example.

The author says that the concept of consequence is an extensionalization of the concept of inference. Semantical consequence is a semantical concept. Syntactical consequence is a device employed in the attempt of capturing semantical consequence. They are not of same nature. Also, the difference between Hypo and the Category Theory treatment by Došen boils down to a difference between a theory of meaning and a mathematical theory of syntactical proofs.

Fourth, the concept of basis, usual in proof-theoretic semantics, plays no role in the core meaning of logical constants from Hypo's perspective. Bases were not introduced until now. Also, it is not urgent to answer the question of what is a basis, if they should be of semantical nature, of syntactical nature or of pragmatical nature.

[25]The case of absurd, and of negation, which is intimately related, is quite interesting. The semantical clauses make it explicit how to use the absurd, but it is the association of absurd with the relation of consequence between certain sentences that will give more content to the absurd. In this sense we can make a distinction between "core sense" (the one explicitated in the clause) and "plus sense" (the one added to the core content by a general relation of consequence).

7 LH sequent calculus

From Structural Hypo we extract *LH sequent calculus*. In the left side of its sequents occur finite syntactical lists that are to be treated as multisets. The calculus contains an extra rule of contraction, since syntactical sequents contain multisets in the left side:

Structural rules

Basic sequents: $C \vdash C$

Thinning on the left: $\Omega \vdash C \implies \Omega, D \vdash C$

Contraction on the left: $\Omega, E, E, \Psi \vdash C \implies \Omega, E, \Psi \vdash C$

Dropping: $(\Omega \vdash C \text{ and } C, \Omega \vdash D) \implies \Omega \vdash D$

Operational rules

Implication on the right: $\Omega \vdash C \to D \iff \Omega, C \vdash D$

Disjunction on the left: $\Omega, C \vee D \vdash E \iff (\Omega, C \vdash E \text{ and } \Omega, D \vdash E)$[26]

Conjunction on the left: $\Omega, C \wedge D \vdash E \iff \Omega, C, D \vdash E$

Absurd axiom for an atomic e: $\bot \vdash e$

Negation is a defined constant: $\neg C \equiv_{\text{df}} C \to \bot$.

The left side of the semantical symbol contains sets of hypotheses and the left side of a sequent contains lists representing multisets. In the semantics "C, C" in the left side designates the same set of hypotheses than "C", but in the syntax "C, C" has to be viewed as a different multiset from "C". Contraction rule in the syntactic calculus brings them together.

LH is another example of a calculus where no rule contains a quantification in the premisses. That is, the calculus only involves expressions of a recursive sentential language. Once soundness and completeness are established, a true simplification on how to use logical constants will be granted. The same observation will hold for LJ^s once proven the equivalence to LH .

8 ISL soundness and completeness

Theorem 10. LJ^s *and* LH *are sound and complete with respect to Hypo.*

Proof. Here is a description of the proof. First, focus Structural Hypo. Second, it is immediate that LH sequent calculus is sound with respect to Structural Hypo, just exchange the semantical symbol by the syntactical one. The contraction rule in LH has identical premise and conclusion from the semantical point of view. Second, prove that LH sequent calculus and LJ^s sequent calculus are equivalent. Third, as a Corollary, LJ^s sequent calculus is sound with respect to Structural Hypo. Fourth, LH is complete with respect of Structural Hypo since all clauses and principles of Structural Hypo are metaproperties of LH .

[26] The "and" can be eliminated if we state three distinct rules.

Fifth, LJ^s *is complete with respect to Structural Hypo*. Since Structural Hypo is equivalent to Hypo, completeness and soundness follow. QED

At this point we have an *argument for defending* (Dec), by using Gentzen's remark about decidability in his seminal paper [Gen35].

Corollary 1. $\Gamma \Vdash C$ *or* $\Gamma \nVdash C$.[27]

Proof. Gentzen [Gen35] observes that ISL is decidable. Therefore, $\Gamma \vdash C$ or $\Gamma \nvdash C$ for LJ^s and, because of equivalence, for LH too. By completeness and soundness, $\Gamma \Vdash C$ or $\Gamma \nVdash C$. That is, (Dec). QED

Of course, this is not a pure semantical argument for the validity of (Dec).[28]

9 Atomic consequence and bases

For completing the semantics it is necessary to employ a notion of consequence for atomic sentences, i.e., atomic bases. But the intention is to leave room for wide variations in bases, admitting even non-atomic sentences in them and avoiding any excessive restrictions on the relations represented in a basis.

Let $\text{Atoms}^+ = \{d \mid d \in \mathfrak{L}_{\text{At}} - \{\bot\}\}$ and $\text{Atoms}^- = \{d \to \bot \mid d \in \mathfrak{L}_{\text{At}} - \{\bot\}\}$. The atomic sentence d is the core of the positive atom d and of the negative atom $d \to \bot$. Atoms are the basic hypotheses which can figure in a consequence relation of an atomic basis. Let $\mathsf{SetOfBasicHypotheses} = \{\gamma \cup \delta \mid \gamma \subseteq \text{Atoms}^+ \text{ and } \delta \subseteq \text{Atoms}^- \text{ such that } \gamma \text{ and } \delta \text{ are finite sets disjoint on their core sentences}\}$. Let $\mathsf{BasicPairsOf}\mathfrak{L} = \{\langle \gamma, c\rangle \mid c \in \mathfrak{L}_{\text{At}} \text{ and } \gamma \in \mathsf{SetOfBasicHypotheses}\}$. For the sake of simplicity let $R_B \subseteq \mathsf{BasicPairsOf}\mathfrak{L}$ such that R_B is characterized constructively. The *atomic consequence relation on basis B over* $\mathfrak{L}$ is defined as follows:

$$\gamma \Vdash^B c \text{ if } \langle \gamma, c\rangle \in R_B$$

A *basis B is inconsistent* when $\Vdash^B \bot$. A *basis B is trivial* when, for any atomic sentence c, $\Vdash^B c$. Normally, none is desirable[29].

The reason for considering sentences of form $\boldsymbol{e} \to \bot$ as (negative) atomic hypotheses is that not all suppositions are of the form "suppose that it were true". They can also be like "suppose that it were false". These should have space in our semantical considerations, since we are taking hypotheses to be a primitive notion in our environment. We follow the usual constructive practice of representing the case of a sentence e being false as $e \to \bot$. Constructively

[27] This can also be proven directly from Hypo's definition, but it would be beside our main point here.

[28] The reason why this is not a semantical argument lies in the fact that Gentzen's proof of syntactical decidability requires induction on the length of syntactical proofs which might contain instances of contraction.

[29] In a basis these are different concepts.

speaking, it could occur that, for some atomic sentence e, $\nVdash^B e$, and, at the same time, $e \to \bot \Vdash^B \bot$, from which we expect that $\vDash^B e$ and $\vDash^B \neg\neg e$ will follow, given the usual definitions of negation and the clauses for implication. This relation $e \to \bot \Vdash^B \bot$ has then to be established at a basic level, in a basis B let's say.

The absurd is here considered a logical constant, but it is also going to be used in atomic bases – in the role of consequent and in negative hypotheses. Relations of basic consequence involving absurd are to be seen as an extension of the core meaning of this constant – core meaning given by semantical clauses.

An atomic basis B' is an *extension* of the atomic basis B whenever: for all $\langle\gamma, c\rangle \in \mathsf{BasicPairsOf}\mathfrak{L}$: $(\gamma \Vdash^B c \implies \gamma \Vdash^{B'} c)$. The extension of an atomic basis B means that the relation of atomic consequence in the extended basis B' is a superset of that in basis B.

The *empty basis* $(\emptyset)$ is the atomic basis defined over the $\mathsf{BasicPairsOf}\mathfrak{L}$ and such that for every $\langle\gamma, c\rangle$, $\gamma \Vdash^{\emptyset} c \iff c \in \gamma$.[30] *Decidable atomic bases* are those atomic bases B, in which R_B is decidable: $\gamma \Vdash^B c$ iff $\langle\gamma, c\rangle \in R_B$.

Theorem 11. *For any $\langle\gamma, c\rangle$ belonging to* $\mathsf{BasicPairsOf}\mathfrak{L}$*: $\gamma \Vdash^{\emptyset} c$ or $\gamma \nVdash^{\emptyset} c$.*[31]

Proof. Directly from the definition of empty basis. QED

Theorem 12. *For any atomic c: $\nVdash^{\emptyset} c$.*

Proof. Directly from the definition of empty basis. QED

Theorem 13. $\nVdash^{\emptyset} \bot$. *(The empty basis is neither inconsistent nor trivial).*

Proof. Immediate. QED

Given a basis B, a theory T_B is the closure of B by logical consequence. *The constructive relation of consequence over B is then the result of extending basis B with the relation of logical consequence* (all clauses are to be read accordingly). The extended relation of consequence is going to be represented as $\Gamma \vDash^B C$.

The focus that has been here put on atomic bases does not imply that non-atomic bases are useless or uninteresting. They can be interpreted as containing an extension of meaning for some logical constants. Atomic bases are defined as a minimal working ground for constructive logical constants. The objective is to be able to present countermodels, specially over the empty basis.

One point of difference between the usual intuitionist conception of logical constants and Hypo lies over disjunction. According to BHK, a disjunction can be asserted if and only if at least one of the components can be asserted.

[30] $\bot$ cannot belong to γ by definition.

[31] Observe that by Corollary 1, already $\gamma \Vdash c$ or $\gamma \nVdash c$.

This is slightly different in Hypo. This statement holds in Hypo only for closed proofs in the empty basis. It does not hold for a non-atomic basis Υ extending the empty basis and in which – for c, d and e distinct atomic sentences – it holds additionally that, for any C: $c, d \to C, e \to C \Vdash^{\Upsilon} C$. In such a non-atomic basis, the following is provable: $c \vDash^{\Upsilon} d \vee e$. But, neither $c \vDash^{\Upsilon} d$ nor $c \vDash^{\Upsilon} e$.

10 A conjecture about the definition of logical constants

If bases are taken as elements that can participate in the semantical explicitation of logical constants, then it might be feasible to give a structural characterization of the rightist disjunction and leftist implication clauses. Considering only atomic bases, a noteworthy conjecture proposed by Gentzen concerning the definition of logical constants by introductions can be answered according to the lines to be argued below.

Theorem 14.

$$\left.\begin{array}{l}\Lambda, \Theta, C \to D \vDash^{\emptyset} E, \textit{ i.e.,} \\ \textit{for any } \Delta, \\ (\Delta, C \Vdash^{\emptyset} D \implies \Lambda, \Theta, \Delta \Vdash^{\emptyset} E)\end{array}\right\} \begin{array}{l}\Longleftarrow \quad \Lambda, D \vDash^{\emptyset} E \textit{ and } \Theta \vDash^{\emptyset} C \\ \not\Longrightarrow \quad \Lambda, D \vDash^{\emptyset} E \textit{ and } \Theta \vDash^{\emptyset} C.\end{array}$$

$$\left.\begin{array}{l}\Gamma \vDash^{\emptyset} C \vee D, \textit{ i.e.,} \\ \textit{for any } E, \\ (\Gamma, C \vDash^{\emptyset} E \textit{ and } \Gamma, D \vDash^{\emptyset} E) \implies \Gamma \vDash^{\emptyset} E\end{array}\right\} \begin{array}{l}\Longleftarrow \quad \Gamma \vDash^{\emptyset} C \textit{ or } \Gamma \vDash^{\emptyset} D \\ \not\Longrightarrow \quad \Gamma \vDash^{\emptyset} C \textit{ or } \Gamma \vDash^{\emptyset} D.\end{array}$$

Proof. "Nonsequiturs" are as follows. Suppose c, d and e to be three distinct atomic sentences. $e, c \to d \vDash^{\emptyset} e$ is provable. Then, either we take Θ empty or equal to $\{e\}$. But $\nVdash^{\emptyset} c$ and $e \nVdash^{\emptyset} c$, since the sentences are atomic and the basis is empty. In the second case, clearly $c \vee d \Vdash^{\emptyset} c \vee d$, but it is not the case that $c \vee d \Vdash^{\emptyset} c$ or $c \vee d \Vdash^{\emptyset} d$. QED

10.1 Inference rules as definitions

According to Theorem 14 the right introduction rule for disjunction of the sequent calculus cannot be taken as the explicitation of the minimal sufficient condition for introducing disjunction. Disjunction right introduction represents a sufficient condition for having a disjunction in the consequent position. Maybe it even represents the weakest sufficient condition that can be stated in structural terms. But it is not the exact precise minimal sufficient condition. Therefore, *right introductions of sequent calculus cannot be taken in block as the definitions of sentential logical constants*.

According to Theorem 14 the left introduction rule for implication of sequent calculus cannot be taken as the explicitation of the maximal necessary condition for implication. Therefore, *left introductions of sequent calculus cannot be taken in block as the definitions of sentential logical constants*.

Gentzen's *dictum* concerning the definition of logical constants is stated for natural deduction introductions. According to him they are like definitions, the eliminations being their consequences thereof. Prawitz in many distinct places has set himself the objective of making such *dictum* precisely justified. In such attempt the notion of validity has the main role, and the introduction rules represent the most basic kind of validity: canonical validity. In [Pra72] the author conjectured that the logic to be validated would be intuitionistic logic.

The subject of validity definitions has been discussed in Sanz & Oliveira [San16] and in Oliveira [Oli19], for the case of Dummett and, for the case of Prawitz, by Piecha et al. [Pie15] and Piecha & Schroeder-Heister[32] [Pie19], which shows that intuitionistic sentential logic is not complete with respect to a standard notion of proof-theoretic semantics.

11 One structure model

The *Ground Hypo Structure* (GHS) of a language $\mathfrak{L}$ is the set of all finite subsets of $\mathfrak{L}$ ordered by inclusion (all possible finite sets of hypotheses), i.e., the semilattice of finite subsets of $\mathfrak{L}$ ordered by inclusion. This structure has a bottom, the empty set, and does not have a top. A *Hypo Semantics Model over a basis B* of $\mathfrak{L}$ ($Hypo_B$) is a semantic model over a GHS structure over $\mathfrak{L}$ in a basis B given by the semantical clauses for logical constants plus the structural consequence (SC) principles above.

A *set of sentences is contradictory* when it contains a sentence C and its negation $\neg C$. It is clear from the definition that a GHS for a language containing a way of expressing sentential negation contains sets of hypotheses that are contradictory.

We say that T_B *is inconsistent* if and only if, $\models^B \bot$. If $\Gamma \models^B C$ and $\Gamma \not\models^B C$, then $\models^B \bot$. Theorem 13 proves that $\not\Vdash^{\emptyset} \bot$. This is not sufficient.

Theorem 15. *$T_\emptyset$ is consistent ($\not\models^{\emptyset} \bot$).*

Proof. Let's consider Structural Hypo in the Empty basis. Attribute the following values V to each expression:

(i) $V(c) = 0$ if c is an atomic sentence.

(ii) $V(C) = \{0, 1\}$ depending on the truth table value of C.

(iii) Let $\Gamma^\wedge$ be the sentence obtained by conjunction of all sentences in Γ.

(iii.i) $V(\Gamma \models C) = 1$ if $V(\Gamma^\wedge) = 0$ and $V(C) = 0$.

(iii.ii) $V(\Gamma \models C) = 1$ if $V(\Gamma^\wedge) = 0$ and $V(C) = 1$.

(iii.iii) $V(\Gamma \models C) = 0$ if $V(\Gamma^\wedge) = 1$ and $V(C) = 0$.

(iii.iv) $V(\Gamma \models C) = 1$ if $V(\Gamma^\wedge) = 1$ and $V(C) = 1$.

[32] In this item the authors disprove that old conjecture by Prawitz that the proof-theoretical notion of validity would result intuitionistic logic.

It is clear that $V(\vDash^{\emptyset} \bot) = 0$. Any (Idempotence) has $V(C \Vdash C) = 1$. For $(\bot^1)$, $V(\Gamma, \bot \Vdash e) = 1$. All other clauses will always result semantical consequence relations with value 1, if the premise(s) has (have) value 1. Therefore it is impossible to recognize a consequence relation with value 0 having an empty set of hypotheses. QED

Corollary 2. *Hypo is consistent ($\nVdash \bot$).*

Proof. Hypo is equivalent to Structural Hypo, i.e, any semantical relation of consequence that can be shown in Structural Hypo can also be shown in $T_{\emptyset}$. But if $\Vdash \bot$, then $\vDash^{\emptyset} \bot$. QED

In a semantics in which hypotheses enter as primitive, sets of hypotheses can be contradictory as we noticed above. We'll say that a set Γ of sentences is *inconsistent* if and only if $\Gamma \vDash^{\emptyset} \bot$, and a set Γ of hypothesis is *trivial* if and only if for all E: $\Gamma \vDash^{\emptyset} E$. Our definition of the absurd above is such that it is known to follow from a set of hypotheses when this set is atomically trivial.[33]

12 Semantical non-recognizability of consequence

The semantical relation of consequence is decidable, Corollary 1 states it. Therefore, non-recognizability implications can be obtained. Starting with structural principles, they are like the following.

Theorem 16.
(unLoad) $\Gamma, D \nVdash C \implies \Gamma \nVdash C$;
(unDrop) $\Gamma \nVdash E \implies (\Gamma \nVdash C$ *or* $\Gamma, C \nVdash E)$.

Proof. (unLoad) and (unDrop) are the contrapositions of (Load) and (Drop) respectively. QED

Now the equivalence clauses.

Theorem 17.
Leftist equivalences:

$(\wedge^l)^-$ $\Gamma, C \wedge D \nVdash E \iff \Gamma, C, D \nVdash E$

$(\vee^l)^-$ $\Gamma, C \vee D \nVdash E \iff \Gamma, C \nVdash E$ *or* $\Gamma, D \nVdash E$

$(\rightarrow^l)^-$ $\Gamma, C \rightarrow D \nVdash E \iff$ (*for any* Δ: $(\Delta, C \Vdash D \implies \Gamma, \Delta \Vdash E)) \implies$ *absurd*

$(\bot^l)^-$ $\Gamma, \bot \nVdash c \iff$ *absurd*

[33] Triviality is then a strategy that can be used for telling when an atomic sentence is false or, what is similar, when its negation is true.

Rightist equivalences:

$(\wedge^r)^-$ $\Gamma \nVdash C \wedge D \iff \Gamma \nVdash C$ *or* $\Gamma \nVdash D$

$(\vee^r)^-$ $\Gamma \nVdash C \vee D \iff$

(*for any* E: $((\Gamma, C \Vdash E$ *and* $\Gamma, D \Vdash E) \implies \Gamma \Vdash E)) \implies$ *absurd*

$(\rightarrow^r)^-$ $\Gamma \nVdash C \rightarrow D \iff \Gamma, C \nVdash D$

$(\bot^r)^-$ $\Gamma \nVdash \bot \iff$ (*for any* c: $\Gamma \Vdash c) \implies$ *absurd*

Proof. Straightforward over Hypo clauses by contraposition and using (Dec). QED

The reading of clause $(\wedge^r)^-$ is the following:

> $C \wedge D$ cannot be recognized to follow from the set of hypotheses Γ *if and only if* C cannot be recognized to follow from Γ, or D cannot be recognized to follow from Γ.

The quantified clauses are stated using "$\implies$ absurd" since the validity of (Dec) – i.e., $\Gamma \Vdash C$ or $\Gamma \nVdash C$ – does not suffice for establishing the equivalence with "There is . . . which is not".[34]

Theorem 18. $\Gamma, \bot \nVdash^B E$ *under no condition.*

Proof. Straightforward. QED

An example of a proof using $(\bot^r)^-$ is the following. Consider the empty basis, and consider d and e to be two distinct atomic sentences and both distinct from $\bot$. Take Γ to be $\{d\}$. The objective is to establish that $d \nvDash^{\emptyset} \bot$. Now comes the reasoning. Suppose that for any c: $d \vDash^{\emptyset} c$. Thus, in particular $d \vDash^{\emptyset} e$. But $d \nvDash^{\emptyset} e$ by the definition of the empty basis, since d and e are distinct. Hence, from the hypothesis an absurd was extracted. Therefore, it is absurd that for any c: $d \nvDash^{\emptyset} c$ – considering this "absurd" as a metalinguistic intuitionistic negation. Thus, $d \nvDash^{\emptyset} \bot$.

13 Classical logic semantics

Everybody knows, classical logic semantics is about truth tables. It is simple, elegant and decidable. The number of lines in the table is 2^n where n is the number of different (atomic) sentences, independently valued, in the whole entire sentence under examination. Each line in the table corresponds to a possible valuation. Each line in the table should correspond to a partition of bases, in our terminology.

It is an interesting fact that *the above semantics for Intuitionistic Sentential*

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[34] If full *tertium non datur* were assumed in the metalanguage, the three clauses would become:

$(\rightarrow^l)^-$ $\Gamma, C \rightarrow D \nVdash E \iff$ There is a $\Delta \geq \Gamma$ such that $(\Delta, C \Vdash D$ and $\Delta \nVdash E)$

$(\vee^r)^-$ $\Gamma \nVdash C \vee D \iff$ There is an E such that $((\Gamma, C \Vdash E$ and $\Gamma, D \Vdash E)$ but $\Gamma \nVdash E)$

$(\bot^r)^-$ $\Gamma \nVdash \bot \iff$ There is an e such that $\Gamma \nVdash e$

Logic can be extended to a semantics for Classical Logic by the addition of only one semantical principle. And it can be done in a way that it does not involve any logical constant:

(Peirce) $\Gamma \Vdash D \iff$ For any Δ: (for any C: $(\Delta, D \Vdash C) \implies \Gamma, \Delta \Vdash D$)

Let's call *HypoC* the semantics obtained from Hypo by the addition of the above principle. The intuitive reading is: for any Δ, if D would follow from Γ, Δ on the supposition that a consequence C whatever follows from Δ, D, then D really follows from Γ. Of course, now, the interpretation of the semantic relation becomes different. We cannot talk anymore of recognition.

Concerning the content of this principle, once we have a proof of $\Gamma \Vdash D$ in Hypo, the principle does not say much. By (Load), for any extension Δ, it will be the case that $\Gamma, \Delta \Vdash D$. Hence, according to (Peirce), $\Gamma \Vdash D$, but that was already established.

However, suppose that $\Delta, \Gamma \Vdash D$ can be established in Hypo on the supposition of having – for a given list $C1, \ldots, Cn$ of sentences – $(\Delta, D \Vdash C_1), \ldots, (\Delta, D \Vdash C_n)$. Notice that it is not necessary to have used all these suppositions in order to derive $\Delta, \Gamma \Vdash D$. Hence, according to (Peirce), this will be enough for saying that we have shown $\Gamma \Vdash D$.

Theorem 19. *Peirce's sentence $((D \to E) \to D) \to D$ is validated in HypoC.*

Proof. Suppose that, for any C, $(\Delta, D \Vdash C)$. By instantiation, $\Delta, D \Vdash E$. Then, $\Delta \Vdash D \to E$. By (Load) $\Delta, (D \to E) \to D \Vdash D \to E$. By (Idempotence) and (Load) $\Delta, (D \to E) \to D \Vdash (D \to E) \to D$. By modus ponens, $\Delta, (D \to E) \to D \Vdash D$. Thus, for any C, $(\Delta, D \Vdash C) \implies \Delta, (D \to E) \to D \Vdash D$, discharging the supposition. Hence, for any Δ, (for any C: $(\Delta, D \Vdash C) \implies \Delta, (D \to E) \to D \Vdash D$). By (Peirce), $(D \to E) \to D \Vdash D$. Finally, by $(\to^r)$, $\Vdash ((D \to E) \to D) \to D$. QED

14 Conclusion

In the above we presented Hypo constructive semantics for Intuitionistic Sentential Logic and HypoC for Classical Sentential Logic. The proposal takes the concept of semantical consequence as primitive, with the inherent primitive concept of hypothesis.

From Hypo's perspective, the semantical distinction between classical and intuitionistic logic at the sentential level does not lie over any particular logical constant. That is, from a formal point of view, clauses for intuitionistic logic and classical logic are the same in their general form. What changes is the underlying relation of semantical consequence. The intuitionists use an epistemological concept of semantical consequence (which requires an act of cognition), the classical logicians use an ontological concept of semantical consequence (which is committed with the existence of a definite answer for every problem). The difference between them is represented by an extra principle for the concept of semantical consequence.

From the perspective of Hypo semantics, Kripke semantics is criticizable. It depends on accepting principles that intuitionists cannot. This is the case of the third middle excluded for the relation of forcing. It is not wrong, but is not justifiable as a constructive principle. Its assumption in this semantics induces an ontological commitment, which is even clearer if we observe that a state of knowledge can depend on an actual infinite set of atomic propositions. This principle is of the form $\alpha \Vdash c$ or $\alpha \nVdash c$, where α is a state of knowledge possibly (crazily) infinite. As we saw, it is the case that $\gamma \Vdash c$ or $\gamma \nVdash c$ in Hypo, but this was not assumed, it was proved.

We also claim that the canonical way of making explicit the meaning of logical constants is by means of clauses stating necessary and sufficient conditions in the context of semantical consequence. No appeal to the notions of truth or of proof was made, although these notions are not considered meaningless. Once the concept of logical consequence is established with the support of the concept of semantical consequence, we can now define the concepts of proof and of valid inference in an easy way that the reader may well figure out.

Appendix A

Hypo's implication clause $(\to^{\mathrm{r}})$ is equivalent to the following alternative formulation:

$(\to^{\mathrm{r}^*})$ $\Gamma \Vdash C \to D \iff$ For all $\Delta \geq \Gamma$: $(\Delta \Vdash C \implies \Delta \Vdash D)$

The proof of equivalence requires (Transitivity):

(i) $(\to^{\mathrm{r}^*})$ from $(\to^{\mathrm{r}})$. Suppose, for all $\Delta \geq \Gamma$, $(\Delta \Vdash C \implies \Delta \Vdash D)$. By instantiation, $\Gamma, C \Vdash C \implies \Gamma, C \Vdash D$. By (Idempotence) and (Load) $\Gamma, C \Vdash C$. By modus ponens, $\Gamma, C \Vdash D$. Hence, by $(\to^{\mathrm{r}})$, $\Gamma \Vdash C \to D$. Now the reverse. Suppose $\Gamma \Vdash C \to D$. By $(\to^{\mathrm{r}})$, $\Gamma, C \Vdash D$. Suppose $\Delta \geq \Gamma$. Suppose $\Delta \Vdash C$. By (Transitivity), $\Delta \Vdash D$. Thus $\Delta \Vdash C \implies \Delta \Vdash D$. Finally, for all $\Delta \geq \Gamma$ $((\Delta \Vdash C \implies \Delta \Vdash D)$.

(ii) $(\to^{\mathrm{r}})$ from $(\to^{\mathrm{r}^*})$. Suppose $\Gamma \Vdash C \to D$. By $(\to^{\mathrm{r}^*})$, for all $\Delta \geq \Gamma$, $(\Delta \Vdash C \implies \Delta \Vdash D)$. By instantiation, $\Gamma, C \Vdash C \implies \Gamma, C \Vdash D$. By (Idempotence) and (Load) $\Gamma, C \Vdash C$. By modus ponens, $\Gamma, C \Vdash D$. Now the reverse. Suppose $\Gamma, C \Vdash D$, $\Delta \geq \Gamma$ and $\Delta \Vdash C$. By (Load), $\Delta, C \Vdash D$. By (Drop) $\Delta \Vdash D$. Hence, for all $\Delta \geq \Gamma$, $(\Delta, A \Vdash D \implies \Delta \Vdash C)$. I.e., by $(\to^{\mathrm{r}^*})$, $\Gamma \Vdash A \to D$.

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On Prawitz' Ecumenical system

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Proof-theoretical semantics, Tübingen, 2019

With a very big help from my friends!

(Valeria de Paiva, Elaine Pimentel, Dag Prawitz, Alberto Naibo, Victor Nascimento, Wagner Sanz, Hermann Haeusler..)

Plan for the talk

1. What is Ecumenism?
2. Prawitz' system
3. One digression
4. A bit of proof theory
5. New ecumenical systems
6. Related and future work

Ecumenism

Ecumenical systems

Main idea: a codification where two or more logics can *coexist in peace.*

Ecumenism

Coexisting in peace: the different (maybe rival!) logics accept and reject the same things (principles, rules,...)

The ecumenical view
Prawitz 2015
Dowek 2015
Krauss 1992

A possible problem for revisionism in Logic

An standard form of disqualifying the conflict between two logics is based on the somewhat reasonable idea that the *litigants* are talking about distinct things (or speaking different things), and that if they are talking about different things, there is not "the same thing" - a rule or a principle - on which they diverge and dispute.

According to this position, it is as if the participants of the conflict spoke different languages and did not realize it.

A possible problem for revisionism in Logic

An easy argument (Quine, 1970):

1. If the deviant/revisionist logician does not accept the general validity of a classical principle of reasoning, then he gives new meanings to the concepts used in the formulation of the principle.
2. If the deviant logician gives new meanings to the concepts used in the formulation of the principle, then the deviant logician and the classical logician are not talking about the same thing (principle).
3. If they are are talking about different things, they cannot disagree!!!
4. The deviant logician does not accept the general validity of the principle.

Thus, they do not disagree!!!!

Dag Prawitz seems to agree with Quine when he says:

"When the classical and intuitionistic codifications attach different meanings to a constant, we need to use different symbols, and I shall use a subscript c for the classical meaning and i for the intuitionistic. The classical and intuitionistic constants can then have *a peaceful coexistence* in a language that contains both."
(Prawitz [2015])

An alternative is to use the idea of Hilbert and Poincaré that axioms and deduction rules define the meaning of the symbols of the language and it is then possible to explain that some judge the proposition $(P \vee \neg P)$ true and others do not because they do not assign the same meaning to the symbols $\vee$, $\neg$, etc.
(Dowek [2015])

Taking this idea seriously, we should not say that the proposition $(P \vee \neg P)$ has a classical proof but no constructive proof, but we should say that the proposition $(P \vee_c \neg_c P)$ has a proof and the proposition $(P \vee \neg P)$ does not, that is we should introduce two symbols for each connective and quantifier, for instance a symbol $\vee$ for the constructive disjunction and a symbol $\vee_c$ for the classical one, instead of introducing two judgments: "has a classical proof" and "has a constructive proof' (Dowek [2015])

What's Prawitz' main idea?

The same meaning explanation for classical logic and intuitionistic logic.

But this does not seem possible!

Gentzen's introduction rule for disjunction (and for implication and the existential quantifier) is too strong! It cannot give the meaning of classical disjunction.

Solution: different introduction rules for classical disjunction

Interesting: two disjunctions, but the same idea of meaning explanation.

The Natural Deduction Ecumenical system

NEc

The language of NEc is defined as follow:
Alphabet

1. Individual variables, individual parameters, predicate letters;
2. logical constants: $\bot$, $\wedge$, $\neg$, $\forall$, $\vee_i$, $\vee_c$, $\rightarrow_i$ and $\rightarrow_c$, $\exists_i$, $\exists_c$;
3. Auxiliary signs: (,) .

The grammar of Ec is the usual.

The Natural Deduction system NEc defined by Prawitz has the following rules of inference:

1. The rules for $\wedge$, $\neg$ and for the intuitionistic operators are the usual Gentzen-Prawitz introduction and elimination for these operators.
2. The intuitionistic absurd rule:

$$\frac{\bot}{A}$$

3. The rules for classical disjunction and classical implication are defined as follows:

$$\dfrac{\begin{matrix}[A] & & [\neg B] \\ & \Pi_1 & \\ & \bot & \end{matrix}}{A \rightarrow_c B} \rightarrow_c\text{-Int}$$

$$\dfrac{A \rightarrow_c B \qquad A \qquad \neg B}{\bot} \rightarrow_c\text{-Elim}$$

$$\dfrac{\begin{matrix}[\neg A] & & [\neg B] \\ & \Pi_1 & \\ & \bot & \end{matrix}}{A \vee_c B} \vee_c\text{-Int}$$

$$\dfrac{A \vee_c B \qquad \neg A \qquad \neg B}{\bot} \vee_c\text{-Elim}$$

$$\dfrac{\begin{matrix}[\forall x \neg A(x)] \\ \Pi \\ \bot\end{matrix}}{\exists_c x A(x)}\ \exists_c - I$$

$$\dfrac{\exists_c x A(x) \qquad \forall x \neg A(x)}{\bot}\ \exists_c - E$$

Classical implication and *modus ponens*

Classical implication: Contrary to what we could expect from any reasonable concept of conditional judgements (hypothetical judgement), the operator $\rightarrow_c$ does not satisfy *modus ponens*. This is due to the fact that the introduction rule for $\rightarrow_c$ is weaker than the introduction for $\rightarrow_i$, since the classical logician is allowed to assert $(A \rightarrow_c B)$ in cases where the intuitionistic logician is not. It is interesting to observe that the general validity of *modus ponens* for $\rightarrow_c$ would not depend solely on the meaning of $\rightarrow_c$, but would also depend on a concept of negation that is not determined by the introduction rule for negation. The classical implication $\rightarrow_c$ clearly satisfies a weak form of *modus ponens*: {A, (A $\rightarrow_c$ B)} $\vdash \neg\neg B$.

Some interesting theorems

1. $\vdash_{NEc} (A \rightarrow_i B) \Rightarrow_i (A \rightarrow_c B)$
2. $\vdash_{NEc} (A \wedge B) \Leftrightarrow_i \neg(\neg A \vee_c \neg B)$
3. $\vdash_{NEc} (A \wedge B) \Leftrightarrow_i \neg(A \rightarrow_c \neg B)$
4. $\vdash_{NEc} \neg(\neg A \wedge \neg B) \Leftrightarrow_i (A \vee_c B)$
5. $\vdash_{NEc} \neg(A \wedge \neg B) \Leftrightarrow_i (A \rightarrow_c B)$

Definition

A formula B is called *classical* if and only if its main operator is classical (we sometimes indicate that B is classical with the notation B^c)

Some more interesting theorems

1. $\vdash_{NEc} (A \rightarrow_c B^c) \rightarrow_i (A \rightarrow_i B^c)$
2. $\{A, (A \rightarrow_c B^c)\} \vdash_{NEc} B^c\}$

Interesting remark: The system NEc does not satisfy the deduction theorem!

One negation or Two negations

In the propositional part of the ecumenical system defined by Prawitz, we have the following logical constants: $\wedge$, $\bot$, $\neg$, $\rightarrow_i$, $\rightarrow_c$, $\vee_i$, and $\vee_c$. The problem now is: *why do we have just one negation, given that we have two implications and the negation of A could be understood as "A implies $\bot$"?* (The "ecumenical" system defined by Peter Krauss uses a single negation, and although the system defined by Gilles Dowek begins with two negations, at some point in the paper (p.232), Dowek concludes that the system can work with just one negation).

Two possible answers:

1. We can prove that $(A \rightarrow_i \bot)$ and $(A \rightarrow_c \bot)$ are "interderivable" in the ecumenical system, in the sense that the equivalences $((A \rightarrow_i \bot) \leftrightarrow_i (A \rightarrow_c \bot))$ and $((A \rightarrow_i \bot) \leftrightarrow_c (A \rightarrow_c \bot))$ are provable in the ecumenical system.
2. We can argue that in fact there's only one way to assert the negation of a proposition A: in order to assert $\neg A$ we have to derive a contradiction from A.

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One could reply:

1. Interderivability is a weak form of equivalence. The fact that all theorems of classical propositional logic are "equivalent" clearly does not imply that we just have one theorem! Although it is not clear how to define a more robust notion of equivalence, it is clear that "material equivalence" alone is not sufficient to justify the use of a single negation
2. We may accept that there's just one way to assert the negation of a proposition A, to wit, to produce a derivation of a contradiction from the assumption A. But we may also accept that there might be different ways to derive a contradiction from A, that there might be classical and intuitionistic derivations of $\bot$ from A, and that this fact would establish two different ways we could use to negate A, and hence that we should have two negations, a classical one and an intuitionistic one.

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1. *Question* Can we find a derivation of $\perp$ from A such that it is "essentially classic", in the sense that it (differently from what happens in the example of Fact 1) "essentially" uses classical reasoning in the derivation of $\perp$ from A? If such a derivation does exist, we would have a very good reason to defend the use of two negations, one classical, one intuitionistic.
2. In the case of propositional logic, the answer is no! Given any classical derivation of $\perp$ from an assumption A, then there is also an intuitionistic derivation of $\perp$ from the assumption A. This is a trivial consequence of Glivenko's theorem and the normalization theorem.

The Joyal collapse and Natural Deduction

The collapse was formulated in categorial terms, but in a natural deduction setting it corresponds to the idea that there aren't different intuitionistic derivations of a formula of the form $\neg A$, or equivalently, there's just one intuitionistic derivation of $\perp$ from the assumption A. Imteresting point: can we obtain the colapse without extra reductions?

The Joyal collapse and Natural DEduction

Consider the following two derivations of $\bot$ from $(A \wedge \neg A)$:

$$\dfrac{\dfrac{(A \wedge \neg A)}{A} \qquad \dfrac{(A \wedge \neg A)}{\neg A}}{\bot}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(A \wedge \neg A)}{A} \qquad \dfrac{(A \wedge \neg A)}{\neg A}}{\bot}}{A}\bot_I \qquad \dfrac{(A \wedge \neg A)}{\neg A}}{\bot}$$

In what sense would these two derivations be equal?

A new Reduction

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \bot\end{matrix}}{A} \qquad \begin{matrix}\Pi_2 \\ (A \rightarrow B)\end{matrix}}{B}$$

Reduces to

$$\dfrac{\begin{matrix}\Pi_1 \\ \bot\end{matrix}}{B}$$

Lot's of things to be done!

1. Proof theory
2. Semantics.

A bit of Proof Theory

Reductions

The reductions for the intuitionistic operators are the usual Prawitz' reductions.
The reductions for the classical operators are defined below:

$$\dfrac{\dfrac{\begin{matrix}[A] & & [\neg B]\\ & \Pi_1 & \\ & \bot & \end{matrix}}{A \to_c B} \qquad \begin{matrix}\Pi_2\\ A\end{matrix} \qquad \begin{matrix}\Pi_3\\ \neg B\end{matrix}}{\bot}$$

Reduces to:

$$\begin{matrix}\Pi_2 & & \Pi_3\\ [A] & & [\neg B]\\ & \Pi_1 & \\ & \bot & \end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[\neg A] \quad [\neg B] \\ \Pi_1 \\ \bot\end{matrix}}{A \vee_c B} \qquad \begin{matrix}\Pi_2 \\ \neg A\end{matrix} \qquad \begin{matrix}\Pi_3 \\ \neg B\end{matrix}}{\bot}$$

Reduces to:

$$\begin{matrix}\Pi_2 \qquad \Pi_3 \\ [\neg A] \quad [\neg B] \\ \Pi_1 \\ \bot\end{matrix}$$

Problem for normalization: inductive measures!

Solution: new measures of complexity!

In the case of the new reductions we immediately see that through the elimination of a maximum formula, new maximum formulas of the same degree may be produced, and because of this the usual normalization strategy does not work anymore. An easy way to solve this difficult is through the modification of the usual definition of the degree of a formula as the number of occurrences logical operators in the formula.

It is clear that in the case of classical disjunction and classical implication there are some *hidden negations*, and that any definition of the complexity of a formula must take this point in consideration. The new measure of complexity of a formula A will be called the ecumenical degree of A, ed(A), and is defined as follows:

E-degree

1. ed(ϕ) = 0
2. ed($\neg A$) = ed(A) + 1
3. ed(A □ B) = ed(A) + ed(B) + 1, if □ is $\wedge$ or an intuitionistic operator.
4. ed(A $\vee_c$ B) = ed(A) + ed(B) + 4
5. ed(A $\rightarrow_c$ B) = ed(A) + ed(B) + 3

The Normalization Theorem

The main definitions are standard.

Definition

A formula A in a derivation Π is a maximum formula if and only if:

1. A is the conclusion of an application of an α-introduction rule and at the same time the major premisse of an α-elimination rule in Π.
   or
2. A is the conclusion of an application of the $\bot$-rule and at the same time the major premisse of an elimination rule in Π.
   or
3. A is the conclusion of an application of the $\vee_i$-elimination rule and at the same time the major premisse of an elimination rule in Π.

Definition

The ecumenical degree of a derivation Π, e$d[\Pi]$ is defined as the max{ ed[A] s. t. A is a maximum formula in Π }.

Definition

A derivation Π is called *critical* iff

1. Π ends with an elimination rule α;
2. The major premiss of α is a maximum formula;
3. For every proper sub-derivation Π' of Π, $ed[\Pi'] \leq ed[\Pi]$.

Lemma

Let $\Pi 1$ / A and A/$\Pi 2$ be two derivations in NEc such that d($\Pi 1$) = n1 and d($\Pi 2$) = n2. Then, d($\Pi 1$/[A]/$\Pi 2$) = max(d[A], n1, n2).

Lemma

Let Π be a critical derivation of $\Gamma \vdash_{NEc} A$. Then, Π reduces to a derivation Π' of $\Delta \subseteq \Gamma \vdash_{NEc} A$ such that d(Π') $< d(\Pi)$.

Lemma

Let Π be a derivation of $\Gamma \vdash_{NEc} A$. Then, Π reduces to a derivation Π' of $\Delta \subseteq \Gamma \vdash_{NEc} A$ such that d(Π') $< d(\Pi)$.

Proof.

Directly from the previous lemma using induction on the length of Π. □

Theorem

Let Π be a derivation of $\Gamma \vdash_{NEc} A$. Then, Π reduces to a normal derivation Π' of $\Delta \subseteq \Gamma \vdash_{NEc} A$.

New Ecumenical systems

Translations

Theorem

Let S_1 and S_2 be two logics formulated in the languages L_1 and L_2 respectively, and let F be a translation from L_2 into L_1 that interprets S_2 into S_1. Let S_3 be an intermediate logic between S_1 and S_2. Then, if F also satisfies the property that A $-||-$ F(A) in S_2, then F is a translation from L_2 into the language L_3 of S_3 that interprets S_2 into S_3.

Translations

Theorem

The translation F of the previous theorem cannot be a translation from S_3 into S_1.

A new ecumenical system - Classical Logic (CL) and the Logic of Constant (CD) Domains

The system FIL

Definition

A (decorated) *sequent* is an expression of the form

$$A_1(n_1), \ldots, A_k(n_k) \Rightarrow B_1/S_1, \ldots, B_m/S_m$$

where

- A_i for $(1 \leq i \leq k)$ and B_j for $(1 \leq j \leq m)$ are formulae of intuitionistic propositional logic;
- n_i for $(1 \leq i \leq k)$ are natural numbers. We say that n_i is the *index* of the formula A_i;
- S_j for $(1 \leq j \leq m)$ are *sets* of natural numbers. We call S_j the *dependency set* of the formula B_j.

THE SYSTEM FIL

$$\text{Axiom}\ \frac{}{A(n) \vdash A\{n\}} \qquad \frac{\Gamma \vdash A/S, \Delta' \qquad A(n), \Gamma' \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta', \Delta^*}\ \text{Cut}$$

$$Ex_L\ \frac{\Gamma, A(n), B(m), \Gamma' \vdash \Delta}{\Gamma, B(m), A(n), \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash A/S, B/S', \Delta}{\Gamma \vdash B/S', A/S, \Delta}\ Ex_R$$

$$W_L\ \frac{\Gamma \vdash \Delta}{\Gamma, A(n) \vdash \Delta^*} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A/\{\ \}, \Delta}\ W_R$$

$$Con_L\ \frac{\Gamma, A(n), A(m) \vdash \Delta}{\Gamma, A(k) \vdash \Delta^*} \qquad \frac{\Gamma \vdash A/S, A/S'\Delta}{\Gamma \vdash A/S \cup S', \Delta}\ Com_R$$

$$\to_L\ \frac{\Gamma \vdash A/S, \Delta \qquad \Gamma', B(n) \vdash \Delta'}{\Gamma, \Gamma', A \to B(n) \vdash \Delta, \Delta'} \qquad \frac{\Gamma, A(n) \vdash B/S, \Delta}{\Gamma \vdash (A \to B)/S - \{n\}, \Delta}\ \to_R$$

THE SYSTEM FIL

$$\wedge_L\ \frac{\Gamma \vdash A/S, \Delta \qquad \Gamma' \vdash B/S'\Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta', (A \wedge B)/S \cup S'} \qquad \frac{\Gamma, A(n), B(m) \vdash \Delta}{\Gamma, (A \wedge B)(k) \vdash \Delta^*}\ \wedge_R$$

$$\frac{\Gamma \vdash \Delta, A/S, B/S'}{\Gamma \vdash \Delta, (A \vee B)/S \cup S'}\ \vee_R \qquad \wedge_L\ \frac{\Gamma, A(n) \vdash \Delta \qquad \Gamma', B(m) \vdash \Delta'}{\Gamma, \Gamma', (A \vee B)(k) \vdash \Delta^*, \Delta'^*}$$

The System CD

The system CD is obtained from the system FIL through the addition of the usual classical rules (now with decorations) for quantifiers:

THE SYSTEM CD

The system CD is obtained from the system FIL through the addition of the usual classical rules (now with decorated) for quantifiers:

$$\forall_L \frac{\Gamma, A(t)(n) \vdash \Delta}{\Gamma, \forall x A(x)(n) \vdash \Delta} \qquad \frac{\Gamma \vdash A(a)/S, \Delta}{\Gamma \vdash \forall x A(x)/S, \Delta} \forall_R$$

$$\exists_L \frac{\Gamma, A(a) \vdash \Delta}{\Gamma, \exists x A(x) \vdash \Delta} \qquad \frac{\Gamma \vdash A(t)/S, \Delta}{\Gamma \vdash \exists x A(x)/S, \Delta} \exists_R$$

THE ECUMENICAL SYSTEM ECD

The ecumenical system ECD is obtained from the system CD through the addition of the new rules (now with decoratations) for $\rightarrow_c$, $\vee_c$ and $\exists_c$:

$$\rightarrow_c -L \; \frac{\Gamma, A(n), \neg B(m) \vdash \bot / S}{\Gamma \vdash (A \rightarrow_c B)/S^*} \qquad \frac{\Gamma \vdash A/S \qquad \Gamma \vdash \neg B/S'}{\Gamma, (A \rightarrow_c B)(m) \vdash \bot / S \cup S' \cup \{m\}} \; \rightarrow_c -R$$

$$\vee_c - R \; \frac{\Gamma, \neg A(n), \neg B(m) \vdash \bot / S}{\Gamma \vdash (A \vee_c B)/S} \qquad \frac{\Gamma \vdash \neg A/S \qquad \Gamma' \vdash \neg B/S'}{\Gamma, (A \vee_c B)(m) \vdash \bot / S \cup S' \cup \{m\}} \; \vee_c - L$$

$$\exists_c - R \; \frac{\Gamma, \forall x \neg A(x)(m) \vdash \bot / S}{\Gamma \vdash \exists_c x A(x)/S^*} \qquad \frac{\Gamma \vdash \forall x \neg A(x)/S}{\Gamma, \exists_c x A(x)(m) \vdash \bot / S \cup \{m\}} \; \exists_c - L$$

A new ecumenical system NEm - Intuitionistic Logic (IL) and Minimal Logic (ML) Domains (Victor Nascimento)

The system NEm

- Two implications: $\rightarrow_i$, $\rightarrow_m$
- Two universal quantifiers: $\forall_i$, $\forall_m$

The ecumenical system NEm

The rules for $\wedge$, $\perp$, $\vee$ and $\exists$ are the usual ones.
The new rules for the new operators are as follows:

$$\frac{\begin{array}{c}[A] \\ \vdots \\ B \vee \bot\end{array}}{A \rightarrow_i B}\ \mathsf{I} \rightarrow_i$$

$$\frac{A \rightarrow_i B \qquad A}{B \vee \bot}\ E \rightarrow_i$$

$$\frac{(A \vee \bot)}{\forall_i x A(x)}\ \forall_i - I$$

$$\frac{\forall_i x A(x)}{(A \vee \bot)}\ \forall_i - E$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^2 \qquad [\neg A]^1}{\bot}}{(B \vee \bot)}}{(\neg A \to_i B)}\,{\scriptstyle 1}\;\; \to_i I}{((\neg A \to_i B) \vee \bot)}}{A \to_i (\neg A \to_i B))}\,{\scriptstyle 2}\;\; \to_i I$$

Related work

The idea of using different signs for the different meanings attached to intuitionistic and classical operators is not new, it was used by P. Krauss in 1992. The same idea was used again in 2015 by Gilles Dowek. Both Krauss and Dowek have classical versions for $\wedge$ and $\forall$. It is interesting to observe that [1] $\wedge_c$ does not satisfy (in general) projections and is not idempotent and that [2] $\forall_c$ does not (in general) satisfy universal instantiation.

Related work

The main motivation of both Krauss and Dowek was to explore the possibility of *hybrid* readings of axioms of mathematical theories. The example discussed by Krauss is the axiom of choice and the example discussed by Dowek is also taken from set theory. The whole point is, in Dowek's own words, to consider that "which mathematical results have a classical formulation that can be proved from the axioms of constructive set theory or constructive type theory and which require a classical formulation of these axioms and a classical notion of entailment remais to be investigated".

Future work

1. We have just indicated the way to obtain a normalization for NEc. Clearly there are lots of things to be done with respect to the *proof theory* of NEc. We know that we do not have as a corollary of normalization the sub-formula principle in its usual form. But can we have a weak sub-formula principle based on the intended meaning of the classical operators? Can we have confluence? Strong Normalization?
2. It would be interesting to explore the intended meaning of the classical operators in order to obtain a Curry-Howard type of result.
3. As we mentioned above, an interesting application of ecumenical systems is related to the analysis of mathematical results that depend on ecumenical readings of axioms (see Krauss and Dowek). It would certainly be interesting to pursue the investigation of other axiomatic theories. .
4. We are also planning to define a sequent calculus and a tableaux system for the Ecumenical modal logic S4.

The system CILP

5. The system CILP - Caleiro & Ramos

Caleiro C., Ramos J. (2007) Combining Classical and Intuitionistic Implications. In: Konev B., Wolter F. (eds) Frontiers of Combining Systems. FroCoS 2007. Lecture Notes in Computer Science, vol 4720. Springer, Berlin, Heidelberg

Definition 3. The axiomatization of *CIPL* consists of the axioms

(C1)	$A \Rightarrow (B \Rightarrow A)$
(C2)	$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
(C3)	$((A \Rightarrow B) \Rightarrow A) \Rightarrow A$
(I1)	$A \rightarrow (B \rightarrow A)$
(I2)	$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
(X1)	$A \rightarrow (B \Rightarrow A)$
(X2)	$(A \Rightarrow B) \rightarrow (A \rightarrow B)$, for A classical
(X3)	$A \rightarrow ((A \Rightarrow B) \rightarrow (A \rightarrow B))$
(X4)	$(X \Rightarrow (A \rightarrow B)) \rightarrow ((X \Rightarrow A) \rightarrow (X \Rightarrow B))$

and the inferences rules

$$\frac{A \quad (A \Rightarrow B)}{B} \ \textbf{(CMP)}$$

$$\frac{A \quad (A \rightarrow B)}{B} \ \textbf{(IMP)}.$$

We denote by $\vdash$ the corresponding deductive consequence relation.

A labelled ecumenical system

$$\dfrac{\begin{matrix}[\omega_i : A] \qquad [\omega_i R \omega_j] \\ \Pi \\ \omega_j : B\end{matrix}}{\omega_i : (A \to B)} \to -I$$

$$\dfrac{\omega_i : (A \to B) \qquad \omega_i R \omega_j \qquad \omega_j : A}{\omega_j : B} \to -E$$

A labelled ecumenical system

$$\dfrac{\begin{matrix}[\omega_0 : A] \\ \Pi \\ \omega_j : B\end{matrix}}{\omega_i : (A \Rightarrow B)} \Rightarrow -I$$

$$\dfrac{\omega_i : (A \Rightarrow B) \qquad \omega_0 : A}{\omega_j : B} \Rightarrow -E$$

An alternative Johansson/Heyting Ecumenical system

6. Still another system

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \rightarrow_i B}\ \mathsf{I}\rightarrow_i 1$$

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{A \rightarrow_i B}\ \mathsf{I}\rightarrow_i 2$$

$$\dfrac{A \rightarrow_i B \qquad A \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[\bot]\\ \vdots \\ C\end{matrix}}{C}\ E\rightarrow_i$$

Semantics

7. The problem of impurity/separability - Too many negations!

J. Murzi, 2018

$$\dfrac{\begin{matrix}[A/\bot]^i & & [B/\bot]^i \\ & \Pi & \\ & \bot & \end{matrix}}{(A \vee_c B)}\; I\vee_c$$

$$\dfrac{(A \vee_c B) \qquad A/\bot \qquad B/\bot}{\bot}\; E\vee_c$$

Luiz Carlos Pereira

Further Reading I

G. Dowek.
On the definitions of the classical connective and quantifiers.
Why is this a proof?
Edward Hermann Haeusler, Wagner Sanz and Bruno Lopes, editors
College Books, 2015.

P. Krauss.
A constructive interpretation of classical mathematics.
Mathematische Schriften Kassel, preprint No. 5/92 (1992)

D. Prawitz.
Classical versus intuitionistic logic.
Why is this a proof?
Edward Hermann Haeusler, Wagner Sanz and Bruno Lopes, editors
College Books, 2015.

Luiz Carlos Pereira

Validity of Inferences Reconsidered*

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1 Introduction

I am returning to the theme 'validity of inferences' because of being dissatisfied with my previous treatment of it. Before explaining what I am dissatisfied with, I roughly indicate what kind of validity I am interested in.

My aim is to explicate a concept of *valid inference* that pertains to our use of inferences in arguments and proofs in order to support assertions. Consequently, for an inference to be *valid* in the sense that I am interested in, it must serve its purpose to *justify* the assertion that occurs as conclusion, given that the premisses are already justified. It is indisputable that inferences have this purpose of justifying assertions, especially when used in arguments or proofs to convince or prove something, and it seems reasonable that when speaking of the validity of inferences, it should pertain to this purpose.

As we all know, the *dominant definition of validity in contemporary logic and philosophy* is quite different and says in effect:

> An inference is *valid*, if and only if, the proposition asserted by the conclusion is a *logical consequence* of the propositions asserted by the premisses.

There may be some variations in how the concept of logical consequence is understood, in terms of necessary truth-preservation or truth-preservation under all variations of the meaning of non-logical terms, but in any case it is obvious that no epistemic aspect is involved. That a proposition is a logical consequence of some other propositions does not in general mean that the assertion of that proposition is justified by reference to the truth of these other propositions being known. We need often long proofs with many inferences in order to establish that the relation of logical consequence holds between propositions.

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*This text comprises the slides of my presentation and parts of my manuscript for the talk integrated and slightly edited. As such it is far from a finished paper.

Historically the now dominant notion has not been so prevalent. There is a long tradition in which the validity of inference has been understood rather in the sense that I want to explicate – a tradition that actually started with the very beginning of logic.

Aristotle's definition of syllogism:

> A *syllogism* is a form of speech in which, certain things being laid down, something follows of necessity from them.

was immediately followed by another definition:

> A *perfect syllogism* is one that needs nothing other than the premisses to make the conclusion evident. (Ross 1949)

It seems clear that Aristotle saw the need to distinguish between saying, on the one hand, that a consequence relation holds and, on the other hand, that an inference makes a conclusion evident. A notion like perfect syllogism is obviously needed to understand Aristotle's view of deductive sciences. Indeed, the whole axiomatic tradition from Aristotle and Euclid where one starts with evident axioms and proceeds by inferences to make other assertions evident presupposes a notion of valid inference like Aristotle's notion of perfect syllogism.

However, there have been few attempts to analyse further what this validity consists in, and in the mainstream of logic and philosophy of logic of today, the concept has simply vanished. In formal systems, all inferences are brought down to the applications of a few rules, but the typical attitude is that the choice of them is arbitrary, a matter of economy and convenience, as long as they preserve truth. Gentzen is here an exception to which I shall soon return.

In contemporary logic, it is especially Per Martin-Löf and Göran Sundholm, as far as I know, who have emphasized the epistemic element of the notion of valid inference. Speaking about "a rule of immediate inference", Martin-Löf (1985) says:

> "you must make the conclusion evident to yourself immediately, without any intervening steps, on the assumption that you know the premises".

After criticising the now dominant notion of valid inference, Sundholm (1998) says:

> "an inference effects a passage from known judgements to a novel judgement that becomes known in virtue of the inference in question".

This important connection between validity of inference and knowledge, when thought of as justified beliefs, is equivalent with characterizing a valid inference by saying as above that it justifies its conclusion.

2 Some remarks to narrow down the concept of validity a bit further

When an inference is challenged, it is a customary practice to try to support it by breaking it down to simpler inferences. The challenged inference becomes accepted when it can be based in that way on simpler, accepted inferences. This uncontroversial practice, called by Dummett (1991) *proof-theoretic justification of the first grade*, is of course not elucidating the concept of valid inference philosophically, unless we can fall back on some inferences whose validity is explained in some other way. The philosophical problem is to explain what it is for an inference to be valid when it cannot be broken down into simpler valid inferences.

Given such a notion of *immediate validity*, we may introduce a notion of *mediate validity*:

> An inference from the premisses $A_1, A_2, \ldots, A_n$ to the conclusion A is *mediately valid* if and only if there is a (deductive) argument from the assumptions $A_1, A_2, \ldots, A_n$ ending with the conclusion A made up of inferences that are all (immediately) valid.

However, it must be noted that an inference that is mediately valid may not in itself (without inserting a number of intermediate inference steps which shows it to be mediately valid) justify its conclusion, even if its premisses have been justified. The situation is the same as the one we were in when we considered the dominant definition of valid inference: What is said by the premisses may not at all make the conclusion evident. Consequently, a mediately valid inference need not be valid in my sense.

Mediate validity is of course a transitive notion: if two inferences from A to B and from B to C are both mediately valid, then so is the inference from A to C. In contrast, validity – or immediate validity, but I will usually not use this attribute "immediate" – can clearly not be a transitive notion. More precisely, although we have not yet explicated the notion of (immediate) validity, it is clear that the relation that holds when A is inferable from $A_1, A_2, \ldots, A_n$ by an (immediately) valid inference *cannot be transitive*.[1]

It has been suggested that the wanted notion of valid inference could easily be explicated by a definition of the following kind:

> An inference is valid if and only if most (normal or competent) people see that the conclusion is a logical consequence of the premisses.

However, what I am after is *not* a socio-psychological notion of that kind but an objective notion of validity such that valid inferences do justify their

[1] Bolzano noted that the notion of immediate ground that he introduced (but did not know how to define) must be intransitive, but otherwise most proposed notions of valid inference have been transitive.

conclusions when the premisses are justified; hopefully, of course, this is also something that we will be able to realize after reflections.

3 Valid arguments

To come now to the shortcomings from my present point of view of what I have proposed as a definition of valid inference, I briefly recapitulate the notion of *valid argument* on which it was built. The notion was inspired by Gentzen's idea that an introduction rule determines the meaning of the proposition asserted in the conclusion of an application of the rule. His idea concerned natural deductions, and to generalize it to inferences in general, arbitrary arguments were considered.

By an argument was meant a tree-formed arrangement of sentences like a natural deduction except that the inferences are not fixed in advance but can be any kind of inferences, valid or invalid. As in natural deduction, an inference may *bind* (discharge) assumptions and free variables.

An argument is *closed* when all assumptions in the argument and all free variables occurring in some sentence of the argument are bound by inferences occurring in the argument; otherwise it is *open*.

An argument is *canonical* if its last inference is an application of an introduction rule.

The notion of *valid* argument was defined by recursion over the complexity of the last sentence of the argument relative to given valid arguments for atomic sentences. The notion was taken up by Michael Dummett and Peter Schroeder-Heister and somewhat modified by them and later also by myself. I summarize the essential content of the different variants below (but the outcomes that I shall soon discuss are independent of which variant we consider):

(1) A closed argument in canonical form is valid, if its immediate subarguments are.

(2) A non-canonical closed argument for A is valid,

   (a) if it can be reduced (Prawitz 1973, Schroeder-Heister 2006) or

   (b) transformed (Dummett 1991)[2] to an argument for A that is valid by clause 1, or

   (c) if a closed argument for A that is valid by clause 1 is contained in the set of its immediate sub-arguments (Prawitz 2015, 2019).

(3) An open argument is valid, if all its closed instances are; open assumptions being replaced by closed valid arguments for them.

[2] For a discussion of some other differences between Dummett's and my definition, see Prawitz (2006); they are however not essential in the present context.

Validity according to clause 2a was combined with an assignment of reduction operations or reduction relations to inferences other than introductions and was relative to such assignments. The arguments that come out as valid when one uses Dummett's clause 2b are the same that come out as valid relative to some assignment of reductions when one uses clause 2a.

Clause 2c gives as result a much narrower class of valid arguments, but unfortunately it is not sufficiently narrow, as we shall see soon.

4 Valid inferences

To describe fully an individual inference that binds assumptions we need to refer to the argument in which it occurs in order to be able to specify which occurrences of assumptions it binds[3]. However, we may describe a generic inference act or an inference figure by specifying its premisses, its conclusion, the forms and locations (with respect to the premisses) of the assumption occurrences that it may (is allowed to) bind, and the variables that it binds. This may be done graphically, writing for instance

$$\frac{\begin{matrix}[\Gamma_1] & [\Gamma_2] & \dots & [\Gamma_n] \\ A_1 & A_2 & \dots & A_n\end{matrix}}{A}\ \xi_1, \xi_2, \dots, \xi_n$$

where $\Gamma_1, \Gamma_2, \dots$, and Γ_n are sets of sentences and $\xi_1, \xi_2, \dots$, and ξ_n are variables. It is to indicate an inference from the premisses $A_1, A_2, \dots, A_n$ to the conclusion A that binds the variables $\xi_1, \xi_2, \dots, \xi_n$ and may bind assumption occurrences in an argument for the premiss A_i of the form of a sentence in Γ_i.

The validity of inferences was defined in terms of valid arguments as follows:

> An inference of the kind indicated above is *valid*, if and only if, any application of the inference to valid arguments $\Pi_1, \Pi_2, \dots$, and Π_n for the premisses $A_1, A_2, \dots$, and A_n, respectively, yields a valid argument for A.
>
> By an *application* of the inference figure in question to the arguments $\Pi_1, \Pi_2, \dots$, and Π_n is meant an argument of the form
>
> $$\frac{\begin{matrix}\Pi_1 & \Pi_2 & \dots & \Pi_n \\ A_1 & A_2 & \dots & A_n\end{matrix}}{A}$$
>
> in which the last inference binds free assumptions and free variables in accordance with how the inference figure is specified.

[3]The corresponding problem concerning the binding of variables may be ignored by assuming a suitable convention about the writing of variables.

If an inference does not bind assumptions or variables, to say that it is valid is equivalent with saying that it constitutes a valid argument.

In this case, we may also restrict ourselves to proofs, that is, closed arguments, and say that the inference is valid if and only if any application of it to proofs of the premisses yields a proof of the conclusion.

There is nothing wrong with the only if part of this equivalence: It must certainly be intuitively right to say that given a valid inference and proofs of its premisses, the construction that results by conjoining the proofs and adding the inference to them constitutes a proof of the conclusion of the inference. But to take this as also a sufficient condition for the validity of an inference has a very bad consequence:

> The condition is vacuously satisfied if one of the premisses of an inference has no proof. Hence, the inference comes then out as automatically valid.

For instance, if a proposition is false and consequently its assertion has no proof, any conclusion can be validly inferred from its assertion.

Why is that a bad outcome? That any proposition is said to be *a logical consequence* of a logically false proposition is right in my opinion. I am not arguing for relevant logic or para-consistent logic. Furthermore, I find it to be in order to say that one can validly infer any conclusion from falsity or from two premisses that openly contradict each other. We can summarize the situation as follows:

> *Desired outcome*: For any B, the inferences below are valid:

$$\frac{A \quad \neg A}{B} \qquad\qquad \frac{\bot}{B}$$

> *Undesired outcome*: If A has no proof, then for any B the following inference is (vacuously) valid

$$\frac{A}{B}$$

The last outcome is against the very idea of a valid inference, as I have described it. To illustrate by a concrete *example*:

> Let F be $(\exists n > 2)(\exists xyz)\, x^n + y^n = z^n$. Its assertion denies Fermat's (Last) Theorem, which according to the definition gets the following neat proof

$$\cfrac{\cfrac{F}{\bot}}{\neg F}$$

Assuming F, we can validly infer anything, since F has no proof. Inferring $\bot$, we can conclude $\neg F$ and discharge the assumption F. But, of course, this is not what we mean by a proof of $\neg F$. As we know from Wiles, the inference of falsity from F is mediately valid. It illustrates why mediately valid inferences are not genuinely valid.

5 Some other definitions of valid inference and proof

There are several other proposed definitions of valid inference and proof with the same outcome. One is the so-called BHK-interpretation proposed by Troelstra and van Dalen (1988), which gives an inductive definition of proof with the following clause for implication:

> A proof of $A \to B$ is a construction which permits us to transform any proof of A into a proof of B.

If A has no proof, the empty construction taking any proof of A into a proof of B is by definition a proof of $A \to B$. Thus, if we let A be the F we had above and let B be $\bot$, the empty construction is a proof of Fermat's Theorem. There is no objection to this if by proof is here meant the same as was later called proof-object in Martin-Löf's type theory. The empty construction is a construction of Fermat's Theorem, and to prove that it is, one has to prove Fermat's Theorem. But by a proof Troelstra clearly means proof in the usual epistemic sense, and then of course this empty construction is again not what we mean by a proof of Fermat's Theorem.

Another example is the notion of inference proposed in the paper quoted above by Martin-Löf (1985) where an inference is identified with a hypothetical proof, which is defined in terms of categorical proofs in essentially the same way as an open argument was defined as valid:

> Something is a hypothetical proof from hypotheses if it becomes a (categorical) proof when supplemented by (categorical) proofs of the hypotheses.

Thus, the outcome is the same: an inference from an assertion that has no proof to any arbitrary assertion comes out as valid.

6 Rejecting and accepting some basic principles

A new approach to the concepts of valid inference and proof is clearly needed. Among others, the relation between the concepts of proof and valid inference must be reconsidered. An idea that does not seem to be viable is that the concept of proof can be defined by recursion over the assertion that is proved and that the validity of inferences can be defined in terms of a notion of proof obtained in that way.

But some principles behind previously proposed definitions may be possible to retain and to generalize. As I already said, one half of the definition of valid inference (when it does not bind anything) is sound, namely the one saying that if we add a valid inference to given proofs, we get a new proof. Generalizing this we get that an argument built up of valid inferences is itself valid. The converse of this must also be right. We may formulate this as a principle about the relation between valid inferences and valid arguments or proofs:

> *Principle 1. An argument is valid if and only if all its inferences are valid.*

A closed valid argument amounts to a *proof*. If we had a definition of the validity of inferences, the principle would serve as a definition of proof, and we would then have reversed the conceptual order between the two notions that appeared in the now rejected definitions. However, it is not certain that the concept of valid inference can be explained without the use of the concept of proof; the two notions may have to be explained simultaneously by a number of principles that relate them to each other.

Another idea that must be rejected is that an open argument is valid if all its closed instances are valid, where free assumptions are being replaced by valid arguments. This was the third clause in the proposed definition of valid argument, which had the bad consequence that all open arguments with no closed instances come out as automatically valid.

However, the converse of that clause is reasonable. It expresses a basic intuition about how to understand an open argument. It may be adopted as a second principle:

> *Principle 2. All substitution instances of a valid argument are valid, when free assumptions are replaced by valid arguments.*

7 Gentzen's idea about meaning and I-rules

Gentzen's idea that his introduction rules determine the meanings of the symbols concerned, which as mentioned inspired the definition of valid argument presented above, seems to me to be still a most fruitful idea for approaching the concept of valid inference, but it has to be used in a different way.

Gentzen phrased his idea in the following often quoted statement:

> "The introductions represent, as it were, the 'definitions' of the symbols concerned".

From his wording, it is clear that he did not mean that the introduction rules were to be seen as literal definitions of the logical constants. Stating his idea less metaphorically, we could say

> The introduction rule for a logical constant c *determines the meaning* of c and of sentences in c-form, that is, sentences that have c as their outermost symbol.

The question is how it determines this meaning. I suggest that the idea can be elaborated as follows:

> *Gentzen's idea elaborated*: To know the *meaning* of a sentence A in c-form is to know that *introductions* of c (that is applications of the introduction rule for c) are the *canonical* ways of inferring A, which is to say: (1) they are valid inferences, and (2) if A can be proved, it can be proved in that way, that is, by a proof whose final step is an application of the introduction rule for c (which we therefore call a *canonical proof*).

An example of the relevance of clause (2) is that Gentzen's introduction rule for $\vee$

$$\frac{A_i}{A_1 \vee A_2} \qquad (i = 1 \text{ or } 2)$$

is *not* an introduction rule for $\vee$ when understood classically. Although all applications of this rule are valid also when $\vee$ is read classically, it is not part of the classical understanding of disjunction that it can be proved only when one of the disjuncts can be proved; on the contrary, $A_1 \vee A_2$ may be possible to prove classically while neither A_1 nor A_2 can be proved classically. As introduction rule for classical disjunction one can choose instead:

$$\begin{array}{c}[\neg A_1, \neg A_2]\\ \dfrac{\bot}{A_1 \vee A_2}\end{array}$$

The full significance of the introductions of c being the canonical ways of establishing sentences in c-form has to do with inferences other than introductions, which I shall soon consider.

Gentzen's introduction rules are for the logical constants. His idea must be generalized to all forms of sentences, if it is to amount to a general principle on which the validity of inferences can be based. An example of how it can be generalized to atomic sentences was provided by Martin-Löf (1971), who proposed the following two introduction rules for the property of being a natural number:

$$N0 \qquad\qquad \frac{Nx}{Ns(x)}$$

The first one is an axiom, an inference rule without premisses, saying that 0 is a natural number, and the second one is a rule allowing the inference from the premiss that t is a natural number to the conclusion that the successor of t is a natural number.

We could formulate this generalization as follows:

> *Generalization of Gentzen's idea about introduction rules.* A language comes with introduction rules for its various forms of sentences giving them their meaning.

The validity of inferences is thus to be relativized to languages, and clause (1) in the above elaboration of Gentzen's idea then gives rise to

> *Principle 3. Introductions (applications of introduction rules) given with a language L are valid inferences relative to L.*

I shall usually leave the relativization to languages implicit in the sequel.

8 Validity of inferences other than introductions

Inferences that are not introductions can be valid only in virtue of the meanings given by the introduction rules. Quite generally, for an inference to be valid, the premisses must guarantee the conclusion in virtue of what the premisses mean, that is, more precisely, since the meaning is articulated in terms of proofs, the existence of *proofs* of the premisses must guarantee the existence of a *proof* of the conclusion.

Now, according to how Gentzen's idea has been elaborated above, if the premisses have proofs, they have canonical proves. It is therefore sufficient to consider only them, and we can therefore sharpen the condition for an inference to be valid:

> *Main idea 1.* For an inference that is not an introduction to be valid, any *canonical* proofs of the premisses must constitute a guarantee that there is 'already' a proof of the conclusion.

This is still rather vague, of course, and we need to be more precise about the kind of guarantee that should be required. It must have some quality of obviousness since one should not require a proof in order to see that there is a guarantee. Furthermore, the guaranteed proof of the conclusion should exist 'already', that is, it is not to be obtained by applying the inference.

Leaving the vagueness aside for the moment, we do not want to take as a sufficient condition for the validity of an inference that canonical proofs of the premisses guarantee a proof of the conclusion, because we would then open again for the vacuous satisfaction of the condition when there is one premiss that has no proof.

The condition can be strengthened, however, by letting it concern not only proofs but all valid arguments, also the open ones. It then becomes a reasonable sufficient condition for validity, because for all sentences except those that have no introduction rule there is a canonical argument from suitable premisses; that the inference becomes vacuously valid when a premiss has no introduction rule is as it should be, as was previously remarked.

We then arrive at the following general idea about validity of inferences other than introductions:

> *Main idea 2.* An inference that is not an introduction is valid, if and only if, for any application of it to valid canonical arguments

$\mathcal{A}_1, \mathcal{A}_2, \ldots$, and $\mathcal{A}_n$ resulting in an argument for A from Γ, the arguments $\mathcal{A}_1, \mathcal{A}_2, \ldots$, and $\mathcal{A}_n$ constitute together a guarantee of the existence of a valid argument for A from Γ.

How can there be a guarantee of the kind required? We can find one in what I have called the *inversion principle*, using a term from Lorenzen. As I shall now show, a set of valid canonical arguments for the premisses of an inference that satisfies the inversion principle *contains* a valid argument for the conclusion from the assumptions in question and gives thereby the guarantee required by Main idea 2.

9 Definitions and corollaries

We need to define the two crucial notions used above, namely, what it is to satisfy the inversion principle and what it is for a set of arguments to contain another argument.

Definition 1. An inference *satisfies the inversion principle* when for any application of it to valid canonical arguments $\mathcal{A}_1, \mathcal{A}_2, \ldots$, and $\mathcal{A}_n$ resulting in an argument for A from Γ, it holds that the set $\{\mathcal{A}_i\}_{i \leq n}$ contains an argument for A from Γ.

When I first stated this inversion principle (Prawitz 1965), the term "contain" was used metaphorically. It was later given a precise definition (Prawitz 2015, 2019).

Definition 2. $\mathcal{A}$ is *contained* in a set S of arguments, if and only if, $\mathcal{A}$ can be extracted from S.

Definition 3. $\mathcal{A}$ can be *extracted from* S, if and only if, there is a sequence of arguments $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ where $\mathcal{A} = \mathcal{A}_n$, and each argument $\mathcal{A}_i$, $i \leq n$, is (i) a subargument of an argument of $S \cup \{\mathcal{A}_j \mid j < i\}$, or (ii) the result of substituting terms for variables in an argument of $S \cup \{\mathcal{A}_j \mid j < i\}$, or (iii) the composition of two arguments in $S \cup \{\mathcal{A}_j \mid j < i\}$.

The following corollary follows now easily from principles 1 and 2:

Corollary 1. An argument contained in a set of valid arguments is valid.

Extractions involve according to definition 3 three kinds of operations on arguments: (i) taking out a subargument, (ii) making substitutions of terms for variables, and (iii) composing two arguments. The first one preserves validity by principle 1, and the two other operations preserve validity by principle 2.

> *Corollary 2.* A set of *valid* canonical arguments for the premisses of an inference satisfying the inversion principle contains a *valid* argument for the conclusion of the inference that depend on the same (or fewer assumptions) as the argument obtained by applying the inference to the canonical arguments in question depends on.

By definition, an inference satisfies the inversion principle when for any application of the inference to canonical arguments for the premisses it holds that the set of the arguments contains an argument for the conclusion of the result of the application which depends on at most the assumptions that the result of the application depends on. By corollary 1, the argument contained in the set of arguments for the premisses is valid, if the ones of the set are valid.

We can conclude that when an inference satisfies the inversion principle and is applied to valid canonical arguments for the premisses, the set of these arguments guarantees the required existence of an argument for the conclusion by *containing* it. This seems to be a quite strong and obvious form of guarantee. According to what was called above Main idea 2, we then get

> *Principle 4. Inferences satisfying the inversion principle are valid.*

10 Examples

It is easily verified that the elimination rules of intuitionistic predicate logic satisfy the inversion principle. Hence, they are valid by principle 4.

As another illustration of principle 4, consider the following two inferences:

$$\frac{A \quad A \to B \quad B \to C}{C} \qquad\qquad \frac{A \to B \quad B \to C}{A \to C}$$

The first is valid by this principle but not the second, because canonical arguments for $A \to B$ and $B \to C$ need not contain an argument for $A \to C$, while canonical arguments for A, $A \to B$, and $B \to C$ do contain an argument for C.

To illustrate that the relation validly inferable which comes out of principles 3 and 4 is not transitive (as required in section 2), we note that the first two inferences below are valid in force of principle 4 and 3, respectively, while the third is not:

$$\frac{A \wedge B}{A} \qquad\qquad \frac{A}{A \vee B} \qquad\qquad \frac{A \wedge B}{A \vee B}$$

11 Concluding remarks

That an inference satisfies the inversion principle seems to be a too strong requirement to be a reasonable necessary condition for validity. One can imagine other ways in which canonical arguments for the premisses of the inference may guarantee a valid argument for the conclusion (from the assumptions in question) than that of containing such an argument. It is an open question whether one can formulate an equally sharp condition as that of satisfying the inversion principle that is reasonable to adopt as a sufficient and necessary condition for validity.

One may also be interested in a notion of logical validity, that is, the validity of an inference that depends only on the meaning of the logical terms involved. All inferences used in intuitionistic predicate logic (of first order) are either valid or mediately valid in force of principles 3 and 4. A kind of completeness question that arises is whether all inferences that are valid or mediately valid in force of these principles are derivable in intuitionistic predicate logic. Another question is whether one could imagine inferences that are logically valid relative to the language of predicate logic but are not valid in force of principles 3 and 4.

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Logic and Ethics

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Note

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A more precise formulation of the title would be Logic and Deontology. Let me begin by referring to the structure of a speech act. We have the act of uttering something and that which is uttered, the utterance in that sense, which we in linguistics call a complete sentence:

$$\longrightarrow \ \text{sentence}$$

When you consider a complete sentence, then the outermost structure of it, the most basic structure, is the mood/content structure:

$$\longrightarrow \ \text{mood content}$$

The view that this is the most basic form of utterance is properly ascribed to Charles Bally, who was a French linguist, successor of Saussure in Geneva: he so to say replaced the subject/predicate form as the most basic form with the mood/content form.

The mood we can characterize at this stage simply as the kind of speech act: if it is an assertion, then we have the assertoric mood; if it is a question, then we have the question mood; if it is a warning, we have the warning mood, etc. But, what about the other part here, the content part? Here I have a proposal. I am going to distinguish between assertoric contents and propositions, or in Dummett's terminology, their ingredient senses. This can be illustrated in the following way, if we have as the mood the assertoric mood, and we take the special case of a content of the form A true, where A is a proposition:

$$\underset{\substack{\uparrow \\ \text{mood}}}{\vdash}\ \underbrace{\overset{\substack{\text{prop} \\ \downarrow}}{A}\ \text{true}}_{\text{content}}$$

Clearly there is a distinction between content and proposition if you view it in this way. Then we have, on the other hand, the BHK-interpretation of the notion of proposition, where a proposition is identified with an expectation or an intention, in Heyting's terms, or with a task, Aufgabe, in Kolmogorov's terminology. We now have two levels here: we have both propositions and assertoric contents, propositions being so to say the content made into an object in your theory. And by making it into an object I only mean that we make assertions of the form that something is a proposition: then they are no longer contents, but precisely what we call propositions. On the other hand, there is also a great similarity between propositions and contents, so it seems like a natural idea, if we explain propositions in the way I just referred to, the BHK-way, that we could try that for assertoric contents also. So we would have expectations, or intentions, or tasks, at two levels so to say: on the content level and on the proposition level. This will underlie my talk here from the beginning to the end. So from this point on I look upon the content here as a task in Kolmogorov's terminology.

What is a task then? Simply, something to do, or – in the passive voice – something to be done. As soon as we have a content in this sense we can speak of fulfilling it, or doing it. And then we have immediately also all the temporal modifications and other modifications to which we can subject our verbs.

Now I have fixed the part

$$A \text{ true}$$

What about the mood? Well, for the mood I have already given the general explanation, and then it only remains, for the specific moods that are going to be considered, to give the special explanation for each of them.

Of course, the first mood is the assertoric mood. I will take as the logical notation for assertion the modernized Frege sign, $\vdash$. To a large extent this talk will be a talk about the explanation of the meaning of $\vdash$. How is it to be explained? I want to adhere to the point of view that what makes something into an assertion is purely formally that its mood is the assertoric one. If we just prefix the mood to a well-defined content, it is already an assertion, that is, you are going to be held responsible for having made an assertion as soon as you have uttered something with the assertoric mood, just as when you make a promise, you are held responsible for its being a promise however unlikely it may seem that you really are going to fulfil it: it is a promise anyway. Similarly with an assertion: it is an assertion as soon as you have this force sign.

There is a huge literature on the notion of assertion, and it has been made much more accessible by Peter Pagin through his contribution under the entry of 'Assertion' to the Stanford Encyclopedia of Philosophy. There, there is at least half a dozen different views of what assertion is, with endless variations on them, so it makes up a paper of no less than 30 pages or something like that. I am not at all going to contribute to that, I do not have the competence, and it is already available. So I will only take up essentially two views here of what an assertion is, namely the so-called knowledge account of assertion, on the one hand, and on the other hand the commitment account of assertion. I hope to clear up sufficiently how they are related to each other.

Concerning the knowledge account, first of all the term here, to speak of different accounts of assertion, that comes from Williamson. And his own preferred view of the nature of assertion is precisely the knowledge account. But the knowledge account goes back to Frege, we must remember. When Frege defined a judgement as the acknowledgement of the truth of a thought, it was clearly a knowledge account, because of the word 'acknowledgement', which has 'knowledge' in it, and it is the same in the German original: die Anerkennung der Wahrheit eines Gedankens. So Frege's account of assertion was a knowledge account.

For Frege, the content was the thought, and now I have replaced that by the notion of task, so the question is, What modification does that necessitate as compared with Frege? For Frege it was the acknowledgement of the truth of a thought, so Frege used the word true, or truth. And truth now, when we understand content in the way that I have suggested, corresponds to fulfillability. (Fulfillability works perfectly for expectation and intention, but less well for task, so maybe performability rather, in case you choose task.) So truth corresponds to fulfillability, and then Frege's acknowledgement of the truth of a thought corresponds to acknowledgement of the fulfillability of an intention. And to acknowledge, that is, to get to know, the fulfillability of an intention, that is interpreted in the plainest possible way, namely, that is to grasp how the content is fulfilled. So, to know the thought to be true becomes simply to know how to fulfil the task which makes up the content. That is how the analysis I am giving here is related to Frege's analysis.

Then we may already formulate what it is natural to call the correctness condition for assertion, namely the condition under which it is right, and here several terms are possible to use: right, correct, proper. I am going to use them in the same sense. The condition under which it is right, or correct, or proper, to make an assertion is that you know how to perform the task which constitutes the content of the assertion. This is what I have called the correctness condition for assertion in my abstract.

For acts in general it is usually illuminating to ask, What is the purpose of the act? In this case, if we accept the correctness condition that I just gave, What is the purpose of making an assertion? Then we have already to bring in that the speech act involves not only the speaker, but also the hearer, the receiver of the speech act. So, what is it that the assertor wants to achieve, what is the purpose of making an assertion? Well, if we stick to this knowledge account of assertion that I am discussing right now, then the purpose is nothing but to convey to the hearer that the speaker knows how to fulfil the content, the task which makes up the content. The speech act of assertion has no other purpose than to transmit from the speaker to the hearer the information that the speaker knows how to fulfil the task which makes up the content of the assertion, and this succeeds because the speaker must adhere to the correctness condition for assertion that I just formulated.

Since the speaker is conveying to the hearer that he knows how to do something, he knows how to fulfil this task, that means that this could be useful to the hearer: well, he knows how to do that, which means that I can go to him and get help with doing this, if I am in need of that help. But, as it is now, there is no mechanism for this, because then we have to introduce some more things first. And that brings me to the commitment account of assertion, because what it does is precisely to bring in these extra bits that are needed.

So, now I come to the commitment account of assertion, which has its origin in Peirce's work during a very early stage of the last century, 1902-03, I think. Peirce's view was that an assertion should be understood as a taking on of responsibility, taking responsibility for the content of the assertion. Responsibility and commitment are not significantly different, and commitment, on the other hand, refers to obligation and duty, so we have

commitment
obligation
duty

which means that now the deontic notions have already come in that I referred to in the more precise title.

On the other hand, there is the correlativity of rights and duties, a very fundamental insight due to Bentham right at the beginning of the 1800s – by the correlativity of rights and duties I just mean this, that if I have an obligation, or duty, towards my neighbour, then my neighbour has a right against me, and vice versa. So it is the same action that is carried out, but from my point of view it is an obligation to do it, and from the other person's view it is something that he benefits from by getting me to do it.

So, there is this correlativity of rights and duties, which means that as soon as we have the notions of obligation and duty, we also have the notions of permission, dual to obligation, and right, dual to duty:

commitment	entitlement
obligation	permission
duty	right

Now you see that much more has come into this structure, namely the hearer in addition to the speaker, and these deontic notions and their duals. The duality comes in precisely because of the duality between speaker and hearer. So now I can give a first formulation of the commitment account of assertion, so that it can be compared with what I just said about the knowledge account of assertion. By making an assertion, the speaker assumes the duty of performing the task which constitutes the content of the assertion at the request of the hearer. Now you see more of this duality has come in, because at the end I said 'at the request of the hearer'. So now we have not only the speaker, who is the assertor, but we also have the hearer, who receives the assertion, and now he is going to play an important role here, namely in that he has the right to ask the speaker to fulfil his obligation. So we have now request also coming in here:

speaker	hearer
assertion	request

Now things are beginning to look much more promising, because if we take other speech acts like question and answer, then we take immediately for granted that question and answer have to be explained together: you cannot explain the one without bringing in the other. And if we have a command, for instance, there must also be obeyings of the command: we cannot explain the command without having someone who is commanded and who is obligated to obey the command. So it seems very natural, and strange that it has not become generally accepted, as far as I know, that there is a speech act that is dual to assertion in precisely the same way, namely request. Assertion and request have to be explained together, as we already saw a moment ago in my formulation of the commitment account of assertion.

Now I want to vary that formulation in the same way that I varied the formulation of the knowledge account of assertion, namely by putting it in an explicitly teleological way, by asking, What is the purpose of making an assertion? What I said in other words a moment ago then becomes this: the purpose of an assertion on the part of the speaker is to give the hearer the right to request the speaker to perform the task which makes up the content of the assertion, whereupon the speaker is compelled to fulfil his duty by actually performing the task in question. It is essentially the same content as I gave a few minutes ago, but now formulated in an explicitly teleological way.

If we accept this, then we are in the lucky situation of having discovered a very basic logical structure. Namely, we have first of all the assertion of the speaker, which has this form:

$$\vdash C$$

The speaker makes an assertion, and then the hearer has the right to ask the speaker to fulfil his ability, to put his knowledge-how into practice, and that is a speech act of request:

$$C?$$

When the assertor is requested in this way, he is put under the obligation, or duty, to fulfil C, or to do C. So the conclusion is that C gets done, or C is fulfilled:

$$\frac{\vdash C \qquad C?}{C \text{ done}}$$
$$C \text{ fulfilled}$$

This way of writing it makes it look as much as possible like an ordinary inference, but you could also put it, perhaps better, in this way:

$$\cfrac{\cfrac{\vdash C}{C?}}{C \text{ done}}$$

We have the assertion followed by the request, and then, because all of this has already occurred, we may proceed further to have the speaker to do C.

This is really not one rule, this is a whole scheme of rules: one for each form of assertoric content C. I should give at least one or two examples to elucidate this logical structure. A completely non-logical example – it is logical, but let's speak of it as a non-logical example – is this: you have a child running to its mother saying, Oh, I can swim! That corresponds to the assertion. Then the mother says, Can you? or Show me! (in which case we have an exclamation mark) or something like that, and then as a result of that request, the child actually swims. With this example you already see that this is a practical inference in Aristotle's sense: it is a rule where the conclusion is the performance of an action. Practical syllogism sounds a bit old-fashioned, but practical inference is a perfectly good term that we can use presently. So that is one name for this kind of logical rule. Another possibility is to call it the manifestation rule, or if you think of tests of the kind that we are all engaged in, or examinations, we could call it the examination rule, or test rule.

Knowledge-how, or an ability, is definitely what philosophers call a disposition. Disposition covers a variety of disparate concepts, but at least it is clear that an ability is a disposition. Hence the terminology that has been introduced for dispositions can be used here, in which case the request

$$C?$$

is called the stimulus condition, and the conclusion

$$C \text{ done}$$

is called the manifestation of the disposition. Now, stimulus has a ring that I am not quite happy about, so one could perhaps use prompting condition instead of stimulus condition.

Here is the new logical structure that this talk is basically about. Something now should be said about how it relates to the knowledge account of assertion that I gave previously. Under the knowledge account of assertion I simply stipulated what the condition is for the assertion to be right, namely that the speaker knows how to fulfil the task in question. That is a stipulation: it is right under that provision. But, if we go from that to the teleological account in terms of purpose, then it is no longer stipulated that what gives the speaker the right to make the assertion is that he knows how to fulfil the content: it is no longer stipulated. It must nevertheless still be so, of course, but it requires now an argument why that is needed in order for the purpose to be fulfilled.

One way it is simple, namely that if the speaker knows how to fulfil the content, the task, then this interaction works properly, because if he knows how to do it and gets challenged, then he simply does it, and it is no problem for him to do that, because he can do it. It is sufficient that he knows how to fulfil that task, but in the other direction, that it is also necessary, you need to invoke the ought-implies-can principle, as I said in my abstract. Because, if he makes the assertion

$$\vdash C$$

then by so doing, he is undertaking a conditional obligation, namely the conditional obligation

$$\frac{C?}{C \ \text{done}}$$

And by the ought-implies-can principle, in order to have the right to undertake an obligation, you must be able to fulfil it. Since you are assuming an obligation, you must be able to fulfil it, and that is precisely the condition that we need for this. So it is both necessary and sufficient that the speaker knows how to fulfil the task that makes up the content.

If we look at the rule

$$\frac{\vdash C \qquad C?}{C \ \text{done}}$$

you see that the major premiss here is connected with *can*, because the speaker must know how to fulfil the intention – know-how and can I make no difference between. Then we have the hearer, he gets the right to challenge the assertion: he gets the right, which means that he *may* challenge the assertor. And when the assertor has been challenged in this way, he is under an obligation, so he *must* do something:

$$\frac{\text{can} \qquad \text{may}}{\text{must}}$$

I have put it this way just to make it plausible that this is a natural analysis, because can, may and must are among the auxiliary verbs, the main modal auxiliaries, and it seems quite natural that they come in a package, so to say: they fit together into this pattern, and you cannot explain one of them without also bringing in the other two.

Dummett proposed to lift the introduction and elimination pattern from its usual place due to Gentzen, to shift it to the level of assertions, or even to utterances in general, since I began with utterances in general. So he distinguished between conditions for an utterance and consequences of an utterance: what follows from an utterance as compared with what the utterance follows from. Now we have something like this, because an ordinary inference has assertions as premisses and an assertion as conclusion:

$$\frac{\vdash C_1 \ \ldots \ \vdash C_n}{\vdash C} \qquad (C\text{-intro})$$

And this we can consider now as an introduction rule for the form of assertoric content, C, that you have in the conclusion. We now have also the dual rule here, namely

$$\frac{\vdash C \quad C?}{C \text{ done}} \qquad (C\text{-elim})$$

This clearly then should be considered as an elimination rule, since $\vdash C$ occurs as major premiss, an elimination rule for the form of content that C has. So you have an introduction and elimination pattern here arising on the level of assertions.

That brings me to my final remark. I began by saying that this whole lecture will be roughly about what the meaning is of the assertion sign. We are used to the fact that when we ask for the meaning of some linguistic construction, it should be visible somehow from the rules that govern that construction, in general Wittgensteinian terms. The first example of this is of course Gentzen's suggestion that the logical operations are defined by their introduction rules. What about the assertion sign? If you did not have this new rule (C-elim), you would only have the usual rules of inference, which are of the form (C-intro). If you were to take the assertion sign to be determined by these rules, the assertion $\vdash C$ could not mean anything than that C has been demonstrated, has been inferred by the usual inference rules. And that is not how Frege introduced the assertion sign, what Frege meant by the assertion sign. I explained that earlier on: it is the acknowledgement of the truth of a content that the assertion sign expresses. So, we simply cannot explain the assertion sign by referring to the rules governing it if you only have the rules (C-intro). But now we are in a better situation, because we also have the rules (C-elim), and they are precisely the rules that are meaning-determining for the assertion sign.

Abstract

The condition under which it is correct (proper) to make an assertion is that the assertor knows how (is able) to perform the task which constitutes the content of the assertion (correctness condition for assertions).

To make an assertion is to commit (obligate) yourself to performing the task which constitutes the content of the assertion (commitment account of assertion).

The condition under which it is correct (proper) to undertake an obligation (make a commitment) is that the obligor knows how (is able) to fulfil it (ought implies can).

The relation between the preceding three principles is simple: the correctness condition for assertions follows from the commitment account of assertion taken together with the ought-implies-can principle. Both the commitment account of assertion and the ought-implies-can principle bring in the notion

of duty (obligation) and hence implicitly, by the correlativity of rights and duties, the notion of right. On the other hand, the notions of right and duty are the key notions of deontological ethics. Thus, all in all, logic has, not only an ontological layer and an epistemological layer, but also a deontological layer underlying the epistemological one. It can be avoided only by treating the notion of knowledge how (can) as a primitive notion, thereby abstaining from relating it to the notions of right and duty (may and must).

VALIDITY

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Valid (adjective): (of an argument or point)

having a sound basis in logic or fact;

reasonable or cogent.

Etym. Latin *validus* strong

Dictionary Definition of Validity:

The quality of being logically or factually

sound; soundness or cogency.

OED VALID
2. a. Of arguments, proofs, assertions, etc.: Well founded and fully applicable to the particular matter or circumstances; sound and to the point; against which no objection can fairly be brought.

German:	Gültig	Gültigkeit	gelten	Geltung
Swedish:	giltig	giltighet	gälla	gällande
English	valid	validity		

Herman Lotze 1817-1881

Geltung;

Truth (of a judgment),

Goodness (of an action),

Obtaining (of a state of affairs),

Validity (of a demonstration)

The menu:

1) Validity of a proof (*demonstration*);
2) Validity of an inference;
3) (Logical) Validity of a wff (proposition);
4) (Logical) Validity of a consequence among wff's (propositions);
5) (Prawitz-)Validity of derivations in Proof-Theoretical Semantics.

The first two are contentful (*inhaltliche*) notions of great philosophical import.
The latter three are meta-mathematical notions applicable for the wff's of formal languages that are being given metamathematical syntax and semantics.

Prawitz's metamathematical notion (5) first appeared in his wide-ranging Oslo 1970 lecture *Ideas and Results in Proof Theory* in order to demonstrate the *Strong Normalization Theorem.* It is his version of a familiar notion known under many names and used for many purposes:

Tait 1963, 1967, 1970 *convertibility*;
Shoenfield 1967 *reducibility*;
Martin-Löf 1969 *computability*;
Girard 1970 *reducibility*

This rose clearly smells as sweet under whatever name….

that is used for showing normalization theorems for terms of typed lambda calculus, or via the Curry-Howard isomorphism, for natural deduction derivations. Trolstra SLN 344 gives a full arithmetization of Prawitz validity.

The terminological choice of ***validity*** is an unfortunate one: after all, the notion was not novel and was *already* variously named. It not only meant an unnecessary proliferation of terminology, but also a conflation of this **technical notion** with notions 1) and 2), which is pernicious.

(3) (Logical) Validity of a wff (proposition)

(i) Variation of *contentual* terms

Bolzano 1837: *Proposition* (in itself) *true*

under all variations of non-logical Ideas

(***logisch analytische Wahrheit***)

Also Ajdukiewicz and Quine 1930's.

(ii) Variation of *objects/worlds*

(a) Wittgenstein *Tractatus*: proposition

true, irrespective of what is the case

Tautologie

Tarski 1935/6 : proposition satisfied

by *any assignment of objects*

(iii) Variaion of *syntactic* terms

Carnap (1934/5/7): proposition true under

all substitutions of *syntactical* terms.

"true under any variation of terms"

(i) conceptual "simple" supposition,

(ii) referential "personal" supposition

(iii) syntactic ("material") supposition

Mutatis mutandis and *ceteris paribus*,
similar things hold also for notion (4)
(Logical) Validity of a consequence:
preservation of truth, under all variations, from antecedents to consequent(s)
Bolzano 1837 and **Tarski** 1936.
Bolzano: compatibility of antecedents and consequents of a multiple consequent sequent are read conjunctively rather that disjunctively
Sub specie aternitatis Bolzano is the inventor of "*logical consequence*".

Tarski's 1936 title common mistake:
On the Concept of Logical Consequence,
(LSM 1956)
On the concept of following logically
clumsy title of novel translation directly from the Polish.
Tarski undoubtedly defines is the modern notion of logical consequence. His ***own*** German title is *Über die logische Folgerung.* ***Folgerung*** is cognate with the German verb ***folgern***, Eng. ***to infer***, whence the *proper* English translation of Tarski's German title is "On Logical Inference", or even "On Logical Inferring".

This is as good as an example as any of the *second "Bolzano reduction"*, namely that of inferential validity to the logical ("in all variations') holding of consequence.

Validity of inference

Inference

(a) **act of getting to know** a novel judgement on the basis of previously granted judgements;

(b) **inference-*figure*** (or –***schema***), blueprint according to which such inference-acts may be carried out.

An inference ***figure***

$$\text{(I)} \quad \frac{J_1 \ldots J_k}{J}$$

in the language of predicate logic, where each judgment J has the Bolzano-Frege form *proposition A is true*, specializes to

$$(*) \quad \frac{A_1 \text{ true}, \ldots A_k \text{ true}}{C \text{ true}}$$

The *relata of inference are epistemic items*, e.g., judgements/assertions to the effect that a proposition is true.

Bolzano (1837), and a century later, *Tractatus*, Ajdukiewicz, Quine, and Tarski(1936), customarily reduce the validity of inferences to the ***logical holding "in all variations"*** (that is, *logical validity*) of the consequence

$[A_1 \ldots A_k \Rightarrow C]$,

or to the matching implicational wff $\mathbf{I_S}$:

$A_1 \& \ldots \& A_k \supset C$

being a *tautology*, that is a "*logical truth*".

A Gentzen sequent of ***pure*** **predicate logic** *holds logically* when the proposition I_S is a tautology (**logical truth**).

Bolzano's reduces the epistemic notion of validity for an inference schema to the (ontolgical "alethic") logical holding of consequence, or alternatively, logical truth. The judgement *that A is true* is **correct** when proposition A **really** is true, and it is "logically analytic" when the proposition A is a logical truth.

Blindness: the Bolzano reductions render *blind* judgement and *blind* inference epistemically correct. (Brentano's excellent terminology.)

[First Bolzano reduction:

the judgement *A is true* is correct (right, ***richtig***) is explained as: proposition A is really true.]

Consequence preserves truth from antecedent propositions to consequent proposition, and ***logical* consequence** does so "*under all variations".* The demonstration of the Prime Number Theorem (PNT) by De la Vallée-Poussin and Hadamard in 1896 can be formalized within NBG, the set theory of Von Neumann, Bernays, and Gödel. Since this theory is finitely axiomatized, we may conjoin its axioms into one proposition NBG and then consider the inference

(**) $\frac{\text{NBG is true}}{\text{PNT is true}}$

Truth is preserved from NBG to PNT, in the in the light of the formalized demonstration. So under the Bolzano reduction (**) is a valid inference, but it **provides no epistemic insight**.

An inference is **actually valid** if a chain of immediately valid inferences linking premises and conclusion has been found and ran through.

An inference is **potentially valid** if a chain of immediately valid inferences linking premises and conclusion can be found.

Validation of immediate, but not directly meaning-given inference.

Consider an *immediate* inference I:

$$\frac{J_1 \;\; J_2 \;\; \dots \;\; J_k}{J,}$$

How does one validate I?

We are granted of a number of items:

(a) The syntactic form of the inference, its premises and conclusion;

(b) The semantic assertion conditions for the premise judgements $J_1 \dots J_k$;

(c) The semantic assertion condition for the conclusion judgement J;

(d) The **epistemic assumption** that assertion conditions for $J_1 \dots J_k$, are fulfilled, that is, the assumption that the premises are granted.

On this, but no further knowledge, the assertion condition for the judgement J has to be fulfilled. Consider a simple immediate and meaning given inference, for instance, &-introduction:

$$\frac{a:A \qquad b:B}{\langle a,b\rangle:A\&B}$$

We assume that the premises are known. By meaning explanation for conjunction & <a,b> is a canonical proof-object for A&B . Hence &-introduction is valid.

When someone does not acknowledge the conclusion we have reached stalemate. There is nothing left to but to say: "I am sorry, but apparently I cannot make you understand."

Frege, as was his wont, phrased matters considerable more harshly:

> *Da haben wir eine bisher unbekannte Art der Verrücktheit*.
>
> GGA I, Vorwort, p. XVI.

Consider now the rule &-elimination:

A&B is true
A is true

In order to explain it we consider its fully explicit form:

c:A&B
p(c):A

When we present this rule to someone it may happen that he does not understand the conclusion even though he is entirely familiar with the meaning explanations of the operator &, the projection p, etc., and the relevant assertion-conditions, etc., of everything in sight here.

Why does p(c) belong to proof(A)? he asks.

We can then ***make things explicit*** in terms of meaning explanations already offered.

Since the inference is an immediate one, there is no other *inference* (in the language under consideration) that will do.

In order to help the "non-understander", I ask him or her to assume that he knows, or is granted, the premise judgement. Accordingly, he and I, "we", know that c is a proof of the proposition A&B. By the meaning explanation of what it is to be a proof of a proposition, this proof-object c evaluates, or "unfolds", deterministically, by means of previously given explanations, to canonical form c=<a,b>, where a:A and b:B. But, as c=<a,b>, by what it means to be a function, p(c)=p(=<a,b>), and so, by the meaning explanation for the operator p, p(<a,b>) = a. By the meaning explanations of definitional identity =, we have transitivity of unfolding, so p(c) = a. But a:A

and so p(c):A by the meaning of definitional identity, and we are done. Everything is accounted for in terms of meaning explanations: the account is *ex vi terminorum*. This should **not** be viewed as a demonstration of some kind of "theorem" that &-elimination is a valid rule of inference. It is meant as an aid to someone who has difficulty in understanding its conclusion. In an analogous fashion one could aid someone, who does not acknowledge an immediate, or *self-evident,* judgement, that is, an *axiom*.

Of course, such aid in no way guarantees that the explanations offered will be understood. The non-understander might be recalcitrant and say: I **still** do not understand. We may then ask him, perhaps, to pinpoint the particular spot where non-comprehension sets in. But in the end, when incomprehension prevails, we have to give in. Frege as we saw reacted harshly; "Verrücktheit", etc.

Per Martin-Löf put the point in a more friendly way in his Hannover LMPS lecture 1979:

> For each of the rules of inference, the reader is asked to try to make the conclusion evident on the presupposition that he knows the premises. This does not mean that further verbal explanations are of no help in bringing about an understanding of the rules, only that this is not the place for such detailed explanations. But there are also certain limits to what verbal explanations can do when it comes to justifying axioms and rules of inference. In the end, every body must understand for himself.

Alternative terminologies are possible here *meaning giving* **and** *meaning given* for the *directly* meaning given inference(-rule)s, that is, the introductions, whereas eliminations, even though they are immediate inferences, are not *meaning giving,* but only *meaning given*.
Another version would be: *Explicitly analytic* versus *implicitly analytic,* mirroring Kant on explicitly and implicitly analytic *judgements*, in the *Jäsche Logik*, §§ 33, 36, and 37.
The introduction inferences are then explicitly analytic, whereas the eliminations are only implicitly analytic.

An inference (-rule), finally, is *valid* if we can interpose a chain of analytic inference[-step]s linking premiss and conclusion judgements: we are allowed to assert that an inference is valid only when we know how to insert such a chain.

Validity of a demonstration (proof, *Beweis*).

(a) Act of demonstrative (mediate, discursive) knowledge

(b) Trace, or " blue-print" of such an act. Martin-Löf 1992

Validity of demonstration:

Instance of *rightness* (Latin *rectitudo*).

Cf "**true** friend", "**false** friend".

We give a demonstration of a mathematical theorem, but do not have to give another one that the first is correct. Only after a mistake is diagnosed do we say:

"the demonstration is *not* valid."

An ***in****valid (**ungültig**)* demonstration will not do and has to be *retracted*.

John Austin: the negative word "wears the trousers".

These *modifying* notions come as parts of pairs.

Their deployment carries a **presupposition** that a mistake is diagnosed or suspected.

Modifying concept, ***restitutive*** or ***restorative*** concept

[Brentano, Marty, Twardowski]

Double presupposition for restitutive concepts.

False (friend), **true** (friend);

Invalid, **valid** with respect to demonstrations;

Lückenhaft, lückenlos (Frege, GLA) with respect to *Schlussketten*

Gappy, gap-free

Today taken up by Philosophers of Mathematics, but with respect to "proofs" (demonstrations, *Beweise*).

Fake, real

Inferential validity and validity of demonstrations do not coincide. A valid demonstration is one where each axiom *really* is an axiom, that is, is self-evident *ex vi terminorum*, and where each inference *really* is valid.

Note here the use of the restituitive **real** w. r. t. the epistemic components of the demonstration.

A New Approach to Proof-Theoretic Semantics for Classical Logic

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Ancient Stoics

Sextus Empiricus (2nd century AD):

> *[Stoics] say, just as in life we do not say that the piece of clothing that is sound in most parts, but torn in a small part, is sound (on the basis of its sound parts, which is most of them), but torn (on the basis of its small torn part), so too the conjunction, even if it has only one false component and majority of true ones, will as a whole be called false on the basis of that one.*

Remark

In classical logic there's no room for partial truths:

- $0 \wedge 1 = 0$
- $0 \wedge 1 \wedge 1 = 0$
- $0 \wedge 1 \wedge 1 \wedge 1 = 0$

$\vdots$

How to wear a torn shirt

The sky is blue & The grass is orange

The sky is blue & The grass is orange = 0

The sky is blue & The grass is orange = $\frac{1}{2}$

The sky is blue & The grass is orange & The grass is green = $\frac{2}{3}$

Overview

Problem

find a way to deal with the notion of **partial truth**, while retaining the **deductive engine of classical logic**

- **strict proof-theoretic treatment of multi-valuedness**, without resorting to alternative algebraic structures

$$[\![\cdot]\!] : \mathcal{F} \mapsto [0, 1] \subset \mathbb{Q}$$

- interpret this approach as a kind of **proof-theoretic semantics for classical logic**

Intuitionistic logic	NJ	introduction rules
Classical logic	G4	multi-valuedness

Kleene's system GS4

Axiom:

$$\frac{}{\vdash \Gamma, p, \overline{p}}\ ax.$$

Logical rules:

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge \qquad\qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee$$

- Weakening is implicit in the generalized axiom
- Contraction is implicit in the additive formulation of the conjunction rule

GS4 maximally extended to $\overline{\overline{\text{GS4}}}$

Axioms:

$$\frac{}{\vdash \Gamma, p, \overline{p}}\ ax. \qquad\qquad \frac{}{\vdash \Delta}\ \overline{ax}.\ (\star)$$

Logical rules:

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge \qquad\qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee$$

$(\star)$ Δ is a multiset of literals s.t. $A \in \Delta \Rightarrow \overline{A} \notin \Delta$

Example

- $[p, q, q, \overline{t}]$ this is a complementary axiom instance
- $[p, t, q, \overline{t}]$ this is not, because it contains a couple of dual literals

Example

$$\dfrac{\dfrac{\dfrac{}{\vdash r, \overline{q}}\,\overline{ax}. \qquad \dfrac{}{\vdash q, \overline{q}}\,ax.}{\vdash r \wedge q, \overline{q}}\,\wedge \qquad \dfrac{\dfrac{}{\vdash r, t}\,\overline{ax}. \qquad \dfrac{}{\vdash q, t}\,\overline{ax}.}{\vdash r \wedge q, t}\,\wedge}{\dfrac{\vdash r \wedge q, \overline{q} \wedge t}{\vdash (r \wedge q) \vee (\overline{q} \wedge t)}\,\vee}\,\wedge$$

Remark

- Anything is provable in $\overline{\overline{\text{GS4}}}$
- In particular, $\varnothing \vdash \varnothing$ is an instance of the complementary axiom

From $\overline{\overline{\text{GS4}}}$ to mGS4

Axioms:

$$\dfrac{}{\left|\frac{1}{1}\right. \Gamma, p, \overline{p}}\,ax. \qquad\qquad \dfrac{}{\left|\frac{0}{1}\right. \overline{\Gamma}, \Delta}\,\overline{ax}.$$

Logical rules:

$$\dfrac{\left|\frac{p}{q}\right. \Gamma, A \qquad \left|\frac{r}{s}\right. \Gamma, B}{\left|\frac{p+r}{q+s}\right. \Gamma, A \wedge B}\,\wedge \qquad\qquad \dfrac{\left|\frac{p}{q}\right. \Gamma, A, B}{\left|\frac{p}{q}\right. \Gamma, A \vee B}\,\vee$$

Remark

- sequents come indexed with **ordered pairs** of naturals
- indices are designed to keep record, along proofs, of the **number of identity axioms out of the total number of axioms**

Example

$$\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} r,\overline{q}}\,\overline{ax.} \qquad \dfrac{}{\vdash^{1}_{1} q,\overline{q}}\,ax.}{\vdash^{1}_{2} r\wedge q,\overline{q}}\wedge \qquad \dfrac{\dfrac{}{\vdash^{0}_{1} r,t}\,\overline{ax.} \qquad \dfrac{}{\vdash^{0}_{1} q,t}\,\overline{ax.}}{\vdash^{0}_{2} r\wedge q,t}\wedge}{\dfrac{\vdash^{1}_{4} r\wedge q,\overline{q}\wedge t}{\vdash^{1}_{4} (r\wedge q)\vee(\overline{q}\wedge t)}\vee}\wedge$$

Remark

The end-sequent is indexed with $\langle 1, 4\rangle$ to mean that its proof displays **one identity axiom out of four axioms in total**

A proof-based (multi-valued) semantics

Definition: $[\![\,\cdot\,]\!] : \mathcal{F} \mapsto [0,1] \subset \mathbb{Q}$

For any A and any cut-free proof π of $\vdash^{p}_{q} A$:

$$[\![A]\!]_\pi = \frac{p}{q}$$

Theorem

If π and ρ are both cut-free proofs of A, then: $[\![A]\!]_\pi = [\![A]\!]_\rho$

Remark

The $[\![\,\cdot\,]\!]$ interpretation is actually a semantics in the sense that, for any formula A, it quotients on the set of A's proofs

The classical Boolean taxonomy

- Atoms: variables over $\{0, 1\}$
- Sentences:
  - Tautologies: verified by *any* valuation
  - Truth-functional contingencies: verified and falsified by *some* valuation
  - Contradictions: falsified by *any* valuation

A new taxonomy

- $[\![\cdot]\!] = 1$ Classical tautologies

$\vdots$

- $[\![\cdot]\!] = 0$ this level doesn't coincide with the set of classical contradictions!
  - Literals: $p, q, \overline{p}, \overline{q}, \ldots$
  - Some contradictions: $p \wedge \overline{p}, q \wedge \overline{q}, \ldots$
  - Other formulas such as: $p \to q$, $p \wedge q$, ...

Remark

- at the level 0, one finds all those formulas that do not contain identities at all
- atoms are no longer variables over $\{0, 1\}$, but mere syntactic placeholders guiding substitution

Example: two contradictions

$$\dfrac{\dfrac{}{\vdash^{0}_{1} p}\ \overline{ax}. \qquad \dfrac{}{\vdash^{0}_{1} \overline{p}}\ \overline{ax}.}{\vdash^{0}_{2} p \wedge \overline{p}}\ \wedge \qquad \text{and so } [\![\, p \wedge \overline{p} \,]\!] = 0$$

$$\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} p}\ \overline{ax}. \qquad \dfrac{}{\vdash^{0}_{1} \overline{p}}\ \overline{ax}.}{\vdash^{0}_{2} p \wedge \overline{p}}\ \wedge \qquad \dfrac{\dfrac{}{\vdash^{1}_{1} p, \overline{p}}\ ax.}{\vdash^{1}_{1} p \vee \overline{p}}\ \vee}{\vdash^{1}_{3} (p \wedge \overline{p}) \wedge (p \vee \overline{p})}\ \wedge \qquad \text{and so } [\![\, p \wedge \overline{p} \wedge (p \vee \overline{p}) \,]\!] = \frac{1}{3}$$

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Non-truth-functionality: negation and disjunction

- (Negation) Consider p and $p \wedge \overline{p}$:
  - $[\![\, p \,]\!] = [\![\, p \wedge \overline{p} \,]\!] = 0$
  - $[\![\, \overline{p} \,]\!] = 0$
  - $[\![\, \overline{p \wedge \overline{p}} \,]\!] = 1$
- (Disjunction) Consider p, $\overline{p}$ and $p \wedge \overline{p}$:
  - $[\![\, p \,]\!] = [\![\, \overline{p} \,]\!] = [\![\, p \wedge \overline{p} \,]\!] = 0$
  - $[\![\, p \vee \overline{p} \,]\!] = 1$
  - $[\![\, p \vee (p \wedge \overline{p} \,]\!] = \frac{1}{2}$

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Non-truth-functionality: conjunction

$$\dfrac{\begin{matrix}\pi \\ \vdots \\ \vdash^{p}_{q} A\end{matrix} \qquad \begin{matrix}\rho \\ \vdots \\ \vdash^{r}_{s} B\end{matrix}}{\vdash^{p+r}_{q+s} A \wedge B}\wedge$$

Remark

$\frac{p}{q} \oplus \frac{r}{s} ::= \frac{p+r}{q+s}$ is not an operation on $\mathbb{Q}$

Example

$\frac{1}{2} = \frac{2}{4}$, but $\frac{1}{2} \oplus \frac{1}{3} \neq \frac{2}{4} \oplus \frac{1}{3}$

Why GS4?

In GS4:

- the generalized axiom allows us to introduce arbitrary contexts
- additive, i.e., context-sharing rules
- this has the effect of **maximizing**:
  1. the number of atoms occurring in axiomatic clauses, and so
  2. the number of identity axioms in proofs

Example

$$\pi: \dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} \overline{q}, p}\,ax. \qquad \dfrac{}{\vdash^{0}_{1} q}\,ax.}{\vdash^{0}_{2} \overline{q}, p \wedge q}\wedge}{\vdash^{0}_{2} \overline{q} \vee (p \wedge q)}\vee \qquad \rho: \dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} \overline{q}, p}\,ax. \qquad \dfrac{}{\vdash^{1}_{1} q, \overline{q}}\,ax.}{\vdash^{1}_{2} \overline{q}, p \wedge q}\wedge}{\vdash^{1}_{2} \overline{q} \vee (p \wedge q)}\vee$$

Strong cut-elimination

Theorem (Girard, 1989)

The cut rule $\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, \overline{A}}{\vdash \Gamma}$ *is redundant in* $\overline{\overline{\mathrm{GS4}}}$ *(and so is in* mGS4*).*

Proof.

- Gentzen's cut-elimination algorithm pushes cut-applications upwards along proof-trees
- Instances of the complementary axiom are closed under elimination of single occurrences of literals:

$$\dfrac{\begin{array}{c}\vdots\\ \vdash \Delta, p\end{array} \qquad \dfrac{}{\vdash \Delta, \overline{p}}\,\overline{ax.}}{\vdash \Delta}\,cut \quad \longrightarrow \quad \dfrac{}{\vdash \Delta}\,\overline{ax.}$$

□

Cuts and cut-elimination

- Cut-applications may 'distort' semantic evaluations of logical formulas

Example

$$\pi : \dfrac{\dfrac{}{\vdash^{1}_{1} p, \overline{p}}\,ax. \qquad \dfrac{}{\vdash^{0}_{1} p, p}\,\overline{ax.}}{\vdash^{1}_{2} p}\,cut \quad \longrightarrow \quad \pi' : \dfrac{}{\vdash^{0}_{1} p}\,\overline{ax.}$$

- Eliminate cuts = compute the function $[\![\cdot]\!]$

Example

$$\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} p, \overline{q}}\,\overline{ax}. \qquad \dfrac{}{\vdash^{1}_{1} p, \overline{p}}\,ax.}{\vdash^{1}_{2} p, \overline{q} \wedge \overline{p}}\,\wedge \qquad \dfrac{\dfrac{}{\vdash^{0}_{1} q, p, p}\,\overline{ax}.}{\vdash^{0}_{1} q \vee p, p}\,\vee}{\vdash^{1}_{3} p}\,cut$$

$$\downarrow$$

$$\dfrac{\dfrac{\dfrac{}{\vdash^{0}_{1} p, p, \overline{q}}\,\overline{ax}. \qquad \dfrac{}{\vdash^{0}_{1} q, p, p}\,\overline{ax}.}{\vdash^{0}_{2} p, p}\,cut \qquad \dfrac{}{\vdash^{1}_{1} p, \overline{p}}\,ax.}{\vdash^{1}_{3} p}\,cut$$

$$\downarrow$$

$$\dfrac{\dfrac{}{\vdash^{0}_{1} p, p}\,\overline{ax}. \qquad \dfrac{}{\vdash^{1}_{1} p, \overline{p}}\,ax.}{\vdash^{1}_{2} p}\,cut$$

$$\downarrow$$

$$\dfrac{}{\vdash^{0}_{1} p}\,\overline{ax}. \qquad \text{and, indeed, } [\![p]\!] = 0$$

Bounded supraclassical logics

Definition ($mGS4_{p/q}$)

Given p and q such that $q \neq 0$ and $p \leqslant q$:

A is a theorem of $mGS4_{p/q}$ $\iff$ $[\![A]\!] \geqslant \dfrac{p}{q}$

DENSITY

For any $p, q, r, s \in \mathbb{N}$:

$$mGS4_{p/q} \subset mGS4_{r/s} \implies mGS4_{p/q} \subset mGS4_{p+r/q+s} \subset mGS4_{r/s}$$

A hierarchy of supraclassical logics

$$
\begin{array}{ccc}
\text{mGS4}_{1} & = & \text{GS4} \\
\cap & & \\
\vdots & & \\
\cap & & \\
\text{mGS4}_{2/3} & & \\
\cap & & \\
\vdots & & \\
\cap & & \\
\text{mGS4}_{1/2} & & \\
\cap & & \\
\vdots & & \\
\cap & & \\
\text{mGS4}_{0} & = & \overline{\text{GS4}}
\end{array}
$$

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Bounded *vs* Makinson's supraclassical logics

	$\text{mGS4}_{p/q}$	Makinson's supraclassical logics
REFLEXIVITY	✓	✓
MONOTONICITY	✓	✓
TRANSITIVITY		✓
STRUCTURALITY	✓	

Example (The cut-rule is not admissible)

- $[\![\, p, (p \vee \overline{p}) \wedge (p \wedge \overline{p}) \,]\!] = \frac{1}{3}$ (provable in $\text{mGS4}_{1/3}$)
- $[\![\, p, \overline{(p \vee \overline{p}) \wedge (p \wedge \overline{p})} \,]\!] = 1$ (provable in $\text{mGS4}_{1/3}$)
- $[\![\, p \,]\!] = 0$ (unprovable in $\text{mGS4}_{1/3}$!)

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Non-transitivity and paradoxes

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\vdash^{1}_{1} T[\lambda], \overline{T[\lambda]}}}{\vdash^{1}_{1} \lambda, \overline{T[\lambda]}}}{\vdash^{1}_{1} \lambda, \lambda}
\qquad
\dfrac{}{\vdash^{1}_{1} \lambda, \overline{\lambda}}
}{\vdash^{2}_{2} \lambda}\,cut
\qquad
\dfrac{\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\vdash^{1}_{1} T[\lambda], \overline{T[\lambda]}}}{\vdash^{1}_{1} \lambda, \overline{T[\lambda]}}}{\vdash^{1}_{1} \lambda, \lambda}
\qquad
\dfrac{}{\vdash^{1}_{1} \lambda, \overline{\lambda}}
}{\vdash^{2}_{2} \lambda}\,cut
}{\vdash^{2}_{2} T[\lambda]}}{\vdash^{2}_{2} \overline{\lambda}}
}{\vdash^{4}_{4} \varepsilon}\,cut
$$

Remark:

- $[\![\,\varepsilon\,]\!] \neq 1$, actually $[\![\,\varepsilon\,]\!] = 0$
- The derivation of the empty sequent proves blocked in any system $\mathrm{mGS4}_q$ with $q > 0$

A kind of paraconsistency

- The set of contradictions is **dense**, i.e., for any $\frac{p}{q}$ there is a contradiction $\bot$ such that $[\![\,\bot\,]\!] = \frac{p}{q}$
- For any bounded system $\mathrm{mGS4}_{p/q}$:
  - there is a contradiction $[\![\,\bot_1\,]\!] \geqslant \frac{p}{q}$
  - there is a contradiction $[\![\,\bot_2\,]\!] < \frac{p}{q}$

A novel approach to proof-theoretic semantics!

- Put aside all you know about classical Boolean semantics...
- Acknowledge as true only the fact that anything implies itself: $p \vdash p$
- Acknowledge the logical rules as soundness mixing / preserving

Key ideas

- A cut-free proof $\pi : T$ measures the quantity of identity present in T
- This measure gives us a semantics in terms of proofs

A novel approach to proof-theoretic semantics?

What is a proof-theoretic semantics?

proof-theoretic semantics $\Rightarrow$ semantics in terms of proofs

semantics in terms of proofs $\not\Rightarrow$ proof-theoretic semantics

Actually:

recognizing identity $A \vdash A$ in classical logic amounts to recognizing the excluded middle $\vdash A, \overline{A}$ which is always a problematic move from the epistemic viewpoint

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Formal Semantics in Modern Type Theories

(MTT-semantics is both model/proof-theoretic)

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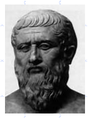
Natural Language Semantics

- Semantics – study of meaning (communicate = convey meaning)
- Various kinds of theories of meaning
  - Meaning is reference ("referential theory")
    - Word meanings are things (abstract/concrete) in the world.
    - c.f., Plato, …



  - Meaning is concept ("internalist theory")
    - Word meanings are ideas in the mind.
    - c.f., Aristotle, …, Chomsky.

  - Meaning is use ("use theory")
    - Word meanings are understood by their uses.
    - c.f., Wittgenstein, …, Dummett.

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Type-Theoretical Semantics

- Montague Semantics (MG)
  - R. Montague (1930–1971)
  - Dominating in linguistic semantics since 1970s
  - Set-theoretic, using simple type theory as intermediate
  - Types ("single-sorted"): e, t, e→t, …

- MTT-semantics: formal semantics in modern type theories
  - Examples of MTTs:
    - Martin-Löf's TT: predicative; non-standard FOL
    - pCIC (Coq) & UTT (Luo 1994): impredicative; HOL
  - Ranta (1994): formal semantics in Martin-Löf's type theory
  - Recent development ➔ full-scale alternative to MG
  - Many types ("many-sorted"): Table, ∑(Man, handsome), Phy•Info

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- Recent development on rich typing in NL semantics
  - MTT-semantics is one of these developments.
  - Chatzikyriakidis and Luo (eds.) Modern Perspectives in Type Theoretical Sem. Springer, 2017. (Collection on rich typing)
  - Chatzikyriakidis and Luo. Formal Semantics in Modern Type Theories. Wiley/ISTE. (Monograph on MTT-sem, to appear)
- Advantages of MTT-semantics, including
  - Both model-theoretic & proof-theoretic – offering a new perspective not available before.
  - Today: focus on this after introducing MTT-semantics.

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MTT-semantics compared with Montague sem.

	Example	Montague semantics	Semantics in MTTs
CN	man, human	$[\![man]\!],\ [\![human]\!] : e \to t$	$[\![man]\!],\ [\![human]\!] : Type$
IV	talk	$[\![talk]\!] : e \to t$	$[\![talk]\!] : [\![human]\!] \to Prop$
Adj	handsome	$[\![handsome]\!] : (e \to t) \to (e \to t)$	$[\![handsome]\!] : [\![man]\!] \to Prop$
MCN	handsome man	$[\![handsome]\!]([\![man]\!])$	$\sum[m : [\![man]\!],\ h : [\![handsome]\!](m)]$
S	A man talks	$\exists m : e.\ [\![man]\!](m) \& [\![talk]\!](m)$	$\exists m : [\![man]\!].\ [\![talk]\!](m)$

E.g., in MTT-semantics, CNs are types rather than predicates:

(*) John is a man.

- Montague: man(j) where man : e→t
- MTT-sem: j : Man where Man : Type

(#) The table talks. – What about talk(t)?

- Well-typed/false in Montague (talk : e→t & t : e)
- Untypable/meaningless in MTT-sem (talk : Human→Prop & t:Table)
- "selectional restriction": meaningfulness v.s. truth

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Modelling Adjective Modifications [CL13, Luo18, XLC18]

Classical classification	Example	Characterisation of Adj(N)	MTT-semantics
intersective	handsome man	N & Adj	∑x:Man.handsome(x)
subsective	large mouse	N (Adj depends on N)	large : ΠA:CN. A→Prop large(mouse) : Mouse→Prop
privative	fake gun	¬N	$G = G_R + G_F$ with $G_R \leq_{inl} G$, $G_F \leq_{inr} G$
non-committal	alleged criminal	nothing implied	∃h:Human. $B_h(\ldots)$

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Note on Subtyping

- Subtyping essential for MTT-semantics
  - Could a “handsome man” talk?
  - Paul talks ➔ talk(p)?
    where talk:Human→Prop and p:[handsome man]
  - talk(p) : Prop, because
    p : [handsome man] = Σ(Man,handsome) $\leq$ Man $\leq$ Human
- Remarks
  - Subtyping is crucial for MTT-semantics.
  - Coercive subtyping [Luo97, XLS12] is adequate for MTTs and we use it in MTT-semantics.

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Advanced features in MTT-semantics: examples

- Anaphora analysis
  - MTTs provide alternative mechanisms for proper treatments via Σ-types [Sundholm 1989] (cf, DRTs, dynamic logic, …)
- Linguistic coercions
  - Coercive subtyping provides a promising mechanism [Asher & Luo 2012]
- Copredication
  - Cf, [Pustejovsky 1995, Asher 2011, Retoré et al 2010]
  - Dot-types [Luo 2009, Xue & Luo 2012, Chatzikyriakidis & Luo 2018]
- Several recent developments
  - Dependent event types in event sem. [Luo & Soloviev (WoLLIC17, TYPES19)]
  - Propositional Forms of Judgemental Interpretations [Xue et al (NLCS18)]
  - CNs as Setoids [Chatzikyriakidis & Luo (J of Oslo meeting 2018)]
  - HoTT-logic for MTT-semantics in Martin-Löf’s TT (LACompLing18)

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MTT-semantics is both model/proof-theoretic

- Model-theoretic semantics (traditional)
  - Meaning as denotation (Tarski, …)
  - Montague: NL → (simple TT) → set theory
- Proof-theoretic semantics
  - Meaning as inferential use (proof/consequence)
  - Gentzen, Prawitz, …, Martin-Löf
  - e.g., Martin-Löf's meaning theory
- MTT-semantics
  - Both model-theoretic and proof-theoretic – in what sense?
  - What does this imply?

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Formal semantics in Modern Type Theories (MTT-semantics) is both model-theoretic and proof-theoretic.

- NL → MTT (representational, model-theoretic)
  - MTT as meaning-carrying language with its <u>types</u> representing collections (or "sets") and <u>signatures</u> representing situations
- MTT → meaning theory (inferential roles, proof-theoretic)
  - MTT-judgements, which are semantic representations, can be understood proof-theoretically by means of their inferential roles
- Z. Luo. Formal Semantics in Modern Type Theories: Is It Model-theoretic, Proof-theoretic, or Both? Invited talk at LACL14.

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MTT-semantics being model-theoretic

- MTTs offer powerful representations.
- Rich type structure
  - Collections represented by types
  - Eg, CNs and their adjective modifications (see earlier slides)
  - Wide coverage – a major advantage of model-theoretic sem
- Useful contextual mechanisms – signatures
  - Various phenomena in linguistic semantics (eg, coercion & infinity)
  - Situations (incomplete world) represented by signatures (next slide)

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MTT-semantics being model-theoretic (cont^ed^)

- Signatures Σ as in (cf, Edin LF [Harper et al 1987])

$$\Gamma \vdash_{\Sigma} a : A$$

  with $\Sigma = c_1{:}A_1, \ldots, c_n{:}A_n$
- New forms besides c:A [Luo LACL14]

$$\ldots, c{:}A, \ldots, A \leq_c B, \ldots, c \sim a : A, \ldots$$

  - Subtyping entries (cf, Lungu's PhD thesis 2018)
  - Manifest entries (can be emulated by coercive subtyping)
- *Theorem (conservativity)*

  The extension with new signature entries preserves the meta-theoretic properties for coherent signatures.

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MTT-semantics being proof-theoretic

- MTTs are representational with proof-theoretic sem
  - Not available before – cf, use theory of meaning
- MTT-based proof technology
  - Reasoning based on MTT-semantics can be carried out in proof assistants like Coq:
    - pretty straightforward but nice application of proof technology to NL reasoning (not-so-straightforward in the past …)
  - Some Coq codes can be found in:
    - Z. Luo. Contextual analysis of word meanings in type-theoretical semantics. Logical Aspects in Computational Linguistics. 2011.
    - S. Chatzikyriakidis & Z. Luo. NL Inference in Coq. JoLLI 23(4). 2014.
    - S. Chatzikyriakidis & Z. Luo. Proof assistants for NL semantics. LACL 2016.
    - T. Xue et al. Propositional Forms of Judgemental Interpretations. NLCS 2018.

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- Why important?
  - Model-theoretic – powerful semantic tools
    - Much richer typing mechanisms for formal semantics
    - Powerful contextual mechanism to model situations
  - Proof-theoretic – practical reasoning on computers
    - Existing proof technology: proof assistants (Coq, Agda, Lego/Plastic, Nuprl)
    - Applications to NL reasoning
  - Leading to both of
    - Wide-range modelling as in model-theoretic semantics
    - Effective inference based on proof-theoretic semantics

Remark: MTT-semantics offers a new perspective – new possibility not available before!

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Abstract Semantic Conditions and the Incompleteness of Intuitionistic Propositional Logic with respect to Proof-Theoretic Semantics

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Joint work with Peter Schroeder-Heister

Third Tübingen Conference on Proof-Theoretic Semantics
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See T. P. & P. S.-H.: Incompleteness of intuitionistic logic with respect to proof-theoretic semantics, *Studia Logica* 107(1), 233–246, 2019.

1. Proof-theoretic validity in an abstract setting

Validity-based proof-theoretic semantics

We consider the intuitionistic propositional calculus (IPC) with the standard constants $\wedge$, $\vee$, $\rightarrow$ and $\neg$.

In validity-based proof-theoretic semantics:

- validity of atomic formulas determined by an atomic system S
- semantical clauses for the logical constants inductively determine validity with respect to S:
  - S-validity $\vDash_S A$ of a formula A
  - S-consequence $\Gamma \vDash_S A$ between a set of formulas Γ and a formula A.

Our abstract semantic setting:

- nature of S is left open
- we just assume that a set $\mathscr{S}$ of entities called bases S is given.

S-consequence $\Gamma \vDash_S A$ and S-validity $\vDash_S A$

We assume that for each base $S \in \mathscr{S}$ a consequence relation

$$\Gamma \vDash_S A$$

is given, called S-consequence.

S-validity of A, that is, $\vDash_S A$ means $\emptyset \vDash_S A$.

We expect that S-validity respects the intended meaning of the logical constants, expressed by the conditions:

($\wedge$) $\vDash_S A \wedge B \iff \vDash_S A$ and $\vDash_S B$.

($\vee$) $\vDash_S A \vee B \iff \vDash_S A$ or $\vDash_S B$.

($\rightarrow$) $\vDash_S A \rightarrow B \iff A \vDash_S B$.

Note: Only $\wedge$, $\vee$, $\rightarrow$ are taken into account.

Logical consequence

Universal or logical consequence is, as usual, understood as transmitting S-validity from the antecedents to the consequent.

This is achieved by assuming that besides $\vDash_S$ there is a consequence relation $\vDash$ available, such that:

($\vDash$) $\Gamma \vDash A \iff$ For all $S \in \mathscr{S}$: ($\vDash_S \Gamma \implies \vDash_S A$).

($\vDash'$) If $\Gamma \vDash A$, then $\Gamma \vDash_S A$ for any S.

Condition ($\vDash'$) expresses that $\vDash$ is a generalization of $\vDash_S$.

($\vDash'$) follows from ($\vDash$), if we assume that ($\vDash_S \Gamma \implies \vDash_S A$) implies $\Gamma \vDash_S A$.

However, we do not presuppose this as a necessary condition.

Abstract notion of a semantics

We speak of a

validity-based proof-theoretic semantics in the abstract sense

in short: a semantics, if

- a non-empty set $\mathscr{S}$ of bases is given
- consequence relations $\vDash_S$ (for each $S \in \mathscr{S}$) and $\vDash$ are given

such that

- the five conditions ($\wedge$), ($\vee$), ($\rightarrow$), ($\vDash$), ($\vDash'$) are met.

Most concrete versions of proof-theoretic semantics are semantics in this abstract sense.

Deviant proof-theoretic semantics challenge the fact that $\vDash_S$ or $\vDash$ are standard consequence relations.

Standard results

Lemma

(i) Harrop's rule

$$\frac{\neg A \rightarrow (B_1 \vee B_2)}{(\neg A \rightarrow B_1) \vee (\neg A \rightarrow B_2)}$$

is not derivable in IPC (though it is admissible).

(ii) For IPC ("$\vdash$") the generalized disjunction property holds:
GDP($\vdash$) If $\Gamma \vdash A \vee B$, where $\vee$ does not occur in Γ, then $\Gamma \vdash A$ or $\Gamma \vdash B$.

(iii) Disjunctions can always be removed from a negated formula:

$$(\vee\text{-}removal) \begin{cases} \neg(A \vee B) \dashv\vdash \neg A \wedge \neg B; \\ \neg(A \wedge B) \dashv\vdash \neg(\neg\neg A \wedge \neg\neg B); \\ \neg(A \rightarrow B) \dashv\vdash \neg\neg A \wedge \neg B. \end{cases}$$

Soundness and completeness of IPC

Definition

Soundness of IPC means:

For any Γ and A: if $\Gamma \vdash A$, then $\Gamma \vDash A$.

Completeness of IPC means:

For any Γ and A: if $\Gamma \vDash A$, then $\Gamma \vdash A$.

Lemma A

In view of

($\vDash'$) If $\Gamma \vDash A$, then $\Gamma \vDash_S A$ for any S,

soundness implies:

For any Γ and A, if $\Gamma \vdash A$, then $\Gamma \vDash_S A$ for any S.

2. Conditions for incompleteness of IPC

Generalized disjunction property

We show that IPC is incomplete, if a semantics satisfies certain special conditions.

The generalized disjunction property is crucial.

Lemma (GDP($\vdash$) for the derivability relation $\vdash$ of IPC)

If $\Gamma \vdash A \vee B$, where $\vee$ does not occur in Γ, then either $\Gamma \vdash A$ or $\Gamma \vdash B$.

We are particularly interested in its semantical version.

Definition (GDP($\Vdash$), for arbitrary consequence relations $\Vdash$)

If $\Gamma \Vdash A \vee B$, where $\vee$ does not occur in Γ, then either $\Gamma \Vdash A$ or $\Gamma \Vdash B$.

We assume:

- A semantics is given according to our abstract semantic setting.
- IPC is sound with respect to this semantics.

GDP($\models_S$) $\Longrightarrow$ Harrop's rule is valid

Lemma

If GDP($\models_S$) for every S, then Harrop's rule is valid.

Thus, if we have GDP($\models_S$) for every S, then completeness fails, since Harrop's rule is not derivable in IPC.

GDP($\models_S$) $\Longrightarrow$ Harrop's rule is valid

Lemma

If GDP($\models_S$) for every S, then Harrop's rule is valid.

Proof.

$$
\begin{aligned}
\models_S \underbrace{\neg A \to (B_1 \vee B_2)}_{\text{premiss of Harrop}} &\Longrightarrow \neg A \models_S B_1 \vee B_2; \text{ by } (\to) \\
&\Longrightarrow A' \models_S B_1 \vee B_2 \text{ for some } \vee\text{-free formula } A' \text{ s.t.} \\
&\qquad A' \dashv\vdash \neg A; \text{ by } (\vee\text{-}\textit{removal}), \text{ Lemma A, trans. of } \models_S \\
&\Longrightarrow A' \models_S B_i \text{ for } i = 1 \text{ or } 2; \text{ by GDP}(\models_S) \\
&\Longrightarrow \neg A \models_S B_i; \text{ by } (\vee\text{-}\textit{removal}), \text{ Lemma A, trans. of } \models_S \\
&\Longrightarrow \models_S \neg A \to B_i; \text{ by } (\to) \\
&\Longrightarrow \models_S \underbrace{(\neg A \to B_1) \vee (\neg A \to B_2)}_{\text{conclusion of Harrop}}; \text{ by } (\vee)
\end{aligned}
$$

Holds for any S, thus Harrop's rule is valid by condition ($\models$). □

Export + completeness $\Longrightarrow$ GDP($\models_S$) for all S

Definition (Export)

For every base S there is a set of $\vee$-free formulas S^* such that for all Γ and A:
$\Gamma \models_S A \iff \Gamma, S^* \models A$.

Lemma

Assume completeness of IPC. Then Export implies GDP($\models_S$) for every S.

Proof.

Suppose completeness holds, and $\vee$ does not occur in Γ.
Then we obtain GDP($\models_S$) as follows:

$$\begin{aligned}
\Gamma \models_S A_1 \vee A_2 &\Longrightarrow \Gamma, S^* \models A_1 \vee A_2; \text{ by Export}\\
&\Longrightarrow \Gamma, S^* \vdash A_1 \vee A_2; \text{ by completeness}\\
&\Longrightarrow \Gamma, S^* \vdash A_i \text{ for } i = 1 \text{ or } 2; \text{ by GDP}(\vdash)\\
&\Longrightarrow \Gamma, S^* \models A_i \text{ for } i = 1 \text{ or } 2; \text{ by soundness}\\
&\Longrightarrow \Gamma \models_S A_i \text{ for } i = 1 \text{ or } 2; \text{ by Export.}
\end{aligned}$$

□

Export + completeness $\Longrightarrow$ GDP($\models_S$) for all S

Definition (Export)

For every base S there is a set of $\vee$-free formulas S^* such that for all Γ and A:
$\Gamma \models_S A \iff \Gamma, S^* \models A$.

Lemma

Assume completeness of IPC. Then Export implies GDP($\models_S$) for every S.

Hence, assuming completeness we obtain that Harrop's rule is valid.

Again assuming completeness, this implies that Harrop's rule is derivable in IPC, which is not the case.

Thus we have refuted completeness.

Note that we have not shown GDP($\models_S$) outright, but only under the assumption of completeness, which is, however, sufficient to refute completeness.

Condition $(\vDash_S)$ $\Longrightarrow_{\text{cl.}}$ GDP$(\vDash_S)$ for all S

Definition (Condition $(\vDash_S)$)

$\Gamma \vDash_S A \iff (\vDash_S \Gamma \implies \vDash_S A)$.

Note: Since $\vDash_S$ is a consequence relation: $\Gamma \vDash_S A \implies (\vDash_S \Gamma \implies \vDash_S A)$.

Lemma

Suppose $(\vDash_S)$.
Then, using classical logic in the metalanguage, GDP$(\vDash_S)$ can be proved.

Proof.

$$
\begin{aligned}
\Gamma \vDash_S A \vee B &\implies (\vDash_S \Gamma \implies \vDash_S A \vee B)\text{; by } (\vDash_S)\\
&\implies (\vDash_S \Gamma \implies (\vDash_S A \text{ or } \vDash_S B))\text{; by } (\vee)\\
&\implies (\vDash_S \Gamma \implies \vDash_S A) \text{ or } (\vDash_S \Gamma \implies \vDash_S B)\text{; classical metalang.}\\
&\implies \Gamma \vDash_S A \text{ or } \Gamma \vDash_S B\text{; by } (\vDash_S).
\end{aligned}
$$

(Here $\vee$ may occur in Γ.) □

Import $\implies$ GDP$(\vDash_S)$ for all S

Definition (Import)

For every S, every $\vee$-free Γ and every A there is a base $S + \Gamma$ such that:
$\Gamma \vDash_S A \iff \vDash_{S+\Gamma} A$.

Any $\vee$-free set of assumptions of logical consequence $\vDash$ can be 'imported' into a base S of non-logical consequence $\vDash_S$.

Lemma

Import implies GDP$(\vDash_S)$.

Proof.

Suppose $\vee$ does not occur in Γ.

$$
\begin{aligned}
\Gamma \vDash_S A \vee B &\implies \vDash_{S+\Gamma} A \vee B\text{; by Import}\\
&\implies \vDash_{S+\Gamma} A \text{ or } \vDash_{S+\Gamma} B\text{; by } (\vee)\\
&\implies \Gamma \vDash_S A \text{ or } \Gamma \vDash_S B\text{; by Import.}
\end{aligned}
$$

□

Results so far

For any semantics with respect to which IPC is sound, we have:

Theorem

(i) GDP($\vDash_S$) for all S $\Longrightarrow$ validity of Harrop's rule,
thus: GDP($\vDash_S$) for all S $\Longrightarrow$ incompleteness.

(ii) Export + completeness $\Longrightarrow$ GDP($\vDash_S$) for all S,
thus: Export $\Longrightarrow$ incompleteness.

(iii) Condition ($\vDash_S$) $\Longrightarrow$ GDP($\vDash_S$) for all S (using classical metalanguage),
thus: Condition ($\vDash_S$) $\Longrightarrow$ incompleteness.

(iv) Import $\Longrightarrow$ GDP($\vDash_S$) for all S,
thus: Import $\Longrightarrow$ incompleteness.

To establish incompleteness of IPC for a semantics, for which IPC is sound, we only need to establish one of the four conditions.

3. Incompleteness results for concrete proof-theoretic semantics

Concrete semantics

Bases S are explicitly specified.

Consequence relations $\vDash$ and $\vDash_S$ defined according to abstract semantic setting.

Specification of $\vDash$ and $\vDash_S$ by explicit or inductive definition.

Alternatively:

Start with a different fundamental concept in terms of which $\vDash_S$ is then defined.

For example: Prawitz defines validity of derivation structures, on which the definition of valid consequence is based.

We consider certain types of concrete semantics.

They are proof-theoretic semantics in the sense that bases are atomic systems generating valid atomic formulas by means of inference rules.

Atomic systems and S-validity of atomic formulas

Definition

An atomic system S is a deductive system with rules of the form $\dfrac{a_1 \quad \ldots \quad a_n}{b}$ where $a_1, \ldots, a_n, b$ are atoms. Notion of derivability: $\vdash_S a$.

Definition (S-validity of atomic formulas)

S-validity of an atomic formula a is defined as derivability of a in S:

(At) $\quad \vDash_S a \;:\Longleftrightarrow\; \vdash_S a.$

The underlying set $\mathscr{S}$ of bases is the set of all atomic systems S.

Atomic systems S are identified with the sets of their rules.

The systems within $\mathscr{S}$ are ordered in the usual way by set inclusion $\subseteq$.

Different kinds of semantics possible, depending on different kinds of atomic systems and different kinds of derivability relations $\vdash_S$.

Extension and non-extension semantics

Extension semantics

S-consequence defined using extensions $S' \supseteq S$, for atomic systems S and S':

($\models_S^{\text{ext}}$) $\Gamma \models_S A \ :\Longleftrightarrow$ For all $S' \supseteq S$: ($\models_{S'} \Gamma \implies \models_{S'} A$)

- Thus $\models_S A \to B \iff$ For all $S' \supseteq S$: ($\models_{S'} A \implies \models_{S'} B$).
- Guarantees monotonicity of $\models_S$: $\Gamma \models_S A \implies \Gamma \models_{S \cup S'} A$ for any S, S'.
- We can strengthen (At) to $a_1, \ldots, a_n \models_S a \iff a_1, \ldots, a_n \vdash_S a$.

Non-extension semantics

S-consequence is defined by

($\models_S^{\text{non-ext}}$) $\Gamma \models_S A \ :\Longleftrightarrow$ ($\models_S \Gamma \implies \models_S A$)

- Thus $\models_S A \to B \iff$ ($\models_S A \implies \models_S B$).
- S-consequence fails to be monotone.
- As we saw, ($\models_S^{\text{non-ext}}$) (= condition ($\models_S$)) classically implies GDP($\models_S$).

Extension semantics, Export

Lemma (Export)

For extension semantics we can establish

(Export) For every S there is a set of $\vee$-free formulas S^* such that for all Γ and A: $\Gamma \models_S A \iff \Gamma, S^* \models A$.

Proof

- involves representation of atomic rules by implicational formulas;
- monotonicity of $\models_S$ w.r.t. atomic systems is crucial.

Extension semantics, Import

Lemma (Import)

For extension semantics we can establish

(Import) For every S, every $\vee$-free Γ and every A there is a base $S+\Gamma$ such that: $\Gamma \vDash_S A \iff \vDash_{S+\Gamma} A$.

Proof requires atomic systems of higher-level rules, since one has to translate left-iterated implications into rules.

Definition (Higher-level atomic system S)

A higher-level atomic system S is a (possibly empty) set of atomic rules of the form

$$\frac{\begin{array}{c}[\Gamma_1]\\ a_1\end{array} \quad \ldots \quad \begin{array}{c}[\Gamma_n]\\ a_n\end{array}}{b}$$

where the a_i and b are atoms, and the Γ_i are finite sets $\{R^i_1, \ldots, R^i_k\}$ of atomic rules, which may be empty; $n = 0$ is allowed.

Incompleteness results

Theorem

(i) Condition $(\vDash_S)$ $(= (\vDash_S^{\text{non-ext}}))$ $\Longrightarrow_{\text{cl.}}$ incompleteness.

(ii) Export $\Longrightarrow$ incompleteness.

(iii) Import $\Longrightarrow$ incompleteness.

Corollary

(i) IPC is incomplete with respect to non-extension semantics. (By classical reasoning.)

(ii) IPC is incomplete with respect to extension semantics.

(iii) IPC is incomplete with respect to extension semantics based on higher-level atomic systems.

(iv) IPC is incomplete with respect to Prawitz semantics based on derivation structures.

The Jacobson Radical of a Propositional Theory

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A Contribution to Ninety Years of Glivenko's Theorem

Abstract

Alongside the analogy between maximal ideals and complete theories, the Jacobson radical carries over from ideals of commutative rings to theories of propositional calculi. This prompts a variant of Lindenbaum's Lemma that relates classical validity and intuitionistic provability, and the syntactical counterpart of which is Glivenko's Theorem. As a by-product it becomes possible to interpret in logic Rinaldi, Schuster and Wessel's axioms-as-rules conservation criterion for Scott-style entailment relations.

MSC 2010: 03F03, 03F65.
Key words: Jacobson radical; propositional theory; Lindenbaum's Lemma; Glivenko's Theorem; axioms as rules; conservation criterion; entailment relation.

1 Introduction

Alongside the analogy between maximal ideals and complete theories, the Jacobson radical carries over from ideals of commutative rings to theories of propositional calculi. This prompts a variant of Lindenbaum's Lemma that relates classical validity and intuitionistic provability, and the syntactical counterpart of which is Glivenko's Theorem. As a by-product it makes possible to interpret in logic Rinaldi, Schuster and Wessel's axioms-as-rules conservation criterion for Scott-style entailment relations [59, 60]. This criterion appears to be the common syntactical core of many semantic conservation theorems occurring in actual mathematical practice, typically as reduction to a special case by Zorn's Lemma.

Unless specified otherwise, we work in a suitable fragment of Aczel and Rathjen's *Constructive Set Theory* (***CZF***) [1, 2] based on intuitionistic logic. When we occasionally need to invoke classical logic or even (a suitable form of) the Axiom of Choice (AC), and thus go beyond ***CZF***, we switch to ***ZF*** and ***ZFC***, respectively.

By a *finite* set we understand a set that can be written as $\{ a_1, \dots, a_n \}$ for some $n \geq 0$. Given any set S, let $\mathrm{Pow}(S)$ (respectively, $\mathrm{Fin}(S)$) consist of the (finite) subsets of S. We refer to [60] for further provisos to carry over to the present note.

2 Entailment relations

Entailment relations, both in their single- and multi-conclusion variant, are at the heart of this note. We briefly recall the basic notions, to which end we closely follow [59, 60].

2.1 Consequence

Let S be a set and $\rhd \subseteq \mathrm{Pow}(S) \times S$. All but one of Tarski's axioms of consequence [73] can be put as

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{\forall b \in U (V \rhd b) \quad U \rhd a}{V \rhd a}\ (\mathrm{T}) \qquad \frac{U \rhd a}{\exists U_0 \in \mathrm{Fin}(U)(U_0 \rhd a)}\ (\mathrm{A})$$

where $U, V \subseteq S$ and $a \in S$. These axioms also characterise a finitary covering or Stone covering in formal topology [63][1]; see further [10, 9, 48, 49, 64, 65]. The notion of consequence has allegedly been described first by Hertz [29, 30, 31]; see also [4, 37]. We do not employ the one of Tarski's axioms by which he requires that S be countable. This axiom aside, Tarski has rather characterised the set of consequences of a set of propositions, which corresponds to the *algebraic closure operator* $U \mapsto U^{\rhd}$ on $\mathrm{Pow}(S)$ correlated to a relation $\rhd$ as above, viz.

$$U^{\rhd} \equiv \{a \in S : U \rhd a\}\,.$$

Rather than with Tarski's notion, we henceforth work with its restriction to finite subsets, that is, the notion of a *single-conclusion entailment relation*. This is a relation $\rhd \subseteq \mathrm{Fin}(S) \times S$ that satisfies

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{V \rhd b \quad V', b \rhd a}{V, V' \rhd a}\ (\mathrm{T}) \qquad \frac{U \rhd a}{U, U' \rhd a}\ (\mathrm{M})$$

for all finite $U, U', V, V' \subseteq S$ and $a, b \in S$, where as usual $U, V \equiv U \cup V$ and $V, b \equiv V \cup \{b\}$. Our focus thus is on *finite* subsets of S, for which we henceforth reserve the letters $U, V, W, \dots$; we also sometimes write $a_1, \dots, a_n$ in place of $\{a_1, \dots, a_n\}$. Redefining

$$T^{\rhd} \equiv \{a \in S : \exists U \in \mathrm{Fin}(T)(U \rhd a)\} \tag{1}$$

[1] This also is from where we have taken the symbol $\rhd$, which is further used [74, 8] to denote a 'consecution' [56].

for *arbitrary* subsets T of S gives back an algebraic closure operator on $\mathrm{Pow}(S)$; we sometimes write $T \rhd a$ to mean $a \in T^{\rhd}$. The single-conclusion entailment relations thus correspond exactly to the relations satisfying Tarski's axioms above.

2.2 Entailment

Let S be a set and $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Fin}(S)$. Scott's [69] axioms of entailment can be put as

$$\frac{U \between W}{U \vdash W}\ (\mathrm{R}) \qquad \frac{V \vdash W, b \quad V', b \vdash W'}{V, V' \vdash W, W'}\ (\mathrm{T}) \qquad \frac{U \vdash W}{U, U' \vdash W, W'}\ (\mathrm{M})$$

for finite $U, V, W \subseteq S$ and $b \in S$, where $U \between W$ means that U and W have an element in common [64]. To be precise, any such $\vdash$ is a *multi-conclusion entailment relation*, where 'multi' includes 'empty'.

This fairly general notion of entailment has been introduced by Scott [68, 69, 70], building on Hertz's and Tarski's work (see above), and of course on Gentzen's sequent calculus [26, 27]. Shoesmith and Smiley [71] trace multi-conclusion entailment relations back to Carnap [5]. Before Scott, Lorenzen has developed analogous concepts formally [42, 43, 44, 45]; he has even listed [43, pp. 84–5] counterparts of the axioms (R), (T) and (M) for single- and multi-conclusion entailment [19, 18].[2] [3] The relevance of the notion of entailment relation to point-free topology and constructive algebra has been pointed out in [6], and has been used very widely, e.g. in[11, 13, 15, 12, 16, 21, 22, 54, 57, 75, 62, 67, 76]. Consequence and entailment have caught interest from various angles [3, 24, 25, 32, 33, 34, 55, 66, 71, 77].

In practice, $\rhd$ and $\vdash$ are *inductively generated* from the axioms of the intended models, which procedure we here take for granted, referring to [6, 60, 61].

3 Conservation

3.1 Conservation in syntax and semantics

Again, let S be a set, and let $a, b, c, \ldots$ and $U, V, W, \ldots$ range over the elements of S and $\mathrm{Fin}(S)$, respectively. Given a multi-conclusion entailment relation $\vdash$ and a single-conclusion entailment relation $\rhd$ on S, respectively, we throughout assume *Extension*:

$$\mathrm{Ext} \qquad U \rhd a \implies U \vdash a.$$

[2] Stefan Neuwirth has kindly pointed this out to us.

[3] As compared with Gentzen's and Lorenzen's approaches, Scott's entailment relation follow the contexts-as-sets paradigm, which has caused reservations [52, 53].

Of major interest to us is the reverse implication, alias *Conservation*:

$$\text{Con} \qquad U \rhd a \impliedby U \vdash a .$$

An *arbitrary* subset P of S is a *model* of $\vdash$ if

$$P \supseteq U \implies V \between P \quad \text{whenever } U \vdash V .$$

The notion of model carries over to single-conclusion $\rhd$ in the apparent manner, such that the models of $\rhd$ are exactly the $P \in \mathrm{Pow}(S)$ which are *closed* under $\rhd$, i.e. for which $P^{\rhd} = P$. Let $\mathrm{Mod}(\vdash)$ and $\mathrm{Mod}(\rhd)$ consist of the models of $\vdash$ and $\rhd$, respectively. By Extension, $\mathrm{Mod}(\vdash) \subseteq \mathrm{Mod}(\rhd)$, which is equivalent to Extension in ***ZFC*** [60, Lemma 9].

Now Con follows from the *Generalised Krull–Lindenbaum Lemma*, viz.

$$\text{GKL} \qquad \forall P \in \mathrm{Mod}(\vdash)(P \supseteq U \implies a \in P) \implies U \rhd a .$$

the converse of which holds by Extension. Again by Extension, GKL implies the *Trace Completeness Theorem*, viz.

$$\text{CT}_0 \qquad \forall P \in \mathrm{Mod}(\vdash)(P \supseteq U \implies a \in P) \implies U \vdash a ,$$

the converse of which holds by the definition of a model of $\vdash$. This CT_0 is a fragment of AC that implies Restricted Excluded Middle [60, Corollary 5].[4]

In ***ZFC***, GKL and Con are equivalent [60, Theorem 6]. More precisely:

Remark 1. GKL is equivalent to the conjunction of Con and CT_0.

In all, GKL is semantic conservation, and Con is its syntactical counterpart.

3.2 Conservation in mathematical practice

In proof practice, GKL is useful for reductions to special cases, for making possible to use $\vdash$ in proofs about $\rhd$, but GKL is of semantic nature and requires some AC. In comparison, Con is equally sufficient for that kind of reduction, is syntactical and has elementary proofs. Many such cases are known in point-free topology such as locale theory and formal topology [6, 7, 11, 14, 46, 47]; in constructive algebra, especially with dynamical methods [17, 23, 38, 39, 40, 41, 78]; and in the proof theory of order relations [52, 54]. Most of these cases concern algebra at large. But what about logic? For instance, does Con regard classical versus intuitionistic logic? One may think of Gentzen's classical multi-succedent sequent calculus as extending his intuitionistic single-succedent variant [26, 27, 72, 51].[5] This extension of course is not conservative as it stands!

The following *conservation criterion* [59, 60] will help to a better understanding of Con:

[4] The proof of [60, Proposition 4] goes through with CT_0 in place of full CT.

[5] While ***LK*** without doubt is Gentzen's acronym for the former, the one for the latter is usually read as ***LJ*** but sometimes [52] as ***LI***.

Theorem 1. *Let $\vdash$ extend $\rhd$ with certain additional axioms $a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell$. Then $\vdash$ and $\rhd$ satisfy* Con *if and only if*

$$\frac{W, b_1 \rhd c \quad \ldots \quad W, b_\ell \rhd c}{W, a_1, \ldots, a_k \rhd c}$$

for every additional axiom $a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell$, all $c \in S$ and all $W \in$ $\mathrm{Fin}(S)$.

This swiftly follows [60, Theorem 2] from a sandwich criterion for conservation given by Scott [69]. Theorem 1 further is a corollary of cut elimination for entailment relations [61] related to cut elimination in the presence of axioms [50], and gathers together the aforementioned syntactical treatments by computational rules of many a form of AC [58].

3.3 Conservation in logic: the case of classical logic

Let S be a propositional language, and let $\rhd$ stand for (deducibility in) an intermediate logic over S. The models of $\rhd$ are the *theories* of $\rhd$, i.e. the (deductively) closed subsets of S. Let $\vdash$ extend $\rhd$ with the following additional axioms, where as usual $\neg\varphi \equiv \varphi \to \bot$:

$$\vdash \varphi, \neg\varphi \qquad (\varphi \in S).$$

The models of $\vdash$ are the theories T of $\rhd$ which are *complete*, i.e. satisfy

$$\forall \varphi \in S (\varphi \in T \vee \neg\varphi \in T)$$

The corresponding conservation criterion (Theorem 1) reads

$$\frac{\Gamma, \varphi \rhd \psi \quad \Gamma, \neg\varphi \rhd \psi}{\Gamma \rhd \psi}$$

with $\Gamma \in \mathrm{Fin}(S)$ and $\varphi, \psi \in S$. To have Con thus means that $\rhd$ satisfy $\rhd\varphi \vee \neg\varphi$, which is to say that $\rhd$ be classical. Hence Con simply means that adding $\vdash \varphi, \neg\varphi$ amounts to adding $\rhd\varphi \vee \neg\varphi$. This of course is well known and of little interest. Can't we do better?

4 Jacobson radicals

4.1 The Jacobson radical in algebra

Let R be a commutative ring with 1, and let $\rhd$ stand for generation (i.e. linear combination) in R. A model of $\rhd$ is nothing but an *ideal* of R, i.e. a subset closed under linear combination. An ideal J of R is *proper* if $1 \notin J$, which is to say that J is a proper subset of R. A (proper) ideal J of R is a *maximal ideal* if and only if

$$\forall r \in R(J \ni r \vee J, r \rhd 1). \tag{2}$$

With this notation in place, the *Jacobson radical of an ideal* J can be defined as

$$\mathrm{Jac}(J) = \{a \in R : \forall b \in R\,(a, b \rhd 1 \implies J, b \rhd 1)\}.$$

We thus follow the first-order definition for distributive lattices [20] rather than the one for commutative rings [41]. In ***ZFC***, the definition above is equivalent to the more customary second-order definition of the Jacobson radical [36]:

Lemma 1 (***ZFC***)**.** *For every ideal J of R,*

$$\bigcap \mathrm{MaxIdl}_J(R) = \mathrm{Jac}(J)$$

where $\mathrm{MaxIdl}_J(R)$ *consists of the maximal ideals* $\mathfrak{m}$ *in* R *with* $J \subseteq \mathfrak{m}$.

Proof. Let $a \in \mathrm{Jac}(J)$, and let $\mathfrak{m}$ be a maximal ideal such that $\mathfrak{m} \supseteq J$. Either $\mathfrak{m} \ni a$ or $\mathfrak{m}, a \rhd 1$. In the former case we are done. In the latter case there is $b \in R$ such that $\mathfrak{m} \rhd b$ (in particular, $b \in \mathfrak{m}$) and $a, b \rhd 1$. Since $a \in \mathrm{Jac}(J)$, we get $J, b \rhd 1$. By (R) and (T) this implies $J, \mathfrak{m} \rhd 1$ and thus $\mathfrak{m} \rhd 1$. Hence $\mathfrak{m} = R$, by which $\mathfrak{m} \ni a$. Conversely, if $a \notin \mathrm{Jac}(J)$, then there exists $b \in R$ for which $a, b \rhd 1$ but $J, b \rhd 1$ fails. By (R) and (T), $(J, b)^{\rhd}$ lacks a. Zorn's Lemma yields a maximal ideal $\mathfrak{m}$ that contains $(J, b)^{\rhd}$ yet misses a. □

4.2 The Jacobson radical in logic

Let again S be a propositional language, and let $\rhd$ stand for (deducibility in) an intermediate logic over S. A theory T of S is *consistent* if $\bot \notin T$, which is to say that T is a proper subset of S. A (consistent) theory T is *complete* if and only if

$$\forall \varphi \in S(T \ni \varphi \vee T, \varphi \rhd \bot). \tag{3}$$

Alongside the analogy between (2) and (3), we define the *Jacobson radical of a theory* T:

$$\begin{aligned}\mathrm{Jac}(T) &= \{\alpha \in S : \forall \beta \in S\,(\alpha, \beta \rhd \bot \implies T, \beta \rhd \bot)\}\\ &= \{\alpha \in S : \forall \beta \in S(\alpha \rhd \neg\beta \implies T \rhd \neg\beta)\}.\end{aligned}$$

Mutatis mutandis the proof of Lemma 1 proves the *Intermediate Lindenbaum Lemma*:

Theorem 2 (***ZFC***)**.** *For every theory T of S,*

$$\text{ILL} \qquad \bigcap \mathrm{ComThy}_T(S) = \mathrm{Jac}(T)$$

where $\mathrm{ComThy}_T(S)$ *consists of the complete theories* C *in* S *with* $T \subseteq C$.

4.3 From Lindenbaum's Lemma to Glivenko's Theorem

We still consider theories T of an intermediate logic $\rhd$ in a propositional language S.

Lemma 2. *Every complete theory C is classical, i.e. deductively closed for classical logic.*

Proof. For every $\varphi \in S$ either $\varphi \in C$ or $\neg\varphi \in C$, and thus $\varphi \vee \neg\varphi \in C$ in any case. ☐

For a deeper discussion with original references we refer to [35] and [51, p. 27].

Hence the left-hand side of ILL is as for the original Lindenbaum Lemma [73], and thus equals in ***ZFC*** the classical deductive closure of T. What about the right-hand side of ILL?

Lemma 3. *For every theory T,*

$$\mathrm{Jac}(T) = \{\alpha \in S : T \ni \neg\neg\alpha\}.$$

Proof. Let $\alpha \in \mathrm{Jac}(T)$. Since $\alpha \rhd \neg\neg\alpha$, we have $T \rhd \neg\neg\alpha$. Conversely, let $\alpha \in S$ such that $T \ni \neg\neg\alpha$, that is, $T, \neg\alpha \rhd \bot$. If $\beta \in S$ is such that $\alpha \rhd \neg\beta$, then $\beta \rhd \neg\alpha$, which cuts with $T, \neg\alpha \rhd \bot$ to yield $T, \beta \rhd \bot$ or equivalently $T \rhd \neg\beta$. ☐

Now let $\rhd$ be intuitionistic logic $\rhd_i$, and write $\rhd_c$ for classical logic. In this case and by the above, ILL is the semantics of *Glivenko's Theorem* [28], well known as purely syntactical:

Theorem 3 (Glivenko 1929)**.** *For all $\Gamma \subseteq S$ and $\varphi \in S$,*

$$\Gamma \rhd_c \varphi \implies \Gamma \rhd_i \neg\neg\varphi.$$

4.4 Glivenko's Theorem as syntactical conservation

Let again $\rhd_i$ and $\rhd_c$ stand for intuitionistic and classical logic, respectively. Define

$$\Gamma \rhd_g \varphi \equiv \Gamma \rhd_i \neg\neg\varphi \qquad \text{and} \qquad \Gamma \vdash_c \Delta \equiv \Gamma \rhd_c \bigvee \Delta\,,$$

which are a single- and a multi-conclusion entailment relation, respectively. By Glivenko's Theorem, $\rhd_g$ is nothing but $\rhd_c$, but let us forget this for the sake of the argument.

Clearly, $\vdash_c$ extends $\rhd_g$ with

$$\vdash_c \varphi, \neg\varphi \qquad (\varphi \in S)$$

as additional axioms, which indeed mean nothing but $\rhd_c\, \varphi \vee \neg\varphi$ for every $\varphi \in S$.

Theorem 4. $\vdash_c$ *is a conservative extension of* $\rhd_g$.

The conservation criterion required for Theorem 4 reads

$$\frac{\Gamma, \varphi \rhd_g \psi \quad \Gamma, \neg\varphi \rhd_g \psi}{\Gamma \rhd_g \psi}$$

with $\Gamma \in \mathrm{Fin}(S)$ and $\varphi, \psi \in S$, and the proof is the proof of Glivenko's Theorem.

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Inquisitive Proof-Theoretic Semantics

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March 2019
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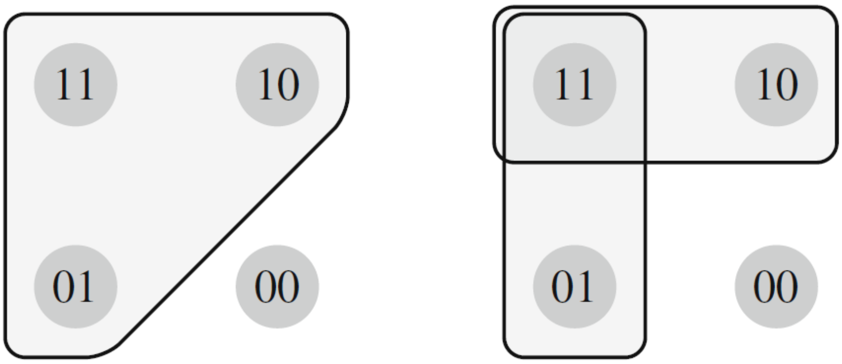

(a) $|p \vee q|$ (b) $[p \vee q]$

Ciardelli and Roelofsen, 2011

4

State

A set of valuations on propositional letters.

5

State

A set of valuations on propositional letters.

Example

Let $v(p) = t$ for all p,

Let $v'(p) = f$ for all p,

$\{v\}, \{v'\}, \{v, v'\}$ are all states.

5

$$s \vDash_{Inq} p$$

6

$$s \vDash_{Inq} p$$
$$\text{for all } v \in s, v(p) = t$$

6

$s \vDash_{Inq} p$	$s \vDash_{Inq} \bot$
for all $v \in s$, $v(p) = t$	$s = \varnothing$

6

$s \vDash_{Inq} p$	$s \vDash_{Inq} \bot$
for all $v \in s$, $v(p) = t$	$s = \varnothing$
$s \vDash_{Inq} \varphi \wedge \psi$	
$s \vDash_{Inq} \varphi$ and $s \vDash_{Inq} \psi$	

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$s \vDash_{Inq} p$	$s \vDash_{Inq} \bot$
for all $v \in s$, $v(p) = t$	$s = \varnothing$
$s \vDash_{Inq} \varphi \wedge \psi$	$s \vDash_{Inq} \varphi \vee \psi$
$s \vDash_{Inq} \varphi$ and $s \vDash_{Inq} \psi$	$s \vDash_{Inq} \varphi$ or $s \vDash_{Inq} \psi$

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$s \vDash_{Inq} p$ for all $v \in s$, $v(p) = t$	$s \vDash_{Inq} \bot$ $s = \varnothing$
$s \vDash_{Inq} \varphi \wedge \psi$ $s \vDash_{Inq} \varphi$ and $s \vDash_{Inq} \psi$	$s \vDash_{Inq} \varphi \vee \psi$ $s \vDash_{Inq} \varphi$ or $s \vDash_{Inq} \psi$

$$s \vDash_{Inq} \varphi \to \psi$$

for all $t \subseteq s$ if $t \vDash_{Inq} \varphi$ then $t \vDash_{Inq} \psi$

6

Inquisitive Logic

7

Weak Logics

Let L be a set of formulas.

8

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L is a *weak logic* if it is closed under modus ponens.

8

Weak Logics

Let L be a set of formulas.

L is a *weak logic* if it is closed under modus ponens.

L is an *(weak) intermediate logic* if $IPL \subseteq L \subseteq CPL$.

8

Inquisitive Logic

Let $InqL$ be the weak logic consisting of IPL and:

$$(\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$$

(Harrop's formula)

$\neg\neg p \to p$ (DNEA)

9

Axiomatizing Inquisitive Semantics

Lemma (Ciardelli and Roelofsen, 2011)

φ is supported by all states

if and only if

$\varphi \in InqL$.

10

Equivalent Logics

Lemma (Ciardelli and Roelofsen, 2011)

Any weak intermediat logic $L = InqL$ if L has:

1. the disjunction property,
2. all formulas equivalent to one in disjunction negative form.

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More Equivalent Logics

Two weak intermediate logics L_1, L_2 are equivalent if each meets one of the conditions:

1. disjunction property and all formulas equivalent to one in disjunction negative form,
2. disjunction property, Harrop's formula and $\neg\neg p \to p$,

12

More Equivalent Logics

Two weak intermediate logics L_1, L_2 are equivalent if each meets one of the conditions:

1. disjunction property and all formulas equivalent to one in disjunction negative form,
2. disjunction property, Harrop's formula and $\neg\neg p \to p$,

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Atomic Systems

13

Level-0

$\overline{p}$ An axiom

14

Level-1

$$\frac{p}{q}$$

Inference from premises

15

Level-2

$$\begin{array}{c}[p]\\ \vdots\\ \dfrac{q}{r}\end{array}$$

Inference from premises discharging an assumption

16

Level-n+2

$$\cfrac{\cfrac{\overset{\vdots}{p}}{q}\ \text{Discharged}}{\cfrac{\overset{\vdots}{q}}{r}}$$

Inference from premises discharging lower level rules

17

Example Proof

$$\overline{p\qquad}$$

18

Example Proof

$$\frac{}{p, \{\bar{p}\}}$$

Example Proof

$$\frac{\dfrac{}{p, \{\bar{p}\}}}{q}$$

Example Proof

$$\dfrac{\overline{p, \{\bar{p}\}}}{q, \{\bar{p}, p/q\}}$$

Example Proof

$$\dfrac{\dfrac{\overline{p, \{\bar{p}\}}}{q, \{\bar{p}, p/q\}}}{r}$$

Example Proof

$$\cfrac{\cfrac{\overline{p, \{\bar{p}\}}}{q, \{\bar{p}, p/q\}}}{r, \{\bar{p},\ \ p/q,\ q/r\}}$$

18

Example Proof

$$\cfrac{\cfrac{\overline{p, \{\bar{p}\}}}{q, \{\bar{p}, p/q\}}}{r, \{\bar{p}, [\ p/q\ ]q/r\}}$$

18

Atomic Systems

Let the set of all atomic rules of any level be denoted as $\mathbb{S}$.

Call a set of atomic rules $S \subseteq \mathbb{S}$ an *atomic system*.

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Supersystems

Call a set $\mathfrak{S} \subseteq \mathcal{P}(\mathbb{S})$ an *atomic supersystem* if it is closed under finite unions and intersections.

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Examples of Supersystems

1. $\mathfrak{S}^M_\infty = \{S \mid S \subseteq \mathbb{S}\}$
2. $\mathfrak{S}_\infty = \{S \subseteq \mathbb{S} \mid (\bot/p) \in S \text{ for all atomic } p\}$
3. $\mathfrak{S}_n = \{S \subseteq \mathbb{S}_n \mid (\bot/p) \in S \text{ for all atomic } p\}$
4. $\mathfrak{S}_\Delta = \{S \in \mathfrak{S}_\infty \mid \Delta \subseteq S\}$

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Proof-Theoretic Validity

22

$$\models_S^{\mathfrak{S}} p$$

23

$$\models_S^{\mathfrak{S}} p$$
$$\vdash_S p$$

23

$\vDash_S^{\mathfrak{S}} p$	$\vDash_S^{\mathfrak{S}} \bot$
$\vdash_S p$	$\vdash_S \bot$

23

$\vDash_S^{\mathfrak{S}} p$	$\vDash_S^{\mathfrak{S}} \bot$
$\vdash_S p$	$\vdash_S \bot$
$\vDash_S^{\mathfrak{S}} \varphi \wedge \psi$	
$\vDash_S^{\mathfrak{S}} \varphi$ and $\vDash_S^{\mathfrak{S}} \psi$	

23

$\models_S^{\mathfrak{S}} p$ $\vdash_S p$	$\models_S^{\mathfrak{S}} \bot$ $\vdash_S \bot$
$\models_S^{\mathfrak{S}} \varphi \wedge \psi$ $\models_S^{\mathfrak{S}} \varphi$ and $\models_S^{\mathfrak{S}} \psi$	$\models_S^{\mathfrak{S}} \varphi \vee \psi$ $\models_S^{\mathfrak{S}} \varphi$ or $\models_S^{\mathfrak{S}} \psi$

23

$\models_S^{\mathfrak{S}} p$ $\vdash_S p$	$\models_S^{\mathfrak{S}} \bot$ $\vdash_S \bot$
$\models_S^{\mathfrak{S}} \varphi \wedge \psi$ $\models_S^{\mathfrak{S}} \varphi$ and $\models_S^{\mathfrak{S}} \psi$	$\models_S^{\mathfrak{S}} \varphi \vee \psi$ $\models_S^{\mathfrak{S}} \varphi$ or $\models_S^{\mathfrak{S}} \psi$

$$\models_S^{\mathfrak{S}} \varphi \to \psi$$
$$\varphi \models_S^{\mathfrak{S}} \psi$$

23

$\vDash_S^{\mathfrak{S}} p$	$\vDash_S^{\mathfrak{S}} \bot$
$\vdash_S p$	$\vdash_S \bot$
$\vDash_S^{\mathfrak{S}} \varphi \wedge \psi$	$\vDash_S^{\mathfrak{S}} \varphi \vee \psi$
$\vDash_S^{\mathfrak{S}} \varphi$ and $\vDash_S^{\mathfrak{S}} \psi$	$\vDash_S^{\mathfrak{S}} \varphi$ or $\vDash_S^{\mathfrak{S}} \psi$

$$\vDash_S^{\mathfrak{S}} \varphi \to \psi$$

$$\varphi \vDash_S^{\mathfrak{S}} \psi$$

$$\Gamma \vDash_S^{\mathfrak{S}} \varphi$$

for all $S' \supseteq S$ if $\vDash_{S'}^{\mathfrak{S}} \Gamma$ then $\vDash_{S'}^{\mathfrak{S}} \varphi$

23

$\vDash_S^{\mathfrak{S}} p$	$\vDash_S^{\mathfrak{S}} \bot$
$\vdash_S p$	$\vdash_S \bot$
$\vDash_S^{\mathfrak{S}} \varphi \wedge \psi$	$\vDash_S^{\mathfrak{S}} \varphi \vee \psi$
$\vDash_S^{\mathfrak{S}} \varphi$ and $\vDash_S^{\mathfrak{S}} \psi$	$\vDash_S^{\mathfrak{S}} \varphi$ or $\vDash_S^{\mathfrak{S}} \psi$

$$\vDash_S^{\mathfrak{S}} \varphi \to \psi$$

for all $S' \supseteq S$ if $\vDash_{S'}^{\mathfrak{S}} \varphi$ then $\vDash_{S'}^{\mathfrak{S}} \psi$

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Validity

$$\Gamma \vDash_{\mathfrak{S}} \varphi \Leftrightarrow \forall S \in \mathfrak{S}, \Gamma \vDash_{S}^{\mathfrak{S}} \varphi$$

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A Proof-Theoretic and Inquisitive Semantics

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$\mathfrak{S}_\infty$ and Harrop's

Theorem (Piecha, Campos Sanz, and Schroeder-Heister, 2015)

$\mathfrak{S}_\infty$ validates Harrop's formula

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Filters Preserve Validity

Lemma

If $\mathfrak{S}'$ is a filter on $\mathfrak{S}$, then

$$\vDash_{\mathfrak{S}} \varphi \Rightarrow \vDash_{\mathfrak{S}'} \varphi$$

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Disjunction Free Sentences

Lemma

For any set of disjunction free formulas Δ there is a $\mathfrak{S}_\Delta$ such that $\mathfrak{S}_\Delta$ is a filter on $\mathfrak{S}_\infty$ and $\vDash_{\mathfrak{S}_\Delta} \Delta$.

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An Inquisitive Proof-Theoretic Semantics

$$\mathfrak{S}_{Inq} = \{S \in \mathfrak{S}_\infty \mid ([p/\bot]\bot/p) \in S \text{ for all atomic } p\}$$

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An Inquisitive Proof-Theoretic Semantics

$$\mathfrak{S}_{Inq} = \{S \in \mathfrak{S}_\infty \mid ([p/\bot]\bot/p) \in S \text{ for all atomic } p\}$$

$$\dfrac{\dfrac{\neg\neg p \quad \dfrac{\dfrac{[p]}{\bot}}{\neg p}}{\bot}}{p}$$

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An Inquisitive Proof-Theoretic Semantics

Lemma

$\mathfrak{S}_{Inq}$ is a filter on $\mathfrak{S}_\infty$ and $\mathfrak{S}_{Inq}$ satisfies both Harrop's rule and $\neg\neg p \to p$ for all atomic p.

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Consequences

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$$InqL = L_{\mathfrak{S}_{Inq}}$$

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KPL=IPL+ Harrop's formulas

ML= the logic of finite problems

Lemma
$KPL \subseteq L_{\mathfrak{S}_\infty} \subseteq InqL$

Lemma (Ciardelli and Roelofsen, 2011)
$Schm(L_{\mathfrak{S}_{Inq}}) = ML$

Lemma
$Schm(L_{\mathfrak{S}_\infty}) \subseteq ML$

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References i

- Ciardelli, Ivano and Floris Roelofsen (2011). "Inquisitive Logic". In: *Journal of Philosophical Logic* 40.1, pp. 55–94.
- Piecha, Thomas, Wagner de Campos Sanz, and Peter Schroeder-Heister (2015). "Failure of Completeness in Proof-Theoretic Semantics". In: *Journal of Philosophical Logic* 44.3, pp. 321–335.

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Questions?

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Notation

1. $\bar{p}$ or $/p$ for the level-0,

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2. $p_0, \ldots, p_n/q$ for the level-1,
3. $[p_{0_0}, \ldots, p_{m_0}]q_0, \ldots, [p_{0_n}, \ldots, p_{m_n}]q_n/r$ for the level-2,

Notation

1. $\bar{p}$ or $/p$ for the level-0,
2. $p_0, \ldots, p_n/q$ for the level-1,
3. $[p_{0_0}, \ldots, p_{m_0}]q_0, \ldots, [p_{0_n}, \ldots, p_{m_n}]q_n/r$ for the level-2,
4. $([R_{0_0}, \ldots, R_{m_0}]q_0), \ldots, ([R_{0_n}, \ldots, R_{m_n}]q_n)/r$ for the level-n+2.

Theorem

Given weak intermediate logics L_1, L_2 such that they both have the disjunction property and prove that for all φ there is a ψ in disjunctive negation form such that $\psi \equiv \varphi$, it follows that $L_1 = L_2$

Proof.

Without loss of generality assume $\varphi \in L_1$. It follows that $\psi \in L_1$. Let ψ be $\neg\varphi_1 \vee \cdots \neg\varphi_n$. As L_1 has the disjunction property there is an $i \leq n$ such that $\neg\varphi_i \in L_1$. Because L_1 is a sublogic of CPC it follows that $\neg\varphi_i$ is a tautology of classical logic and so by Glivenko's theorem $\neg\neg\neg\varphi_i$ is a tautology of IPC and so $\neg\varphi_i$ is a tautology of IPC. It follows then that $\neg\varphi_1 \vee \cdots \neg\varphi_n$ is a tautology of IPC by disjunction introduction. And as IPC is a sublogic of L_2 it follows that $\neg\varphi_1 \vee \cdots \vee \neg\varphi_n \in L_2$ and so $\varphi \in L_2$ as L_2 proves these are equivalent. So L_1 is a sublogic of L_2. And as the exact reasoning goes through with L_1 switched with L_2 it follows that the two logics must be equal. □

Lemma

If $\mathfrak{S}' \subseteq \mathfrak{S}$ is such that if $S \in \mathfrak{S}'$, $S' \in \mathfrak{S}$ and $S \subseteq S'$ then $S' \in \mathfrak{S}'$, that is it is closed under supersets, then for all $S \in \mathfrak{S}'$

$$\vDash_S^{\mathfrak{S}} \varphi \Leftrightarrow \vDash_S^{\mathfrak{S}'} \varphi$$

Proof.
Induction on φ. All cases except for $\rightarrow$ are covered by Lemma **??**.

For the induction hypothesis, assume we have for all $S \in \mathfrak{S}'$ that $\vDash_S^{\mathfrak{S}} \varphi \Leftrightarrow \vDash_S^{\mathfrak{S}'} \varphi$ and $\vDash_S^{\mathfrak{S}} \psi \Leftrightarrow \vDash_S^{\mathfrak{S}'} \psi$. Assume $\vDash_S^{\mathfrak{S}} \varphi \rightarrow \psi$, then by the weak deductive theorem and monotanicity of Definition **??** it follows that for all $S' \in \mathfrak{S}$ extending S such that $\vDash_{S'}^{\mathfrak{S}} \varphi$ it follows that $\vDash_{S'}^{\mathfrak{S}} \psi$. Take $S' \in \mathfrak{S}'$ extending S such that $\vDash_{S'}^{\mathfrak{S}'} \varphi$. Now by the induction hypothesis $\vDash_{S'}^{\mathfrak{S}} \varphi$ from which it follows that $\vDash_{S'}^{\mathfrak{S}} \psi$ and so again by the induction hypothesis $\vDash_{S'}^{\mathfrak{S}'} \psi$.

Assume $\vDash_S^{\mathfrak{S}'} \varphi \rightarrow \psi$, then for all $S' \in \mathfrak{S}'$ extending S such that $\vDash_{S'}^{\mathfrak{S}'} \varphi$ it follows that $\vDash_{S'}^{\mathfrak{S}'} \psi$. Take $S' \in \mathfrak{S}$ extending S such that $\vDash_{S'}^{\mathfrak{S}} \varphi$. As S' extends $S \in \mathfrak{S}'$ it follows by assumption that $S' \in \mathfrak{S}'$. Now by the induction hypothesis $\vDash_{S'}^{\mathfrak{S}'} \varphi$ from which it follows that $\vDash_{S'}^{\mathfrak{S}'} \psi$ and so again by the induction hypothesis $\vDash_{S'}^{\mathfrak{S}} \psi$. $\square$

Lemma

If $\mathfrak{S}'$ is such that if $S \in \mathfrak{S}'$, $S' \in \mathfrak{S}$ and $S \subseteq S'$ then $S' \in \mathfrak{S}'$, then

$$\vDash_{\mathfrak{S}} \varphi \Rightarrow \vDash_{\mathfrak{S}'} \varphi$$

Proof.

Assume $\vDash_{\mathfrak{S}} \varphi$ then for all $S \in \mathfrak{S}$, $\vDash_S^{\mathfrak{S}} \varphi$, as all $S' \in \mathfrak{S}'$ are also in $\mathfrak{S}$ it follows so by Lemma 11 that $\vDash_{S'}^{\mathfrak{S}'} \varphi$ and so $\vDash_{\mathfrak{S}'} \varphi$. □

Lemma

Given a set of rule-formula Δ there is a $\mathfrak{S}$ such that $\mathfrak{S}_{\Delta}$ is a filter on $\mathfrak{S}_{\infty}$ and $\vDash_{\mathfrak{S}_{\Delta}} \Delta$.

Proof.

Let $\mathfrak{S}_{\Delta} = \{S \in \mathfrak{S}_{\infty} \mid \Delta^{+} \subseteq S\}$. Clearly this is a filter on $\mathfrak{S}_{\infty}$ so by Lemma 11 we have for all $S \in \mathfrak{S}_{\Delta}$ that $\vDash_S^{\mathfrak{S}_{\infty}} \Delta \Rightarrow \vDash_S^{\mathfrak{S}_{\Delta}} \Delta$. Note that for all $S \in \mathfrak{S}_{\infty}$, $\Delta \vDash_S^{\mathfrak{S}_{\infty}} \Delta$. So by Lemma **??** it follows that $\vDash_{S \cup \Delta^{+}}^{\mathfrak{S}_{\infty}} \Delta$. Let $S \in \mathfrak{S}_{\Delta}$ then $S = S' \cup \Delta^{+}$ so $\vDash_{S' \cup \Delta^{+}}^{\mathfrak{S}_{\infty}} \Delta$ which implies $\vDash_{S' \cup \Delta^{+}}^{\mathfrak{S}_{\Delta}} \Delta$ and so $\vDash_S^{\mathfrak{S}_{\Delta}} \Delta$. From which it follows that $\vDash_{\mathfrak{S}_{\Delta}} \Delta$. □

Lemma

$\mathfrak{S}_{Inq}$ is a filter on $\mathfrak{S}_\infty$ and $\mathfrak{S}_{Inq}$ satisfies both Harrop's rule and $\neg\neg p \to p$ for all atomic p.

Proof.

As $\mathfrak{S}_{Inq} = \mathfrak{S}_{\{\neg\neg p \to p | p \text{atomic}\}}$ it follows by Lemma 13. □

Nested sequent calculi for Lewis' counterfactual logics

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Proof theoretic semantics
Tübingen, 27-30 March 2019

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Counterfactual logics Lewis, 1973

$$A \Box\!\!\rightarrow B$$

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Counterfactual logics Lewis, 1973

$$A \mathrel{\Box\!\!\rightarrow} B$$

If Oswald hadn't shoot Kennedy, someone else would have.

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Counterfactual logics Lewis, 1973

$$A \mathrel{\Box\!\!\rightarrow} B$$

If Oswald hadn't shoot Kennedy, someone else would have.

If Trump had read the Pentagon report on climate change, he would believe the planet is warming.

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Comparative plausibility Lewis, 1973

$$\mathcal{F}^{\preccurlyeq} : p \mid \bot \mid A \to B \mid A \preccurlyeq B$$

- $A \preccurlyeq B$, meaning "A is at least as plausible as B"
- $A \mathrel{\Box\!\!\rightarrow} B \;\equiv\; (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \wedge B))$

Axiomatization Lewis, 1973

$\mathbb{V}$ Axiomatization of classical propositional logic plus

(CPR) $\dfrac{B \to A}{A \preccurlyeq B}$

(CPA) $(A \preccurlyeq (A \vee B)) \vee (B \preccurlyeq (A \vee B))$

(TR) $(A \preccurlyeq B) \wedge (B \preccurlyeq C) \to (A \preccurlyeq C)$

(CO) $(A \preccurlyeq B) \vee (B \preccurlyeq A)$

Ext. Axiomatization of $\mathbb{V}$ plus

(N) $\neg(\bot \preccurlyeq \top)$

(T) $(\bot \preccurlyeq \neg A) \to A$

(W) $A \to (A \preccurlyeq \top)$

(C) $(A \preccurlyeq \top) \to A$

(U$_1$) $\neg(\bot \preccurlyeq A) \to (\bot \preccurlyeq (\bot \preccurlyeq A))$

(U$_2$) $(\bot \preccurlyeq \neg A) \to (\bot \preccurlyeq \neg(\bot \preccurlyeq \neg A))$

(A$_1$) $\neg(A \preccurlyeq B) \to (\bot \preccurlyeq (A \preccurlyeq B))$

(A$_2$) $(A \preccurlyeq B) \to (\bot \preccurlyeq \neg(A \preccurlyeq B))$

Sphere models Lewis, 1973

$$\mathcal{M} = \langle W, N, [\![\ ]\!] \rangle$$

- W, non-empty set of elements;
- $N : W \to \mathcal{P}(\mathcal{P}(W))$, function assigning to each x a set $N(x)$;
- $[\![\ ]\!] : Atm \to \mathcal{P}(W)$ propositional evaluation.

Properties of N

- *Non-emptiness*: For all $\alpha \in N(x)$, $\alpha \neq \emptyset$
- *Nesting*: For all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

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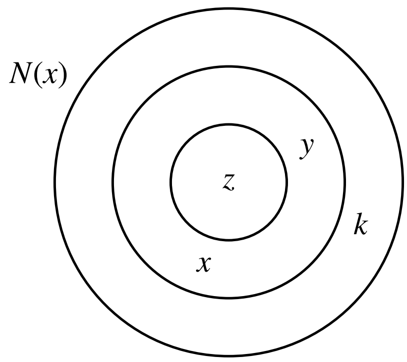
Sphere models

$\mathcal{M}, x \Vdash A \preccurlyeq B$ iff for all $\alpha \in N(x)$ if $\alpha \Vdash^{\exists} B$ then $\alpha \Vdash^{\exists} A$



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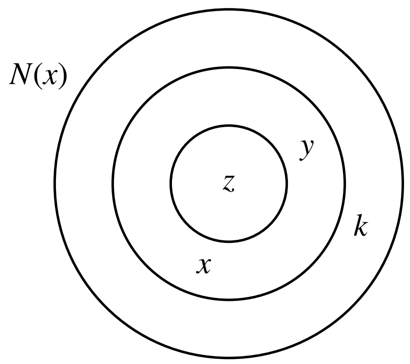
Sphere models

$\mathcal{M}, x \Vdash A \preccurlyeq B$ iff for all $\alpha \in N(x)$ if $\alpha \Vdash^{\exists} B$ then $\alpha \Vdash^{\exists} A$



$\alpha \Vdash^{\exists} A \rightsquigarrow$ there is $y \in \alpha$ such that $y \Vdash A$

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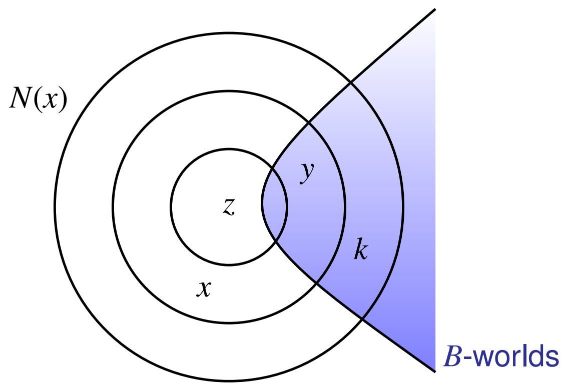
Sphere models

$\mathcal{M}, x \Vdash A \preccurlyeq B$ iff for all $\alpha \in N(x)$ if $\alpha \Vdash^{\exists} B$ then $\alpha \Vdash^{\exists} A$



$\alpha \Vdash^{\exists} A \rightsquigarrow$ there is $y \in \alpha$ such that $y \Vdash A$

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Sphere models

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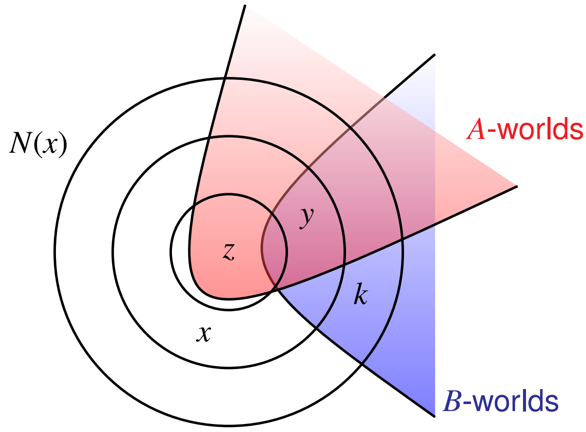

$\alpha \Vdash^{\exists} A \rightsquigarrow$ there is $y \in \alpha$ such that $y \Vdash A$

Sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$

Sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$

Olivetti and Pozzato 2015; Girlando, Lellmann, Olivetti, Pozzato 2016

Conditional blocks

$$(A_1, \ldots, A_m \lhd B)$$

$$(A_1 \vee \cdots \vee A_m) \preccurlyeq B$$

Conditional sequents and formula interpretation

$$\Gamma \Rightarrow \Delta, (\Sigma_1 \lhd B_1), \ldots, (\Sigma_n \lhd B_n)$$

$$\text{meaning } \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \bigvee_{1 \leq i \leq n} (\bigvee \Sigma_i \preccurlyeq B_i)$$

Example

$$A \Rightarrow B, (C, D \lhd E) \quad \text{means} \quad A \rightarrow B \vee ((C \vee D) \preccurlyeq E))$$

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Sequent calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$

Initial sequents

$$\frac{}{\Gamma, p \Rightarrow \Delta, p}\ \text{init} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_L$$

Propositional rules

$$\frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \rightarrow B \Rightarrow \Delta}\ \mathsf{L} \rightarrow \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ \mathsf{R} \rightarrow$$

Conditional rules

$$\frac{\Gamma \Rightarrow \Delta, (A \lhd B)}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}\ \mathsf{R} \preccurlyeq^{\mathsf{i}}$$

$$\frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, (B, \Sigma \lhd C) \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, (\Sigma \lhd A), (\Sigma \lhd C)}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, (\Sigma \lhd C)}\ \mathsf{L} \preccurlyeq^{\mathsf{i}}$$

$$\frac{\Gamma \Rightarrow \Delta, (\Sigma_1, \Sigma_2 \lhd A), (\Sigma_2 \lhd B) \quad \Gamma \Rightarrow \Delta, (\Sigma_1 \lhd A), (\Sigma_1, \Sigma_2 \lhd B)}{\Gamma \Rightarrow \Delta, (\Sigma_1 \lhd A), (\Sigma_2 \lhd B)}\ \text{com}$$

$$\frac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, (\Sigma \lhd A)}\ \text{jump}$$

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Example

Derivation of axiom (CO) in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, B}{\Rightarrow (A, B \lhd B), (B \lhd A)}\,\text{jump} \qquad \dfrac{A \Rightarrow A, B}{\Rightarrow (A \lhd B), (A, B \lhd A)}\,\text{jump}}{\Rightarrow (A \lhd B), (B \lhd A)}\,\text{com}}{\Rightarrow (A \lhd B), B \preccurlyeq A}\,\text{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\,\text{R}\preccurlyeq^{\mathsf{i}}}{\Rightarrow (A \preccurlyeq B) \lor (B \preccurlyeq A)}\,\text{R}\lor$$

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Features of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$

Nice

- Cut elimination for $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$(cut)
- Optimal complexity (PSPACE, Halpern and Friedman 1994)

Not so nice

- The jump rule is not invertible
- The calculus does not allow for a direct countermodel construction: *it is not possible to extract a countermodel from one failed derivation*

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Nested calculus $\mathcal{N}_{\mathbb{V}}$

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Conditional nested sequents

Conditional sequent

$$\Gamma \Rightarrow \Delta, (\Sigma_1 \lhd B_1), \ldots, (\Sigma_n \lhd B_n)$$

meaning $\bigwedge \Gamma \to \bigvee \Delta \vee \bigvee_{1 \leq i \leq n} (\bigvee \Sigma_i \preccurlyeq B_i)$

Conditional nested sequent

$$\Gamma \Rightarrow \Delta, [S_1], \ldots, [S_n]$$

With $\Box A = \bot \preccurlyeq \neg A$:

$$(\Gamma \Rightarrow \Delta, [S_1], \ldots, [S_n])^{\text{int}} = \bigwedge \Gamma \to \bigvee \Delta \vee \Box(S_1)^{\text{int}} \vee \cdots \vee \Box(S_n)^{\text{int}}$$

Example

$A \Rightarrow [B \Rightarrow (D \lhd F), [G \Rightarrow H]]$ means $A \to \Box(B \to (D \preccurlyeq F) \vee \Box(G \to H))$

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Context

The notions of *context* and *filling a context* are recursively defined in the standard way.

$$S\{\ \}$$

$$\Gamma \Rightarrow \Delta, [S\{\ \}]$$

$$\Gamma \Rightarrow \Delta, [\Phi \Rightarrow \Omega]$$

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Rules of $\mathcal{N}_{\mathbb{V}}$

Initial sequents

$$\frac{}{G\{p, \Gamma \Rightarrow \Delta, p\}}\ \text{init} \qquad \frac{}{G\{\bot, \Gamma \Rightarrow \Delta\}}\ \bot$$

Propositional rules

$$\frac{G\{\Gamma \Rightarrow \Delta, A\} \quad G\{B, \Gamma \Rightarrow \Delta\}}{G\{A \to B, \Gamma \Rightarrow \Delta\}}\ \mathsf{L} \to \qquad \frac{G\{A, \Gamma \Rightarrow \Delta, B\}}{G\{\Gamma \Rightarrow \Delta, A \to B\}}\ \mathsf{R} \to$$

Conditional rules

$$\frac{G\{\Gamma \Rightarrow \Delta, (A \lhd B)\}}{G\{\Gamma \Rightarrow \Delta, A \preccurlyeq B\}}\ \mathsf{R} \preccurlyeq$$

$$\frac{G\{A \preccurlyeq B, \Gamma \Rightarrow \Delta, (B, \Sigma \lhd C)\} \quad G\{A \preccurlyeq B, \Gamma \Rightarrow \Delta, (\Sigma \lhd C), (\Sigma \lhd A)\}}{G\{A \preccurlyeq B, \Gamma \Rightarrow \Delta, (\Sigma \lhd C)\}}\ \mathsf{L} \preccurlyeq$$

$$\frac{G\{\Gamma \Rightarrow \Delta, (\Sigma_1, \Sigma_2 \lhd A), (\Sigma_2 \lhd B)\} \quad G\{\Gamma \Rightarrow \Delta, (\Sigma_1 \lhd A), (\Sigma_1, \Sigma_2 \lhd B)\}}{G\{\Gamma \Rightarrow \Delta, (\Sigma_1 \lhd A), (\Sigma_2 \lhd B)\}}\ \mathsf{com}$$

$$\frac{G\{\Gamma \Rightarrow \Delta, (\Sigma \lhd C), [C \Rightarrow \Sigma]\}}{G\{\Gamma \Rightarrow \Delta, (\Sigma \lhd C)\}}\ \mathsf{jump}$$

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Example

Derivation of axiom (CO) in $\mathcal{N}_{\mathbb{V}}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow (A, B \lhd B), (B \lhd A), [B \Rightarrow A, B]}{\Rightarrow (A, B \lhd B), (B \lhd A)}\ \text{jump} \qquad \dfrac{\Rightarrow (A \lhd B), (A, B \lhd A), [A \Rightarrow A, B]}{\Rightarrow (A \lhd B), (A, B \lhd A)}\ \text{jump}}{\Rightarrow (A \lhd B), (B \lhd A)}\ \text{com}}{\Rightarrow (A \lhd B), B \preccurlyeq A}\ \text{R}\preccurlyeq}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \text{R}\preccurlyeq}{\Rightarrow (A \preccurlyeq B) \vee (B \preccurlyeq A)}\ \text{R}\vee$$

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Features of $\mathcal{N}_{\mathbb{V}}$

Nice

- Invertibility of the jump rule

$$\dfrac{G\{\Gamma \Rightarrow \Delta, (\Sigma \lhd C), [C \Rightarrow \Sigma]\}}{G\{\Gamma \Rightarrow \Delta, (\Sigma \lhd C)\}}\ \text{jump}$$

- Direct countermodel construction:
  a countermodel can be constructed form a single failed derivation

Not so nice

- We loose optimal complexity

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Completeness: Proof sketch

Cumulative version of the rules

$$\frac{G\{A \to B, \Gamma \Rightarrow \Delta, A\} \quad G\{A \to B, B, \Gamma \Rightarrow \Delta\}}{G\{A \to B, \Gamma \Rightarrow \Delta\}}\ \mathsf{L} \to^c \qquad \frac{G\{A, \Gamma \Rightarrow \Delta, A \to B, B\}}{G\{\Gamma \Rightarrow \Delta, A \to B\}}\ \mathsf{R} \to^c$$

$$\frac{G\{\Gamma \Rightarrow \Delta, A \preccurlyeq B, (A \lhd B)\}}{G\{\Gamma \Rightarrow \Delta, A \preccurlyeq B\}}\ \mathsf{R} \preccurlyeq^c$$

$$\frac{G\{\Gamma \Rightarrow \Delta, (\Sigma_1, \Sigma_2 \lhd A), (\Sigma_1 \lhd A), (\Sigma_2 \lhd B)\} \quad G\{\Gamma \Rightarrow \Delta, (\Sigma_1 \lhd A), (\Sigma_2 \lhd B), (\Sigma_1, \Sigma_2 \lhd B)\}}{G\{\Gamma \Rightarrow \Delta, (\Sigma_1 \lhd A), (\Sigma_2 \lhd B)\}}\ \mathsf{com}^c$$

- Definition of *saturated sequent*, a sequent which in not an initial sequent and to which all the rules have been non-redundantly applied
- A countermodel can be constructed from a saturated sequent

Interlude

Membership in a nested sequent

For S_1, S_2 conditional nested sequents, we say S_2 occurs in S_1, in symbols $S_2 \,\tilde{\in}\, S_1$, if

- $S_1 = S_2$
- $S_1 = \Gamma \Rightarrow \Delta, [S_3]$, and $S_2 \,\tilde{\in}\, S_3$

Example $\quad S = A \Rightarrow B, [D \Rightarrow E, [C \Rightarrow D]] \qquad C \Rightarrow D \,\tilde{\in}\, S$

Completeness: Proof sketch (continues)

Saturated nested sequent

$$S = \Gamma \Rightarrow \Delta, (\Sigma_1 \lhd C_1), \dots, (\Sigma_k \lhd C_k), \overbrace{[\Phi_1 \Rightarrow \Omega_1]}^{S_1}, \dots, \overbrace{[\Phi_k \Rightarrow \Omega_k]}^{S_k}$$

- By saturation condition com, $\Sigma_1 \subseteq \dots \subseteq \Sigma_k$
- By saturation condition jump, to each $(\Sigma_i \lhd C_i)$ there corresponds a sequent $S_i = [\Phi_i \Rightarrow \Omega_i]$, with $\Sigma_i \subseteq \Omega_i$ and $C_i \in \Phi_i$

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Completeness: Proof sketch (continues)

Saturated nested sequent

$$S = \Gamma \Rightarrow \Delta, (\Sigma_1 \lhd C_1), \dots, (\Sigma_k \lhd C_k), \overbrace{[\Phi_1 \Rightarrow \Omega_1]}^{S_1}, \dots, \overbrace{[\Phi_k \Rightarrow \Omega_k]}^{S_k}$$

- By saturation condition com, $\Sigma_1 \subseteq \dots \subseteq \Sigma_k$
- By saturation condition jump, to each $(\Sigma_i \lhd C_i)$ there corresponds a sequent $S_i = [\Phi_i \Rightarrow \Omega_i]$, with $\Sigma_i \subseteq \Omega_i$ and $C_i \in \Phi_i$

Countermodel construction

- $W = \{S_j \mid S_j \,\tilde{\in}\, S\}$
- $[\![p]\!] = \{S_j \mid p \in \Phi_j\}$
- $N(S_j) = \{\{S_k\}, \{S_k, S_{k-1}\}, \dots, \{S_k, S_{k-1}, \dots, S_1\}\}$

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Example

Saturated nested sequent

$$\Pi = (p_1 \lhd r), (p_1 \lhd s), (p_1, p_2 \lhd t), (p_1, p_2, p_3 \lhd u)$$

$$S = c \Rightarrow \Pi, \overbrace{[r \Rightarrow p_1]}^{S_1}, \overbrace{[s \Rightarrow p_1]}^{S_2}, \overbrace{[t \Rightarrow p_1, p_2]}^{S_3}, \overbrace{[u \Rightarrow p_1, p_2, p_3, (a \lhd b), \underbrace{[b \Rightarrow a]}_{S_5}]}^{S_4}$$

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Example

Saturated nested sequent

$$\Pi = (p_1 \lhd r), (p_1 \lhd s), (p_1, p_2 \lhd t), (p_1, p_2, p_3 \lhd u)$$

$$S = c \Rightarrow \Pi, \overbrace{[r \Rightarrow p_1]}^{S_1}, \overbrace{[s \Rightarrow p_1]}^{S_2}, \overbrace{[t \Rightarrow p_1, p_2]}^{S_3}, \overbrace{[u \Rightarrow p_1, p_2, p_3, (a \lhd b), \underbrace{[b \Rightarrow a]}_{S_5}]}^{S_4}$$

Countermodel construction

- $W = \{S, S_1, S_2, S_3, S_4, S_5\}$
- $[\![p_1]\!] = \emptyset$; $[\![r]\!] = \{S_1\}$; ...
- $N(S) = \{\{S_4\}, \{S_4, S_3\}, \{S_4, S_3, S_2, S_1\}\}$
  $N(S_4) = \{\{S_5\}\}$

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Example

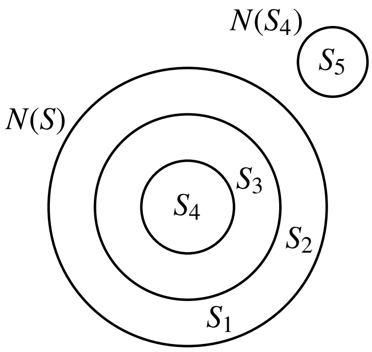
Saturated nested sequent

$$\Pi = (p_1 \lhd r), (p_1 \lhd s), (p_1, p_2 \lhd t), (p_1, p_2, p_3 \lhd u)$$

$$S = \ c \Rightarrow \Pi, \overbrace{[r \Rightarrow p_1]}^{S_1}, \overbrace{[s \Rightarrow p_1]}^{S_2}, \overbrace{[t \Rightarrow p_1, p_2]}^{S_3}, \overbrace{[u \Rightarrow p_1, p_2, p_3, (a \lhd b), \underbrace{[b \Rightarrow a]}_{S_5}]}^{S_4}$$

Countermodel construction

- $W = \{S, S_1, S_2, S_3, S_4, S_5\}$
- $[\![p_1]\!] = \emptyset$; $[\![r]\!] = \{S_1\}$; ...
- $N(S) = \{\{S_4\}, \{S_4, S_3\}, \{S_4, S_3, S_2, S_1\}\}$
  $N(S_4) = \{\{S_5\}\}$



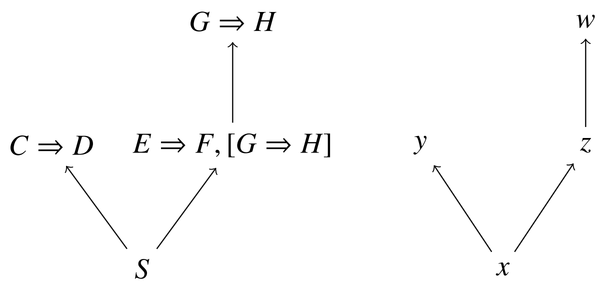
Tree-like property

Tree-like models and netsed sequents

Nested sequents are often associated with a tree-like property of models.

Example (modal logic)

$$S = A \Rightarrow [C \Rightarrow D], [E \Rightarrow F, [G \Rightarrow H]]$$



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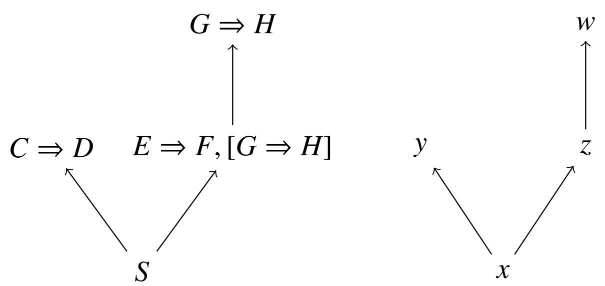
Tree-like models and netsed sequents

Nested sequents are often associated with a tree-like property of models.

Example (modal logic)

$$S = A \Rightarrow [C \Rightarrow D], [E \Rightarrow F, [G \Rightarrow H]]$$



Is there a tree-like semantics for conditional nested sequents?

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Preferential models Lewis 1973

$$\mathcal{M} = \langle W, R, \{\leqslant_x\}_{x \in W}, [\![\ ]\!] \rangle$$

where

- W is a non-empty set of elements;
- R is a binary relation among worlds;
- for all $x \in W$, $\leqslant_x$ is a reflexive and transitive binary relation between worlds x, y such that xRy and xRz;
- $[\![\ ]\!]$ is the propositional evaluation.

Relation $\leqslant_x$ satisfies the following property:

Connectedness: if xRy and xRz, either $y \leqslant_x z$ or $z \leqslant_x y$.

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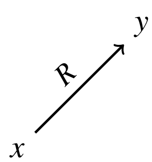
Preferential models

$x \Vdash A \preccurlyeq B$ *iff* for all y such that xRy,



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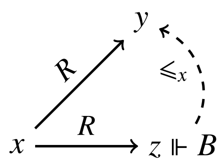
Preferential models

$x \Vdash A \preccurlyeq B$ *iff* for all y such that xRy,
if there exists $z \leqslant_x y$ such that xRz and $z \Vdash B$



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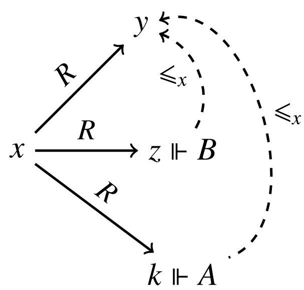
Preferential models

$x \Vdash A \preccurlyeq B$ *iff* for all y such that xRy,
if there exists $z \leqslant_x y$ such that xRz and $z \Vdash B$
then there exists $k \leqslant_x y$ such that xRk and $k \Vdash A$



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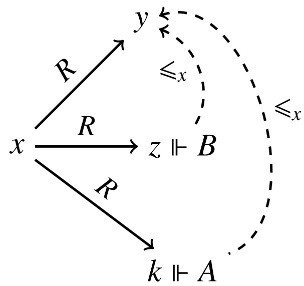
Preferential models

$x \Vdash A \preccurlyeq B$ *iff* for all y such that xRy,
if there exists $z \leqslant_x y$ such that xRz and $z \Vdash B$
then there exists $k \leqslant_x y$ such that xRk and $k \Vdash A$



A preferential model is tree-like if R is tree-like.

Completeness of $\mathcal{N}_{\mathbb{V}}$ w.r.t. preferential models

Saturated nested sequent

$$S = \Gamma \Rightarrow \Delta, (\Sigma_1 \lhd C_1), \dots, (\Sigma_k \lhd C_k), \overbrace{[\Phi_1 \Rightarrow \Omega_1]}^{S_1}, \dots, \overbrace{[\Phi_k \Rightarrow \Omega_k]}^{S_k}$$

Completeness of $\mathcal{N}_{\mathbb{V}}$ w.r.t. preferential models

Saturated nested sequent

$$S = \Gamma \Rightarrow \Delta, (\Sigma_1 \lhd C_1), \dots, (\Sigma_k \lhd C_k), \overbrace{[\Phi_1 \Rightarrow \Omega_1]}^{S_1}, \dots, \overbrace{[\Phi_k \Rightarrow \Omega_k]}^{S_k}$$

Countermodel construction

- $W = \{S_i \mid S_i \tilde{\in} S\}$
- $[\![p]\!] = \{S_i \in W \mid p \in \Phi_i\}$
- $S_1 \, R \, S_2$ *if* $S_1 = \Gamma \Rightarrow \Delta, [S_2]$
- $S_3 \leqslant_{S_1} S_2$ *if* $S_1 \, R \, S_2, \; S_1 \, R \, S_3$ and $\Sigma_2 \subseteq \Sigma_3$

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Example

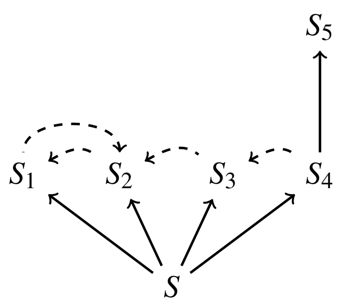
Saturated nested sequent

$$\Pi = (p_1 \lhd r), (p_1 \lhd s), (p_1, p_2 \lhd t), (p_1, p_2, p_3 \lhd u)$$

$$S = c \Rightarrow \Pi, \overbrace{[r \Rightarrow p_1]}^{S_1}, \overbrace{[s \Rightarrow p_1]}^{S_2}, \overbrace{[t \Rightarrow p_1, p_2]}^{S_3}, \overbrace{[u \Rightarrow p_1, p_2, p_3, (a \lhd b), \underbrace{[b \Rightarrow a]}_{S_5}]}^{S_4}$$

Countermodel construction

- $W = \{S, S_1, S_2, S_3, S_4, S_5\}$
- $[\![p_1]\!] = \emptyset, \; [\![r]\!] = \{S_1\}, \; \dots$
- $S \, R \, S_1, \; S \, R \, S_2, \; S \, R \, S_3, \; S \, R \, S_4$
  $S_4 \, R \, S_5$
- $S_1 \leqslant_S S_2, \; S_2 \leqslant_S S_1, \; S_3 \leqslant_S S_2,$
  $S_3 \leqslant_S S_1, \; S_4 \leqslant_S S_3$



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Conclusions

To sum up

- The nested calculus allows for direct countermodel construction
- A saturated conditional nested sequents yields a tree-like finite preferential model
- The nested calculus suggests an alternative (natural) semantics: *the proof theory guides the choice of the semantics*

Further work

- Nested calculi for weaker and stronger conditional logics
- Relationship with labelled calculi based on preferential semantics (work in progress)

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Thank you!

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Appendix

Context

Context

- $S\{\ \} = \Gamma \Rightarrow \Delta, \{\ \}$ is a context;
- if $F\{\ \}$ is a context, then $S\{\ \} = \Gamma \Rightarrow \Delta, [F\{\ \}]$ is a context.

The result of filling a context $S\{\ \}$ with a nested sequent $\Gamma \Rightarrow \Delta$ is denoted as $G\{\Gamma \Rightarrow \Delta\}$ and defined as:

- If $S\{\ \} = \Sigma \Rightarrow \Pi, \{\ \}$, then $S\{\Gamma \Rightarrow \Delta\} = \Gamma, \Sigma \Rightarrow \Pi, \Delta$;
- If $S\{\ \} = \Sigma \Rightarrow \Pi, [F\{\ \}]$ then $S\{\Gamma \Rightarrow \Delta\} = \Sigma \Rightarrow \Pi, [F\{\Gamma \Rightarrow \Delta\}]$.

Preferential models Lewis 1973

- Sound and complete with respect to $\mathbb{V}$ (Lewis)
- There is a mutual correspondence between the models, via Alexandroff topologies:

$$\text{Downwards closed set } \downarrow^{\leqslant_x} w = \{z \mid xRz \text{ and } z \leqslant_x w\}$$

$$N(x) = \{\downarrow^{\leqslant_x} w \mid xRw\}$$

(and vice versa)

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Outer modal logics Lewis 1973

Lewis' conditional logic	*Outer modal logic*
$\mathbb{V}$	**K**
$\mathbb{VN}$	**D**
$\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$	**T**
$\mathbb{VU}$, $\mathbb{VA}$	**K45**
$\mathbb{VNU}$, $\mathbb{VNA}$	**D45**
$\mathbb{VTU}$, $\mathbb{VWU}$, $\mathbb{VCU}$, $\mathbb{VTA}$, $\mathbb{VWA}$	**S5**
$\mathbb{VCA}$	Classical logic

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Outer modal logics Lewis 1973

Lewis' conditional logic	*Outer modal logic*
$\mathbb{V}$	**K**
$\mathbb{VN}$	**D**
$\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$	**T**
$\mathbb{VU}$, $\mathbb{VA}$	**K45**
$\mathbb{VNU}$, $\mathbb{VNA}$	**D45**
$\mathbb{VTU}$, $\mathbb{VWU}$, $\mathbb{VCU}$, $\mathbb{VTA}$, $\mathbb{VWA}$	**S5**
$\mathbb{VCA}$	Classical logic

$$\Box A := \bot \preccurlyeq \neg A$$
$$\Diamond A := \neg(\bot \preccurlyeq A)$$

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Session
in memoriam Roy Dyckhoff

† 24 August 2018

PTS3 In Memoriam Roy 28.3.2019 – p. 1

PTS3 In Memoriam Roy 28.3.2019 – p. 2

What Roy would have presented here

From email of 21st July 2018:

“Recent work has been my paper (with SB [Susanne Bobzien], in Studia Logica) on (the statics of) Stoic Logic, a 2 page paper on the G4 calculus (JSL, next issue), a 15 page paper on Syllogisms (BSL, accepted); and a paper on the dynamics of Stoic logic (disguised as a Hertz-Gentzen system...) ....”

PTS3 In Memoriam Roy 28.3.2019 – p. 3

What Roy would have presented here

On the next day, when offered a presentation via Skype in case he would not be able to come:

> The topics on syllogism and stoic logic you are
> mentioning in your mail would all be perfect.

Thank you. At least you now know the score.

I can probably manage Skype, if I’m spared. (That’s a Scottish phrase meaning “so long as the Good Lord has not taken me to his bosom” (or allowed the Devil to take me to his...)).

Best wishes, Roy

PTS3 In Memoriam Roy 28.3.2019 – p. 4

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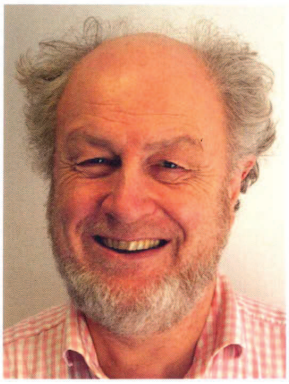
Service of Thanksgiving for
Dr Roy Dyckhoff



ST SALVATOR'S CHAPEL
Saturday 10 November 2018

PTS3 In Memoriam Roy 28.3.2019 – p. 5

Roy's scientific achievements

Particularly interesting to the community represented here:

- Sequent-style rules for implication in natural deduction ('generalized', 'parallelized' rules, but first-level)
  $\Longrightarrow$ von Plato, Tennant
- Contraction-free sequent calculus, which is particularly well-suited for automated theorem proving
  $\Longrightarrow$ Hudelmaier
- Proof-editing, theorem proving, didactical presentation of proofs — "MacLogic", EU-project GENTZEN
  $\Longrightarrow$ Abrusci
- Proof-theoretic approaches to logic programming — Conferences on "Extensions of Logic Programming"
  $\Longrightarrow$ Miller, Hudelmaier
- Many other topics, including topology and categorial algebra

PTS3 In Memoriam Roy 28.3.2019 – p. 6

Our speakers today

Sara Negri has published with Roy on geometric rules, on cut-free sequent systems, on intermediate logic and special Heyting algebras ...

Nissim Francez has published with Roy on harmony principles in proof-theoretic semantics, on proof-theoretic semantics of natural language, on the proof-theoretic semantics of subsentential phrases ...

Hence it is most appropriate that they speak in this session.

PTS3 In Memoriam Roy 28.3.2019 – p. 7

In memory of Roy Dyckhoff, friend and coauthor

Proof-Theoretic Semantics for natural language

Nissim Francez
Computer Science dept., the Technion-IIT, Haifa, Israel
(francez@cs.technion.ac.il)

introduction

– This is a position paper, not presenting any new results. Its purpose is to draw the attention of the community (logic, philosophy, linguistics) to *Proof-Theoretic Semantics (PTS)* as an alternatives to the traditional *model-theoretic semantics (MTS)*, as applicable to natural language semantics.

The talk has two parts:

1. A brief exposition of PTS, not necessarily in connection to NL.

2. A review of an application of PTS to NL with an indication of advantages of PTS as such theory of meaning for NL.

Overview of PTS

– In a nutshell, the core idea of PTS programme as a theory of meaning for defining sentential proof-theoretic meanings is the following.
- *Replace the received approach of taking sentential meanings as* **truth-conditions** *(in arbitrary models) by taking their meanings as* **canonical derivability conditions** *(from suitable assumptions) within a* meaning-conferring *natural-deduction proof-system in which the derivability conditions are formulated.*
– In a sense, the proof system should reflect the "use" (inferential practices) of the sentences in the considered fragments, and should allow recovering pre-theoretic properties of the meanings of sentences such as *entailment* and *consequence drawing (inference)*.

Meaning-conferring proof-systems

– Meaning is determined by a *meaning-conferring* natural-deduction proof-system, say $\mathcal{N}$, *a definitional tool*, having two kinds of rules:
- **Introduction rules (I-rules)**:] their conclusion is a formula (sentence) governed by an operator *introduced* by (an application of) the rule.
I-rules establish the way the formula can be deduced from the premises of the rule in the *most direct* way.
- **Elimination rules (E-rules)**: their *major premise* is a formula (sentence) governed by an operator *eliminated* by (an application of) the rule. E-rules establish the *most direct* way a conclusion may be drawn *from* the major premise.
- Both kind of rules may *discharge* assumptions when applied, rendering the conclusion independent of some temporarily assumed assumptions.

Examples

– I/E-rules for *conjunction*.

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}(\wedge I) \qquad \frac{\varphi \wedge \psi}{\varphi}(\wedge E_1) \qquad \frac{\varphi \wedge \psi}{\psi}(\wedge E_2)$$

No discharge of assumption is involved.
Assumption discharge: the I-rule for material implication.

$$\frac{\begin{array}{c}[\varphi]_i \\ \vdots \\ \psi\end{array}}{\varphi \supset \psi}(\supset I^i) \qquad \frac{\varphi \quad \varphi \supset \psi}{\psi}(\supset E)$$

($\supset I$) says that in order to derive an implication $\varphi \supset \psi$, temporarily assume φ, with square brackets to indicate discharge, indexed i for connecting with a discharging application of the rule indexed i, and derive ψ. Once ψ has been derived, the assumption is discharged. The E-rule is the familiar modus-ponens.
– The claim of PTS is that those rules *determine [in what sense?] the full meaning* of conjunction and material implication, independently of the model-theoretic definition using truth-tables.

Canonical derivation

– An *$\mathcal{N}$-derivation* $\mathcal{D}$ (of ψ from Γ) is defined recursively by iterating $\mathcal{N}$-rules applications starting from assumptions Γ and reaching a conclusion ψ.
- The *derivability* of φ from Γ in $\mathcal{N}$ is denoted $\vdash_{\mathcal{N}} \Gamma : \varphi$.
Definition: A $\mathcal{N}$-derivation $\mathcal{D}$ for $\Gamma : \psi$ is *canonical* iff
1. The last rule applied in $\mathcal{D}$ is an I-rule, *or*
2. The last rule applied in $\mathcal{D}$ is an assumption-discharging E-rule, the major premise of which is some $\varphi \in \Gamma$, and its encompassed minor premise sub-derivations $\mathcal{D}_1, \cdots, \mathcal{D}_n$ are all canonical derivations of ψ.
– Denote by $\vdash^c_{\mathcal{N}}$ canonical derivability.
– The recursion always terminates via (1.), an *essential application* of an I-rule.
- Canonicity preserves *directness* of inference, but ψ *propagates* through application of E-rules to open assumptions in Γ.

The role of canonicity

$$\frac{\alpha \quad (\alpha\rightarrow(\varphi\wedge\psi))}{\varphi\wedge\psi}(\rightarrow E)$$

– is a derivation of a conjunction – but not a canonical one, as it does not end with an application of $(\wedge I)$, nor does it have an essential application of it.
- Thus, the conjunction here was *not* derived according to its meaning! As far as this derivation is concerned, it could mean anything, e.g., disjunction.
- The following derivation *is* according to the conjunction's meaning, being canonical.

$$\frac{\dfrac{\alpha \quad \alpha\rightarrow\varphi}{\varphi}(\rightarrow E) \qquad \dfrac{\beta \quad \beta\rightarrow\psi}{\psi}(\rightarrow E)}{\varphi\wedge\psi}(\wedge I)$$

Sentential meaning

Definition: For a compound $\varphi\in L$, its *meaning* (contributed semantic value) $[\![\varphi]\!]$, is:

$$[\![\varphi]\!] = \lambda\Gamma.[\![\varphi]\!]^c_\Gamma$$

– A useful by-product of this notion of meaning, underlying a natural proof-theoretic consequence definition, is that of *grounds for assertion*.
Definition:

$$GA[\![\varphi]\!] = \{\Gamma \mid \Gamma\vdash^c\varphi\}$$

- Thus, any Γ that canonically derives φ serves as grounds for assertion for φ.

- Note that both sentential meanings and grounds for assertion are *proof-theoretic* (syntactic) objects, not related at all to models and denotations therein. So, those are notions more easily amenable to computational treatment.

Proof-theoretic consequences (entailment)

Definition: ψ is a *proof-theoretic consequence* of Γ ($\Gamma \Vdash_c \psi$) iff $GA[\![\Gamma]\!] \subseteq GA[\![\psi]\!]$.

- That is, every grounds for asserting (all of) Γ are already grounds for asserting ψ.

– I will skip here the discussion of *sub-sentential proof-theoretic meanings*, to replace *denotations (extensions)* in models.

An application of PTS to NL: two kinds of transitive verbs

– I present a PTS for (a small fragment of) NL sentences with transitive verbs. I highlight the semantics with an exemplary major distinction:
- Extensional vs. intensional objects
– To simplify, I disregard here a typical issue accompanying transitive verbs: quantifier scope ambiguity.
– The kind of any given verb is specified in the lexicon.
– In the NL fragment considered here, I/E-rules introduce/eliminate a Determiner Phrase (DP) into/from the subject or the object positions.

Extensional vs. intensional verbs

– Extensional transitive verbs (ETVs) occur in sentences like

Every boy loves some girl

understood as every boy loves (one or more) *specific* girls, thereby expressing *a relation* between boys and girls they love.
The object some girl is said to have *a specific* reading.
– Intensional transitive verbs (ITVs) occur in sentences like

Every lawyer seeks some/a secretary

understood as every lawyer seeks (one or more) *non-specific* secretaries, thereby expressing *a relation* between lawyers and some abstract notion, here of a secretary. The object some secretary is said to have *a non-specific* (or notional) reading.

Characteristics of ITVs

– An adequate semantics for ITVs has to reveal the following three distinguishing characteristics:
1. Admittance of non-specific (notional) objects.
This difference will be revealed by the corresponding meaning-conferring systems having different I/E-rules.
2. Resistance to substitutability of co-extensives.
This difference can be exemplified by the following two similarly looking arguments, where the first is valid while the second is not.

$$\checkmark\ \frac{\text{John loves Mary} \quad \text{Mary is the daughter of the dean}}{\text{John loves the daughter of the dean}}$$

$$(*)\ \frac{\text{John seeks Mary} \quad \text{Mary is the daughter of the dean}}{\text{John seeks the daughter of the dean}}$$

Characteristics of ITVs – continued

3. Suspension of existential commitment.
This difference is manifested by

John loves some girl

presupposing the existence of girls, allowing the passivisation inference

$$\checkmark\ \frac{\text{John loves some girl}}{\text{some girl is loved by John}}$$

while

John seeks a unicorn

does not carry a presupposition of the existence of unicorns, invalidating

$$(*)\ \frac{\text{John seeks a unicorn}}{\text{some unicorn is sought by John}}$$

Difficulties in MTS - I

– **extensional verbs:** The kind of models employed for such verbs are very much like traditional models for 1st-order logic, based on a *domain* (of quantification), populated by elements of an arbitrary set of *unqualified entities*.
- Such sets are an abstraction suitable for modelling mathematical structures, but their use for modelling NL is controversial (at least). Natural language does not "speak" about unqualified entities. The quantification used is *restricted quantification*, such as every girl or some boy. The unqualified entities are only marginally expressible like everything or something.
- Furthermore, all the complications involved in set-theory regarding the sizes of sets (vs. classes) are carried over to the semantics, complications unneeded for specifying meaning in for NL sentences.

Difficulties in MTS - II

intensional verbs:
– Here, the question of what populates models suitable for specifying adequate truth-conditions for ITVs is even more complicated, and there is no consensus about it within the community; not even regarding the semantic type of the object of the intensional verb:
- a quantifier
- a property
- a minimal situation
- indirect interpretation via *decomposition* of the intensional verb: try to find
- Davidsonian event semantics and its thematic roles
- Recently, a *dynamic* semantics was proposed in, with an update via ITVs.

The proof-theoretic semantics: the meaning-conferring proof-system

– The I/E-rules introduce and eliminate *determiners* (expressing quantifiers) into subjects and objects positions.
- Let the meta-variables R and $\hat{R}$ range over extensional and intensional verbs, respectively. For simplicity, I consider here only two determiners: every and some.
- Let X range over nouns.
- There are also two kinds of copulas: isa and is being, explained below.

– The distinction is accounted for merely by introducing *two kinds of parameters*, syntactic objects into the meaning-conferring proof-system, not carrying any ontological burden.

Parameters - I

individual parameters: ranged over by **i, j, k**. They occur in *pseudo-sentences* (a slight extension of the natural language) in the form of $\mathbf{j}\ R\ \mathbf{k}$, $\mathbf{j}$ isa X
- For example: **j** loves **k**, **j** loves every girl and **j** isa girl.
- Such parameters are a kind of place holders for noun-phrases with determiners introduced/eliminated into/from positions the parameters are located in.
notional parameters: ranged over by $\nu,\ \mu$. They occur in pseudo-sentences like ν is being a X and $\mathbf{j}\ \hat{R}\ \nu$. For example, ν is being a secretary and **j** seeks a secretary.
– Ground Pseudo-sentences contain parameters only (of both kinds), filling the role of atomic sentences in logic. Their meaning is given from outside the meaning-conferring system.

Parameters - II

- The distinction between two kind of parameters replaces the distinction of unqualified entities and whatever is taken as denotation of notions in the model-theoretic approach mentioned above, with *no ontological burden*! They just have different deductive roles in the meaning-conferring system, as seen by the rules below.

Extensional verbs

– S ranges over sentences (including pseudo-sentences). $S[\mathbf{j}]$ means S with a distinguished position filled by **j**, and $S[\text{every/some } X]$ means that **j** was replaced by every/some X.

$$\frac{}{\Gamma, S : S}(Ax)$$

$$\frac{\Gamma, \mathbf{j} \text{ isa } X \ : S[\mathbf{j}]}{\Gamma : S[(\text{every } X)]}(eI) \qquad \frac{\Gamma : \mathbf{j} \text{ isa } X \quad \Gamma : S[\mathbf{j}]}{\Gamma \vdash S[(\text{some } X)]}(sI)$$

$$\frac{\Gamma : S[(\text{every } X)] \quad \Gamma : \mathbf{j} \text{ isa } X \quad \Gamma, S[\mathbf{j}] : S'}{\Gamma : S'}(eE)$$

$$\frac{\Gamma : S[(\text{some } X)] \quad \Gamma, \mathbf{j} \text{ isa } X, S[\mathbf{j}] : S'}{\Gamma : S'}(sE)$$

where **j** is *fresh* in (eI), and (sE).

A convenient, easier to understand, *derived* E-rule.

$$\frac{\Gamma : S[(\text{every } X)] \quad \Gamma : \mathbf{j} \text{ isa } X}{\Gamma : S[\mathbf{j}]}(e\hat{E})$$

For example,

$$\frac{\Gamma : \text{every girl smiled} \quad \Gamma : \mathbf{j} \text{ isa girl}}{\Gamma : \mathbf{j} \text{ smiled}}(e\hat{E})$$

Remarks about the rules – I

– eI: similar to the I-rule for universal quantification in 1st-order logic. It allows inferring $S[\text{every } X]$ from some Γ if $S[\mathbf{j}]$ can be inferred from Γ and the additional assumption **j** isa X; the freshness of **j** guarantees the *arbitrariness* of **j**, thereby assuring that the derivation applies to any X. For example:

$$\frac{\Gamma, \mathbf{j} \text{ isa girl} : \mathbf{j} \text{ loves } \mathbf{k}}{\Gamma : \text{every girl loves } \mathbf{k}}(eI) \quad \text{or} \quad \frac{\Gamma, \mathbf{j} \text{ isa girl} : \mathbf{k} \text{ loves } \mathbf{j}}{\Gamma : \mathbf{k} \text{ loves every girl}}(eI)$$

$e\hat{E}$: The usual *instantiation* rule, allowing to instantiate (in S) every X by any individual parameter.

sI: similar to the I-rule of an existential quantification in 1st-order logic. It allows inferring $S[(\text{some } X)]$ from a *witness* $S[\mathbf{j}]$ that has been proved to be an X in the other premise. For example:

$$\frac{\Gamma : \mathbf{j} \text{ isa girl} \quad \mathbf{j} \text{ loves } \mathbf{k}}{\Gamma : \text{some girl loves } \mathbf{k}}(sI) \quad \text{or} \quad \frac{\Gamma : \mathbf{j} \text{ isa girl} \quad \mathbf{k} \text{ loves } \mathbf{j}}{\Gamma : \mathbf{k} \text{ loves some girl loves}}(sI$$

sE: An arbitrary conclusion S' can be drawn from $S[(\text{some } X)]$ provided it can be drawn from an *arbitrary witness* for which it is only assumed it is a X.

intensional verbs

– The I/E-rules for the intensional sentences are:

$$\frac{\Gamma : \nu \text{ is being a } X \quad \Gamma : \mathbf{j} \ \widehat{R} \ \nu}{\Gamma : \mathbf{j} \ \widehat{R} \text{ some } X} (s_n I)$$

$$\frac{\Gamma : \mathbf{j} \ \widehat{R} \text{ some } X \quad \begin{matrix} [\nu \text{ is being a } X]_i, [\mathbf{j} \ \widehat{R} \ \nu]_j \\ \vdots \\ S' \end{matrix}}{\Gamma : S'} (s_n E^{i,j}) \qquad \nu \text{ fresh}$$

Remarks about the rules – I

– I explain the intensional rules and their difference from the corresponding extensional rules.
- Note that only some X (or, more colloquially, a X) can be introduced notionally, and only into the object position.
non-specificity: The non-specificity of some X is reflected by the absence (in $s_n I$) of a predicative premise of the form $\mathbf{j}$ isa X, replaced with a non-predicative premise ν is being a X.
- This is in contrast to the introduction of some X in its specific reading via the sI of the extensional fragment.
For example:

$$\frac{\Gamma : \nu \text{ is being a secretary} \quad \Gamma : \mathbf{j} \text{ seeks } \nu}{\Gamma : \mathbf{j} \text{ seeks some/a secretary}} (s_n I)$$

– The difference of interpretation of the transitive verbs resides in the way parameters are associated with nouns.

Remarks about the rules – II

Absence of existential commitment:

– Existential commitment is expressed via a use of an individual parameter. So, suspension of existence is manifested directly by the form of the $(s_n I)$-rule, introducing a non-specific indefinite determinate-phrase into the object position, that does not have a premise involving an individual parameter.

- Suppose secretaries exist and unicorns do not exist. This would be embodied at the level of ground sentences, so that **k** is a secretary could surface as a possible premise, while **k** is a unicorn could not.

- This difference between secretaries and unicorns has no bearing on the notions being a secretary and being a unicorn. At the atomic level, there is no difference in the status of ν is being a secretary and ν is being a unicorn as premises.

- Associating a name to a notion has no existential commitment.

More on existential committment

– Note: **j** seeking a (non-specific) secretary and **j** seeking a (non-specific) unicorn are derived in exactly the same way:

$$\frac{\nu \text{ is being a secretary} \quad \mathbf{j} \text{ seeks } \nu}{\mathbf{j} \text{ seeks some secretary}}\,(s_n I)$$

$$\frac{\nu \text{ is being a unicorn} \quad \mathbf{j} \text{ seeks } \nu}{\mathbf{j} \text{ seek some unicorn}}\,(s_n I)$$

- This contrasts with the derivation of **j** finds a unicorn, which needs a premise appealing to an individual parameter.

$$\frac{\mathbf{k} \text{ isa unicorn} \quad \mathbf{j} \text{ finds } \mathbf{k}}{\mathbf{j} \text{ finds some unicorn}}\,(sI)$$

- Under the supposition about unicorns, the premises **k** isa unicorn and **j** finds **k** will not be simultaneously derivable from any grounds Γ.

– The lack of existential commitment is also manifested by the inference

$$\frac{\mathbf{j} \text{ seeks a unicorn}}{\mathbf{j} \text{ seeks something}}$$

being valid.

Other applications of PTS to NL

– **Explicit/Implicit extensional objects:**
-$[\![John\ ate]\!] = [\![John\ ate\ something]\!]$
- *Reducing an argument = [proof-theoretically] omission of a premise in I- rules*.
- *Sameness of meaning is based on sameness of I-rules, implying sameness of canonical derivations*.

– **Adjective-noun modification:**
- Distinguishing *intersective/subsective/privative* adjectives via *sub-structurality*.

$$(\checkmark)\ \frac{\text{j isa grey elephant}}{\text{j is grey}} \qquad (\checkmark)\ \frac{\text{j isa grey elephant}}{\text{j isa elephant}}$$

$$(\checkmark)\ \frac{\text{j isa small elephant}}{\text{j isa elephant}} \qquad (*)\ \frac{\text{j isa small elephant}}{\text{j is small (?)}}$$

$$(\checkmark)\ \frac{\text{j isa fake gun}}{\text{j is fake}} \qquad (*)\ \frac{\text{j isa fake gun}}{\text{j isa gun}}$$

– **Contextual domain restriction:**
every bottle is empty - in the universe?
Context = additional premise with a free variable.

Geometric rules in infinitary logic

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Based on joint work with the late Roy Dyckhoff

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Background

- Proof-theoretical semantics builds on the goals of *general proof theory*: shift from the so called reductionist study of mathematics (Hilbert's program) to the analysis of proofs in their own right (Gentzen, Prawitz).
- Basic requirements to achieve these goals include
  1. A precise definition of formal systems of derivation
  2. Establishing structural properties, subformula property
  3. Establishing the meaning-conferring nature of the rules of deduction
- Achieved already by Gentzen for sequent calculi for *purely logical systems* (1933) and for *arithmetic* (1935), and by Prawitz (1973) for natural deduction [also, Gentzen 2008]. Considered an impossibility for *extra-logical* axioms (Girard 1987).

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Background (cont.)

- The program of **proof analysis** extends the goals of PTS to elementary mathematical theories such as
  1. Theories with universal axioms (N and von Plato 1998)
  2. Finitary geometric theories or *coherent theories* (N 2003, Simpson 1994 in ND-style)
  3. Generalized geometric theories (N 2016)
  4. Arbitrary first-order theories (Dyckhoff and N 2015)
- As a payoff of 2, 3, and 4 used in conjunction with the *labelled* formalism, we obtained uniform proof systems for a a wide class of modal, non-classical, conditional logics (N 2005, Dyckhoff and N 2011, Boretti and N 2009, Hakli and N 2011, 2012, Maffezioli, Naibo and N 2013, N and Sbardolini 2014, N and Olivetti 2015, Girlando, N, Olivetti 2018, ...)
- More concrete application of proof analysis: Extraction of computational content of classical proofs

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Coherent and geometric implications

A formula is Horn iff built from atoms (and $\top$) using only $\wedge$.

A formula is coherent, *or "positive",* iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$ and $\exists$.

A formula is geometric iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$, $\exists$ and infinitary disjunctions.

A sentence is a coherent implication iff of the form $\forall \mathbf{x}.C \supset D$, where C, D are coherent [$\forall \mathbf{x}.D$ also regarded as a coherent implication, with $\top \equiv C$].

A sentence is a geometric implication iff of the form $\forall \mathbf{x}.C \supset D$, where C, D are geometric.

Theorem: Any coherent implication is equivalent to a finite conjunction of sentences of the form $\forall \mathbf{x}.C \supset D$ where C is Horn and D is a (finite) disjunction of existentially quantified Horn formulae.

Theorem: Any geometric implication is equivalent to a (possibly infinite) conjunction of sentences of the form $\forall \mathbf{x}.C \supset D$ where C is Horn and D is a (possibly infinite) disjunction of existentially quantified Horn formulae.

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Examples of coherent theories

Universal formulas $\forall \mathbf{x}.A$ can be written as finite conjunctions of coherent implications.

Theory of *fields* : $\forall x(x = 0 \vee \exists y(xy = 1))$

Theory of *local rings* : $\forall x.\exists y(xy = 1) \vee \exists y((1 - x)y = 1)$

Theory of *transitive relations* : $\forall xyz.(xRy \wedge yRz) \supset xRz$

Theory of *partial order* : $\forall xy.\ (x \leq y \wedge y \leq x) \supset x = y$

Theory of *strongly directed relations* : $\forall xyz.(xRy \wedge xRz) \supset \exists u.yRu \wedge zRu$

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Examples of geometric theories

(Infinitary) theory of *torsion abelian groups* : $\forall x. \bigvee_{n>1}(nx = 0)$

Theory of *Archimedean ordered fields* : $\forall x. \bigvee_{n\geq 1}(x < n)$

Theory of *connected graphs* :

$\forall xy.x = y \vee \bigvee_{n\geq 1} \exists z_0 \dots \exists z_n(x = z_0 \,\&\, y = z_n \,\&\, z_0 R z_1 \,\&\, \dots \,\&\, z_{n-1} R z_n)$

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Common knowledge as a geometric theory

Common knowledge among a set of agents $\{a_1, \dots, a_k\}$ is defined as

$$\mathcal{C}A \equiv \bigwedge_n \mathcal{E}^n(A)$$

$$\mathcal{E}(A) \equiv \mathcal{K}_{a_1}(A) \wedge \dots \wedge \mathcal{K}_{a_k}(A)$$

everybody knows, everybody knows that everybody knows, etc.

In Kripkean terms, each $\mathcal{K}_{a_i}$ is a modality with its own accessibility relation R_{a_i}; the accessibility relation $R_{\mathcal{E}}$ associated to $\mathcal{E}$ is the *union* of all the R_{a_i}, i.e.

$$xR_{\mathcal{E}}y \equiv xR_{a_1}y \vee \dots \vee xR_{a_k}y$$

the accessibility relation for $\mathcal{C}$ is the **transitive closure** of $R_{\mathcal{E}}$, that is,

$$xR_{\mathcal{C}}y \equiv (\exists n \in \mathbb{N})(\exists y_1 \dots y_{n-1} \in W)(xR_{\mathcal{E}}y_1 \;\;\&\;\; \dots \;\;\&\;\; y_{n-1}R_{\mathcal{E}}y)$$

By defining $R^n_{\mathcal{E}} \equiv (\exists y_1 \dots y_{n-1} \in W)(xR_{\mathcal{E}}y_1 \;\;\&\;\; \dots y_{n-1}R_{\mathcal{E}}y)$ the above is, in infinitary logic,

$$xR_{\mathcal{C}}y \equiv xR^1_{\mathcal{E}}y \vee \dots \vee xR^n_{\mathcal{E}}y \dots$$

The truth condition for the common knowledge operator $\mathcal{C}$ is

$$x \Vdash \mathcal{C}A \text{ iff for all } y,\; xR_{\mathcal{C}}y \text{ implies } y \Vdash A$$

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Other examples

Reformulation of the axioms and **choice of basic concepts** may be crucial for obtaining a geometric axiomatization.

Fields: $\neg\, a = 0 \supset \exists y\; a \cdot y = 1$ is not geometric, but the equivalent $a = 0 \vee \exists y\; a \cdot y = 1$ is.

Robinson arithmetic: $\neg\, a = 0 \supset \exists y\; a = s(y)$ is not geometric, but the equivalent $a = 0 \vee \exists y\; a = s(y)$ is.

Real-closed fields:
$\neg\, a_{2n+1} = 0 \supset \exists y\; a_{2n+1} \cdot y^{2n+1} + a_{2n} \cdot y^{2n} + \ldots a_1 \cdot y + a_0 = 0$ is not geometric, but the equivalent
$a_{2n+1} = 0 \vee \exists y\; a_{2n+1} \cdot y^{2n+1} + a_{2n} \cdot y^{2n} + \ldots a_1 \cdot y + a_0 = 0$ is.

Classical projective geometry: Not a geometric theory!
Axiom of existence of three non-collinear points.
$\exists x \exists y \exists z(\neg\, x = y \;\&\; \neg\, z \in ln(x, y))$
if the basic notions are replaced by the constructive notions of apartness between points and lines and "outsideness" of a point from a line, a geometric axiomatization is found (von Plato 1995):
$\exists x \exists y \exists z(x \neq y \;\&\; z \notin ln(x, y))$

Infinitary logics: syntax

- Atomic formulae P
- Countably many predicates and function symbols, and equality
- *Formulas* A are built up using, in addition to the standard connectives and quantifiers, countable disjunctions $\bigvee_{n>0} A_n$ and countable conjunctions $\bigwedge_{n>0} A_n$

The language is not minimal: Even if binary conjunctions and disjunctions are special cases of the countable ones, and the latter are interdefinable, it is convenient (for reasons that will become clear below) to consider the full language.

The *depth* $d(A)$ of a formula is defined inductively on the formation of A:

- $d(\bot) = d(P) \equiv 1$
- For compound formulas A, $d(A) \equiv sup_{B \in IS(A)} d(B) + 1$, where $B \in IS(A)$ iff B is an *immediate subformula* of A.

If A' is a proper subformula of A, then $d(A') < d(A)$.

Infinitary logics as contraction-free sequent calculi

Sequents are expressions of the form $\Gamma \Rightarrow \Delta$ where Γ, Δ are *finite multisets* of formulas.

Infinitary rules for disjunction:

$$\frac{\{\Gamma, A_n \Rightarrow \Delta \mid n > 0\}}{\Gamma, \bigvee_{n>0} A_n \Rightarrow \Delta}\ L\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\ R\bigvee_k$$

- $L\bigvee$ has countably many premisses, one for each $n > 0$.
- Derivations built using these rules are thus (in general) infinite trees, with countable branching but where (as may be proved by induction on the definition of derivation) each branch has finite length.
- The *leaves* of the trees are those $\Gamma \Rightarrow \Delta$ where Γ contains $\bot$ or Γ and Δ have an atomic formula in common.

Derivations in infinitary sequent calculi

Formal definition of the notion of *derivation* $\mathcal{D}$ and *height* $ht(\mathcal{D})$ and its *end-sequent*:

1. Any sequent $\Gamma \Rightarrow \Delta$, where some atomic formula occurs in both Γ and Δ, is a derivation, of *height* 0 and with *end-sequent* $\Gamma \Rightarrow \Delta$.
2. Any sequent $\Gamma \Rightarrow \Delta$ where $\bot$ occurs in Γ and Δ, is a derivation, of *height* 1 and with end-sequent $\Gamma \Rightarrow \Delta$.
3. If each $\mathcal{D}_n$ is a derivation, of height α_n, with end-sequent $\Gamma_n \Rightarrow \Delta_n$ and

$$\frac{\dots \quad \Gamma_n \Rightarrow \Delta_n \quad \dots}{\Gamma \Rightarrow \Delta}\ R$$

is an inference (i.e. an instance of a rule), then

$$\frac{\dots \quad \dfrac{\mathcal{D}_n}{\Gamma_n \Rightarrow \Delta_n} \quad \dots}{\Gamma \Rightarrow \Delta}\ R$$

is a derivation, of *height* the countable ordinal $sup_n(\alpha_n) + 1$ and with *end-sequent* $\Gamma \Rightarrow \Delta$.

Derivations in infinitary sequent calculi (cont.)

It follows from the definition that:

- Each derivation has a countable ordinal *height* (the successor of the supremum of the heights of its immediate subderivations).
- If $\mathcal{D}'$ is a subderivation of $\mathcal{D}$, then $ht(\mathcal{D}') < ht(\mathcal{D})$.

Remark: The definitions of depth and height differ from those in Feferman (1968): we use the successor of a supremum rather than the supremum of the successors: note that

$$sup_{n>0}(n+1) = \omega \neq \omega + 1 = (sup_{n>0}(n)) + 1$$

The calculus $\mathbf{G3c}_\omega$

$$P, \Gamma \Rightarrow \Delta, P \qquad\qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A_k, \bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}{\bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}\ L\bigwedge_k \qquad\qquad \frac{\{\Gamma \Rightarrow \Delta, A_n \mid n > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{n>0} A_n}\ R\bigwedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{\{\Gamma, A_n \Rightarrow \Delta \mid n > 0\}}{\Gamma, \bigvee_{n>0} A_n \Rightarrow \Delta}\ L\bigvee \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\ R\bigvee_k$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad\qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall\ (y\ fresh)$$

$$\frac{A(y/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists\ (y\ fresh) \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A(t/x)}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

Proposition. The following sequents (the "infinitary De Morgan laws") are derivable, where $\neg A \equiv A \supset \bot$:

1. $\neg \bigwedge_{n>0} A_n \Rightarrow \bigvee \neg_{n>0} \neg A_n$
2. $\bigvee_{n>0} \neg A_n \Rightarrow \ \neg \bigwedge_{n>0} A_n$
3. $\neg \bigvee_{n>0} A_n \Rightarrow \bigwedge_{n>0} \neg A_n$
4. $\bigwedge_{n>0} \neg A_n \Rightarrow \ \neg \bigvee_{n>0} A_n$

$$\dfrac{\dfrac{\ldots \quad \dfrac{\dfrac{A_n \Rightarrow A_n, \bigvee_{n>0} \neg A_n}{\Rightarrow A_n, \neg A_n, \bigvee_{n>0} \neg A_n}\,R\supset}{\Rightarrow A_n, \bigvee_{n>0} \neg A_n}\,R\bigvee \quad \ldots}{\Rightarrow \bigwedge_{n>0} A_n, \bigvee_{n>0} \neg A_n}\,R\bigwedge \qquad \bot, \Rightarrow \bigvee_{n>0} \neg A_n}{\neg \bigwedge_{n>0} A_n \Rightarrow \bigvee_{n>0} \neg A_n}\,L\supset$$

Extensions with rules for geometric theories

Extension of **G3c** with rules for finitary geometric theories (N 2003) maintains the structural properties of the ground sequent calculus.

Generalize to infinitary geometric theories

Recall that a *geometric implication* is a sentence G of the form

$$\forall \mathbf{x}.\, C \supset D$$

where

- the quantifer binds all free variables of $C \supset D$;
- $C \equiv P_1 \wedge \cdots \wedge P_k$ and D is a *finite or countably infinite* disjunction $\bigvee E_n$ where $E_n \equiv \exists \mathbf{y}_n (Q_{n1} \wedge \cdots \wedge Q_{nm_n})$;
- even if D is an infinite disjunction, it only has finitely many free variables.

Such a sentence G determines a (finitary or infinitary) *geometric rule*

$$\dfrac{\ldots \quad Q_{n1}(\mathbf{x}, \mathbf{y}_n), \ldots, Q_{nm_n}(\mathbf{x}, \mathbf{y}_n), P_1(\mathbf{x}), \ldots, P_k(\mathbf{x}), \Gamma \Rightarrow \Delta \quad \ldots}{P_1(\mathbf{x}), \ldots, P_k(\mathbf{x}), \Gamma \Rightarrow \Delta}\,R_G$$

with one premiss for each of the countably many disjuncts E_n of D. The variables in $\mathbf{y}_n$ are *fresh*, i.e. are not in the conclusion.

Examples of geometric rules

Axiom of **torsion abelian groups**, $\forall x. \bigvee_{n>1}(nx = 0)$, becomes the rule

$$\frac{\dots \quad nx = 0, \Gamma \Rightarrow \Delta \quad \dots}{\Gamma \Rightarrow \Delta} R_{Tor}$$

Axiom of **Archimedean ordered fields**, $\forall x. \bigvee_{n\geq 1}(x < n)$, becomes the rule

$$\frac{\dots \quad x < n, \Gamma \Rightarrow \Delta \quad \dots}{\Gamma \Rightarrow \Delta} R_{Arc}$$

Axiom of **connected graphs**,

$$\forall xy.x = y \vee \bigvee_{n\geq 1} \exists z_0 \dots \exists z_n(x = z_0 \,\&\, y = z_n \,\&\, z_0 R z_1 \,\&\, \dots \,\&\, z_{n-1} R z_n)$$

becomes the rule

$$\frac{x = y, \Gamma \Rightarrow \Delta \quad xRy, \Gamma \Rightarrow \Delta \quad \dots \quad x = z_0, y = z_n, z_0 R z_1, \dots, z_{n-1} R z_n, \Gamma \Rightarrow \Delta \quad \dots}{\Gamma \Rightarrow \Delta}$$

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Examples of geometric rules (cont.)

Definition of **transitive closure**,

$$xR_{\mathcal{C}}y \supset\subset (xR_{\mathcal{E}}y \vee \dots \vee xR^n_{\mathcal{E}}y \dots)$$

gives the geometric rules

$$\frac{xR_{\mathcal{E}}y, \Gamma \Rightarrow \Delta \quad \dots \quad xR^n_{\mathcal{E}}y, \Gamma \Rightarrow \Delta \dots}{xR_{\mathcal{C}}y, \Gamma \Rightarrow \Delta} T^{\omega}$$

$$\frac{xR_{\mathcal{C}}y, \Gamma \Rightarrow \Delta}{xR^n_{\mathcal{E}}y, \Gamma \Rightarrow \Delta} Inc$$

On the modal side, common knowledge is captured by an ω rule (Jäger et al. 2007)

$$\frac{\dots \mathcal{E}^n A, \Gamma \dots \text{ for all } n}{\mathcal{C}A, \Gamma} \omega\mathcal{C}$$

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From rules to axioms

If we add to the basic system for infinitary classical logic a finite or infinite family of rules R_G, then we can prove all the geometric sentences G from which they were determined.

$$\dfrac{\ldots \quad \dfrac{\dfrac{\dfrac{\ldots \quad \dfrac{}{Q_{j1},\ldots,Q_{jm_j},P_1,\ldots,P_k \Rightarrow \bigvee E_n, Q_{jl}}Ax \quad \ldots}{Q_{j1},\ldots,Q_{jm_j},P_1,\ldots,P_k \Rightarrow \bigvee E_n, Q_{j1}\wedge\cdots\wedge Q_{jm_j}}R\wedge^{m_j-1}}{Q_{j1},\ldots,Q_{jm_j},P_1,\ldots,P_k \Rightarrow \bigvee E_n, E_j}R\exists}{Q_{j1},\ldots,Q_{jm_j},P_1,\ldots,P_k \Rightarrow \bigvee E_n}R\bigvee_j \quad \ldots}{\dfrac{\dfrac{P_1,\ldots,P_k \Rightarrow \bigvee E_n}{\Rightarrow (P_1\wedge\cdots\wedge P_k)\supset \bigvee E_n}R\supset, L\wedge^{k-1}}{\Rightarrow \forall\mathbf{x}.(P_1\wedge\cdots\wedge P_k)\supset\bigvee E_n}R\forall^*}R_G$$

To go from **axioms to rules** we need to establish admissibility of the structural rules.

Structural properties of $\mathbf{G3c}^*_\omega$

Denote with $\mathbf{G3c}^*_\omega$ any extension with a finite or infinite family of such rules R_G.

Proposition. For every formula A, and for all Γ and Δ, the sequent $A,\Gamma \Rightarrow \Delta, A$ is derivable in $\mathbf{G3c}_\omega$.

Proof. By (transfinite) induction on the depth of the formula. If $A = \bigvee_{n>0} A_n$, then we have

$$\dfrac{\ldots \quad \dfrac{\dfrac{\ldots}{A_i,\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_i}Ind.Hyp.}{A_i,\Gamma \Rightarrow \Delta,\bigvee_{n>0} A_n}R\bigvee_i \quad \ldots}{\bigvee_{n>0} A_n,\Gamma \Rightarrow \Delta,\bigvee_{n>0} A_n}L\bigvee$$

from which the result follows by transfinite induction, each A_i being of less depth than A.

Proposition. The rules of left and right weakening are height-preserving (hp)-admissible $\mathbf{G3c}^*_\omega$.

Proof. By a straightforward (transfinite) induction of the height of the derivation of the premiss of each rule.

Structural properties of $\mathbf{G3c}^*_\omega$ (cont.)

Proposition. All the rules of $\mathbf{G3c}^*_\omega$ are hp-invertible.

Proposition. The rules of left and right contraction

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ RC$$

are hp-admissible.

Admissibility of cut for $\mathbf{G3c}^*_\omega$

Admissibility of

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

for the infinitary calculus proved using finite *sets* is shown by a Gentzen-style argument in Feferman (1968) and by Tait (1968) using single-sided sequents. Takeuti (1975) uses infinitary sequents. Lopez-Escobar (1965) infinitary sequents as sets.

Here finite *multisets* and extension with rules for geometric implications

Admissibility of cut for $\mathbf{G3c}^*_\omega$ (cont.)

Inductive parameters
Rank $\pi(I)$ of an instance I of *Cut* with cut-free premisses $\mathcal{D}$ and $\mathcal{D}'$ is the pair

$$(\delta, \sigma)$$

where

- $\delta \equiv d(A) \equiv$ *depth of A*
- $\sigma \equiv h(\mathcal{D}) \# h(\mathcal{D}') \equiv$*natural sum of the heights of the premisses* $\equiv$*total height*

Pairs are *lexicographically* ordered.

Here $\#$ is the standard notion of (natural or Hessenberg) sum $\alpha \# \beta$ for ordinals α and β

1. $\#$ is commutative.
2. If $\alpha < \alpha'$ then $\alpha \# \beta < \alpha' \# \beta$.

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Admissibility of cut for $\mathbf{G3c}^*_\omega$ (cont.)

Lemma. In

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \; Cut$$

if the premisses have cut-free derivations then so has the conclusion.

Proof. By transfinite lexicographic induction on the rank of instances of *Cut* and case analysis. We just show a representative sample of the reductions used. As usual, we distinguish between:

1. Cuts with cut formula principal in both premisses, i.e., *principal cuts*
2. Cuts with cut formula non-principal in at least one premiss, i.e., *non-principal cuts*

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Admissibility of cut for $\mathbf{G3c}^*_\omega$ (cont.)

If the cut formula $\bigvee_{n>0} A_n$ is **principal in both premisses**, we have

$$\dfrac{\dfrac{\vdots \atop \Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\, R\bigvee_k \qquad \dfrac{\vdots \atop \dots \quad A_n, \Gamma' \Rightarrow \Delta' \quad \dots}{\bigvee_{n>0} A_n, \Gamma' \Rightarrow \Delta'}\, L\bigvee}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut$$

which we transform into

$$\dfrac{\dfrac{\dfrac{\vdots \atop \Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k \qquad \dfrac{\vdots \atop \dots \quad A_n, \Gamma' \Rightarrow \Delta' \quad \dots}{\bigvee_{n>0} A_n, \Gamma' \Rightarrow \Delta'}\, L\bigvee}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A_k}\, Cut \qquad {\vdots \atop A_k, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}\, Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Ctr^*$$

The rank of both cuts is reduced: first cut has lower height, second has lower depth, so the induction hypothesis applies to both
The contractions are admissible.

Admissibility of cut for $\mathbf{G3c}^*_\omega$ (cont.)

With C as cut formula **non-principal in at least one premiss**, e.g.

$$\dfrac{\dfrac{\vdots \atop \Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k, C}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, C}\, R\bigvee_k \qquad {\vdots \atop C, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \bigvee_{n>0} A_n, \Delta'}\, Cut$$

can be transformed to

$$\dfrac{\dfrac{{\vdots \atop \Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k, C} \qquad {\vdots \atop C, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k, \Delta'}\, Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \bigvee_{n>0} A_n, \Delta'}\, R\bigvee_k$$

with unchanged cut formula C and reduced total height.

Admissibility of cut for $\mathbf{G3c}^*_\omega$ (cont.)

$$\dfrac{\dfrac{\ldots \quad \overset{\vdots}{A_n, \Gamma \Rightarrow \Delta, C} \quad \ldots}{\Gamma, \bigvee_{n>0} A_n \Rightarrow \Delta, C} L\bigvee \quad \overset{\vdots}{C, \Gamma' \Rightarrow \Delta'}}{\Gamma, \bigvee_{n>0} A_n, \Gamma' \Rightarrow \Delta, \Delta'} \mathit{Cut}$$

can be transformed to

$$\dfrac{\ldots \quad \dfrac{\overset{\vdots}{A_n, \Gamma \Rightarrow \Delta, C} \quad \overset{\vdots}{C, \Gamma' \Rightarrow \Delta'}}{\Gamma, A_n, \Gamma' \Rightarrow \Delta, \Delta'} \mathit{Cut} \quad \ldots}{\Gamma, \bigvee_{n>0} A_n, \Gamma' \Rightarrow \Delta, \Delta'} L\bigvee$$

with countably many cuts, each of lower rank (cut formula C unchanged but total height in each cut reduced).

Admissibility of cut for $\mathbf{G3c}^*_\omega$ (cont.)

Theorem. The *Cut* rule is admissible.

Proof. Show that an arbitrary derivation using instances (possibly infinite in number) of the *Cut* rule can be transformed to a cut-free derivation.

By transfinite induction on the height of the derivation:

Consider a derivation $\mathcal{D}$; if it does not end in a cut, but with a step by the rule R, then, by inductive hypothesis, each premiss (which has height less than $ht(\mathcal{D})$) can be transformed to a cut-free derivation (with conclusion unchanged), and thus so, by adding an R-step, can $\mathcal{D}$.

Otherwise, if $\mathcal{D}$ ends with a cut, the derivations of its premisses both have height less than $ht(\mathcal{D})$; by inductive hypothesis, each can be transformed to a cut-free derivation (with conclusion unchanged). We now use the Lemma to obtain a cut-free derivation of the conclusion of $\mathcal{D}$. **QED**

G3i$_\omega$, an infinitary intuitionistic calculus

The intuitionistic infinitary calculus is obtained from the classical one by

1. Taking both infinitary disjunction and conjunction as primitive (observe that the classical calculus can be defined with a sparing choice of logical constants and this holds also for the infinitary case)
2. Imposing to the right rule of infinitary conjunction the same restrictions that are needed for rule $R\forall$
3. Having both binary and infinitary conjunction (By 2., cannot have context-independent binary conjunction rules as special cases of the infinitary one)

$$\frac{\{\Gamma \Rightarrow A_n \mid n > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{n>0} A_n} R\bigwedge \qquad \frac{A_k, \bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}{\bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta} L\bigwedge_k$$

$$\frac{A \supset B, \Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta} L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B} R\supset$$

$$\frac{\forall x A, A(t/x), \Gamma \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta} L\forall \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A} R\forall$$

Structural properties of G3i$_\omega$

The proofs of the structural properties for **G3i**$_\omega$ involve some “Dragalin-style” subtleties, similar to those in use for the *finitary* intuitionistic multisuccedent calculus.

We consider right away any extension **G3i**$^*_\omega$ of **G3i**$_\omega$ with rules following the infinitary geometric rule scheme.

- Left and right weakening are hp-admissible in **G3i**$^*_\omega$.
- All the rules of **G3i**$^*_\omega$ **except** $R\bigwedge$**,** $R\supset$**, and** $R\forall$ are hp-invertible in **G3i**$^*_\omega$.
- Left and right contraction are hp-admissible in **G3i**$^*_\omega$.
- Cut is admissible in **G3i**$^*_\omega$.

A proof of the infinitary Barr theorem

First-order Barr's theorem: If a (finitary) geometric implication is provable classically in a geometric theory, it is provable also intuitionistically.

Several proofs in the literature for the finitary case: Orevkov (1968), Palmgren (1998), Coste and Coste (1975), Nadathur (2001) .... ; for the infinitary Rathjen (2016). We extend the method of Negri (2003).

1. Consider a classical theory T axiomatized by finitary or infinitary geometric implications.
2. Convert the geometric axioms into infinitary geometric rules.
3. Transform the classical theory into a contraction- and cut-free sequent calculus, denoted by $\mathbf{G3c}_\omega\mathbf{T}$.
4. Denote by $\mathbf{G3i}_\omega\mathbf{T}$ the corresponding intuitionistic extension of $\mathbf{G3i}_\omega$.

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A proof of the infinitary Barr theorem (cont.)

Theorem. If a finitary or infinitary geometric implication is derivable in $\mathbf{G3c}_\omega\mathbf{T}$, it is derivable in $\mathbf{G3i}_\omega\mathbf{T}$.

Proof. *Almost nothing to prove.* Any derivation in $\mathbf{G3c}_\omega\mathbf{T}$ uses only rules that follow the (infinitary) geometric rule scheme and logical rules. Because of the shape of the conclusion, no instance of the rules that violates the intuitionistic restrictions is used, so the derivation directly gives (through the addition, where needed, of the missing implications in steps of $L\supset$) a derivation in $\mathbf{G3i}_\omega\mathbf{T}$ of the same conclusion.

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Conclusion

By the methodology developed (axioms as rules in contraction and cut-free calculi, as similar as possible for classical and intuitionistic logic) conservativity results for mathematical theories are established in a uniform and general way.

Further work includes the extension of these results to infinite generalized geometric theories (with arbitrary quantifier alternations).

Generalization from “infinitary” as meaning “indexed by integers” to meaning “indexed by a set in the sense of **CZF**”.

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Reductive logic & proof-theoretic semantics: a coalgebraic perspective

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UCL & The Alan Turing Institute

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Assessment and Future Perspectives
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I want to say some quite simple things, trying to make some connections between

- reductive logic & tactical proof construction,
- proof-theoretic semantics, and
- coalgebraic semantics, as a candidate unifying approach.

Reductive logic: proof theory and proof-search

Here, we have the most basic idea: we read inference rules as *reduction operators*, from conclusion to premisses. Instead of the *deduction*

$$\frac{\text{Premiss}_1 \quad \ldots \quad \text{Premiss}_k}{\text{Conclusion}} \Downarrow,$$

the *reduction*

$$\frac{\text{Sufficient Premiss}_1 \quad \ldots \quad \text{Sufficient Premiss}_k}{\text{Putative Conclusion}} \Uparrow$$

Here, failure to construct a proof derives from failure to reduce to axiom sequents, say. For example,

$$\frac{\Gamma \vdash \mathrm{p}}{\ldots} \Uparrow$$

and there is no unpacking of Γ that exposes an occurrence of p.

Reductive logic: Kripke semantics and model-checking

Model-theoretic satisfaction relations also work like this, of course:

$w \models_{\mathcal{M}} \phi \wedge \psi$ iff $w \models_{\mathcal{M}} \phi$ and $w \models_{\mathcal{M}} \psi$

$w \models_{\mathcal{M}} \phi * \psi$ iff there are worlds u and v s.t. $R(u, v, w)$ and $u \models_{\mathcal{M}} \phi$ and $v \models_{\mathcal{M}} \psi$

and so on.

Here, failure to construct a realizer derives from failure to reduce to satisfiable atoms:

$$w \models_{\mathcal{M}} \mathrm{p} \quad \text{iff} \quad w \in \mathcal{V}(\mathrm{p})$$

That is, we reduce to atoms that do not satisfy this condition.

Reductive logic

Towards a semantic perspective.

- Reductive logic:
  - Proof-search
    - Syntactic Reductions (e.g., for logic programming, theorem proving)
    - Bigger space than proofs
  - Truth-functional semantics
    - Semantic Reductions (e.g., for model-checking)
    - Bigger space than realizers
  - Distinction at axioms and atomics, respectively.
- But these larger spaces are not sufficient alone to characterize reductive logics: typically, reductions are one-to-many, whereas, typically, deductions are many-to-one.
- Let's unpack this a bit, following Pym & Ritter, *Reductive Logic and Proof-search: Proof Theory, Semantics, and Control*, Oxford Logic Guides, 2004, has a detailed set-up for classical and intuitionistic logic.

Theoretical backstory

- A *reduction model* is a fibred structure $\mathcal{R}$ — in the sense of the use of fibred and indexed categories, and doctrines, in categorical logic — interpreting propositions and proofs — relative to indeterminates, interpreted using polynomial constructions, which stand for terms and propositions that remain to be calculated.
- Along with this, we need a semantic judgement, defined relative to the model,

$$W \models_{\Theta} (\Phi : \phi)\Gamma$$

between worlds W, indeterminates in Θ, sequents $\Gamma \mathrel{?\text{-}} \phi$, and reductions, Φ.
- At world W, relative to Θ, Φ is a reduction of $\Gamma \mathrel{?\text{-}} \phi$:

$$\Gamma \mathrel{?\text{-}} \Phi : \phi$$

Theoretical backstory

- In this truth-functional sense, *soundness* means that all $\Gamma \mathrel{?\text{-}} \phi$ for which a reduction can be calculated are true in the model and *completeness* means that there is a (term) reduction model for which all true $\Gamma \mathrel{?\text{-}} \phi$ have reductions.
- Again, Pym & Ritter, Oxford Logic Guides, 2004, has the details.

Theoretical backstory

- A reduction Φ is interpreted as a map

$$[\![\Gamma]\!]^W_\Theta \stackrel{[\![\Phi]\!]^W_\Theta}{\to} [\![\Delta]\!]^W_\Theta$$

- *Soundness* means that every reduction that can be calculated can be so interpreted in the model.
- *Completeness* means that there is a (term) model consisting of exactly the reductions that can be calculated.
- In this denotational setting, we seek to interpret not only the realizer of a consequence but the control process.

Theoretical backstory

- For example, Prolog's strategy of left-to-right clause selection with depth-first traversal and Cut, and the input-output model.
- A control process is associated with the realizer Φ, constructed using the process E:

$$[\![\Gamma]\!]_{\Theta}^{W} \stackrel{[\![E:\Phi]\!]_{\Theta}^{W}}{\rightarrow} [\![\Delta]\!]_{\Theta}^{W}$$

- There is a well-developed theory of (bi)simulation (equality) of processes.
- We can explore examples of game-theoretic models that are able to account for both the structural and operational aspects of reductive logic.
- Again, Pym & Ritter, Oxford Logic Guides, 2004.

Milner and LCF: a concrete theory of proof-search

In the late 70s and early 80s, Robin Milner and colleagues worked on machine-assisted proof, producing the 'Stanford LCF' and 'Edinburgh LCF' systems.

This work was about 'goal-oriented reasoning' — that is, proof-search.

- R. Milner. The use of machines to assist in rigorous proof. *Phil. Trans. R. Soc. Lond.* A 312, 411–422 (1984).
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Goals, theorems, and procedures

Following Milner:

- A *theorem* is a (proved) sequent $\Gamma \vdash \phi$.
- A *goal* G is a sequent $\Gamma \ ?\!\!- \ \phi$.

Then:

- An *event* E is [the proving of] a theorem.
- An event $\Delta \vdash \psi$ *achieves* a goal $\Gamma \ ?\!\!- \ \phi$ if, for some $\Theta \subseteq \Gamma$, $\Theta \ ?\!\!- \ \phi \quad \simeq \quad \Delta \ ?\!\!- \ \psi$, for some equivalence (generalizing Milner a bit here).
- A *procedure* is a partial function

$$\rho : (\text{list of theorems}) \rightharpoonup \text{theorem}$$

Tactics

- A *tactic* is a partial function that takes a goal and returns a list of goals and a procedure:

$$\text{tactic} : \text{goal} \rightharpoonup \text{goal list} \times \text{procedure}$$

- Elementary tactics are given by the the reduction operators that correspond to the 'inverses' of inference rules

$$\frac{\text{Premiss}_1 \quad \dots \quad \text{Premiss}_k}{\text{Conclusion}} \Downarrow$$

$$\frac{\text{Subgoal}_1 \quad \dots \quad \text{Subgoal}_k}{\text{Goal}} \Uparrow$$

- A tactic T is *valid* if, whenever

$$T(G) = ([G_1, \dots G_n)], \rho)$$

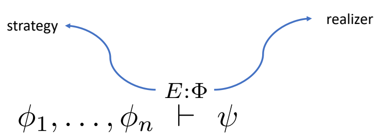
is defined and whenever $[E_1, \dots, E_n]$ respectively achieve the goals $[G_1, \dots, G_n]$, then the event $\rho([E_1, \dots, E_n])$ achieves G.

Tacticals

- Complex goals require, in practice, complex strategies.
- Need combinators, called *tacticals*, for composing tactics.
- Tactical combinations of tactics are themselves tactics.
- Examples would include:
  - Basic sequencing
  - The definition of *uniform proof* in the sense of Miller
  - In some sense, this is the origin of the ML ('Meta-Language') family of programming languages.

Relate to theoretical backstory

- Recall
$$[\![\phi_1, \ldots, \phi_n]\!]^W_\Theta \overset{[\![E:\Phi]\!]^W_\Theta}{\to} [\![\psi]\!]^W_\Theta$$
- Here, abusing notation a bit



- A strategy is a tactical combination of tactics
- A procedure converts/reduces realizers to proofs, working up to $\simeq$.

Proof-theoretic semantics

Proof-theoretic used to be known as the *theory of meaning* — that is, of logical of constructs — in the sense of Prawitz, Martin-Löf, Sundholm, and others, mainly in the Scandinavian logical school.

- Basic idea: provide theory of logical validity that is based on proof-theoretic structures instead model-theoretic structures.
- We can think of this as working with mathematical structures that are built out of proof systems (inference rules, meaningfully organized) instead of satisfaction relations (truth in models).

Proof-theoretic semantics

Some questions that proof-theoretic semantics asks.

- In order to understand how it is that proof characterizes meaning, of structures are proofs delineating examples?
- One answer, inspired by truth semantics!, is that inferences rules are special cases of relations on sequents (or other basic units of a proof system).
- Then, within this bigger space of constructions, how can proofs — or, more generally, things equivalent to proofs — be identified? There's a kind of subtext of constructivism here.

I'll try to look at all this in terms of Milner's theory of tactical proof.

Relate to proof-theoretic semantics

We can see, informally, for now, a correspondence between Milner's analysis and current ideas in proof-theoretic semantics.

- Let's work with something like the Prawitz, Schröder-Heister, ... approach (e.g., Proof-theoretic vs model-theoretic consequence, 2008).
- *Proof structures* $\mathcal{D}$ that are tree-like arrangements of *sequents*.
- A *justification system* $\mathcal{J}$ that maps structures to structures.
- Idea is that justifications pick out those structures that correspond to proofs in a 'ground' system of proof-theoretic rules.

Relate to proof-theoretic semantics

- Validity with respect to $\mathcal{J}$ and atomic system S (inference restricted to propositional atoms):

$$\phi_1, \ldots, \phi_n \models \psi \quad \text{iff} \quad \begin{array}{l}\text{there is a } \mathcal{J} \text{ s.t. for every } S \\ \text{and all } \mathcal{J}_1, \ldots, \mathcal{J}_n, \text{ if} \\ (\mathcal{J}_1, S) \models \phi_1, \ldots, (\mathcal{J}_n, S) \models \phi_n, \\ \text{then } (\mathcal{J}, S) \models \psi\end{array}$$

- $\mathcal{J}(\mathcal{J}_1, \ldots, \mathcal{J}_n)$ amounts to the procedure, relative to possibly generalized S ('ground' inferences).
- Strategy not included in this picture (as I understand it).

Towards a coalgebraic approach

- A *coalgebra* for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is a morphism $\alpha : X \to FX$ in $\mathcal{C}$, usually written (X, α).
- Intuitively, F assigns structure to a state space X, while α describes the dynamics for a system that traverses this structured space.
- This concept subsumes and generalizes phenomena as wide-ranging as automata, context-free grammars, datatypes, games, program semantics, and transition systems.

This approach provides an algebraic framework within which to generalize the theoretical backstory.

Towards a coalgebraic approach — first, a logic

- Kripke semantics can be seen coalgebraically.
- Example: BI, the logic of bunched implications (O'Hearn & Pym, BSL 1999) the basis of Separation Logic.
  - Essentially, freely combines IL and MILL in a bunched proof-theoretic framework.
  - Name comes from sequent calculus: bunched contexts, separating intuitionistic and linear parts.
  - Very different logic from LL: for example, $\phi \to \psi = !\phi \multimap \psi$ does not hold.

Towards a coalgebraic approach — first, a logic

- Ordered partial monoid $(R, \sqsubseteq, \circ, e)$ of worlds, r, s, t, ...

$$
\begin{array}{rcl}
r \models \mathrm{p} & \text{iff} & r \in \mathcal{V}(\mathrm{p}) \\
r \models \bot & \text{never} & \\
r \models \top & \text{always} & \\
r \models \phi \vee \psi & \text{iff} & r \models \phi \text{ or } r \models \psi \\
r \models \phi \wedge \psi & \text{iff} & r \models \phi \text{ and } r \models \psi \\
r \models \phi \rightarrow \psi & \text{iff} & \text{for all } s \sqsubseteq r,\ s \models \phi \text{ implies } s \models \psi \\
 & & \\
r \models I & \text{iff} & r \sqsubseteq e \\
r \models \phi * \psi & \text{iff} & \text{there are worlds } s \text{ and } t \text{ such that} \\
 & & r \sqsubseteq (s \circ t)\downarrow \text{ and } s \models \phi \text{ and } t \models \psi \\
r \models \phi \mathrel{-\!\!*} \psi & \text{iff} & \text{for all } s \text{ such that } (r \circ s)\downarrow \text{ and } s \models \phi, \\
 & & r \circ s \models \psi
\end{array}
$$

Truth-functional semantics, coalgebraically

- BI can be given by coalgebras for the functor $T : \mathcal{C} \rightarrow \mathcal{C}$,

$$TX = \mathbb{2} \times \mathrm{P}_c(X \times X) \times \mathrm{P}_c(X^{op} \times X)$$

  where $\mathcal{C}$ is the category of posets, $\mathbb{2}$ the two element poset and P_c the convex powerset functor (Egli–Milner order).
- The first component interprets of the unit constant I, the second $*$, and the third $\mathrel{-\!\!*}$.
- Given a monoid $(R, \circ, e)$, a poset is given by setting $r \sqsubseteq s$ iff there exists r' such that $r \circ r' = s$.
- Then the coalgebra $\alpha : R \rightarrow \mathbb{2} \times \mathrm{P}_c(R \times R) \times \mathrm{P}_c(R^{op} \times R)$ is:
  - $\pi_0(\alpha(r))$ is 1 if $r = e$ and 0 if $r \neq e$ — for I
  - $\pi_1(\alpha(r)) = \{(s, t) \mid s \circ t \leq r\}$ — for $*$
  - $\pi_2(\alpha(r)) = \{(s, t) \mid r \circ s = t\}$ — for $\mathrel{-\!\!*}$

- The coalgebraic interpretation of the logic is given, essentially, by a natural transformation δ from a functor that forms the formulae of the BI to the functor T.
- In the specific case of $*$, given interpretations for ϕ and ψ, we obtain the interpretation

$$\delta_X(\phi * \psi) = \{t \in TX \mid \exists (x, y) \in \pi_2(t), x \in \delta_X(\phi), y \in \delta_X(\psi)\}$$

- In the coalgebra associated to a monoid, this corresponds precisely to the standard truth-functional clause for $*$, but the class of coalgebraic models strictly extends the class of truth-functional models.

What's the use of all this?

Proof-search with substructural connectives

- From the computational perspective, the reduction operators that correspond to the inference rules for multiplicative connectives, such as $\otimes$ and $\multimap$, and $*$ and $-\!\!*$, are problematic.
- For example,

$$\frac{\Gamma_1 \vdash \phi_1 \quad \Gamma_2 \vdash \phi_2}{\Gamma \vdash \phi_1 * \phi_2} \quad \Gamma = \Gamma_1, \Gamma_2$$

- How to calculate the Γs?
- Iterates through the search: suppose $\phi_1 = \psi_1 * \psi_2$, then need $\Gamma_1 = \Delta_1, \Delta_2 \ldots$.
- Computationally expensive (potentially both time and space).
- Not just right rules,

$$\frac{\Gamma_1 \vdash \phi \quad \Gamma_2, \psi \vdash \chi}{\Gamma, \phi -\!\!* \psi \vdash \chi} \quad \Gamma = \Gamma_1, \Gamma_2$$

The input–output model

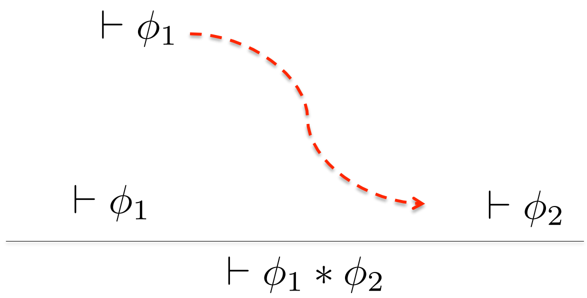


Figure 1: The input–output model (Miller)

- All 'resources' are sent up the first branch.
- Those required to close the branch (if possible) are retained on the branch, with what remains being sent to the next branch.

Back to coalgebra, for the input-output model

Coalgebraically, we can see this as a further structuring of the search space TX by updating to $Bool \times In \times TX$.
Then the coalgebra $\alpha : X \to Bool \times In \times TX$ works as follows:

- at a reduction with a multiplicative conjunction leaf $\Gamma \vdash \phi_1 * \phi_2$, α is designed to choose to reduce the left-hand premiss;
- *In* outputs a list of the formulae required for the current proof of ϕ_1;
- *Bool* is a test for termination of that branch;
- if a proof is found, the next step of computation defined by α is to begin reducing the right-hand premiss with respect to the context given by Γ minus the current value of *In*;
- *In* is then reset to the empty list and *Bool* to false.

Why coalgebra?

- The motivation for adopting a coalgebraic approach is strong; it handles both
  - Kripke semantics, as a framework for defining logics, and
  - proof-search and model-checking procedures.
- The latter point perhaps deserves some expansion.
- Search procedures are *not* naturally functional, but are naturally stateful. ML, the programming language initially developed as language for specifying tactics and tacticals in LCF, is not a purely functional language. Rather, it makes explicit use of imperative *exceptions*.
- Exceptions are used to handle failure and continuation/ resumption — essential features of search procedures.
- Thus while deduction naturally has functional accounts, reduction does not.

Generalizing

At this level of generality — remaining agnostic about the exact nature of the termination test — it is easy to see how this coalgebraic description could incorporate more general examples like the resource-distribution model of Harland & Pym (Resource-distribution via Boolean constraints, ACM ToCL, 2000), where the test is solutions to Boolean constraints.

More generally still, this can be seen as the use of the classical (sequent) calculus, as a meta-calculus for the reductive (proof-search) view of non-classical logics, L:

$$\text{L-search} = \text{LK-search} + \text{Conditions}$$

- Dummett's restriction of multiple-conclusion sequent calculus for IL;
- Essentially modal conditions;
- Resource-distribution in substructural logics ... .

Actually, it's the and–or combinatorics that matter, with negation a sometimes-convenient tool.

A general approach to resource distribution

- We consider a sequent calculus for, for example, Linear Logic in which the non-deterministic splitting of contexts at multiplicative reductions is explicit.
- This allows us to set up a calculus which is independent of the choice of strategy that is used to distribute formulae, but which makes the necessary constraints explicit.
- To motivate/frame the approach, let's start with examples of key strategies for calculating multiplicative splitting.
- Method is very general, actually ...
- Let's begin with a simple example: the provable Linear Logic sequent
$$p, p, q, q \vdash (p \otimes q) \otimes (p \otimes q)$$

Lazy distribution

First pass all of context to a chosen (lefmost, say) branch, calculate which formulae are required to close the branch, and pass the remaining formulae to the next branch.

- So, first

$$\dfrac{\dfrac{\dfrac{}{\cancel{p}, p, \cancel{q}, \cancel{q} \vdash p} \quad \dfrac{}{X_2 \vdash q}}{p, p, q, q \vdash p \otimes q} \quad \dfrac{}{X_1 \vdash p \otimes q}}{p, p, q, q \vdash (p \otimes q) \otimes (p \otimes q)}$$

- Then X_2 gets p, q, q and uses a q and so X_1 gets p and q.
- Repeat for the leftmost remaining branch, and the proof

$$\dfrac{\dfrac{\dfrac{}{\cancel{p}, p, \cancel{q}, \cancel{q} \vdash p} \quad \dfrac{}{\cancel{p}, q, \cancel{q} \vdash q}}{p, p, q, q \vdash p \otimes q} \quad \dfrac{\dfrac{}{p, \cancel{q} \vdash p} \quad \dfrac{}{\cancel{p}, q \vdash q}}{p, q \vdash p \otimes q}}{p, p, q, q \vdash (p \otimes q) \otimes (p \otimes q)}$$

Lazy distribution

- Let's see how a lazy search can fail.
- So, first

$$\dfrac{\dfrac{\dfrac{}{\cancel{p}, p, \cancel{q}, \cancel{q} \vdash p} \quad \dfrac{}{X_2 \vdash q}}{p, p, q, q \vdash p \otimes q} \quad \dfrac{}{p, p, q, q \vdash p \otimes q}}{p, p, q, q \vdash (p \otimes q) \,\&\, (p \otimes q)}$$

- Again, X_2 gets p, q, q and uses a q.
- Now we fail: all of the leaves on the left-hand branch have been closed. unused formulae remain, and there is nowhere to send them (the branching point below is additive).

Eager distribution

- Pass all formulae to all possible branches

$$\dfrac{\dfrac{\overline{p, \cancel{p}, \cancel{q}, \cancel{q} \vdash p} \quad \overline{\cancel{p}, \cancel{p}, q, \cancel{q} \vdash q}}{X_1 \vdash p \otimes q} \qquad \dfrac{\overline{\cancel{p}, p, \cancel{q}, \cancel{q} \vdash p} \quad \overline{\cancel{p}, \cancel{p}, \cancel{q}, q \vdash q}}{X_2 \vdash p \otimes q}}{p, p, q, q \vdash (p \otimes q) \otimes (p \otimes q)}$$

- p , p , q , q has to all leaves.
- Requirements to close each leaf can be solved simultaneously.
- So a proof is determined.

Intermediate distribution

- Lazy and eager are extreme points in the space of possible strategies here.
- In between, we have 'intermediate' strategies.
- That is, where many multiplicative branch is explored simultaneously *up to some specified maximum number*.
- Lazy is depth-first, eager is breadth-first, intermediate is bounded depth-first (cf. 'iterative deepening').

The generality of the approach

See Harland and Pym, Resource-distribution via Boolean constraints, ACM ToCL 2000 for the following:

- Formulation of general sequent calculus with Boolean constraints;
- Soundness and completeness results;
- The method appllies to full Linear Logic;
- The method applies to BI's sequent calculus;
- The method applies to the full family of relevant logics as described in Read's *Relevant Logic*, Blackwell, 1988.

Additionally, we conjecture that the method applies to the families of layered graph logics and the families of bunched modal and epistemic logics sketched in these lectures.

We also conjecture fits into the coalgebraic framework.

Summary to take away

- General, theoretically supported view of reductive logic — proof-theoretically and model-theoretically.
- Uniform coalgebraic treatment of structure and control.
- Sits in the framework of proof-theoretic semantics.
- Provides a setting for systematic understanding of

$$\text{Reductive Logic} = \text{Structure} + \text{Control}$$

Proof-Theoretic Semantics
Some Open Problems

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What I will not speak about

- Reductive approaches, in particular dialogue semantics
- Substructural issues, even though they influence all aspects of proof-theoretic semantics
- The format of reasoning in general, and thus the ‘dogmas’ of semantics
- The problem of negation, in particular of indirect negation vs. direct negation (‘denial’)

Which means that I stay within the original framework outlined by Prawitz and Martin-Löf, and also by myself in earlier work, and point to some problems within this framework.

Outline:

- Is there a Tarski-style approach to proof-theoretic semantics?
  - What are the potential ways out of the problem of semantical completeness?
- What means local harmony, when the problem of proof identity is taken into account?
  - Are the generalized elimination rules the *non plus ultra* in proof-theoretic semantics?
- Beyond logic: Atomic reasoning
  - The issue of generality and substitution

Problem 1: A Tarski-style approach to proof-theoretic semantics?

By Tarski-style I mean the following:

We define a semantical concept of truth, and then in a second step justify a formal system as being adequate with respect to this concept.

Such a concept has been advocated by Prawitz under the name 'validity' since the late 1960s.

Inspired by technical concepts of 'convertibility' and 'computability' in proofs of normalization (Tait, Martin-Löf, Girard).

Proof-theoretic validity

Basic ingredients:

- The classical role of structures is taken by atomic systems ('bases').
- The candidates for semantic attribution are not propositions, but proof structures ('arguments').
- Proof reductions play the role of justifying procedures.

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The definition of proof-theoretic validity

- A closed argument for an atom is valid, if this atom is derivable in the base under consideration.
- A closed argument for a non-atomic formula is valid, if it is in introduction form or reduces to introduction form by means of the reductions under consideration.
- An open argument is valid, if every result of substituting closed arguments for its open assumptions is valid.

B follows logically from *A* if there is a set of reductions such that for every base there is a valid argument for *B* with open assumption *A*.

This semantics has its plausibility. See my contribution to PTS1 and the supplement to my SEP-entry.

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The plausibility of proof-theoretic validity

- Semantics for proofs in terms of proof
- Plausible concept of basis: Atomic systems, which are a general form of Herbrand bases. Inductive definitions.
- Proper rendering of direct vs. indirect justifications
- Proper rendering of open proofs

The completeness problem

Since Heyting, the standard formalism of constructive logic is formal intuitionistic logic, that is at the propositional level: IPC.

This is why in mainstream constructive logic people aimed at justifying IPC, that is, rendering it complete.

Unfortunately, Prawitz's validity-based semantics does not make IPC semantically complete.

The semantic incompleteness of IPC

The validity based semantics satisfies the general disjunction property:

If $\Gamma \models A \vee B$, where $\vee$ does not occur in Γ, then $\Gamma \models A$ or $\Gamma \models B$.

As presented by Thomas Piecha, from this property (and some other very natural conditions) it follows that IPC is incomplete for validity-based semantics. The Prawitz completeness conjecture of 1971 is not true.

Ways out of the completeness problem

First option: Give up the formalism of intuitionistic logic

Problem of intuitionistic logic: It is not structurally complete, as soon as implication and disjunction are both present.

However, validity based semantics, as well as other semantics approaches (BHK, Realizability) have the flavour of an admissibility semantics for implication.

This is due to the procedural interpretation of implication. Perhaps intuitionistic logic is not the desirable logic after all.

Perhaps axiomatisations of admissible inferences ($\Longrightarrow$ Iemhoff) *...*

Ways out of the completeness problem

Second option: Change the semantics

Validity-semantics looks like the canonical-model-semantics for intuitionistic implicational logic, where one only needs to consider a single model.

Kripke semantics needs to consider not just one single tree based on set inclusion, but possibly more than one frame.

How to capture this idea proof-theoretically?

Ways out of the completeness problem

Two-layer semantics analogous to Kripke semantics. Consider special kind of base systems, which are able to imitate a Kripke frame ('boundary rules', Goldfarb).

Use second sort of base systems, which correspond to our knowledge base.

This keeps the spirit of Prawitz's validity based semantics, but, due to the additional basic rules, allows one to prove completeness.

This might also solve another problem with atomic systems: Are they definitional or factual? Are they collections of definitions or do they represent our knowledge base?

Problem 2: Local harmony rather than global validity?

Validity is a global concept which concerns proof-structures ('arguments') as a whole.

Harmony is a relation between specific rules.

Basing proof-theoretic semantics on harmony means to base it on rules rather than proofs.

Semantics of proofs via a semantics of rules.
A proper proof is built from semantically justified rules.

Basic principle: The concept of harmony should take the problem of proof identity into account.

Proof theory is interested in proofs, not only provability, and this applies even more to proof-theoretic semantics.

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Problems of harmony

Standard example:

$$\frac{A \quad B}{A \wedge B} \qquad \frac{A \wedge B}{A} \quad \frac{A \wedge B}{B}$$

Equivalent:

$$\frac{A \quad B}{A \wedge B} \qquad \frac{A \wedge B}{A} \quad \frac{A \wedge B \quad A}{B}$$

Only extensional, but not intensional harmony:

$A \wedge B$ and $A \wedge (A \to B)$ are not proof-theoretically isomorphic.

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Proof-theoretic isomorphism

The proof

$$\begin{array}{c} A \wedge (A \to B) \\ \vdots \\ A \wedge B \\ \vdots \\ A \wedge (A \to B) \end{array} \qquad \text{does not reduce to} \qquad A \wedge (A \to B)$$

This is a big problem for general definitions of harmony.

In particular for those definitions which use a translation into second-order logic.

The problem with generalized elimination rules

Generalized rules for conjunction:

$$\frac{A \quad B}{A \wedge B} \qquad\qquad \frac{A \wedge B \qquad \begin{array}{c}[A \quad B] \\ C\end{array}}{C}$$

$A \wedge B$ and $(\forall C)(((A \wedge B) \to C) \to C)$ are not proof-theoretically isomorphic.

The problem with generalized elimination rules

An even simpler example:

$$\frac{A}{+A} \qquad\qquad \frac{+A}{A} \quad \text{vs.} \quad \frac{+A \quad \begin{matrix}[A]\\ C\end{matrix}}{C}$$

A and $(\forall C)((A \to C) \to C)$ are not proof-theoretically isomorphic.

Perhaps Schroeder-Heister, Dyckhoff, Tennant, von Plato, Lopez-Escobar were all misguided when considering the general elimination rules to be the appropriate model of elimination rules.

Perhaps the direct elimination rules are the proper basis of proof-theoretic harmony.

Perhaps disjunction is the exception and not the rule.

The problem with generalized elimination rules

This is independent of whether one uses higher-level rules or sequent-style generalized rules for implication:

$$\frac{A \to B \qquad \begin{matrix}[A \Rightarrow B]\\ C\end{matrix}}{C} \qquad\qquad \frac{A \to B \quad A \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

The first is type-theoretically preferable, but still suffers from the same defect.

I share these second thoughts about generalized rules with Roy Dyckhoff.

See his contribution to the PTS2 volume.

The special problem of disjunction

Rules for disjunction:

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad \frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

$A \vee B$ and $(\forall C)(((A \to C) \wedge (B \to C)) \to C)$ are not proof-theoretically isomorphic.

In principle, we can do without disjunction, but not without implication.

Perhaps we need some dual to η-reduction:

$$\dfrac{\dfrac{c}{c^E}}{c} \triangleright c\ (\eta) \qquad\qquad \dfrac{\dfrac{c^E}{c}}{c^E} \triangleright c^E \text{ (dual to } \eta)$$

This would save the general elimination rules.

The special problem of disjunction

This is satisfied, if we look for canonical introduction rules to given elimination rules, which are then normally of higher level.

In the case of disjunction:

$$\frac{\left(\dfrac{A \Rightarrow C \quad B \Rightarrow C}{C}\right)_C}{A \vee B} \qquad\qquad \frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

Even if we have several elimination rules, this is not problem, as they are understood conjunctively, for example:

$$\frac{A \quad \begin{matrix}[A]\\ B\end{matrix}}{A \wedge B} \qquad\qquad \frac{A \wedge B}{A} \qquad \frac{A \wedge B \quad A}{B}$$

Intensional harmony based on elimination rules

Elimination rules: $(c\ \text{E})\ \dfrac{c \quad \Delta_1}{A_1} \quad \ldots \quad \dfrac{c \quad \Delta_m}{A_m}$

Canonical introduction rules: $(c\ \text{I})_{can}\ \dfrac{\begin{matrix}[\Delta_1] \\ A_1\end{matrix} \quad \ldots \quad \begin{matrix}[\Delta_m] \\ A_m\end{matrix}}{c}$

The intensional approach speaks for elimination rules as basis.

The nondeterminism in introductions disappears.

There are no indirect rules.

Problem IIA. Negation in proof-theoretic semantics

In the intuitionistic framework $\neg A$ is defined as $A \rightarrow \bot$.

Giving a proof of $\neg A$ consists in showing that every proof of A can be transformed into a proof of $\bot$.

Since by definition there is no proof of $\bot$:
Giving a proof of $\neg A$ consists in showing that there is no proof of A.

Thus we need to climb up to a proof of a negative existence claim at the metalevel.

Is this indirect interpretation of negation plausible?

How many proofs of negative propositions can be given?

According to the standard setup: "...in an intuitionistic propositional calculus there is at most one proof $A \to \bot$, up to equivalence of proofs" (Lambek/Scott, Prop. 8.3).

And even this single proof is only available if A is isomorphic to $\bot$.

This reflects the indirect character of negation.

For positive propositions there can be different proofs.

Prawitz's example were different proofs of $(A{\to}A) \to ((B{\to}(A{\to}A)))$

Is this the most natural notion of negation?

Or better: Is this the only natural notion of negation?

Way out: Direct negation

Denial as as a form of judgement.

There are corresponding formalisms, the earliest one by Nelson.

There are various ways of introducing both assertion and denial into intuitionistic logic.

In particular by dualizing implication.

See work by Tranchini and by Wansing.

What about a system with both sorts of negation, direct and indirect?

This leads to a system which also incorporates two modes of assertion.

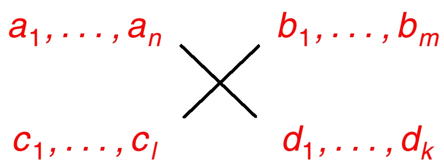
- Direct operators
  - Direct assertion: A
  - Direct denial: $\sim A$
- Indirect operators:
  - Indirect denial (denial by failure): $-A$
  - Indirect assertion (assertion by failure): $+A$

This gives rise to corresponding inversion principles.

However, I have not studied what this means from the intensional point of view.

Calculus of squares

The square of opposition as a form of judgement:



as expressing

$$\vdash a_1, \ldots, a_n, \sim b_1, \ldots, \sim b_m, +c_1, \ldots, +c_l, -d_1, \ldots, -d_k$$

where the comma is understood disjunctively.

A generalised hypothetical judgement.

Problem 3: Beyond logic — Generalized inversion, generality and substitution

Proof-theoretic semantics has been almost exclusively occupied with logical constants.

There is a realm beyond logic:

- Subatomic proof-theoretic semantics ($\Longrightarrow$ linguistics, see Francez, Wieckowski and others)
- Atomic proof-theoretic semantics

I will be concerned with the latter. It has direct relations with recent research on paradoxes, non-logical axioms etc.

This has been my topic for decades. ($\Longrightarrow$ Dale Miller). Here I will focus on issues of generality and substitution.

Production rules as introduction rules

Production rules occur explicitly in:

- inductive definitions
- logic programs

Orientation:

- Lorenzen
- Logic programming

They look at rules in a fairly general form

Other approaches:

- Martin-Löf's iterated IDs
- Brotherston/Simpson

Generality and substitution

A double-line principle for generality and substitution

$$\frac{\{\Delta(t) \vdash C(t)\}_t}{\Delta(x) \vdash C(x)}$$

This is a double-line rule, which affects both sides of a sequent.

The top-bottom direction is infinitary.

In natural deduction:

$$\frac{\Delta(x) \quad \left\{\begin{matrix}[\Delta(t)] \\ C(t)\end{matrix}\right\}_t}{C(x)} \qquad \frac{\Delta(t) \quad \begin{matrix}[\Delta(x)] \\ C(x)\end{matrix}}{C(t)}$$

This can be made finitary in certain circumstances.

Definitional clauses for atoms

$$\left\{ B \Leftarrow \Delta \right.$$

First generalization: As in logic programming, no well-foundedness needed.

For example: $\left\{ p \Leftarrow \neg p \right.$

Here: Clauses with variables, as in logic programming.

$$\left\{\begin{array}{rcl} P(g(x)) & \Leftarrow & R(h(x)) \\ P(f(x)) & \Leftarrow & R(g(x)) \\ P(f(a)) & \Leftarrow & R(a) \\ C & \Leftarrow & R(x) \end{array}\right.$$

Atoms are finitarily (and partially) defined.

An example for generalization

$$\begin{cases} P(1) & \Leftarrow & \Delta_1(1) \\ P(2) & \Leftarrow & \Delta_2(2) \\ C(1) & \Leftarrow & \Gamma_1(1) \\ C(2) & \Leftarrow & \Gamma_2(2) \end{cases}$$

From that we obtain the following generalization rule:

$$\frac{P(x) \quad \begin{matrix}[P(1)] \\ C(1)\end{matrix} \quad \begin{matrix}[P(2)] \\ C(2)\end{matrix}}{C(x)}$$

This even works, if P(1) and P(2) are trivially defined:

$$\begin{cases} P(1) & \Leftarrow & P(1) \\ P(2) & \Leftarrow & P(2) \\ & \vdots & \end{cases}$$

The general generalization principle.

Given a definition $\mathbb{D}$ with clauses of the form:

$B \Leftarrow \Delta$

Generalization for an atom A:

$$\frac{A \quad \left\{ \begin{matrix}[A\sigma] \\ C\sigma\end{matrix} : \sigma = mgu(A, B) : (B \Leftarrow \Delta) \in \mathbb{D} \right\}}{C}$$

This is not inversion!

The standard example: Free equality

The unification based theory:

$$\frac{}{\vdash t \doteq t} \qquad \frac{s \doteq t \quad C\sigma}{C}\ \sigma = mgu(s,t)$$

$$\frac{s \doteq t}{C}\ s \text{ and } t \text{ not unifiable}$$

Derivable in our context:

$$\left\{\ x \doteq x \ \Leftarrow\ \Delta \right. \qquad \frac{s \doteq t \qquad \begin{array}{c}[(s \doteq t)\sigma] \\ C\sigma\end{array}}{C}$$

$$\sigma = mgu(s,t) = mgu(x \doteq x, s \doteq t)$$

The converse depends on the form of Δ.

Result: There is generalisation without inversion.

Conclusion

There is a plethora of problems and potential solutions when methods of proof-theoretic semantics are applied to the reasoning with atomic formulas.

This approach can even be dualized, when considering refutation rules ...

Another application might be geometric rules ...

General conclusion

What is the proper approach to proof-theoretic semantics?

Through proofs to rules? — Validity —

Through rules to proofs? — Harmony —

Through atomic rules to logical rules?
— Inductive definitions —

Via indirect negation to refutation? — Assertion and denial —

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Applying a linear logic perspective to arithmetic

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Proof-Theoretic Semantics, Tübingen 29 March 2019

Goals of this talk:

1. Describe what we have learned from linear logic that has been useful in the proof theory of classical and intuitionistic logics.
2. Describe our first steps in applying those lessons to arithmetic. (Work in progress. Joint with Matteo Manighetti.)

Under the Proof-Theoretic Semantics (PTS) umbrella

What formal devices and techniques can we identify, apply, and teach?

Kahle quotes Schroeder-Heister: "PTS has an intuitionistic bias."

PTS also as a natural deduction bias. My perspective:

- The sequent calculus is a more general setting for PTS.
- Linear logic is a useful tool for exploiting the sequent calculus.

Linear Logic

Girard proposed linear logic in 1987. Broadly speaking, it has had two kinds of impact.

As a new logic, it provided

- the λ-calculus (and functional programs) with *new* types;
- logic programming with *new* programs; and
- new proof structures, such as proof nets.

As the "logic behind (computational) logic", it introduced into classical and intuitionistic proof systems

- *polarization*,
- *focused* proofs, and
- new controls on contraction and weakening.

My PTS tool box

Notation: $\mathbf{1}$, $\otimes$, $\mathbf{0}$, $\oplus$, $\top$, $\&$, $\bot$, $⅋$, $\multimap$, $!$, $?$, $(-)^{\perp}$

Terminology:

- *additive* connectives: $\mathbf{0}$, $\oplus$, $\top$, $\&$
- *multiplicative* connectives: $\mathbf{1}$, $\otimes$, $\bot$, $⅋$, $\multimap$
- *exponentials*: $!$, $?$
- *negative polarity*: $\top$, $\&$, $\bot$, $⅋$, $\multimap$, $?$, $\forall$
- *positive polarity*: $\mathbf{1}$, $\otimes$, $\mathbf{0}$, $\oplus$, $!$, $\exists$

Consider the right introduction rule of a logical connective.

- If it is invertible, the connective has *negative* polarity.
- If it is not invertible, the connective has *positive* polarity.

Linear logic negation flips polarities!

Example: Linear logic behind the LK vs LJ distinction

Gentzen accounted for intuitionistic logic by restricting sequents to have at most one formula:

$$\Gamma \vdash \Delta \quad \text{where } \Delta \text{ has zero or one formula.}$$

This restriction is equivalence to the following 2 conditions.

1. No contraction on the right.
2. In the (multiplicative) implication-left rule,

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \supset B \vdash C} \supset L$$

   the formula occurrence C cannot appear in the left premise.

In linear logic terms, Γ is encoded as $!\,\Gamma$ and $A \supset B$ is encoded using two connectives $(!\,A) \multimap B$.

Example: Different information content in proofs

Classical, propositional logic with atoms, negated atoms, $\vee$, and $\wedge$.

Invertible rules

$$\frac{\vdash \Delta, B_1, B_2}{\vdash \Delta, B_1 \vee B_2} \parr \qquad \frac{\vdash \Delta, B_1 \quad \vdash \Delta, B_2}{\vdash \Delta, B_1 \wedge B_2} \&$$

Proof search proceeds by expanding into conjunctive normal form.

- Straightforward computation.
- Order of inference rules is not important.
- No contractions appear in proof.
- Weakening at leaves (only of literals).
- *Exponential* procedure.

Example: Different information content in proofs (con't)

Non-invertible rules

$$\frac{\vdash \Delta, B_1}{\vdash \Delta, B_1 \vee B_2}\ \oplus_1 \qquad \frac{\vdash \Delta, B_2}{\vdash \Delta, B_1 \vee B_2}\ \oplus_2 \qquad \frac{\vdash \Delta_1, B_1 \quad \vdash \Delta_2, B_2}{\vdash \Delta_1, \Delta_2, B_1 \wedge B_2}\ \otimes^{\dagger}$$

The search for a proof of $\vdash B$ generates sequents of the form $\vdash B, C, \mathcal{L}$ where C is a subformula of B and $\mathcal{L}$ is a collection of literals.

- † In classical logic, we can take $\Delta = \Delta_1 = \Delta_2 = \Delta_1, \Delta_2$.
- Contraction is needed but only on B.
- Proof construction consumes an external bit to decide $\oplus_i$.

Proofs can be short since an oracle might contains some "clever" information.

Example: A short proof consuming three bits

Let C have several alternations of conjunction and disjunction and let $B = (p \vee C) \vee \neg p$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, p \qquad , \neg p}\ init}{\vdash B, p \vee C \qquad , \neg p}\ *}{\vdash B, (p \vee C) \vee \neg p \quad , \neg p}\ *}{\vdash B, \neg p}\ contract}{\vdash B, (p \vee C) \vee \neg p}\ *}{\vdash B}\ contract$$

The subformula C is avoided. Clever choices $*$ are injected at these points: right, left, left.

Focusing simply explained: proof search for $\Gamma \vdash \Delta$

- Do invertible introductions in any order, to exhaustion: positive connective on left; negative connective on right.
- Use the *decide* rule to pick a *focus* (includes the only case of contraction in intuitionistic logic).

$$\frac{\Gamma \Downarrow N \vdash \Delta}{\Gamma, N \vdash \Delta} \qquad \frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \vdash \Delta} \qquad \frac{\Gamma \vdash P \Downarrow}{\Gamma \vdash P}$$

- If the polarity flips in the focus, then use the *release* rule.

$$\frac{\Gamma, P \vdash \Delta}{\Gamma \Downarrow P \vdash \Delta} \qquad \frac{\Gamma \vdash N}{\Gamma \vdash N \Downarrow}$$

- Chose an *introduction* rule for non-atomic focus. Ask an oracle for help or consider backtracking. All premises are marked with $\Downarrow$.
- The remaining cases are the *initial* rules.

$$\frac{}{\Gamma \Downarrow N_a \vdash N_a}\ N_a \text{ neg atom} \qquad \frac{}{\Gamma, P_a \vdash P_a \Downarrow}\ P_a \text{ pos atom}$$

Atoms can have a (non-canonical) polarity

Polarity can be assigned to atoms in a fixed but arbitrary fashion.

$$\cfrac{\cfrac{\begin{array}{c}\Xi_1\\ \Gamma \vdash Rab \Downarrow\end{array} \quad \cfrac{\begin{array}{c}\Xi_2\\ \Gamma \vdash Rbc \Downarrow\end{array} \quad \begin{array}{c}\Xi_3\\ \Gamma \Downarrow Rac \vdash \Delta\end{array}}{\Gamma \Downarrow Rbc \supset Rac \vdash \Delta}\ \supset L}{\Gamma \Downarrow Rab \supset Rbc \supset Rac \vdash \Delta}\ \supset L}{\Gamma \Downarrow \forall x \forall y \forall z (Rxy \supset Ryz \supset Rxz) \vdash \Delta}\ \forall L \times 3$$

If R-atoms have neg polarity, then Ξ_3 is initial and Δ is Rac. Also, Ξ_1 and Ξ_2 are release. The synthetic rules is *back-chaining*.

$$\frac{\Gamma \vdash Rab \quad \Gamma \vdash Rbc}{\Gamma \vdash Rac}$$

If R-atoms have pos polarity, then Ξ_3 is release and Ξ_1, Ξ_2 are initial and Γ is Rab, Rbc, Γ'. The synthetic rules is *forward-chaining*.

$$\frac{\Gamma', Rab, Rbc, Rac \vdash \Delta}{\Gamma', Rab, Rbc \vdash \Delta}$$

Synthetic inference rules

In this way, geometric formulas yield inference rules that mention only atomic formulas: no logical connectives are visible in the rule.

See, for example, Negri's "from axioms to inference rules".

Synthetic rules built using focusing automatically satisfy cut-elimination.

Focused proofs provide a means for taking Gentzen's "atoms of inference rules" and building macro-level / synthetic inference rules ("molecules of inference").

Carry these PTS tools to arithmetic

By arithmetic, I mean, more generally, both induction and co-induction (least and greatest fixed points) for general inductive definitions.

In this talk, I will not consider co-induction.

The logic and much of the proof theory described here is part of the Abella theorem prover.

Abella

`http://abella-prover.org/`
`http://abella-prover.org/tutorial/try/`
runs in your browser

Arithmetic as a theory in logic

Peano's axioms fall into three groups.

- Equality is an equivalence relation.
- Zero and successors are constructors.
- Induction scheme

Peano Arithmetic is the classical logic treatment of these axioms.

Heyting Arithmetic is the intuitionistic logic treatment of these axioms.

Before we consider a linear logic treatment of arithmetic, it seems best to update this perspective on arithmetic more generally.

We first move away from Frege/Hilbert proofs to sequent calculus.

Arithmetic as a sequent calculus

We shall consider equality as a logical connectives with left and right introduction rules.

Similarly, the least-fixed point operator μ will also have left and right introduction rules.

A fixed point operator was (in principle) also considered by J-YG and PS-H, but they only considered the unfolding of fixed points (unfolding using the definition).

To capture *least* fixed points, an induction scheme is needed.

Various intuitionistic logics involving least and greatest fixed points have been consider in several papers during 1997-2011 by Gacek, McDowell, Miller, Momigliano, Nadathur, and Tiu.

Baelde and Miller have considered a linear logic variant as well.

Three ways to move beyond MALL

A quick synopsis for the expert in linear logic:

MALL is a propositional logic without contraction and weakening: $\otimes, \mathbf{1}, \oplus, \mathbf{0}, \wp, \bot, \&, \top$. It is decidable.

1. Girard [1987] added the *exponentials* $(!, ?)$ to get linear logic.
2. Liang and M [2009] added *classical and intuitionistic connectives* to get LKU. (Exponentials are behind this design.)
3. Baelde and M [2007] added *fixed points* to get μMALL.

Our examples will illustrate how μMALL seems better suited for model checking and (co)inductive theorem proving than linear logic. Note:

- Fixed point unfolding resembles contraction: $\mu B\bar{t} = B(\mu B)\bar{t}$.
- If B is *purely positive*, then $B \equiv\ !\,B$. In MALL: no interesting such formulas. In μMALL: a rich collection of such formulas.

Equality as a logical connective

When t and s are not unifiable:

$$\frac{}{\mathcal{X}\,;\,\Gamma, t = s \vdash \Delta}$$

Here, $\mathcal{X}$ is the set of eigenvariables. Otherwise, set $\theta = \mathrm{mgu}(t, s)$:

$$\frac{\theta\mathcal{X}\,;\,\theta\Gamma \vdash \theta\Delta}{\mathcal{X}\,;\,\Gamma, t = s \vdash \Delta}$$

Here, $\theta\mathcal{X}$ is the result of removing from $\mathcal{X}$ variables in the domain of θ and then adding the variables free in the codomain of θ.

This treatment of equality was developed independently by Schroeder-Heister and Girard in [1991/92].

Unification is a black box attached to sequent calculus. A failure (of unification) can be turn into a success.

Proving the subset relation for two finite sets

Abbreviate z, $(s\ z)$, $(s\ (s\ z))$, $(s\ (s\ (s\ z)))$, etc by $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, etc.

Let the sets $A = \{\mathbf{0}, \mathbf{1}\}$ and $B = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ be encoded as

$$\lambda x.\ x = \mathbf{0} \lor x = \mathbf{1} \quad \text{and} \quad \lambda x.\ x = \mathbf{0} \lor x = \mathbf{1} \lor x = \mathbf{2}.$$

To prove that A is a subset of B requires proving the formula $\forall x. Ax \supset Bx$ is provable.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot\,;\,\cdot \vdash \mathbf{0} = \mathbf{0}}}{\cdot\,;\,\cdot \vdash \mathbf{0} = \mathbf{0} \lor \mathbf{0} = \mathbf{1} \lor \mathbf{0} = \mathbf{2}}}{x;\ x = \mathbf{0} \vdash x = \mathbf{0} \lor x = \mathbf{1} \lor x = \mathbf{2}} \quad \dfrac{\dfrac{\dfrac{}{\cdot\,;\,\cdot \vdash \mathbf{1} = \mathbf{1}}}{\cdot\,;\,\cdot \vdash \mathbf{1} = \mathbf{0} \lor \mathbf{1} = \mathbf{1} \lor \mathbf{1} = \mathbf{2}}}{x;\ x = \mathbf{1} \vdash x = \mathbf{0} \lor x = \mathbf{1} \lor x = \mathbf{2}}}{\dfrac{x\,;\ x = \mathbf{0} \lor x = \mathbf{1} \vdash x = \mathbf{0} \lor x = \mathbf{1} \lor x = \mathbf{2}}{\cdot\,;\,\cdot \vdash \forall x.(x = \mathbf{0} \lor x = \mathbf{1}) \supset (x = \mathbf{0} \lor x = \mathbf{1} \lor x = \mathbf{2})}}$$

Exercise: Prove $\neg\forall x. Bx \supset Ax$.

Fixed points

The least fixed point μ is a series of operators indexed by their arity. We leave this arity implicit. Unfolding $\mu B t_1 \ldots t_n$ yields $B(\mu B) t_1 \ldots t_n$. Also, μ has positive bias.

$$\dfrac{\Gamma \vdash B(\mu B)\bar{t}, \Delta}{\Gamma \vdash \mu B \bar{t}, \Delta}\ \mu R \qquad \dfrac{\Gamma, B(\mu B)\bar{t} \vdash \Delta}{\Gamma, \mu B \bar{t} \vdash \Delta}\ \mu L$$

The induction rule scheme (S is a higher-order variable).

$$\dfrac{\Gamma, S\bar{t} \vdash \Delta \qquad BS\bar{x} \vdash S\bar{x}}{\Gamma, \mu B \bar{t} \vdash \Delta}\ \mathit{Ind}$$

The rule for μL rule is admissible given the *Ind* rule.

Baelde [ToCL 2012] proved that μMALL satisfies cut-elimination and has a focused proof system μMALLF.

We set aside the induction rule (*Ind*) until the very end.

Examples of fixed point definitions

As a Horn clause theory

```
nat z.
nat (s X) :- nat X.
plus z X X.
plus (s X) Y (s Z) :- plus X Y Z.
```

These can be seen as definitions in the Hallnäs & Schroeder-Heister sense. However, we convert them into the following μ-expressions.

As fixed point definitions

$$nat = \mu\lambda N\lambda n(n = \mathbf{0} \oplus \exists n'(n = s\ n' \otimes N\ n'))$$

$$\begin{aligned} plus = &\mu\lambda P\lambda n\lambda m\lambda p.(n = \mathbf{0} \otimes m = p) \oplus \\ &\exists n'\exists p'(n = s\ n' \otimes p = s\ p' \otimes P\ n'\ m\ p') \end{aligned}$$

Note that μ and $=$ are positive, as are $\otimes$, $\oplus$, and $\exists$. These are *purely positive* expressions.

Example: computing during the invertible phase

Consider searching for a proof of $\Gamma, plus\ \mathbf{2}\ \mathbf{3}\ x \vdash (Q\ x)$.
Using μL yields

$$\Gamma, ((\mathbf{2} = \mathbf{0} \otimes \mathbf{3} = x) \oplus \exists n'\exists x'(\mathbf{2} = s\ n' \otimes x = s\ x' \otimes plus\ n'\ \mathbf{3}\ x')) \vdash (Q\ x).$$

The disjunction introduction rule yields two premises:
(1) $\Gamma, (\mathbf{2} = \mathbf{0} \otimes \mathbf{3} = x) \vdash (Q\ x)$ is proved immediately.
(2)

$$\dfrac{\dfrac{\Gamma, plus\ \mathbf{1}\ \mathbf{3}\ x' \vdash (Q\ (s\ x'))}{\Gamma, (\mathbf{2} = s\ n' \otimes x = s\ x' \otimes plus\ n'\ \mathbf{3}\ x') \vdash (Q\ x)}}{\Gamma, (\exists n'\exists x'(\mathbf{2} = s\ n' \otimes x = s\ x' \otimes plus\ n'\ \mathbf{3}\ x')) \vdash (Q\ x)}$$

The invertible phase terminates with the premise

$$\Gamma \vdash (Q\ \mathbf{5})$$

Abstracting away the invertible phase, we obtain the following synthetic rule:

$$\frac{\vdash Q(\mathbf{5})}{plus\ \mathbf{2}\ \mathbf{3}\ x \vdash Q(x)}$$

The polarity ambiguity of singleton sets

Let P be a predicate of one argument such that

$$\vdash (\exists x.P(x)) \wedge (\forall x \forall y.P(x) \supset P(y) \supset x = y)$$

Thus, $\exists x.P(x) \otimes Q(x) \equiv \forall x.P(x) \multimap Q(x) \equiv Q(\iota P)$.

Assume that P is a purely positive formula.

A proof of $\Gamma \vdash \exists x.(P(x) \otimes Q(x)) \Downarrow$ *guesses* a term t and then proves $\Gamma \vdash P(t) \Downarrow$ and $\Gamma \vdash Q(t) \Downarrow$.

A proof of $\Gamma \vdash \forall x.P(x) \multimap Q(x)$ *computes* the value that satisfies P, starting with proving $\Gamma, P(y) \vdash Q(y)$. The completed phase has the premise $\Gamma \vdash Q(t)$.

When relations denote functions, we have singletons

For example, the predicate (*plus* **2** **3**) denotes the singleton set containing only **5**.

Thus, unlike Church and Hilbert who used choice operators (ϵ, ι) to convert some predicates to functions, proof search during the invertible fashion computes functions.

For more, see [Gérard & M, CSL 2017].

More examples: paths in graph

Horn clauses (Prolog) can be encoded as purely positive fixed point expressions. For example, for specifying a (tiny) graph and its transitive closure:

```
step a b.   step b c.   step c b.
path X Z :- step X Z.
path X Z :- step X Y, path Y Z.
```

Write the `step` as the least fixed point expression

$$\mu(\lambda A \lambda x \lambda y.\, (x = a \otimes y = b) \oplus (x = b \otimes y = c) \oplus (x = c \otimes y = b))$$

Likewise, `path` can be encoded as the relation $path(\cdot, \cdot)$:

$$\mu(\lambda A \lambda x \lambda z.\, \texttt{step}\ x\ z \oplus (\exists y.\, \texttt{step}\ x\ y \otimes A\, y\, z)).$$

These expressions use only positive connectives and no non-logical predicates.

Examples: reachability

There is no proof that there is a step from a to c.

$$\dfrac{\dfrac{\mathit{fail}}{\vdash (a = a \wedge^+ c = b) \vee (a = b \wedge^+ c = c) \vee (a = c \wedge^+ c = b)}}{\vdash \mathtt{step}\ a\ c}$$

There is a proof that there is a path from a to c.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathtt{step}\ a\ b \qquad \vdash \mathit{path}\, b\, c}{\vdash \mathtt{step}\ a\ b \wedge^+ \mathit{path}\, b\, c}}{\vdash \exists y.\, \mathtt{step}\ a\ y \wedge^+ \mathit{path}\, y\, c}}{\vdash \mathtt{step}\ a\ c \vee (\exists y.\, \mathtt{step}\ a\ y \wedge^+ \mathit{path}\, y\, c)}}{\vdash \mathit{path}(a, c)}$$

Examples: reachability (con't)

Below is a proof that the node a is not adjacent to c.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a = a, c = b \vdash \cdot}}{a = a \wedge^+ c = b \vdash \cdot} \quad \dfrac{\dfrac{}{a = b, c = c \vdash \cdot}}{a = b \wedge^+ c = c \vdash \cdot} \quad \dfrac{\dfrac{}{a = c, c = b \vdash \cdot}}{a = c \wedge^+ c = b \vdash \cdot}}{(a = a \wedge^+ c = b) \vee (a = b \wedge^+ c = c) \vee (a = c \wedge^+ c = b) \vdash \cdot}}{\mathtt{step}\ a\ c \vdash \cdot}$$

In general, proofs by negation-as-finite-failure yield sequent calculus proofs in this setting. (Hallnäs & S-H, 1990)

Example: simulation

Let $P \xrightarrow{A} Q$ be a labeled transition system between processes and actions. Assume it is defined as a purely positive expression.

Let ν be the de Morgan dual of μ. Since we are only unfolding fixed points, μ and ν are extensionally the same although the polarity of ν is negative.

The following expressions denote simulation and bisimulation for this label transition systems.

$$\nu\big(\lambda S \lambda p \lambda q.\, \forall a \forall p'.\, p \xrightarrow{a} p' \multimap \exists q'.\, q \xrightarrow{a} q' \otimes S\, p'\, q'\big)$$

$$\begin{aligned} \nu(\lambda B \lambda p \lambda q.\quad & (\forall a \forall p'.\, p \xrightarrow{a} p' \multimap \exists q'.\, q \xrightarrow{a} q' \otimes B\, p'\, q') \\ & \&\ (\forall a \forall q'.\, q \xrightarrow{a} q' \multimap \exists p'.\, p \xrightarrow{a} p' \otimes B\, q'\, p')) \end{aligned}$$

Note that bisimulation has both conjunctions. Stirling's games for bisimulation [1996] are directly encoded in these focused proofs.

An example of a synthetic inference rules

$$\dfrac{\dfrac{\cdots \quad \dfrac{\dfrac{\dfrac{\vdash sim(p_i, q_i)}{\vdash sim(p_i, q_i) \Downarrow}}{\vdash \exists Q'.q_0 \xrightarrow{a_i} Q' \otimes sim(p_i, Q') \Downarrow}\ C}{\vdash \exists Q'.q_0 \xrightarrow{a_i} Q' \otimes sim(p_i, Q')} \quad \cdots}{P', A : p_0 \xrightarrow{A} P' \vdash \exists Q'.q_0 \xrightarrow{A} Q' \otimes sim(P', Q')}\ B}{\vdash sim(p_0, q_0)}\ A$$

A contain introduction rules for $\forall$ and $\multimap$.

B consists of left invertible rules which generate all a_i and p_i such that $p_0 \xrightarrow{a_i} p_i$.

C is a sequence of $\Downarrow$ rules that proves that $q_0 \xrightarrow{a_i} q_i$.

Finally, the top-most inference rule is a release rule.

A proof theory for model checking

μMALL can provide a proof theory for model checking.

See [Heath & M, J. Automated Reasoning 2018].

Focusing can be used to design proof certificates for some common model checking problems.

- A path in a graph can be proof certificate for *reachability*.
- Connected components can be a proof certificate for *non-reachability*.
- A bisimulation can be a proof certificate for bisimilarity.
- A Hennessy-Milner modal formula can be a proof certificate for *non-bisimilarity*.

Next steps

Turing machines are easy to code in (pure) Prolog. Thus, we can defining predicates as purely positive expression which capture general notions of computability.
The next challenges involve the induction scheme.

- What predicates can be proved total?
- Relate the arithmetic hierarchy (involving quantifier alternations) to focusing polarity.
- Design μLJ and μLJF and prove cut-elimination and completeness of focusing (mostly done).
- Design μLK and μLKF and establish cut-elimination and completeness (maybe impossible). In the most natural settings, completeness of focusing for μLKF would provide a simple method for extracting computational content of classical proofs of Π_2^0 formulas (something we do not expect).

Going Without: A Linear Modality and its Role

Valeria de Paiva
Proof Theoretic Semantics 3, Tübingen

Mar, 2019

Thank you Peter Schroeder-Heister and Thomas Piecha for the invitation!

Thanks to Samsung Research America for letting me come!

Thank you Luiz Carlos and Dag for all the teaching!

Introduction

I'm a logician, a proof-theorist and a category theorist.
I work in industry, have done so for the last 20 years, applying the purest of pure mathematics, in surprising ways.
Today I want to talk about

- Categorical Proof Theory
- Linear Type Theories
- A new application, perhaps...

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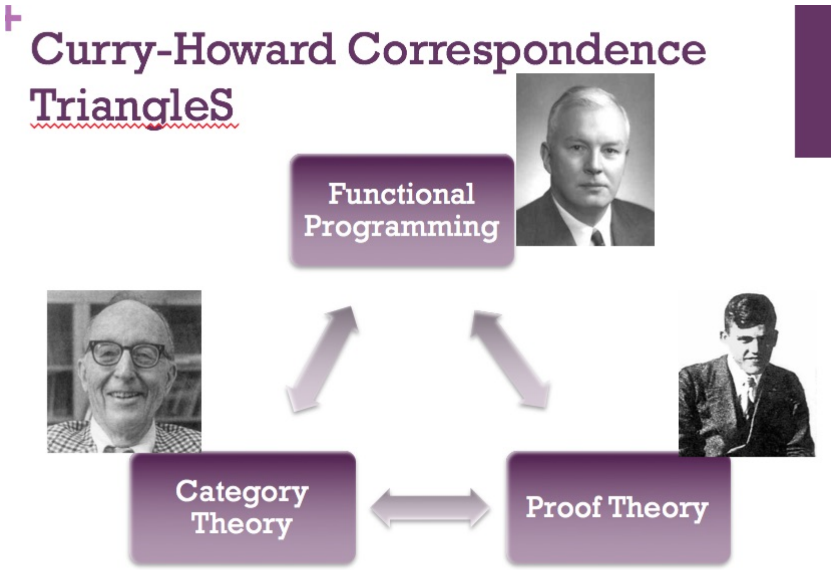
Introduction

This is (old) joint work with Milly Maietti, Paola Maneggia and Eike Ritter

Relating Categorical Semantics for Intuitionistic Linear Logic (Applied Categorical Structures, volume 13(1):1–36, 2005)
Categorical Models for Linear and Intuitionistic Type Theory (FOSSACS, LNCS Springer, vol 1784, 2000).

Curry-Howard Enablers



Proof Theory using Categories...

Category: a collection of objects and of morphisms, satisfying obvious composition laws
Functors: the natural notion of morphism between categories
Natural transformations: the natural notion of morphisms between functors

Constructors: products, sums, limits, duals....
Adjunctions: an abstract version of equality

How does this relate to logic?
Where are the theorems?

A long time coming:
Schoenfinkel (1920s), Curry and Feys (1958), Howard (1969, published in 1980), etc.

Curry-Howard for Implication

Natural deduction rules for implication (without λ-terms)

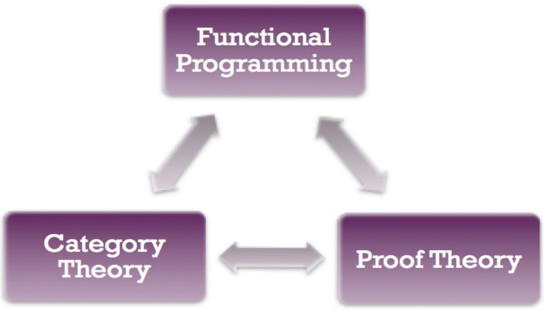
$$\frac{A \to B \quad A}{B} \qquad \frac{\begin{array}{c}[A]\\ \vdots\, \pi \\ B\end{array}}{A \to B}$$

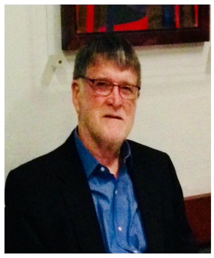
Natural deduction rules for implication (with λ-terms)

$$\frac{M\colon A \to B \quad N\colon A}{M(N)\colon B} \qquad \frac{\begin{array}{c}[x\colon A]\\ \vdots\, \pi \\ M\colon B\end{array}}{\lambda x.M\colon A \to B}$$

function application abstraction

Categorical Proofs are Programs





Types are formulae/objects in **appropriate** category,
Terms/programs are proofs/morphisms in the **appropriate** category,
Logical constructors are **appropriate** categorical constructions.
Most important: Reduction is proof normalization (Tait)
Outcome: Transfer results/tools from logic to CT to CSci, etc

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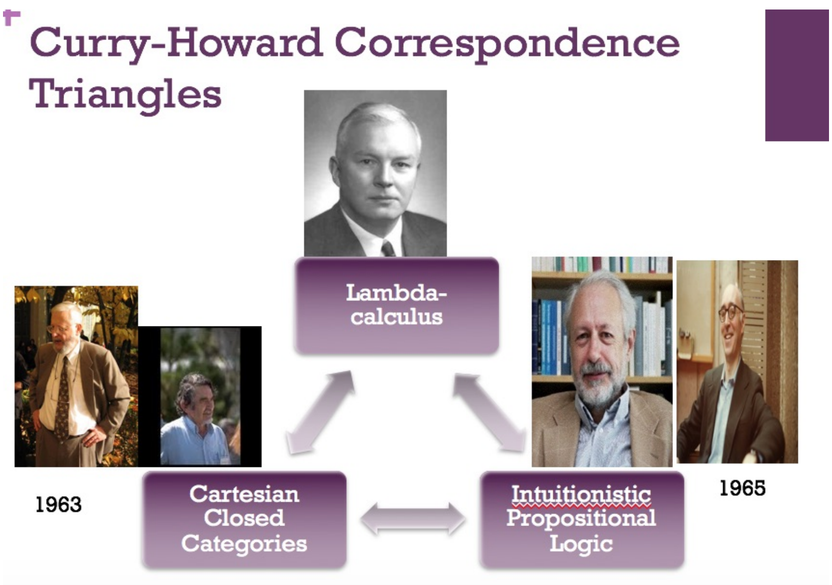

Categorical Proof Theory

- Model derivations/proofs, not whether theorems are true or not
- Proofs definitely first-class citizens
- How? Uses extended Curry-Howard correspondence
- Why is it good? Modeling derivations useful where you need proofs themselves, in linguistics, functional programming, compilers, etc.
- Why is it important? Widespread use of logic/algebra in CS means new important problems to solve with our favorite tools.

Why so little impact on maths, CS or logic?

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How many Curry-Howard Correspondences?

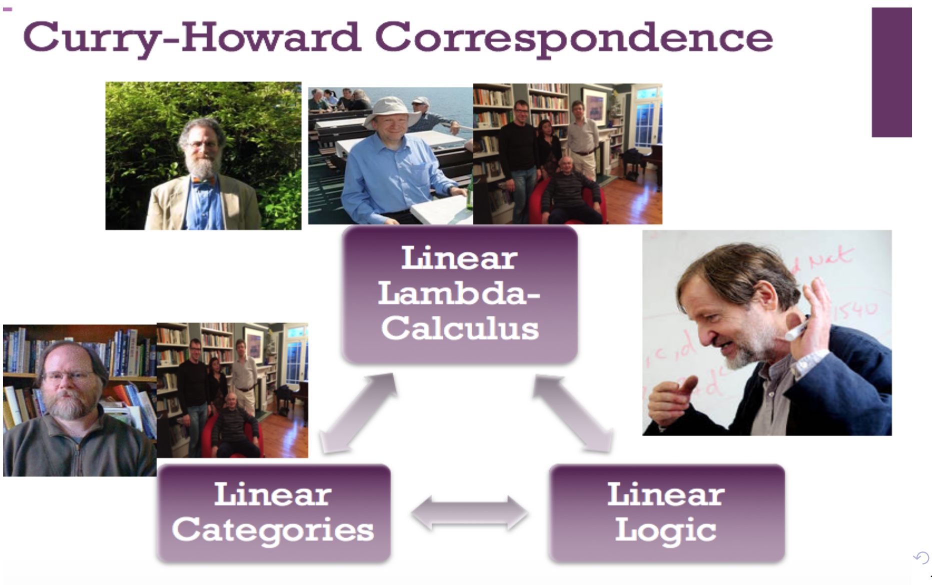
Many!!!

Intuitionistic Propositional Logic, System F, Dependent Type Theories (Martin-Löf, Calc of Constructions,...), Linear Logic, Constructive Modal Logics, various versions of Classical Logic (since the early 90's)

The programs corresponding to these logical systems are 'futuristic' programs.

The logics inform the design of new type systems, that can be used in new applications.

Linear Type Theory



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Linear Logic: a tool for semantics

- A proof theoretic logic described by Girard in 1986.
- Basic idea: assumptions cannot be discarded or duplicated. They must be used exactly once – just like dollar bills (except when they're marked by a modality !)
- Other approaches to accounting for logical resources before. **Relevance Logic!**
- Great win of Linear Logic: Account for resources when you want to, otherwise fall back to traditional logic via translation $A \to B$ iff $!A \multimap B$

Linear Implication and (Multiplicative) Conjunction

Traditional implication: $A, A \rightarrow B \vdash B$
$A, A \rightarrow B \vdash A \wedge B$ Re-use A

Linear implication: $A, A \multimap B \vdash B$
$A, A \multimap B \nvdash A \otimes B$ Cannot re-use A

Traditional conjunction: $A \wedge B \vdash A$ Discard B

Linear conjunction: $A \otimes B \nvdash A$ Cannot discard B

Of course: $!A \vdash !A \otimes !A$ Re-use
$!(A) \otimes B \vdash B$ Discard

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The challenges of modeling Linear Logic

Traditional modeling of intuitionistic logic:

- formula $A \rightsquigarrow$ object A of appropriate category
- $A \wedge B \rightsquigarrow A \times B$ (real product)
- $A \rightarrow B \rightsquigarrow B^A$ (set of functions from A to B) and
- they relate via adjunction
  $A \wedge B \rightarrow C \iff A \rightarrow Hom(B, C) = C^B$
- These are real products, have projections ($A \times B \rightarrow A$) and diagonals ($A \rightarrow A \times A$) corresponding to deletion and duplication of resources.

Not linear!!!
Need to use *tensor products* and *internal homs*.
Hard to define the "make-everything-usual" operator "!".

Varieties of Linear Logic and Type Theory

- Girard and Lafont Combinators (1988)
- Abramsky (1993) and Mackie Lilac (JFP 1994)
- Wadler's There's no substitute for linear logic (1991)
- Benton et al Linear Lambda Calculus (1992)
- Barber's DILL (1996)
- Benton LNL (Linear-Non-Linear Calculus 1995)
- Mints (1998), Roversi and Della Rocca (1994), Lincoln and Mitchell (1992), Negri(2002), Pfenning (2001), Martini and Masini (1995), etc.

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Linear Type Theories with Categorical Models

- Linear λ-calculus (Benton, Bierman, de Paiva, Hyland, 1992) https://www.cl.cam.ac.uk/techreports/UCAM-CL-TR-262.html
- DILL system (Plotkin and Barber, 1996) http://www.lfcs.inf.ed.ac.uk/reports/96/ECS-LFCS-96-347/
- LNL system (Benton, 1995) `https://www.researchgate.net/publication/221558077_A_Mixed_Linear_and_Non-Linear_Logic_Proofs_Terms_and_Models`

Linear λ-calculus

$$x : A \vdash x : A$$

$$\frac{\Gamma \vdash e : A \qquad \Delta, x : A \vdash f : B}{\Gamma, \Delta \vdash f[e/x] : B}\ \mathit{Cut}$$

$$\frac{\Gamma \vdash e : A \qquad \Delta, x : B \vdash f : C}{\Gamma, g : A \multimap B, \Delta \vdash f[(ge)/x] : C}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x.e : A \multimap B}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma \vdash e : A}{\Gamma, x : I \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e : A}\ (I_{\mathcal{L}}) \qquad \frac{}{\vdash * : I}\ (I_{\mathcal{R}})$$

$$\frac{\Delta, x : A, y : B \vdash f : C}{\Delta, z : A \otimes B \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f : C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \vdash e : A \qquad \Delta \vdash f : B}{\Gamma, \Delta \vdash e \otimes f : A \otimes B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \vdash e : B}{\Gamma, z : !A \vdash \mathsf{discard}\ z\ \mathsf{in}\ e : B}\ \mathit{Weakening} \qquad \frac{\Gamma, x : !A, y : !A \vdash e : B}{\Gamma, z : !A \vdash \mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ e : B}\ \mathit{Contraction}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma, z : !A \vdash e[\mathsf{derelict}(z)/x] : B}\ \mathit{Dereliction}$$

$$\frac{\overline{x} : !\Gamma \vdash e : A}{\overline{y} : !\Gamma \vdash \mathsf{promote}\ \overline{y}\ \mathsf{for}\ \overline{x}\ \mathsf{in}\ e : !A}\ \mathit{Promotion}$$

DILL

$$(\mathit{Int}-\mathit{Ax})\ \Gamma, x : A; \_ \vdash x : A \qquad (\mathit{Lin}-\mathit{Ax})\ \Gamma; x : A \vdash x : A$$

$$(\mathit{Graph})\ \frac{\Gamma; \Delta \vdash t : A}{\Gamma; \_ \vdash f(t) : B}\ (f \in G(A, B))$$

$$(I-I)\ \Gamma; \_ \vdash * : I \qquad (I-E)\ \frac{\Gamma; \Delta_1 \vdash t : I \quad \Gamma; \Delta_2 \vdash u : A}{\Gamma; \Delta_1, \Delta_2 \vdash \mathsf{let}\ *\ \mathsf{be}\ t\ \mathsf{in}\ u : A}$$

$$(\otimes - I)\ \frac{\Gamma; \Delta_1 \vdash t : A \quad \Gamma; \Delta_2 \vdash u : B}{\Gamma; \Delta_1, \Delta_2 \vdash t \otimes u : A \otimes B} \qquad (\otimes - E)\ \frac{\Gamma; \Delta_1 \vdash u : A \otimes B \quad \Gamma; \Delta_2, x : A, y : B \vdash t : C}{\Gamma; \Delta_1, \Delta_2 \vdash \mathsf{let}\ x \otimes y : A \otimes B\ \mathsf{be}\ u\ \mathsf{in}\ t : C}$$

$$(\multimap I)\ \frac{\Gamma; \Delta, x : A \vdash t : B}{\Gamma; \Delta \vdash (\lambda x : A.t) : (A \multimap B)} \qquad (\multimap E)\ \frac{\Gamma; \Delta_1 \vdash u : A \multimap B \quad \Gamma; \Delta_2 \vdash t : A}{\Gamma; \Delta_1, \Delta_2 \vdash (ut) : B}$$

$$(! - I)\ \frac{\Gamma; \_ \vdash t : A}{\Gamma; \_ \vdash !t : !A} \qquad (! - E)\ \frac{\Gamma; \Delta_1 \vdash u : !A \quad \Gamma, x : A; \Delta_2 \vdash t : B}{\Gamma; \Delta_1, \Delta_2 \vdash \mathsf{let}\ !x : A\ \mathsf{be}\ u\ \mathsf{in}\ t : B}$$

LNL

$$\Theta; a{:}\, A \vdash_{\mathcal{L}} a{:}\, A \qquad \Theta, x{:}\, X \vdash_{\mathcal{C}} x{:}\, X$$

$$\frac{\Theta \vdash_{\mathcal{C}} s{:}\, X \qquad \Theta \vdash_{\mathcal{C}} t{:}\, Y}{\Theta \vdash_{\mathcal{C}} (s, t){:}\, X \times Y} \qquad \frac{}{\Theta \vdash_{\mathcal{C}} (){:}\, 1}$$

$$\frac{\Theta \vdash_{\mathcal{C}} s{:}\, X \times Y}{\Theta \vdash_{\mathcal{C}} \mathsf{fst}(s){:}\, X} \qquad \frac{\Theta \vdash_{\mathcal{C}} s{:}\, X \times Y}{\Theta \vdash_{\mathcal{C}} \mathsf{snd}(s){:}\, Y}$$

$$\frac{\Theta; \Gamma \vdash_{\mathcal{L}} e{:}\, A \qquad \Theta; \Delta \vdash_{\mathcal{L}} f{:}\, B}{\Theta; \Gamma, \Delta \vdash_{\mathcal{L}} e \otimes f{:}\, A \otimes B} \qquad \frac{\Theta; \Gamma \vdash_{\mathcal{L}} e{:}\, A \otimes B \qquad \Theta; \Delta, a{:}\, A, b{:}\, B \vdash_{\mathcal{L}} f{:}\, C}{\Theta; \Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\ a \otimes b = e\ \mathsf{in}\ f{:}\, C}$$

$$\frac{}{\Theta \vdash_{\mathcal{L}} *{:}\, I} \qquad \frac{\Theta; \Gamma \vdash_{\mathcal{L}} e{:}\, I \qquad \Theta; \Delta \vdash_{\mathcal{L}} f{:}\, A}{\Theta; \Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\ * = e\ \mathsf{in}\ f{:}\, A}$$

$$\frac{\Theta, x{:}\, X \vdash_{\mathcal{C}} s{:}\, Y}{\Theta \vdash_{\mathcal{C}} (\lambda x{:}\, X.s){:}\, X \to Y} \qquad \frac{\Theta \vdash_{\mathcal{C}} s{:}\, X \to Y \qquad \Theta \vdash_{\mathcal{C}} t{:}\, X}{\Theta \vdash_{\mathcal{C}} s\, t{:}\, Y}$$

$$\frac{\Theta; \Gamma, a{:}\, A \vdash_{\mathcal{L}} e{:}\, B}{\Theta; \Gamma \vdash_{\mathcal{L}} (\lambda a{:}\, A.e){:}\, A \multimap B} \qquad \frac{\Theta; \Gamma \vdash_{\mathcal{L}} e{:}\, A \multimap B \qquad \Theta; \Delta \vdash_{\mathcal{L}} f{:}\, A}{\Theta; \Gamma, \Delta \vdash_{\mathcal{L}} e\, f{:}\, B}$$

$$\frac{\Theta \vdash_{\mathcal{C}} s{:}\, X}{\Theta \vdash_{\mathcal{L}} \mathsf{F}(s){:}\, FX} \qquad \frac{\Theta; \Gamma \vdash_{\mathcal{L}} e{:}\, FX \qquad \Theta, x{:}\, X; \Delta \vdash_{\mathcal{L}} f{:}\, A}{\Theta; \Gamma, \Delta \vdash_{\mathcal{L}} \mathsf{let}\ \mathsf{F}(x) = e\ \mathsf{in}\ f{:}\, A}$$

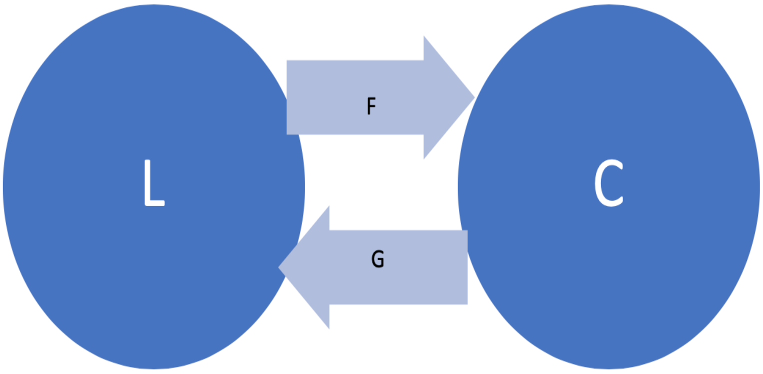
$$\frac{\Theta \vdash_{\mathcal{L}} e{:}\, A}{\Theta \vdash_{\mathcal{C}} \mathsf{G}(e){:}\, GA} \qquad \frac{\Theta \vdash_{\mathcal{C}} s{:}\, GA}{\Theta \vdash_{\mathcal{L}} \mathsf{derelict}(s){:}\, A}$$

Categorical Models

- Linear λ-calculus (ILL)
  A linear category is a symmetric monoidal closed (smc) category equipped with a linear monoidal comonad, such that the co-Kleisli category of the comonad is a Cartesian closed Category (CCC). (loads of conditions)
- DILL system
  A DILL-category is a pair, a symmetric monoidal closed category and a cartesian category, related by a monoidal adjunction.
- LNL system
  An LNL-model is a (symmetric) monoidal adjunction between a smcc and a ccc.

In pictures

Monoidal Adjunctions:
L is smcc, C is ccc, F-|G, both monoidal



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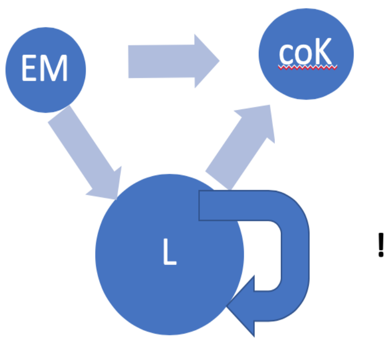
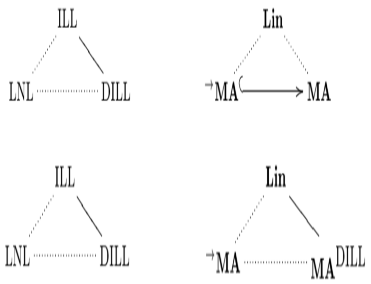
In pictures

Linear Categories
L smcc, !=linear exponential comonad



Linear Curry-Howard Isos

"Relating Categorical Semantics for Intuitionistic Linear Logic" (with P. Maneggia, M. Maietti and E. Ritter), Applied Categorical Structures, vol 13(1):1–36, 2005.



Take home: Categorical models need to be more than sound and complete. They need to provide *internal languages* for the theories they model.

Why a new calculus?

(ILT, Fossacs 2000)

Choosing between the three type theories:

- DILL is best, less verbose than ILL, but closer to what we want to do than LNL.
- Easy formulation of the promotion rule
- Contains the usual lambda-calculus as a subsystem
- however, most FPers would prefer to use a variant of DILL where instead of !, one has two function spaces, $\to$ and $\multimap$
- But then what's the categorical model?

Intuitionistic and Linear Type Theory

(ILT, Fossacs 2000)

If we have two function spaces, but no modality !, how can we model it? All the models discussed before have a notion of !, created by the adjunction.
Well, we need to use deeper mathematics, i.e. **fibrations** or indexed categories.

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Have calculus ILT
Categorical models for intuitionistic and linear type theory (Maietti et al, 2000)

$$\Gamma \mid a : A \vdash a : A \qquad\qquad \Gamma, x : A \mid \_ \vdash x : A$$

$$\frac{\Gamma \mid \Delta, a : A \vdash M : B}{\Gamma \mid \Delta \vdash \lambda a^A.M : A \multimap B} \qquad \frac{\Gamma, x : A \mid \Delta \vdash M : B}{\Gamma \mid \Delta \vdash \lambda x^A.M : A \to B}$$

$$\frac{\Gamma \mid \Delta_1 \vdash M : A \multimap B \quad \Gamma \mid \Delta_2 \vdash N : A}{\Gamma \mid \Delta \vdash M_l N : B} \qquad \frac{\Gamma \mid \Delta \vdash M : A \to B \quad \Gamma \mid \_ \vdash N : A}{\Gamma \mid \Delta \vdash M_i N : B}$$

Have ILT MODELS (Maietti et al, 2000)

Idea goes back to Lawvere's hyperdoctrines satisfying the comprehension axiom.
Model of ILT should modify this setting to capture the separation between intuitionistic and linear variables.
A base category B, which models the intuitionistic contexts of ILT, objects in B model contexts ($\Gamma|\_$).
Each fibre over an object in B modelling a context models terms $\Gamma|\Delta \vdash M : A$ for any context Δ
The fibres are now symmetric monoidal closed categories with finite products and model the linear constructions of ILT

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Have ILT MODELS and have internal language full theorems

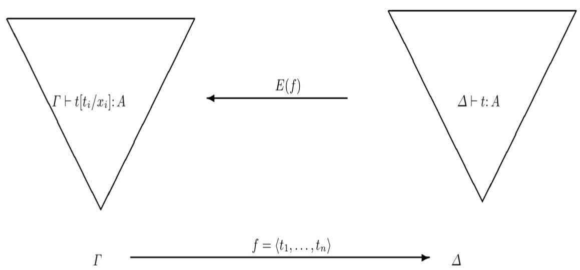


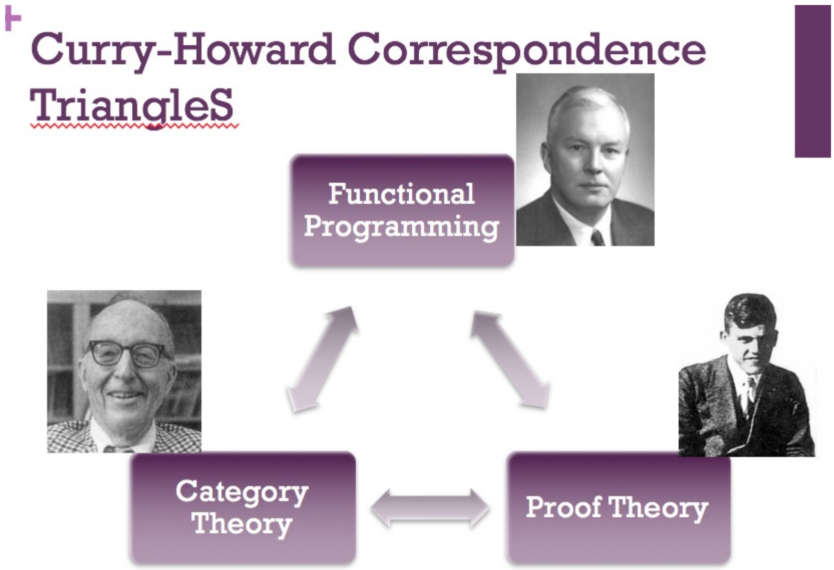
Fig. 1. Modelling the simply-typed λ-calculus in a D-category

Missing an understanding of what is essential, what can be changed, and limitations of the methods.

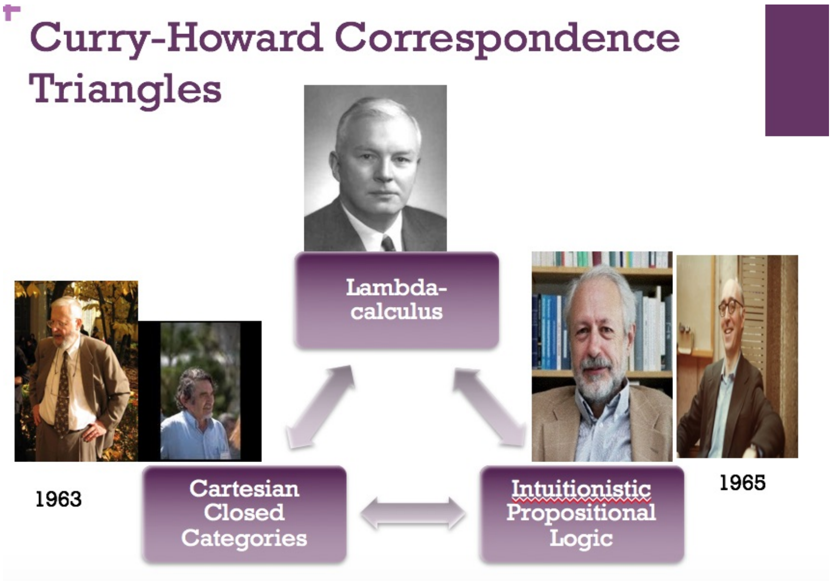
Speculation

Can we do the same for Modal Logic S4?
Can we do the same for Relevant Logic?
I hope so. But have not managed yet.
Meanwhile pictures do help to remember the main messages.

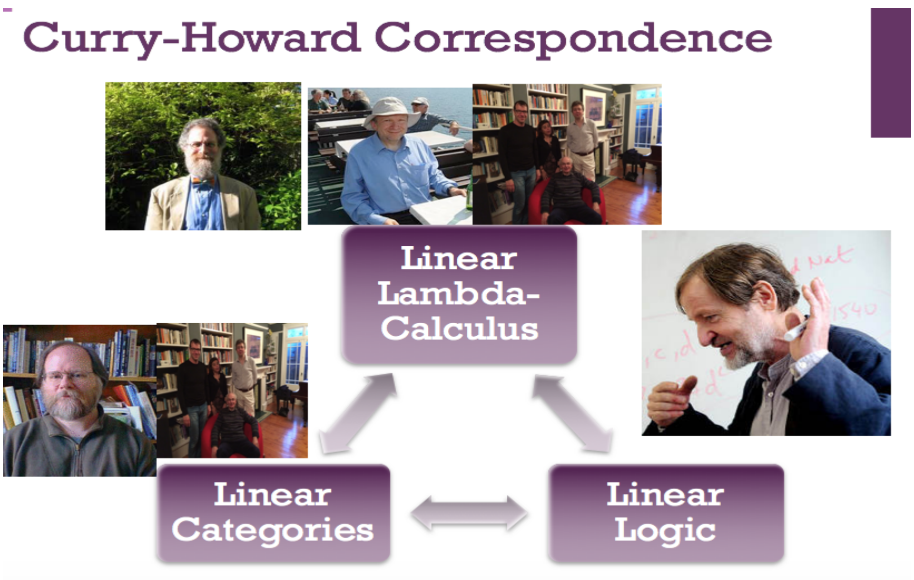
Curry-Howard Enablers

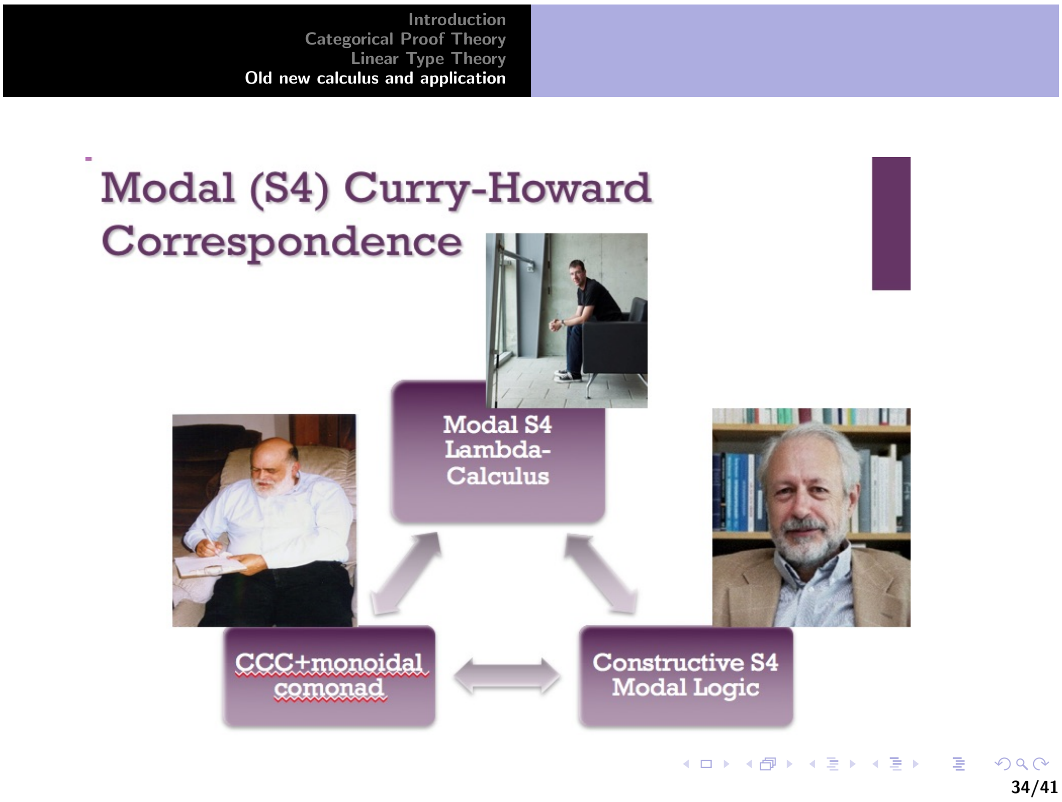


Intuitionistic Type Theory



Linear Type Theory

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CCC+monoidal comonad
Constructive S4 Modal Logic
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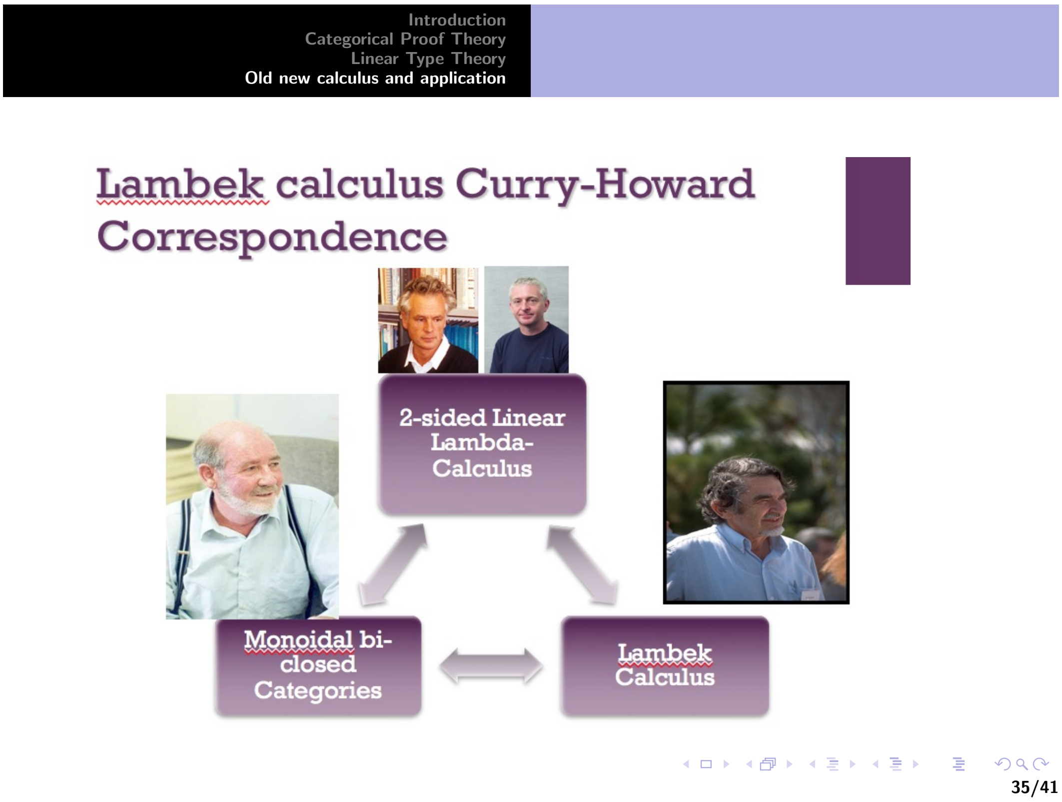
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2-sided Linear Lambda-Calculus
Monoidal bi-closed Categories
Lambek Calculus
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Thank you!

Explaining Internal Languages

Thanks Milly Maietti for slides!

for a given calculus $\mathcal{T}$ and a class of complete models

- **Organize models of $\mathcal{T}$ into a categories $\mathrm{Mod}(\mathcal{T})$**
  (find suitable model morphisms)
- **Organize theories of $\mathcal{T}$ into a category $\mathrm{Th}(\mathcal{T})$**
  (find suitable theory morphisms)
- **Check validity of Internal language theorem:**

  $\mathrm{Mod}(\mathcal{T})$ is equivalent to $\mathrm{Th}(\mathcal{T})$

  if so, the category of best models of $\mathcal{T}$ are determined up to equivalences

Explaining Internal Languages

Thanks Milly Maietti for slides!

Idea of an internal language theorem

- Define internal language out of a model of τ :

$$Int: \mathrm{Mod}(\tau) \to \mathrm{Th}(\tau)$$

- Define syntactic category out of a theory of τ :

$$Syn: \mathrm{Th}(\tau) \to \mathrm{Mod}(\tau)$$

- such that

for any model M $\quad\quad$ for any theory T

$$\mathsf{M} \simeq Syn(Int(\mathsf{M})) \quad \& \quad \mathsf{T} \simeq Int(Syn(\mathsf{T}))$$

$\simeq$ is isomorphism
(works for simple type theories)

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Explaining Internal Languages

Thanks Milly Maietti for slides!

- Define internal language out of a model of τ :

$$Int: \mathrm{Mod}(\tau) \to \mathrm{Th}(\tau)$$

- Define syntactic category out of a theory of τ :

$$Syn: \mathrm{Th}(\tau) \to \mathrm{Mod}(\tau)$$

- such that

for any model M $\quad\quad$ for any theory T

$$\mathsf{M} \simeq Syn(Int(\mathsf{M})) \quad \& \quad \mathsf{T} \simeq Int(Syn(\mathsf{T}))$$

$\simeq$ is equivalence of categories
(only this works for dependent type theories)

Explaining Internal Languages

Thanks Milly Maietti for slides!

Soundness and completeness may just give

- an internal language functor:

$$Int\colon \mathrm{Mod}(\tau) \to \mathrm{Th}(\tau)$$

- a syntactic category functor:

$$Syn\colon \mathrm{Th}(\tau) \to \mathrm{Mod}(\tau)$$

- sometimes a monic natural trasformation for any theory T

$$\mathsf{T} \hookrightarrow Int(Syn(\mathsf{T}))$$

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Explaining Internal Languages

Thanks Milly Maietti for slides!

various formulations for Intuitionistic Linear Logic

- Benton-Bierman-de Paiva-Hyland '93 ILL type calculus : based on usual sequents $\Gamma \vdash A$
- Barber-Plotkin '97 Dual Intuitionistic Linear Logic (DILL) : based on double-context sequents $\Gamma \mid \Delta \vdash A$ such that

Γ has intuitionistic assumptions
Δ has linear assumptions

$$\Gamma, B \mid \Delta \vdash A \quad \text{iff} \quad \Gamma \mid !B, \Delta \vdash A$$

ILL is not equivalent to DILL via a bijective translation of proofs:
ILL-sequents correspond only to DILL-sequents of the form $\_ \mid \Delta \vdash A$

but

$$\mathrm{Th}(DILL) \simeq \mathrm{Th}(ILL)$$

Proof-theoretic semantics and paradoxical languages

Mattia Petrolo

Federal University of ABC

3rd Tübingen Conference on Proof-theoretic semantics

27-30 March 2019

Joint work with Paolo Pistone

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Outline

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Outline

1. Two approaches to proof-theoretic validity

2. Two analysis of paradoxality

3. Back to proof-theoretic semantics

Introduction rules as basic

- One knows the meaning of a sentence ϕ when one knows what it is to *verify* the sentence;
- The introduction rules for a connective $\circ$ define what it is to *canonically verify* a sentence with $\circ$ as the principal connective;
- A canonical argument is an argument terminating with an introduction rule.
- An argument from Γ to ϕ is *valid* iff any canonical argument to the premisses Γ can be transformed into a canonical argument to ϕ. (Prawitz [1971])

Elimination rules as basic

- One knows the meaning of a sentence ϕ if one knows what can be ultimately *deduced* from it.
- The elimination rules for a connective $\circ$ define what it is to *canonically deduce* a sentence with $\circ$ as the principal connective;
- A canonical argument is an argument terminating with an elimination rule.
- An argument from Γ to ϕ is *valid* iff any canonical argument to the premisses Γ can be transformed into a canonical argument to ϕ. (Prawitz [1971])

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Symmetry and its limit

- In the context of usual systems for intuitionistic logic, both the introduction-based and the elimination-based approach to validity allow to show that all derivations are valid.
- (Soundness) For any Γ and A : if $\Gamma \vdash A$, then $\Gamma \models A$
- These PTS enjoy an important symmetry, corresponding to the fact that one can justify elimination rules starting from the hypothesis that introduction rules are valid by definition and, conversely, justify introduction rules starting from the hypothesis that elimination rules are valid by definition.
- However, this symmetry breaks in the case of proof systems dealing with paradoxical derivations.

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Outline

Prawitz on paradoxes (I)

- Prawitz [1965] considered a system for naive set theory by extending minimal logic with an introduction and elimination rule for formulas of the form $t \in \{x : A\}$ for set-theoretical comprehension

$$\frac{A[t/x]}{t \in \{x : A\}} \in I \qquad\qquad \frac{t \in \{x : A\}}{A[t/x]} \in E$$

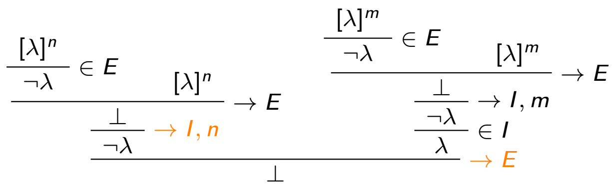
- An application of $\in I$ immediately followed by $\in E$ constitutes a redundancy which can be eliminated by a $\in$-reduction

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ A[t/x]\end{array}}{t \in \{x : A\}}}{A[t/x]} \quad \rightsquigarrow_{\in} \quad \begin{array}{c}\mathcal{D}\\ A[t/x]\end{array}$$

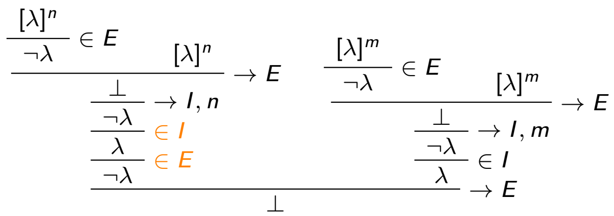
- Take λ to be $r \in r$, where r is the Russell term $\{x : x \notin x\}$.
  An application of $\in E$ allows one to pass from $\neg\lambda$ to λ; an application of $\in I$ allows one to pass from λ to $\neg\lambda$.

Prawitz on paradoxes (II)

- Russell's paradox has the form of the following derivation of absurdity



- By applying an implication reduction $\rightsquigarrow_{\to}$, one obtains the following



- By applying an $\in$ reduction $\rightsquigarrow_{\in}$, one obtains the first derivation.
  <u>All</u> possible reduction sequences loop.

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Tennant on paradoxes

- Tennant [1982] considered a wide range of examples and showed that all prominent mathematical and logical paradoxes follow this pattern. The steps playing the role of $\in I$ and $\in E$ are called *id est* inferences, as they result from extra-logical principles.

- He conjectures that the reduction loops are the distinguishing feature of these paradoxes and proposes the test of non-terminating reduction sequences as the criterion for paradoxicality.

  > *[...] enumerate proofs of absurdity; start normalizing those that are not in normal form; and check to see whether the reduction sequences ever enter loops, or manifest any other conclusive evidence that they will not terminate. As soon as a reduction sequence does enter a loop, or manifest such evidence, one can check off the proof concerned as a 'paradoxical' proof.*

- Tennant [1995] broadens the test to non-terminating reduction sequences, which covers paradoxes such as Yablo's.

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Tennant's criterion (I)

- Let $\mathcal{S}$ be any extension by means of *id est* rules of an adequate system.

  A T(ennant)-paradox is a derivation $\mathcal{D}$ in a system $\mathcal{S}$ when:

  (i) $\mathcal{D}$ is closed;

  (ii) $\mathcal{D}$ employs *id est* rules;

  (iii) $\mathcal{D}$ has no normal form;

  (iv) the conclusion of $\mathcal{D}$ is the absurdity ($\perp$).

- Condition (iii) is a straightforward consequence of the fact that Tennant characterizes a paradox as a derivation of absurdity whose reduction sequence does not terminate.

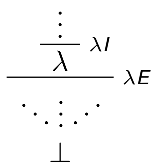
- Thus, following Tennant, the lack of a normal form is a necessary condition for paradoxicality.

Tennant's criterion (II)

- Because of (iii), a paradox contains at least one redundancy.

- Since all derivations in standard natural deduction without *id est* inferences have a normal form, the reduction of this redundancy must produce a derivation containing a redundancy which introduces and immediately eliminates a formula λ whose behavior is determined by the *id est* inferences.

- Under Tennant's hypotheses, a paradoxical derivation $\mathcal{D}$, possibly after some standard reductions, can be depicted as follows:



An alternative perspective

- One can normalize derivations with *ad hoc* open assumptions by exploiting a technique put forward by Kreisel & Takeuti [1974]:

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}[A]^n\\ \vdots\\ B\end{matrix}}{A\to B}\to I,n \qquad \begin{matrix}\vdots\\ A\end{matrix}}{B}\to E\\ \vdots\end{matrix}$$

is transformed into

$$\begin{matrix}\dfrac{[A\to A]^m \qquad \begin{matrix}\vdots\\ A\end{matrix}}{A}\to E\\ \vdots\\ B\\ \vdots\end{matrix}$$

Paradoxes in normal form (I)

- Idea: by exploiting KT technique, a *normal* paradoxical derivation can be constructed (see Petrolo & Pistone [2018]).
- The critical redundancy formed by the *id est* rules can be blocked by introducing a new assumption of the form $\lambda \to \lambda$, a trivially valid formula. The new configuration is the following:

$$\dfrac{\begin{matrix}\dfrac{\dfrac{[\lambda\to\lambda]^n \qquad \dfrac{\vdots}{\lambda}\lambda I}{\lambda}\to E}{\quad\quad}\lambda E\\ \ddots\vdots\cdot^{\cdot^{\cdot}}\\ \bot\end{matrix}}{(\lambda\to\lambda)\to\bot}\to I,n$$

- By discharging such an assumption at the end of the derivation via a $\to$-introduction rule, one obtains a closed derivation in normal form of a formula which is false in every interpretation (i.e. a contradiction).

Paradoxes in normal form (II)

- The KT transformation of T-paradoxes suggests an alternative test for paradoxicality (see Petrolo & Pistone [2018]).
- Let $\mathcal{S}$ be any extension by means of *id est* rules of an adequate system.

  A N(ormal)-paradox is a derivation $\mathcal{D}$ in a system $\mathcal{S}$ when:

  (i) $\mathcal{D}$ is closed;

  (ii) $\mathcal{D}$ employs *id est* rules;

  (iii) $\mathcal{D}$ is normal;

  (iv) if A is the conclusion of $\mathcal{D}$, then either $A \to \bot$ or $\neg A$ can be proved in $\mathcal{S}$.

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Normalization

- Add to usual natural deduction a new formula λ and
  - either assume that *every* derivation has an open assumption $[\neg\lambda \to \neg\lambda]^n$;
  - or take $\neg\lambda \to \neg\lambda$ as an axiom
- Then, given usual rules for λ:

$$\frac{\neg\lambda}{\lambda}\,\lambda I \qquad\qquad \frac{\lambda}{\neg\lambda}\,\lambda E$$

we can define an *ad hoc* reduction designed to block the detours on the formula $\neg\lambda$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \neg\lambda\end{array}}{\lambda}\,\lambda I}{\begin{array}{c}\neg\lambda\\ \vdots\end{array}}\,\lambda E \quad\rightsquigarrow\quad \dfrac{[\neg\lambda \to \neg\lambda] \qquad \begin{array}{c}\vdots\\ \neg\lambda\end{array}}{\begin{array}{c}\neg\lambda\\ \vdots\end{array}} \to E$$

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Outline

1. Two approaches to proof-theoretic validity
2. Two analysis of paradoxality
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Introduction-based approach

- In the introduction-based approach, a T-paradox is an example of a non-canonical derivation (as its last rule is an elimination rule) which cannot be reduced in canonical form (as no closed derivation of the absurdity can end by an introduction rule);

$$\dfrac{\dfrac{\vdots}{\lambda}\,\lambda I \qquad \dfrac{\vdots}{\neg\lambda}}{\bot} \to E$$

- T-paradoxes show that elimination rules do not preserve validity and, thus, they cannot be justified starting from introduction rules;
- In an introduction-based model of a paradoxical system, *modus ponens* (i.e. function composition) might fail to be correct, since elimination rules are not justified by introduction rules.
    - There exist semantics of proofs implementing this intuition (three-valued semantics as in Girard [1976], dinatural transformations as in Girard *et al.* [1992], etc.).

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Elimination-based approach

- In the elimination-based approach, a N-paradox is an example of a non-canonical derivation (as its last rule is an introduction rule) which, when applied to a valid derivation of $\rightarrow$, yields a derivation which cannot be reduced to a valid one.

$$\dfrac{\dfrac{\vdots}{(\lambda \rightarrow \lambda) \rightarrow \bot} \rightarrow I \qquad \lambda \rightarrow \lambda}{\bot} \rightarrow E$$

- N-paradoxes show that introduction rules do not preserve validity and, thus, they cannot be justified starting from elimination rules.
- However, since the "application" of an N-paradox to a valid derivation of $\lambda \rightarrow \lambda$ yields a non-terminating derivation (a T-paradox), it follows that non-terminating derivations must have a meaning.
  - There exist semantics of proofs implementing this intuition (e.g., Scott domains, see Scott [1976]). In a functional interpretation, proofs correspond to partial functions.

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Conclusion

- We showed that the PTS of paradoxical languages arising from the introduction-based and the elimination-based approaches is not univocal.
- Paradoxality can be interpreted
  - In an introduction-based approach: as the failure of some compositional principle (e.g. *modus ponens*);
  - In an elimination-based approach: as some notion of partial function.
- We argue that an investigation on the PTS of paradoxes should be complemented by a deeper analysis of the models of type theory.

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The Role of Structural Reasoning in the Genesis of Graph Theory

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Section 1

Introduction

Graphs in Proof Theory
Statement of Intent
Dramatis Personae – Paul Hertz
Dramatis Personae – Dénes Kőnig

Graphs in Proof Theory

A few examples:

- Proof nets (Girard; 1987)
- Interaction nets (Lafont; 1990)
- Graphs for proofs (Alves, Fernández, Mackie. 2011)
- λ-graphs (Ariola, Klop; 1994)
- Deduction graphs (Geuvers, Loeb; 2007)
- Proof-graphs (Quispe-Cruz, Haeusler, Gordeev; 2013)
- Flow graphs (Buss; 1991)
- Combinatorial proofs (Hughes; 2006)

Graphs in Proof Theory

The mentioned contributions to the logical literature avail themselves of very little more than the most basic notions of graph theory.

- A graph consists of a set of vertices and a set of directed or undirected edges connecting these vertices.
- Some kind of labelling function assigns to each vertex (or edge) some formula in the considered logic, some logical symbol, or some mixture of the two.

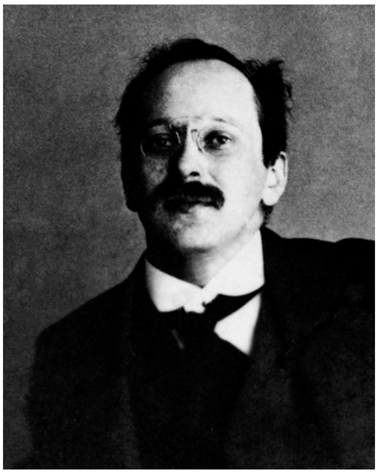
Beyond these basic notions, there is usually some study of the dynamics of simplifying graph transformations.

Statement of Intent

The aims of this talk are as follows:

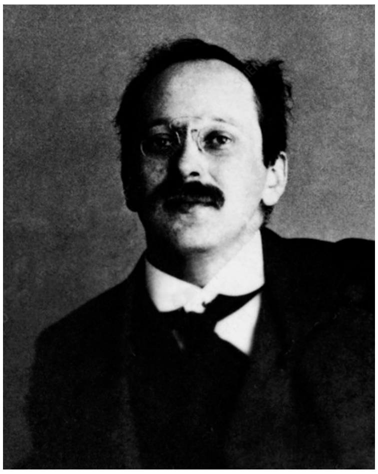
1. Give a brief overview over Hertz' work on structural reasoning.
2. Summarize Paul Hertz' 1922 article on structural reasoning, *"Über Axiomensysteme für beliebige Satzsysteme. I. Teil. Sätze ersten Grades"*, about sentences (clauses) of the form $a \to b$.
3. Demonstrate how Dénes Kőnig integrates the notions and results presented therein in his seminal book on graph theory, *"Theorie der endlichen und unendlichen Graphen"*, published in the year 1936.
4. Point out the independent reinvention of Hertz' results within the graph theoretical community.

Dramatis Personae – Paul Hertz



- Born July 29, 1881, in Hamburg.
- He studied physics, mathematics and philosophy at the universities of Heidelberg, Leipzig and Göttingen.
- In the year 1904 he received his PhD in the field of theoretical physics in Göttingen.
- His formal doctoral advisor was David Hilbert, but he closely worked with Max Abraham.
- He obtained his *Habilitation* in Heidelberg in 1908/1909.

Dramatis Personae – Paul Hertz



- He returned to Göttingen in the year 1912 as an associate professor.
- He became extraordinary professor for "methods of exact science" in 1921.
- His *venia legendi* was revoked in September 1933 in the wake of the national socialists' rise to power.
- After short stays in Geneva and Prague, he emigrated to the USA.
- He died only a few years later, on March 24, 1940, in Pittsburgh.

Dramatis Personae – Dénes Kőnig

- Born September 21, 1884, in Budapest.
- His father, Gyula Kőnig, was a professor for mathematics at the universitiy of Budapest.
- He studied mathematics in Budapest and (from 1904 onward) in Göttingen.
- He received his doctorate in the field of geometry 1907 in Göttingen.
- His doctoral advisors were Hermann Minkowski and József Kürschák.

Dramatis Personae – Dénes Kőnig

- He returned to Budapest to work at the universitiy as an assistant for four years.
- He subsequently was a lecturer for more than 20 years.
- He was appointed extraordinary professor in 1932.
- He was appointed full professor in 1935.
- On October 19, 1944, after the Hungarian national socialists' rise to power, he committed suicide.

Section 2

The Logic of Paul Hertz

Hertz' Publications on Logic

After his appointment to extraordinary professor for "methods of exact science" in 1921, Hertz became interested in the epistemology and methodology of science, particularly the topic of reasoning.

This is evidenced by the series of publications between the years of 1921 and 1939:

- *Über die Minimalzahl von Axiomen für ein System von Sätzen und den Begriff des idealen Elementes*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 30 (1921), p. 98.
- *Über Axiomensysteme für beliebige Satzsysteme. I. Teil. Sätze ersten Grades*, Mathematische Annalen, 87 (1922), pp. 246–269.
- *Über Axiomensysteme für beliebige Satzsysteme. II. Teil. Sätze höheren Grades*, Mathematische Annalen, 89 (1923), pp. 76–100.
- *Reichen die üblichen syllogistischen Regeln für das Schließen in der positiven Logik elementarer Sätze aus?*, Annalen der Philosophie, 7 (1928), pp. 272–277.

Hertz' Publications on Logic

- *Über Axiomensysteme beliebiger Satzsysteme*, Annalen der Philosophie, 8 (1929), pp. 178–204.
- *Über Axiomensysteme für beliebige Satzsysteme*, Mathematische Annalen, 101 (1929), pp. 457–514.
- *Über Axiomensysteme von Satzsystemen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 38 (1929), pp. 45–46.
- *Über den Kausalbegriff im Makroskopischen, besonders in der klassischen Physik*, Erkenntnis, 1 (1930), pp. 211–227.
- *Vom Wesen des Logischen, insbesondere der Bedeutung des modus barbara*, Erkenntnis, 2 (1931), pp. 369–392.
- *Über das Wesen der Logik und der logischen Urteilsformen*, Abhandlungen der Friesschen Schule, 6 (1935), pp. 227–272.
- *Sprache und Logik*, Erkenntnis, 7 (1939), pp. 309–324.

Hertz' Structural Reasoning

The best known of Hertz' articles is the 1923 *Über Axiomensysteme für beliebige Satzsysteme. II. Teil. Sätze höheren Grades*.

There he studies systems of sentences of the form $(a_1, \ldots, a_n) \to b$.

The calculus he considers has two rules of inference.

The first one, *immediate inference*, allows for the enlarging of the antecedent complex:

$$\frac{(a_1, \ldots, a_n) \to b}{(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \to b}$$

Hertz' Structural Reasoning

The second rule is *syllogism*:

$$\frac{\begin{array}{r} (a^1_1, \ldots, a^1_{n_1}) \to b_1 \\ \vdots \\ (a^k_1, \ldots, a^k_{n_k}) \to b_k \\ {}'\| \, (b_1, \ldots, b_k, a_1, \ldots, a_m) \to c \end{array}}{{}'\| \, (a^1_1, \ldots, a^1_{n_1}, \ldots, a^k_1, \ldots, a^k_{n_k}, a_1, \ldots, a_m) \to c}$$

The first k premises are the *system of minor sentences*, the remaining premise is the *major sentence*.

The notation "${}'\|$" expresses the operation on complexes which removes all duplicate elements so that every listed element is a singleton.

Gentzen's work on Hertz' Systems

Hertz' calculus became the basis of Gentzen's sequent calculus.

In his first article, *"Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen"*, published in 1933, Gerhard Gentzen extends Hertz' results of the 1923 article.

In order to deal more elegantly with Hertz' systems of sentences, Gentzen simplifies the rules of inference.

Using capital letters $L, M, \ldots$ for complexes of elements and $u, v, \ldots$ for elements, he renames immediate inference to *thinning*, thus indicating the effect of the inference step:

$$\frac{L \to v}{ML \to v}$$

Gentzen's work on Hertz' Systems

More significantly, Gentzen breaks up Hertz' syllogism rule into a much simpler and more elegant rule which he calls *cut*:

$$\frac{L \to u \qquad Mu \to v}{LM \to v}$$

The proviso to the application of this rule is that the cut element u must not occur in M.

Gentzen then proceeds to demonstrate very briefly that any application of syllogism of k minor sentences can be mimicked by k applications of cut or thinning.

The Significance of Hertz' Work

Hertz' two part study of sentence systems formulate the fundamental tenets of structural reasoning, and this early work arguably constitutes the first studies in structural proof theory.

He studied relationships of (complex) antecedents and succedents (that cannot be nested).

Systems of sentences are to be closed under rules allowing the arbitrary weakening of the antecedent (immediate inference, thinning) as well as the stepping over intermediary elements (syllogism, cut).

The notion of removing multiple occurrences of the same element (contraction) is present as a side effect of the syllogism rule.

These are the structural rules of the sequent calculus that Gentzen later introduced in his *Untersuchungen über das logische Schließen*.

The Significance of Hertz' Work

Hertz introduces the study of the structure of derivations (what he calls *systems of inferences* and *genealogical trees*) as formal objects.

Among other things he addresses the question of the existence of *normal derivations* for sentences and provides a positive answer.

The originality of Hertz' vision cannot be overemphasized, especially considering the fact that he was residing in Göttingen at the very same time when Hilbert, Bernays, Ackermann and others were pursuing the program of formulating a consistent foundation of arithmetic.

Despite the fact that Paul Hertz regularly discussed such matters with Bernays, his conception of reasoning was not influenced by the logical industry that was taking place next doors.

Section 3

Presentation of Hertz 1922

Overview

The article has a total of 24 pages, and it consists of 4 sections.

- An introduction of 3 pages.

The remaining sections are comprised of a enumerated sequence of definitions and propositions.

- §1. Problem statement – items 1 to 9.
  A brief introduction of the basic notions. (The problem statement is tucked onto these definitions.)
- §2. Unique choice of axiom systems – items 10 to 52.
  Dedicated to the question of the uniqueness of axiom systems for closed sentence systems.
- §3. Ideal elements – items 53 to 86.
  Concerned with the question of reducing the number of axioms in given axiom systems by introducing ideal elements.

Hertz 1922 – Introduction

Hertz points out that this article is the precursor to an investigation of axiom systems for a more general form of sentences: $(a_1, \ldots, a_n) \to b$.

On what such a sentence might express, he merely remarks:

> *"If $(a_1, \ldots, a_n)$ altogether holds, so does b."*

Immediately afterwards, he restricts the notion of sentence to sentences of the form $a \to b$, the simplified notation for $(a) \to b$.

Hertz refuses to go into any great detail as far as the meaning of a sentence is concerned, and goes as far as stating that:

> *"However, what is actually meant by such a sentence, what the symbol $\to$ means in the combination of characters $a \to b$ or the word 'if' in the corresponding linguistic formulation, does not have to be indicated here."*

Hertz 1922 – Introduction

Hertz studies *sentence systems*, sets of what might be viewed as atomic sequents, each relating some *element* (propositional symbol) to another.

$$\{\, a \to b,\ b \to c,\ c \to a,\ a \to c,\ c \to b,\ b \to a \,\}$$

An *axiom system* is a subsystem of a system of sentences that allows the reconstruction of the entire system by means of a basic inference rule.

$$\{\, a \to b,\ b \to c,\ c \to a \,\}$$

In the case of the 1922 article the (unnamed) single rule of inference is:

$$\frac{\begin{array}{c} a \to b \\ b \to c \end{array}}{a \to c}$$

Hertz 1922 – Introduction

Hertz motivates his investigation as using axioms to reduce the number of valid sentences "that have to be committed to memory".

He argues that, while the choice of axioms may be arbitrary to some degree, the main criterion is to obtain the minimal amount of axioms.

The investigation aims to describe a procedure for obtaining axiom systems of a minimal number of sentences.

Hertz uses diagrams to appeal to the readers' "geometric apprehension" in the introduction, which he avoids in the main part of the article.

Hertz 1922 – Introduction

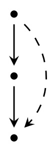


Fig. 1

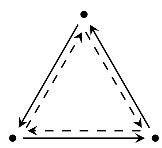


Fig. 2a

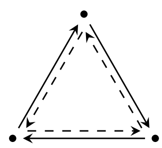


Fig. 2b

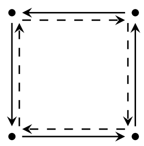


Fig. 3a

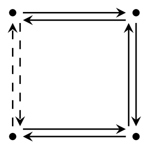


Fig. 3b

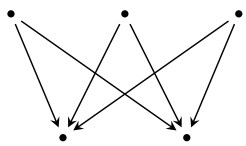


Fig. 4a

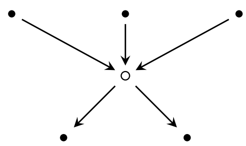


Fig. 4b

Hertz 1922 – Problem Statement

The first main section serves as both a list of formal definitions, followed by the propositions of two basic properties.

The notions of *element*, *complex of elements*, *antecedent*, *succedent*, *sentence of arbitrary degree*, and *sentence of first degree* are defined.

$$(a_1, \ldots, a_n) \to b$$

This is followed by the definition of the notions of *inference*, *minor* and *major sentences* (premises) and *conclusion*.

$$\begin{array}{c} a \to b \\ b \to c \\ \hline a \to c \end{array}$$

Hertz 1922 – Problem Statement

A *chain of inferences* wrt. a sentence system $\mathfrak{T}$ is a sequence of inferences in which every sentence that does not belong to $\mathfrak{T}$ must be the conclusion of a preceding inference.

This notion gives rise to that of sentences being (properly or improperly) *provable* from $\mathfrak{T}$.

A sentence system is *closed* if every sentence that is provable from it is improperly provable from it, i.e. is already contained in it.

An *axiom system* $\mathfrak{A}$ of a sentence system $\mathfrak{S}$ is a system of sentences from which every sentence in $\mathfrak{S}$ is provable.

An axiom system is *independent* if none of its axioms is properly provable from it.

Hertz 1922 – Problem Statement

The first proposition is the generalized transitivity of the provability relationship:

If every sentence of a system $\mathfrak{C}$ is provable from $\mathfrak{B}$ and sentence $\mathfrak{a}$ is provable from $\mathfrak{C}$, then $\mathfrak{a}$ is provable from $\mathfrak{B}$.

The second proposition states the existence of an independent axiom system:

For every closed sentence system there exists at least one independent axiom system.

This is established by successively removing provable sentences. As systems are finite, the procedure is guaranteed to terminate.

Hertz 1922 – Uniqueness of Axiom Systems

This section is mostly concerned with characterizations of sentence systems with the purpose of providing specific conditions that guarantee the uniqueness of axiom systems.

Hertz shows that every sentence system can be decomposed into disjoint *nets* and *chains*.

He subsequently establishes that chains can be further separated into overlaying *sequences*.

Hertz 1922 – Uniqueness of Axiom Systems

A *net* is a sentence system which, for each pair of different elements x, y, contains both sentences $x \to y$ and $y \to x$.

This notion immediately results in several propositions.

Every net is a closed sentence system.

Every net of n elements has a minimal independent axiom system of n cyclic sentences that connect all of the elements of the net.

If two nets have a common element, then they are parts of a common net.

A *maximal net* is a net that is not a proper part of another net.

Hertz 1922 – Uniqueness of Axiom Systems

Two elements of a sentence system are called *unconnected*, if all the elements can be separated into two sets such that there is no sentence connecting an element of one set to an element to the other, and the two elements belong to separate sets, otherwise they are called *connected*.

Two sentences of a sentence system are called *unconnected*, if all the sentences can be separated into two sets that do not share any elements, and the two sentences belong to separate sets, otherwise they are called *connected*.

A *chain* is a system of connected sentences that does not contain any sentence that belongs to a net, a *maximal chain* is a chain that is not properly contained in any other chain.

Hertz 1922 – Uniqueness of Axiom Systems

These notions bring forth a number of propositions.

Two chains that share an element are, when considered together, also a chain.

In a closed sentence system, the conclusion of two sentences occurring in a maximal chain itself belongs to this maximal chain. Hence, in a closed sentence system, a maximal chain is itself closed.

No sentence can belong to two different maximal chains.

(In other words, two different maximal chains in a closed sentence system are unconnected.)

Hertz 1922 – Uniqueness of Axiom Systems

Hertz is now set up to describe closed sentence systems.

The most eminent property is this:

In a closed sentence system, every sentence either belongs to a maximal net or to a maximal chain.

Consequently, a closed sentence system consists of maximal nets and maximal chains (and nothing else).

Closure of a system implies maximality of all of its subsystems.

Hertz 1922 – Uniqueness of Axiom Systems

This is already enough understanding of sentence systems for Hertz to state what he calls *Hauptsatz*:

Any closed sentence system that does not contain any net has a unique independent axiom system.

Existence has already been established at the end of the problem statement, so what remains at this point is to demonstrate uniqueness.

Hertz does this by considering the proof of the axiom of a second axiom system.

A case analysis results for each considered case in the requirement that the sentence system must contain a net, contrary to the assumption.

Hertz 1922 – Uniqueness of Axiom Systems

Two crucial observations are thus worth pointing out:

- The presence of nets breaks the uniqueness of axiom systems.
- Nets are essentially sentence systems that "equate" all of their elements, i.e. their sentences do not really represent valuable information.

This leads Hertz to consider *reduced systems* for closed systems.

Hertz 1922 – Uniqueness of Axiom Systems

A reduced system $\mathfrak{S}'$ is obtained from a closed system $\mathfrak{S}$ as follows:

1. All of the elements of any maximal net are identified with a single new element.
2. All of the sentences whose antecedents (respectively succedents) lie outside of such a maximal net and whose succedents (respectively antecedents) lie inside are identified with a single sentence, and so are all of the sentences whose antecedents lie inside one maximal net and whose succedents lie inside another maximal net.

With this simple operation, Hertz removes all redundancies from closed sentence systems.

Hertz 1922 – Uniqueness of Axiom Systems

A reduced system to a closed sentence system does not contain any nets and is itself a closed system.

A reduced system for a closed sentence system consists of one or more maximal chains that are mutually unconnected.

This establishes the uniqueness of axiom systems for reduced systems.

Hertz can now formulate a procedure for producing independent axiom systems for any closed sentence system:

Simply construct an independent axiom system for the reduced system and then add an independent axiom system for each maximal net.

Hertz 1922 – Ideal Elements

The final section, which constitutes the second large part of the article, is concerned with *extensions* to given sentence systems.

Hertz' motivation is to further simplify axiom systems by reducing their size.

This reduction in size is achieved by the introduction of new *ideal* elements and the subsequent replacement of certain groups of axioms by new sentences that utilize these new elements.

Hertz 1922 – Ideal Elements

A closed sentence system $\widehat{\mathfrak{S}}$ is an *extension* of a given closed sentence system $\mathfrak{S}$ if the following properties hold:

- Every element a of $\mathfrak{S}$ corresponds to an element $\hat{a}$ of $\widehat{\mathfrak{S}}$; these are called the *real* elements of $\widehat{\mathfrak{S}}$; any other elements of $\widehat{\mathfrak{S}}$ are called *ideal*.
- Every sentence $a \rightarrow b$ of $\mathfrak{S}$ corresponds to a sentence $\hat{a} \rightarrow \hat{b}$ between the corresponding real elements of $\widehat{\mathfrak{S}}$.
- Every sentence $\hat{a} \rightarrow \hat{b}$ of $\widehat{\mathfrak{S}}$ corresponds to a sentence $a \rightarrow b$ of $\mathfrak{S}$ between the corresponding elements.

The extended sentence system may contain additional sentences between ideal elements or between real and ideal elements to which no sentences in the original sentence system correspond.

Hertz 1922 – Ideal Elements

Consider six independent axioms of a system $\mathfrak{S}$:

$$a_1 \to b_1,\ a_2 \to b_1,\ a_3 \to b_1,\ a_1 \to b_2,\ a_2 \to b_2,\ a_3 \to b_2$$

Introduce an extended system that contains the ideal element γ and the following five additional sentences:

$$a_1 \to \gamma,\ a_2 \to \gamma,\ a_3 \to \gamma,\ \gamma \to b_1,\ \gamma \to b_2$$

In the extended system, the original axioms are consequences of the new sentences, which can therefore be assumed as new axioms.

Hertz 1922 – Ideal Elements

In connection to this example, Hertz remarks:

> *"By introducing ideal elements we circumvent the usage of the word 'or' in the antecedent, as well as the usage of the word 'and' in the succedent."*

Clearly, Hertz is aware of the fact that multiple sentences having the same succedent and different antecedents could be expressed as a disjunction of the antecedents wrt. that succedent, and that multiple sentences having the same antecedent and different succedents could be expressed as a conjunction of the succedents wrt. that antecedent.

The original six sentences could be expressed by the combined single sentence $(a_1 \text{ or } a_2 \text{ or } a_3) \to (b_1 \text{ and } b_2)$. On the other hand, the five new axioms could be stated as $(a_1 \text{ or } a_2 \text{ or } a_3) \to \gamma$ as well as $\gamma \to (b_1 \text{ and } b_2)$.

Hertz 1922 – Ideal Elements

A *group pair* is a pair of sets of elements $\langle A, B\rangle$ (our notation) in a closed sentence system with a given independent axiom system, such that for each element in A there exists an axiom to each element in B.

A group pair is *maximal*, if there is no antecedent element that does not belong to A from which axioms connect to all elements in B, and if there is no succedent element that does not belong to B to which axioms connect from all elements in A.

Trivially, every axiom connects elements of at least one group pair, and, consequently, of at least one maximal group pair.

A maximal group pair is *small*, if either A or B are singletons; it is *medium*, if A and B have two elements each; it is *large*, if it is neither small nor medium.

Hertz 1922 – Ideal Elements

A group pair is *closed*, if there is no antecedent element except for those in A that connects via an axiom to an element in B, and if there is no succedent element, except for those in B, to which an axiom connects from an element in A.

Every closed group pair is maximal.

Neither the antecedent groups nor the succedent groups of two closed group pairs can share any elements.

A closed sentence system is *simple* with regard to an independent axiom system, if it does not contain any net and if all of its maximal group pairs are closed.

This notion is useful, because in every simple sentence system, every axiom belongs to exactly one maximal group pair.

Hertz 1922 – Ideal Elements

If $\mathfrak{S}$ is a simple sentence system and $\widehat{\mathfrak{S}}$ an extension, an independent axiom system of the latter in which no axiom connects two ideal elements is called axiom system of the *first degree*.

If a belongs to an antecedent group and b to a succedent group of a maximal group pair of $\mathfrak{S}$, i.e. there is an axiom $a \to b$, then the axiom system of $\widehat{\mathfrak{S}}$ contains either an axiom $\hat{a} \to \hat{b}$, or there exists an ideal element γ such that $\widehat{\mathfrak{S}}$ contains axioms $\hat{a} \to \gamma$ and $\gamma \to \hat{b}$.

Hertz 1922 – Ideal Elements

A *connection system* for a disjoint pair of sets of elements $\langle A, B \rangle$ consists of a set of elements Γ and sentences between these elements such that:

1. For every pair of elements a_i of A and b_j of B, there is either a sentence $a_i \to b_j$ or two sentences $a_i \to \gamma_{ij}$ and $\gamma_{ij} \to b_j$.
2. Every *connecting element* γ_{ij} is only succedent of sentences with antecedents in A and is only antecedent of sentences with succedents in B.

A connection system without connecting elements is called *disparate*.

A connection system with only a single connecting element that is succedent to all elements of A and antecedent to all elements of B (and that contains no other sentences) is called a *centralized* connection system.

Hertz 1922 – Ideal Elements

For every maximal group pair $\langle A, B \rangle$ of a simple sentence system $\mathfrak{S}$, any axiom system $\widehat{\mathfrak{A}}$ of first degree for an extended system $\widehat{\mathfrak{S}}$ contains a connection system, and all of its connecting elements are ideal elements of $\widehat{\mathfrak{S}}$.

The connection systems of two different maximal group pairs in a simple sentence system have no connecting ideal element in common.

If A and B are sets of elements of a sentence system $\mathfrak{T}$, then we can assume the existence of a sentence system $\widehat{\mathfrak{T}}$ that contains a connection system for $\langle A, B \rangle$.

Instead of constructing a new sentence system with corresponding elements and sentences, we consider $\mathfrak{T}$ to be modified by adding that connection system in such a manner as to obtain $\widehat{\mathfrak{T}}$.

Hertz 1922 – Ideal Elements

For any maximal group pair $\langle A, B \rangle$ in a simple sentence system $\mathfrak{S}$ that has an axiom system $\widehat{\mathfrak{A}}$ of first degree, if $\widehat{\mathfrak{B}}$ is a connection system for $\langle A, B \rangle$ and $\widehat{\mathfrak{B}}^*$ is another connection system for the pair that is added, then to every sentence between real elements that can be proven from $\widehat{\mathfrak{A}}$ and $\widehat{\mathfrak{B}}^*$ there corresponds a sentence in $\mathfrak{S}$.

The same holds if the sentences of the connection system $\widehat{\mathfrak{B}}$ are completely removed and replaced by the sentences of $\widehat{\mathfrak{B}}^*$. Call the thus modified axiom system $\widehat{\mathfrak{A}}^*$, then the system of sentences $\widehat{\mathfrak{S}}^*$ that are provable from $\widehat{\mathfrak{A}}^*$ is an extension of $\mathfrak{S}$.

Hertz 1922 – Ideal Elements

A pair of sets of elements $\langle A, B\rangle$ is a small *pair, if one of the sets is a singleton; it is a* medium *pair, if both sets contain exactly two elements; otherwise it is a* large *pair.*

The following concerns the size of connection systems for pairs $\langle A, B\rangle$.

If $|A| = m$ and $|B| = n$, then the pair has a connection system of at least $m + n$ sentences; it is exactly $m + n$ if the connection system is centralized.

If $\langle A, B\rangle$ is a medium pair, then any centralized and any disparate connection system has exactly 4 sentences. If it is a small pair, then any connection system has at least m sentences; it is exactly m if the connection system is disparate.

Hertz 1922 – Ideal Elements

An axiom system of first degree for a closed sentence system $\mathfrak{S}$ is *minimal*, if $\mathfrak{S}$ has no other independent axiom system of first degree that contains less sentences.

The *canonical sentence system* for a simple sentence system $\mathfrak{S}$ without any nets consists of the following connection systems for maximal group pairs of $\mathfrak{S}$:

- for any large or medium maximal group pair a centralized connection system;
- for any small maximal group pair a disparate connection system.

In the case of medium maximal group pairs, Hertz actually allows a choice between centralized and disparate connection systems. While this has no impact on the size of the resulting system, in the spirit of canonicity we suggest to use centralized connection systems whenever possible.

Hertz 1922 – Ideal Elements

The main result of this section is the following:

Any minimal axiom system of first degree for a simple sentence system is canonical, and any canonical system is a minimal axiom system of first degree.

This final result concludes Paul Hertz' discussion of axiom systems for sentence systems in the case of sentences of first degree.

Discussion

Taken by itself, the connection of this article to logic is not easy to see, especially beyond the introductory remarks and the problem statement, when Hertz proclaims and addresses the problems that interest him.

At the same time, his terminology that talks of "axioms", "sentences" and "inference rules" makes it equally difficult to look beyond the topic of logic and to consider his work in a more general context.

While Hertz talks about logic, his problems and results appear to have nothing to do with what might at the time of publication have been considered to be significant (especially for foundational questions).

This might explain the striking lack of reception of Hertz' work at the time.

Discussion

We leave the question of logical interpretations aside and turn to a mathematical description of his investigations.

The fundamental observation is that the sentence arrow is really nothing other than a binary relation symbol, and a system of sentences is therefore a binary relation on a finite set of elements.

The inference rule serves the purpose of inducing transitivity on any such relation, and, consequently, the closure of a sentence systems is the transitive closure of a given relation.

From the perspective of properties of binary relations, Hertz describes an algorithm for computing a transitive reduction of a given binary relation, i.e. a subset of that relation that allows the restoration of the entire set by the application of the transitive closure operation.

Discussion

Hertz identifies the presence of nets as a problem as far as uniqueness is concerned.

In the terminology of binary relations, a net is a subset that is closed under symmetry.

Since he only concerns himself with transitive closures of such relations, all of the elements of a net can thus be considered to be equivalent.

Hertz describes a method for removing partially equivalent subsets of relations and replacing them by single elements while bundling the relation pairs connecting equivalent elements to elements that do not belong to the net into a single pair.

He is thus left with a transitive relation that is irreflexive and asymmetric, and for this he can uniquely generate the transitive reduction.

Discussion

In the last part of the article Hertz addresses the matter of maximal group pairs, pairs of maximal element sets $\{a_i\}_{1\leq i\leq m}$ and $\{b_j\}_{1\leq j\leq n}$ such that for each pair (i, j) the pair (a_i, b_j) belongs to the relation.

Maximal group pairs of sizes m and n result in $m \cdot n$ relationship pairs.

The pairs above can be replaced by the pairs (a_i, γ) and (γ, b_j). Thus, the $m \cdot n$ parallel pairs are sequentialized into $m + n$ pairs.

The price to pay for this reduction of the number of pairs is having to extend the relation in the sense that a new element γ must be introduced.

The new element only relates locally, however, i.e. only to elements in the maximal group pair.

Discussion

It is at this point that we can turn to graph theory.

Recall that a directed graph consists of a set of *vertices* V together with a set of *directed edges* $E \subseteq V \times V$.

Thus, the set of edges of a directed graph is nothing other than a binary relation (a set of ordered pairs) on vertices.

Consequently, Hertz' sentence systems, which are apparently binary relations, can be interpreted by graphs.

Dénes Kőnig not only recognized this fact but also understood that Hertz' results establish fundamental results for the theory of directed graphs.

Section 4

Presentation of Kőnig 1936

Overview

"Theorie der endlichen und unendlichen Graphen" is a comprehensive collection of research into the field of graphs.

Apart from presenting his own previously published results, Kőnig provides a comprehensive bibliography which lists 110 works concerned with or utilizing graphs or related notions.

He states in the introduction that he puts a particular emphasis on providing a complete overview of the available literature.

Throughout the text he meticulously points out the origins of the notions and results that he presents.

Overview

Already in the introduction to the book does Kőnig credit Hertz.

> *"Through a work of P. Hertz (1922) formal logic was also at the source of graph theoretical research."*

This claim is substantiated in chapters VII, §§2–4, and VIII, §§1–2, the chapters on elementary problems of directed graphs and on applications of directed graphs.

A note on Kőnig's notation: Using $P, Q, \ldots$ for *vertices*, possibly with number indices, his notation for an *edge* from P to Q is $\overrightarrow{PQ}$, and he sometimes uses $\overleftarrow{PQ}$ for an edge in the opposite direction.

Kőnig 1936 - Chapter VII, §1

We first turn to chapter VII, §1, where Kőnig's presentation of directed graphs first addresses vertices and properties of sets of vertices (with regard to their connectivity).

He defines the notions of basic set, source and vertex basis, and he states some easy propositions.

As Hertz' focus was put on sentences, and he rarely remarked on elements or sets of elements, his work is not mentioned in the four pages of this first section.

Kőnig 1936 - Chapter VII, §2

Kőnig commences by introducing the notion of an *edge basis* B for a directed graph G as a subset of the set of edges of G such that:

1. If $\overrightarrow{PQ}$ is an edge of G not occurring in B, then there exists a path in B from P to Q.
2. If $\overrightarrow{PQ}$ is an edge occurring in B, then there exists no path from P to Q in B consisting of edges in B other than $\overrightarrow{PQ}$.

Immediately following this definition, which corresponds to Hertz' notion of independent axiom system for a sentence system, Kőnig explicitly refers to Hertz' article:

> *"The theory of edge bases to be considered now – so far as it relates to finite graphs – is essentially the graph theoretical interpretation of those investigations, which P. Hertz [. . .] carried out for certain problems of logic (axiomatic theory). In his introduction Hertz already referred to this 'geometrical' interpretation [. . .]."*

Kőnig 1936 - Chapter VII, §2

Kőnig provides slightly modified versions of Fig. 1, Fig. 2a, Fig. 2b, Fig. 3a and Fig. 3b of Hertz' introduction, using solid arrows for edges belonging to an edge basis and dashed arrows for other edges.

The changes are that the points of his version of Fig. 1 are arranged in a triangle, and the reverse arrows in his versions of Fig. 2a, Fig. 2b, Fig. 3a and Fig. 3b are not straight but curved.

Otherwise Kőnig's examples depict the same graphs, except for one significant modification of Fig. 2b, for which an edge basis of four edges is given instead of just using the reverse edges of Fig. 2a as Hertz did.

The descriptions accompanying these figures mimic those given by Hertz.

Kőnig 1936 - Chapter VII, §2

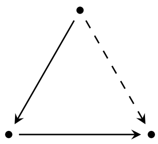


Fig. 39

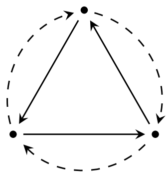


Fig. 40 a)

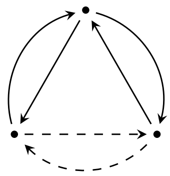


Fig. 40 b)

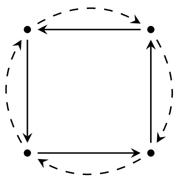


Fig. 41 a)

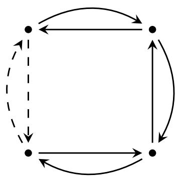


Fig. 41 b)

Kőnig 1936 - Chapter VII, §2

Kőnig introduces the notion of a *transitive graph* as a graph that contains an edge $\overrightarrow{PR}$ whenever it contains both edges $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ for $P \neq R$. He notes in a footnote that this notion corresponds to what Hertz calls "closed".

Kőnig remarks that any graph can be completed into a transitive graph by adding all the edges required by the definition.

Next, Kőnig defines that, if vertices P and Q are only connected by either $\overrightarrow{PQ}$ or $\overleftarrow{PQ}$, the edge is a *simple edge*, if they are connected in both directions, the edges, considered together, are *double edges*.

The definitions of *net* and *maximal net* are based on the notion of double edges and exactly correspond to Hertz' definitions.

Kőnig 1936 - Chapter VII, §2

Kőnig subsequently gives a characterization of maximal nets of a transitive graph as the connected components of the sub-graph that is obtained by considering those vertices that are incident to double edges.

The characterization is followed by the proposition that a transitive graph that only contains simple edges cannot have two different edge bases.

This proposition expresses Hertz' *Hauptsatz* about the unique existence of axiom systems for closed sentence systems in Kőnig's terminology.

Note, however, that Kőnig's result importantly does not address the question of existence at this point, which means that he instead only precludes the existence of two different edge bases.

Kőnig 1936 - Chapter VII, §2

Kőnig then turns to the notion of the *condensation* of a transitive graph, which he defines by following Hertz' method of replacing all maximal nets by single vertices.

In the original text, Kőnig explicitly refers to Paul Hertz by attaching his name to the notion:

> *"Nun definieren wir den Hertzschen reduzierten Graphen G_{red} des transitiven Graphen G folgendermaßen."*

This reference to Hertz is omitted in the 1990 translation:

> *"We now define the condensation of the transitive graph G in the following way."*

Kőnig 1936 - Chapter VII, §2

Three propositions about condensations match those given by Hertz.

First, Kőnig states that the condensation of a transitive graph only contains simple edges. This corresponds to Hertz' proposition 38 stating that the reduced system to a closed system does not contain any nets.

Then Kőnig states that the condensation to a given transitive graph is itself transitive, which corresponds to Hertz' proposition 40 about the fact that the reduced system to a given closed system is closed.

Finally, Kőnig proposes that the condensation of a transitive graph cannot have two different edge bases. This reflects Hertz' proposition 41, except for the fact that Hertz positively talks about the uniqueness of axiom systems.

Kőnig 1936 - Chapter VII, §3

In §3 Kőnig turns to the question of reducing the problem of an edge basis for an arbitrary transitive graph to that of edge bases for its maximal nets and the edge basis for its condensation.

This is discussed in considerable detail on four pages.

His two propositions correspond to Hertz' propositions 42, stating the same reduction, and 44, which states that there cannot exist any edge bases that are not obtained in this manner.

Kőnig 1936 - Chapter VII, §4

§4 addresses the topics of existence and minimality of edge bases.

As Kőnig's book is not restricted to finite graphs, the question of existence is not trivial, which is why he states his results corresponding to those of Hertz in §2 and §3 in a manner that does not imply existence.

Indeed, after the proposition that every finite graph does have an edge basis, he develops a simple example for an infinite transitive graph comprised of simple edges that does not have an edge basis:

$$\begin{array}{ll} \overrightarrow{P_i Q} & \text{for all } i \\ \overrightarrow{P_i P_j} & \text{for all } i < j \end{array}$$

Kőnig 1936 - Chapter VII, §4

The example is a simplified version of one that Gerhard Gentzen gave in his 1933 article *"Über die Existenz von Axiomensystemen für unendliche Satzsysteme"* on extending Hertz' analysis to infinite sentence systems:

$$\begin{array}{rl} (a_\nu b) \to c & \text{for all } \nu \\ (a_\lambda) \to a_\mu & \text{for all } \lambda < \mu \end{array}$$

It is remarkable that, despite the striking similarity, Kőnig does not reference Gentzen's research into Hertz' sentence systems, nor does he list that article in his otherwise extensive bibliography.

Kőnig 1936 - Chapter VII, §4

Next, Kőnig addresses edge bases of infinite and finite nets.

He first proposes the even more general property that every graph consisting entirely of double lines has an edge basis.

This implies that all nets, being transitive graphs consisting entirely of double lines, have an edge basis.

Kőnig's proposition that every finite net of n vertices has an edge basis consisting of n edges, but none of less than n edges, corresponds to Hertz' proposition 13.

Kőnig 1936 - Chapter VII, §4

The issue of minimality is addressed by a single paragraph in which Kőnig merely sums up the previously stated propositions of §2, §3 and §4.

If the condensation of a graph has an edge basis, it is unique and therefore minimal.

Moreover, it is always possible to determine the minimal edge bases for the maximal nets contained in a graph.

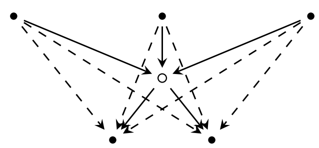
Kőnig 1936 - Chapter VII, §4

Kőnig concludes §4 and thereby chapter VII with a brief remark that summarizes Hertz' work on the introduction of ideal elements for the purpose of further minimizing axiom systems:

> *"Finally we would like to point out something which was also remarked by Hertz [. . .]. Graph theoretically this remark of Hertz can be formulated in the following way. A graph can have a smaller edge basis than a sub-graph of itself."*

Kőnig 1936 - Chapter VII, §4

The example Kőnig provides depicts Hertz' Fig. 4a and Fig. 4b overlayed, where the edges of the former are solid, those of the latter are dashed.



This is Kőnig's somewhat brief summary:

> *"By adding new (ideal) vertices and edges it is possible that the minimal number of edges of edge bases can be reduced. [. . .] This problem was also posed by Hertz [. . .] and its solution worked on."*

Kőnig 1936 - Chapter VII, §4

It is not entirely clear why Kőnig gives such a brief and vague summary of Hertz' systematic and concise solution to the problem instead of embracing it and reformulating it in graph theoretical terms.

Considering the fact that Kőnig introduces the notion of a bipartite graph later in the book, one could imagine a generalization of Hertz' method that generates a bipartite graph of real and ideal vertices by adding an ideal vertex for every pair of sets of vertices corresponding to Hertz' maximal group pairs.

Such a construction could serve as an intermediate step towards Hertz' minimization result.

Kőnig 1936 - Chapter VIII, §1

In chapter VIII, Kőnig turns towards applications of directed graphs.

The first application, given in §1, is dedicated to logic, more specifically, axiomatic theory.

On roughly two pages he recounts Hertz' motivation for his research, albeit with several small but noteworthy modifications.

Kőnig 1936 - Chapter VIII, §1

Kőnig interprets vertices as statements and edges $\overrightarrow{A_i A_j}$ as expressing the fact that statement A_j follows from statement A_i, for which he adopts Hertz' notation $A_i \to A_j$.

Just like Paul Hertz when introducing his notation, Kőnig refuses to commit to a logical interpretation of what this notation should express:

> *"What we mean here by 'follow' we do not need to explain any more precisely."*

However, in the following sentence he admits that $\to$ actually designates a binary relation and postulates that it must observe transitivity.

It is apparent that Kőnig is actually talking about a consequence relation.

Kőnig 1936 - Chapter VIII, §1

Kőnig restates Hertz' idea of axiom systems.

In contrast to Hertz, he first considers sets of statements and observes that the notion of a vertex basis (a set of vertices from which all other vertices of the graph can be reached via a single edge) corresponds to that of a set of statements that do not follow from each other, but from which all other statements follow.

Kőnig calls such statements "axioms" and uses the phrase "independent axiom system" for such sets of statements.

In connection with these notions he introduces the more general form of relationship between statements $(A_1 \,\&\, A_2 \,\&\, A_3 \,\&\, \ldots) \to B$, which resembles Hertz' sentences of higher degree.

Kőnig 1936 - Chapter VIII, §1

This is followed by a paragraph in which Kőnig begins to turn to Hertz' statement of intent.

Unfortunately, he does so in an uncharacteristically confusing manner, because he now wants to be able to talk about independent axiom systems in the sense of Paul Hertz.

For this purpose he shifts the perspective from relations on statements to relationships between relations on statements.

He introduces the abbreviation k_{ij} to signify a logical relation $A_i \to A_j$ and proceeds to address questions of logical dependency or in-dependency of such k_{ij} from others.

Kőnig 1936 - Chapter VIII, §1

For this purpose he introduces the abbreviation $(k_1 \ \& \ k_2 \ \& \ k_3 \ \& \ldots) \to k$ to mean that there exists a chain inference in Hertz' sense from sentences $k_1, k_2, k_3, \ldots$ that yields the sentence k.

This most likely reflects the notation $\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3 \ldots \rightrightarrows \mathfrak{b}$, where the $\mathfrak{a}_i$ and $\mathfrak{b}$ represent sentences, which was introduced by Hertz in a 1929 article for a similar purpose.

This shift in perspective allows Kőnig to transfer his notion of independent axiom system from sets of vertices to sets of edges.

It is far from obvious what Kőnig is doing in this paragraph, and, assuredly, it most likely could have been accomplished in a much more elegant manner.

Kőnig 1936 - Chapter VIII, §1

In the last paragraph of §1, Kőnig addresses the implications of the results of chapter VII, §§2–4 for the theory of axiom systems as it was formulated by Hertz.

Kőnig points out that closed sentence systems can be interpreted by transitive graphs and their minimal independent axiom systems by minimal edge bases.

In other words, after having used the results of Hertz' analysis of the unique existence of axiom systems for his purpose of talking about edge bases of transitive graphs, he now points out that Hertz' work is an example of applying these graph theoretical notions to questions of logic.

Kőnig 1936 - Chapter VIII, §2

In the following §2, Kőnig discusses matters regarding binary relations.

Despite the fact that he introduces this section as being concerned with "a still more general application of directed graphs to logic", he only refers to Hertz in two footnotes.

After having introduced his notion of ordered graph as a transitive connected graph that only contains simple edges, he refers to Hertz' definition of a sequence as the logical structure corresponding to this.

Following the statement that the unique edge base of an ordered graph is given by the path that follows the simple ordering of its vertices, he refers to Hertz' corresponding result for sequences.

Discussion

This concludes the scope of Kőnig's references to Hertz in "*Theorie der endlichen und unendlichen Graphen*".

Closing with discussions of applications of directed graphs to representations of games and of groups and their actions, chapter VIII also concludes Kőnig's presentation of the elementary notions and the basic problems of graph theory.

It is thus apparent that, save for matters of vertex bases, the entire chapter VII on basic problems of directed graphs as well as a significant part of chapter VIII on applications of directed graphs was informed by the work of Paul Hertz, specifically, by his 1922 article on axiom systems for sentence systems in the case of sentences of the first degree.

Discussion

Despite the fact that Hertz states his opposition to an appeal to the geometrical apprehension of the reader in his 1922 article, he depicts what Kőnig later calls directed graphs for the sole purpose of making his intuitions easier to grasp.

This, in turn, must have attracted Kőnig to his article, who discovers a trove of results which can not only be restated in graph theoretical terms, but which he deems to be the fundamental problems for directed graphs.

It is highly unfortunate that this direct and immediate recognition was not picked up in the subsequent literature on graph theory.

Section 5

Reinventions of Hertz' Results

The Success of Graph Theory

After the publication of the first modern English language text books on graph theory in the 1960s, interest in the field proliferated.

- C. Berge, *The Theory of Graphs and its Applications*, John Wiley & Sons, 1964.
- F. Harary, R. Z. Norman, and D. Cartwright, *Structural Models: An Introduction to the Theory of Directed Graphs*, John Wiley & Sons, 1965.
- F. Harary, *Graph Theory*, Addison-Wesley, 1969.

During the 1970s more than two dozen text books on the subject were published.

F. Harary established the *Journal of Graph Theory* in the year 1977.

1965 – Line Bases of Digraphs

In the first book specifically dedicated to directed graphs (or digraphs) by Harary, Norman and Cartwright HNC65, published in 1965, the authors remark in a section captioned "removal of a set of lines":

> *"A more general problem is to investigate the effects of removing a set of lines from a digraph. Unfortunately, a thorough investigation of these effects becomes extremely complicated."*

Indeed, while the authors do address the matter of removing *lines* (the term they use for directed edges), from a directed graph, they merely describe the properties of *line bases* (their notion corresponding to edge bases).

1965 – Line Bases of Digraphs

The authors state several characterizing theorems, such as:

> *"Theorem 7.20. The sub-graph generated by a line basis of a digraph D has the same transitive closure as D."*

> *"Theorem 7.21. Every digraph has a line basis."*

> *"Theorem 7.24. Every acyclic digraph has a unique line basis containing all of its basic lines."*

Despite the somewhat unfamiliar terminology, these results are recognizable as what had previously been stated by both Hertz and Kőnig.

1965 – Line Bases of Digraphs

However, no systematic procedure for constructing a line basis for a given graph is described.

This is quite surprising, especially considering the fact that the footnote to theorem 7.21 states:

> *"This theorem and other related results on line bases are developed in König (1936, Ch. VII, §2)."*

As illustrated above, König did certainly describe a method for obtaining the line bases, albeit not in a straightforward algorithmic manner.

It remains unclear why the authors do not expound that presentation in their own terms, and why they consider the matter to be "extremely complicated".

1965 – Line Bases of Digraphs

What is also remarkable is the fact that, although Kőnig profusely refers to Hertz' article, even attaching the adjective "Hertzschen" to the notion of condensation, no reference to Hertz' 1922 article is made in the bibliography of this 1965 book.

Since an English translation of Kőnig was not published until 1990, there can be no doubt that it is this omission that disconnected the original work of Paul Hertz from the subsequent literature on graph theory.

1969 – Minimum Equivalent Graphs

Hertz' method of generating axiom systems for sentence systems was reinvented in the year 1969 in the article *"An algorithm for finding a minimum equivalent graph of a digraph"* by Moyles and Thompson.

The authors point out in the introduction to their article that the problem had been discussed only a few years earlier.

A noteworthy difference to Hertz' approach is that Moyles and Thompson don't explicitly consider the transitive closure of a directed graph, but apply their method to arbitrary directed graphs.

1969 – Minimum Equivalent Graphs

The authors define the *reachability relation* on a graph by the requirement that there exist a path between the respective two vertices, and they observe that this relation is transitive (and reflexive, since they do not embrace an exemption like that of Hertz).

They can thus talk about properties related to transitive closure without explicitly having to add every edge between such pairs of vertices connected by a path to the given graph.

They further observe that there may be *mutually reachable* vertices, and that mutual reachability is an equivalence relation on the set of vertices.

Non-singleton equivalence classes contain mutually reachable vertices, and the authors call the corresponding subgraphs *strong components* (which correspond to Hertz' maximal nets).

1969 – Minimum Equivalent Graphs

Edges between different strong components are called *parallel*.

The authors further introduce the notion of *complete sequence* in a graph as a finite directed path which has the same initial and terminal vertex and passes through every vertex at least once.

About these they observe that

- every strong component possesses a minimum complete sequence;
- a minimum complete sequence is a minimum equivalent graph for a strong component.

1969 – Minimum Equivalent Graphs

Finally, the authors obtain a *condensed graph* by replacing strong components by single vertices.

They provide the following procedure for constructing a minimum equivalent graph to any given directed graph:

1. Identify all strong components.
2. Give a minimum complete sequence for each strong component.
3. Remove all but one edge in each set of parallel edges.
4. Give the minimum equivalent graph for the condensed graph.

1969 – Minimum Equivalent Graphs

Compare this to Hertz' method, summarized in his proposition 43:

> *"An independent axiom system $\mathfrak{A}$ for a given closed sentence system $\mathfrak{S}$ is obtained as follows: Generate the independent axiom system $\mathfrak{A}'$ for [the reduced system] $\mathfrak{S}'$, then take for each sentence in $\mathfrak{A}'$ the corresponding sentence in $\mathfrak{S}$ [as an axiom] and add an independent axiom system for every maximal net contained in $\mathfrak{S}$."*

Furthermore, proposition 13 states:

> *"[. . .] an [independent] axiom system for a net of n elements is the following: $a_1 \to a_2$, $a_2 \to a_3, \dots, a_n \to a_1$."*

The latter specifies a minimum complete sequence, and the former constructs a minimum equivalent graph for the condensed graph with no parallel edges (as Hertz' notion of a reduced system encompasses both).

1972 – Transitive Reduction of a Directed Graph

A similar but slightly more general result is published by Aho, Garey and Ullman in their 1972 article *"The transitive reduction of a directed graph"*.

They observe that the method given by Moyles and Thompson can be improved in such a manner that the minimum equivalent graph may contain fewer edges in the case of graphs containing strong components.

As an example for a critical case they state that, if a directed graph contains the strong component $\{(v_1, v_2), (v_2, v_3), (v_3, v_2), (v_2, v_1)\}$, then this is already its minimum complete sequence.

The authors observe that the result of the preceding article can be improved by replacing two of the edges, for example (v_3, v_2) and (v_2, v_1) by a single edge (v_3, v_1) without affecting reachability.

Thus, the first reinvention of Hertz' method in ML69 was not as effective as the original.

1972 – Transitive Reduction of a Directed Graph

Using the notion G^T for the *transitive closure* of a directed graph G, Aho, Garey and Ullman give a succinct characterization of the *transitive reduction* G^t of a directed graph G as the unique graph G which satisfies:

1. $(G^t)^T = G^T$.
2. If $H^T = G^T$, then H contains at least as many edges as G.
3. If G is not acyclic, then G^t is the canonical cyclic expansion of the transitive reduction of the equivalent acyclic graph for G.

1972 – Transitive Reduction of a Directed Graph

What the authors call an *equivalent acyclic graph* corresponds to Hertz' reduced system and the previous authors' condensed graph, whereas the *canonical cyclic expansion* merely expands each loop and vertex corresponding to a multi-member equivalence class into an ordered simple cycle, with all arcs between equivalence classes transformed into arcs between the least members of the equivalence classes.

That is, these authors do not simply call for "some minimum complete sequence for each strong component" as the previous authors do, but instead give a canonical minimum complete sequence.

1989 – Edge Concentration for Directed Graphs

Paul Hertz' method of introducing new elements for the purpose of reducing the number of required axioms reappears in graph theory as late as 1989 in the context of graph drawing.

The more edges a graph contains, the less readable it is. A measure that captures the intuition of readability is the *density* of a graph, which measures the number of its edges in relation to the maximum number of edges that a graph with a given number of vertices could support.

However, graphs generally do not possess a uniform density. Certain sub-graphs of a given graph can be denser than the rest.

Clustering is the general term describing the issue of discovering denser regions within a given graph and thinning them out without losing desirable properties the graph possesses.

1989 – Edge Concentration for Directed Graphs

In her 1989 article *"Edge concentration: A method for clustering directed graphs"*, Newbury reviews several clustering algorithms from the early 1970s onward, and she observes of one interesting candidate:

> *"[it] does not always result in a graph with the fewest number of edges in the concentrated graph. Our goal is to find a solution with the fewest number of edges [. . .]."*

She then explains the crucial part of the algorithm that she proposes:

> *"The suggested approach finds sets of directed edges such that all of their source nodes are connected to all of their target nodes. Each set of edges is replaced by a specially marked node whose 'fan-in' is all of the source nodes and whose 'fan-out' is all of the target nodes in the set of edges. The set of edges will be called an* edge concentration*, and the special node used to depict it will be called an* edge concentration node*."*

1989 – Edge Concentration for Directed Graphs

It is apparent that what the author describes corresponds to Hertz' construction of a canonical axiom system for the closed sentence system by means of introducing an ideal element.

It must be stated, however, that Newbury's algorithm for solving the edge concentration problem is somewhat more general than Hertz' method, as it can be applied to sets of vertices corresponding to arbitrary group pairs, not just maximal group pairs.

By using multiple edge concentration nodes in order to concentrate edges coming from (or going to) sets of vertices having a sufficiently large intersection, the algorithm can minimize the number of edges even in cases in which Hertz' stricter criterion would preclude the introduction of a single ideal element. In this sense, Newbury's algorithm takes Hertz' undertaking beyond what he was able to conceptualize.

Are the Open-Ended Rules for Negation Categorical?

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Proof-Theoretic Semantics: Assessment and Future Perspectives
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The Categoricity Problem –*Background*

- PTS is inferential. According to logical inferentialism, the formal axioms/rules of inference determine the meanings of the logical terms. In the case of classical logic, however, these meanings are already fixed and the problem is whether the rules really succeed in uniquely determining them, i.e., whether the rules are categorical.
  - In *Formalization of Logic (1943)*, Carnap proved that the standard formalizations of classical propositional and predicate logic allow for non-normal interpretations, i.e., interpretations for which the calculi remain sound, but in which the logical terms have different meanings than the standard ones. The existence of such interpretations shows that the standard calculi do not fully formalize all the logical properties of the logical terms and, thus, fail in uniquely determining their meaning. To eliminate this asymmetry between syntax and semantics, Carnap aimed to obtain a *full formalization* of classical logic, i.e., a formalization that represents all the semantic properties of the logical terms.

  - In his 1944 review to Carnap's book, Church argued that no purely syntactic solution would work and criticized Carnap's full formalization of a "concealed use of semantics". Some logicians have followed Carnap's strategy to strengthen the proof systems (Shoesmith & Smiley 1978, Smiley 1996, Rumfitt 2000) and some have adopted Church's attitude (Koslow 2010, Bonnay & Westerståhl 2015). My presentation analyzes Van McGee's inferentialist approach to classical logic and, in particular, I argue that his open-ended formalization of classical logic fails to be a full formalization, i.e., a categorical one.

Overall View

- Classical propositional and predicate logics are non-categorical, i.e., they allow non-normal interpretations of the logical terms.

- V. McGee has recently argued that Belnap's criteria constrain the formal rules of classical natural deduction to uniquely determine the semantic values of the logical connectives and quantifiers *if the rules are taken to be open-ended*, i.e., if they are truth preserving within any mathematically possible extension of the original language.

- An assumption of his argument is that *for any class of models there is a mathematically possible language in which there is a sentence true in just those models.*

- I argue that this assumption is problematic for the class of models of classical propositional logic. In particular, I show that the existence of non-normal models for negation undermines McGee's argument.

Plan of the Talk

I. The Non-Categoricity of Classical Logic

- Non-normal Interpretations of Classical Logic.
- Rules and Meanings

II. McGee's *Open-Ended Formalization* of Logic

- Conservativeness, Uniqueness and Open-endedness
- *The Open-ended Formalization of Logic*

III. Are the Open-Ended Rules for Negation Categorical?

- The Model-Theoretic Assumption (MTA)
- MTA and the Non-normal Models of Negation

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Carnap's *Formalization of Logic*

"The task of the formalization of any theory, i.e. of its representation by a formal system or calculus, belongs to syntax, not to semantics. On the other hand, the question of whether a proposed calculus formalizes a given theory adequately and completely is a matter of the relations between a calculus and an interpreted system, and hence requires semantics in addition to syntax." [Carnap 1943: vi]

- A semantical property of an expression is formalized by a calculus if, and only if, the expression has that property in any interpretation for which the calculus is sound.
- Logical truth and logical deduction are formalized by propositional calculi, but not all logical properties of the propositional operators are formalized by standard propositional calculi.
- "In a stricter sense of completeness, PC is not complete" [Carnap 1943:73]

Carnap's *Formalization of Logic*

- In *Formalization of Logic*, Carnap proved that the standard formalizations of classical propositional and predicate logic allow for non-normal interpretations, i.e., interpretations for which the calculi remain sound and complete, but in which the logical constants have different meanings than the standard ones.
- The existence of such interpretations shows that the standard calculi do not fully formalize all the logical properties of the logical terms and, thus, fail in uniquely determining their meaning.

Carnap's *Formalization of Logic*

- The rules for negation, disjunction, material implication, and quantifiers, in contrast with those for conjunction, do not determine all the properties of these operators as defined by standard semantics.
- There are two mutual kinds of non-normal interpretations for logical operators that exhaust all possibilities for propositional calculi:

  (I) all sentences are true and –and, thus, a sentence and its negation are both true.

  (II) at least one sentence is false – a sentence and its negation are both false here and, thus, their disjunction is true).
- There are non-normal interpretations in which which "(∀x)Fx" could be interpreted as "every individual is F, and b is G", where "b" is an individual constant, and, likewise, "(∃x)Fx" could be interpreted as "at least one individual is F, or b is G".

Conjunction: PC & NTT

(&I) $\dfrac{p \quad q}{p \,\&\, q}$ (&E)’ $\dfrac{p \,\&\, q}{p}$ (&E)’’ $\dfrac{p \,\&\, q}{q}$

	p	q	p & q
C1	$\top$	$\top$	$\top$
C2	$\top$	$\bot$	$\bot$
C3	$\bot$	$\top$	$\bot$
C4	$\bot$	$\bot$	$\bot$

(&I) : C1
(&E)’ : C3, C4
(&E)’’ : C2, C4

Disjunction: PC & NTT

(vI)’ $\dfrac{p}{p \vee q}$ (vI)’’ $\dfrac{q}{p \vee q}$ (vE) $\dfrac{p \vee q \quad \begin{matrix}[p] \\ r\end{matrix} \quad \begin{matrix}[q] \\ r\end{matrix}}{r}$

	p	q	p v q
D1	$\top$	$\top$	$\top$
D2	$\top$	$\bot$	$\top$
D3	$\bot$	$\top$	$\top$
D4	$\bot$	$\bot$	?

(vI) : D1, D2
(vI)’ : D1, D3
(vE) : -------

Material Implication: PC & NTT

(→I)

$$\frac{\begin{matrix}[p]\\ q\end{matrix}}{p \rightarrow q}$$

(→E)

$$\frac{p \quad p \rightarrow q}{q}$$

	p	q	p → q
I1	⊤	⊤	⊤
I2	⊤	⊥	⊥
I3	⊥	⊤	⊤
I4	⊥	⊥	?

(→ I) : I2, I3
(→ E) : I1, I2

Negation: PC & NTT

(~I)

$$\frac{\begin{matrix}[p]\\ ⅄\end{matrix}}{\sim p}$$

(~E)

$$\frac{p \quad \sim p}{⅄}$$

	p	~p
N1	⊤	?
N2	⊥	?

(~I) : .....
(~E): .....

Carnap's general diagnosis:

- Standard calculi state conditions for C-implicate and C-truth only. Thus, they can formalize only those L-concepts definable on the basis of L-implication (L-truth being one of them).
- However, L-exclusive and L-disjunct are not definable on the basis of L-implication and, thus, they are not formalized.
- *L-exclusive and L-disjunct occur in the principles of non-contradiction and excluded middle; thus, these principles are not represented in PC.*
- In the first kind of non-normal interpretations, the first principle is violated; in the second kind, the second principle is violated. Neither the validity, nor the invalidity of these two principles is assured by the rules of PC. In some interpretations they hold, in others they do not hold.

Quantification: Rules and Meanings

(∀ I)

$$\frac{\phi c}{(\forall x)\phi x}$$

where c does not occur in any premise or assumption on which ϕc depends.

(∀E)

$$\frac{(\forall x)\phi x}{\phi c}$$

(∃I)

$$\frac{\phi c}{(\exists x)\phi x}$$

(∃E)

$$\frac{(\exists x)\phi x \quad \begin{matrix}[\phi c] \\ \psi\end{matrix}}{\psi}$$

where c does not occur in any premise or assumption on which ϕc depends, in (∃x)ϕx and in ψ.

- There are non-normal interpretations of the quantifiers even thought the classical propositional connectives have only normal interpretations.
- In particular, there are sound interpretations of quantificational logic in which "(∀x)Fx" could be interpreted as "every individual is F, and b is G", where "b" is an individual constant.
- Likewise, "(∃x)Fx" could be interpreted as "at least one individual is F, or b is G".
- The possibility of these non-normal interpretations arises because, in the standard formalizations of quantificational logic, a universal sentence is not deductively equivalent (C-equivalent, in Carnap's terms) with the class formed by the conjunction of all the instances of the operand and an existential sentence is not C-equivalent with the class of the disjunction of all the instances of the operand.

A Full Formalization of Logic

A Full Formalization of Propositional Logic:

(1) $A \vee B \vdash \{A, B\}^{v}$

(2) $V^{\&} \vdash \Lambda^{v}$

- Multiple Conclusion Logic
- (1) is a disjunctive rule of inference and (2) is a rule of refutation.
- $V^{\&}$ is the universal conjunctive and Λ^{v} is the null disjunctive. $V^{\&}$ is always true and Λ^{v} is always false.

A Full Formalization of Predicate Logic:

"The use of indefinite rules referring to transfinite junctives will be necessary for solving the task of a full formalization of functional logic." [Carnap 1943: 142]

"Indefinite calculi seem to be admissible and convenient and even necessary for certain purposes." "An example of a task which cannot be solved without the use of indefinite rules is that of constructing an L-exhaustive calculus for arithmetic." [Carnap 1943: 143]

(1) and (2) from above.

(3) $\{\phi v\}^{\&} \vdash \phi t$

- Rule (3) refers to a transfinite conjunctive.
- $(\forall v)\phi v \vdash \{\phi v\}^{\&}$, $\{\phi v\}^{\&} \vdash (\forall v)\phi v$, $\{\phi v\}^{\&} \dashv\vdash (\forall v)\phi v$
- "The use of indefinite rules referring to transfinite junctives is necessary "
- "A rule of this kind was first used by Tarski (1927) and Hilbert (1931)" [Carnap 1943:145]

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McGee's *Open-Ended Formalization* of Logic

- [V. McGee 2000, 2015] has recently argued that the formal rules of classical natural deduction uniquely determine the semantic values of the logical connectives and quantifiers if these rules are open-ended, i.e., if they are sound not only within a certain language, but they remain sound in any mathematically possible extension of that language.
- The requirement of open-endedness is meant to supplement [N. Belnap 1960]'s criteria (conservativeness and uniqueness) that a rule should satisfy for the acceptability of the connective that it introduces.
  - Conservativeness guarantees that the addition of a new connective creates a conservative extension of the initial language (i.e., it adds no new truths about the initial language).
  - Uniqueness guarantees that a rule which introduces a new connective is such that it allows precisely one *inferential role* for that connective (i.e., if there are two syntactical connectives, $*_1$ and $*_2$, that obey the same formal rules and σ' is a sentence obtained from σ by replacing each occurrence of $*_1$ with $*_2$, then σ and σ' are interderivable).

McGee's *Open-Ended Formalization* of Logic

- [J. H. Harris 1982] proved that the formal rules of natural deduction for classical propositional connectives and quantifiers, indeed, do satisfy the uniqueness condition.

- [McGee 2000: 67] takes this result as showing that the rules of classical natural deduction "uniquely pin down the semantic role of the connectives and quantifiers", if they are open-ended. The semantic role of a sentence is taken by McGee to be determined, "uniquely up to logical equivalence, by indicating the models in which the sentence is true."

- McGee's proposal could be seen as an attempt to offer what [Carnap 1943] called a full formalization of classical logic, i.e., a formalization that uniquely represents all the semantic properties of the logical terms.

McGee's *Open-Ended Formalization* of Logic

- Does an open-ended formalization of propositional classical logic, however, eliminate the non-normal interpretations?
- In particular, since [Carnap 1943: 84, T16-3] proved that if negation has a normal interpretation, then all the other propositional connectives also have a normal interpretation, the problem that has to be analyzed is whether the open-ended rules for negation are categorical.
  - In a non-normal interpretation, disjunction violates the fourth row from the NTT. Nevertheless, if negation is normal, then disjunction is also normal, otherwise the Disjunctive Syllogism Rule (AvB, ~A ⊢ B) would become unsound (i.e., if "A" and "B" are false and negation is normal (thus, "~A" is true), then "AvB" cannot be true). However, since negation and disjunction form a functionally complete set of connectives, then all the other connectives will be normal.

  - Some of Harris' results had been already known by [Carnap 1943: 32-33, T8-9]. He showed that, under ordinary conditions, if a propositional calculus contains two signs for negation ('$\sim_1$' and '$\sim_2$'), then for any closed sentence σ, $\sim_1\sigma$ and $\sim_2\sigma$ are syntactically interderivable and interchangeable (C-equivalent and C-interchangeable in Carnap's terms).

McGee's *Open-Ended Formalization* of Logic

- There are two mutually exclusive kinds of non-normal interpretations for propositional calculi (and in particular for classical negation):
  - non-normal interpretations in which a sentence and its negation are both true (and, thus, all the sentences are true)
  - and non-normal interpretations in which they are both false (and their disjunction is true).

  - [McGee 2000: 71] assumes that there is no model in which all sentences are true and, thus, excludes by stipulation the first kind of non-normal interpretations.
  - However, he argues that if the rules are open-ended, then there is no model in which a sentence and its negation are both false (i.e., the second kind of non-normal interpretations is not possible).

Are the Open-Ended Rules for Negation Categorical?

- McGee's argument goes as follows:

(1) Let θ be a sentence that is true in just those models in which neither ϕ nor $\sim\phi$ is true.

(2) There are no models in which θ and ϕ are both true. **(1)**

(3) $\{\theta, \phi\} \vdash \curlywedge$ **(2)**

(4) $\{\theta\} \vdash \sim\phi$ **(3) (Rule: If $\Gamma \cup \{\phi\} \vdash \curlywedge$, then $\Gamma \vdash \sim\phi$)**

(5) There are no models in which θ and $\sim\phi$ are both true. **(1)**

(6) $\{\theta, \sim\phi\} \vdash \curlywedge$ **(5)**

(7) $\{\theta\} \vdash \sim\sim\phi$ **(6) (Rule: If $\Gamma \cup \{\phi\} \vdash \curlywedge$, then $\Gamma \vdash \sim\phi$)**

(8) $\{\sim\phi, \sim\sim\phi\} \vdash \curlywedge$

(9) $\{\theta\} \vdash \curlywedge$ **(4), (7), (8)**

(10) There are no models in which θ is true. **(9)**

(11) In every model, either ϕ is true or $\sim\phi$ is true. **(1), (10)**

- The general structure of the argument could be seen as a *reductio ad absurdum*.
- It is assumed that the language of propositional logic is extended by adding a sentence that is true *just* in those models in which neither ϕ nor $\sim\phi$ is true.
- This is an instance of [McGee 2000: 70]'s general assumption that "for any class of models, there is a mathematically possible language in which there is a sentence true just in those models."
- On this assumption, by validly reasoning in the meta-theory, it follows (at line 9) that θ is inconsistent and, thus, that it has no model (at line 10).
- Therefore, McGee concludes that in every model, either ϕ is true or $\sim\phi$ is true.

Are the Open-Ended Rules for Negation Categorical?

'Everything' 71

The open-endedness of the rules gives us the classical truth condition for negation, namely,

$\neg\phi$ is true in a model if and only if ϕ is not true in that model.

That ϕ and $\neg\phi$ are not both true in any model follows immediately from rule 6, given our assumption that there is no model in which every sentence is true. To see that one or the other of them must be true in each model, introduce a new sentence θ, true in just those models in which neither ϕ nor $\neg\phi$ is true. Then, there are no models in which θ and ϕ are both true, so that '$\perp$' is a logical consequence of $\{\theta, \phi\}$. It follows by rule 7a that $\neg\phi$ is a logical consequence of $\{\psi\}$. Likewise, there are no models in which θ and $\neg\phi$ are both true, so that '$\perp$' is a logical consequence of $\{\theta, \neg\phi\}$ and, by rule 7a, $\neg\neg\phi$ is a logical consequence of $\{\theta\}$. Since both $\neg\phi$ and $\neg\neg\phi$ are logical consequences of $\{\theta\}$ and, by rule 6, '$\perp$' is a logical consequence of $\{\neg\phi, \neg\neg\phi\}$, '$\perp$' is a logical consequence of $\{\theta\}$. Thus there are no models in which θ is true, and so, in every model, either ϕ is true or $\neg\phi$ is true.

Plan of the Talk

I. **The Non-Categoricity of Classical Logic**
- Non-normal Interpretations of Classical Logic.
- Negation and Non-Categoricity

II. **McGee's *Open-Ended Formalization* of Logic**
- Conservativeness, Uniqueness and Open-endedness
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III. **Are the Open-Ended Rules for Negation Categorical?**
- The Model-Theoretic Assumption (MTA)
- MTA and the Non-normal Models of Negation

Are the Open-Ended Rules for Negation Categorical?

- McGee's argument goes as follows:

  (1) Let θ be a sentence that is true in just those models in which neither ϕ nor $\sim\phi$ is true.
  (2) There are no models in which θ and ϕ are both true. **(1)**
  (3) $\{\theta, \phi\} \vdash \curlywedge$ **(2)**
  (4) $\{\theta\} \vdash \sim\phi$ **(3) (Rule: If $\Gamma \cup \{\phi\} \vdash \curlywedge$, then $\Gamma \vdash \sim\phi$)**
  (5) There are no models in which θ and $\sim\phi$ are both true. **(1)**
  (6) $\{\theta, \sim\phi\} \vdash \curlywedge$ **(5)**
  (7) $\{\theta\} \vdash \sim\sim\phi$ **(6) (Rule: If $\Gamma \cup \{\phi\} \vdash \curlywedge$, then $\Gamma \vdash \sim\phi$)**
  (8) $\{\sim\phi, \sim\sim\phi\} \vdash \curlywedge$
  (9) $\{\theta\} \vdash \curlywedge$ **(4), (7), (8)**
  (10) There are no models in which θ is true. **(9)**
  (11) In every model, either ϕ is true or $\sim\phi$ is true. **(1), (10)**

- The general structure of the argument could be seen as a *reductio ad absurdum*.
- It is assumed that the language of propositional logic is extended by adding a sentence that is true *just* in those models in which neither ϕ nor $\sim\phi$ is true.
- This is an instance of [McGee 2000: 70]'s general assumption that "for any class of models, there is a mathematically possible language in which there is a sentence true just in those models."
- On this assumption, by validly reasoning in the meta-theory, it follows (at line 9) that θ is inconsistent and, thus, that it has no model (at line 10).
- Therefore, McGee concludes that in every model, either ϕ is true or $\sim\phi$ is true.

Are the Open-Ended Rules for Negation Categorical?

- It seems to me that the derivation of (10') is a *non-sequitur.*

- What we could legitimately do after line (10), when we implicitly find out that there is a contradiction between (1) and (10), is to deny assumption (1), i.e., to derive that: (10`) *it is not the case that* θ is a sentence that is true just in those models in which neither ϕ nor $\sim\phi$ is true.

- However, (10`) could be true in two cases: (a) θ is a sentence that is true in those models in which neither ϕ nor $\sim\phi$ is true, but not only in them, or (b) there are no models in which neither ϕ nor $\sim\phi$ is true and, thus, *a fortiori*, θ could not be true.

- McGee offers no reason for excluding option (a). However, due to Carnap's results, I argue that (a) is what actually makes (10`) true.

Are the Open-Ended Rules for Negation Categorical?

- The propositional calculus is sound with respect to a model in which neither ϕ nor $\sim\phi$ is true and, as a matter of *mathematical fact*, there exists such a model, i.e., the one that satisfies each and every theorem of the calculus and satisfies no non-theorem (let us note with N this model).
- To see that the propositional calculus is sound in this model, let Γ be an arbitrary set of premises and σ an arbitrary sentence in the language of propositional calculus, and let us further suppose that $\Gamma \vdash \sigma$.
- There are two cases to be considered: Γ contains only theorems or it contains at least one non-theorem. If Γ contains only theorems, then σ will also be a theorem and, thus, true in the model N. If Γ contains at least one non-theorem, then the sequent $\Gamma \vdash \sigma$ will be valid even if σ is false.

Are the Open-Ended Rules for Negation Categorical?

- McGee assumes that θ is true in just those models in which neither ϕ nor $\sim\phi$ is true (let M be this class of models).
- Since neither ϕ nor $\sim\phi$ is true in N, it follows that N is a member of M. Thus, due to assumption (1), θ is true in N. But if θ is true in N, it follows that θ is a theorem, because only the theorems are true in N.
- However, if θ is a theorem of the propositional calculus, it cannot be true *just* in M, but in all the models of the propositional calculus.
- Thus, the starting assumption of McGee is false; θ is not true *just* in M. We see thus that the existence of the model N falsifies assumption (1).

Are the Open-Ended Rules for Negation Categorical?

- Now, since we have a logical reason to take assumption (1) to be false, we may reconsider McGee's argument.
- Naturally, since assumption (1) is false, it leads to a contradiction.
- Formulated explicitly, assumption (1) is a conjunction of "θ is true in M" and "there are no other models, besides those from M, in which θ is true". Since (1) leads to a contradiction, we have to deny it.
- Hence, by DeMorgan's rules, we obtain the disjunction of:

(A) "it is not the case that θ is a true sentence in M"
(B) "there is at least one model, besides M, that satisfies θ".

- This disjunction is indeed true, because θ could be either a theorem, or a non-theorem.
  - If θ is a theorem, then (B) is true, because all the theorems of propositional calculus are true in all the models of propositional calculus, not only in M, and, thus, (10`) is true.
  - If θ is not a theorem, then (A) is true, because all non-theorems are false in N and, since N belongs to M, θ will not be generally true in M. Hence, (10`) is true. Therefore, McGee's argument from (1) to (10) is valid, but the derivation of (10') is a *non-sequitur*.

Are the Open-Ended Rules for Negation Categorical?

- Another way of looking to McGee's is to use the resources of set theory. We can read McGee's argument as starting from the universal sentence (1), where W is an arbitrary model.

1.	$(\forall W)[(W\vDash\theta) \leftrightarrow ((W \nvDash \varphi) \& (W\nvDash \sim \varphi))]$	(McGee's starting premise)
2.	$(\forall W) [(W \nvDash \varphi) \& (W\nvDash \sim \varphi)] \rightarrow (\forall W)(W\vDash\theta)$	(from 1, by the distribution of ∀ over →)
3.	$(\forall W) [(W \nvDash \varphi)\& (W\nvDash \sim \varphi)] \rightarrow (\exists W)(W\vDash\theta)$	(from 2, a weker statement)
4.	$(\exists W)(W\vDash\theta) \vdash \curlywedge$	(by McGee's argument)
5.	$\sim (\forall W) [(W \nvDash \varphi)\& (W\nvDash \sim \varphi)]$	(from 4, by contraposition)
6.	$(\exists W)((W\vDash \varphi) \vee (W\vDash \sim \varphi)]$	(from 5, by QS and DeMorgan)

- What follows from (5) is an existential statement, and not an universal one, as McGee would want. This last statement, however, is perfectly compatible with the existence of non-normal interpretations of classical negation.

Are the Open-Ended Rules for Negation Categorical?

- As [McGee 2000: 72] puts it, correctly I think, we have to go beyond language in order to determine whether a sentence is true in a particular model (so to say: "we have to take a look at the model").
- We cannot simply stipulate that in whatever context, there is a sentence that is true *just* in a certain class of models. Actually, since the soundness theorem for classical propositional logic is insufficient to pin down uniquely the intended meanings of all its connectives, the open-endedness understanding of the rules will neither work for this job.
- An open-ended understanding of the rules for negation will not be able to eliminate a model in which a sentence and its negation are both false and their disjunction is true. The propositional calculus is sound and complete with respect to this model.
- Thus, an extension of the propositional language with a sentence that is true *just* in the class of models in which a sentence and its negation are false is not a "mathematically possible extension".
- The existence of the non-normal interpretations of the second kind for classical propositional calculi shows that [McGee 2000: 70]'s assumption that "for any class of models, there is a mathematically possible language in which there is a sentence true just those models" is not universally true.

Thank you for your attention!

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Stability in Sequent Calculus

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Harmony and stability

- One of the basic ideas of proof-theoretic semantics is that the behavior of logical constants should be completely characterized by their inference rules.
- In order to obtain this complete characterization, inference rules are asked to satisfy some specific properties.
- According to Dummett (1991), two of the most basic properties are those of *harmony* and *stability*.
- Harmony has been extensively studied, and it is now quite common to understand it in terms of Prawitz's inversion principle.
  Only recently, instead, some attempts to formally capture stability have been proposed: Francez (2017), Jacinto & Read (2017) Tranchini (2016, 2018).
- These attempts share many similarities. Here, we made the choice to focus on just one of them, namely, Tranchini's account.

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Dummett's stability

- According to Dummett, harmony is not enough in order to completely determine the behavior of a connective through its inference rules. This leads him to add the requirement of stability.
- He presents it through an example. He considers standard disjunction

$$\frac{A}{A \vee B}\vee_{I_1} \qquad \frac{B}{A \vee B}\vee_{I_2} \qquad \frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}\vee_E$$

  and quantum disjunction

$$\frac{A}{A \,\bar{\vee}\, B}\bar{\vee}_{I_1} \qquad \frac{B}{A \,\bar{\vee}\, B}\vee_{I_2} \qquad \frac{A \,\bar{\vee}\, B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}\bar{\vee}_E$$

  with the proviso that *no side assumptions* have to be present in the sub-derivations of the two minor premisses of the $\bar{\vee}_E$ rule.
- Both of these connectives are characterized by harmonious rules, but only the rules of standard disjunction are stable.

Tranchini's account of stability

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- Tranchini calls this proof-transformation a *generalized expansion*.

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since the sub-derivation $\mathcal{D}'$ could contain side assumptions, and thus the application of the rule $\bar{\vee}_E$ would not be correct.

From simple expansion to generalized expansion

- A *simple expansion* can be understood as an expansions made on the conclusion of a derivation, i.e.

$$\begin{array}{c} \mathcal{D} \\ A \vee B \end{array} \quad \text{becomes} \quad \dfrac{\begin{array}{c} \mathcal{D} \\ A \vee B \end{array} \qquad \dfrac{\overset{m}{A}}{A \vee B}\,\vee_{I_1} \qquad \dfrac{\overset{n}{B}}{A \vee B}\,\vee_{I_2}}{A \vee B}\,\vee_E(m,n)$$

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- By composing the previous derivation with a derivation

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one obtains that

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$$\begin{array}{c}A\vee B\\ \mathcal{D}'\\ C\end{array}$$

and by permuting then the latter upward with respect to the minor premisses of $\vee_E$, one obtains that

$$\begin{array}{c}\mathcal{D}\\ [A\vee B]\\ \mathcal{D}'\\ C\end{array} \quad\text{becomes}\quad \dfrac{\begin{array}{c}\mathcal{D}\\ A\vee B\end{array} \qquad \begin{array}{c}\dfrac{\overset{m}{A}}{[A\vee B]}\vee_{I_1}\\ \mathcal{D}'\\ C\end{array} \qquad \begin{array}{c}\dfrac{\overset{n}{B}}{[A\vee B]}\vee_{I_2}\\ \mathcal{D}'\\ C\end{array}}{C}\vee_E(m,n)$$

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- Consider the derivation:

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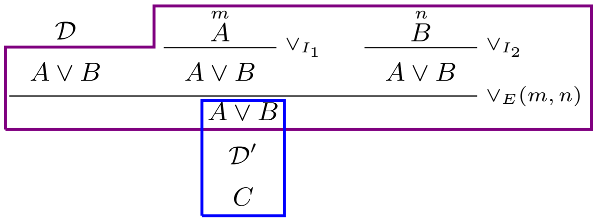

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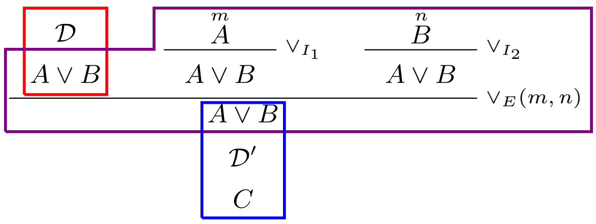

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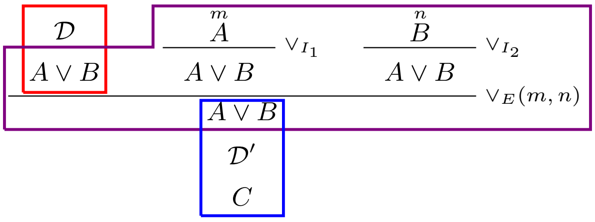
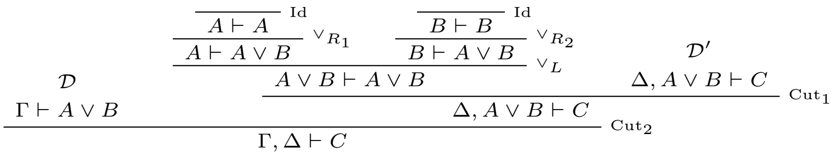


- In the sequent calculus setting, the cut rule allows one to make explicit the order in which the (three) sub-derivations have been composed.



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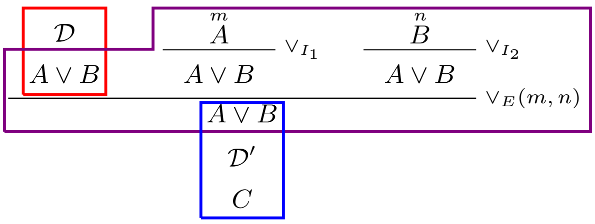

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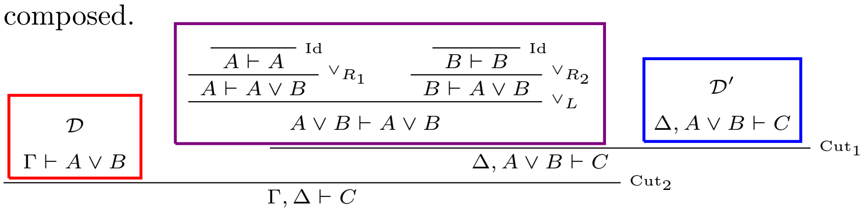

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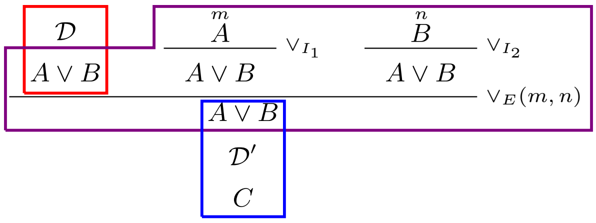

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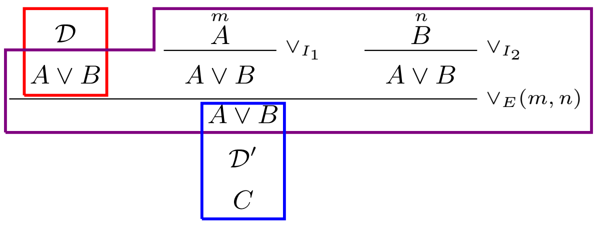

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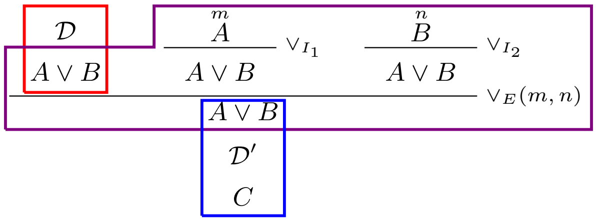

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- Stability corresponds then to a specific kind of reduction of the *height* of a cut.

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- A possible solution can be found by considering a sort of dual of stability.

The dual of stability

- Consider again the derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A \to B\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Id} \qquad \dfrac{}{B \vdash B}\,\text{Id}}{A \to B, A \vdash B}\,{\to_L}}{A \to B \vdash A \to B}\,{\to_R} \qquad \begin{array}{c}\mathcal{D}'\\ \Delta, A \to B \vdash C\end{array}}{\Delta, A \to B \vdash C}\,\text{Cut}_1}{\Gamma, \Delta \vdash C}\,\text{Cut}_2$$

  and switch the order of the cuts (i.e. let Cut_2 pass over Cut_1):

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A \to B\end{array} \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Id} \qquad \dfrac{}{B \vdash B}\,\text{Id}}{A, A \to B \vdash B}\,{\to_L}}{A \to B \vdash A \to B}\,{\to_R}}{\Gamma \vdash A \to B}\,\text{Cut}_2 \qquad \begin{array}{c}\mathcal{D}'\\ \Delta, A \to B \vdash C\end{array}}{\Gamma, \Delta \vdash C}\,\text{Cut}_1$$

- Cut_2 can be permuted upward with respect to the rule $\to_R$, i.e.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A \to B\end{array} \qquad \dfrac{\dfrac{}{A \vdash A}\,\text{Id} \qquad \dfrac{}{B \vdash B}\,\text{Id}}{A, A \to B \vdash B}\,{\to_L}}{\Gamma, A \vdash B}\,\text{Cut}_2}{\Gamma \vdash A \to B}\,{\to_R} \qquad \begin{array}{c}\mathcal{D}'\\ \Delta, A \to B \vdash C\end{array}}{\Gamma, \Delta \vdash C}\,\text{Cut}_1$$

- The same holds for $\parr$. By considering stability and its dual together, MLL becomes an acceptable system.

Intuitionistic implication and higher-level rules

- By considering the sequent calculus version of Schroeder-Heister's general elimination rules, as presented by Avron (1990), one can prove stability also for standard intuitionistic implication.
- Consider a language containing:
  - Formulas: $A ::= p \,|\, \bot \,|\, A \wedge A \,|\, A \vee A \,|\, A \to A$
  - Rules: $R ::= A \,|\, R_1, \ldots, R_n \Rightarrow A$
- For implication we have the following inference rules:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\to_R \qquad \frac{\Gamma, A \Rightarrow B \vdash C}{\Gamma, A \to B \vdash C}\to_L$$

where the behavior of $\Rightarrow$ is governed by the inference rule:

$$\frac{\Delta_1, \Gamma_1 \vdash A_1 \qquad \ldots \qquad \Delta_n, \Gamma_n \vdash A_n}{\Delta_1, \ldots, \Delta_n, ((\Gamma_1 \Rightarrow A_1), \ldots, (\Gamma_n \Rightarrow A_n) \Rightarrow A) \vdash A}\Rightarrow\vdash$$

- The following rule is admissible from the rule $\Rightarrow\vdash$:

$$\frac{\Delta \vdash A \qquad \Sigma, B \vdash C}{\Delta, \Sigma, A \Rightarrow B \vdash C}\Rightarrow_L$$

Intuitionistic implication and higher-level rules

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- And stability holds:

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma \vdash A \to B\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{Id} \qquad \dfrac{}{B \vdash B}\,\mathrm{Id}}{A, A \Rightarrow B \vdash B}\,{\Rightarrow_L}}{A \Rightarrow B \vdash A \to B}\,{\to_R}}{A \to B \vdash A \to B}\,{\to_L} \qquad \begin{matrix}\mathcal{D}'\\ \Delta, A \to B \vdash C\end{matrix}}{\Delta, A \to B \vdash C}\,\mathrm{Cut}_1}{\Gamma, \Delta \vdash C}\,\mathrm{Cut}_2$$

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Unifying perspective through sequent calculus

- According to Dummett, harmony and stability are two desirable properties for the rules governing the behavior of a connective.
- We have translated these properties into sequent calculus, and the way in which sequent calculus is designed suggested us to consider another property, the dual of stability.
- These properties, in sequent calculus, can be understood as formal operations traditionally used in cut elimination proofs.
  - Harmony corresponds to the operation of reducing a principal cut, namely, a cut in which the two cut formulas are both principal and coming from a logical rule.
  - Stability and its dual correspond to the operation of reducing a particular kind of non-principal cut, namely, a cut in which one of the two cut formulas is not principal in the conclusion of an identity derivation.

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- The foregoing $\rightarrow_R$ rule does not go via $\Rightarrow$: this may be seen as a lack of symmetry from an inferential point of view.

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$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow_R \qquad \dfrac{\Gamma \vdash A \Rightarrow B}{\Gamma \vdash A \rightarrow B} \rightarrow_R$$

- Consider the principal case for $\rightarrow$:

$$\dfrac{\dfrac{\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow_R}{\Gamma \vdash A \rightarrow B} \rightarrow_R \qquad \dfrac{\dfrac{\Delta_1 \vdash A \qquad \Delta_2, B \vdash C}{\Delta_1, \Delta_2, A \Rightarrow B \vdash C} \Rightarrow_L}{\Delta_1, \Delta_2, A \rightarrow B \vdash C} \rightarrow_L}{\Gamma, \Delta_1, \Delta_2 \vdash C} \text{Cut}$$

A potential conceptual problem for implication

- The foregoing $\rightarrow_R$ rule does not go via $\Rightarrow$: this may be seen as a lack of symmetry from an inferential point of view.
- Symmetry can be re-established by using the following two rules:

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow_R \qquad \dfrac{\Gamma \vdash A \Rightarrow B}{\Gamma \vdash A \rightarrow B} \rightarrow_R$$

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Its reduction has to be done in two steps:

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12 / 14

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Its reduction has to be done in two steps:

$$\dfrac{\dfrac{\Gamma, A \vdash B \qquad \Delta_1 \vdash A}{\Gamma, \Delta_1 \vdash B}\text{Cut} \qquad \Delta_2, B \vdash C}{\Gamma, \Delta_1, \Delta_2 \vdash C}\text{Cut}$$

- The first step of the reduction generates a principal case for $\Rightarrow$.

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- If $\Rightarrow$ is treated like a connective, then, contrary to what usually happens with the reduction of the principal cases, the first step does not reduce the complexity of the cut formula.

A potential conceptual problem for implication

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- Symmetry can be re-established by using the following two rules:

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- What is reduced instead is the height of the cut, as it happens for the non-principal cases.

A potential conceptual problem for implication

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- The distinction between the cut elimination cases used before becomes then blurred.

Closing remarks

- Some cases of Dummett's stability – once they are understood in terms of Tranchini's generalized expansion – can be rephrased in sequent calculus as the permutation of a cut on the succedent of the pertinent identity sequent. E.g. additive disjunction.
- There are other cases that are difficult to understand in these terms. E.g. multiplicative disjunction, standard (i.e. multiplicative) intuitionistic implication.
- However, we have seen that there are a few ways of accommodating the situation in sequent calculus.

  As it concerns implication, these ways are:
  - The dual of stability: permuting cut on the antecedent of the pertinent identity sequent.
  - Introducing $\rightarrow$ to the antecedent via the higher-level rule $L_{\Rightarrow}$.

  For the other connectives (conjunction and disjunction):
  - Using structural rules in the derivation of the identity sequent.
- Is there one of these solutions that should be preferred to the others?

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Identity and definition in natural deduction

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Philosophical Institute, Czech Academy of Sciences

Proof-Theoretic Semantics
Tübingen, 29 March 2019

: 1

Natural deduction rules for identity

Introduction rule:

$$t = t$$

Elimination rule:

$$\frac{t = u \qquad A[x, x]}{A[t, u]}$$

An application of this rule binds the variable x in the subderivation ending in $A[x, x]$.

Reduction procedure:

$$\frac{t = t \qquad \begin{matrix}\mathcal{D}[x] \\ A[x, x]\end{matrix}}{A[t, t]} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}[t] \\ A[t, t]\end{matrix}$$

: 2

Two apparent problems with this treatment of identity

1. The major premiss of the elimination rule is $t = u$, whereas the conclusion of the introduction rule is $t = t$.
2. Consider Peano arithmetic formalized in natural deduction. It seems that the axioms

$$x + 0 = x$$
$$x + \mathbf{s}(y) = \mathbf{s}(x + y)$$

must be regarded as introduction rules for the identity predicate.

: 3

Prawitz's notion of validity of derivations

Certain closed derivations $\mathcal{D}$ are deemed valid outright provided the immediate subderivations of $\mathcal{D}$ are valid.

These are called *canonical* derivations.

A non-canonical closed derivation is valid if it reduces to a valid canonical derivation.

An open derivation $\mathcal{D}$ is valid if any closed instance of $\mathcal{D}$ is valid.

: 4

Canonical derivations

Gentzen suggested that we regard the introduction rules for an operator as its definition.

Following this suggestion, it is natural to say that $\mathcal{D}$ is a canonical derivation if its ends in an introduction rule.

The validity of rules

A rule ρ is valid if any derivation $\mathcal{D}$ that ends in ρ is valid whenever the immediate subderivations of $\mathcal{D}$ are valid.

Introduction rules are therefore valid by definition.

But elimination rules require justification.

Attempted justification of =-elimination

Assume that we have valid derivations

$$\begin{array}{c} \mathcal{D}_1 \\ t = u \end{array} \qquad \begin{array}{c} \mathcal{D}_2 \\ A[x,x] \end{array}$$

We must show that the derivation

$$\frac{\begin{array}{c} \mathcal{D}_1 \\ t = u \end{array} \qquad \begin{array}{c} \mathcal{D}_2 \\ A[x,x] \end{array}}{A[t,u]}$$

is valid.

We may assume $\mathcal{D}_1$ to be closed and valid; hence it reduces to a valid canonical derivation

$$\begin{array}{c} \mathcal{D}_1' \\ t = u \end{array}$$

But our stipulations fail to specify what a canonical derivation of $t = u$ looks like.

The two problems again

1. We cannot justify =-elimination on the basis of =-introduction.
2. The meaning of the identity predicate will change from theory to theory, since its introduction rule(s) will change. Hence, identity is not topic-neutral.

Definitional identity

Definitional identity $a \equiv b$ is an equivalence relation on expressions generated by the rules:

$$\textit{definiendum} \equiv \textit{definiens}$$

$$\frac{a \equiv b}{a[t/x] \equiv b[t/x]} \qquad \frac{a \equiv b[c]_! \qquad c \equiv c'}{a \equiv b[c']_!}$$

In richer languages also other rules may be considered.

Adjoining definitional identity to natural deduction

A theory of definitional identity may be adjoined to natural deduction via the following rule:

$$\frac{\begin{array}{c}\mathcal{D}\\ A[b]_!\end{array} \qquad \begin{array}{c}\mathfrak{D}\\ b \equiv c\end{array}}{A[c]_!}$$

We call this rule *formula conversion*.

A novel conception of canonical derivation

A derivation $\mathcal{D}$ is canonical if it ends in an introduction rule followed by any number, possibly zero, of formula conversions.

Thus a canonical derivation has the following form:

$$\dfrac{\dfrac{\mathcal{D}_1 \quad \dots \quad \mathcal{D}_n}{\Phi\bar{x}.(\bar{A},\bar{t})}\ \Phi\text{-introduction}}{B}\ \text{formula conversions}$$

The conclusion B is just a rewriting of $\Phi\bar{x}.(\bar{A},\bar{t})$.

Novel reductions 1

It is immaterial precisely which conversions are applied in getting from $\Phi\bar{x}.(\bar{A},\bar{t})$ to B.

We are therefore led to postulate all reductions of the following form:

$$\begin{array}{c} \mathcal{D} \\ A \\ \Delta \\ B \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \mathcal{D} \\ A \\ \Delta' \\ B \end{array} \qquad (\Delta\text{-red})$$

Here Δ and Δ' are derivations consisting entirely of conversions.

Novel reductions 2

Prawitz's reductions are defined for derivations ending in

introduction rule + elimination rule

We must define reduction for derivations ending in

introduction rule + formula conversions + elimination rule

For instance:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ B\end{array}}{A \wedge B} \quad \begin{array}{c}\mathfrak{D}_1\\ A \equiv A'\end{array}}{A' \wedge B} \quad \begin{array}{c}\mathfrak{D}_2\\ B \equiv B'\end{array}}{A' \wedge B'}}{A'} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A\end{array} \quad \begin{array}{c}\mathfrak{D}_1\\ A \equiv A'\end{array}}{A'}$$

Identity revisited

Definitions are dealt with in a separate theory of definitional identity.

Hence, definitions do not threaten the topic-neutrality of identity.

Moreover, a canonical derivation of $t = u$ may be assumed to have the form

$$\dfrac{\dfrac{t' = t' \quad \begin{array}{c}\mathfrak{D}_1\\ t' \equiv t\end{array}}{t = t'} \quad \begin{array}{c}\mathfrak{D}_2\\ t' \equiv u\end{array}}{t = u}$$

Novel reductions 3

For a derivation ending in

$$=\text{-introduction} + \text{formula conversions} + =\text{-elimination}$$

we stipulate the following reduction:

$$\dfrac{\dfrac{\dfrac{t'=t' \qquad \begin{array}{c}\mathfrak{D}_1\\ t'\equiv t\end{array}}{t=t'} \qquad \begin{array}{c}\mathfrak{D}_2\\ t'\equiv u\end{array}}{t=u} \qquad \begin{array}{c}\mathcal{D}[x]\\ A[x,x]\end{array}}{A[t,u]}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}[t']\\ A[t',t']\end{array} \qquad \begin{array}{c}\mathfrak{D}_1\\ t'\equiv t\end{array}}{A[t,t']} \qquad \begin{array}{c}\mathfrak{D}_2\\ t'\equiv u\end{array}}{A[t,u]}$$

Justification of =-elimination

Assume that we have valid derivations

$$\begin{array}{c}\mathcal{D}_1\\ t=u\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ A[x,x]\end{array}$$

We must show that the derivation

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ t=u\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ A[x,x]\end{array}}{A[t,u]}$$

is valid.

We may assume this derivation to be closed. Hence we must show that it reduces to a valid canonical derivation.

Justification of =-elimination, contd.

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ t = u\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ A[x,x]\end{matrix}}{A[t,u]}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{t' = t' \qquad \begin{matrix}\mathfrak{D}_1\\ t' \equiv t\end{matrix}}{t = t'} \qquad \begin{matrix}\mathfrak{D}_2\\ t' \equiv u\end{matrix}}{t = u} \qquad \begin{matrix}\mathcal{D}_2[x]\\ A[x,x]\end{matrix}}{A[t,u]}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_2[t']\\ A[t',t']\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\\ t' \equiv t\end{matrix}}{A[t,t']} \qquad \begin{matrix}\mathfrak{D}_2\\ t' \equiv u\end{matrix}}{A[t,u]}$$

Definitional identity in arithmetic

We begin with a language containing only 0, **s**, and variables.

New symbols are defined according to the following schemes.

1. If t is a closed term and c is fresh, then we may stipulate

$$c \equiv t$$

2. If the free variables of t are among $\overline{x}$, and f is fresh, then we may stipulate

$$f(\overline{x}) \equiv t$$

3. If the free variables of t are among $\overline{x}$, the free variables of u are among $\overline{x}, y, z$, and f is fresh, then we may stipulate

$$f(\overline{x}, 0) \equiv t$$
$$f(\overline{x}, \mathbf{s}(y)) \equiv u[\overline{x}, y, f(\overline{x}, y)]$$

An Inferentialist Semantics for Atomics, Predicates, and Names

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Proof Theoretic Semantics Tübingen 29/03/19
Monash University

1

Aims

- Contribute to the literature on PTS for non-logical vocabulary;
- Formalise some of Robert Brandom's ideas about material inferences, and the distinction between names and predicates;
- Use Greg Restall's bilateralist interpretation of the classical multiple conclusion sequent calculus as the basis for a semantics for a language with atomics, predicates, and names;
- Construct a system so that:
  - the inference rules for non-logical vocabulary are of a similar kind to logical vocabulary;
  - neither left (elimination) nor right (introduction) rules are prioritise.

2

Outline

3

Background

Inferentialism

Inferentialism
The meaning of expressions are constituted by their inferential relations.

- Contrast this with views which take notions like reference and representation to be central.
- Thinking of meaning in terms of inference natural lends itself to a proof-theoretic rather than model-theoretical semantics.

4

Bilateralism

Normative Pragmatism
Meaning is determined by norms of use

Bilateralism
The relevant meaning determinate norms are those of both assertion and denial.

- Interpret multiple conclusion sequence calculus rules in terms of these norms.

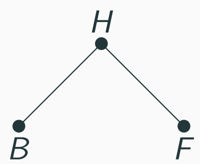
$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\neg\mathrm{L} \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\neg\mathrm{R}$$

5

Example Inference Rules

Humans

- **H** Human
- **B** Biped
- **F** Featherless



$$\frac{X \vdash B\alpha, Y \quad X \vdash F\alpha, Y}{X \vdash H\alpha, Y}\, H\text{R}$$

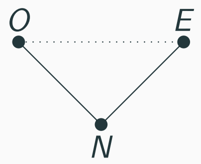
6

Numbers

N Number

O Odd

E Even



$$\frac{X \vdash O\alpha, Y}{X \vdash N\alpha, Y}\,N\mathrm{R}_1 \qquad \frac{X \vdash E\alpha, Y}{X \vdash N\alpha, Y}\,N\mathrm{R}_2$$

$$\frac{X \vdash E\alpha, Y}{X, O\alpha \vdash Y}\,O\mathrm{L} \qquad \frac{X \vdash O\alpha, Y}{X, E\alpha \vdash Y}\,E\mathrm{L}$$

7

Super

S Superman

C Clark

$$\frac{X, \Phi c \vdash Y}{X, \Phi s \vdash Y}\,s\mathrm{L} \qquad \frac{X \vdash \Phi c, Y}{X \vdash \Phi s, Y}\,c\mathrm{R}$$

8

Subatomic Systems

Semantics

Subatomic System
A proof system which assigns inference rules to subatomic vocabulary, such as names and predicates.

Definition
A *subatomic system* is a triple of a language $\mathfrak{L}$, a set of rules R, and a valuation function v.

- $\mathfrak{L}$ is made up of names, predicates, and atomic sentences (formed from the first two).
- R is made up inference rules linking expressions of the same syntactic category in $\mathfrak{L}$.
- v assigns rules from R to expressions in $\mathfrak{L}$.

9

Restricting v 1: Compositionality

Compositionality

The meaning of complex expressions is a function of the meaning of their constituents and the way they combine.

Compositionality Restriction

The rules assigned to atomic sentences are a result of substituting the name(s) in the sentence into the rules assigned to the predicate, and the the predicate into the rules assigned to the name(s).

- E.g The sentence Gb

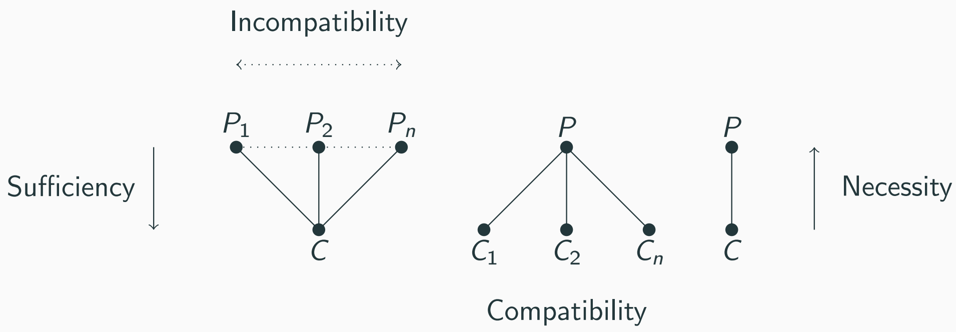
$$\frac{X \vdash F\alpha, Y}{X \vdash G\alpha, Y}\,G\mathrm{R} \quad \Longrightarrow \quad \frac{X \vdash Fb, Y}{X \vdash Gb, Y}\,Gb\mathrm{R}_1$$

$$\frac{X \vdash \Phi a, Y}{X \vdash \Phi b, Y}\,b\mathrm{R} \quad \Longrightarrow \quad \frac{X \vdash Ga, Y}{X \vdash Gb, Y}\,Gb\mathrm{R}_2$$

10

Trees and General Rules

General Trees



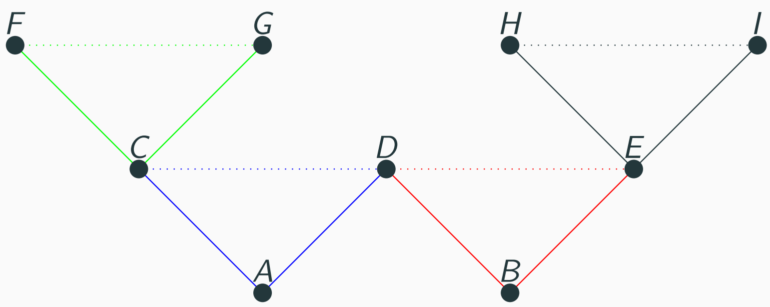
Parent An expression above another connected by a vertical line.

Child An expression below another connected by a vertical line.

Concept Cluster Pick an expression. This expression and all the expressions linked to it by arrows in one direction are all members of the concept cluster.

11

An Example Tree



12

Restricting v 2: General Rules

Figure 1: First Rules

$$\frac{X \vdash P_i, Y}{X \vdash C_i, Y}\,\text{CR} \qquad \frac{X, C_1, \ldots, C_n \vdash Y}{X, P_i \vdash Y}\,\text{PL}_1 \qquad \frac{X \vdash P_i, Y}{X, P_j \vdash Y}\,\text{PL}_2$$

Figure 2: Second Rules

$$\frac{X, P_1 \vdash Y \quad \ldots \quad X, P_n \vdash Y}{X, C_1, \ldots, C_n \vdash Y}\,\text{CL}$$

$$\frac{X \vdash C_1, Y \quad \ldots \quad X \vdash C_n, Y \quad X, P_i \vdash Y \quad \ldots \quad X, P_n \vdash Y}{X \vdash P_j, Y}\,\text{PR}$$

13

Example Rules: {N, O, E} Cluster

$$\frac{X \vdash O, Y}{X \vdash N, Y}\,\text{NR}_1 \qquad \frac{X \vdash E, Y}{X \vdash N, Y}\,\text{NR}_2$$

$$\frac{X, N \vdash Y}{X, O \vdash Y}\,\text{OL}_1 \qquad \frac{X, N \vdash Y}{X, E \vdash Y}\,\text{EL}_1$$

$$\frac{X \vdash E, Y}{X, O \vdash Y}\,\text{OL}_2 \qquad \frac{X \vdash O, Y}{X, E \vdash Y}\,\text{EL}_2$$

$$\frac{X, E \vdash Y \quad X, O \vdash Y}{X, N \vdash Y}\,\text{NL}$$

$$\frac{X \vdash N, Y \quad X, E \vdash Y}{X \vdash O, Y}\,\text{OR} \qquad \frac{X \vdash N, Y \quad X, O \vdash Y}{X \vdash E, Y}\,\text{ER}$$

14

Local Soundness & Completeness

Local Soundness
Our rules are not too strong. What results from applying the second rules to the outputs of the first, is what was used as inputs for our application of the first rules.

Local Completeness
Our rules are not too weak. What results from applying the first rules to the outputs of the second, is what was used as inputs for our application of the second rules.

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Local Soundness I

$$\dfrac{\dfrac{\overset{\vdots\,\pi}{X \vdash P_i, Y}}{X \vdash C_1}\,C_1\mathrm{R} \quad \dfrac{\overset{\vdots\,\pi}{X \vdash P_i, Y}}{X \vdash C_n}\,C_n\mathrm{R} \quad \dfrac{\overset{\vdots\,\pi}{X \vdash P_i, Y}}{X, P_j \vdash Y}\,P_j\mathrm{L}_2 \quad \dfrac{\overset{\vdots\,\pi}{X \vdash P_i, Y}}{X, P_{n-i} \vdash Y}\,P_n\mathrm{L}_2}{X \vdash P_i, Y}\,P\mathrm{R}$$

$$\Longrightarrow \overset{\vdots\,\pi}{X \vdash P_i, Y}$$

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Local Soundness II

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi\\ X, C_1, ..., C_n \vdash Y\end{array}}{X, P_1 \vdash Y}\,P_1L_1 \qquad \dfrac{\begin{array}{c}\vdots\ \pi\\ X, C_1, ..., C_n \vdash Y\end{array}}{X, P_n \vdash Y}\,P_nL_1}{X, C_1, ..., C_n \vdash Y}\,CL$$

$$\Longrightarrow \begin{array}{c}\vdots\ \pi\\ X, C_1, ..., Cn \vdash Y\end{array}$$

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Local Completeness

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ X \vdash C_1, Y\end{array} \quad \begin{array}{c}\vdots\ \pi_n\\ X \vdash C_n, Y\end{array} \quad \begin{array}{c}\vdots\ \delta_i\\ X, P_i \vdash Y\end{array} \quad \begin{array}{c}\vdots\ \delta_{n-j}\\ X, P_{n-j} \vdash Y\end{array}}{X \vdash P_j, Y}\,P\mathrm{R}}{X \vdash C_1}\,C_1\mathrm{R}$$

$$\Longrightarrow \begin{array}{c}\vdots\ \pi_1\\ X, \vdash C_1, Y\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ X \vdash C_1, Y\end{array} \quad \begin{array}{c}\vdots\ \pi_n\\ X \vdash C_n, Y\end{array} \quad \begin{array}{c}\vdots\ \delta_i\\ X, P_i \vdash Y\end{array} \quad \begin{array}{c}\vdots\ \delta_{n-j}\\ X, P_{n-j} \vdash Y\end{array}}{X \vdash P_j, Y}\,P\mathrm{R}}{X, P_i \vdash Y}\,P_i\mathrm{L}_2$$

$$\Longrightarrow \begin{array}{c}\vdots\ \delta_1\\ X, P_i \vdash Y\end{array}$$

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Cut Elimination

$$\frac{X \vdash S, Y \quad X', S \vdash Y'}{X, X' \vdash Y, Y'}\,\mathrm{Cut}$$

- Cut is a structural rule applying to sequents in general.
- A cut-elimination proof shows that any sequent that is derivable using Cut is derivable without Cut .
    - A consequence of cut-elimination is that the proof system is globally sound.
- Cut-elimination is provable for the above general rules.

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Identity

$$\frac{}{p \vdash p}\,\mathrm{Id}$$

- Axiom in sequent systems;
- Usually define [Id] for atomic formulas and prove that it holds for complex ones;
    - Corresponds to global completeness.
- Assuming identity for parents, identity for children is provable (and vice versa).

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Predicates & Names

Restricting v 3: Predicates & Names

- Differentiate between predicates and names in terms of their inferential structure.
- Restrict v to reflect this.
- Names are assigned symmetric inference rules. These correspond to concept clusters without branching.
- Predicates are assigned asymmetric inference rules. These correspond to concept clusters with branching, upwards or downwards.

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Name Rules

Figure 3: Non-branching cluster

$$\frac{X, \Phi a \vdash Y}{X, \Phi b \vdash Y}\, CL \qquad \frac{X \vdash \Phi a, Y}{X \vdash \Phi b, Y}\, CR$$

$$\frac{X, \Phi b \vdash Y}{X, \Phi a \vdash Y}\, PL \qquad \frac{X \vdash \Phi b, Y}{X \vdash \Phi a, Y}\, PR$$

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Subatomic Identity

- Names which stand in no substitution relations still have meanings;
- Take the identity axiom to apply not to atomic propositions but instead to names in arbitrary predicate contexts:

$$\frac{}{\Phi\alpha \vdash \Phi\alpha}\, Id_s$$

- The identity axiom captures the simple sense of asserting and denying the same thing being out of bounds.

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Limit Positions & Models

Limit Positions

Positions
$X : Y$ is a position iff $X \nvdash Y$.

Extensions
The $X' : Y'$ is an extension of the $X : Y$ iff $X \subseteq X'$ and $Y \subseteq Y'$.

Limit Positions
Limit positions maximally extended positions. Given a language $\mathfrak{L}$, the partition $\mathfrak{X} : \mathfrak{Y}$ is a limit position iff:
(a) whenever $X \subset \mathfrak{X}$ and $Y \subset \mathfrak{Y}$, $X \nvdash Y$; and
(b) $\mathfrak{X} \cup \mathfrak{Y} = \mathfrak{L}$.

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Limit Position Fact

Limit Position Fact
FACT: For any limit position LP:
(i) A parent P_i is to the left of LP iff all its children $C_1 ... C_n$ are to the left of LP and all other parents $P_j ... P_{n-i}$ are to the right of LP;
(i') A parent P_i is to the right of LP iff either some child C_i is to the right of LP or some other parent P_j is to the left of LP;
(ii) A child C is to the left of LP iff a parent P is to the left of LP;
(ii') A child C is to the right of LP iff all its parents $P_1 ... P_n$ are to the right of LP.

- For name concept clusters, because there is only one parent and one child, a parent will be to the left [right] iff its child is to the left [right]. They are symmetric.
- Because predicate concept clusters have one of multiple parents or children they are asymmetric.

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Models

- Limit Positions let us read off model theoretic extensions.
  - Single objects for names; and
  - Sets of objects for predicates.
- Inference rules between names [predicates] dermine relations between their extensions, just like meaning postulates.

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Conclusions

Conclusions

1. Names and predicates are distinguished by the structure of their inference rules (Symmetry & Asymmetry);
2. The rules for atomic sentences are determined by those for their constituents (Compositionality);
3. The system is globally & locally sound and complete, without prioritising either left or right rules (LSC);
4. Standard model-theoretic extensions can be "read off" the inference rules (Limit Positions).

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Thank You!

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Harmony, Higher-Order Rules, and the Curry-Howard-Lambek Correspondence

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Theories of Meaning

Two theories of meaning:

- The referentialist one; meaning is conferred via outward reference to reality or the world.
  - Underlies Model-Theoretic Semantics. Meaning qua denotation or truth-value.
- The inferentialist one; meaning is autonomous within language; nothing outside involved.
  - Underlies Proof-Theoretic Semantics. Meaning is inherent within the internal structure of inferences.

Categorical Semantics can be either of them, suggesting the third way (discussed in my article on categorical harmony in *Adv. in PTS*, 2016).

Synopsis

- Starting with (general elimination) harmony, we elucidate the universal architecture of the Curry-Howard correspondence.
- Given a class of logical constants, harmony allows us to derive the corresponding Curry-Howard isomorphism.
- The scope of CH, nonetheless, is broader than that of harmony, as witnessed, e.g., by Prior's tonk, which admits a CH isomorphism.

Remarks

Curry-Howard-Lambek means the trinity of logic, types, and cats.

- In this talk I focus upon an iso. b/w logic and cats.
  - Type theory is basically a syntax-dependent presentation of cats.
- This work might make better sense of categorical harmony than my work in *Advances in Proof-Theoretic Semantics* (2016).
  - M., Categorical Harmony and Paradoxes in Proof-Theoretic Semantics, chapter of the book, 2016. Discussed adjunction-based harmony.

Lawvere: logical constants as adjojnt functors.
Došen: logical constants as adjonts with respect to structural functors.

Došen (2006)

Došen (2006):

> *However, what I did brings something which I think should be added to Lawvere's thesis: namely, the functor carrying the logical constant should be adjoint to a structural functor recording some features of deduction. With this amendment the thesis might serve to separate logical constants from other expressions.*

NB: he was primarily concerned with logicality rather than harmony.

Yet Another Remark

I do not discuss classical logic, the proof identity of which trivialises (Joyal Lemma), even in the absence of full contraction (Monoidal JL).

- CCC with dualising $\perp$ (i.e. DNE) are Boolean algebras.
  - No proof-relevant categorification of Boolean algebras; cf. bCCC as the categorification of Heyting algebras.
  - Hyperdoctrine-style proof-irrelevant categorical semantics is available for any substructural logic over FL (M. 2013; 2017).
- MCC with contr. dualising $\perp$ and terminal I are ordered monoids.
  - Some believe both contraction and weakening are crucial in JL; yet contraction is only needed for $\perp$ alone (M. 2012).

The latter is actually a version of No-Deleting Theorem in CQM.

General-Elimination Harmony

General-Elimination Harmony:

- GE harmony is a fundamental principle to generate elim. from intro. rules.
- To that end, one needs higher-order rules, in which rules may be discharged as well as propositions. Fix n-ary S.
- I-rules $\Gamma \vdash \Phi_i(A_1, ..., A_n) \Rightarrow \Gamma \vdash S(A_1, ..., A_n)$ for $i = 1, ..., m$ yield E-rule $\Phi_1 \vdash A, ..., \Phi_m \vdash A \Rightarrow S(A_1, ..., A_n) \vdash A$ (contexts omitted).
- When you have no I-rule (resp. E-rule) you have got $\bot$ (resp. $\top$).

Ref: Schroeder-Heister, A Natural Extension of Natural Deduction, 1984.

Examples

Let us have a look at a couple of examples.

- GE harmoninous rules for $\vee$. I-rules: $\Gamma \vdash A_1 \Rightarrow \Gamma \vdash A_1 \vee A_2$; $\Gamma \vdash A_2 \Rightarrow \Gamma \vdash A_1 \vee A_2$. E-rule: $A_1 \vdash A, A_2 \vdash A \Rightarrow A_1 \vee A_2 \vdash A$
- Conjunction. I-rule: $\Gamma \vdash A_1, A_2 \Rightarrow \Gamma \vdash A_1 \wedge A_2$. (Monoidal) E-rule: $A_1, A_2 \vdash A \Rightarrow A_1 \wedge A_2 \vdash A$.
  - Cartesian E-rule: $A_1 \vdash A \Rightarrow A_1 \wedge A_2 \vdash A$ and $A_2 \vdash A \Rightarrow A_1 \wedge A_2 \vdash A$.
- Rules for implication are higher-order. $[A, ..., B]$ denotes a derivation from A to B. I-rule: $\Gamma \vdash [A, ..., B] \Rightarrow \Gamma \vdash A \to B$. E-rule: $[A, ..., B] \vdash C \Rightarrow A \to B \vdash C$.
- Liar logic. I-rule: $[R, ..., \bot] \vdash R$. E-rule: $[R, ..., \bot] \vdash A \Rightarrow R \vdash A$.

Cf. Sambin's reflection principle and Došen's logical constants as punctuation marks. I compared them with categorical harmony in my article above in *Adv. in PTS* (2016).

Three Layers of Structural Proof Theory

Fundamental proof theory, presumably, must be three-layered.

- Background-theory: what you have to presuppose to make it possible to express rules.
  - Part of this is format-theory: e.g., the theory of ',' and '[...]' (MCC); we control structural rules at this level rather than the object-level.
- Object-theory: a formal system expressed on the basis of the background theory. E.g., intuitionisitc logic (bCCC).
  - In this case the format-theory (CCC) embeds into the object-theory (bCCC); yet this is coincidence.
- Meta-theory: what you rely on to reason about properties of the formal system over the background theory. E.g., finitism or set th.

Hilbert: meta-theory = background-theory, both being finitism (PRA).

Putnam on Quine and Wittgenstein (ultimately, Carroll)

It would be impossible to fully explicate our background-theory in view of the fundamental circularity lurking behind the scene:

> *The 'exciting' thesis that logic is true by convention reduces to the unexciting claim that logic is true by convention plus logic. (Putnam 1979)*

We can still explicate the underlying format-theory, which is indeed crucial for categorical semantics because categories must model the structure of the format-theory (',', '[...]', type contexts, etc.) as well.

The Universal Curry-Howard Correspondence

The categorical construction corresponding to S is such that, given $\{f_i : \Gamma \to \Phi_i\}_{i=1,...,m}$ and $\{g_i : \Phi_i \to A\}_{i=1,...,m}$, there are arrows

$$S_r(f_i) : \Gamma \to S$$

and

$$S_l(\{g_i\}_{i=1,...,m}) : S \to A$$

satisfying the following β and η equations:

$$S_l(\{g_i\}_{i=1,...,m}) \circ S_r(f_i) = g_i \circ f_i.$$

$$S_l(\{S_r(id_{\Phi_i})\}_{i=1,..,m}) = id_S.$$

',' and '[...]' are replaced by monoidal prod. $\otimes$ and internal hom [-, -].

Remarks

- You can recover the equational characterisation of CCC from the universal CH correspondence above.
- This includes both positive and negative constructions, giving a logically symmetrised account of different categorical constructions.
- In general, the above (equationally characterised) S-construction cannot be characterised via universality, as monoidal products or modalities cannot. They are purely additional structures.

The Liar Logic and its Categorical Models

The universal CH gives you a CH iso. for logic with the Liar sentence.

- I-rule: $[R, ..., \bot] \vdash R$. E-rule: $[R, ..., \bot] \vdash A \Rightarrow R \vdash A$.
  - It is just another way to say $R \leftrightarrow \neg R$.
  - This Liar equation $R \leftrightarrow \neg R$ can be solved by self-duality $A \simeq A^*$.
- The corresponding "Liar categories" include any compact closed category with self-duality (i.e., $A \simeq A^*$ coherently for any A).
  - A compact closed category is a category with dual objects (namely, A^*, $ev : A \otimes A^* \to I$, and $cv : I \to A \otimes A^*$ for any object A).
  - E.g., $A = A^*$ in **Rel**. Almost the same as structures in CQM.

If "linear logic" means the logic of vector spaces, it is inconsistent (and yet you can make sense of the inconsistency as in comp. closed cats).

Prior's tonk

The scope of CH is wider than that of GE as witnessed by Prior's tonk.

- I-rules: $\Gamma \vdash A_i \Rightarrow \Gamma \vdash A_1 \mathrm{tonk} A_2$. E-rule: $A_i \vdash A \Rightarrow A_1 \mathrm{tonk} A_2 \vdash A$.
  - Causes explosion: $A \vdash B$ for any A, B.
- This pair of rules is not harmonious, and yet admits the CH iso. in the presence of a zero object (terminal and initial at once).
  - This is not an instance of the universal CH above.
- Tonk can be modelled by biproduct, which presupposes a zero object 0, which, in turn, yields a zero proof $0 : A \to B$ for any A, B.
  - In **Vect**, any V indeed follows from any W.
  - cf. Došen-Petric (2007). Zero proofs for classical proof identity.

Inconsistency is not a limitation for the Curry-Howard correspondence.
This particular case of Prior's tonk is discussed in my *Synthese* paper (2016).

Concluding Remarks

Harmony does not imply much (conservativity, normalisability, etc.; Read 2010).

- Yet GE harmony induces a generic family of CH iso's, including, e.g., the one b/w Liar logic and cats. with self-duality.
- The converse does not hold: the GE-harmonious CH iso's are not all of the CH iso's, as witnessed by Prior's tonk.

The machinery tells us categorical constructions share a common logical structure, hinting at something like logical category theory.

Appendix 1: Consistency

- A rule is pure iff it includes no logical constant in the assumptions; it is impure otherwise.
- Purity is the key to ensuring consistency; otherwise there is room for inconsistency, which is, however, not that bad.
- If you keep the logic linear, you could keep consistency even in the presence of impure rules. No proof in a general setting yet.

Appendix 2: Levels of Inconsistency

You have to be careful of different levels of inconsistency (provability-level, proof-level, object-level, and their combinations).

- The Joyal lemma is about the proof-level inconsistency.
- The object-level inconsistency includes the collapse of cartesian comp. closed cats. and the collapse of toposes with fixpoints.
  - Collapsing phenomena are interesting on their own.

NB: all this is proof/type theory in categorical disguise. Where exactly does category theory deviate from proof/type theory?

Modes of assumptions and moods of implications

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Introduction

Standard analysis of counterfactuals: Similarity analysis

$A > B$ is true at world w just in case B is true in all the possible worlds in which A is true that are most *similar* to w.

Similarity semantics: systems of spheres, preference relations, selection functions

Standard counterfactual logics extend classical logic (Lewis's chart [4])

Introduction

Structural proof theory for standard counterfactual logics:

Labelled proof systems:
incorporate the model-theoretic similarity semantics into the syntax by means of labels and relations on them which are alien to the language of the logic studied

e.g., S. Negri and N. Olivetti [5], S. Negri and G. Sbardolini [6], F. Poggiolesi [10]

Internal proof systems:
imitate model-theoretic structures by means of specific structural rules without enriching the language of the logic studied

e.g., B. Lellmann and D. Pattinson [3], N. Olivetti and G. L. Pozzato [9]

Translations:
M. Girlando, S. Negri, and N. Olivetti [2]

Focus constructions, intuitions, and aims

Focus constructions 1: *would*-counterfactuals

'If it were the case that A, then it would be the case that B.'

Intuition: subordinated clause is not true

Counterfactual implication: $A \supset_c B$

Focus constructions 2: reason giving *since*-subordinator sentences

'Since A is the case, it is the case that B.'

Intuition: subordinated clause is true

Factual implication: $A \supset_f B$

Aim:

A proof-theoretic semantics for factual and counterfactual implication which is:

- *foundationally autarkic*:
  not defined on the basis of a formal semantics of a different kind
  (e.g., some similarity semantics)
- *intuitionistically acceptable*:
  admits of a BHK-interpretation for its operators

A proof-theoretic semantics based on a labelled or on an internal proof system for a standard counterfactual logic is neither autarkic nor intuitionistically acceptable.

Proof system

Reference system S:

Let S be a natural deduction system for intuitionistic first-order logic which contains rules for the introduction and elimination of atomic and identity sentences, and which enjoys normalization and the subformula property (e.g., [21]).

Theses of S:

Let a thesis of S be a formula which has been derived canonically in S (i.e., by means of an application of an I-rule in the last derivation step) and let Θ_S be the set of theses.

Proof system

Modes of assumptions in IFC:

- $|A|$ indicates that A is assumed in the *factual mode*, where $A \in \Theta_S \cup \Phi$ (with $\Phi = \{B \supset_f C, B \supset_c C\}$).
- $\wr A \wr$ indicates that A is assumed in the *counterfactual mode*, where $A \in \Theta_S^c \cup \Phi$ with $\Theta_S^c =_{def} Fml \setminus \Theta_S$.
- A (no markers) indicates that A is assumed in the *independent mode*, that is, independently of whether it is contained in Θ_S or in its complement Θ_S^c.

Fml is the set of formulae of the language of S.
$/A/$ is used to indicate that A is assumed in one of the three modes.

Proof system

Derivations in IFC-systems:

Basic step. Any formula A assumed in the factual (resp., counterfactual, independent) mode $|A|$ ($\wr A \wr$, A) is a derivation from the open factual (counterfactual, independent) assumption of A.
Induction step. If $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are derivations, then a derivation can be constructed by means of the rules listed below.

$$\dfrac{\begin{matrix}[|A|]^{(u)} \\ \mathcal{D}_1 \\ B\end{matrix}}{A \supset_f B}\,(\supset_f\mathrm{I}),\ u \qquad \dfrac{\begin{matrix}\mathcal{D}_1 \\ A \supset_f B\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ A\end{matrix}}{B}\,(\supset_f\mathrm{E})$$

Side conditions:

1. $\supset_f$I: no empty cancellation.
2. $\supset_f$E: minor premiss depends only on factual assumptions.

Proof system

Derivations in IFC-systems (contnd.):

$$\dfrac{\begin{matrix}[\wr A \wr]^{(u)}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset_c B}\,(\supset_c\mathrm{I}),\ u \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \supset_c B\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\,(\supset_c\mathrm{E})$$

Side conditions:

3. $\supset_c$I:
   (a) no empty cancellation.
   (b) $\mathcal{D}_1$ contains no formula which is both the conclusion of an $\supset_c$E-application and a minor premiss of an $\supset_c$E-application (*break formula*).
4. $\supset_c$E: minor premiss A
   (a) depends on at least one counterfactual assumption;
   (b) contains no eliminated conjuncts (i.e., conclusions of &E*i*-applications);
   (c) contains no introduced disjuncts (i.e., conclusions of $\vee$I*i*-applications).

Proof system

Derivations in IFC-systems (contnd.):

$$\dfrac{\begin{matrix}[A]^{(u)}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset B}\,(\supset\mathrm{I}),\ u \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \supset B\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\,(\supset\mathrm{E})$$

Side conditions:

5. $\supset$E: minor premiss depends only on independent assumptions.

Proof system

Derivations in IFC-systems (contnd.):

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ B\end{matrix}}{A\&B}\,(\&\mathrm{I})\qquad\dfrac{\begin{matrix}\mathcal{D}_1\\ A\&B\end{matrix}}{A}\,(\&\mathrm{E1})\qquad\dfrac{\begin{matrix}\mathcal{D}_1\\ A\&B\end{matrix}}{B}\,(\&\mathrm{E2})$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix}}{A\vee B}\,(\vee\mathrm{I1})\qquad\dfrac{\begin{matrix}\mathcal{D}_1\\ B\end{matrix}}{A\vee B}\,(\vee\mathrm{I2})\qquad\dfrac{\begin{matrix}\mathcal{D}_1\\ A\vee B\end{matrix}\qquad\begin{matrix}[/A/]^{(u)}\\ \mathcal{D}_2\\ C\end{matrix}\qquad\begin{matrix}[/B/]^{(v)}\\ \mathcal{D}_3\\ C\end{matrix}}{C}\,(\vee\mathrm{E}),\ u,v$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \bot\end{matrix}}{A}\,(\bot\mathrm{i})$$

Side conditions:

6. $\vee$E: if the major premiss depends only on factual assumptions [on at least one counterfactual assumption], at least one [both] of its disjuncts has [have] to be assumed factually [counterfactually]; otherwise at least one of the disjuncts has to be assumed in the independent mode.

Proof system

Some theorems of any IFC-system: Mixed

- All theorems of intuitionistic propositional logic.

1 $(A \supset_f B) \supset (\neg_f B \supset_f \neg_f A)$

2 $\neg_f \neg_f (A \vee \neg_f A)$

3 $(A \supset_f (B \supset_f C)) \leftrightarrow ((A\&B) \supset_f C)$

4 $(A \supset_f (B \supset_c C)) \leftrightarrow ((A\&B) \supset_c C)$

5 $(A \supset_c (B \supset_f C)) \leftrightarrow ((A\&B) \supset_c C)$

6 $(A \supset_c (B \supset_c C)) \leftrightarrow ((A\&B) \supset_c C)$

7 $\neg_f (A \vee B) \leftrightarrow (\neg_f A \& \neg_f B)$

8 $(\neg_f A \& \neg_c B) \supset \neg_f (A \vee B)$

9 $(\neg_c A \& \neg_c B) \supset \neg_c (A \vee B)$

Proof system

Example: 8

$$
\dfrac{\dfrac{[|A \vee B|]^{(2)} \qquad \dfrac{\dfrac{[\neg_f A \& \neg_c B]^{(1)}}{\neg_f A} \qquad [|A|]^{(3)}}{\bot}\,(\supset_f \mathrm{E}) \qquad \dfrac{\dfrac{[\neg_f A \& \neg_c B]^{(1)}}{\neg_c B} \qquad [\wr B \wr]^{(4)}}{\bot}\,(\supset_c \mathrm{E})}{\dfrac{\bot}{\neg_f (A \vee B)}\,(\supset_f \mathrm{I}),\ 2}\,(\vee \mathrm{E}),\ 3,\ 4}{(\neg_f A \& \neg_c B) \supset \neg_f (A \vee B)}\,(\supset \mathrm{I}),\ 1 \qquad (1)
$$

Proof system

Some theorems of any IFC-system: Counterfactual

1 $A \supset_c A$ (ID)

2 $((A \supset_c B) \& (A \supset_c C)) \supset (A \supset_c (B \& C))$ (CC)

3 $((A \supset_c C) \& (B \supset_c C)) \supset ((A \vee B) \supset_c C)$ (CA)

4 $((A \supset_c (B \& C)) \supset ((A \supset_c B) \& (A \supset_c C))$ (CM)

5 $((A \& B) \supset_c C) \supset ((A \supset_c B) \supset (A \supset_c C))$ (RT)

6 $(A \supset_c (B \supset C)) \supset ((A \supset_c B) \supset (A \supset_c C))$ (CK)

7 $(A \supset_c (B \supset_c C)) \supset ((A \supset_c B) \supset (A \supset_c C))$

Proof system

Example: CC

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[(A \supset_c B)\&(A \supset_c C)]^{(1)}}{A \supset_c B} \qquad [\wr A \wr]^{(2)}}{B} \qquad \cfrac{\cfrac{[(A \supset_c B)\&(A \supset_c C)]^{(1)}}{A \supset_c C} \qquad [\wr A \wr]^{(2)}}{C}}{B\&C}\,(\&\mathrm{I})}{A \supset_c (B\&C)}\,(\supset_c\mathrm{I}),\ 2}{((A \supset_c B)\&(A \supset_c C)) \supset (A \supset_c (B\&C))}\,(\supset\mathrm{I}),\ 1 \tag{2}$$

Proof system

Example: RT

$$\cfrac{\cfrac{\cfrac{\cfrac{[(A\&B) \supset_c C]^{(1)} \qquad \cfrac{[\wr A \wr]^{(3)} \qquad \cfrac{[A \supset_c B]^{(2)} \qquad [\wr A \wr]^{(3)}}{B}\,(\supset_c\mathrm{E})}{A\&B}\,(\&\mathrm{I})}{C}\,(\supset_c\mathrm{E})}{A \supset_c C}\,(\supset_c\mathrm{I}),\ 3}{(A \supset_c B) \supset (A \supset_c C)}\,(\supset_c\mathrm{I}),\ 2}{((A\&B) \supset_c C) \supset ((A \supset_c B) \supset (A \supset_c C))}\,(\supset_c\mathrm{I}),\ 1 \tag{3}$$

Proof system

Detour conversions:

$$\dfrac{\dfrac{\begin{matrix}[|A|]^{(u)}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset_f B}\,(\supset_f\mathrm{I}),\,u \qquad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\,(\supset_f\mathrm{E}) \qquad \text{conv} \qquad \begin{matrix}\mathcal{D}_2\\ [|A|]\\ \mathcal{D}_1\\ B\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[\wr A\wr]^{(u)}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset_c B}\,(\supset_c\mathrm{I}),\,u \qquad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\,(\supset_c\mathrm{E}) \qquad \text{conv} \qquad \begin{matrix}\mathcal{D}_2\\ [\wr A\wr]\\ \mathcal{D}_1\\ B\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A_i\end{matrix}}{A_1 \vee A_2}\,(\vee\mathrm{I}i) \qquad \begin{matrix}[/A_1/]^{(u)}\\ \mathcal{D}_1\\ C\end{matrix} \qquad \begin{matrix}[/A_2/]^{(v)}\\ \mathcal{D}_2\\ C\end{matrix}}{C}\,(\vee\mathrm{E}),\,u,v \qquad \text{conv} \qquad \begin{matrix}\mathcal{D}\\ [/A_i/]\\ \mathcal{D}_i\\ C\end{matrix} \qquad i \in \{1,2\}$$

Proof system

Theorem 1: Normalization

Any derivation $\mathcal{D}$ in an IFC-system can be transformed into a normal IFC-derivation.

Theorem 2: Subformula property

If $\mathcal{D}$ is a normal IFC-derivation of formula A from a set of formulae Γ, then each formula in $\mathcal{D}$ is a subformula of a formula in $\Gamma \cup \{A\}$.

Theorem 3: Separation

If $\mathcal{D}$ is a normal derivation of A from Γ in an IFC-system, then the only derivation rules which are applied in $\mathcal{D}$ are the IFC-derivation rules for the logical operators which occur in $\Gamma \cup \{A\}$.

Proof system

Meaning:

An IFC-derivation $\mathcal{D}$ of a formula A from a (possibly empty) set of formulae Γ is a *canonical derivation* just in case it applies an I-rule for the main logical operator of A in the last rule application of $\mathcal{D}$.

The *meaning* of a formula is given by the set of all normal canonical IFC-derivations of that formula.

TR, SA, and CP

Theorem 4: Underivability of TR, SA, and CP.

The following are underivable in any IFC-system:

TR $((A \supset_c B) \& (B \supset_c C)) \supset (A \supset_c C)$
SA $(A \supset_c B) \supset ((A \& C) \supset_c B)$
CP $(A \supset_c B) \supset (\neg B \supset_c \neg A)$

Proof: Relying on Theorems 1 and 2, we construct *counter-derivations* bottom-up. We first use the I/E-rules for $\supset_f$ and $\supset_c$ ignoring the side conditions on them. In doing so we may use the classical absurdity rule $\bot$c, a rule which is not part of any IFC-system. In a second step we check whether side conditions on the I/E-rules of the full system have been violated and whether the $\bot$c-rule has been used. If this applies, we have a counter-derivation, and the formula cannot be derived in an IFC-system. If none of the side conditions has been violated and the $\bot$c-rule has not been applied, we have a normal IFC-derivation.

TR, SA, and CP

TR (Transitivity):

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A \supset_c B)\&(B \supset_c C)]^{(1)}}{B \supset_c C} \quad \dfrac{\dfrac{[(A \supset_c B)\&(B \supset_c C)]^{(1)}}{A \supset_c B} \quad [\wr A \wr]^{(2)}}{B}\,(\supset_c\text{E})}{C}\,(\supset_c\text{E})}{A \supset_c C}\,(\supset_c\text{I}),\ 2 \text{ illegal}}{((A \supset_c B)\&(B \supset_c C)) \supset (A \supset_c C)}\,(\supset\text{I}),\ 1 \tag{4}$$

Comments:
This counter-derivation violates side condition 3b, since only the counterfactuality of A is guaranteed in the derivation but not that of B.

TR, SA, and CP

SA (Strengthening of the antecedent):

$$\dfrac{\dfrac{\dfrac{[A \supset_c B]^{(1)} \quad \dfrac{[\wr A\& C \wr]^{(2)}}{A}\,(\&\text{E1})}{B}\,(\supset_c\text{E}) \text{ illegal}}{(A\& C) \supset_c B}\,(\supset_c\text{I}),\ 2}{(A \supset_c B) \supset ((A\& C) \supset_c B)}\,(\supset\text{I}),\ 1 \tag{5}$$

Comments:
This counter-derivation violates side condition 4b, since there is no guarantee that A rather than C does not count among the facts.

TR, SA, and CP

CP (Contraposition):

$$\dfrac{\dfrac{\dfrac{\dfrac{[\wr B \supset \bot \wr]^{(2)} \quad \dfrac{[A \supset_c B]^{(1)} \quad [\wr A \wr]^{(3)}}{B}\,(\supset_c\mathrm{E})}{\bot}\,(\supset\mathrm{E})}{A \supset \bot}\,(\supset\mathrm{I}),\ 3\ \text{illegal}}{(B \supset \bot) \supset_c (A \supset \bot)}\,(\supset_c\mathrm{I}),\ 2}{(A \supset_c B) \supset ((B \supset \bot) \supset_c (A \supset \bot))}\,(\supset\mathrm{I}),\ 1 \qquad (6)$$

Comments:
This counter-derivation violates the implicit side condition on the ⊃I-rule, since for a standard implication the assumption of A has to be made in the independent mode.

Comparisons

Some familiar axioms:

Name	Axiom	S	L	B	IFC
TAUT	All truth-functional tautologies.	√	√	√	×
ID	$A \supset_c A$	√	√	√	√
CSO	$((A \supset_c B)\&(B \supset_c A)) \supset ((A \supset_c C) \leftrightarrow (B \supset_c C))$	√	√	√	×
MP	$(A \supset_c B) \supset (A \supset B)$	√	√	×	×
MOD	$(\neg A \supset_c A) \supset (B \supset_c A)$	√	√	×	×
CV	$((A \supset_c B)\&\neg(A \supset_c \neg C)) \supset ((A\&C) \supset_c B)$	√	√	×	×
CEM	$(A \supset_c B) \vee (A \supset_c \neg B)$	√	×	×	×
CS	$(A\&B) \supset (A \supset_c B)$	×	√	×	×
CC	$((A \supset_c B)\&(A \supset_c C)) \supset (A \supset_c (B\&C))$	×	×	√	√
CA	$((A \supset_c C)\&(B \supset_c C)) \supset ((A \vee B) \supset_c C)$	×	×	√	√

Comparisons

CSO:

Violation of side condition 3b. Let $X = (A \supset_c B) \& (B \supset_c A)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \supset_c C]^2 \quad \dfrac{\dfrac{[X]^1}{B \supset_c A} \quad \dfrac{\dfrac{[X]^1}{A \supset_c B} \quad \dfrac{\dfrac{[X]^1}{B \supset_c A} \quad [\wr B \wr]^3}{A}}{B}}{A}}{C}}{B \supset_c C}\,(\supset_c\mathrm{I}),\ 3\ \text{illeg.}}{(A \supset_c C) \supset (B \supset_c C)}\,(\supset\mathrm{I}),\ 2 \quad \dfrac{\dfrac{\dfrac{[B \supset_c C]^4 \quad \dfrac{\dfrac{[X]^1}{A \supset_c B} \quad \dfrac{\dfrac{[X]^1}{B \supset_c A} \quad \dfrac{\dfrac{[X]^1}{A \supset_c B} \quad [\wr A \wr]^5}{B}}{A}}{B}}{C}}{A \supset_c C}\,(\supset_c\mathrm{I}),\ 5\ \text{illeg.}}{(B \supset_c C) \supset (A \supset_c C)}\,(\supset\mathrm{I}),\ 4}{\dfrac{(A \supset_c C) \leftrightarrow (B \supset_c C)}{((A \supset_c B) \& (B \supset_c A)) \supset ((A \supset_c C) \leftrightarrow (B \supset_c C))}\,(\supset\mathrm{I}),\ 1}\,(\&\mathrm{I}) \tag{7}$$

Comparisons

MP:

Violation of side condition 4a.

$$\dfrac{\dfrac{\dfrac{[A \supset_c B]^{(1)} \qquad [A]^{(2)}}{B}\,(\supset_c\mathrm{E})\ \text{illegal}}{A \supset B}\,(\supset\mathrm{I}),\ 2}{(A \supset_c B) \supset (A \supset B)}\,(\supset\mathrm{I}),\ 1 \tag{8}$$

Comparisons

MOD:

Violation of side conditions 3a and 4a, and use of classical absurdity. Let $X = (((A \supset \bot) \supset_c A) \supset (B \supset_c A)) \supset \bot$.

$$\dfrac{[X]^{(1)} \quad \dfrac{\dfrac{\dfrac{[(B \supset_c A) \supset \bot]^{(3)} \quad \dfrac{\dfrac{\dfrac{[A \supset \bot]^{(4)} \quad \dfrac{[(A \supset \bot) \supset_c A]^{(2)} \quad [A \supset \bot]^{(4)}}{A}\,(\supset_c\text{E})\text{ illeg.}}{\bot}\,(\supset\text{E})}{A}\,(\bot\text{c}),\ 4\text{ illeg.}}{B \supset_c A}\,(\supset_c\text{I})\text{ illeg.}}{\bot}}{B \supset_c A}\,(\bot\text{c}),\ 3\text{ illeg.}}{((A \supset \bot) \supset_c A) \supset (B \supset_c A)}\,(\supset\text{I}),\ 2}{\dfrac{\bot}{((A \supset \bot) \supset_c A) \supset (B \supset_c A)}\,(\bot\text{c}),\ 1\text{ illeg.}}\,(\supset\text{E}) \tag{9}$$

Comparisons

CV:

Violation of side condition 4b.

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A \supset_c B)\&((A \supset_c (C \supset \bot)) \supset \bot)]^{(1)}}{A \supset_c B} \quad \dfrac{[\wr A\&C\wr]^{(2)}}{A}\,(\&\text{E1})}{B}\,(\supset_c\text{E})\text{ illegal}}{(A\&C) \supset_c B}\,(\supset_c\text{I}),\ 2}{((A \supset_c B)\&\neg(A \supset_c \neg C)) \supset ((A\&C) \supset_c B)}\,(\supset\text{I}),\ 1 \tag{10}$$

Comparisons

CEM:

Violation of side condition 3a and use of classical absurdity rule. Let $X = ((A \supset_c B) \lor (A \supset_c (B \supset \bot))) \supset \bot$.

$$\dfrac{[X]^{(1)} \qquad \dfrac{\dfrac{\bot}{A \supset_c B}\,(\bot c),\ 2\ \text{illeg.}}{(A \supset_c B) \lor (A \supset_c (B \supset \bot))}\,(\lor I1)}{\dfrac{\bot}{(A \supset_c B) \lor (A \supset_c (B \supset \bot))}\,(\bot c),\ 1}\,(\supset E)$$

where the premise $\bot$ of $(\bot c),\ 2$ is derived by

$$\dfrac{[(A \supset_c B) \supset \bot]^{(2)} \qquad \dfrac{\dfrac{\dfrac{[X]^{(1)} \qquad \dfrac{\dfrac{[B \supset \bot]^{(3)}}{A \supset_c (B \supset \bot)}\,(\supset_c I)\ \text{illeg.}}{(A \supset_c B) \lor (A \supset_c (B \supset \bot))}\,(\lor I2)}{\bot}\,(\supset E)}{B}\,(\bot c),\ 3\ \text{illeg.}}{A \supset_c B}\,(\supset_c I)\ \text{illeg.}}{\bot}\,(\supset E) \qquad (11)$$

Comparisons

CS:

Violation of side condition 3a.

$$\dfrac{\dfrac{\dfrac{[A\&B]^{(1)}}{B}\,(\&E2)}{A \supset_c B}\,(\supset_c I)\ \text{illegal}}{(A\&B) \supset (A \supset_c B)}\,(\supset I),\ 1 \qquad (12)$$

Comparisons

Converse of CA:

Violation of side condition 4c.

$$\dfrac{\dfrac{\dfrac{[(A \vee B) \supset_c C]^{(1)} \qquad \dfrac{[\wr A \wr]^{(2)}}{A \vee B}\,(\supset_c\text{E})\text{ illeg.}}{C}}{A \supset_c C}\,(\supset_c \text{I}), 2 \qquad \dfrac{\dfrac{[(A \vee B) \supset_c C]^{(1)} \qquad \dfrac{[\wr B \wr]^{(3)}}{A \vee B}}{C}\,(\supset_c\text{E})\text{ illeg.}}{B \supset_c C}\,(\supset_c\text{I}), 3}{\dfrac{(A \supset_c C)\&(B \supset_c C)}{((A \vee B) \supset_c C) \supset ((A \supset_c C)\&(B \supset_c C))}\,(\supset\text{I}), 1} \tag{13}$$

Proof interpretation

Clauses for $\supset_f$ and $\supset_c$:

FI. A proof of $A \supset_f B$ is a construction which permits us to transform any proof of A available in a reference system into a proof of B.

CI. A proof of $A \supset_c B$ is a construction which permits us to transform any proof of A not available in a reference system into a proof of B.

Modes of assumptions and moods of implications

Summary:

IFC-systems

- rely for their assumptions on a *reference system* which determines the facts
- distinguish different *modes of assumptions*: factual, counterfactual, independent
- distinguish different *moods of implications*: factual, counterfactual, standard
- admit of a *foundationally autarkic* and *intuitionistically acceptable* proof-theoretic semantics

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Proof-Theoretic Semantics of Natural Deduction Based on Inversion

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Abstract

The article presents a full Proof-Theoretic Semantics of Natural Deduction based on an extended inversion principle: the elimination rule for an operator q may invert the introduction rule for q, but also vice versa, the introduction rule for a connective q may invert the elimination rule for q. Such an inversion – extending Prawitz' concept of inversion – gives the following theorem: Inversion for 2 rules of operator q (intro rule, elim rule) exists iff a reduction of a maximum formula for q exists. The inversion theorem is specified to two logics: Lambek Calculus, LC and Intuitionistic Linear Logic, ILL with 4 propositional connectives: 2 multiplicatives, implication and conjunction, 2 additives, conjunction and disjunction, with 2 quantifiers and 2 modals. LC is defined by using elimination rules by composition. ILL is defined by using usual general elimination rules. Elimination rules by composition are an exciting alternative to general elimination rules, in some cases they do not need permutations.

1 Inversion for natural deduction – rethought

Proof-Theoretic Semantics is a large family of theories, encompassing quite different frameworks, some are based on validity concepts, others on inversion concepts and further on structural concepts. For an overview see Schroeder-Heister (2012). Among these theories the discussions on Inversion may be historically the first, because they go back to certain remarks of Gentzen (1934/35), and were picked up by Prawitz for Natural Deduction in 1965. It are the investigations began by Prawitz, which are continued in the present essay, but where Prawitz considers intuitionistic logic, it is argued in the present essay that logics below intuitionistic logic are more appropriate for such investigations. First of all logics below intuitionistic logic in the substructural hierarchy dispense at least some of the structural rules, and so inversion properties may be investigated without using structural rules in the context of intuitionistic linear logic, a logic defined by Girard (1987), which uses permutation as its one and only structural rule, or in the context of intuitionistic non-commutative linear logic, i.e. Lambek Calculus, which is intuitionistic logic without any structural rule, a logic going back to Lambek (1958).

And secondly logics below intuitionistic logic have a larger variety of logical operators, since the difference of multiplicative and additive operators has to be taken into account. So, in the sequel inversion is investigated for intuitionistic multiplicative implication and conjunction, for intuitionistic additive disjunction and conjunction, for universal and existence quantifier and for modals necessity and possibility.

However, why should inversion be investigated at all? It is well known that considerations of inversion can provide a local criterion of justification, of verification, of grounding of rules of logic, and in the context of Natural Deduction, of pairs of rules of logic – introduction rules and elimination rules – for a specific logical operator. And exactly this promise of a local criterion of justification, verification, grounding of rules of logic is investigated in the sequel.

The work, in which the idea of inversion was worked out most detailed, as mentioned, is Prawitz' *Natural Deduction*. In this very book Prawitz writes on p. 33:

> Inversion Principle. Observe that an elimination rule is, in a sense, the inverse of the corresponding introduction rule: by an application of an elimination rule one essentially only restores what had already been established if the major premiss of the application was inferred by an application of an introduction rule.

So, I guess it is not unfair to sum up the considerations of Prawitz shortly in the phrase: an elimination rule for a logical operator inverts an introduction rule for the very same operator.

In the sequel it is shown that this understanding of inversion is broad enough to cover intuitionistic propositional logic, but that it is not sufficient for substructural logics or for predicate logic. Therefore an extended understanding of inversion is used, which simply allows both directions of inversion. So, an elimination rule may invert an introduction rule, but also an introduction rule may invert an elimination rule, and inversion restores all and exactly the same assumptions and the conclusion already established by constructing a derivation with a maximum formula – if the major premiss of the inversion rule is inferred by the rule to be inverted or if the major premiss of the rule to be inverted is inferred by the inversion rule.

But, still, how is inversion properly defined? This question is best answered after a look to an other important metalogical property, after a look to reduction. Although it is clear that reductions or conversions remove maximum formulas, the proper definition of reduction is not given by a metalanguage description. Instead the proper definition of reduction is given simply by pairs of derivations, one pair for every logical operation. And so we do for inversion. Inversion is defined by pairs of derivations, one pair for every operator.

Interestingly it turns out that it throws some light on reduction, if inversion defined by pairs of derivations is compared with reduction defined by pairs of derivations. The relation of reduction and inversion can be simply put into the slogan that inversion equals reduction, as will be shown in some detail. This is an exciting insight, since the two operations act differently but in a certain since in a dual way: reductions remove maximum formulas and preserve conclusions and assumptions, but inversions introduce maximum formulas for the purpose of restoring a former derivation with its conclusion and assumptions.

Since preservation of assumptions plays an important role in the considerations on inversions once more intuitionistic logic may not be regarded as the logic of choice, since intuitionistic logic has structural properties like exchange (permutation), contraction and weakening (thinning) for free, all acting on assumptions, and in Natural Deduction these structural properties are hidden in the rules for operators. So, Lambek Calculus and Linear Logic, which do preserve assumptions, are more appropriate for investigations on inversion.

2 Two calculi in natural deduction – bottom up

Intuitionistic linear logic with general elimination rules

Collections of assumptions in a deduction tree are interpreted as multisets.

$$A \quad \text{(Base Rule)}$$

$$\frac{\begin{matrix} A^u \\ \vdots \\ B \end{matrix}}{A \to B} \to I\ u \qquad\qquad \frac{\begin{matrix} \vdots \\ A \to B \end{matrix} \quad \begin{matrix} \vdots \\ A \end{matrix}}{B} \to E$$

$$\frac{\begin{matrix} \Gamma \\ \vdots \\ A \end{matrix} \quad \begin{matrix} \Gamma^v \\ \vdots \\ B \end{matrix}}{A \wedge B} \wedge I\ v \qquad\qquad \frac{\begin{matrix} \vdots \\ A \wedge B \end{matrix}}{A} \wedge E \qquad \frac{\begin{matrix} \vdots \\ A \wedge B \end{matrix}}{B} \wedge E$$

$$\frac{\begin{matrix} \vdots \\ A \end{matrix}}{A \vee B} \vee I \qquad \frac{\begin{matrix} \vdots \\ B \end{matrix}}{A \vee B} \vee I \qquad\qquad \frac{\begin{matrix} \vdots \\ A \vee B \end{matrix} \quad \begin{matrix} A^v\ \Gamma \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B^v\ \Gamma^v \\ \vdots \\ C \end{matrix}}{C} \vee E\ v$$

$$\frac{\begin{array}{c}\vdots\\ A\end{array}}{\forall x A}\,\forall I$$

x not free in A or x not free in open assumptions of A

$$\frac{\begin{array}{c}\vdots\\ \forall x A\end{array}}{A(x/t)}\,\forall E$$

$$\frac{\begin{array}{c}\vdots\\ A(x/t)\end{array}}{\exists x A}\,\exists I$$

$$\frac{\begin{array}{c}\vdots\\ \exists x A\end{array}\quad \begin{array}{c}A^{v}\\ \vdots\\ C\end{array}}{C}\,\exists E\ v$$

x not free in A or x not free in C and its open assumptions except A

$$\frac{\begin{array}{c}\vdots\\ A\end{array}}{\Diamond A}\,\Diamond I$$

$$\frac{\begin{array}{c}\vdots\\ \Diamond A\end{array}\quad \begin{array}{c}\vdots\\ \Box\Gamma \quad A^{v}\\ \vdots\\ \Diamond C\end{array}}{\Diamond C}\,\Diamond E\ v$$

every branch, except A, has a $\Box$ node

$$\frac{\begin{array}{c}\vdots\\ \Box\Gamma\\ \vdots\\ B\end{array}}{\Box B}\,\Box I$$

every branch has a $\Box$ node

$$\frac{\begin{array}{c}\vdots\\ \Box A\end{array}}{A}\,\Box E$$

Remarks

(i) For Intuitionistic Linear Logic in Natural Deduction compare Avron (1988), Troelstra (1995) and Negri (2002), all with different treatment of Additives. Avron has both additives, but no technical device of treating context. Troelstra does not consider additives. And Negri introduces a joining label for pairs of context in additives, which is a function of course different from the usual discharge function. Zimmermann (2007), presents the treatment of additives used here: with one and the same discharge function active formulas can be discharged and context can be discharged, but substitutions in contexts during reductions have to be carefully carried out as a at least double substitution with adapted discharge.

(ii) General elimination rules were proposed first by Schroeder-Heister (1984).

(iii) For the treatment of modals the ideas of Bierman & de Paiva (2000), for modal necessity in Natural Deduction in sequent style are transferred to real Natural Deduction and to modal possibility.

Lambek Calculus with Elimination by Composition

Collections of assumptions in a deduction tree are interpreted as sequences – in their tree order. The notation "+1" assigns to a rule instance a natural number, which is larger than any other natural number assigned to given rule instances by 1. Elimination rules by composition are explained best by the example of disjunction elimination. Disjunction elimination by composition presupposes the same three derivations as the usual disjunction elimination rule: one derivation with a disjunctive conclusion $A \vee B$, one derivation from assumptions A to conclusion C, and one derivation from assumptions B to conclusion C. But these three derivations are put together at the new derivation line in different styles. Usual disjunction elimination sets all three derivations above the new derivation line, elimination by composition sets only two derivations above and one below the new derivation line, below is one derivation with conclusion C. Since in calculi with elimination rules by composition the last rule applied is not necessarily the very rule at the bottom of the derivation, consecutive step numbers of rule applications identify the last rule applied in a derivation, it is simply the rule with the largest number. So, every instance of a rule in a derivation, i.e. every new derivation line, receives a step number, which is by 1 larger than every given step number of rule instances in given derivations.

$$A \quad \text{(Base Rule)}$$

$$\begin{array}{c} A^{+1} \quad \emptyset \\ \vdots \\ \dfrac{B}{A \rightarrow B} \rightarrow I \ +1 \end{array} \qquad\qquad \begin{array}{c} \vdots \qquad \vdots \\ \dfrac{A \rightarrow B \quad A}{B} \rightarrow E \ +1 \end{array}$$

$$\begin{array}{c} \emptyset \quad A^{+1} \\ \vdots \\ \dfrac{B}{A \Rightarrow B} \Rightarrow I \ +1 \end{array} \qquad\qquad \begin{array}{c} \vdots \qquad \vdots \\ \dfrac{A \quad A \Rightarrow B}{B} \Rightarrow E \ +1 \end{array}$$

$$\begin{array}{c} \vdots \\ \dfrac{A \bullet B}{A} \end{array} \quad \begin{array}{c} \emptyset \quad B^{+1} \\ \vdots \\ C \end{array} \bullet E \ +1 \qquad\qquad \begin{array}{c} \vdots \quad \vdots \\ \dfrac{A \quad B}{A \bullet B} \bullet I \ +1 \end{array}$$

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\qquad\begin{array}{c}\Gamma^{+1}\\ \vdots\\ B\end{array}}{A\wedge B}\wedge I\ \ +1$$

$$\dfrac{\begin{array}{c}\vdots\\ A\wedge B\end{array}}{A}\wedge EL\ \ +1\qquad\dfrac{\begin{array}{c}\vdots\\ A\wedge B\end{array}}{B}\wedge ER\ \ +1$$

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A\vee B}\vee IL\ \ +1\qquad\dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A\vee B}\vee IR\ \ +1$$

$$\Gamma\quad\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array}\qquad\begin{array}{c}\Gamma^{+1}B^{+1}\Delta^{+1}\\ \vdots\\ C\end{array}}{A}\\ \vdots\\ C\end{array}\quad\Delta\quad\vee E\ \ +1$$

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\forall xA}\forall I\ \ +1$$

x not free in A or x not free in open assumptions of A

$$\dfrac{\begin{array}{c}\vdots\\ \forall xA\end{array}}{A(x/t)}\forall E\ \ +1$$

$$\dfrac{\begin{array}{c}\vdots\\ A(x/t)\end{array}}{\exists xA}\exists I\ \ +1$$

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \exists xA\end{array}}{A}\exists E\ \ +1\\ \vdots\\ C\end{array}$$

x not free in A or x not free in C and its open assumptions except A

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\Diamond A}\Diamond I\ \ +1$$

$$\begin{array}{c}\vdots\\ \Box\Gamma\end{array}\quad\begin{array}{c}\vdots\\ \dfrac{\Diamond A}{A}\\ \vdots\\ \Diamond B\end{array}\quad\begin{array}{c}\vdots\\ \Box\Delta\end{array}\quad\Diamond E\ \ +1$$

every branch, except A, has a $\Box$ node

$$\dfrac{\begin{array}{c}\vdots\\ \Box\Gamma\\ \vdots\\ B\end{array}}{\Box B}\Box I\ \ +1$$

every branch has a $\Box$ node

$$\dfrac{\begin{array}{c}\vdots\\ \Box A\end{array}}{A}\Box E\ \ +1$$

Remarks

(i) Of course, defined Lambek Calculus can be extended to Intuitionistic Linear Logic – with Elimination by Composition – simply by interpreting collections of assumptions in a deduction tree as multisets.

(ii) Lambek Calculus in Natural Deduction is defined by Zimmermann (2010), with General Elimination Rules, in contrast to Elimination by Composition, shown here.

(iii) Elimination Rules by Composition were proposed first by Zimmermann (2017). The motivation was to remove some phenomena of inconfluence caused by General Elimination Rules in Lambek Calculus.

(iv) Advantages of Elimination Rules by Composition:

- No permutations for $\bullet, \exists, \Diamond$; nevertheless, permutations for additives $\vee, \wedge$, if one of the pairwise lower or upper contexts is a max formula.
- Improvement of performance in derivations, no repetitions of context formulas as in General Elimination Rules, a deficiency which several times was critically highlighted by Girard, see 1989.
- Nevertheless, in intuitionistic logic Elimination by Composition for disjunction again gives inconfluences – due to empty discharge in presence of structural property weakening. So, in intuitionistic logic General Elimination Rules for disjunction are indispensable.

(v) Recently there are doubts concerning general elimination rules formulated by Dyckhoff (2016), even disadvantages such as: ‘too many deductions’, ‘disharmonious mess’, and others.

Examples

The derivations show how rules for additives and elimination rules by composition do work in Natural Deduction for Lambek Calculus.

(1) Prefixing of multiplicative implication $\to$:

$$\cfrac{\cfrac{\cfrac{\cfrac{A \to B^5 \qquad \cfrac{C \to A^4 \qquad C^3}{A}\,{\to}E\ 1}{B}\,{\to}E\ 2}{A \to B}\,{\to}I\ 3}{(C \to A) \to (C \to B)}\,{\to}I\ 4}{(A \to B) \to ((C \to A) \to (C \to B))}\,{\to}I\ 5$$

(2) Associativity of additive disjunction $\vee$ from right to left:

$$\dfrac{\dfrac{\dfrac{A \vee (B \vee C) \qquad \dfrac{\dfrac{\dfrac{B \vee C^4 \qquad \dfrac{C^3}{(A \vee B) \vee C}\,\vee IR\,1}{B}\,\vee E\,3}{A \vee B}\,\vee IR\,1}{(A \vee B) \vee C}\,\vee IL\,2}{A}\,\vee E\,4}{A \vee B}\,\vee IL\,1}{(A \vee B) \vee C}\,\vee IL\,2$$

(3) Associativity of additive conjunction $\wedge$ from right to left:

$$\dfrac{\dfrac{\dfrac{A \wedge (B \wedge C)}{A}\,\wedge EL\,1 \qquad \dfrac{\dfrac{A \wedge (B \wedge C)^3}{B \wedge C}\,\wedge ER\,1}{B}\,\wedge EL\,2}{A \wedge B}\,\wedge I\,3 \qquad \dfrac{\dfrac{A \wedge (B \wedge C)^4}{B \wedge C}\,\wedge ER\,1}{C}\,\wedge ER\,2}{(A \wedge B) \wedge C}\,\wedge I\,4$$

(4) Associativity of multiplicative conjunction $\bullet$ from left to right:

$$\dfrac{\dfrac{\dfrac{(A \bullet B) \bullet C}{A \bullet B}\,\bullet E\,4}{A}\,\bullet E\,3 \qquad \dfrac{B^3 \qquad C^4}{B \bullet C}\,\bullet I\,1}{A \bullet (B \bullet C)}\,\bullet I\,2$$

3 Inversions

Multiplicative implication

$$\dfrac{\begin{array}{c} A^k \quad \emptyset \\ \vdots \\ B \end{array}}{A \to B}\,{\to}I\,k \qquad \text{Inversion} \qquad \dfrac{\dfrac{\begin{array}{c} A^k \quad \emptyset \\ \vdots \\ B \end{array}}{A \to B}\,{\to}I\,k \qquad A}{B}\,{\to}E\,k+1$$

Multiplicative implication $\to$: full introduction rule $\to I$ is inverted by an instance of elimination rule $\to E$, generating a max formula $\to$, open assumption A and conclusion B of the deduction $\to I$ is applied on are restored.

But if this inversion argument is detailed a bit, it can be seen at once that inversion presupposes something, a deductive argument, and that it allows immediately a reductive argument, so the full picture are 3 pairs of derivation:

$$\begin{array}{c} A \quad \emptyset \\ \vdots \\ B \end{array} \qquad \overset{\mathbf{D}}{\Longrightarrow} \qquad \dfrac{\begin{array}{c} A^k \quad \emptyset \\ \vdots \\ B \end{array}}{A \to B}\,{\to}I\,k$$

$$\overset{\mathbf{I}}{\Longrightarrow} \qquad \dfrac{\dfrac{\begin{matrix}A^k & \emptyset\\ \vdots & \\ B & \end{matrix}}{A \to B}\to I\,k \qquad A}{B}\to E\,k+1 \qquad \overset{\mathbf{R}}{\Longrightarrow}$$

So, inversion is pair **I**, reduction is pair **R** and deduction is pair **D**.

There can be stated simple propositions about the 3 pairs:

Propositions

(i) $\mathbf{I}_{\to}$ presupposes $\mathbf{D}_{\to}$.

(ii) The first component of $\mathbf{D}_{\to}$ is the reduct of $\mathbf{R}_{\to}$.

(iii) So: reduction $\mathbf{R}_{\to}$ exists, if pairs $\mathbf{D}_{\to}$ and $\mathbf{I}_{\to}$ exist – and they do.

(iv) Given reduction $\mathbf{R}_{\to}$ the reduct can be continued to pair $\mathbf{D}_{\to}$.

(v) The first component of $\mathbf{R}_{\to}$ is the inverted derivation of $\mathbf{I}_{\to}$.

(vi) So: inversion $\mathbf{I}_{\to}$ exists, if pairs $\mathbf{R}_{\to}$ and $\mathbf{D}_{\to}$ exist – and they do.

Proof of the propositions by inspection of the derivations.

In fact, the propositions prove:

Inversion theorem

Inversion **I** of operator $\to$ exists iff reduction **R** for operator $\to$ exists.

Of course, the inversion theorem for multiplicative implication is shown in Lambek Calculus and holds for every extension of it.

In the sequel the 3 pairs **D**, **I**, **R** are shown for the operators $\wedge, \bullet, \vee, \forall, \exists, \Box, \Diamond$ respectively, such that the mentioned propositions and the theorem can be shown exactly as done for $\to$. To save space the inversion pair is not shown twofold as 'inversion' and '**I**', but simply as '**I**'.

Additive conjunction

$$\begin{matrix}\Gamma & \Gamma\\ \vdots & \vdots\\ A & B\end{matrix} \qquad \overset{\mathbf{D}}{\Longrightarrow} \qquad \dfrac{\begin{matrix}\Gamma & \Gamma^k\\ \vdots & \vdots\\ A & B\end{matrix}}{A \wedge B}\wedge I\,k$$

$$\overset{\mathbf{I}}{\Longrightarrow} \qquad \dfrac{\dfrac{\begin{matrix}\Gamma & \Gamma^k\\ \vdots & \vdots\\ A & B\end{matrix}}{A \wedge B}\wedge I\,k}{A}\wedge EL\,k+1 \qquad \dfrac{\dfrac{\begin{matrix}\Gamma & \Gamma^k\\ \vdots & \vdots\\ A & B\end{matrix}}{A \wedge B}\wedge I\,k}{B}\wedge ER\,k+1 \qquad \overset{\mathbf{R}}{\Longrightarrow}$$

Additive conjunction $\wedge$: full introduction rule $\wedge I$ is inverted by two instances of elimination rules $\wedge E$ generating a max formula $\wedge$, conclusions A, B of the deductions $\wedge I$ is applied on are restored.

The inversion argument for additive conjunction is shown in Lambek Calculus and holds for every extension of it.

Multiplicative conjunction with general elimination rules

$$\begin{array}{c} A \quad B \\ \vdots \\ C \end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \dfrac{A \bullet B \qquad \begin{array}{c} A^k \quad B^k \\ \vdots \\ C \end{array}}{C} \bullet E\, k$$

$$\overset{\mathbf{I}}{\Longrightarrow} \quad \dfrac{\dfrac{A \quad B}{A \bullet B} \bullet I \qquad \begin{array}{c} A^k \quad B^k \\ \vdots \\ C \end{array}}{C} \bullet E\, k \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Multiplicative conjunction $\bullet$: general elimination rule $\bullet E$ is inverted by an instance of introduction rule $\bullet I$ generating a max formula $\bullet$, open assumptions A, B of the deduction $\bullet E$ is applied on are restored.

The first inversion argument for multiplicative conjunction is shown in Intuitionistic Linear Logic and holds for every extension of it.

Multiplicative conjunction with elimination by composition

$$\begin{array}{ccc} A & \emptyset & B \\ & \vdots & \\ & C & \end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \begin{array}{cccc} \dfrac{A \bullet B}{A} & \emptyset & B^{k+1} & \bullet E\; k+1 \\ & \vdots & & \\ & C & & \end{array}$$

$$\overset{\mathbf{I}}{\Longrightarrow} \quad \begin{array}{cccc} \dfrac{\dfrac{A \quad B}{A \bullet B} \bullet I\; 1}{A} & \emptyset & B^{k+1} & \bullet E\; k+1 \\ & \vdots & & \\ & C & & \end{array} \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Multiplicative conjunction $\bullet$: elimination rule $\bullet E$ with the connective formula as open assumption is inverted by an instance of introduction rule $\bullet I$ generating a max formula $\bullet$, open assumptions A, B of the deduction $\bullet E$ is applied on are restored.

The second inversion argument for multiplicative conjunction is shown in Lambek Calculus and holds for every extension of it.

Additive disjunction with general elimination rules

$$\begin{array}{cc} A\ \Gamma & B\ \Gamma \\ \vdots & \vdots \\ C & C \end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \dfrac{A \vee B \qquad \begin{array}{c} A^k\ \Gamma^k \\ \vdots \\ C \end{array} \qquad \begin{array}{c} B^k\ \Gamma^k \\ \vdots \\ C \end{array}}{C} \vee E\ k$$

$$\overset{\mathbf{I}}{\Longrightarrow} \quad \dfrac{\dfrac{A}{A \vee B} \vee IL \qquad \begin{array}{c} A^k\ \Gamma^k \\ \vdots \\ C \end{array} \qquad \begin{array}{c} B^k\ \Gamma^k \\ \vdots \\ C \end{array}}{C} \vee E\ k$$

$$\dfrac{\dfrac{A}{A \vee B} \vee IR \qquad \begin{array}{c} A^k\ \Gamma^k \\ \vdots \\ C \end{array} \qquad \begin{array}{c} B^k\ \Gamma^k \\ \vdots \\ C \end{array}}{C} \vee E\ k \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Additive disjunction $\vee$: general elimination rule $\vee E$ is inverted by instances of introduction rules $\vee I$ generating a max formula $\vee$, open assumptions A_1, A_2 of the deduction $\vee E$ is applied on are restored.

The first inversion argument for additive disjunction is shown in Intuitionistic Linear Logic and holds for every extension of it.

Additive disjunction with elimination by composition

$$\begin{array}{cc} \Gamma\ A\ \Delta & \Gamma\ B\ \Delta \\ \vdots & \vdots \\ C & C \end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \Gamma \quad \begin{array}{c} \dfrac{A \vee B \qquad \begin{array}{c} [\Gamma\ B\ \Delta]^{k+1} \\ \vdots \\ C \end{array}}{A} \\ \vdots \\ C \end{array} \quad \Delta \quad \vee E\ k+1$$

$$\overset{\mathbf{I}}{\Longrightarrow} \quad \Gamma \quad \begin{array}{c} \dfrac{\dfrac{A}{A \vee B} \vee IL1 \qquad \begin{array}{c} [\Gamma\ B\ \Delta]^{k+1} \\ \vdots \\ C \end{array}}{A} \\ \vdots \\ C \end{array} \quad \Delta \quad \vee E\ k+1$$

$$\Gamma \quad \begin{array}{c} \dfrac{\dfrac{B}{A \vee B} \vee IR1 \qquad \begin{array}{c} [\Gamma\ B\ \Delta]^{k+1} \\ \vdots \\ C \end{array}}{A} \\ \vdots \\ C \end{array} \quad \Delta \quad \vee E\ k+1 \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Additive disjunction $\vee$: elimination rule $\vee E$ with the connective formula as open assumption is inverted by instances of introduction rules $\vee I$ generating a max formula $\vee$, open assumptions A, B of the deductions $\vee E$ is applied on are restored.

The second inversion argument for additive disjunction is shown in Lambek Calculus and holds for every extension of it.

Universal quantification

$$\begin{array}{c}\vdots\\ A\end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\forall y A}\,\forall I\,k \quad \overset{\mathbf{I}}{\Longrightarrow} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\forall y A}\,\forall I\,k}{A(y/y)}\,\forall E\,k+1 \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Universal quantifier $\forall$: introduction rule $\forall I$ is inverted by elimination rule $\forall E$ by generating a max formula $\forall$, conclusion A of the deduction $\forall I$ is applied is restored.

The inversion argument for universal quantification is shown in Lambek Calculus and holds for every extension of it.

Existential quantification with general elimination rules

$$\begin{array}{c}A\\ \vdots\\ C\end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \dfrac{\exists x A \qquad \begin{array}{c}A^k\\ \vdots\\ C\end{array}}{C}\,\exists E\,k \quad \overset{\mathbf{I}}{\Longrightarrow} \quad \dfrac{\dfrac{A(x/x)}{\exists x A}\,\exists I \qquad \begin{array}{c}A^k\\ \vdots\\ C\end{array}}{C}\,\exists E\,k \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Existential quantifier $\exists$: elimination rule $\exists E$ with the quantifier formula as open assumption is inverted by introduction rule $\exists I$ generating a max formula $\exists$, open assumption A of the deduction $\exists E$ is applied on is restored.

The first inversion argument for existential quantification is shown in Intuitionistic Linear Logic and holds for every extension of it.

Existential quantification with elimination by composition

$$\begin{array}{c}A\\ \vdots\\ C\end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \begin{array}{c}\dfrac{\exists x A}{A}\,\exists E\,k+1\\ \vdots\\ C\end{array} \quad \overset{\mathbf{I}}{\Longrightarrow} \quad \begin{array}{c}\dfrac{\dfrac{A(x/x)}{\exists x A}\,\exists I\,1}{A}\,\exists E\,k+1\\ \vdots\\ C\end{array} \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Existential quantifier $\exists$: elimination rule $\exists E$ with the quantifier formula as open assumption is inverted by introduction rule $\exists I$ generating a max formula $\exists$, open assumption A of the deduction $\exists E$ is applied on is restored.

The second inversion argument for existential quantification is shown in Lambek Calculus and holds for every extension of it.

Modal necessity

$$\begin{array}{c}\vdots\\ \Box\Gamma\\ \vdots\\ A\end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \begin{array}{c}\vdots\\ \Box\Gamma\\ \vdots\\ \dfrac{A}{\Box A}\,\Box I\,k\end{array} \quad \overset{\mathbf{I}}{\Longrightarrow} \quad \begin{array}{c}\vdots\\ \Box\Gamma\\ \vdots\\ \dfrac{\dfrac{A}{\Box A}\,\Box I\,k}{A}\,\Box E\,k+1\end{array} \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Modality $\Box$: introduction rule $\Box I$ is inverted by elimination rule $\Box E$ by generating a max formula $\Box$, conclusion A of the deduction $\Box I$ is applied is restored.

The inversion argument for modal necessity is shown in Lambek Calculus and holds for every extension of it.

Modal necessity

$$\begin{array}{c} \quad\vdots\\ A \quad \Box\Gamma\\ \vdots\\ \Diamond C\end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \dfrac{\Diamond A \qquad \begin{array}{c} \quad\vdots\\ A^k \quad \Box\Gamma\\ \vdots\\ \Diamond C\end{array}}{\Diamond C}\,\Diamond E\,k$$

$$\overset{\mathbf{I}}{\Longrightarrow} \quad \dfrac{\dfrac{A}{\Diamond A}\,\Diamond I \qquad \begin{array}{c} \quad\vdots\\ A^k \quad \Box\Gamma\\ \vdots\\ \Diamond C\end{array}}{\Diamond C}\,\Diamond E\,k \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Modality $\Diamond$: elimination rule $\Diamond I$ with the connective formula as open assumption is inverted by introduction rule $\Diamond E$ generating a max formula $\Diamond$, open assumption A of the deduction $\Diamond E$ is applied on is restored.

The first inversion argument for modal possibility is shown in Intuitionistic Linear logic and holds for every extension of it.

Modal possibility with elimination by composition

$$\begin{array}{ccc}\vdots & & \vdots\\ \Box\Gamma & A & \Box\Delta\\ & \vdots & \\ & \Diamond C & \end{array} \quad \overset{\mathbf{D}}{\Longrightarrow} \quad \begin{array}{ccc}\vdots & & \vdots\\ \Box\Gamma & \dfrac{\Diamond A}{A} & \Box\Delta\\ & & \vdots\\ & & \Diamond C\end{array}\;\Diamond E\,k+1$$

$$\overset{\mathbf{I}}{\Longrightarrow} \quad \begin{array}{ccc}\vdots & & \vdots\\ \Box\Gamma & \dfrac{\dfrac{A}{\Diamond A}\,\Diamond I\,1}{A} & \Box\Delta\\ & & \vdots\\ & & \Diamond C\end{array}\;\Diamond E\,k+1 \quad \overset{\mathbf{R}}{\Longrightarrow}$$

Modality $\Diamond$: elimination rule $\Diamond I$ with the connective formula as open assumption is inverted by introduction rule $\Diamond E$ generating a max formula $\Diamond$, open assumption A of the deduction $\Diamond E$ is applied on is restored.

The second inversion argument for modal possibility is shown in Lambek Calculus and holds for every extension of it.

4 Inversion – not arbitrary

So far, in logics like Intuitionistic Linear Logic or Lambek Calculus, i.e. Non-Commutative Intuitionistic Linear Logic, the picture of inversion is clear and distinct: every operator has one direction of inversion, not both, either elimination inverting introduction or introduction inverting elimination. This picture changes as soon as more structural properties enter the scene. Especially in full intuitionistic logic, where all the structural properties exchange, contraction, weakening are present, further inversions are available.

To start with intuitionistic conjunction, the collapse of multiplicative and additive conjunction gives immediately inversion in both directions. For disjunction, which is additive in intuitionistic logic, it was already shown that introduction inverts elimination, but the following argument shows the other direction too with the usual general elimination rule, nevertheless the argument does not depend on general elimination rules:

$$\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{A \vee B}\,\vee I \qquad \text{Inversion} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{A \vee B}\,\vee I \quad [A] \quad \begin{matrix}\vdots\\ A\end{matrix}}{A}\,\vee E$$

Conclusion A, on which $\vee I$ is applied, is restored by $\vee E$ using structural rules: Implicit weakening of $\vee E$ is used by empty discharge of active formula B. And implicit contraction of $\vee E$ is used since subdeductions and assumptions within are doubled and have to be discharged twice.

So, for intuitionistic disjunction introduction inverts elimination and elimination inverts introduction, but the last only by using structural properties.

For intuitionistic implication, which is multiplicative, the picture is even more complex. It was already shown that elimination inverts introduction, so it rests to be shown that introduction inverts elimination. We start with an argument using general elimination rules for implication:

$$\dfrac{A \to B \quad \begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\to EG \qquad \text{Inversion} \qquad \dfrac{\dfrac{A}{A \to B}\,\to I \quad \begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\to EG$$

Assumption B, discharged by $\to E$, is restored by an $\to I$ using structural rules: Implicit weakening of $\to I$ is used by empty discharge of A. And implicit contraction of $\rightarrow I$ is used if assumption class $[B]$ is larger than a singleton. So again structural properties assure this inversion, restoring assumption B. But how to restore conclusion A? This is only classically available, what can be seen best in the calculus of sequents.

In sequent calculus of classical logic implication may be defined as an additive or as a multiplicative, using the devices of Došen (1989) and Sambin (2000), and in the sequel additive implication is used to invert introduction to the left.

$$\Gamma \vdash A \rightarrow B, \Delta \text{ iff } \Gamma, A \vdash B, \Delta \qquad \text{multiplicative implication;}$$
$$\Gamma, A \rightarrow B \vdash \Delta \text{ iff } \Gamma \vdash A, \Delta \text{ and } \Gamma, B \vdash \Delta \qquad \text{additive implication.}$$

The definition of additive implication in terms of sequents amounts to the following left introduction rule and axioms:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \qquad \vdash A, A \rightarrow B \text{ and } B \vdash A \rightarrow B.$$

The left introduction rule of additive implication is twofold invertible by using the cut-rule: the first inversion is similar to the argument shown in Natural Deduction, restoring assumption B; and the second inversion is purely classic, restoring conclusion A:

$$\frac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \qquad B \vdash A \rightarrow B}{\Gamma, B \vdash \Delta}$$

$$\frac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \qquad \vdash A, A \rightarrow B}{\Gamma \vdash A, \Delta}$$

So for intuitionistic implication elimination inverts introduction, but introduction inverts elimination only partially and only by use of structural rules.

Finally the quantifiers and the modalities are left. They do not allow other directions of inversion additionally to the shown directions of inversion. This can be easily checked, since inversion arguments consist of exactly two instances of rules, one instance of an introduction rule and one instance of an elimination rule for the very same logical operator, and the formula with the logical operator is by one rule instance either a bottom formula or a top formula. The other rule instance is the inversion.

5 Summary

Let's sum up what has been done in the present paper. The considerations start by extending Prawitz' notion of inversion. Extended Inversion says: elimination inverts introduction or introduction inverts elimination. This extended notion of inversion gives a full proof-theoretic semantic of Natural Deduction, covering all intuitionistic operators of logic, propositional, quantificational, modal, and covering all intuitionistic logics in the substructural hierarchy from the bottom to the top, starting with no structural property, Lambek Calculus and ending with all structural properties, intuitionistic logic.

Propositional constants are not discussed in this article, but they often are disputed as being cases for elimination and introduction rules, see for example Prawitz remark on the falsum from 1965, p. 34.

As the main outcome it is shown that inversion equals reduction, i.e. inversion for the logical operators in question proves reduction for the logical operators in question and vice versa reduction proves inversion. This is an interesting insight into Natural Deduction, because with this insight reductions do not fall out of the sky. Reductions are linked to the inversion process and inversion intimately shows properties of pairs of rules and justifies pairs of rules.

As an important further observation one has to state that inversion in substructural logics – linear, Lambekian – is not arbitrary. Instead inversion is clear and distinct, it reveals distinct intrinsic semantic properties of operators.

$\rightarrow, \wedge, \forall, \Box$: elimination inverts introduction, not vice versa.

$\bullet, \vee, \exists, \Diamond$: introduction inverts elimination, not vice versa.

Of course in intuitionistic logic, due to structural rules, further directions of inversion are available, for instance in intuitionistic propositional logic all eliminations invert introductions, but nevertheless not all introductions invert eliminations, counterexample is implication. And in intuitionistic predicate logic even this breaks down, since not all eliminations invert introductions, counterexample is the existence quantifier.

Finally, what's wrong with Tonk? 1960, a couple of years ago, Prior set up an influential argument to baffle all rule based meaning theories. He argued that one might run into difficulties if one would regard it as necessary and sufficient for the meaning of a language particle to be introduced by a rule and to be eliminated by an other rule. As a compromising example he defined the operator 'Tonk' with an introduction rule and an elimination rule as follows. I-Tonk: from A conclude to A Tonk B; E-Tonk: From A Tonk B conclude to B.

Obviously 'Tonk' leads to troubles. But what is wrong with Tonk from the perspective of the present proof-theoretic semantics? It is quite simple: Given I-Tonk, E-Tonk does not restore exactly the same assumptions and conclusion already established, and vice versa. Another rule E-Tonk′ would perfectly invert I-Tonk: From A Tonk B conclude to A.

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Session
in memoriam Kosta Došen

† 21 October 2017

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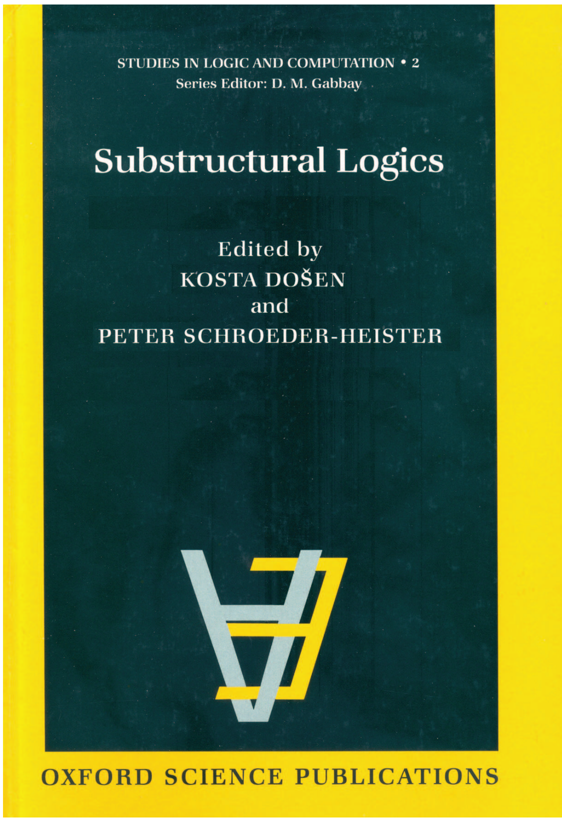

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Kosta's further scientific achievements

Particularly interesting to the community represented here:

- Theory of logical constants (DPhil with Dummett and Scott): Double-line rules
- Categorial proof theory
  - Cut elimination in categories
  - Proof-theoretical coherence
- General proof theory, with particular emphasis on the identity of proofs
  - Identity as reduction-based
  - Identity as generality-based
- He thus founded what might be called "intensional proof theory".

The proper format of consequence statements

When proofs are made explicit by means of terms, the proper notation for a hypothetical judgement leading from A to B is something like

$$f : (A \vdash B)$$

and not

$$x : A \vdash t(x) : B$$

Conceptually, this means that the hypothetical is prior to the categorial.

Technically, this means that the categorial approach to proof theory is primary.

СЕМИНАР ЗА ОПШТУ ТЕОРИЈУ ДОКАЗА

Математички институт, Београд

$$f\colon A \vdash B$$

На Семинару за општу теорију доказа у понедељак 5. јуна 2017, у 18^{00}, у сали 301f Математичког института (Кнез Михаилова 36, III спрат), треба да се одржи следеће предавање:

Peter Schroeder-Heister
Eberhard Karls Universität Tübingen

The completeness problem in proof-theoretic semantics

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Our speakers today

Phil Scott is one of the leading figures in categorial proof-theory, co-author (with Joachim Lambek) of the classic textbook in the field. He worked closely with Kosta

Greg Restall is an expert on substructural logics, where he has published the first textbook, and has recently made identity of proofs from a categorial perspective a central topic of his work in proof-theoretic semantics.

Hence it is most appropriate that they speak in this session.

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What are equations between proofs?

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Why equations between proofs?

Hilbert's 24th problem (never published) raised the issue:

> *The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof. Quite generally, if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite evident what variant conditions (and aids) have been used in the two proofs ...*

"Hilbert's Twenty-Fourth Problem", Rüdiger Thiele, *American Mathematical Monthly*, January 2003

Categorical Proof Theory: the 1960's

Lawvere: introduced categorical approaches to universal algebra, logic, and proof theory in the 1960's. Two fundamental papers:

- F. W. Lawvere, Adjointness in Foundations, *Dialectica*, 1969.
- F. W. Lawvere, Equality in hyperdoctrines and comprehension schema as an adjoint functor, *AMS Symp. Categorical Algebra*,1970.

Lambek: Introduced categorical proof theory. He employed Gentzen's methods in (i) linguistics, (ii) Coherence Problems, (iii) notions of equality of proofs. Early fundamental papers include:

- J. Lambek: On the calculus of syntactic types, AMS Symp. Appl. Math.1961.
- J. Lambek: Deductive Systems and Categories I, J. Math. Syst. Theory, 1968.
- J. Lambek: Deductive Systems and Categories II, III : SLNM 86, (1969) SLNM 274, (1972).

Categorical Proof Theory: questions which arise.

- Are there "natural" equations between proofs?
- Criteria for equality based on standard categorical/mathematical practice.
- The categorical viewpoint may diverge from traditional logical concerns, e.g. status of CR, extensionality (internal vs external), emphasis on "naturality", etc.

Categories:= directed multigraphs + equations

1. Nodes (objects) are types (or sorts), closed under various operations.
2. Edges (Arrows) are "programs" (= equivalence classes of proof terms or labelled Genzen sequents). E.g. arrow $A \xrightarrow{f} B$ = a program or proof $x : A \vdash f(x) : B$.
3. Composition = substitution of terms (or the interpretation of cut). Equality of arrows is provable equality in the logic of proof terms (or equality of provable sequents in Gentzen proof theory).
4. Additional types, terms, equations .

Deductive Systems & Categories

Definition

A *(Labelled) Deductive System* is a directed multigraph whose nodes are called *objects* (or *formulas*) and whose directed edges are called *arrows* (or *proofs*), with the following structure:

- For each object A, there is a specified arrow $A \xrightarrow{id_A} A$.
- There is a binary rule of *composition* generating new arrows:

$$\frac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$$

Definition

A *Category* is a deductive system with equations between arrows:

$$f \circ id_A = f = id_B \circ f \text{ and } h \circ (g \circ f) = (h \circ g) \circ f$$

for all $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$.

Deductive Systems and Equations between Proofs

- In general, deductive systems may be large (i.e. have proper classes of formulas and proofs).
- In categorical proof theory, we tend to look at countable, freely generated deductive systems (whose formulas and proofs are freely generated).
- This leads to *free categories of proofs*, satisfying appropriate universal properties (e.g. initiality in categories of deductive systems with strict structure-preserving maps.)
- An equation between proofs (i.e. between arrows which have the same source and target) is called a *commutative diagram*.

Cartesian Categories (= the proof theory of $\wedge, \top$)

Cartesian Categories have explicit Cartesian Products of objects.

1. Objects: $1, A, B, \ldots$, closed under $\times$.
2. Distinguished arrows: $A \stackrel{!^A}{\longrightarrow} 1$, $A \times B \stackrel{\pi_1^{A,B}}{\longrightarrow} A$, $A \times B \stackrel{\pi_2^{A,B}}{\longrightarrow} B$.
3. Distinguished binary operation on arrows

$$\frac{C \stackrel{f}{\longrightarrow} A \quad C \stackrel{g}{\longrightarrow} B}{C \stackrel{\langle f,g\rangle}{\longrightarrow} A \times B}$$

4. Equations:
   - Equations of a category
   - $f = !_A$ for any $A \stackrel{f}{\longrightarrow} 1$.
   - $\pi_1 \circ \langle f, g\rangle = f$, $\pi_2 \circ \langle f, g\rangle = g$.
   - $\langle \pi_1 \circ h, \pi_2 \circ h\rangle = h$, for $C \stackrel{h}{\longrightarrow} A \times B$. (surjective pairing)

Think of 1 as $\top$ and $A \times B$ as $A \wedge B$.

Monoidal Categories (MCs), the proof theory of $\otimes, I$

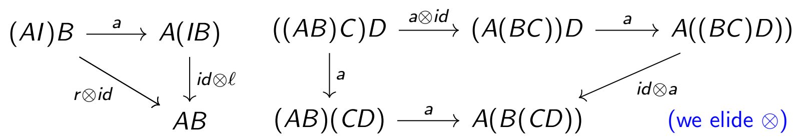
Definition (Monoidal/tensor categories)

A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object I, and specified natural isomorphisms:

$$a_{ABC} : (A \otimes B) \otimes C \xrightarrow{\simeq} A \otimes (B \otimes C)$$

$$\ell_A : I \otimes A \xrightarrow{\simeq} A \quad \text{and} \quad r_A : A \otimes I \xrightarrow{\simeq} A$$

with coherence equations below: the unit and *associativity coherence (Mac Lane's Pentagon)*. $\mathcal{C}$ is Strict if $\simeq$ is $=$.



Symmetric Monoidal (**SMCs**): add $c_{AB} : A \otimes B \to B \otimes A$ and equations: natural in A, B, $c_{AB} \circ c_{BA} = id$, more coherence eqns.

Monoidal Categories II (= the proof theory of $\otimes, I$)

Definition (*A monoidal category* as a Deductive System)

- Objects: $I, \ldots, A, B \ldots$, closed under $\otimes$
- Arrows: $a_{ABC} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$,
  $\ell_A : I \otimes A \longrightarrow A$, $r_A : A \otimes I \longrightarrow A$,
  and their formal inverses: $a^{-1}, \ell^{-1}, r^{-1}$
- Rule: $\dfrac{A \xrightarrow{f} C \quad B \xrightarrow{g} D}{A \otimes B \xrightarrow{f \otimes g} C \otimes D}$ for objects A, B, C, D.

Satisfying many equations:

- category rules + equations
- $\otimes$ is a bifunctor,
- a_{ABC}, ℓ_A, r_A (with inverses) are natural isos (in A, B, C)
- coherence equations

(similarly for *symmetric monoidal categories*, **SMC**s)

Cartesian Closed Categories (= proofs in $\wedge, \Rightarrow, \top$)

CCCs are Cartesian categories with function spaces B^A and a natural isomorphism $Hom_{\mathcal{C}}(C \times A, B) \cong Hom_{\mathcal{C}}(C, B^A)$.

Definition (Deductive System for CCCs)

- Objects (formulas) closed under $\times$ and $(\ )^{(\ )}$: if A, B objects, so are $\mathbf{1}$, $A \times B$ and B^A (also written $A \Rightarrow B$).
- Cartesian Category arrows and rules.
- Distinguished arrows $ev_{A,B} : B^A \times A \to B$,

  closed under the Currying rule: $\dfrac{C \times A \stackrel{f}{\longrightarrow} B}{C \stackrel{f^*}{\longrightarrow} B^A}$
- Equations:
  - Cartesian Category
  - (β) $ev\circ\langle f^*\circ\pi_1, \pi_2\rangle = f : C \times A \to B$.
  - (η) $(ev\circ\langle g\circ\pi_1, \pi_2\rangle)^* = g : C \to B^A$

Symmetric monoidal closed (= proofs in ILL: $\otimes, \multimap, I$)

SMCCs are symmetric monoidal categories with linear function spaces $A \multimap B$ and a natural Currying iso (bijection of homsets)

$$(*) \quad \dfrac{C \otimes A \longrightarrow B}{C \longrightarrow A \multimap B} \qquad f \longmapsto f^*$$

Definition (*SMCCs as a Deductive System*)

- Objects: $I, \ldots, A, B \ldots$, closed under $\otimes, \multimap$
- Arrows & Rules: as in symmetric monoidal, plus
  $ev_{AB} : (A \multimap B) \otimes A \to B$, for all A, B.
  $d_{CA} : C \to (A \multimap C \otimes A)$ for all A, C.
  $f^* : C \longrightarrow (A \multimap B)$ (for each $f : C \otimes A \longrightarrow B$)

Satisfying many equations:

- symmetric monoidal categories.
- $ev, d, (\ )^*$ set up natural Currying iso satisfying $(*)$.
- many coherence equations (cf. Kelly-Mac Lane).

Some consequences of this viewpoint

- Syntax gives *freely generated deductive systems* satisfying usual universal properties, e.g. *free categories of proofs* $\mathcal{F}$.
- *Model theory* (semantics) of proofs. E.g. functorial interpretations $[\![-]\!] : \mathcal{F} \longrightarrow \mathcal{M}$ from free categories of proofs $\mathcal{F}$ into (concrete) models $\mathcal{M}$, preserving structure.
- *Full Completeness theorems*: representation theory of proofs. (i.e. Completeness Theorems at the level of proofs).

  A (concrete) model $\mathcal{M}$ is *fully complete for* $\mathcal{F}$ if the canonical free functor $[\![-]\!] : \mathcal{F} \longrightarrow \mathcal{M}$ is full (and hopefully faithful).

  This says, any morphism $[\![A]\!] \to [\![B]\!] \in \mathcal{M}$ arises as $[\![A]\!] \xrightarrow{[\![\pi]\!]} [\![B]\!]$, for a (unique) proof π of $A \vdash B$.

  (Arose in 1990's for fragments of linear logic; analogous to earlier Plotkin, Statman, et.al. invariance theorems for logical relations in 1970s-'80s).

Coherence theorems, proof-theoretically

Mathematical problem: can we identify "isomorphism" with "equality" ? This is a long standing question, studied in category theory and (most recently) Homotopy Type Theory.

Mac Lane in [CWM] says coherence should be: *In freely generated structured categories* $\mathcal{C}$, *every diagram (of canonical morphisms) commutes.*

I.e. for all objects (formulas) $A, B \in \mathcal{C}$ obeying suitable conditions, there is at most one proof of $A \vdash B$, up to equality of proofs.

Lambek reformulated the question more generally as follows:

(i) Given a freely generated structured category $\mathcal{C}$, how do we effectively generate the hom-sets $Hom_{\mathcal{C}}(A, B)$?

(ii) For such $\mathcal{C}$, find an effective method to solve the word problem for $Hom_{\mathcal{C}}(A, B)$.

Lambek's answers: Use Gentzen's methods. For (i), proof search. For (ii), cut-elimination/normalization.

Versions of Coherence: $\simeq$ versus $=$?

- Joyal-Street [Braided Tensor Categories, *Adv. Math*, 1993]
- T. Leinster [*Higher Operads, Higher Categories*, 2003]

1. All formal diagrams in a monoidal category built from a, ℓ, r by tensoring, substituting, inverting and composing commute. (Mac Lane, 1960s) [More generally, as equality of natural transformations, not just arrows]
2. Any free monoidal category is strictly equivalent to a *strict* one (where canonical maps a, ℓ, r are identities).
3. Mac Lane & Barr said: Prove a Representation Theorem!

   Mac Lane Conjecture (1963?): A diagram commutes in a free SMC category (and hence in all SMC categories) iff all instantiations by vector spaces give a commutative diagram.

   Proved by Soloviev (APAL, 1997), using Gentzen proof theory. **Note:** this may be interpreted as a faithful representation of a free SMC into Vector Spaces.

Coherence Theorems: very brief history

- Mac Lane: Coherence for Monoidal categories, 1960's.
- Lambek: Coherence for structured monoidal categories & monads, 1969-72.
- Kelly-Mac Lane: Coherence for SMCCs [JPAA, 1971].
- Mints: Coherence for MCs, MCCs, SMCCs & CCCs (redoes Lambek's program, via ND & normalizing λ terms), 1970s.

  Mac Lane: "Why commutative diagrams coincide with equiv. proofs" [*Contemp. Math*, 82] proves in detail that the Kelly-Mac Lane proof (coherence for SMCCs) essentially equals the Mints proof (wrt different rank functions for normalization).
- Coherence in closed categories: [La Plaza, Voreadu] (1977).
- Lambek-Scott 1986: used simply typed lambda calculus and normalization. Units: handwaving!
- Čubrić 1993: handled units by theory of expansionary rewriting, Friedman's thm. (correcting Mints).

Coherence Theorems: more recent approaches

After Joyal-Street's string diagrams and Girard's introduction of proof-nets in linear logic (1986-87), many categorical proof-theorists moved to graphical calculi. Among logicians:

- Blute, Cockett, Seely (and coworkers): coherence theorems for a vast number of categories arising from linear logic. They pioneered the use of (2-sided) proof nets and correctness criteria for units. BCST: *-autonomous cats with units.
- Došen, Petrić: Extends the Lambek program in extensive, detailed equational studies of cut-elimination, adjunctions, coherence and proof-net categories for a wide range of categories in linear logic and algebra and topology.
- D. Hughes: new proof-net approaches to coherence for *-autonomous categories with units (JPAA 2012).
- W.Heijltjes, R. Houston "Proof Equivalence in MLL (with units) is PSpace-Complete", 2016.

Lambek's response

"I don't like proof nets, commutative diagrams, etc. I follow Descartes' advice: replace geometry by algebra.

PROPOSAL. Someone should rewrite the paper of Joyal and Street without the horrible diagrams. "

(J. Lambek, Lecture on Bicategories, Feb. 2003)

Complications of Using Normalization Methods

- Recall, **Coherence Theorems** by usual logicians' method: to prove an equation $f = g$, reduce each side to a normal form, with a decidable equality on normal forms.
- For simply typed λ-calculus with $\mathbf{1}$, CR fails.
- Adding Units causes major problems:
    - W.Heijltjes & R. Houston: complexity issues even for MLL.
    - Free CCC (with $\mathbf{1}$) is decidable: Cubric introduced elaborate theory of η-expansions.
    - Free CCC with $\mathbf{1}$ is decidable using NBE via categorical Yoneda. (Normalization and the Yoneda Lemma (Cubric, Dybjer, S.), MSCS, 1998.)
    - Free Bi-CCC with $\mathbf{1}$ is decidable: used NBE techniques (Altenkirch,Dybjer, Hofmann, S.) (LICS 2001): problems with $\mathbf{0}$ only recently resolved.
- Add Squires' Homological Monoids to typed λ-calculus (as base type). How to solve decision problems, since word problem for such monoids is **not** solvable by terminating rewriting? Use NBE [CDS, 98]

Other notions of equivalence of proofs and coherence, I

K. Dosen "Identity of proofs based on normalization and generality", BSL 9 (2003).

Lambek's original Normalization/Cut-Elimination criteria for studying equivalence of proofs arose in solving coherence problems. We identify proofs of the same sequent whose normal forms are equal (modulo the equations of a structured category). It has limitations, but is consistent with Lawvere's adjoint-functor representation of logic (Dosen) and intuitionistic systems.

Dosen, above, looks at another Lambek notion: *having the same generality*. E.g. the two projections $\pi_1, \pi_2 : p \wedge p \longrightarrow p$ (as sequent proofs) can be generalized to: $\pi_1^{p,q} : p \wedge q \longrightarrow p$ and $\pi_2^{r,p} : r \wedge p \longrightarrow p$, which should clearly be distinct! Roughly, two proofs are equivalent when their generalizations wrt diversifying variables (without changing the rules of inference) produce proofs with the same source and target, up to a renaming of variables.

Other notions of equivalence of proofs and coherence, II

Lambek (69-72) worked a lot with "generality", but reluctantly abandoned it for various technical problems. Dosen points out, if expressed graphically, generality seems to be related to various faithful representation theorems of a free category into graphical, algebraic or topological structures (already familiar to experts in coherence theory, linear and classical logics). Hard to use, though.

A third notion, which we do not discuss, is the "Australian category school's" method of proving coherence theorems: using a kind of higher Yoneda lemma to embed a free category into a stricter one, from which certain "abstract normal forms" can be obtained. It is related to NBE (Normalization by Evaluation) proofs, and is illustrated in our [CDS98]. The equality of proofs we obtain is a kind of generalized version of the "Normalization Method" above.

Lambek's Multicategories I

Lambek's paper "Deductive Systems and Categories II" : SLNM 86 (1969) introduced many fundamental new ideas.

- Precise uses of Gentzen's methods for proving coherence theorems in monoidal categories and categories with monads.
- Introduction of Multicategories, whose arrows $\Gamma \longrightarrow B$ correspond to (derivable) intuitionist sequents, modulo equations between proofs.
- Sketched a new interpretation of "equality of proofs": having the same "generality" (not directly based on equality of normal forms, à la Gentzen): discussed earlier.
- In Lambek's paper "Multicategories revisited" (Contemp. Math, **92** (1989) pp. 217-239, he introduced a formal *internal language* of proof terms for multicategories, and functional completeness, and discussed their connections to algebra and linguistics.

Multicategories II

Lambek defined a *context-free (recognition) grammar* as a "derivation" $f : A_1 \dots A_n \longrightarrow B$ where $A_i, B \in \mathcal{V}$ ($\mathcal{V}$ a vocabulary). We think of f as a multi-sorted algebraic operation. We begin with specific derivations:

Identity $1_A : A \to A$ **Cut** $\dfrac{f : \Lambda \longrightarrow A \qquad g : \Gamma A \Delta \longrightarrow B}{g\langle f \rangle : \Gamma \Lambda \Delta \longrightarrow B}$

where $\Lambda = L_1 \dots L_n$, $\Gamma = C_1 \dots C_m$, $\Delta = D_1 \dots D_\ell \in \mathcal{V}^*$.

A *multicategory* is a context-free grammar with equality between proofs (derivations):

1. $1_A\langle f \rangle = f$, $g\langle 1_A \rangle = g$ (Identity Laws)
2. $h\langle g\langle f \rangle \rangle = h\langle g \rangle \langle f \rangle$ (Associative Law)
3. $h\langle g \rangle \langle f \rangle = h\langle f \rangle \langle g \rangle$ (Commutative Law)

Multicategories IIa

Multicategories := *multigraphs* plus equations , where a multigraph is a class of *arrows* (sequents) and *objects* (formulas) & two maps **source**: {arrows}$\longrightarrow$ {objects}*, **target**: {arrows}$\longrightarrow$ {objects}.

Examples of the equations:

(1) $$\dfrac{g : \Lambda \longrightarrow A \quad 1_A : A \longrightarrow A}{g\langle 1_A \rangle : \Lambda \longrightarrow A} \quad = \quad g : \Lambda \longrightarrow A$$

(2) $$\dfrac{\dfrac{f : \Lambda \longrightarrow A \quad g : \Gamma A \Delta \longrightarrow B}{g\langle f \rangle : \Gamma \Lambda \Delta \longrightarrow B} \qquad h : \Phi B \Psi \longrightarrow C}{h\langle g\langle f \rangle \rangle : \Phi \Gamma \Lambda \Delta \Psi \longrightarrow C} \quad =$$

$$\dfrac{f : \Lambda \longrightarrow A \qquad \dfrac{g : \Gamma A \Delta \longrightarrow B \quad h : \Phi B \Psi \longrightarrow C}{h\langle g \rangle : \Phi \Gamma A \Delta \Psi \longrightarrow C}}{h\langle g \rangle \langle f \rangle : \Phi \Gamma \Lambda \Delta \Psi \longrightarrow C}$$

Multicategories III: Structured Multicategories

Lambek introduced *structured multicategories* and their internal languages: LL + *no* structural rules + equations between proofs.

1. E.g. to introduce a tensor product $\otimes$, follow Bourbaki; add a binary operation $m_{AB} : AB \longrightarrow A \otimes B$ plus equations forcing $Mult(\Gamma AB\Delta, C) \cong Mult(\Gamma A \otimes B\Delta, C)$.

   (cf. C. Hermida: Representable Multicategories. (*Adv. Math* **151**, *2000*.)
2. Similarly, can equationally introduce connectives $I, \multimap, \circ\!-$ etc.
3. *Internal languages for multicategories:* Add infinitely many variables $x_i^A : A$ for each formula A. Freely generate terms from variables (and constants), e.g. $f\vec{x} : B$ for each derivation $f : A_1 \ldots A_n \longrightarrow B$, where x_i are distinct typed variables (even of the same type).
4. Introduce congruence relation $f =_X g$ on terms in "context" $X = \{x_1 : A_1 \ldots x_n : An\}$. E.g. Interpret Cut Rule by substitution: $g\langle f\rangle\vec{u}\vec{x}\vec{v} =_X g\vec{u}[x^A := f\vec{x}]\vec{v}$, etc.

Multicategories IV: Why Multicategories?

- Can introduce structural rules (Perm, Contraction, Weakening) with associated (usual Curry-Howard) equations.
- Lambek proved a generalized Functional Completeness for multicategories: for all terms $\varphi(\vec{x})$ containing distinct variables $x_i : A_i$ in that order, $\exists! f : A_1 \ldots A_n \longrightarrow B$ s.t. $f\vec{x} =_X \varphi(\vec{x})$.
- Lambek pointed out we can, in some sense, avoid coherence:

Theorem (Avoiding Coherence!)

All properties/equations of $\otimes$ and I follow (ditto other connectives)

- *$\otimes$ is functorial*
- *There's natural isomorphism* $\alpha_{ABC} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$.
- *Naturality Isos* $\lambda_A : I \otimes A \to A$, $\rho_A : A \otimes I \to A$
- *Mac Lane's Pentagon*
- *Mac Lane's Triangle Conditions.*

Application: "Generalized connectives"

Leinster's book on Operads pointed out another view of Lambek's observation: if we present $\otimes$-categories with n-ary tensors ("generalized connectives") instead of binary tensors, why do coherence problems virtually disappear? Let us call these Generalized Monoidal Categories.

- Generalized Monoidal Categories $\simeq$ strict (ordinary) ones.
- Also, there's an equivalence:

  Multicategories $\simeq$ Generalized Monoidal Cats :

$$\frac{A_1 A_2 \cdots A_n \longrightarrow B}{\overline{A_1 \otimes A_2 \cdots \otimes A_n \longrightarrow B}}$$

- Usual coherence (Mac Lane) for monoidal cats presented with binary tensors becomes a specialized problem:

  Ordinary monoidal cats $\simeq$ strict (ordinary) ones

Formulas-as-types, categorically

In late 60's, Lambek had his version of Curry-Howard, extended to equations between proofs. In a precise form, the key example:

Theorem (L-S, 1986)

There is a strong equivalence between the following categories:

$$\boxed{\textbf{CCC}} \xrightarrow{\simeq} \boxed{\textbf{Typed } \lambda\textbf{-Calc}}$$

given by functors $L : \mathcal{C} \longmapsto \mathcal{L}_{\mathcal{C}}$ *(the internal language of* $\mathcal{C}$*)*
$C : C(\mathcal{L}) \longleftarrow \mathcal{L}$ *(cat. generated by* $\mathcal{L}$ *= term model)*

Can add NNO's **N**, other data types. Detailed extensions include:

1. Cartesian Categories (Cockett-Seely, Dosen-Petric)
2. Monoidal/Closed/$\cdots$ (Jay,Blute-Cockett-Seely,Dosen-Petric)
3. Adding Negation and classical logic (Selinger,B-C-S-T,Dosen-Petric, Strassberger)

Recent Results:

- Uustalu, Veltri, Zeilberger: The sequent calculus of Szlachányi's Skew-Monoidal Categories (where a, ℓ, r are n.t.'s, not isos). [Also arises as a subset of axioms of BCS's *contextual categories*]. From Quantum Algebra &Topology.

  Based on a focussed calculus of intuitionistic non-commutative linear logic, modulo equivalence of proofs. Construction of free skew monoidal categories and algorithms for word problems, as well as non-emptiness of hom-sets.
- Castellan, Clairambault, Dybjer: *Categories with Families.* Sets up various 2-categories of **Cwf** (categories with families: syntactical contexts and context morphisms, with structure of types, terms, comprehensions). Proves bi-equivalences, e.g. (**FL** = finite limit categories; **LCC** = locally cartesian closed)
  1. $\mathbf{FL}^{op} \simeq \mathbf{Cwf}^{\,I_{ext},\Sigma}$
  2. $\mathbf{LCC}^{op} \simeq \mathbf{Cwf}^{\,I_{ext},\Sigma,\Pi}$

What are computable functions in categories?

When I first came to McGill as a postdoc, I asked Lambek "shouldn't we learn about what the recursion theorists are doing?"

Lambek said: "No. We have our own natural notions of computation: the function(al)s computable in various free categories, e.g. in free cartesian categories, free CCCs, the free topos, etc. Let's try to understand them first, then compare."

Definition (Lawvere)

A Natural Numbers Object *(NNO) in a (cartesian closed) category is a diagram* $\mathbf{1} \xrightarrow{0} N \xrightarrow{S} N$ *initial among diagrams* $\mathbf{1} \xrightarrow{a} A \xrightarrow{h} A$. *i.e., there exists a unique* $I_{ah} : N \to A$ *satisfying:*

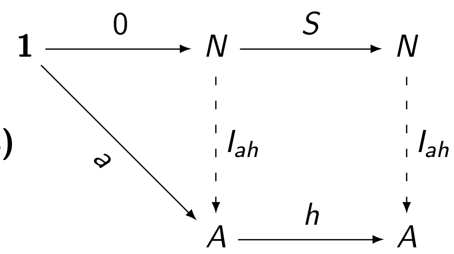
$$I_{ah}0 = a \quad , \quad I_{ah}S = hI_{ah}$$

Existence, without uniqueness, of I_{ah} *called* weak NNO

Philip Scott University of Ottawa What are equations between proofs?

Lawvere's NNOs

Lawvere's NNO (for CCC's)



Can get uniqueness of I_{ah} equationally:

Theorem (Lambek, '89)

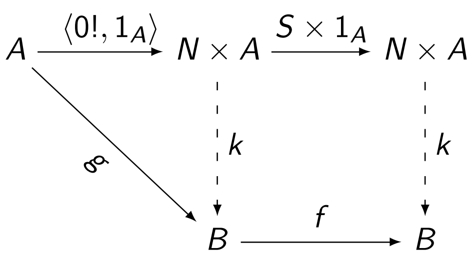
If all types A of a lambda calculus (with N) have a Mal'cev operator $m_A : A^3 \to A$, then we can equationally express the uniqueness of the iterator I_{ah}.

In particular, this applies to simply typed lambda calculus. The equations are complicated (see Okada-Scott 1999).

Primitive Recursion with parameters (weak NNO's)

Lawvere's NNO (for CCC's) implies several useful variants. The following (provable) instance of NNO is useful in weaker settings. Again, the *weak* case assumes the existence, but not uniqueness, of k, below.

Parametrized NNO: For $A \xrightarrow{g} B$, $B \xrightarrow{f} B$, $\exists!\, k : N \times A \to B$.



In Sets

$k(0, a) = g(a)$

$k(n+1, a) = f(k(n, a))$

Representing Numerical Functions in Categories

Definition (L-S, 1986)

Let $\mathcal{C}$ be a category with a weak NNO $\mathbf{1} \xrightarrow{0} N \xrightarrow{S} N$.
$f : \mathbb{N}^k \to \mathbb{N}$ *is* representable *in $\mathcal{C}$ if there is an arrow* $F : N^k \to N$ *such that* $F\langle \bar{n}_1 \cdots \bar{n}_k \rangle = \overline{f(n_1, \cdots, n_k)}$, *where* $\bar{n} = S^n 0$.

A category is *Cartesian* if it has finite cartesian products.

Theorem (L. Roman, 1989)

The representable numerical functions in $\mathcal{F}_c$, the free cartesian category with parametrized NNO, are exactly the primitive recursive ones.

Hence the unique representation functor $\mathcal{F}_c \longrightarrow \mathbf{Set}$ has image = the subcategory of sets with objects = powers $\mathbb{N}^n$ and whose maps are tuples of primitive recursive functions.

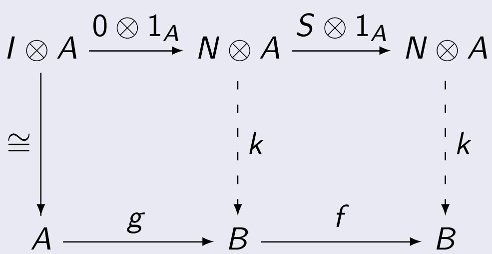
Aside: NNO's in Monoidal Categories?

Cartesian categories, as deductive systems, correspond to conjunction calculus $\mathcal{L} = \{\wedge, \top\}$. What if we move to a substructural, or linear, logic $\mathcal{L} = \{\otimes, I\}$? (Cf. interesting paper, Paré-Roman: *Studia Logica*, 1989.)

Don't even assume symmetry (or permutation) $A \otimes B \xrightarrow{\sigma} B \otimes A$. Get *Left* and *Right* NNO's.

Definition (Left Parametrized NNO:)

For $A \xrightarrow{g} B$, $B \xrightarrow{f} B$, $\exists \;\; k : N \otimes A \to B$.



Paré-Roman's Theorem

$\otimes$ does not have explicit projections. Nevertheless:

Theorem (Paré-Roman)

(i) The primitive recursive functions are representable in any monoidal category with LNNO.
(ii) Indeed, the free monoidal category with LNNO exists and is isomorphic to the free cartesian category with parametrized NNO.

Why? *We can code projections (weakening) and diagonals between tensor powers* $N^{\otimes k}$. Proved by a sophisticated categorical argument by Paré-Roman. Plotkin[2013] gives a direct coding proof of (i).

How do we get more? Increase the logical strength (the types) from the logic of $\{\wedge, \top\}$ to $\{\wedge, \Rightarrow, \top\}$, as follows.

Free CCC with N: syntactically

The *free CCC with* N is generated from simply typed lambda calculus, with iterators.

Objects:= Types $A, B ::= \mathbf{1} \mid N \mid A \times B \mid A \Rightarrow B$

Arrows:= Terms (modulo equations):
e.g. $(x : A, t(x) : B) : A \longrightarrow B$.
Typical terms

$\mathbf{1}$	N	$A \times B$	$A \Rightarrow B$
$*$	0	$\langle a, b\rangle$	$\lambda x : A.\varphi(x)$
	Sn		

together with *variables, evaluation* $ev(f, a)$ (where $f : A \Rightarrow B$ and $a : A$) , *projections, and iterators of every type* $It_{a,h} : N \Rightarrow A$.

Equations: add the minimal amount needed to have a CCC with weak NNO. (essentially : products, β, η, iteration)

Representable Functions in the Free CCC with N

The following is a theorem in simply typed lambda calculus, translated into the language of CCCs:

Theorem (L-S, 1986)

In the free ccc $\mathcal{C}$ with weak NNO N, we have

1. *All primitive recursive functions and the Ackerman function are representable (in fact, if there's a strong NNO, by their usual free variable equations)*
2. *We represent a proper subclass of the total recursive functions.*

There is also a version of Gödel's Incompleteness for this language.

Which functionals are representable in the free ccc?

- These closed lambda terms represent a version of Gödel's Dialectica Functionals ($=$ *the primitive recursive functionals of finite type*).
- Thus we represent exactly the provably total functions of classical (first-order) Peano Arithmetic, i.e. those satisfying $\vdash \forall x \exists ! y A(x, y)$, for $A \in \Delta$. These correspond to the ε_0-recursive functions, a proper subclass of the total recursive functions.

Can we get still more numerical functions and functionals?

Q: What if we add equalizers? (from Burroni's work, we can do this equationally: see L-S, 1986). What is the free ccc with weak NNO and equalizers? What does it mean to add equalizers to the Dialectica functionals? What happens at higher types?

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Thank You

ISOMORPHISMS IN A CATEGORY OF PROOFS

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I aim to show how a category of propositional formulas and *classical* proofs can give rise to finely grained hyperintensional notions of sameness of content. One notion is *very* finely grained (distinguishing p and $p \wedge p$), others less so. I show that one notion amounts to equivalence in Richard Angell's logic of analytic containment [1].

1. A CATEGORY OF CLASSICAL PROOFS

Four different *derivations*, and two *proofs*.

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,{\scriptstyle \wedge L}}{p \wedge q \succ p \vee q}\,{\scriptstyle \vee R} \;\approx\; \dfrac{\dfrac{p \wedge q}{p}}{p \vee q} \;\approx\; \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,{\scriptstyle \vee R}}{p \wedge q \succ p \vee q}\,{\scriptstyle \wedge L}$$

$$\dfrac{\dfrac{q \succ q}{p \wedge q \succ q}\,{\scriptstyle \wedge L}}{p \wedge q \succ p \vee q}\,{\scriptstyle \vee R} \;\approx\; \dfrac{\dfrac{p \wedge q}{q}}{p \vee q} \;\approx\; \dfrac{\dfrac{q \succ q}{q \succ p \vee q}\,{\scriptstyle \vee R}}{p \wedge q \succ p \vee q}\,{\scriptstyle \wedge L}$$

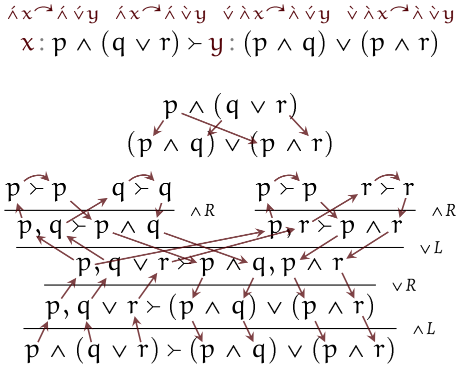
MOTIVATING IDEA: *Proof terms* are an *invariant* for derivations under rule permutation. δ_1 and δ_2 have the same *term* iff some permutation sends δ_1 to δ_2.

$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : p \succ y : p}}{\overset{\acute{\wedge} x \curvearrowright y}{x : p \wedge q \succ y : p}}\,{\scriptstyle \wedge L}}{\overset{\acute{\wedge} x \curvearrowright \acute{\vee} y}{x : p \wedge q \succ y : p \vee q}}\,{\scriptstyle \vee R} \qquad \acute{\wedge} x \curvearrowright \acute{\vee} y \qquad \dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ y : p}}{\overset{x \curvearrowright \acute{\vee} y}{x : p \succ y : p \vee q}}\,{\scriptstyle \vee R}}{\overset{\acute{\wedge} x \curvearrowright \acute{\vee} y}{x : p \wedge q \succ y : p \vee q}}\,{\scriptstyle \wedge L}$$

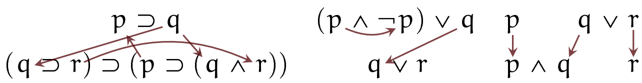
$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{\grave{\wedge} x \curvearrowright y}{x : p \wedge q \succ y : q}}\,{\scriptstyle \wedge L}}{\overset{\grave{\wedge} x \curvearrowright \grave{\vee} y}{x : p \wedge q \succ y : p \vee q}}\,{\scriptstyle \vee R} \qquad \grave{\wedge} x \curvearrowright \grave{\vee} y \qquad \dfrac{\dfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{x \curvearrowright \grave{\vee} y}{x : q \succ y : p \vee q}}\,{\scriptstyle \vee R}}{\overset{\grave{\wedge} x \curvearrowright \grave{\vee} y}{x : p \wedge q \succ y : p \vee q}}\,{\scriptstyle \wedge L}$$

[*]Thanks to *Rohan French*, *Lloyd Humberstone*, *Dave Ripley*, *Shawn Standefer* and audiences at the University of Melbourne, the CUNY Graduate Center, CMU and MIT for helpful feedback on this material. ¶ In memory of Kosta Došen. ¶ This research is supported by the Australian Research Council, through Grant DP150103801.

A *proof term* for $\Sigma \succ \Delta$ encodes the flow of information in a proof of $\Sigma \succ \Delta$. They are certain directed graphs on sequents [2, 10–12].



More examples:



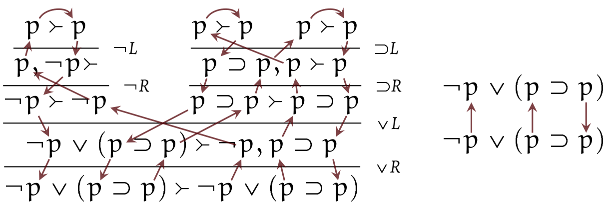
Links wholly internal to a *premise* or a *conclusion* are called *cups* (⤻) and *caps* (⤼).

FACTS: Not every directed graph on occurrences of atoms in a sequent is a proof term. ¶ They *typecheck*. [An occurrence of p is linked only with an occurrence of p.] ¶ They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.] ¶ They must satisfy an "enough connections" condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise $p \vee q$ and conclusion $p \wedge q$ is not connected enough to be a proof term.]

Cut is chaining of proof terms, composition of graphs.

Cut elimination is *confluent* and *terminating*. [So it can be understood as a kind of *evaluation*.] ¶ Cut elimination for proof terms is *local*. [So it is easily made parallel.]

$\mathfrak{C}$ is the *Category of Classical Proofs*. *OBJECTS*: Formulas. *ARROWS*: Cut-Free Proof Terms. *COMPOSITION*: Composition of derivations with the elimination of *Cut* — If $\pi : A \succ B$ and $\tau : B \succ C$ then $\tau \circ \pi : A \succ C$. *IDENTITY*: Canonical identity proofs — Id(A) : A ≻ A.



The category $\mathfrak{C}$ is *symmetric monoidal* and *star autonomous*, but not *Cartesian*, with structural *monoids* and *comonoids*, and is enriched in *SLat* (the category of semi-lattices) [9]. Being enriched in *SLat* means that proof terms come ordered by $\subseteq$, and compose under $\cup$, and these interact sensibly with composition.

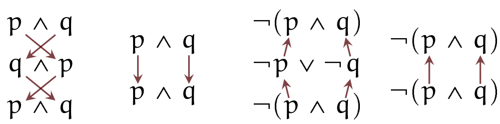
$$
\begin{aligned}
\pi \subseteq \pi' &\Rightarrow \pi \circ \tau \subseteq \pi' \circ \tau \\
\tau \subseteq \tau' &\Rightarrow \pi \circ \tau \subseteq \pi \circ \tau' \\
\pi \circ (\tau \cup \tau') &= (\pi \circ \tau) \cup (\pi \circ \tau') \\
(\pi \cup \pi') \circ \tau &= (\pi \circ \tau) \cup (\pi' \circ \tau)
\end{aligned}
$$

$\mathfrak{C}$ is *classical* propositional logic, in a categorical setting. (The sequent calculus plays no special role. Proof terms can be defined on *natural deduction*, *Hilbert proofs*, *tableaux*, *resolution*, etc.)

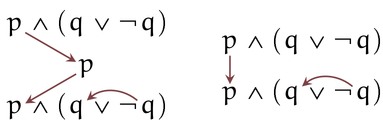
2. ISOMORPHISMS

$f : A \to B$ is an *isomorphism* in a category iff it has an *inverse* $g : B \to A$, where $g \circ f = id_A : A \to A$ and $f \circ g = id_B : B \to B$. If A and B are isomorphic in a category $\mathcal{C}$, then what we can do with A (in $\mathcal{C}$) we can do with B, too. If A and B are isomorphic in $\mathfrak{C}$, then they agree not only on *provability*, but also, on *proofs*. The distinctions drawn when you analyse how something is *proved* (from premises), are not far from what you want to understand when you ask how something is *made true*.

Isomorphisms in $\mathfrak{C}$: $p \wedge q \cong q \wedge p$; $\neg(p \wedge q) \cong \neg p \vee \neg q$



Non-isomorphisms in $\mathfrak{C}$: $p \wedge (q \vee \neg q) \not\cong p$; $p \wedge p \not\cong p$; $p \wedge (q \vee r) \not\cong (p \wedge q) \vee (p \wedge r)$; $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$.



Occurrence Polarity Condition: If A is isomorphic to B in $\mathfrak{C}$ then each variable occurs positively [negatively] in A exactly as many times as it occurs positively [negatively] in B. (This condition is *necessary*, not *sufficient*: $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$.)

A is *isomorphic to* B iff A and B are equivalent in the following calculus:

$$A \wedge B \leftrightarrow B \wedge A, \qquad A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C.$$
$$A \vee B \leftrightarrow B \vee A, \qquad A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C.$$
$$\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B, \qquad \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B.$$
$$\neg\neg A \leftrightarrow A. \qquad A \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B).$$

(This allows for a *negation normal form*, but not DNF or CNF. These are the equivalences in multiplicative linear logic, where $\wedge$ is understood as $\otimes$, $\vee$ is $\parr$, and $\neg$ is linear negation.)

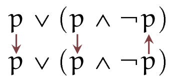
Proof Sketch (Došen and Petrić, 2012 [3]).

If $A \leftrightarrow B$ holds in the calculus, A and B are isomorphic. ¶ A is isomorphic to B iff there are *diversified* A' and B' where A' and B' are isomorphic, and $A = \sigma A'$ and $B = \sigma B'$ for some substitution σ. ¶ A is isomorphic to B iff their negation normal forms are isomorphic. (If A is diversified, so is its negation normal form.) ¶ If A and B are diversified, isomorphic, and in negation normal form, if $l \wedge m$ is a conjunction in A (l and m, literals), a substitution argument (substituting $\top$ and $\bot$ for the *other* atoms) shows that l and m must be conjunctively joined in B, too. The same goes for $l \vee m$. ¶ Replace $l \wedge m$ by a new atom in both A and B, and repeat. ¶ This shows how to reconstruct a proof of equivalence for A and B in the syntactic calculus for isomorphic formulas.

Isomorphism is a very tight constraint: If A and B are isomorphic, they can play *essentially* the same role in proof. ¶ Replacing A by B in a proof (as a premise or conclusion) not only gives you something that is also a proof (mere logical equivalence would do *that*), but it gives you a proof which is *essentially the same*. ¶ Not even A and $A \wedge A$ are equivalent in *this* sense. ¶ Yet, A and $A \wedge A$ seem to have identical *subject matter* (insofar as we understand that notion). ¶ Can we use the fine-grained tools of proof theoretical analysis to generalise this notion of subject matter to arbitrary statements?

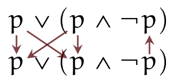
3. more proofs from A to A

Id$(p \vee (p \wedge \neg p))$



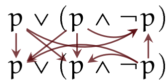
In $\mathrm{Id}(A)$, each occurrence of an atom in the premise is linked with every its corresponding occurrence of in the conclusion. *Different occurrences of atoms in A are treated differently*.

Hz$(p \vee (p \wedge \neg p))$



In $Hz(A)$, each positive [negative] occurrence of an atom in the premise is linked with every positive [negative] occurrence in the conclusion. *We treat occurrences of an atom in* A*—with the same polarity—equally.*

$Mx(p \vee (p \wedge \neg p))$



In $Mx(A)$, each syntactically possible linking is included. *We treat all occurrences of an atom in* A *equally.*

Note: $Hz(A)$ is $Mx(A)$ with the caps and cups removed.

Let's look at relations like isomorphism, but which erase distinctions, up to *Hz* or *Mx*.

Let's say that A and B *Hz*-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$. We write "$\approx_{Hz}$" for the *Hz*-matching relation, and we write "$\pi, \pi' : A \approx_{Hz} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Hz*-match between A and B.

Let's say that A and B *Mx*-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\tau \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$. We write "$\approx_{Mx}$" for the *Mx*-matching relation, and we write "$\pi, \pi' : A \approx_{Mx} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Mx*-match between A and B.

Isomorphism $\subseteq$ *Hz-Matching*: If $\pi : A \succ B$ and $\pi^{-1} : B \succ A$, then consider $\pi' = Hz(B) \circ \pi \circ Hz(A)$ and $\tau' = Hz(A) \circ \pi^{-1} \circ Hz(B)$. These satisfy the *Hz*-matching criteria, $\tau' \circ \pi' = Hz(A)$ and $\pi' \circ \tau' = Hz(B)$.

Hz-Matching $\subseteq$ *Mx-Matching*: If $\pi, \pi' : A \approx_{Hz} B$, then consider $\tau = Mx(B) \circ \pi \circ Mx(A)$ and $\tau' = Mx(A) \circ \pi' \circ Mx(B)$. These satisfy the *Mx*-matching criteria, $\tau' \circ \pi' = Mx(A)$ and $\pi' \circ \tau' = Mx(B)$.

Mx-Matching $\subseteq$ *Logical Equivalence*: If $A \approx_{Mx} B$ then there are proofs $\pi : A \succ B$ and $\tau : B \succ A$.

Matching Relations are Equivalences: *REFLEXIVE* $Hz(A), Hz(A) : A \approx_{Hz} A$. $Mx(A), Mx(A) : A \approx_{Mx} A$. ¶ *SYMMETRIC* If $\pi, \pi' : A \approx_{Hz} B$, then $\pi', \pi : B \approx_{Hz} A$. If $\pi, \pi' : A \approx_{Mx} B$, then $\pi', \pi : B \approx_{Mx} A$. ¶ *TRANSITIVE* If $\pi, \pi' : A \approx_{Hz} B$ and $\tau, \tau' : B \approx_{Hz} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Hz} C$. If $\pi, \pi' : A \approx_{Mx} B$ and $\tau, \tau' : B \approx_{Mx} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Mx} C$.

Matchings: $p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$; $p \wedge (q \vee r) \approx_{Hz} (p \wedge q) \vee (p \wedge r)$.

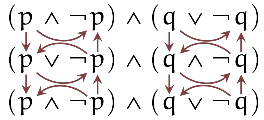
Mx-Matching $\subset$ *Logical Equivalence*: If an atom p occurs positively [negatively] in A but not in B, then A and B do not *Mx*-match.

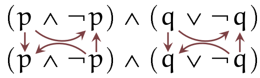
PROOF: $Mx(A) : A \succ A$ contains the link from [to] that occurrence of p in the premise A to [from] its corresponding occurrence in the conclusion A. ¶ No proof from A to B contains a link from [to] that occurrence to anything in B (since there is no positive [negative] occurrence in B at all). ¶ So, in the composition proof from A to A, there is no link from [to] the premise occurrence to

[from] the conclusion occurrence. No proof from A to B and back can recreate $Mx(A)$.

COROLLARY: $p \not\approx_{Mx} p \wedge (q \vee \neg q)$; $p \wedge \neg p \not\approx_{Mx} q \wedge \neg q$.

Hz-matching ⊂ *Mx-matching*: $(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{Mx} (p \vee \neg p) \wedge (q \wedge \neg q)$.





However, $(p \wedge \neg p) \wedge (q \vee \neg q) \not\approx_{Hz} (p \vee \neg p) \wedge (q \wedge \neg q)$. So:

Isomorphism ⊂ *Hz-Matching* ⊂ *Mx-Matching* ⊂ *Logical Equivalence*

4. MATCHING & LOGICS OF ANALYTIC CONTAINMENT

Angell's Logic of Analytic Containment: [AC1] $A \leftrightarrow \neg\neg A$ [AC2] $A \leftrightarrow (A \wedge A)$ [AC3] $(A \wedge B) \leftrightarrow (B \wedge A)$ [AC4] $A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$ [AC5] $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$ [R1] $A \leftrightarrow B, C(A) \Rightarrow C(B)$

Here, $A \vee B$ is shorthand for $\neg(\neg A \wedge \neg B)$. You can define $A \rightarrow B$ as $A \leftrightarrow (A \wedge B)$.

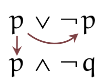
The first degree fragment of *Parry's* Logic of Analytic Containment is found by adding $(A \vee (B \wedge \neg B)) \rightarrow A$ to Angell's Logic. (Parry's logic still satisfies this relevance constraint: $A \rightarrow B$ is provable only when the atoms in B are present in A.)

First Degree Entailment (FDE) is found by adding $A \rightarrow (A \vee B)$ to Angell's Logic. ¶ FDE allows for a disjunctive (or conjunctive) normal form. The only difference from classical logic is that $p \vee \neg p$, and $q \wedge \neg q$ are both non-trivial, and ineliminable. ¶ A simple translation encodes FDE inside classical logic. Choose, for each atom p, a fresh atom p', its *shadow*. For each FDE formula A, its translation is the formula A' found by replacing the negative occurrences of atoms p in A by their shadows. An argument is FDE valid iff its translation is classically valid.

DEFINITION: $Mx(A, B)$ is the set of all possible linkings which could occur in any proof from A to B. ¶ That is, it contains a link from {positive atoms in A, negative atoms in B} to matching {positive atoms in B, negative atoms in A}.

FACT: $Mx(A, B)$ is a proof iff there is some proof from A to B. (And if so, it is the maximal such proof.)

$Mx(p \vee \neg p, p \wedge \neg q)$ is not a proof:



LEMMA: If $A \approx_{Mx} B$, then $Mx(A, B)$ and $Mx(B, A)$ are proofs, and $Mx(A, B)$, $Mx(B, A) : A \approx_{Mx} B$.

PROOF: If $\pi, \pi' : A \approx_{Mx} B$, then $\pi \subseteq Mx(A, B)$ and $\pi' \subseteq Mx(B, A)$, so $Mx(A, B)$ and $Mx(B, A)$ are both proofs. ¶ Since $\pi' \circ \pi = Mx(A)$, we have $Mx(A) = \pi' \circ \pi \subseteq Mx(B, A) \circ Mx(A, B) \subseteq Mx(A)$, and similarly, $Mx(B) = Mx(A, B) \circ Mx(A)$, so $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

FACT: If A is classically equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and *vice versa*, then A and B *Mx*-match—and conversely.

PROOF: If A is logically equivalent to B, then $Mx(A, B)$ and $Mx(B, A)$ are both proofs. ¶ It suffices to show that $Mx(B, A) \circ Mx(A, B) = Mx(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Mx(A, B)$ composed with a link in $Mx(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Mx(A, B)$ and $Mx(B, A)$. ¶ Conversely, if $A \approx_{Mx} B$, we have already seen that A and B must be equivalent, and no atom occurs positively [negatively] in A but not B.

This is *not* Equivalence in Parry's Logic. A is equivalent to B in Parry's logic of analytic containment iff A is classically equivalent to B and the atoms present in A are present in B and *vice versa*. ¶ $(p \wedge \neg p) \wedge q \not\approx_{Mx} (p \wedge \neg p) \wedge \neg q$, but this pair satisfies Parry's variable sharing criterion.

QUESTION: Does the equivalence relation of *Mx*-matching occur elsewhere in the literature?

DEFINITION: $Hz(A, B)$ is the set of all possible linkings which could occur in any proof from A to B, excluding caps and cups. ¶ That is, it contains a link from positive atoms in A to corresponding positive atoms in B and negative atoms in A to corresponding negative atoms in B.

$Hz(p \wedge \neg p, q \vee \neg q)$ contains no links. $Hz(p \wedge \neg p, p \vee \neg p)$ is a proof, but not the maximal one:

$$\begin{array}{ccc} p & \wedge & \neg p \\ \downarrow & & \uparrow \\ p & \vee & \neg p \end{array}$$

FACT: $Hz(A, B)$ is a proof iff A entails B in FDE.

PROOF: From FDE-validity to *Hz*-proof: straightforward induction on an FDE-axiomatisation. ¶ From the *Hz*-proof $Hz(A, B)$ to FDE-validity: Notice that no negative occurrences of atoms in A or B are linked to any positive occurrences of atoms in A or B. So, there is another *Hz*-proof $Hz(A', B')$ for the FDE translations for A and B.

LEMMA: If $A \approx_{Hz} B$, then $Hz(A, B)$ and $Hz(B, A)$ are proofs, and $Hz(A, B)$, $Hz(B, A) : A \approx_{Hz} B$.

PROOF: If $\pi, \pi' : A \approx_{Hz} B$, then then since $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$, π and π' are cap- and cup-free, so $\pi \subseteq Hz(A, B)$ and $\pi' \subseteq Hz(B, A)$,

so $Hz(A, B)$ and $Hz(B, A)$ are both proofs. ¶ Since $\pi' \circ \pi = Hz(A)$, we have $Mx(A) = \pi' \circ \pi \subseteq Hz(B, A) \circ Hz(A, B) \subseteq Hz(A)$,
and similarly, $Hz(B) = Hz(A, B) \circ Hz(A)$, so $Hz(A, B), Hz(B, A) : A \approx_{Mx} B$.

FACT: If A is FDE-equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and *vice versa*, then A and B *Hz*-match — and conversely.

PROOF: If A is FDE-equivalent to B, then $Hz(A, B)$ and $Hz(B, A)$ are both proofs. ¶ It suffices to show that $Hz(B, A) \circ Hz(A, B) = Hz(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Hz(A, B)$ composed with a link in $Hz(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Hz(A, B)$ and $Hz(B, A)$. ¶ Conversely, if $A \approx_{Hz} B$, we have already seen that A and B must be FDE-equivalent, and no atom occurs positively [negatively] in A but not B.

FACT: (Ferguson 2016 [4]; Fine 2016 [5]) A is equivalent to B in Angell's logic of analytic containment iff A is FDE equivalent to B, and any atom occurs positively [negatively] in A iff it occurs positively [negatively] in B.

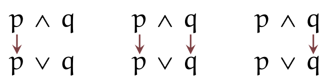
So, *Hz-matching* ≡ *Angellic Equivalence*.

5. MATCHING AS ISOMORPHISM

$Hz(A)$ and $Mx(A)$ are Idempotents: $Hz(A) \circ Hz(A) = Hz(A)$, $Mx(A) \circ Mx(A) = Mx(A)$.

For any category $\mathcal{C}$, if i_A is an idempotent for each object A, we can form a new category $\mathcal{C}_i$ with the same objects as $\mathcal{C}$, and with arrows $i_B \circ f \circ i_A : A \to B$. ¶ In this new category, the idempotents i_A are the new identity arrows. ¶ So, $\mathfrak{C}_{Hz}$ and $\mathfrak{C}_{Mx}$ are both categories — like $\mathfrak{C}$, but less discriminating, with fewer arrows.

Hz-matching is *isomorphism* in $\mathfrak{C}_{Hz}$. *Mx*-matching is *isomorphism* in $\mathfrak{C}_{Mx}$. $\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$ are nontrivial, nonetheless.



These are each different proofs in $\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$.

6. IN CONCLUSION

¶ These results allow for genuinely hyperintensional distinctions to be drawn, using tools that are native to classical proof theory. Proof theoretical resources indigenous to *classical logic* provide tools for fine-grained hyperintensional distinctions, and some of these tools slice at exactly the same joints as have been discerned using very different techniques. It is encouraging to see how non-classical logics like FDE and Angell's logic of analytic containment arise out of proof theoretical considerations in classical logic. (This is not unprecedented. In Chapter I.3 of *Proof Theory and Logical Complexity*, Girard shows how the sequent calculus, under another guise, gives rise to Kleene's 3-valued logic [6].) Here, we have started with the hyperintensionality of the phrase "… proves that …" and shown this has an underlying logical structure and coherence deeper than the surface syntax of a particular representation system for proofs.

¶ Extending these results to include the units $\top$ and $\bot$ are not difficult. (They were left out only to ease the presentation). In short, we allow for degenerate edges for proofs involving the units. For $\succ\top$ we have a link with $\top$ as the target, but with no source. There are *no* links with $\top$ as a source. So, in the identity arrow from $\top$ to $\top$, there is a degenerate link into the conclusion $\top$, and nothing leaving the premise. The situation is reversed for $\bot$. For $\bot\succ$ we have a link *from* $\bot$ going nowhere. This link features in the identity proof for $\bot \succ \bot$.

As for isomorphisms in the calculus with $\top$ and $\bot$, it turns out that $A \vee \bot \approx A \approx A \wedge \top$, $\neg\top \approx \bot$, and $\neg\bot \approx \top$. However, $A \wedge \bot \not\approx \bot$, in general, since this would violate the variable occurrence condition (which still holds). Nonetheless, $\bot \wedge \bot \approx \bot$ and $\bot \vee \bot \approx \bot$ and $\top \wedge \top \approx \top$.

¶ One open question is how to relate these results to *models* of logics of content. Is there a way to move from the family of different proofs for A (from different premises) to *situations* making A true in any rich sense? An immediate issue to be confronted is that proofs—and proof terms—wear their premises and their conclusions on their face. A proof from A to B is not *also* a proof from a different C to a different D. Even though proof terms abstract away from some of the syntactic details of derivations or proofs, they don't abstract away the *premise* and the *conclusion*.

Situations, even though they can be more local and discriminating than possible worlds (or models assigning a truth value to every formula in the language), generally make more than one thing true. To construct situations from proof terms, we must bridge this gap in some way or other.

¶ Another step to consider is whether we can expand these results to first order logic. Some recent work of Dominic Hughes on unification nets for first order multiplicative linear logic [8] brings to light an important distinction for different approaches to proof terms for predicate logic. It is clear that these two derivations here correspond to the one natural deduction proof, and should have the same proof term:

$$\dfrac{\dfrac{Ft \succ Ft}{Ft \succ \exists xFx}\,{\scriptstyle \exists R}}{\forall xFx \succ \exists xFx}\,{\scriptstyle \forall L} \;\approx\; \dfrac{\dfrac{\forall xFx}{Ft}\,{\scriptstyle \forall E}}{\exists xFx}\,{\scriptstyle \exists I} \;\approx\; \dfrac{\dfrac{Ft \succ Ft}{\forall xFx \succ Ft}\,{\scriptstyle \forall L}}{\forall xFx \succ \exists xFx}\,{\scriptstyle \exists R}$$

But what about two different derivations going through two different intermediate terms, t_1 and t_2? Girard's proof nets for first order MLL take these to be *different* [7]. There is one clear sense, proof theoretically, that the information flows from $\forall x Fx$ to $\exists x Fx$ in the same way regardless of which term used, so Hughes' unification nets (which abstract away from the identity of the particular unifiers used) seem well motivated on proof theoretic grounds.

However, when it comes to the metaphysics of grounding and subject matter, it seems that there is good reason allow each object that makes $\exists x Fx$ true contribute in its own, individual, way. This much seems clear. Different objects witness quantifiers in different ways, and this should be reflected in the detail of truthmakers. However, the *logic* of such distinctions is yet to be understood clearly. Perhaps tools from proof theory will be able to help clarify some of the options to further explore.

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Natural Language Processing
by natural deduction in
Transparent Intensional Logic

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Contents

- Transparent Intensional Logic (TIL)
  - why TIL? *Procedural semantics*
  - method of analysis → all the semantically salient features of NL are formalized, made explicit
  - three kinds of context: extensional, intensional and hyperintensional
- Natural deduction in TIL
- Natural language processing
  - Special hard phenomena in NL
    - ***ambiguities***; quantifying *into* a hyperintensional context; ***anaphora*** resolution; ***modifiers***; ***presuppositions*** vs. ***mere entailment***; definite descriptions; ***partiality***; narrow-scope vs. wide scope negation; ***de dicto*** vs. ***de re***; …
  - proof systems for NL in TIL

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Natural Language Processing

- The most important applications of TIL
  - Analysis of natural language
  - Natural language reasoning
  - Multi-agent systems;
    agent's communication and reasoning
- The ***TIL-Script*** *functional programming language*
- Duží M., Jespersen B., Materna P. (2010): *Procedural Semantics for Hyperintensional Logic.* Springer

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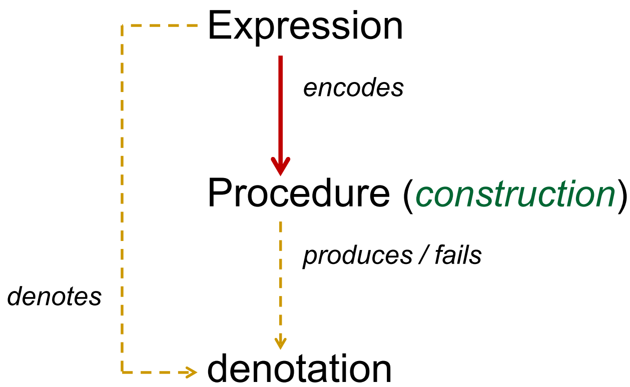
Procedural semantics of TIL



Ontology of TIL: ramified hierarchy of types

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TIL Ontology (types of order 1)

(non-procedural objects)

- ***Basic types***
  truth-values {T, F} **(ο)**
  universe of discourse {individuals} **(ι)**
  times or real numbers **(τ)**
  possible worlds **(ω)**
- ***Functional types*** **$(\beta\ \alpha_1 \ldots \alpha_n)$**
  partial functions $(\alpha_1 \times \ldots \times \alpha_n) \rightarrow \beta$
  - ***PWS Intensions*** – *entities of type* **((ατ)ω)**; $\alpha_{\tau\omega}$

Functional approach; sets and relations

- All the denoted objects are *functions*, possibly in an extreme case 0-ary functions without arguments, i.e. atomic objects like individuals of type ι or numbers of type τ
- ***sets* and *relations*(-in-extension) are modelled by *characteristic functions*.**
  - a set of α-elements is an object of type (οα), Binary relation between α- and β-objects is an object of type (οαβ)
- *examples*.
  - The set of prime numbers is an object of type (οτ); in symbols *Prime*/(οτ)
  - The set of solutions of the equation *Sin(x)* = 0, i.e. the set of multiples of π is also an object of type (οτ)
  - The relation > defined on numbers is an object of type (οττ)

Possible worlds

- No multiple universes; no impossibilia
- *Universe of discourse*: the collection of *bare individuals* – abstract hangers (determined just by an ID) to hang particular traits and relations on
- *Possible world*: chronology of maximal consistent distributions of these basic traits among individuals
- *PWS-intensions* / $((\alpha\tau)\omega)$; or $\alpha_{\tau\omega}$ for short

Examples of PWS-intensions

- *propositions* of type $((o\tau)\omega)$ or $o_{\tau\omega}$ for short;
  - denoted by sentences like "Tom is a student"
- *properties* of individuals of type $(((o\iota)\tau)\omega)$ or $(o\iota)_{\tau\omega}$ for short;
  - denoted by nouns or adjectives like '(being a) student', 'round', …
- *binary relations*-in-intension between individuals of type $(o\iota\iota)_{\tau\omega}$;
  - denoted by transitive verbs like 'to kick', 'to like', …
- *individual offices* (or roles) of type $\iota_{\tau\omega}$;
  - denoted by definite descriptions like 'the Pope', 'the US president', 'Miss World 2019', 'No. 1 in WTA ranking', …
- *Attributes* of type $(\alpha\beta)_{\tau\omega}$;
  - denoted by terms 'temperature in …', 'president of …', 'the mayor of …'

Examples of extensions (no domain ω)

- Logical objects like *truth-functions* and *quantifiers* are extensional
- **∧ (conjunction), ∨ (disjunction) and ⊃ (implication)** are of type (οοο), and **¬ (Boolean negation)** of type (οο).
- ***Quantifiers*** $\forall^\alpha$, $\exists^\alpha$ are type-theoretically polymorphic total functions of type (ο(οα)), for an arbitrary type α, defined as follows.
- The ***universal quantifier*** $\forall^\alpha$ is a function that associates a class *A* of α-elements with **T** if *A* contains ***all*** elements of the type α, otherwise with **F**.
- The ***existential quantifier*** $\exists^\alpha$ is a function that associates a class *A* of α-elements with **T** if *A* is a ***non-empty*** class, otherwise with **F**.
- Mathematical objects like the function + of type (τττ), trigonometric functions are objects of type (ττ), relations between numbers (οττ)

Constructions

- *Variables* *x, y, p, w, t,* ... *v*-construct
- *Trivialization* ^0C constructs *C* (of any type)
  - a *fixed pointer* to *C* and the *dereference* of the pointer.
  - In order to operate on *C*, *C* needs to be grabbed, or 'called', first. Trivialization is such a grabbing mechanism.
- *Closure* $[\lambda x_1 \ldots x_n\, X] \rightarrow (\beta\, \alpha_1 \ldots \alpha_n)$, where $x_1 : \alpha_1$, $x_n : \alpha_n$, $X : \beta$
- *Composition* $[F\; X_1 \ldots X_n] \rightarrow \beta$, where $F : (\beta\, \alpha_1 \ldots \alpha_n)$, $X_1 : \alpha_1$, $X_n : \alpha_n$
- *Execution* ^1X, *Double Execution* ^2X

Extensional vs. intensional (opaque) context

- When is the context extensional?
- The context is extensional if the extensional rules like *substitution of identicals* and *existential generalization* are valid
- And when are these rules valid?
- In an extensional context
  - hmmm … ?
- We stir clear of this circle by
  - Defining *three kinds of context* positively
  - Defining universally valid rules of inference

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Fundamental dichotomy

- Construction *C* can occur

- *Executed* (ordinary contexts): the object produced by *C* is an argument …
  - λ-calculi based on simple theory of types distinguish between *function (in-extension)* and *functional value*
- *Displayed* (hyperintensional contexts): the construction *C* itself is an argument …
  - Hyperintensional λ-calculus based on ramified theory of types distinguishes between *procedure ('functions-in-intension')* producing functions (or its values, if any)
- TIL is a hyperintensional typed λ-calculus of partial functions

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Hyperintensional context of displayed constructions

"Tom computes *Cotg(π)*"

- *Tom*/ι;
- ***Compute***/$(o\iota\mathbf{?})_{\tau\omega}$; computing is an empirical relation(-in-intension) of an individual to what?
- It cannot be a relation to a number, because
  1. there is no such number here
  2. It makes no sense to compute a number without any arithmetic operation
- It is the relation of an individual *to the very meaning procedure* encoded by '*Cotg(π)*'; Tom wants to execute the procedure of applying the function *Cotangent* to the number π
- *Cotg*/(ττ); π/τ; [^{0}*Cotg* ^{0}π] → τ; $^{\mathbf{0}}$[^{0}*Cotg* ^{0}π] → ?

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TIL Ontology (**higher-order types** of procedural objects)

- *Constructions of order 1* ($*_1$)
  - → construct entities belonging to a type of order 1
  - / belong to $*_1$: *type of order **2***
- *Constructions of order 2* ($*_2$)
  - → construct entities belonging to a type of order 2 or 1
  - / belong to $*_2$: *type of order **3***
- *Constructions of order n* ($*_n$)
  - → construct entities belonging to a type of order $n \geq 1$
  - / belong to $*_n$: *type of order **n + 1***
- ***Functional entities***: $(\beta\ \alpha_1 \ldots \alpha_n)$ / belong to $*_n$

  (n: the highest of the types to which $\beta, \alpha_1, \ldots, \alpha_n$ belong)

  And so on, *ad infinitum*

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explicit intensionalization and temporalization

- constructions of possible-world intensions directly encoded in the logical syntax of natural language:

$$\lambda w\, \lambda t\, [\ldots w \ldots . t \ldots]$$

- $w \to \omega$; $t \to \tau$; ${}^0Happy \to (o\iota)_{\tau\omega}$; ${}^0Pope \to \iota_{\tau\omega}$

$$\lambda w\, \lambda t\, [{}^0Happy_{wt}\ {}^0Pope_{wt}] \to o_{\tau\omega}$$

- Instruction: in any possible world (λw) at any time (λt):
  - Take the property of being happy (${}^0Happy$)
  - Take the papal office (${}^0Pope$)
  - Extensionalize both of them (${}^0Happy_{wt}$, ${}^0Pope_{wt}$)
  - Check whether the holder of the Papal office is happy at that *w*, *t* of evaluation ($[{}^0Happy_{wt}\ {}^0Pope_{wt}]$)

"Tom computes Cotg(π)"

- *Tom*/ι; *Compute*/$(o\iota *_n)_{\tau\omega}$;

$$\lambda w \lambda t\, [{}^0Compute_{wt}\ {}^0Tom\ {}^0[{}^0Cotg\ {}^0\pi]] \to o_{\tau\omega}$$

${}^0[{}^0Cotg\ {}^0\pi] \to *_n$; *is a constituent*

$[{}^0Cotg\ {}^0\pi] \to \tau$; *is **not** a constituent,*

it figures just as an argument of the *Compute* relation

"Tom computes *something*"

$\lambda w\, \lambda t\, \exists c\, [{}^0Compute_{wt}\, {}^0Tom\ c]$ $\quad c \to *_n$

"There is a number *x* such that Tom computes *Cotg(x)*"

- $\lambda w\, \lambda t\, \exists x\, [{}^0Compute_{wt}\, {}^0Tom\ {}^0[{}^0Cotg\ x]]$ $\quad x \to \tau$
  - Incorrect; *this procedure is not entailed*
- $\lambda w\, \lambda t\, \exists x\, [{}^0Compute_{wt}\, {}^0Tom\ [{}^0Sub\ [{}^0Tr\ x]\ {}^0y\ {}^0[{}^0Cotg\ y]]]$ $\quad x \to \tau$
  - correct; *this procedure is entailed*

Method of analysis

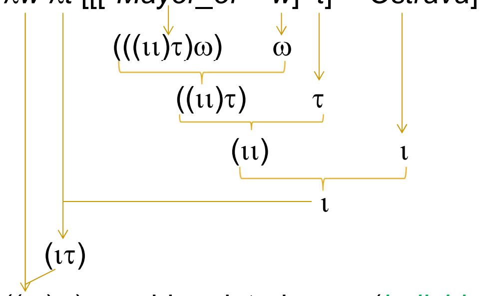
1. Assign *types* to objects that are mentioned by *E*, i.e. to the objects denoted by subexpressions of *E*
2. *Compose constructions* of objects ad 1) to construct the object denoted by *E*

   Semantically simple expressions (including idioms) are furnished with Trivialization of the denoted object as their meaning
3. *Type checking* usually by drawing a derivation tree

Example: 'The Mayor of Ostrava'

- Types: *Mayor_of*/(((ιι)τ)ω) – abbr. $(\iota\iota)_{\tau\omega}$: attribute; *Ostrava*/ι, *Mayor_of_Ostrava*/((ιτ)ω) – abbr. $\iota_{\tau\omega}$
- Synthesis: $\lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Ostrava]$
- Type checking:

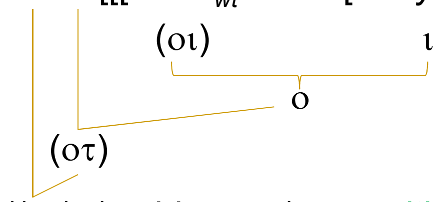
$\lambda w\ \lambda t\ [[[{}^0Mayor\_of\ \ w]\ \ t]\ \ {}^0Ostrava]$



((ιτ)ω) abbreviated as $\iota_{\tau\omega}$ (*individual office*)

"The Mayor of Ostrava is rich"

- Additional type: *Rich*/$(\omicron\iota)_{\tau\omega}$
- Synthesis:
  $\lambda w \lambda t\ [{}^0Rich_{wt}\ \lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Ostrava]]_{wt}]$
- Typechecking (shortened):
  $\lambda w\ \lambda t\ [[[{}^0Rich_{wt}\ \ \lambda w \lambda t\ [{}^0Mayor\_of_{wt}\ {}^0Ostrava]]_{wt}]$



$((\omicron\tau)\omega)$ abbr. $\omicron_{\tau\omega}$ (*proposition*)

TIL vs. Montague's IL

- IL is an extensional logic, since the axiom of extensionality is valid:
  $\forall x\ (Ax = Bx) \rightarrow A = B$.
- This is a good thing. However, the price exacted for the simplification of the language (due to ghost variables) is too high;
  - the law of universal instantiation, lambda conversion and Leibniz's Law do not generally hold, all of which is rather unattractive.
- Worse, IL does *not* validate the Church-Rosser 'diamond'. It is a well-known fact that an ordinary typed λ-calculus will have this property. Given a term $\lambda x(A)B$ (the *redex*), we can simplify the term to the form $[B/x]A$, and *the order in which we reduce particular redexes does not matter*. The resulting term is uniquely determined up to α-renaming variables.
- TIL does not have these defects; it validates the Church-Rosser property (though it works with *n*-ary partial functions) for β-reduction ***by value*** rather than by name
  - the functions of TY_2 are restricted to *unary total* functions (Schönfinkel)
- Moreover, TIL adopts hyperintensionality by displaying procedures
  - For details, see Duží et al. (2010, § 2.4.3)

TIL: *logical core*

- *constructions* + *type hierarchy* *(simple and ramified)*
- The ramified type hierarchy organizes all higher-order objects: constructions (types $*_n$), as well as functions with domain or range in constructions.
- The simple type hierarchy organizes first-order objects: non-constructions like extensions (individuals, numbers, sets, etc.), possible-world intensions (functions from possible worlds) and their arguments and values.

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Three kinds of context

- *hyperintensional context*: a meaning construction occurs *displayed*
  - so that the very *construction is an object of predication*
  - though a construction at least one order higher need to be executed in order to produce the displayed construction
- *intensional context*: a meaning construction occurs *executed* in order to produce a function *f* *(Tom wants to be the Pope)*
  - so that *the whole function f is an object of predication*
  - moreover, the executed construction does not occur within another displayed construction
- *extensional context*: the meaning construction is *executed* in order to produce a particular value of the so-constructed function *f* at its argument (*Tom is the Pope*)
  - so that *the value of the function f is an object of predication*
  - moreover, the executed construction does not occur within another intensional or hyperintensional context.

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Hyperintensionality

- was born out of negative needs, to *block invalid inferences*
  - Carnap (1947, §§13*ff*); there are contexts that are neither extensional nor intensional (attitudes)
  - Cresswell; any context in which substitution of necessary equivalent terms fails is hyperintensional
- ***TIL definition is positive***:
  a context is *hyperintensional* if the very meaning *procedure* is displayed as an object of predication
- *Which inferences are valid in a hyperintensional context* ?
  - Substitution method

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Hyperintensionality

- *Extensional logic* of hyperintensions
- *Transparency*: no context is opaque
- The same (extensional) logical rules are valid in all kinds of context;
  - Leibniz's substitution of identicals, existential quantification even into hyperintensional contexts, …
- Only the types of objects these rules are applicable at differ according to a context
- Anti-contextualism: constructions are assigned to expressions as their context-invariant meanings (no reference shift)

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Natural deduction in TIL

- The I/E rules follow the general pattern and are unproblematic in a non-hyperintensional context; *applicable to constituents*
  - the rules dealing with *truth-functions* are applicable in extensional contexts.
  - the rules for *quantifiers* → we have to take into account a context in which a given construction occurs and the type of an entity that is quantified over.
    - Applying over a constituent is unproblematic
    - Quantifying *into* a hyperintensional context – ***substitution method***
  - When dealing with *empirical propositions*, the first steps of each proof are λ-elimination (λ-E) and the last ones λ-introduction (λ-I) of the left-most $\lambda w\, \lambda t$, because the whole proof sequence must be truth-preserving in any world *w* and time *t*.

example

John is sick or went to the theatre.
If he is sick then he calls a doctor.
But he doesn't call a doctor.

John went to the theatre.

1. $\lambda w \lambda t\ [[{}^0Sick_{wt}\ {}^0John] \vee [{}^0Went_{wt}\ {}^0\text{John}\ {}^0Theatre]]$ ∅
2. $\lambda w \lambda t\ [[{}^0Sick_{wt}\ {}^0John] \supset [{}^0Call_{wt}\ {}^0\text{John}\ {}^0Doctor]]$ ∅
3. $\lambda w \lambda t\ [\neg[{}^0Call_{wt}\ {}^0\text{John}\ {}^0Doctor]]$ ∅
4. $[[{}^0Sick_{wt}\ {}^0John] \vee [{}^0Went_{wt}\ {}^0\text{John}\ {}^0Theatre]]$ 1, λ-E
5. $[[{}^0Sick_{wt}\ {}^0John] \supset [{}^0Call_{wt}\ {}^0\text{John}\ {}^0Doctor]]$ 2, λ-E
6. $\neg[{}^0Call_{wt}\ {}^0\text{John}\ {}^0Doctor]$ 3, λ-E
7. $\neg[{}^0Sick_{wt}\ {}^0John]$ 5,6 MTT
8. $[{}^0Went_{wt}\ {}^0John\ {}^0Theatre]$ 4,7 DS
9. $\lambda w \lambda t\ [{}^0Went_{wt}\ {}^0John\ {}^0Theatre]$ 8, λ-I

Simple concepts; literal analysis & refining

- In TIL the meaning procedure is "primary", a proof is "secondary"
- Literal analysis:
  - semantically simple terms → ^{0}*Object*
- Refinement of analysis:
  - ontological definition of the *Object* is substituted for ^{0}*Object (machine learning methods)*
  - *Example.*
    - The Trivialization ^{0}*Prime* is in fact the least informative procedure for producing the set of prime numbers.
    - Using particular definitions of the set of primes, we can *refine* the simple concept ^{0}*Prime* in many ways, including:

Simple concepts; literal analysis & refining

- $\lambda x\ [{}^0Card\ \lambda y\ [{}^0Divide\ y\ x] = {}^0 2]$,
- $\lambda x\ [[x \neq {}^0 1] \wedge \forall y\ [[{}^0Divide\ y\ x] \supset [[y = {}^0 1] \vee [y = x]]]]$,
- $\lambda x\ [[x > {}^0 1] \wedge \neg\exists y\ [[y > {}^0 1] \wedge [y < x] \wedge [{}^0Divide\ y\ x]]$.

Def. (*refinement of a construction*) Let ^0X be a simple concept of an object X and let ^0X occur as a constituent of C_1. If C_2 differs from C_1 only by containing in lieu of ^0X an ontological definition of X, then C_2 is a *refinement of* C_1. … + transitivity

- *Corollary.* If C_2 is a refinement of C_1, then C_1, C_2 are equivalent but *not* procedurally isomorphic.
  - the term 'prime' is not synonymous with its equivalents like 'the set of naturals with just two factors', 'the set of naturals distinct from 1 that are divisible just by the number 1 and themselves', because the meanings of *synonymous terms* are procedurally isomorphic. Rather, 'prime' is only *equivalent* to these definitions.

Proving analytic truth

"*No bachelor is married*" → TRUE

- *Bachelor*, *Married*/$(o\iota)_{\tau\omega}$; *No*/$((o(o\iota))(o\iota))$

  $\lambda w \lambda t\ [[^0No\ ^0Bachelor_{wt}]\ ^0Married_{wt}]$

How to prove its analyticity? Let us define and refine:

- $m, n/*_1 \rightarrow_v (o\iota)$, $x \rightarrow_v \iota$:
  $^0No = \lambda m\ \lambda n\ \neg\exists x\ [[m\ x] \wedge [n\ x]]$, $[[^0No\ m]\ n] = \neg\exists x\ [[m\ x] \wedge [n\ x]]$.
- $^0Bachelor = \lambda w \lambda t\ \lambda x\ [\neg[^0Married_{wt}\ x] \wedge [^0Man_{wt}\ x]]$.
  - (to be unmarried and man are *requisites* of the property of being a bachelor)
- $[[^0No\ ^0Bachelor_{wt}]\ ^0Married_{wt}] =$
  $\neg\exists x\ [[^0Bachelor_{wt}\ x] \wedge [^0Married_{wt}\ x]] =$
  $\neg\exists x\ [\neg[^0Married_{wt}\ x] \wedge [^0Man_{wt}\ x] \wedge [^0Married_{wt}\ x]]$.
  - For every valuation of *w, t v*-constructs **T**, thus we can generalize:
- $\forall \boldsymbol{w} \forall \boldsymbol{t}\ \neg\exists x\ [\neg[^0Married_{wt}\ x] \wedge [^0Man_{wt}\ x] \wedge [^0Married_{wt}\ x]]$.

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Refining, proving, calculating

- The only fake banknote that is a banknote
- The quickest runner Achilles who never overtakes the slowest tortoise if Achilles gives the tortoise a head start $n > 0$ (and both run in a constant speed)
- By refining we prove that these are *inconsistent concepts* that produce *the impossible office*, i.e. the office that goes necessarily vacant

Tübingen 2019 30

Refining, proving, calculating

$\lambda w \lambda t\ [^0\mathrm{I}\lambda x\ [[^0Banknote_{wt}\ x] \wedge [[^0Fake\ ^0Banknote]_{wt}\ x]]]$

Step 1. Req^*_0: $\{^0Banknote, [^0Fake\ ^0Banknote]\}$

- $[[^0Banknote_{wt}\ x] \wedge [[^0Fake\ ^0Banknote]_{wt}\ x]]$
- $[^0Banknote_{wt}\ x]$ — $\wedge$E, 3
- $[[^0Fake\ ^0Banknote]_{wt}\ x]$ — $\wedge$E, 3

Step 2. refinement of $^0Banknote$;
"a *banknote* is a promissory note issued by a bank payable to the bearer on demand without interest and acceptable as money, a financial instrument to settle debt".

$[[^0Fake\ ^0Banknote]_{wt}\ x] \vdash \neg[[^0Acceptable\text{-}as_{wt}\ ^0Money]\ x]$

- $\{^0Promissory\_note, \lambda w \lambda t\ \lambda x\ [[^0Issued\_by_{wt}\ ^0Bank]\ x],$
  $\lambda w \lambda t\ \lambda x\ [[^0Acceptable\text{-}as_{wt}\ ^0Money]\ x], \ldots,$
  $\lambda w \lambda t\ \lambda x\ \neg[[^0Acceptable\text{-}as_{wt}\ ^0Money]\ x], \ldots\}$

Quine's paradox (modalities)

Necessarily, 8 > 5
The number of planets = 8
———————————————— ???
Necessarily, the number of planets > 5

$\forall w \forall t\ [^0\!>\ ^08\ ^05]$ (analytical necessity)
$\lambda w \lambda t\ [[^0Number\_of\ ^0Planet_{wt}] = {}^08]$ (empirical fact)
————————————————
$\lambda w \lambda t\ [^0\!>\ [^0Number\_of\ ^0Planet_{wt}]\ ^05]$ (empirical fact, not necessary)

Types. Number_of/($\tau(o\iota)$): the number of elements of a set; *Planet*/$(o\iota)_{\tau\omega}$; >, =/($o\tau\tau$)

Proof:

1) $[^0\!>\ ^08\ ^05]$ — 1. assumption, $\forall$E
2) $[[^0Number\_of\ ^0Planet_{wt}] = {}^08]$ — 2. assumption, λE
3) $[^0\!>\ [^0Number\_of\ ^0Planet_{wt}]\ ^05]$ — 1, 2 Leibniz, subst. of identicals
4) $\lambda w \lambda t\ [^0\!>\ [^0Number\_of\ ^0Planet_{wt}]\ ^05]$ — 3, λI

In step (4) we must not introduce $\forall$, for the variables *w*, *t* were bound by λ rather than $\forall$

Additional rules in NLP

- Modifiers / $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$: turn a property to another property
  - Intersective, subsective, privative
  - The rules for left- and ritght- subsectivity
- *Left subsectivity* is valid for all kinds of modifiers;
  - $[[M\ P]_{wt}\ a] \vdash [M^*_{wt}\ a]$ where $M^* =_{df} \lambda w \lambda t\ \lambda x\ \exists p\ [[[M\ p]_{wt}\ x]]$; $p \rightarrow (o\iota)_{\tau\omega}$
    - A skillful surgeon is skillful (as a surgeon)
    - A round peg is round ("absolutely")
    - A fake banknote is fake (as a banknote)
- *Right subsectivity*
  - $[[M\ P]_{wt}\ a] \vdash [P_{wt}\ a]$ for intersective and subsective modifiers;
    - a skillful surgeon is a surgeon; a round peg is a peg
  - $[[M\ P]_{wt}\ a] \vdash [non\text{-}P_{wt}\ a]$ for privative modifiers
    - a forged banknote is a non-banknote;
    - non-banknote is *contrary* rather than contradictory to banknote

Additional rules in NLP

- Presuppositions (If-then-else-fail)
- Existential – triggered in extensional contexts by topic phrases
  - Strawson's (1952, pp. 173ff) example

    "All *John's children* are asleep."

    If John has any children ***then*** check whether each and every one of them is asleep ***else*** fail to produce a truth-value.
- $\lambda w \lambda t\ [If\ [^0\exists\ [^0Children\_of_{wt}\ ^0John]\ then\ [[^0All\ [^0Children\_of_{wt}\ ^0John]]\ ^0Sleep_{wt}]\ else\ fail]$

Presupposition triggers

- Topic/focus articulation
  - Topic → presupposition
  - Focus → mere entailment
- Interruption/termination of an activity
  - Presupposes that the activity was taking place
- Future/past tenses with a time of reference
  - The time of evaluation *t* must be prior to/later than the reference time *t'*
- Factive attitudes
  - knowing that, regretting that, …
- Exclusive-or (XOR)
  - Exactly one alternative

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'If-then-else' strict definition

"If *P* then *C* else *D*"

a two phase instruction:

$$^{2}[^{0}\iota^{*}\ \lambda c\ [[P \wedge [c = {}^{0}C]] \vee [\neg P \wedge [c = {}^{0}D]]]]$$

"If *P* then *C* else fail"

$$^{2}[^{0}\iota^{*}\ \lambda c\ [P \wedge [c = {}^{0}C]]]$$

- General schema for a sentence *S* with a presupposition *P*

$$\lambda w \lambda t\ [^{0}\textit{if-then-else-fail}\ P_{wt}\ {}^{0}[S_{wt}]] =$$
$$\lambda w \lambda t\ {}^{2}[^{0}\iota^{*}\ \lambda c\ [P_{wt} \wedge [c = {}^{0}[S_{wt}]]]]$$
$$\lambda w \lambda t\ [\textit{If}\ P_{wt}\ \textit{then}\ S_{wt}\ \text{else fail}]$$

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Factive verbs (‘know’, ‘regret’, …)

- Since knowing is a factivum, what is known must be true. Not only that, it is a ***presupposition*** of knowing.
- Hence, the rules are these ($c \rightarrow *_n$, ${}^2c \rightarrow o_{\tau\omega}$):

$$[{}^0Know_{wt}\ x\ c] \vdash [{}^0True_{wt}\ {}^2c]$$
$$\neg[{}^0Know_{wt}\ x\ c] \vdash [{}^0True_{wt}\ {}^2c]$$

- the rule that for any construction *C* Double Execution cancels the effect of Trivialization:

$${}^{20}C = C$$

- the rule for elimination of the property of propositions *True*

$$[{}^0True_{wt}\ P] \vdash P_{wt}$$

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“The Mayor of Ostrava doesn’t know that Tom knows that he (the mayor) is not going to the Alps”.

$MO/\iota_{\tau\omega}$: the individual office occupied by at most one ind.

1. $\neg[{}^0Know_{wt}\ {}^0MO_{wt}\ {}^0[\lambda w\lambda t\ [{}^0Know_{wt}\ {}^0Tom\ [{}^0Sub\ [{}^0Tr\ {}^0MO_{wt}]\ {}^0he\ {}^0[\lambda w\lambda t\ \neg[{}^0Go_{wt}\ he\ {}^0Alps]]]]]] \vdash$
2. $[{}^0True_{wt}\ {}^{20}[\lambda w\lambda t\ [{}^0Know_{wt}\ {}^0Tom\ [{}^0Sub\ [{}^0Tr\ {}^0MO_{wt}]\ {}^0he\ {}^0[\lambda w\lambda t\ \neg[{}^0Go_{wt}\ he\ {}^0Alps]]]]]] \vdash$
3. $[{}^0True_{wt}\ [\lambda w\lambda t\ [{}^0Know_{wt}\ {}^0Tom\ [{}^0Sub\ [{}^0Tr\ {}^0MO_{wt}]\ {}^0he\ {}^0[\lambda w\lambda t\ \neg[{}^0Go_{wt}\ he\ {}^0Alps]]]]]] \vdash$
4. $[{}^0Know_{wt}\ {}^0Tom\ [{}^0Sub\ [{}^0Tr\ {}^0MO_{wt}]\ {}^0he\ {}^0[\lambda w\lambda t\ \neg[{}^0Go_{wt}\ he\ {}^0Alps]]]] \vdash$
5. $[{}^0True_{wt}\ {}^2[{}^0Sub\ [{}^0Tr\ {}^0MO_{wt}]\ {}^0he\ {}^0[\lambda w\lambda t\ \neg[{}^0Go_{wt}\ he\ {}^0Alps]]]] \vdash$
6. ${}^2[{}^0Sub\ [{}^0Tr\ {}^0MO_{wt}]\ {}^0he\ {}^0[\lambda w\lambda t\ \neg[{}^0Go_{wt}\ he\ {}^0Alps]]]_{wt}$

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Factive verbs ('know', 'regret', …)

- ^0MO occurs *extensionally, de re*
- *Two principles de re are valid*
  - *Substitution of v-congruent constructions*
  - *Existential presupposition*
- Let $[^0Peter = {}^0MO_{wt}]$. Then the evaluation of the substitution comes down to this:

$^2[^0Sub\ [^0Tr\ {}^0MO_{wt}]\ {}^0he\ {}^0[\lambda w \lambda t\ \neg[^0Go_{wt}\ he\ {}^0Alps]]]_{wt}$

(anaphora resolution)

"… that *he (the Mayor of Ostrava) doesn't go …*"

$= {}^{20}[\lambda w \lambda t\ \neg[^0Go_{wt}\ {}^0Peter\ {}^0Alps]]_{wt}$

$= \neg[^0Go_{wt}\ {}^0Peter\ {}^0Alps]$

Question answering system

Communication not respecting presuppositions

Did all the trucks deliver their cargo?

System based on **FOL**:

$$\forall \mathbf{x}\ [\textbf{\textit{Truck}}(\boldsymbol{x}) \supset \textbf{\textit{Delivered\_Cargo}}(\boldsymbol{x})]$$

Under every interpretation that assigns an ***empty set*** to '*Truck*' the formula is evaluated as True

Yes (because there are **no trucks** delivering anything)

Hence, all the cargo has been delivered?

Yes (because **no cargo** has been delivered)

OK, I will inform the sellers that the goods have arrived.

💣☠

Answering in TIL

- Prior to asking a question, the agent must first check whether any presuppositions of a question are true; if they are not, then the correct answer is the negation of the false presuppositions. Schematically:
- ***If*** presupposition

  then direct answer

  else negated presupposition

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Answering in TIL

- ***If*** there are some trucks delivering their cargo,
  then **T** or **F** according as all of them delivered it,
  else no trucks *delivering* (negated presupposition)

$\lambda w \lambda t$ [*If* [[^{0}*Exist* ^{0}*Truck*$_{wt}$] ^{0}*Delivering*$_{wt}$]
then [[^{0}*All* ^{0}*Truck*$_{wt}$] ^{0}*Delivered*$_{wt}$]
else [[^{0}*No* ^{0}*Truck*$_{wt}$] ^{0}*Delivering*$_{wt}$]]

Exist, All, No / $((o(o\iota))(o\iota))$: restricted quantifiers

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Thank you for your attention

If questions
then answers ☺
else Fail ☹

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Proof-Theoretic Approach to Definite Descriptions

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DEFINITE DESCRIPTIONS – PROPER versus IMPROPER

Examples:

1. the least natural number;
2. the capital of Poland;
3. the capital of Poland in 2018;
4. the older sister of Ben;
5. the oldest son of Ben;
6. the youngest son of Ben;
7. the present King of France;
8. the round square;

DEFINITE DESCRIPTIONS

A survey of approaches:

- Eliminating approach – Russell;
- Descriptions as genuine terms:
  - Classical Logic: W-Frege: elimination of improper descriptions $\Rightarrow$ Hilbert and Bernays, Stenlund;
  - Classical Logic: S-Frege: the chosen object theory $\Rightarrow$ Quine, Carnap, Rosser, Bernays, Kalish and Montague;
  - Free Logics: Hintikka, Lambert, Scott, van Fraasen;
  - Partial Logics: Blamey;
  - Modal Logics: Carnap, Thomason, Fitting and Mendelsohn, Garson, Goldblatt.

DEFINITE DESCRIPTIONS

Russell versus Frege:

R: $\psi(\imath x\varphi(x)) \leftrightarrow \exists y(\forall x(\varphi(x) \leftrightarrow x = y) \wedge \psi(y))$

WF: $\exists y(\forall x(\varphi(x) \leftrightarrow x = y) \wedge \psi(y)) \rightarrow \psi(\imath x\varphi(x))$

SF $\psi(\imath x\varphi(x)) \leftrightarrow$
$\exists y(\forall x(\varphi(x) \leftrightarrow x = y) \wedge \psi(y)) \vee \neg\exists y(\forall x(\varphi(x) \leftrightarrow x = y) \wedge \psi(i))$

where i is a special individual name for the designatum of all improper descriptions.

DEFINITE DESCRIPTIONS

Free Logic:

Hintikka schema:

$t = \imath x\varphi(x) \leftrightarrow \forall x(\varphi(x) \leftrightarrow x = t)$

too strong: leads to contradiction.

FDM – Minimal Free Theory of descriptions (Lambert):

$\forall y(y = \imath x\varphi(x) \leftrightarrow \forall x(\varphi(x) \leftrightarrow x = y)$

Note that in FL we have:

$\forall x\varphi(x), Et \vdash \varphi[x/t]$

Hence we avoid contradiction.

DEFINITE DESCRIPTIONS

Modal Logic:

1. Thomason and Garson – modal extension of FDM.
2. Goldblatt – different solution in two-sorted language.
3. Fitting and Mendelsohn – the most subtle theory but rather complicated.

DEFINITE DESCRIPTIONS

Relative Strength of Modal Theories of DD:

H: $t = \imath x\varphi \leftrightarrow \forall x(\varphi \leftrightarrow x = t)$ (Fitting and Mendelsohn)

L: $Et \rightarrow (t = \imath x\varphi \leftrightarrow \forall x(\varphi \leftrightarrow x = t))$ (Thomason, Garson)

G: $t = \imath x\varphi \leftrightarrow Et \wedge \forall x(\varphi \leftrightarrow x = t)$ (Goldblatt)

Note that:

- H $\vdash$ L but L $\nvdash$ H
- G $\vdash$ L but L $\nvdash$ G (yet L $\vdash$ $G^{\leftarrow}$)
- H $\nvdash$ G but H $\vdash$ $G^{\leftarrow}$
- G $\nvdash$ H but G $\vdash$ $H^{\rightarrow}$

PROOF SYSTEMS FOR DEFINITE DESCRIPTIONS

Some known (to me) approaches:

- Natural Deduction: Kalish and Montague, Słupecki and Borkowski, Stenlund, Tennant, Garson, Carlström, Francez and Więckowski, Kürbis;
- Tableaux: Bencivenga, Lambert and van Fraassen, Gumb, Bostock, Fitting and Mendelson:
- Sequent Calculi: Czermak, Gratzl.

Stenlund's System

Formal account of (Weak) Fregean theory:

Syntactically: ND for FOLI in the spirit of Martin-Löf's type theory.

- three kinds of expressions: $t : I \quad \varphi : F \quad \varphi;$
- assumptions of the form: $t : I$ and φ (provided $\varphi : F$)
- additional introduction rules for deduction of $t : I$ and $\varphi : F$.

Rules for definite descriptions:

1): $\exists y \forall x(\varphi(x) \leftrightarrow x = y) \vdash \imath x\varphi(x) : I$

2): $\exists y \forall x(\varphi(x) \leftrightarrow x = y) \vdash \varphi(\imath x\varphi(x))$

Kalish and Montague System

Formal account of (Strong) Fregean theory:

Syntactically: ND or axiomatic system for FOLI with:

PD: $\exists y \forall x(\varphi(x) \leftrightarrow x = y) \vdash \varphi(\imath x\varphi(x))$

ID: $\neg\exists y \forall x(\varphi(x) \leftrightarrow x = y) \vdash \imath x\varphi(x) = \imath x(x \neq x)$

Garson's System

Modal generalization of Lambert's theory:

In the language all simple terms are rigid and all definite descriptions are nonrigid terms.

DN-system for some modal logic with:

$(\forall E)$ $\forall x\varphi \ \vdash \ Ea \to \varphi[x/a]$, where a is any (rigid) constant.
$(\forall I)$ $\ Ea \to \varphi[x/a] \ \vdash \ \forall x\varphi$, where a is not in active assumptions and in φ.
$(=I)$ $\vdash t = t$
$(=E)$ $\varphi[x/t_1], t_1 = t_2, \ \vdash \ \varphi[x/t_2]$
$(\exists i)$ $\ a \neq d \ \vdash \ \bot$, where a is not in active assumptions and in d.
$(=\Box)$ $\varphi \ \vdash \ \Box\varphi$, where φ is $a = b$ or $a \neq b$
$(\imath E)$ $Ea, a = \imath x\varphi(x) \vdash \varphi(a) \wedge \forall x(\varphi(x) \to x = a)$
$(\imath I)$ $Ea, \varphi(a) \wedge \forall x(\varphi(x) \to x = a) \vdash a = \imath x\varphi(x)$

SEQUENT CALCULUS FOR DEFINITE DESCRIPTIONS

Aims and Problems:

Provide cut-free SC with rules for DD possibly close to standard ones.

Three problems and possible choices:

1. the choice of principal formula;
2. the choice of side formulae;
3. one-sided rules or symmetric rules.

SEQUENT CALCULUS FOR DEFINITE DESCRIPTIONS

The basis for the rules:

1. The choice of principal formula:

$\psi(\iota x\varphi)$ or $t = \iota x\varphi$

2. The choice of side formulae:

$t = \imath x\varphi(x) \leftrightarrow \varphi[x/t] \wedge \forall x(\varphi(x) \rightarrow x = t)$

or

$t = \imath x\varphi(x) \leftrightarrow \forall x(\varphi(x) \leftrightarrow x = t)$

SEQUENT CALCULUS FOR DEFINITE DESCRIPTIONS

Construction of rules for descriptions:

From Hintikka axiom:

$t = \imath x\varphi(x) \leftrightarrow \forall x(\varphi(x) \leftrightarrow x = t)$

we obtain two sequents:

$t = \imath x\varphi(x) \Rightarrow \forall x(\varphi(x) \leftrightarrow x = t)$
$\forall x(\varphi(x) \leftrightarrow x = t) \Rightarrow t = \imath x\varphi(x)$

Each may be changed into introduction rule by RM theorem:

$$\frac{\Gamma \Rightarrow \Delta, \forall x(\varphi(x) \leftrightarrow x = t)}{\Gamma \Rightarrow \Delta, t = \imath x\varphi(x)}$$

and

$$\frac{\forall x(\varphi(x) \leftrightarrow x = t), \Gamma \Rightarrow \Delta}{t = \imath x\varphi(x), \Gamma \Rightarrow \Delta}$$

SEQUENT CALCULUS FOR DEFINITE DESCRIPTIONS

Rules for descriptions:

We continue with decomposition of side-formula obtaining:

$$\frac{\varphi(a), \Gamma_1 \Rightarrow \Delta_2, a = t \qquad a = t, \Gamma_2 \Rightarrow \Delta_2, \varphi(a)}{\Gamma \Rightarrow \Delta, t = \imath x\varphi(x)}$$

where a is not in Γ, Δ, φ; and

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \varphi(a), a = t \qquad \varphi(a), a = t, \Gamma_2 \Rightarrow \Delta_2}{t = \imath x\varphi(x), \Gamma \Rightarrow \Delta}$$

Both rules satisfy subformula property and are reductive.

PROBLEM 1

Clash of DD and identity:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \;\ldots\; \Gamma_k \Rightarrow \Delta_k}{\Gamma \Rightarrow \Delta, d = t} \qquad \dfrac{\Pi_1 \Rightarrow \Sigma_1 \;\ldots\; \Pi_n \Rightarrow \Sigma_n}{d = t, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (Cut)$$

Reductivity is lost if one premiss is obtained via rule for DD and the second via rule for $=$.

PROBLEM 2

Unrestricted rules for quantifiers:

The problem with ($\forall \Rightarrow$) and ($\Rightarrow \exists$) if we admit descriptions as instantiated terms.

Example: From $\forall x A x$ we can infer $A(\imath x(\exists y(Bxy \rightarrow \neg Cxy)))$.

In the framework of SC we have:

$$\frac{A(\imath x(\exists y(Bxy \rightarrow \neg Cxy))), \Gamma \Rightarrow \Delta}{\forall x A x, \Gamma \Rightarrow \Delta}$$

Subformula property lost and induction on the complexity fails.

PROBLEMS

How to avoid the troubles?

1. Either we keep unrestricted rules for quantifiers, but we must change definition of complexity and provide rules for DD which are one-sided.
$\Longrightarrow$ SC for both Fregean systems.

or

2. We provide symmetric rules for DD and keep standard definition of complexity but we must restrict rules for quantifiers.
$\Longrightarrow$ SC for Garson's free modal system.

SC FOR SF

Rules for identity:

SC for FOL +

$$(=\Rightarrow)\ \frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\Rightarrow=)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/t_1] \qquad \Pi \Rightarrow \Sigma, t_1 = t_2}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi[x/t_2]}$$

SC FOR SF

Rules for DD:

KM axioms for descriptions may be expressed by the following sequents:

$\Rightarrow \forall y(\forall x(\varphi \leftrightarrow x = y) \to \imath x\varphi = y)$
$\Rightarrow \neg\exists y\forall x(\varphi \leftrightarrow x = y) \to \imath x\varphi = i$

which are transformed into:

$$(\Rightarrow \imath)^1\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta, t = a \qquad t = a, \Pi \Rightarrow \Sigma, \varphi[x/a]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, t = \imath x\varphi}$$

1. where a is not in $\Gamma, \Delta, \Pi, \Sigma, \varphi$ and t is not i.

$$(\Rightarrow i1)^2\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta,}{\Gamma \Rightarrow \Delta, i = \imath x\varphi}$$

2. where a is not in Γ, Δ, φ.

$$(\Rightarrow i2)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/t_1] \qquad \Gamma_2 \Rightarrow \Delta_2, \varphi[x/t_2] \qquad t_1 = t_2, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, i = \imath x\varphi}$$

SC FOR SF

Equivalence with KM:

Theorem

If $\Gamma \vdash \varphi$ *in KM system, then* $\vdash \Gamma \Rightarrow \varphi$.

and

Theorem

If $\vdash \Gamma \Rightarrow \Delta$, *then* $\Gamma \vdash \vee\Delta$ *in KM system.*

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SC FOR SF

Cut Elimination:

Theorem (Indrzejczak LLP 2018)

Cut is eliminable in SC for SF.

Because:

- complexity measure is modified (problem 2);
- there is no clash between DD and identity rules since they are one-sided (problem 1).

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SC FOR SF

Identities as pricipal cut-formulas:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \ \dots \ \Gamma_k \Rightarrow \Delta_k}{\Gamma \Rightarrow \Delta, d = t} \qquad \dfrac{\Pi_1 \Rightarrow \Sigma_1 \ \dots \ \Pi_n \Rightarrow \Sigma_n}{d = t, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \ (Cut)$$

Note: $d = t$ in the right premiss always parametric!

SC FOR WF

SC countepart of Stenlund's ND:

Three types of sequents:

ordinary: $\Gamma \Rightarrow \Delta$
I-sequents: $\Gamma \Rightarrow t : I$
F-sequents: $\Gamma \Rightarrow \varphi : F$

where Γ consists of formulae and I-expressions and Δ only from formulae in ordinary sequents.

SC FOR WF

Structural rules:

$(AX)\quad t:I \Rightarrow t:I$ $\qquad (Cut)\ \dfrac{\Gamma\Rightarrow \Delta,\varphi \quad \varphi,\Pi\Rightarrow \Sigma}{\Gamma,\Pi\Rightarrow \Delta,\Sigma}$

$(WI\Rightarrow)\ \dfrac{\Gamma\Rightarrow \Delta}{t:I,\Gamma\Rightarrow \Delta}$ $\qquad (\Rightarrow W)\ \dfrac{\Gamma\Rightarrow \varphi:F}{\Gamma,\varphi\Rightarrow \varphi}$

$(\Rightarrow I)\ \dfrac{\Gamma_1\Rightarrow t_1:I,\dots,\Gamma_n\Rightarrow t_n:I}{\Gamma_1,\dots,\Gamma_n\Rightarrow f(t_1,\dots,t_n):I}$

$(\Rightarrow F)\ \dfrac{\Gamma_1\Rightarrow t_1:I,\dots,\Gamma_n\Rightarrow t_n:I}{\Gamma_1,\dots,\Gamma_n\Rightarrow A(t_1,\dots,t_n):F}$

$(C\Rightarrow)\ \dfrac{\varphi,\varphi,\Gamma\Rightarrow \Delta}{\varphi,\Gamma\Rightarrow \Delta}$ $\qquad (\Rightarrow C)\ \dfrac{\Gamma\Rightarrow \Delta,\varphi,\varphi}{\Gamma\Rightarrow \Delta,\varphi}$

$(\Rightarrow \neg F)\ \dfrac{\Gamma\Rightarrow \varphi:F}{\Gamma\Rightarrow \neg\varphi:F}$ $\qquad (\Rightarrow \oplus F)\ \dfrac{\Gamma\Rightarrow \varphi:F \quad \Pi\Rightarrow \psi:F}{\Gamma,\Pi\Rightarrow \varphi\oplus\psi:F}$

$(W\Rightarrow)\ \dfrac{\Gamma\Rightarrow \varphi:F \quad \Pi\Rightarrow \Sigma}{\varphi,\Gamma,\Pi\Rightarrow \Sigma}$ $\qquad (\Rightarrow W)\ \dfrac{\Gamma\Rightarrow \varphi:F \quad \Pi\Rightarrow \Sigma}{\Gamma,\Pi\Rightarrow \Sigma,\varphi}$

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SC FOR WF

Quantifier and identity rules:

$(\Rightarrow\forall F)\ \dfrac{a:I,\Gamma\Rightarrow \Delta,\varphi[x/a]}{\Gamma\Rightarrow \Delta,\forall x\varphi:F}$ $\qquad (\Rightarrow\exists F)\ \dfrac{a:I,\Gamma\Rightarrow \Delta,\varphi[x/a]}{\Gamma\Rightarrow \Delta,\exists x\varphi:F}$

$(\forall\Rightarrow)\ \dfrac{\Gamma\Rightarrow a:I \quad \varphi[x/a],\Pi\Rightarrow \Sigma}{\forall x\varphi,\Gamma,\Pi\Rightarrow \Delta,\Sigma}$ $\qquad (\Rightarrow\forall)^1\ \dfrac{a:I,\Gamma\Rightarrow \Delta,\varphi[x/a]}{\Gamma\Rightarrow \Delta,\forall x\varphi}$

$(\exists\Rightarrow)^1\ \dfrac{a:I,\varphi[x/a],\Gamma\Rightarrow \Delta}{\exists x\varphi,\Gamma\Rightarrow \Delta}$ $\qquad (\Rightarrow\exists)\ \dfrac{\Gamma\Rightarrow a:I \quad \Pi\Rightarrow \Sigma,\varphi[x/a]}{\Gamma,\Pi\Rightarrow \Delta,\Sigma,\exists x\varphi}$

1. where a is not in Γ,Δ and φ.

$(\Rightarrow=)\ \dfrac{\Gamma\Rightarrow t:I}{\Gamma\Rightarrow t=t}$ $\qquad (\Rightarrow=)\ \dfrac{\Gamma\Rightarrow \Delta,t_1=t_2 \quad \Pi\Rightarrow \Sigma,\varphi[x/t_1]}{\Gamma,\Pi\Rightarrow \Delta,\Sigma,\varphi[x/t_2]}$

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SC FOR WF

Definite descriptions rules:

From:

1. $a : I, a = t, \Gamma_1 \Rightarrow \Delta_1, \varphi(a)$
2. $a : I, \varphi(a), \Gamma_2 \Rightarrow \Delta_2, a = t$
3. $\Gamma_3 \Rightarrow t : I$
4. $a : I, \Gamma_4 \Rightarrow \varphi(a) : F$
5. $a : I, \Gamma_5 \Rightarrow a = t : F$

derive:

$\Gamma \Rightarrow \Delta, \imath x\varphi(x) : I$

or

$\Gamma \Rightarrow \Delta, \varphi(\imath x\varphi(x))$

where a is not in Γ, Δ and φ.

SC for Garson's System

Rules for propositional part:

To SC for CPL we add some modal rules, for example for T:

$(\Box\Rightarrow)\ \dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}$ $\qquad$ $(\Rightarrow\Box)\ \dfrac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$

But beware of S5!

SC for Garson's System

Rules for free quantifiers:

$$(\forall\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta, Ea \quad \varphi[x/a],\Pi\Rightarrow\Sigma}{\forall x\varphi,\Gamma,\Pi\Rightarrow\Delta,\Sigma} \qquad (\Rightarrow\forall)^1\ \frac{Ea,\Gamma\Rightarrow\Delta,\varphi[x/a]}{\Gamma\Rightarrow\Delta,\forall x\varphi}$$

$$(\exists\Rightarrow)^1\ \frac{Ea,\varphi[x/a],\Gamma\Rightarrow\Delta}{\exists x\varphi,\Gamma\Rightarrow\Delta} \qquad (\Rightarrow\exists)\ \frac{\Gamma\Rightarrow\Delta, Ea \quad \Pi\Rightarrow\Sigma,\varphi[x/a]}{\Gamma,\Pi\Rightarrow\Delta,\Sigma,\exists x\varphi}$$

side conditions:

1. where a is not in Γ, Δ and φ.

Note that only rigid constants are instantiated! (problem 2 is solved)

SC for Garson's System

Rules for descriptions:

$$\frac{\Gamma_1\Rightarrow\Delta_1, Ea \qquad Eb,\varphi(b),\Gamma_2\Rightarrow\Delta_2, a=b \qquad Eb, a=b,\Gamma_3\Rightarrow\Delta_3,\varphi(b)}{\Gamma\Rightarrow\Delta, a=\imath x\varphi(x)}$$

where b is not in Γ, Δ and φ.

$$\frac{\Gamma_1\Rightarrow\Delta_1, Ea \qquad \Gamma_2\Rightarrow\Delta_2, Eb \qquad \Gamma_3\Rightarrow\Delta_3,\varphi(b), a=b \qquad \varphi(b), a=b,\Gamma_4\Rightarrow\Delta_4}{a=\imath x\varphi(x),\Gamma\Rightarrow\Delta}$$

Warning! identities (with DD) may occur in both premisses of cut as principal fromulas (problem 1).

SC for Garson's System

Rules for identity:

$(=\Rightarrow)\ \dfrac{t=t,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}$ $\qquad (=d\Rightarrow)^1\ \dfrac{a=d,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}$

$$(\Rightarrow=)^2\ \frac{\Gamma_1\Rightarrow\Delta_1,\varphi[x/t_1] \qquad \Gamma_2\Rightarrow\Delta_2,t_1=t_2 \qquad \varphi[x/t_2],\Gamma_3\Rightarrow\Delta_3}{\Gamma\Rightarrow\Delta}$$

$(=\Box)\ \dfrac{\Gamma\Rightarrow\Delta,a=b \quad \Box a=b,\Pi\Rightarrow\Sigma}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}$ $\qquad (\neq\Box)\ \dfrac{a=b,\Gamma\Rightarrow\Delta \quad \Box a\neq b,\Pi\Rightarrow\Sigma}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}$

side conditions:

1. where a is not in Γ,Δ and φ.
2. where φ is atomic.

Note that we avoid a clash of DD and identity rules.

SC for Garson's System

Equivalence with Garson's System:

Theorem

If $\Gamma\vdash\varphi$ in Garson's system, then $\vdash\Gamma\Rightarrow\varphi$.

and

Theorem

If $\vdash\Gamma\Rightarrow\Delta$, then $\Gamma\vdash\bigvee\Delta$ in Garson's system.

SC for Garson's System

Cut Elimination:

Theorem (Indrzejczak AiML 2018)

Cut is eliminable in SC for Garson's system.

Because:

- rules for quantifiers are restricted (problem 2);
- there is no clash between DD and identity rules (problem 1);
- rules for DD are reductive (induction on the complexity holds).

Extensions

Possible enrichments:

1. changes in the background modal logic;
2. enriching the language;
3. strenghtening the theory of descriptions.

Extensions

Richer language – lambda abstraction:

$\lambda x \varphi(x)(a) \leftrightarrow \varphi(a)$

where $\lambda x \varphi(x)$ is a predicate abstracted from a formula φ.

In the setting of sequent calculus it is enough to add two rules:

$$(\Rightarrow \lambda) \ \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \lambda x \varphi(x)(a)} \qquad (\lambda \Rightarrow) \ \frac{\varphi(a), \Gamma \Rightarrow \Delta}{\lambda x \varphi(x)(a), \Gamma \Rightarrow \Delta}$$

Extensions

Stronger theory of DD:

1. FD1 is an extension of MFD by means of the addition of the cancellation law: $(t = \imath x \varphi) = t$

in SC add:

$$\frac{(t = \imath x \varphi) = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Extensions

Stronger theory of DD:

2. FDV of van Fraasen = MFD + the law of extensionality:
$\forall x(\varphi \leftrightarrow \psi) \rightarrow \imath x\varphi = \imath x\psi$

The same effect can be obtained in SC by the addition of the rule:

$$(\Rightarrow d_1 = d_2)\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta, \psi[x/a] \qquad \psi[x/a], \Pi \Rightarrow \Sigma, \varphi[x/a]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \imath x\varphi = \imath x\psi}$$

where a is not in $\varphi, \psi, \Gamma, \Delta, \Pi, \Sigma$.

Extensions

Stronger theory of DD:

3. Scott logic: MFD + $\neg Ed \rightarrow d = \imath x(x \neq x)$.

Since a converse of this implication also holds for Scott's logic we can express it in the setting of SC by means of two rules:

$$(\Rightarrow id)\ \frac{Ed, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, d = \imath x(x \neq x)} \qquad (id \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, Ed}{d = \imath x(x \neq x), \Gamma \Rightarrow \Delta}$$

Alternatives

Goldblatt's theory of DD:

based on the axiom: $\imath x\varphi = t \leftrightarrow Et \wedge \forall x(\varphi \leftrightarrow x = t)$,
where t is rigid and does not have free x
We can easily obtain equivalent formalization of Goldblatt's axiom in standard sequent calculus by means of the following rules:

$$(\Rightarrow \imath)^1 \ \frac{\Gamma_1 \Rightarrow \Delta_1, Et \qquad Ea, \varphi[x/a], \Gamma_2 \Rightarrow \Delta_2, t = a \qquad Ea, t = a, \Gamma_3 \Rightarrow \Delta_3, \varphi[x/a]}{\Gamma \Rightarrow \Delta, t = \imath x\varphi}$$

1. where a is not in Γ, Δ, φ

$$(\imath \Rightarrow) \ \frac{Et, \Gamma_1 \Rightarrow \Delta_1, Ea \qquad Et, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a], t = a \qquad Et, \varphi[x/a], t = a, \Gamma_3 \Rightarrow \Delta_3}{t = \imath x\varphi, \Gamma \Rightarrow \Delta}$$

Alternatives

Fitting and Mendelsohn's theory of DD:

Based on Hintikka axiom and different form of rigidification of nonrigid terms.

Advantages – subtle and rich:

- does not equate existence and designation;
- makes distinctions between different kinds of improper descriptions.

Disadvantages:

complex machinery of labels attached not only to formulae but also to nonrigid terms to fix their denotations in possible worlds.

Andrzej Indrzejczak Proof-Theoretic Approach to Definite Descriptions

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Huge Propositional Proofs Are Redundant

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March 2019

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Propositional proofs (I)

Natural Deduction

$$\dfrac{\dfrac{\dfrac{[A]^1 \qquad A \supset B}{B} \qquad B \supset C}{C}}{A \supset C}\,1$$

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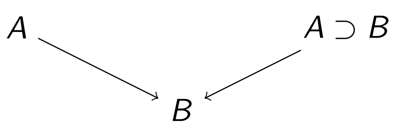
Propositional proofs (II)

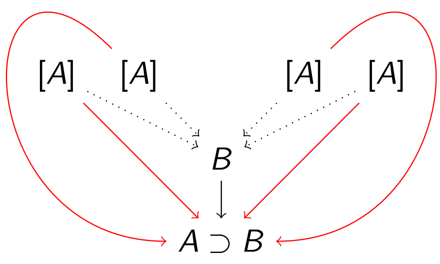
Natural Deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^1}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]^2}{\bot}}{\neg A}\,1}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]^2}{\dfrac{\bot}{A \vee \neg A}\,2}$$

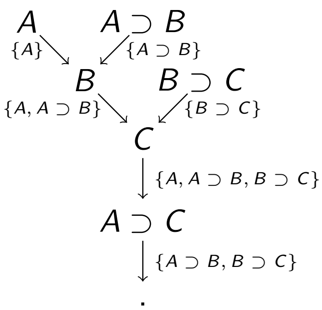
A convenient representation of $M_{\supset}$ in graphs

$$\dfrac{A \quad A \supset B}{B} \supset\text{-e} \qquad\qquad \dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B} \supset\text{-i}$$



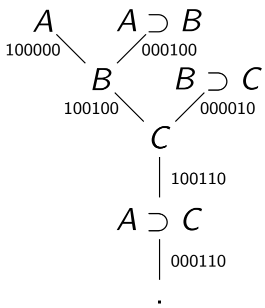


Using dependency sets to eliminate the red (discharge) edges



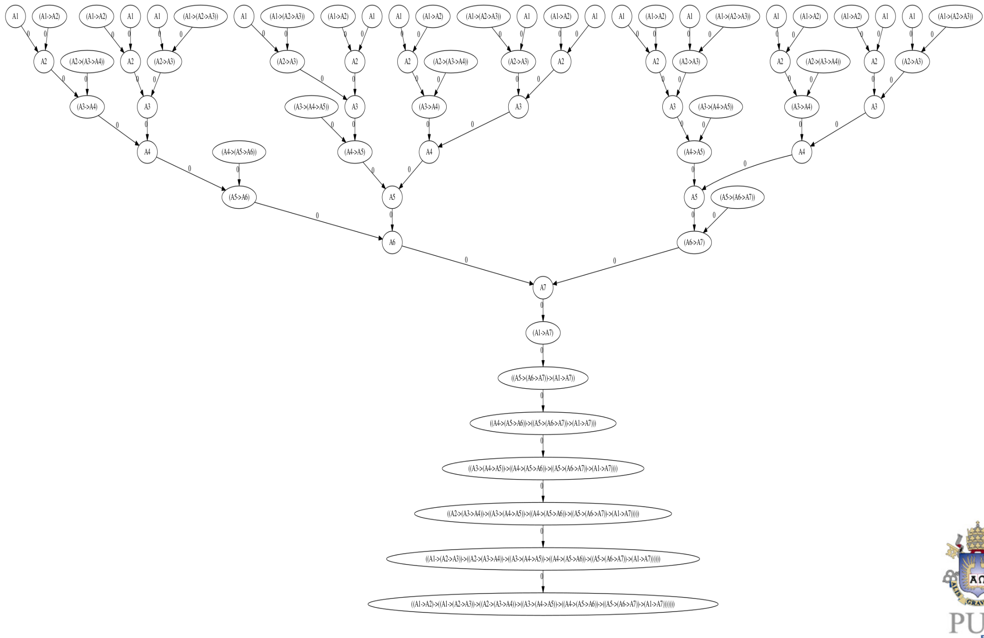
Using bitstrings to eliminate the red (discharge) edges

Considering a total order on formulas (any)
$A \prec B \prec C \prec A \supset B \prec B \supset C \prec A \supset C$



Given the total order and the labeled tree, verifying that the conclusion is a $M_{\supset}$ tautology is polytime on the number of nodes in the tree.

Big propositional proofs



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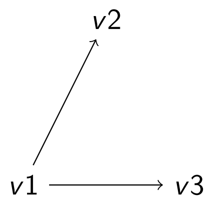
The graph G_3 has no hamiltonian cycle



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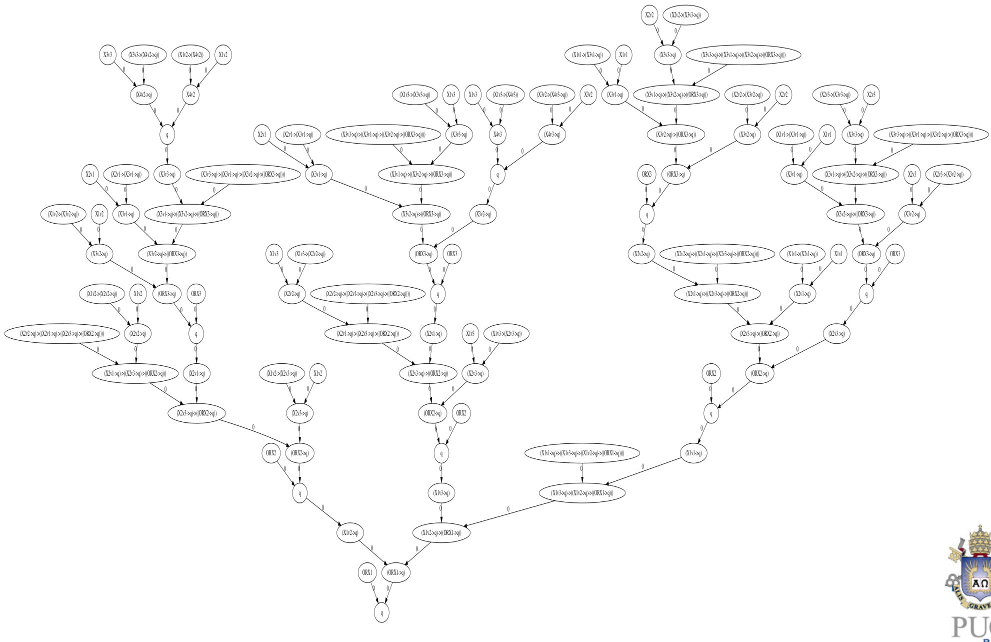
A proof in $M_{\supset}$ that G_3 has no hamiltonian cycle



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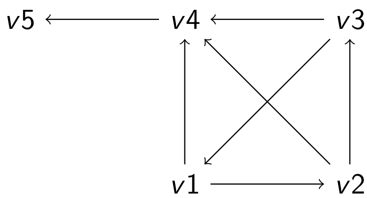
G_5 has no hamiltonian cycle



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A proof in $M_{\supset}$ that G_5 has no hamiltonian cycle

$\Rightarrow G_5$ has no hamiltonian cycle

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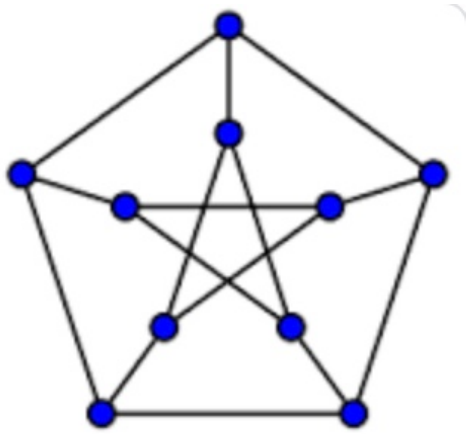
The Petersen Graph has no hamiltonian cycle



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A proof in $M_{\supset}$ that Petersen graph has no hamiltonian cycle

$\Rightarrow$ Petersen Graph has no hamiltonian cycle

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Estimating the size of proofs of non-hamiltonian cycles

(1) The size of the graph is $|G| = n$

(2) The size of the formula α_G, valid iff G does not have hamiltonian cycles is n^3

(3) The size of the naive proof is n^n

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A family of huge proofs in $M_{\supset}$

Consider the formulas:

- $\eta = A_1 \supset A_2$
- $\sigma_k = A_{k-2} \supset (A_{k-1} \supset A_k)$, $k > 2$.

A cut free proof of $A_1 \supset A_n$ from $\eta, \sigma_1, \ldots, \sigma_n$ has size $\geq$ *Fibonnaci*(n).

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[A_1] \\ A_1 \supset A_2 \\ A_1 \supset (A_2 \supset A_3) \\ \Pi_3 \\ A_3\end{array} \quad \dfrac{\dfrac{[A_1] \quad A_1 \supset A_2}{A_2} \quad A_2 \supset (A_3 \supset A_4)}{A_3 \supset A_4}}{A_4} \quad \dfrac{\begin{array}{c}[A_1] \\ A_1 \supset A_2 \\ A_1 \supset (A_2 \supset A_3) \\ \Pi_3 \\ A_3\end{array} \quad A_3 \supset (A_4 \supset A_5)}{A_4 \supset A_5}}{A_5}}{A_1 \supset A_5}$$

A family of huge proofs in $M_{\supset}$

Consider the formulas:

- $\eta = A_1 \supset A_2$
- $\sigma_k = A_{k-2} \supset (A_{k-1} \supset A_k)$, $k > 2$.

A cut free proof of $A_1 \supset A_n$ from $\eta, \sigma_1, \ldots, \sigma_n$ has size $\geq$ *Fibonnaci*(n). In general, for each $5 \leq k$

$$\dfrac{\dfrac{\begin{array}{c}[A_1] \\ \eta \\ \sigma_3, \ldots, \sigma_{k-1} \\ \Pi_{k-1} \\ A_{k-1}\end{array} \quad \dfrac{\begin{array}{c}[A_1] \\ \eta \\ \sigma_3, \ldots, \sigma_{k-2} \\ \Pi_{k-2} \\ A_{k-2}\end{array} \quad A_{k-2} \supset (A_{k-1} \supset A_k)}{A_{k-1} \supset A_k}}{A_k}}{A_1 \supset A_k}$$

$$\begin{aligned} l(\Pi_2) &= 1 \\ l(\Pi_3) &= l(\Pi_2) + 1 \\ l(\Pi_k) &= l(\Pi_{k-2}) + l(\Pi_{k-1}) + 2 \end{aligned}$$

$$\frac{\phi^k}{\sqrt{5}} \approx \mathit{Fibonacci}(k) \leq l(\Pi_k)$$

$\phi = 1.618$

Fundamental facts on Natural Deduction for Minimal Logic

Analyticity

- Every (normal) proof of α from Γ has only occurrences of sub-formulas of Γ and α (Sub-formula Principle $\mathcal{SFP}$).
- Important observation: The amount of sub-formulas of a propositional formula is linearly bound by the size of α.

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On the height of minimal logic proofs (best case)

Every tautology α in $M_{\supset} = \{\supset\text{-I}, \supset\text{-E}\}$ has a normal proof Π, such that $h(\Pi) < Poly(size(\alpha))$

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Minimal tautologies proofs are height-poly-bounded (Hudelmaier S.C. for $M_{\supset}$)

$$(\mathrm{M}A): \quad \Gamma, p \Rightarrow p$$

$$(\mathrm{M}I1\rightarrow): \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} [(\nexists\gamma): (\alpha \rightarrow \beta) \rightarrow \gamma \in \Gamma]$$

$$(\mathrm{M}I2\rightarrow): \quad \frac{\Gamma, \alpha, \beta \rightarrow \gamma \Rightarrow \beta}{\Gamma, (\alpha \rightarrow \beta) \rightarrow \gamma \Rightarrow \alpha \rightarrow \beta}$$

$$(\mathrm{M}E\rightarrow P): \quad \frac{\Gamma, p, \gamma \Rightarrow q}{\Gamma, p, p \rightarrow \gamma \Rightarrow q} \quad [q \in \mathrm{VAR}(\Gamma, \gamma), p \neq q]$$

$$(\mathrm{M}E\rightarrow\rightarrow): \quad \frac{\Gamma, \alpha, \beta \rightarrow \gamma \Rightarrow \beta \quad \Gamma, \gamma \Rightarrow q}{\Gamma, (\alpha \rightarrow \beta) \rightarrow \gamma \Rightarrow q} \quad [q \in \mathrm{VAR}(\Gamma, \gamma)]$$

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Minimal tautologies proofs are height-poly-bounded (Hudelmaier S.C. for $M_{\supset}$)

Theorem 1. $\mathrm{LM}_{\rightarrow}$ is sound and complete with respect to minimal propositional logic and tree-like deducibility. Thus any given formula ρ is valid in the minimal logic iff it (i.e. sequent $\Rightarrow \rho$) is tree-like deducible in $\mathrm{LM}_{\rightarrow}$. Moreover, for any tree-like $\mathrm{LM}_{\rightarrow}$ deduction ∂ of sequent S:

1. The height of ∂ is linear in $|S|$. In particular if S is $\Rightarrow \rho$, then $h(\partial) \leq 3|\rho|$.
2. The foundation of ∂ tree-like $\mathrm{LM}_{\rightarrow}$ deduction ∂ of sequent S is at most quadratic in $|S|$. In particular if S is $\Rightarrow \rho$, then $\phi(\partial) \leq (|\rho| + 1)^2$.

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Minimal tautologies are height-poly-bounded in $NM_{\supset}$

Theorem 2 [Gordeev & Haeusler 2018]. There exists a recursive operator $\mathcal{F}$ that transforms any given tree-like $\mathrm{LM}_{\rightarrow}$ deduction ∂ of $\Gamma \Rightarrow \rho$ into a tree-like $\mathrm{NM}_{\rightarrow}$ deduction $\mathcal{F}(\partial)$ with root-formula ρ and assumptions occurring in Γ. Moreover ∂ and $\mathcal{F}(\partial)$ share the semi-subformula property, linearity of the height and polynomial upper bounds on the foundation. In particular if $\Gamma = \emptyset$, then $\mathcal{F}(\partial)$ is a $\mathrm{NM}_{\rightarrow}$ proof of ρ such that $h(\mathcal{F}(\partial)) \leq 18|\rho|$ and $\phi(\mathcal{F}(\partial)) < (|\rho|+1)^2(|\rho|+2)$.

A quite relevant fact

We have to worry only about the fact that a super-polynomially sized proof of α_G in $\mathbf{M}_{\supset}$is only widely super-polynomial. Without loss of generality the height can be taken as polynomial, in fact linear, on the conclusion.

When a derivation/proof is huge

Def. Let Π be a derivation of α from $\{\gamma_1, \ldots, \gamma_k\}$, and $m = size(\alpha) + \sum_{i=1,k} size(\gamma_i)$. Π is a huge proof, iff, $size(\Pi) > p^{m^q}$, for some $p \in \mathbb{R}$, $p > 1$ and $q \in \mathbb{N}$, $q > 0$.

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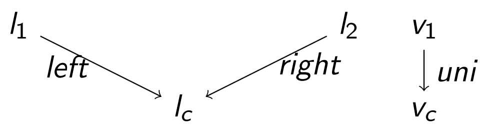
Abstracting on derivations: Thinking of derivations as labeled trees

$\mathcal{B}$-ordered binary trees ($\mathcal{B}$OL-tree)

A partially ordered set $\mathcal{B} = \langle B, \preceq \rangle$ of labels abstracts the formulae-as-labels notion

Two pattern-graphs abstracting $\supset$-Intro and $\supset$-Elim.



proviso: $l_1 \preceq l_2$, $l_c \preceq l_2$ and $v_1 \preceq v_c$

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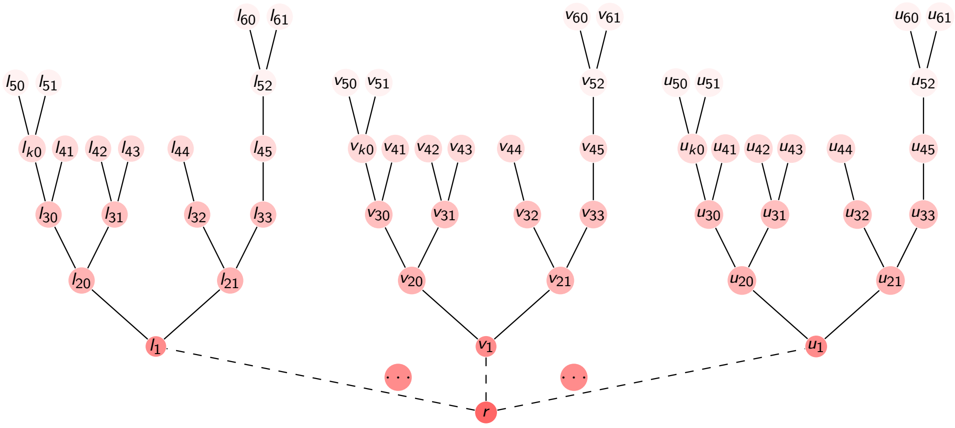
Huge linearly height-bounded $\mathcal{B}$OL-trees are redundant (I)



$size(\mathcal{B}) = m$, $height(tree) = k \times m$, $size(tree) = p^{m^q}$, $p > 1$ and $q > 0$, $p \in \mathbb{R}$, $q \in \mathbb{N}$

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A basic combinatory fact

If $\boxed{\sum_{i=1,m} size(S_i) > p^{m^q}}$, $p > 1$ and $q > 0$, $p \in \mathbb{R}$, $q \in \mathbb{N}$

then there is $1 \leq j \leq m$, such that $\boxed{size(S_j) > r^{m^t}}$, $r \in \mathbb{R}$, $t \in \mathbb{N}$

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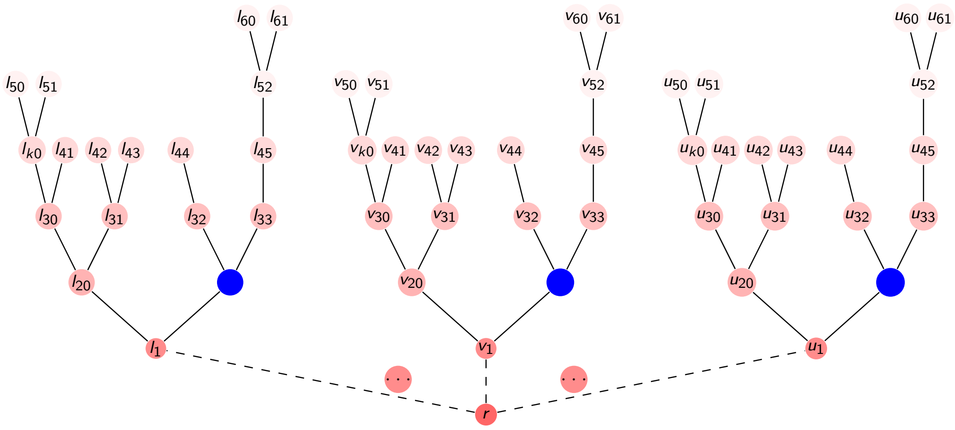
Huge linearly height-bounded $\mathcal{B}$OL-trees are redundant (II)



$V_L^c = \{v : v$ at level L and $l(v) = c\}$, $|tree| = \bigcup_{L \leq km} \bigcup_{c=1,m} |V_L^c|$

There are $L \leq height(tree)$ and c, $1 \leq c \leq m$, such that $\boxed{size(V_L^c) > r^{m^t}}$, $r \in \mathbb{R}$, $t \in \mathbb{N}$

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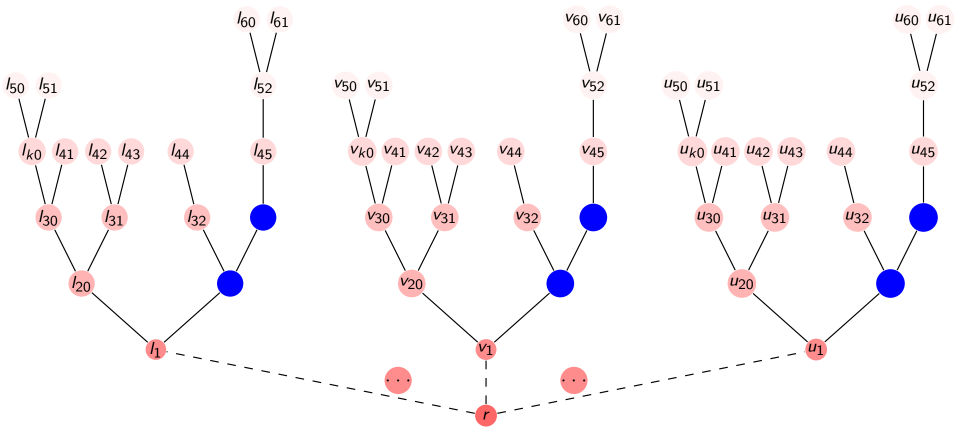
Huge linearly h-bounded $\mathcal{B}$OL-trees are redundant (III)



$size(V_L^c) = size(U_{right}(V_L^c) \cup U_{uni}(V_L^c) \cup Leaves(V_L^c))$

There is a label d, $1 \leq d \leq m$, such that $\boxed{size(V_{L+1}^d) > r^{m^t}}$ for some $r \in \mathbb{R}$, $t \in \mathbb{N}$, $V_{L+1}^d \subseteq U_{left}(V_L^c) \cap U_{right}(V_L^c)$ or $V_{L+1}^d \subseteq U_{uni}(V_L^c)$ or $V_{L+1}^d \subseteq Leaves(V_L^c)$

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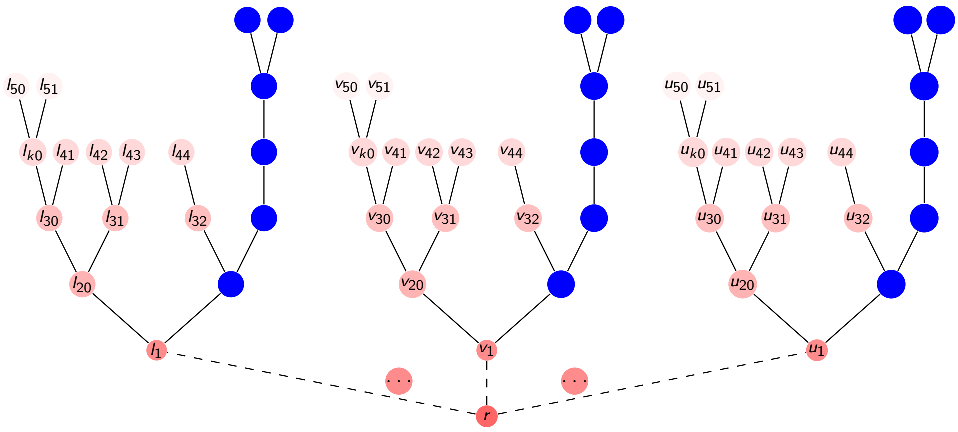
Huge linearly height-bounded $\mathcal{B}$OL-trees are redundant (IV)



Repeating the reasoning . . .

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Main Lemma

Lemma Every huge tree Π has a subtree Π_{sub} that occurs exponentially many times in Π

Applied Lemma Every huge proof Π has a sub-proof Π_{sub} that occurs exponentially many times in Π

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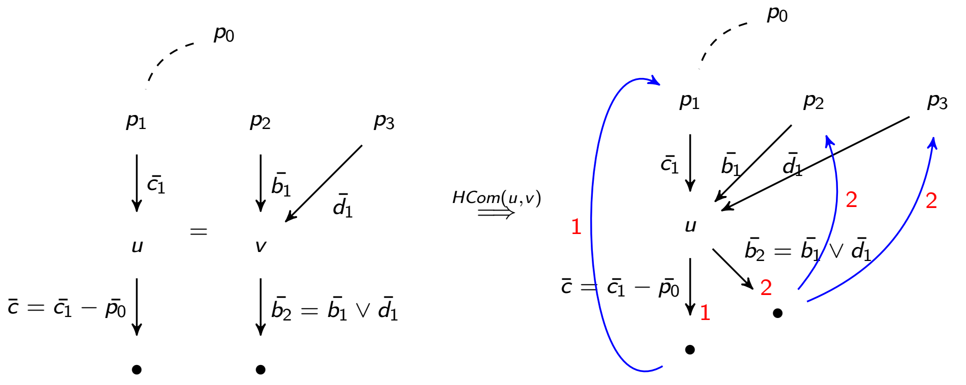
Towards polynomial representation: Using directed-acyclic-graphs (*dags*)

The ancestor edge: How it works (I)



(a) u and v collapse

(b) After collapsing u and v

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Getting an advantageous position

1. If Π is huge then the *main lemma* implies that:

$$R_{level}(\Pi) = \left\{ L : \begin{array}{l} v \text{ is the root of an expo-repeated} \\ \text{sub-tree in } \Pi \text{ and } v \text{ is at level } L \end{array} \right\};$$

2. The level "$min(R_{level}(\Pi)) - 1$" has polynomially only many nodes;

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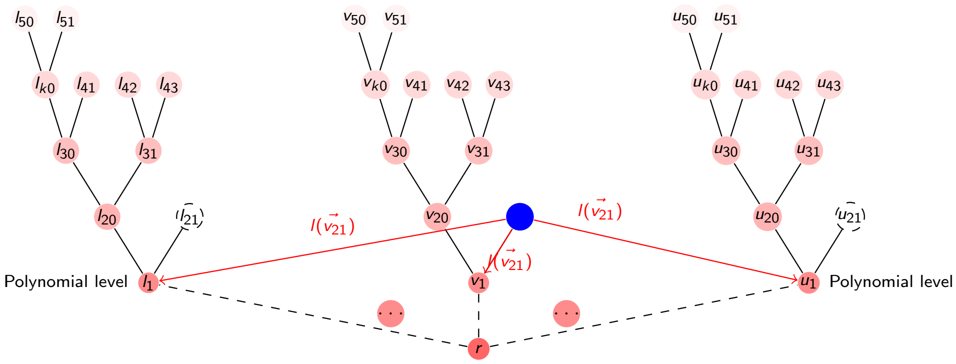
Reducing the size of the proof (|repeated tree| = 1)



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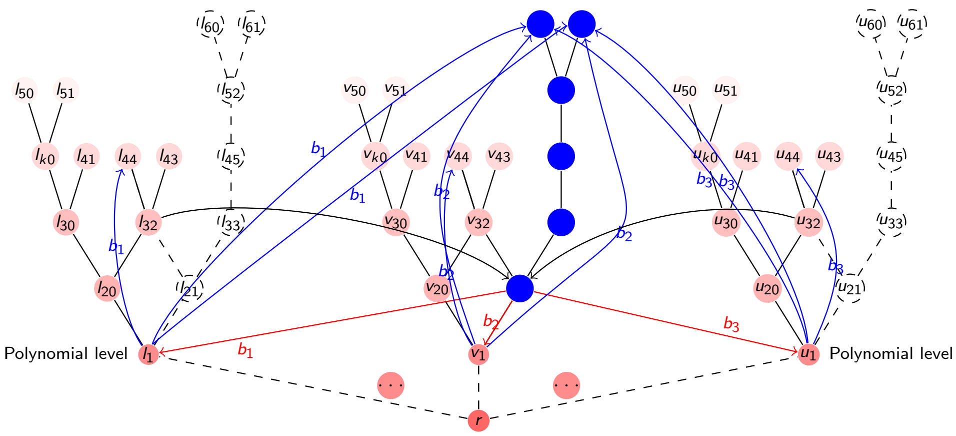
Reducing the size of the proof (|repeated tree| > 1)

Collapsing all exponentially repeated sub-proofs in only one



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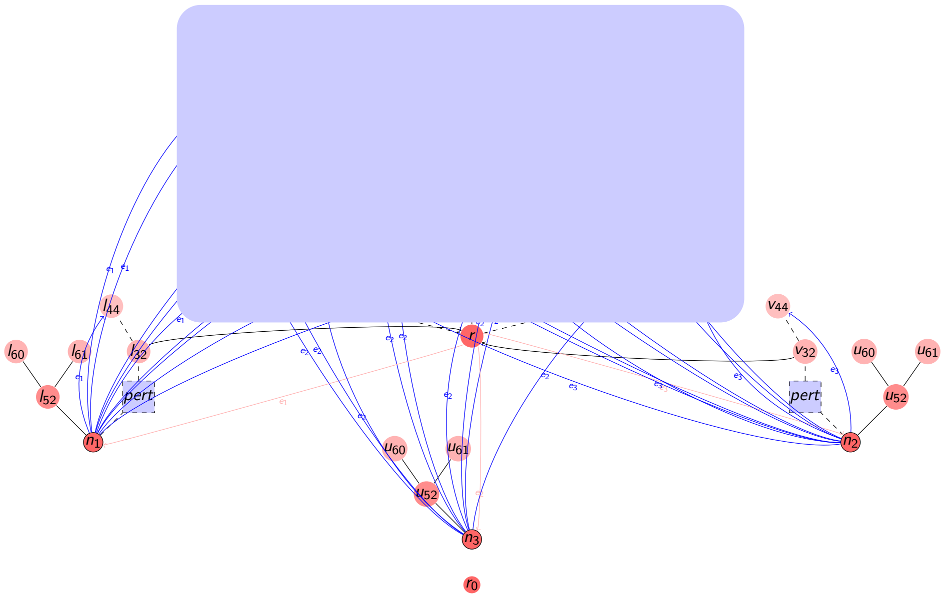
Iterate the lemma to reduce proofs to Dag-proofs of polynomial size (Ia)



pert = "place of the exponentially repeated tree"

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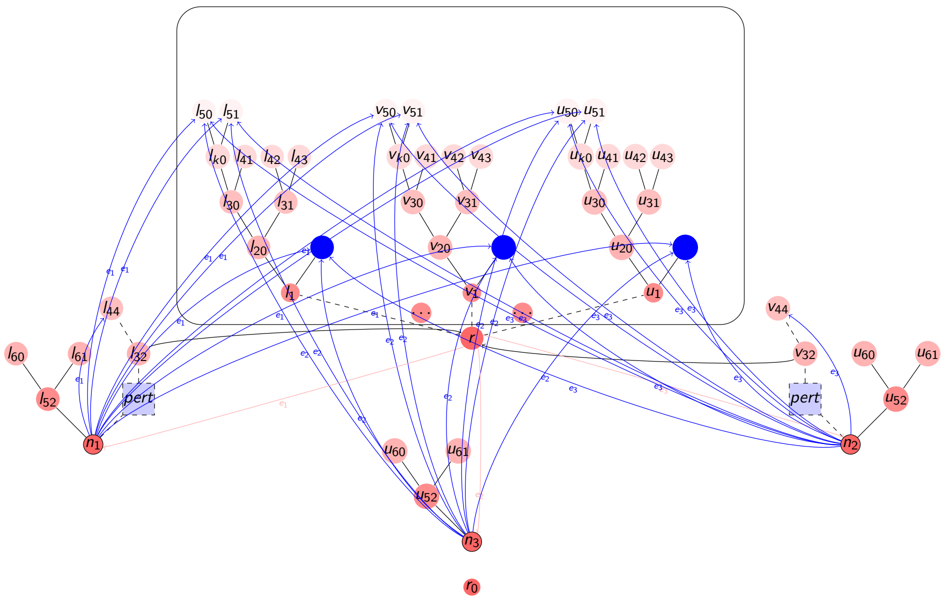
Iterate the lemma to reduce proofs to Dag-proofs of polynomial size (Ib)



pert = "place of the exponentially repeated tree"

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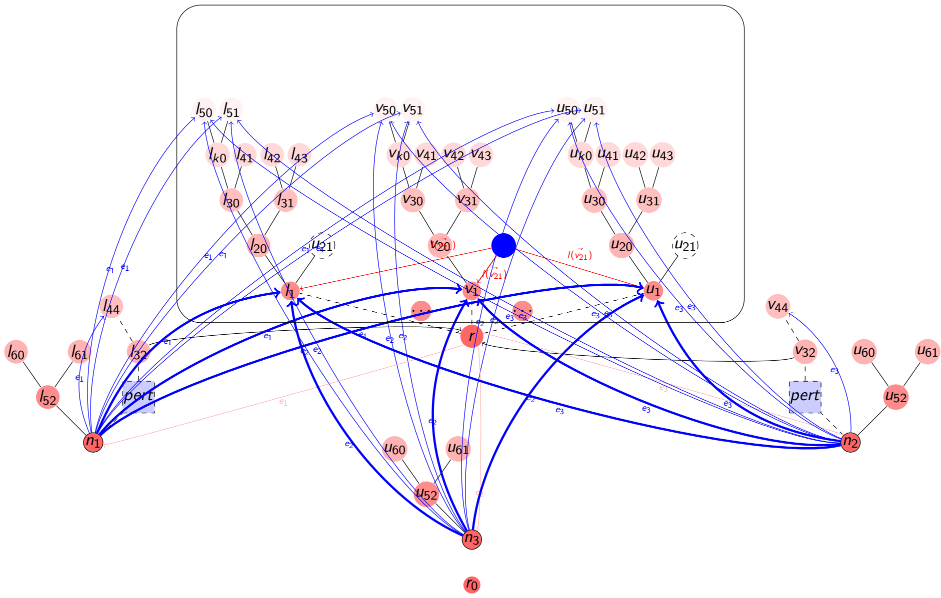
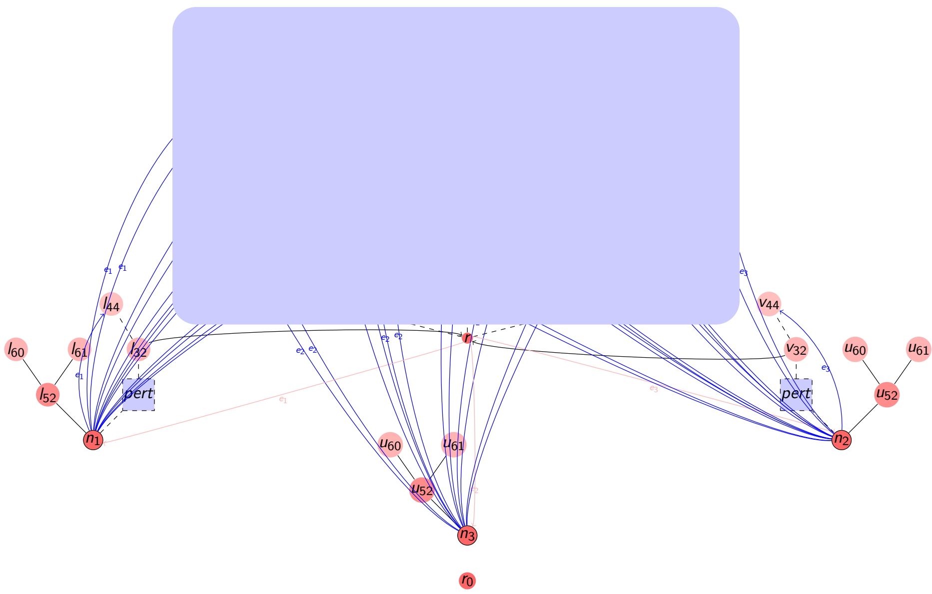
Iterate the lemma to reduce proofs to Dag-proofs of polynomial size (Ic)



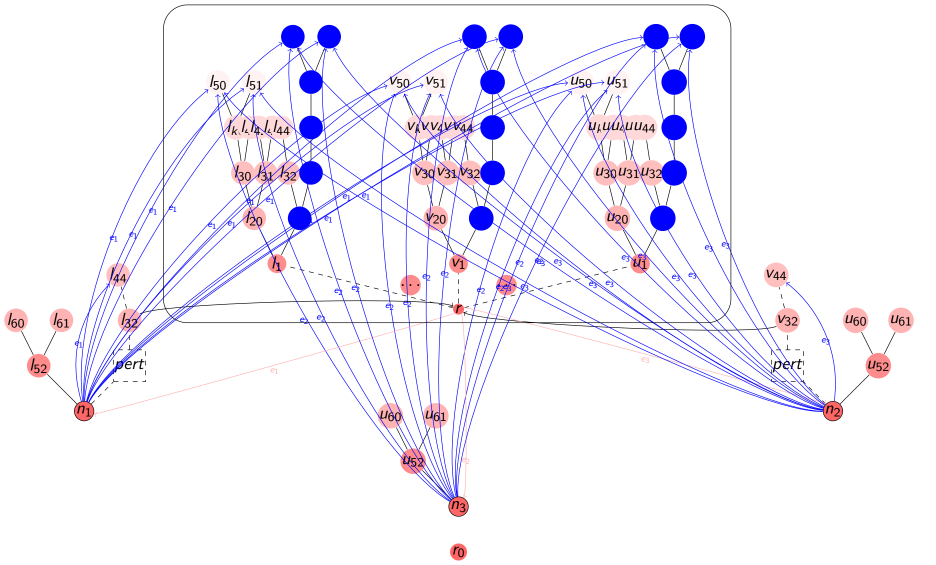
pert = "place of the exponentially repeated tree"

Iterate the lemma to reduce proofs to Dag-proofs of polynomial size (IIa)



pert = "place of the exponentially repeated tree"

Iterate the lemma to reduce proofs to Dag-proofs of polynomial size (IIb)



pert = "place of the exponentially repeated tree"

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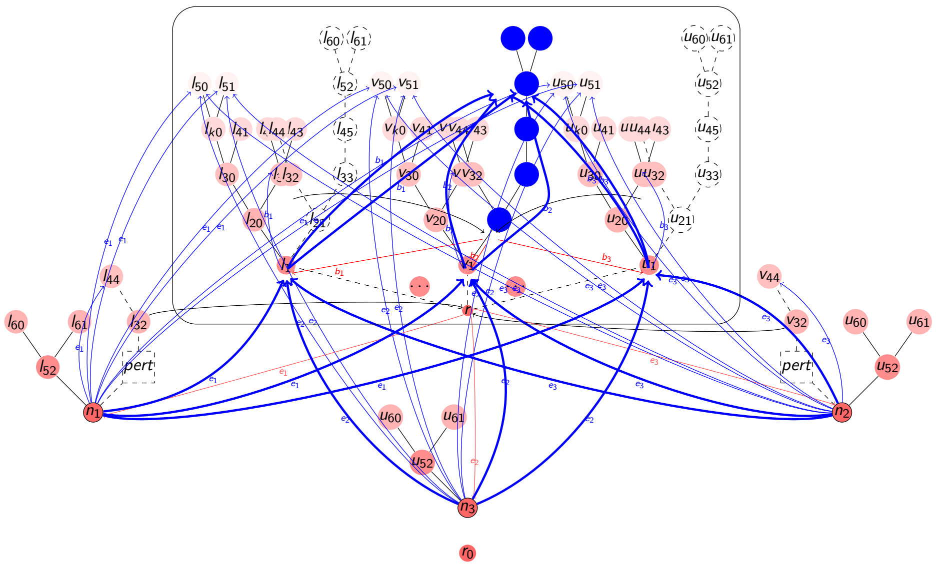
Iterate the lemma to reduce trees to colored-dags of polynomial size (IIc)



pert = "place of the exponentially repeated tree"

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Compressing huge linearly height-bounded $\mathcal{B}$OL-trees into polynomial ones

The Method $\mathcal{C}$. Input: A huge linearly bounded $\mathcal{B}$OL-tree

1. Apply the **Main lemma** observing its proof in order to find the least level where an exponentially repeated sub-tree occurs. Let this level be L_{min}.
2. Build the set $\mathcal{E}$ of all **specimens** of exponentially repeated sub-trees with root at L_{min}.
3. For each Π_{min} in $\mathcal{E}$, collapse all of its *individuals* on only one individual, using the schemata already shown.
4. For each collapsed sub-tree Σ that it is exponentially sized, apply $\mathcal{C}$ to it.
5. Repeat steps 1 to 3 to every exponentially sized sub-tree at level $L_{min} + 1$, if any.

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By induction on the size of the input we prove that the method stops providing a polynomially sized directed acyclic colored graph (colored dag) D_Π that preserves and reflects the stepped ancestrality of Π.

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Some important facts: Soundness of DLDS

The structures obtained by the compression is called Dag-Like Derivability Structure (A colored Dag-proof)

- The Dag-proof obtained by the iteration of finding exponentially sized sub-proofs is polynomial. However *the degree of the polynomial depends always from the application of the main lemma. In this process it cannot be fixed in advance*;
- *Each path in the original tree is segmented according its composition in terms of its (possible) participation in sub-paths that occur exponentially many times in the tree*, because they take part in the sub-tree that occurs exponentially many times;
- For each compression step the paths from hypothesis to conclusion are preserved when related to the paths in the original proof-tree. No additional path is included in the Dag-proof.

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The Dag-proof obtained is a simple graph regarded to deduction edges and to ancestor edges as well. This ensures its polynomial encoding, if the set of vertexes is polynomial.

The set of edges is upper bounded by $|V| \times |V| \times 2|V|^2$;

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The graph-based structure to proofs

Obs: DAG = Directed Acyclic Graph (Colored)
A **D**ag-**L**ike **D**erivability **S**tructure, **DLDS**: $\langle V, (E^i_D)_{i \in \mathcal{O}^i_\Gamma}, E_A, r, l, L, P \rangle$

1. V is a non-empty set of nodes;
2. For each $i \in \mathcal{O}^0_\Gamma$, $E^i_D \subseteq V \times V$, deduction edges of color i;
3. $E_A \subseteq V \times V$, ancestrality edges;
4. $r \in V$ is the root of the **DLDS**;
5. $l : V \to \Gamma$, for $v \in V$, $l(v)$ is the (formula) label of v;
6. $L : \bigcup\limits_{i \in \mathcal{O}^0_\Gamma} E^i_D \to \mathcal{B}(\mathcal{O}_S)$ is a function, such that for every $\langle u, v \rangle \in E^i_D$, $L(\langle u, v \rangle)$ says from which formulas the i-th colored deduction edge $\langle u, v \rangle$ carries its dependency;
7. $P : E_A \to \{1, \ldots, || \Gamma ||\}$.

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What is a correct (valid) DLDS

1. The underlying graph induced by the deduction edges (the set $\bigcup_{i \in \mathcal{O}^0_\Gamma} E^i_D$) is a connected simple-path rooted (hence leveled) Dag.
2. Every node $v \in V$ that has incoming deduction edges represents a bunch of disjoint correct applications. Use of Flow, a partial function, $F : Edges \to (\mathcal{O}^0_\Gamma \to (\mathcal{B}(\mathcal{O}_S) \times \mathcal{O}_S))$ to check this.
3. For every $e \in E_A$, there is $\langle v_1, \ldots, v_n \rangle$, such that $v_1 = target(e)$, $v_n = source(e)$, there is no i, $1 < i < n$, such that there is e' with $target(e) = v_i$, and there is a unique $d \in E^{P(e)}_D$, such that $source(d) = v_n$.
4. If the dynamic dependency sets equals static dependency sets, i.e., $L(e) = \pi_1(F(e)(0))$, for every deduction edge $e \in E^0_D$, for an adequate (see item 2 above).

A DLDS that sums up a dependency set Δ for the (max. two) edges arriving at the root r is a valid certificate for $\Delta \vdash_{M_\supset} l(r)$

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Soundness of the compression of tree-proofs into DLDS

(1) Each Collapse of exponentially repeated derivation preserves validity. The obtained proof-graph is sound. A inductive proof can be provided as well. Completeness is trivial

(2) N.D. proofs are valid DLDS

(3) Every DLDS obtained by collapsing sub-derivations bottom-up preserves minimal-logic validity.

(4) DLDSs are genuine certificates for $M_{\supset}$ validity.

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The polytime verification of DLDS

The validity of a DLDS of size k is verifiable by an algorithm in time k^3: Proof by a recursive reasoning on the collapsed derivations plus following the paths driven by the labels in the ancestor edges

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The algorithm for verifying validity of DLDS

$Flow : E_{Ded} \rightarrow (\mathcal{O} \rightarrow (\mathcal{B}(\mathcal{O}_S)))$

Require: A DLDS $\mathcal{D}$
Ensure: Whether $\mathcal{D}$ is a valid DLDS or not

```
Flows F1 ← ∅;F2 ← ∅
for lev from top-level to 0 do
    for e in lev do
        if Hypo(e) then
            if targetAncestor(e) then
                F2(e) ← ⟨labelAncestor(e), Bitstring(l(source(e)))⟩
            else
                F2(e) ← ⟨λ, Bitstring(l(source(e)))⟩
            end if
        else
            for b ∈ O do
                try F2(e) = UpdateVerifiedRule(e, F1) raise {"not a valid DLDS"}
            end for
        end if
    end for
    F1 ← F2
end for
```

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Our main result (would be ?)

We show arguments towards $Taut_{MIL}$ is in **NP**, and hence

PSPACE=NP

$M_{\supset}$={$\supset$-I,$\supset$-E} (would/should) has short/succinct (polynomially bounded) Dag-proofs for every tautology α in $M_{\supset}$.

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What if the approach has an essential drawback?

We already have a compression method for derivations and proofs in $M_{\supset}$ with a compressing ratio bigger than Huffman's ratio.

⇒ Comparison between GH and Huffman

⇒ G_5 has no hamiltonian cycle

⇒ The compressed proof for G5

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THANK YOU !!

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A Dialogical Reconstruction of Brouwer's Creating Subject?

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Abstract

The concept of a *creating subject* has been widely rejected amongst mathematicians due to its vagueness. Even Kreisel's attempt (Kreisel 1967) to reconstruct it through an axiomatic theory, using a new connective, namely $\Sigma \vdash_n$, has been recently confronted with the difficulty of supplying it with genuine *meaning explanations*, except maybe by postulating a non-constructive proof-object (Sundholm 2014).We sketch here a dialogical reconstruction of one 1928 argument which is based on this controversial concept, by replacing axioms with some specific interaction rules, in such a way that it can be given a suitable (rule-based) semantic.

1 Early form of Brouwer's argument

In his second Vienna Lecture – later published as "die Struktur des Kontinuums" (Brouwer 1975, pp. 429–440) –, Brouwer proposes an argument which holds, in his view, against the implication from inequality to apartness. Arguments of this kind will be later called "creating subject arguments", referring to the article "Essentially negative properties" (Brouwer 1948), in which the notion of a creating subject is made explicit (but not defined).

In its early form, the argument is based on defining:

- a *fleeing property* (f), which is a property for which, in the case of each natural number, one can prove either that it exists or that it is absurd, while one cannot calculate a particular number that has the property, nor one can prove the absurdity of the property for all natural numbers.
- its *critical number* (λ_f), which is the (hypothetical) smallest natural number that possesses this fleeing property.

Brouwer's own words are the following:

> Let p be an element determined by the convergent sequence $c_1, c_2 \ldots$, for which I choose c_1 to be the zero point and every $c_{\nu+1} = c_\nu$ with one exception: As soon as I find a critical number λ_f of a certain fleeing property f, I choose the next c_ν to be equal to $-2^{-\nu-1}$ and as soon as I find a proof of the absurdity of such a critical number, I choose the next c_ν to be equal to $2^{-\nu-1}$. This element is distinct from zero, and yet it is neither smaller nor greater than zero. (Brouwer 1930)

2 Informal reconstruction based on Kreisel's axioms

It is somehow surprising to find an indexical expression such that "I" within a mathematical argument. It is actually not just a way of speech here, because the discovery of the targeted critical number (or of its absurdity) depends on the subjective ordering of proof-attempts, in such a way that *for someone else, the ordering could be different*. In order to make sense of it, we may eventually consider that it holds for implicit axioms (Kreisel 1967, p. 160) referring to a "(correctly) thinking subject".

Amongst these axioms, we have

$$A \to \neg\neg\exists n \Sigma_n A$$

which is to be read: "If A is constructively provable, then it is absurd that there be no moment at which a correctly thinking subject will come to have evidence for it" and from which we have

$$\neg\neg\neg\exists n \Sigma_n A \to \neg A$$

whence constructively

$$\neg\exists n \Sigma_n A \to \neg A$$

Now, let us assume that

$$p = 0.$$

From this assumption, the "correctly thinking subject" can infer that he will never come to know that a λ_f exists, which is equivalent to saying that

$$\neg\exists n \Sigma_n \ (\exists x : \mathbb{N})(x = \lambda_f)$$

whence

$$\neg(\exists x : \mathbb{N})(x = \lambda_f)$$

But from the same assumption that $p = 0$, the aforementioned subject may also infer that he will never come to know that

$$\neg(\exists x : \mathbb{N})(x = \lambda_f)$$

which is equivalent to saying that

$$\neg\exists n\Sigma_n\ \neg(\exists x : \mathbb{N})(x = \lambda_f)$$

whence

$$\neg\neg(\exists x : \mathbb{N})(x = \lambda_f)$$

and

$$p = 0 \rightarrow \bot$$

Now, let us consider that any propositional connective must be associated with a well specified canonical proof-object in order to be genuinely meaningful. Now the question is about how can one ensure the propositionality of an utterance which is based on such a connective[1]? Such an attempt has been made by (Sundholm 2014), by "via the reduction of the theory CS to the Brouwer-Kripke Principle", with the paradoxical result that it reveals itself as *classically* valid, "a somewhat surprising result since 'it represents the extreme consequences of intuitionist subjectivism'" (p. 27).

3 Dialogical reconstruction

We suggest here another reconstruction of Brouwer's argument than one based on Kreisel's axioms, by locating it within the framework of the Dialogical Logic, as it has been recently linked with P. Martin-Löf's Constructive Type Theory (Rahman, McConaughey, et al. 2018). In other words, our reconstruction is based on another view on propositions, namely a reductive one according to which an utterance is a proposition as long as one can build a dialogue out of it, according to well-specified rules.

3.1 Theoretical background

One of the main features of the Dialogical Logic consists in defining formality through a *formal* or *socratic rule*, such that, if the Proponent (**P**) makes an elementary statement, then he must be able to justify it by using a justification (called *local reason*) which has been previously introduced into the game by the Opponent (**O**)[2].

In our view, the indexical "I", or the intuitionist mathematician who deploys his counter-example, must be thought of as an Opponent to the classical view on the continuum, which consists in stating that being different implies being apart. Accordingly, the argument consists in embedding, within a *classical*

[1] "[. . .] neither Kreisel, or Van Rootselaar, nor, to the best of my knowledge, anyone else, has offered a suggestion towards a 'BHK' account for what canonical proof-objects might be for propositions built by means of such 'Creating Subject' connectives, and it is by no means obvious how to provide one" (Sundholm 2014, p. 17)

[2] On *local reasons*, see (Rahman, McConaughey, et al. 2018, pp. 111–129), on the *socratic rule*, see (ibid., p. 141).

play (i.e. a play in which a player is allowed to rectify the way he has defended himself against a previous attack, by taking into account things that have been brought forward later in the play[3]), a construction carried out by **O**, such that **P** loses even though he is using a formally winning strategy, i.e. a strategy through which he should win the play if the *content* of **O**'s constructions, brought forward during the play, were not to be taken into account.

In other words, our claim is that Brouwer's argument is based on forcing the Proponent to admit that a certain construction is possible, such that reducing it to the abstract denotation of a formal reasoning is misleading. It is equivalent to saying that it is possible to argue, by using a mathematical construction, against the validity of a merely *logical truth*.

By referring to Peircian semiotical categories (*icon*, *index*, *symbol*), as re-elaborated by (Scherer 1984), we consider that a formal dialogue corresponds to a determinate scheme, whose appropriation by a concrete player must pass through a semiotic process, which is irreducible to any *symbolical definiteness*, i.e. that the meaning which is to be given to the formal scheme must not be thought of as being ultimately based on a domain of potential constructions rigidly denoted by elementary propositions. For instance, there may be other types of denotation than the usual ones to the concept of a series satisfying a Cauchy criterium, namely if we decide to take into account a time parameter as Brouwer actually does. Now, "denotation" is to be thought of as *immanent to dialogues*, or, more precisely, to the justification process that arises in it, stemming from **O**'s contribution, as regulated through original rules. There would exist accordingly a *semiotic process* which extends the abstract reference of formal reasoning by introducing pragmatically new ways of interacting. Within such a semiotic process, one should distinguish ways of signifying which are to be located below the constitution of symbols:

- *iconicity*, that is the result of an active construction, based on witnessing a concrete singular action, and holding accordingly as the actualization of a scheme. The appropriation of the formal reasoning consists in replaying it, so to say *diagrammatically*.
- *indexicality* is the product of anchoring the diagram within a creative and temporally anchored situation, such that (potentially infinite) theorematical constructions may be developed out of it, in a singular and unpredictable way.

By using these words, we are now saying that Brouwer's argument consists in considering the indexical parameter of the act of construction as being constructively relevant and as enabling him to extend the set $\mathbb{R}$. In our dialogical reconstruction, which is to be thought of as dynamical, there is a place for

[3]On the difference between *classical* and *intuitionist* rules, see (Rahman, McConaughey, et al. 2018, p. 139) Let us point out that, in our reading of Brouwer's argument, the intuitionist *is not playing intuitionistically*, which seems to us consistent with Brouwer's view that "Mathematics is independent of Logic" (Brouwer 1975, p. 99).

indexicality, whose logical expression takes actually the form of dots within brackets $\langle \ldots \rangle$ that **O** can bring forward, corresponding to the hidden content of a box *that* **P** *is entitled to ask for* in case he needs a justification for a certain statement of its own (see below, moves 14 and 18).

3.2 Diagrammatical reconstruction of the argument

	O			**P**	
				$! \; (\forall x : \mathbb{R})(x \neq 0 \rightarrow (x < 0 \vee x > 0))$	0
1	$m = 1$			$n = 2$	2
3	$\langle \, \rangle^{\mathbf{O}}_{p} : \mathbb{R}$	0		$a(\langle \, \rangle^{\mathbf{O}}_{p}) : p \neq 0 \rightarrow ((p < 0) \vee (p > 0))$	4
5	$L^{\rightarrow}(a(\langle \, \rangle^{\mathbf{O}}_{p})) : p \neq 0$	4		$R^{\rightarrow}(a(\langle \, \rangle^{\mathbf{O}}_{p})) : (p < 0) \vee (p > 0)$	8
7	$b(\langle [i] \rangle^{\mathbf{O}}_{p}) : p \neq 0$			$?^{\cdots}/L^{\rightarrow}(a(\langle \, \rangle^{\mathbf{O}}_{p}))$	6
9	$?^{\cdots}/R^{\rightarrow}(a(\langle \, \rangle^{\mathbf{O}}_{p}))$	8		$c(\langle [i] \rangle^{\mathbf{O}}_{p}) : (p < 0) \vee (p > 0)$	10
11	$?^{\vee}$	10		$L^{\vee}(c(\langle \ldots, [\lambda_f]^{?}, \ldots \rangle_p)) : p < 0$	12
13	$?^{\cdots}/L^{\vee}(c(\langle \ldots, [\lambda_f], \ldots \rangle_p))$	12			
15	$\lambda_f \notin \langle [i]^{\mathbf{O}} \rangle_p$		7	**Open** $\langle [i]^{\mathbf{O}}_{n} \rangle_p$	14
11′	$(?^{\vee})$	8		$R^{\vee}(c(\langle 0, \ldots, n, [e : \lambda_f \rightarrow \bot]^{?} \rangle_p)) : p > 0$	16
17	$?^{\cdots}/R^{\vee}(b(\langle 0, \ldots, n, [e : \lambda_f \rightarrow \bot]^{?} \rangle_p))$	16			
19	$e \notin \langle 0, \ldots, n, [i]^{\mathbf{O}} \rangle_p$		7	**Open** $\langle 0, \ldots, n, [i]^{\mathbf{O}}_{n+k} \rangle_p$	18

3.3 Step by step explanations of the implicit interactions between O and P involved in the argument

(0) **P**! $(\forall x : \mathbb{R})(x \neq 0 \rightarrow (x < 0 \vee x > 0))$: **P** states that for any entity which is definable on $\mathbb{R}$, it is universally the case that, if such an entity is different from zero, then it is either less or greater than zero.

(1) **O** $m = 1$.

(2) **P** $n = 2$: The first moves to be made by both player are the choices of a "repetition rank", which is intended to make any play finite, in such a way that propositionality be equivalent to "dialogue definiteness" (see Clerbout 2014, pp. 19–29).

(3) **O** $\langle\ \rangle_p^{\mathbf{O}} : \mathbb{R}$ $[0, ?]$: **O** attacks the universal by choosing an entity p which he defines as depending on a converging series – satisfying a Cauchy criterium – whose complete specification depends on his own future free choices, because it is indexed upon a temporal scale along which he will order his successive attempts of proving the existence of a critical number λ_f or its absurdity[4]. This number is not to be considered as completely given through its definition, because this definition has to be completed along the potentially infinite development of the accorded sequence. It is assumed that such a development will be partly executed along the play, in such a way that it must remain hidden to **P** unless **P** specifies, during the play, a determinate limit n until which he will ask **O** to supply a public initial segment of it. Hence the sign $\mathbb{R}$ is to be understood as an icon whose dynamical indexation process essentially depends on the way a concrete player actualizes, through his own choices, the potential horizon which is to be located beyond the symbolical definiteness of $\mathbb{R}$. As such, this potential horizon is not to be thought of as symbolically definable: no corresponding entity can be thought of as being "already there". The index **O** on $\langle\ \rangle_p^{\mathbf{O}}$ indicates that the internal development of the accorded sequence will depend on **O**'s choices.

(4) **P** $a(\langle\ \rangle_p^{\mathbf{O}}) : p \neq 0 \rightarrow ((p < 0) \vee (p > 0))$ $[3, !]$: **P** defends himself by stating that he possesses a justification of the implication if he substitutes the entity chosen by **O** within his own universal statement. Such a justification is to be thought of as a function of it. It means that, even though the entity $\langle\ \rangle_p^{\mathbf{O}}$ is still to be specified on being indexed upon **O**'s choice, **P** claims that he possesses a method to proof the truth of consequent, provided that **O** concedes the antecedent, by assuming that he has a construction which backs it up.

(5) **O** $L^{\rightarrow}(a(\langle\ \rangle_p^{\mathbf{O}})) : p \neq 0$ $[4, ?]$: **O** attacks the conditional by defining an instruction enabling him to select the left part of the conditional (i.e.

[4]The crucial point is that the justification for **O**'s statement that $p \neq 0$ stems from the way **O** has been defining p, i.e. such that stating its equality to zero would involve a restriction on the open horizon of its future development. Accordingly, in order to state that $p \neq 0$, **O** does not need to furnish a positive justification of its apartness from zero. There are thus two levels to be distinguished: a purely logical level, at which absurdity is to be precluded; a properly *intuitive* level, at which the instantiation of *empty* consistent logical forms (icons), is to be carried out within an open space of play through an indexical process of fulfillment, which is not to be thought of as being connected from the outset to predefined entities. There must be a difference between the possibility to forge a counter-example to $p = 0$ and the possibility to derive an absurdity from it. From the knowledge that a proposition is absurd, one is not allowed to assume that there exists a counter-example, that instantiates another proposition, contradicting the former. The formal derivation of the absurdity of a determinate property is not equivalent to the material instantiation of the contradictory property. Intuitionist negation is weaker than classical negation. Our dialogical reconstruction illustrates this point by linking it to a semiotic perspective, to the extent that the definiteness of an icon (the classical dialogue) cannot preclude its potential indexation on contradictory instantiations, depending on the choices a player will carry out, by locating his own future constructions within the accorded formal frame.

$p \neq 0)$, by building a local reason from the justification $a(\langle\ \rangle_p^{\mathbf{O}})$ which has been furnished by **P** at move 4.

(6) **P** $?^{\cdots}/L^{\rightarrow}(a(\langle\ \rangle_p^{\mathbf{O}}))$ [5, ?]: **P** counter-attacks by asking **O** to resolve this instruction, which means developing the sequence that corresponds to the entity p.

(7) **O** $b(\langle[i]\rangle_p^{\mathbf{O}}) : p \neq 0$ [6, !]: **O** defends himself: he resolves the instruction by supplying a local reason to $p \neq 0$. It is important to note that $p \neq 0$ is a statement that results directly from the definition of p, because were $p = 0$, then it would mean that both the existence and the absurdity of the existence of λ_f would hold, which is absurd. This impossibility corresponds to the way $p \neq 0$ has to be played with: it cannot be considered as a logically complex proposition. Thus, even though **O** is required to develop partly the sequence corresponding to p, the impossibility of p being equal to zero stems indeed from its very definition. It is not an application of the so-called PIN-principle ("from perpetual ignorance to negation") – which is an immediate consequence of the third axiom of Kreisel $(A \rightarrow \exists n \Sigma_n A)$; it is rather depending on the way the space of play is structured from the outset, when based on admitting such an entity as p: its equality to zero is to be thought of as representing an unplayable move. Accordingly, **O** resolves the instruction, but his resolution of it does not depend on the particular content of the sequence. Hence, this content stays *private*: it may be freely unfolded along the play, according to **O**'s future free choices.

O attacks the consequent stated at move 6, by asking to resolve the corresponding instruction.

(8) **P** $R^{\rightarrow}(a(\langle\ \rangle_p^{\mathbf{O}})) : (p < 0) \vee (p > 0)$ [5, !]: **P** defends himself against the attack launched at move 3 by defining an instruction enabling one to back up the right part of the conditional by operating upon the justification of the whole conditional.

(9) **O** $?^{\cdots}/R^{\rightarrow}(a(\langle\ \rangle_p^{\mathbf{O}}))$ [8, ?]: **O** attacks the consequent stated at move 6, by asking to resolve the corresponding instruction.

(10) **P** $c(\langle[i]\rangle_p^{\mathbf{O}}) : (p < 0) \vee (p > 0)$ [9, !]: **P** defends himself, by resolving the instruction through an operator c applied on the (privately) developed sequence, on which p is being indexed. It means that, by using this operator, **P** claims that he can prove one determinate part of the disjunct.

(11) **O** $?^{\vee}$ [10, ?]: **O** attacks the disjunction, by asking *which* part.

(12) **P** $L^{\vee}(c(\langle \ldots, [\lambda_f]^?, \ldots \rangle_p)) : p < 0$ [11, !]: **P** defines an instruction, selecting the *left* part, provided that λ_f arises in the development of the sequence $\langle[i]\rangle_p^{\mathbf{O}}$ whose content is thus hypothetically anticipated.

(13) **O** $?^{\cdots}/L^{\vee}(c(\langle \ldots, [\lambda_f]^?, \ldots \rangle_p))$ [12, ?]: **O** asks **P** to resolve the instruction.

(14) **P** Open$\langle [i]^{\mathbf{O}}_{n} \rangle_p$ [13, !]: **P** counter-attacks, by asking **O** to make public the actual development of the sequence corresponding to p until the n-th choice.

(15) **O** $\langle [i]^{\mathbf{O}} = a_1, \ldots, a_n \rangle_p [\forall i, 1 \leq i \leq n, a_i \neq \lambda_f]$ [14, !]: **O** defends himself by exposing the initial segment he has been developing until n, within which there is no necessity that λ_f appears.

P cannot defend himself against the attack launched by **O** at move 13.

But, by playing classically, he asserts, from now, the other part of the disjunction, and defends himself once again against the attack that had been launched at move 11, namely ***O*** $?^{\vee}$ [10, ?]

(16) **P** $R^{\vee}(c(\langle 0, \ldots, n, [e : \lambda_f \to \bot]^{?} \rangle_p)) : p > 0$ [11, !]: **P** rectifies his choice, carried out at move 10, of the left part of the disjunct, by defining another instruction, enabling one to select the right part of the disjunct, *provided* that a proof of the absurdity of λ_f be given in the development of p.

(17) **O** $?^{\cdots}/R^{\vee}(b(\langle 0, \ldots, n, [e : \lambda_f \to \bot]^{?} \rangle_p))$ [16, ?]: **O** asks now to resolve this other instruction.

(18) **P** Open$\langle 0, \ldots, n, [i]^{\mathbf{O}}_{n+k} \rangle_p$ [17, ?]: **P** counter-attacks, by asking **O** to make public the actual development of the sequence corresponding to p until the $n + k$-th choice.

(19) **O** $\langle [i]^{\mathbf{O}} = a_1, \ldots, a_n + k \rangle_p [\forall i, 1 \leq i \leq n + k, a_i \neq e : \lambda_f \to \bot]$ [18, !]: **O** defends himself by exposing the initial segment he has been developing until $n + k$, within which there is no necessity that a proof e of the absurdity of λ_f appears. **P** cannot make use of the development of the sequence supplied by **O** in order to resolve any of the instructions that he claimed enabling him to justify one or the other part of the disjunct. Therefore, there remains no possible move to him and he loses the play.

P *cannot defend himself against the attack launched at move 17.*

O *wins.*

4 Conclusion

We recall the essential distinction that is to be made between the *play* and the *strategic* levels[5].

From the fact that **O** wins here, we cannot conclude the existence of a winning *counter-strategy* for **O** (i.e. that **O** could not build a universal winning strategy for the negation of the implication from difference to apartness). The

[5]On this distinction, see (Rahman, McConaughey, et al. 2018, pp. 75–109), and in particular (p. 89): "The strategy standpoint is [. . .] a systematic exposition of all the relevant variants of a game – the relevancy of the variants being determined from the viewpoint of one of the two players."

definiteness of a winning strategy presupposes that one can run through all possible choices of the other player.

But a play, which is based on an indexical process, essentially cannot be enclosed within a symbolically defined domain of possible moves.

Accordingly, Kreisel's connective would correspond to the purely hypothetical closed set of all the indexical processes through which an Opponent may instantiate a formal play starting with a universal thesis quantifying on $\mathbb{R}$. But there is no reason to believe that such a set should be constructively, i.e. intensionally definable (whence the difficulty which has been pointed to by Sundholm 2014). We may also conclude that an intuitionist view on mathematics is not to be necessarily loaded with subjectivism[6].

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[6] As already pointed out by (Rahman, Redmond, and Clerbout 2016).

Extra-logical proof-theoretic semantics in HoTT

Andrei Rodin (RAS/HSE/SPBU)

Proof-Theoretic Semantics 3, Tübingen 27-30 March, 2019

Historical background: Kant, Hilbert, Markov

MLTT & HoTT

Models of HoTT and the Initiality Conjecture

Conclusions

Kant on Geometrical Proofs

Give a philosopher the concept of triangle and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on his concept as long as he wants, yet he will never produce anything new. He can analyse and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.

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Kant on Geometrical Proofs

But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle and obtains two adjacent angles that together are equal to the two right ones. [..] In such a way through a chain of inferences that is always guided by intuition, he arrives at a fully illuminated and at the same time general solution of the question." (KRV : A 716 / B 744)

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Kant on Discursive and Constructive Reasoning

In these examples we have only attempted to make distinct what a great difference there is between the discursive use of reason in accordance with concepts and its intuitive use through the construction of concepts. Now the question naturally arises, what is the cause that makes such a twofold use of reason necessary, and by means of which conditions can one know whether only the first or also the second takes place? (KRV : A 719 / B 747)

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Hilbert and Bernays on Axiomatic and Genetic Theories

The term axiomatic will be used partly in a broader and partly in a narrower sense.We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics.

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Hilbert and Bernays on Axiomatic and Genetic Theories

In Hilbert's Foundations of Geometry [of 1899] the axiomatic standpoint received a sharpening regarding the axiomatic development of a theory: From the factual and conceptual subject matter that gives rise to the basic notions of the theory, we retain only the essence that is formulated in the axioms, and ignore all other content. Finally, for axiomatics in the narrowest sense, the existential form comes in as an additional factor. This marks the difference between the axiomatic method and the constructive or genetic method of grounding a theory.

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Hilbert and Bernays on Axiomatic and Genetic Theories

While the constructive method introduces the objects of a theory only as a genus of things, an axiomatic theory refers to a fixed system of things (or several such systems), and for all predicates of the propositions of the theory, this fixed system of things constitutes a delimited domain of subjects, about which hold propositions of the given theory. There is the assumption that the domain of individuals is given as a whole. Except for the trivial cases where the theory deals only with a finite and fixed set of things, this is an idealising assumption that properly augments the assumptions formulated in the axioms.

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Hilbert and Bernays on Axiomatic and Genetic Theories

> *We will call this sharpened form of axiomatics (where the subject matter is ignored and the existential form comes in) formal axiomatics for short. (Hilbert & Bernays 1934, Intro)*

Markov Jr. on Constructive Mathematics

> *The constructive trend in mathematics significantly developed during the last years. It's essence is that only constructive objects along with the abstraction of their potential realisation (without the abstraction of actual infinity) are considered. Purely existential theorems are rejected since the existence proofs require a specification of potentially realisable method of constructing an object with required properties. Constructive objects are figures built from elementary figures, i.e., elementary constructive objects. A simple example of constructive object is a word constructed with a fixed alphabet. A word in a given alphabet is a sequence of letters of this alphabet. (1962)*

Constructive syntax or constructive semantics?

MLTT: Syntax

- 4 basic forms of judgement:
  (i) $A : TYPE$;
  (ii) $A \equiv_{TYPE} B$;
  (iii) $a : A$;
  (iv) $a \equiv_A a'$
- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

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MLTT: Semantics of $t : T$ (Martin-Löf 1984)

- t is an element of set T
- t is a proof (construction) of proposition T ("propositions-as-types")
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T (BHK-style semantics)

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MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{refl_x : x =_A y}$$

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Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is <u>not</u> a theorem in the (intensional) MLTT (Streicher & Hofmann 1995).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- ...

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HoTT: the Idea

Types in MLTT are modelled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in Category theory (in which case one thinks of n-groupoids as higher homotopy groupoids of an appropriate topological space).

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Homotopical interpretation of Intensional MLTT

- $x, y : A$
  x, y are points in space A
- $x', y' : x =_A y$
  x', y' are paths between points x, y; $x =_A y$ is the space of all such paths
- $x'', y'' : x' =_{x=_A y} y'$
  x'', y'' are homotopies between paths x', y'; $x' =_{x=_A y} y'$ is the space of all such homotopies
- ...

Point

Definition

Space S is called contractible or space of h-level (-2) when there is point $p : S$ connected by a path with each point $x : A$ in such a way that all these paths are homotopic (i.e., there exists a homotopy between any two such paths).

Homotopy Levels

Definition

We say that S is a space of h-level $n+1$ if for all its points x, y path spaces $x =_S y$ are of h-level n.

Cummulative Hierarchy of Homotopy Types

- -2-type: single point *pt*;
- -1-type: the empty space $\emptyset$ and the point *pt*: truth-values aka (mere) propositions
- 0-type: sets: points in space with no (non-trivial) paths
- 1-type: flat groupoids: points and paths in space with no (non-trivial) homotopies
- 2-type: 2-groupoids: points and paths and homotopies of paths in space with no (non-trivial) 2-homotopies
- ...

Remark

In addition to the syntactic bottom-up intuition about higher types (cf. paper by Paolo Pistone & Luca Tranchini at this conference) HoTT provides this useful top-down geometrical intuition: n-types finitary approximate the topological structure of the base space, which by default is an ω-type. Cf. the Taylor expansion of a given differentiable function.

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ entails $x \equiv y$ (definitionally).

(Internal criterion of logicality)

Truncation

Each type is transformed into a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- "Merely" logical rules (i.e., rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as *constructive* theory in the sense that besides of propositions it comprises non-propositional objects (on equal footing with propositions rather than "packed into" propositions as usual!) and formal rules for managing such objects (in particular, for constructing new objects from given ones). In fact, HoTT comprises rules with apply *both* to propositional and non-propositional types.

Qs & As

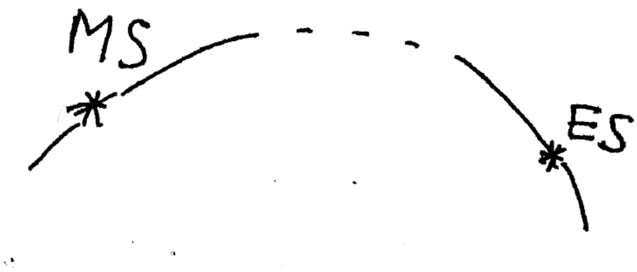
- Q: What is a possible PTS for non-logical theories? (Sara Negri, Peter Schröder-Heister)
- A: It is a geometrical (to wit, homotopical) semantics, which interprets HoTT rules for non-propositional types as rules for geometrical constructions. Cf. Euclid's rules aka postulates for constructing figures by the ruler and compass.
- Q: Does this semantics really qualify as proof-theoretic?
- A: Yes because the higher-order extra-logical constructions prove their underlying propositions (obtained via the propositional truncation).
- Q: What validates these rules in the cases of their extra-logical application?
- A: Geometrical Intuition!

Cassirer 1907 on mathematical intuition

> *The principle according to which our concepts should be sourced in intuitions means that they should be sourced in the mathematical physics and should prove effective in this field. Logical and mathematical concepts must no longer produce instruments for building a metaphysical "world of thought": their proper function and their proper application is only within the empirical science itself. (p. 43-44)*

IF HoTT/UF is a satisfactory formal foundation of mathematics and IF mathematics (rather than pure logic) is a formal foundation of all science then the intuitive grounding of HoTT rules qualifies as topic-neutral also in extra-logical applications of these rules.

Back to *Principia Mathematica*?



Venus Homotopically http://philsci-archive.pitt.edu/12116/

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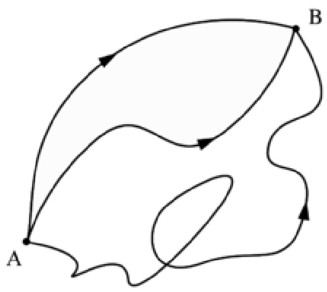
also in the Quantum Realm?



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PTS does not replace the Model theory...

(Göran Sundholm, Zhaohui Luo)

Interpretation of rules: MTS or PTS?

Standard version:
Interpretation m is a model of rule R

$$\frac{A_1^m, \ldots, A_n^m}{B^m} \tag{1}$$

when the following holds: whenever $A_1^m, \ldots, A_n^m$ are true statements B^m is also true statement. (Tsementzis 2017)

Models of HoTT according to Voevodsky

Idea: Theories are generic models (Lawvere 1963 : Functorial Semantics)

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Models of HoTT according to Voevodsky

(1) Construct a general model of given type theory **T** (MLTT or its variant) as a category $\mathcal{C}$ with additional structures which model **T**-rules. For that purpose the authors use the notion of *contextual category* due to Cartmell 1978; in later works Voevodsky uses a modified version of this concept named by the author a *C-system*.

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Models of HoTT according to Voevodsky

(2) Construct a particular contextual category (variant: a C-system) $\mathcal{C}(\mathbf{T})$ of syntactic character, which is called *term model*. Objects of $\mathcal{C}(\mathbf{T})$ are MLTT-contexts, i.e., expressions of form

$$[x_1 : A_1, \ldots, x_n : A_n]$$

taken up to the definitional equality and the renaming of free variables and its morphisms are substitutions (of the contexts into $\mathbf{T}$-rule schemata) also identified up to the definitional equality and the renaming of variables). More precisely, morphisms of $\mathcal{C}(T)$ are of form

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$$f : [x_1 : A_1, \ldots, x_n : A_n] \to [y_1 : B_1, \ldots, y_m : B_m]$$

where f is represented by a sequent of terms $f_1, \ldots, f_m$ such that
$x_1 : A_1, \ldots, x_n : A_n \vdash f_1 : B_1$
$\vdots$
$x_1 : A_1, \ldots, x_n : A_n \vdash f_m : B_m(f_1, \ldots, f_m)$
Thus morphisms of $\mathcal{C}(T)$ represent derivations in $\mathbf{T}$.

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Models of HoTT after Voevodsky

- Define an appropriate notion of morphism between contextual categories (C-systems) and form category $CTXT$ of such categories.
- Show that $\mathcal{C}(\mathbf{T})$ is initial in $CTXT$, that is, that for any object $\mathcal{C}$ of $CTXT$ there is precisely one morphism (functor) of form $\mathcal{C}(\mathbf{T}) \rightarrow \mathcal{C}$.

The last item is the **Initiality Conjecture** that presently stands open.

Conclusion 1

HoTT is as a "genetic" theory in Hilbert's sense. It involves an extra-logical homotopical semantics as well as a clear logical structure and logical semantics. This semantics qualifies as proof-theoretic.

Conclusion 2

The model theory of HoTT remains a work in progress. The novel notions of theory and model, which emerge in this context, require a further sharpening both mathematically and philosophically.

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Thank You! Danke! Спасибо!

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Interpolation in Singular Geometric Theories

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Abstract

We present a generalization of Maehara's lemma to show that the extensions of classical and intuitionistic first-order logic with a special type of geometric axioms, called singular geometric axioms, have Craig's interpolation property. As a corollary, we obtain a direct proof of interpolation for (classical and intuitionistic) first-order logic with identity, as well as interpolation for several mathematical theories, including the theory of equivalence relations, (strict) partial and linear orders, and various intuitionistic order theories such as apartness and positive partial and linear orders.

Craig's interpolation theorem [1] is a central result in first-order logic. It asserts that for any theorem $A \to B$ there exists a formula C, called *interpolant*, such that $A \to C$ and $C \to B$ are also theorems and C only contains non-logical symbols that are contained in both A and B (and if A and B have no non-logical symbols in common, then either $\neg A$ is a theorem or B is). The aim of this paper is to extend interpolation beyond first-order logic. In particular, we show how to prove interpolation in extensions of intuitionistic and classical sequent calculi with *singular geometric rules*, a special case of geometric rules investigated in [7]. Interpolation for singular geometric rules will be obtained by generalizing a standard result, reportedly due to Maehara in [12] and known as "Maehara's lemma" [5].

Proving Maehara-style interpolation for extensions of first-order logic is not at all straightforward, since the standard proof normally relies on the fact that such extensions admit a cut-free systematization in Gentzen's sequent calculus – which is in general not the case. To overcome this obstacle we shall build on previous work by Negri and von Plato who have shown (in a series of papers starting from [8]) how to recover cut elimination (as well as the admissibility of other structural rules) for extensions of the calculi G3c and m-G3i for classical and intuitionistic first-order logic. Of particular interest for the present work are the extensions with geometric rules, investigated in [7]. Once cut elimination is recovered in this way, we impose a singularity condition on geometric rules to isolate those containing at most one non-logical predicate (identity will be counted as logical). Our main result is to show that Maehara's lemma holds when G3c and G3i are extended with singular geometric rules (Lemma 9). Then interpolation follows easily from the generalized Maehara's lemma (Theorem 10). Finally, we consider applications of Theorem 10 and we show that singular geometric rules include many interesting extensions of intuitionistic and classical first-order logic, especially (classical and intuitionistic) first-order logic with identity, the theory of equivalence relations, (strict) partial and linear orders, the theory of apartness and the theory of positive partial and linear orders. We shall omit the proofs altogether and indicate reference to the existing literature when necessary.

1 Classical and intuitionistic sequent calculi

The language $\mathcal{L}$ is a first-order language with individual constants and no functional symbols. Moreover, let $\mathsf{FV}(A)$ be the set of free variables of a formula A and let $\mathsf{Con}(A)$ be the set of its individual constants. We agree that the set of terms $\mathsf{Ter}(A)$ of A is $\mathsf{FV}(A) \cup \mathsf{Con}(A)$. Moreover, if $\mathsf{Rel}(A)$ is the set of non-logical predicates of A then we define the language $\mathcal{L}(A)$ of A as $\mathsf{Ter}(A) \cup \mathsf{Rel}(A)$. Notice that $= \notin \mathcal{L}(A)$, for all A. Such notions are immediately extended to multisets of formulas Γ, by letting $\mathsf{FV}(\Gamma)$ to be defined as $\bigcup_{A\in\Gamma} \mathsf{FV}(A)$, and analogously for $\mathsf{Con}(\Gamma)$, $\mathsf{Ter}(\Gamma)$, $\mathsf{Rel}(\Gamma)$ and $\mathcal{L}(\Gamma)$.

The calculus Gc (Gi) is a variant of LK (LI) for classical (intuitionistic, respectively) first-order logic, originally introduced by Gentzen in [2]. In the literature, especially in [13] and [9], Gc and Gi are commonly referred to as G3c and G3i but we will omit '3' in the interest of readability. Moreover, we will write G to refer to either Gc or Gi. The rules are the standard ones and the reader interested is referred to [13] and [9].

The key feature of G is that the structural rules, including cut, are all admissible in it.

Theorem 1. *Cut elimination holds in* G.

2 Geometric theories

Extensions of G are not, in general, cut free; this means that Theorem 1 does not necessarily hold in the presence of new initial sequents or rules. It does hold, however, in the presence of rules following a certain pattern. To see this, we recall basic results from [7].

A *geometric axiom* is a formula following the *geometric axiom scheme* below:

$$\forall \bar{x}(P_1 \wedge \ldots \wedge P_n \to \exists \bar{y}_1 M_1 \vee \ldots \vee \exists \bar{y}_m M_m)$$

where each P_j is an atom and each M_i is a conjunction of a list of atoms $Q_{i_1}, \ldots, Q_{i_\ell}$ and none of the variables in any $\bar{y}_i$ are free in the P_js. We shall conveniently abbreviate $Q_{i_1}, \ldots, Q_{i_\ell}$ in $\boldsymbol{Q}_i$. In a geometric axiom, if $m = 0$ then the consequent of $\to$ becomes $\bot$, whereas if $n = 0$ the antecedent of $\to$ becomes $\top$. A *geometric theory* is a theory containing only geometric axioms. An m-premise *geometric rule*, for $m \geq 0$, is a rule following the *geometric rule scheme* below:

$$\frac{\boldsymbol{Q}_1^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta \quad \cdots \quad \boldsymbol{Q}_m^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}\ R$$

where each $\boldsymbol{Q}_i^*$ is obtained from $\boldsymbol{Q}_i$ by replacing every variable in $\bar{y}_i$ with a variable which does not occur free in the conclusion. Such variables will be called the *eigenvariables* of R. Without loss of generality, we assume that each $\bar{y}_i$ consists of a single variable. In sequent calculus a geometric theory can be formulated by adding on top of G finitely many geometric rules (recall that Δ contains exactly one formula in Gi). Moreover, geometric rules are assumed to satisfy the well-known closure property for contraction (see [9, 6.1.7]). Let G^{g} be any extension of G with finitely many geometric rules satisfying the closure condition (from now on, we will tacitly assume that the closure condition is always met). Cut elimination and the admissibility of the structural rules hold in G^{g}. Although we will heavily rely on [7], we start by introducing a more general notion of substitution that allows an arbitrary term u (possibly a constant) to be replaced by a term t. In the presence of such general substitutions, special care is needed in order to maintain the height-preserving admissibility of substitutions. In particular, general substitutions are height-preserving admissible, provided that the replaced term u does not occur essentially in the calculus. Intuitively, a term u occurs essentially in a rule R when u cannot be replaced (by an arbitrary term), namely when u is a constant and u already occurs in the axiom from which R is obtained. More precisely,

Definition 2. A constant u occurs essentially in a geometric axiom A if and only if, for some $t \not\equiv u$, $A[{}^t_u]$ is not an instance of the axiom A.

We also agree that a term u occurs essentially in a geometric rule R when it does so in the corresponding axiom. For example, in the geometric axiom $\neg 1 \leqslant 0$ of non-degenerate partial orders (see [10, p. 116]) both 1 and 0 occur essentially; hence they also occur essentially in the corresponding geometric rule *Non-deg*:

$$\frac{}{1 \leqslant 0, \Gamma \Rightarrow \Delta} \; \textit{Non-deg}$$

The general substitution $[\,^t_u\,]$ is height-preserving admissible in $\mathsf{G^g}$, provided that u occurs essentially in none of its geometric rules.

Lemma 3. *In $\mathsf{G^g}$, if $\vdash^n \Gamma \Rightarrow \Delta$, t is free for u in Γ, Δ, and u does not occur essentially in any rule of $\mathsf{G^g}$, then $\vdash^n \Gamma[\,^t_u\,] \Rightarrow \Delta[\,^t_u\,]$.*

It is also easy to show that:

Theorem 4. *Cut elimination holds for $\mathsf{G^g}$.*

Proof. See [7] for $\mathsf{Gc^g}$ and [3] for $\mathsf{Gi^g}$. QED

3 Singular geometric theories

To prove interpolation in extensions of first-order logic, the class of geometric rules seems too large. Thus, we restrict our attention to a proper sub-class of it and we introduce the class of singular geometric theories. In the next section we will state (Lemma 9) that Maehara's lemma holds for singular geometric extensions of first-order logic.

A *singular geometric axiom* is a geometric axiom with at most one non-logical predicate and no constant occurring essentially. A *singular geometric theory* is a theory containing only singular geometric axioms. In sequent calculus a singular geometric theory can be formulated by extending G with finitely many geometric rules of form:

$$\frac{\boldsymbol{Q}_1^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta \quad \cdots \quad \boldsymbol{Q}_m^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_n, \Gamma \Rightarrow \Delta} \; R$$

where no constant occurs essentially and that satisfy the following singularity condition:

$$|\mathsf{Rel}(\boldsymbol{Q}_1^*, \ldots, \boldsymbol{Q}_m^*, P_1, \ldots, P_n)| \leq 1 \qquad (\star)$$

It is evident that a number of important classical and intuitionistic mathematical theories are singular geometric. Regarding the classical ones, the theory of partial orders (R is reflexive, transitive and anti-symmetric), the theory of linear orders (R is a linear partial order), as well as the theories of strict partial orders (R is irreflexive and transitive) and strict linear orders (R is a trichotomic strict partial order) are singular geometric. Constructive singular geometric theories, on the other hand, include von Plato's theories of positive partial orders [11] (R is irreflexive and co-transitive) and positive linear orders (R is an asymmetric positive partial order), as well as the theory of apartness (R is irreflexive and splitting). Also the theory of equivalence relations (R is reflexive, transitive and symmetric) falls within the class of singular geometric theories. Finally, the fact that a relation R is functional (total and right-unique) can be axiomatized using singular geometric axioms. Singular geometric axioms are important in logic, too. Specifically, the axioms of identity are singular geometric.

$=$ is reflexive	$\forall x(x = x)$
$=$ satisfies the indiscernibility of identicals	$\forall x \forall y(x = y \wedge P[\frac{x}{z}] \rightarrow P[\frac{x}{y}])$

Notice that the indiscernibility of identicals satisfies the singularity condition ($\star$) because identity is a logical predicate. Hence, first-order logic with identity is a singular geometric theory.

Cut elimination for singular geometric rules clearly follows from cut elimination for geometric rules. More precisely, let $\mathsf{G^s}$ be any extension of G with a finite set of singular geometric rules. Then:

Theorem 5. *Cut elimination holds in* $\mathsf{G^s}$.

4 Interpolation with singular geometric rules

The standard proof of interpolation in sequent calculi rests on a result due to Maehara which appeared (in Japanese) in [5] and was later made available to international readership by Takeuti in his [12]. While interpolation is a result about logic, regardless the formal system (sequent calculus, natural deduction, axiom system, etc), Maehara's lemma is a "sequent-calculus version" of interpolation. Although originally Maehara proved his lemma for LK, it is easy to adapt the proof so that it holds also in G (cf. [13, §4.4]). We recall from [13] some basic definitions.

Definition 6 (partition, split-interpolant)**.** A *partition of a sequent* $\Gamma \Rightarrow \Delta$ is an expression $\Gamma_1 ; \Gamma_2 \Rightarrow \Delta_1 ; \Delta_2$, where $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$ (for $=$ the multiset-identity). A *split-interpolant* of a partition $\Gamma_1 ; \Gamma_2 \Rightarrow \Delta_1 ; \Delta_2$ is a formula C such that:

I $\quad \vdash \Gamma_1 \Rightarrow \Delta_1, C$

II $\quad \vdash C, \Gamma_2 \Rightarrow \Delta_2$

III $\quad \mathcal{L}(C) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2)$

We use $\Gamma_1 ; \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 ; \Delta_2$ to indicate that C is a split-interpolant for $\Gamma_1 ; \Gamma_2 \Rightarrow \Delta_1 ; \Delta_2$.

Lemma 7 (Maehara)**.** *In* Gc *every partition* $\Gamma_1 ; \Gamma_2 \Rightarrow \Delta_1 ; \Delta_2$ *of a derivable sequent* $\Gamma \Rightarrow \Delta$ *has a split-interpolant. In* Gi *every partition* $\Gamma_1 ; \Gamma_2 \Rightarrow ; A$ *of a derivable sequent* $\Gamma \Rightarrow A$ *has a split-interpolant.*

From Maehara's lemma it is immediate to prove Craig's interpolation theorem.

Theorem 8 (Craig)**.** *If $A \Rightarrow B$ is derivable in* G *then there exists a C such that $\vdash A \Rightarrow C$ and $\vdash C \Rightarrow B$ and $\mathcal{L}(C) \subseteq \mathcal{L}(A) \cap \mathcal{L}(B)$.*

Of any calculus for which Theorem 8 holds, we say that it has the interpolation property. Now we extend Lemma 7 to extensions of G with singular geometric rules.

Lemma 9. *In* Gc^{s} *every partition $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$ of a derivable sequent $\Gamma \Rightarrow \Delta$ has a split-interpolant. In* Gi^{s} *every partition $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \ ; \ A$ of a derivable sequent $\Gamma \Rightarrow A$ has a split-interpolant.*

Proof. The reader is referred to [3]. QED

From Lemma 9 it is immediate to conclude that singular geometric extensions of classical and intuitionistic logic satisfy the interpolation theorem, namely:

Theorem 10. G^{s} *has the interpolation property.*

5 Applications

We now consider some corollaries of Theorem 10 in which the strategy for building interpolants provided in Lemma 9 is applied. Notice that in the theories considered in this section all contracted instances are admissible and, hence, we can ignore them.

5.1 First-order logic with identity

We start with first-order logic with identity. Recall that a cut-free calculus for classical first-order logic with identity has been presented in [8] by adding on top of Gc the rules $Ref_=$ and $Repl_=$ corresponding to the reflexivity of $=$ and Leibniz's principle of indiscernibility of identicals, respectively.

$$\frac{s = s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref_= \qquad\qquad \frac{P[\,{}^{t}_{x}\,], P[\,{}^{s}_{x}\,], t = s, \Gamma \Rightarrow \Delta}{P[\,{}^{s}_{x}\,], t = s, \Gamma \Rightarrow \Delta}\ Repl_=$$

In intuitionistic theories, on the other hand, identity is often treated differently and we will provide a constructively more acceptable treatment of identity later in dealing with apartness. In general, however, nothing prevents us from building intuitionistic first-order logic with identity in a parallel fashion to the classical case. This is, for example, the route taken in [13] and we will follow suit.

Corollary 11. $\mathsf{G}^{=}$ *has the interpolation property.*

5.2 Equivalence relations

In a perfectly parallel fashion, we obtain the theory of equivalence relations by adding to G the rules corresponding to the reflexivity, transitivity and symmetry of a binary relation $\sim$. Thus, $\mathsf{EQ} = \mathsf{G} + \{Ref_{\sim}, Trans_{\sim}, Sym_{\sim}\}$.

$$\frac{s \sim s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref_{\sim} \qquad \frac{s \sim u, s \sim t, t \sim u, \Gamma \Rightarrow \Delta}{s \sim t, t \sim u, \Gamma \Rightarrow \Delta}\ Trans_{\sim}$$

$$\frac{t \sim s, s \sim t, \Gamma \Rightarrow \Delta}{s \sim t, \Gamma \Rightarrow \Delta}\ Sym_{\sim}$$

From the fact that these rules are singular geometric, it follows that:

Corollary 12. EQ *has the interpolation property.*

5.3 Partial and linear orders

Now we consider some well-known order theories. We start with partial orders. In sequent calculus, the theory of partial orders is obtained by extending $\mathsf{Gc}^=$ with the following rules corresponding to the axioms of reflexivity, transitivity and anti-symmetry of a binary relation $\leqslant$. Thus, let $\mathsf{PO} = \mathsf{Gc}^= + \{Ref_{\leqslant}, Trans_{\leqslant}, Anti\text{-}sym_{\leqslant}\}$:

$$\frac{s \leqslant s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref_{\leqslant} \qquad \frac{s \leqslant u, s \leqslant t, t \leqslant u, \Gamma \Rightarrow \Delta}{s \leqslant t, t \leqslant u, \Gamma \Rightarrow \Delta}\ Trans_{\leqslant}$$

$$\frac{s = t, s \leqslant t, t \leqslant s, \Gamma \Rightarrow \Delta}{s \leqslant t, t \leqslant s, \Gamma \Rightarrow \Delta}\ Anti\text{-}sym_{\leqslant}$$

Linear orders are obtained by assuming that the partial order $\leqslant$ is also linear, i.e. $\mathsf{LO} = \mathsf{PO} + \{Lin_{\leqslant}\}$.

$$\frac{s \leqslant t, \Gamma \Rightarrow \Delta \quad t \leqslant s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Lin_{\leqslant}$$

Both PO and LO are singular geometric theories, hence:

Corollary 13. LO *(hence,* PO*) has the interpolation property.*

Unlike $\mathsf{G}^=$ and EQ, the underlying logical calculus of both PO and LO is the classical one. The reason is that linearity is intuitionistically contentious and normally it requires a different, more constructively acceptable, axiomatization that will be considered in Section 5.6.

5.4 Strict partial and linear orders

The theory of strict partial orders consists of the axioms of first-order logic with identity plus the irreflexivity and transitivity of <. As we did for PO and LO, we consider this theory to be based on classical logic, i.e. by adding on top of $\mathsf{Gc}^{=}$ the following rules:

$$\frac{}{s < s, \Gamma \Rightarrow \Delta}\ \mathit{Irref}_{<} \qquad \frac{s < u, s < t, t < u, \Gamma \Rightarrow \Delta}{s < t, t < u, \Gamma \Rightarrow \Delta}\ \mathit{Trans}_{<}$$

Let SPO be $\mathsf{Gc}^{=} + \{\mathit{Irref}_{<}, \mathit{Trans}_{<}\}$. Total strict partial orders are then obtained assuming that $<$ is also trichotomic, i.e. $\mathsf{SLO} = \mathsf{SPO} + \{\mathit{Trich}_{<}\}$:

$$\frac{s = t, \Gamma \Rightarrow \Delta \quad s < t, \Gamma \Rightarrow \Delta \quad t < s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Trich}_{<}$$

Corollary 14. SLO *(hence,* SPO*) has the interpolation property.*

5.5 Apartness

We noticed earlier that in intuitionistic theories the identity relation is not always treated as in classical logic. In particular, identity is defined in terms of the more constructively acceptable relation of apartness. Apartness was originally introduced by Brouwer (and later axiomatized by Heyting in [4]) to express inequality between real numbers in the constructive analysis of the continuum: whereas saying that two real numbers a and b are unequal only means that the assumption $a = b$ is contradictory, to say that a and b are apart expresses the constructively stronger requirement that their distance on the real line can be effectively measured, i.e. that $|\, a - b \,| > 0$ has a constructive proof. Classically, inequality and apartness coincide, but intuitionistically two real numbers can be unequal without being apart. The theory of apartness consists of intuitionistic first-order logic plus the irreflexivity and splitting of $\neq$. Following [6], the theory of apartness is formulated by adding on top of Gi the following rules:[1]

$$\frac{}{s \neq s, \Gamma \Rightarrow A}\ \mathit{Irref}_{\neq} \qquad \frac{s \neq u, s \neq t, \Gamma \Rightarrow A \quad t \neq u, s \neq t, \Gamma \Rightarrow A}{s \neq t, \Gamma \Rightarrow A}\ \mathit{Split}_{\neq}$$

Let $\mathsf{AP} = \mathsf{Gi} + \{\mathit{Irref}_{\neq}, \mathit{Split}_{\neq}\}$. Given that these two rules are singular geometric rules, it follows that:

Corollary 15. AP *has the interpolation property.*

[1] Notice that Negri's underlying calculus is a quantifier-free version of Gi.

5.6 Positive partial and linear orders

Just like apartness is a positive version of inequality, so excess $\nleqslant$ is a positive version of the negation of a partial order $\leqslant$. The excess relation was introduced by von Plato in [11] and has been further investigated by Negri in [6]. The theory of positive partial orders consists of intuitionistic first-order logic plus the irreflexivity and co-transitivity of $\nleqslant$.[2] Let $\mathsf{PPO} = \mathsf{Gi} + \{Irref_{\nleqslant}, \; Co\text{-}trans_{\nleqslant}\}$

$$\frac{}{s \nleqslant s, \Gamma \Rightarrow A}\ Irref_{\nleqslant} \qquad \frac{s \nleqslant u, s \nleqslant t, \Gamma \Rightarrow A \quad u \nleqslant t, s \nleqslant t, \Gamma \Rightarrow A}{s \nleqslant t, \Gamma \Rightarrow A}\ Co\text{-}trans_{\nleqslant}$$

The theory of positive linear orders extends the theory of positive partial orders with the asymmetry of $\nleqslant$. Specifically, let $\mathsf{PLO} = \mathsf{PPO} + \{Asym_{\nleqslant}\}$:

$$\frac{}{s \nleqslant t, t \nleqslant s, \Gamma \Rightarrow A}\ Asym_{\nleqslant}$$

Given that all these rules are singular geometric, from Theorem 10 it follows that

Corollary 16. PPO *and in* PLO *have the interpolation property.*

To conclude, we have shown (Lemma 9) how to extend Maehara's lemma to extensions of classical and intuitionistic sequent calculi with singular geometric rules and provided a number of interesting examples of singular geometric rules that are important both in logic and mathematics, especially in order theories. In particular, we have shown that Lemma 9 covers first-order logic with identity and its extension with the theory of (strict) partial and linear orders. We have also seen that the same holds for the intuitionistic theories of apartness, as well as for positive partial and linear order.

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[2] Co-transitivity and splitting should not be confused. In particular, splitting (along with irreflexivity) gives symmetry, whereas co-transitivity does not. This is what distinguishes apartness (which is symmetric) from excess (which in general is not).

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PROOF NETS
Tools for studying equivalence between proofs and proof-theoretic semantics

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Third Conference on PROOF-THEORETIC SEMANTICS
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Abstract

I will discuss how proof-nets (geometrical representations of linear logic proofs) may be considered as related to the themes investigated in the domain of proof theoretic semantics and may be tools for future works in this field.

Indeed, proof-nets are strictly related to the theme of the equivalence between proofs (represented by sequent calculus derivations) and to the theme of the identity of proofs. Moreover, proof.nets are strictly related to the methodological idea that the meaning of a logical constant is given by the behaviour inside proofs. Finally, important theorems about proof-nets (splitting theorem, sequentialisation theorem together with focalization theorem) are strictly related to the investigation about reversibility of proof rules and allow to better understand "what is a proof".

Introduction

1. Thanks to organizers, thanks for the invitation.

2. Thanks to Dale Miller, Valeria de Paiva and Phil Scott: in their talks, a very good presentation of Linear Logic has been given.

3. My talk:
- a short presentation of proof nets of Linear Logic

- some motivations of proof nets, interesting for proof theoretic semantics

- some results on proof Nets, useful for proof theoretic semantics a

- future perspectives

Part I
PRESENTATION OF PROOF NET

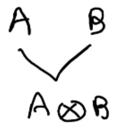
in the simplest case, proof nets for MLL

π is a proof net iff

· π is a graph of occurrences of formulas constructed by means of the following links

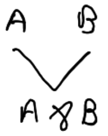
$A \quad A^{\perp}$ (id) $A, A^{\perp}$ conclusions

$A \quad A^{\perp}$ (cut) $A, A^{\perp}$ premisses



(⊗) A, B premisses

A⊗B conclusion



(⅋) A, B premisses

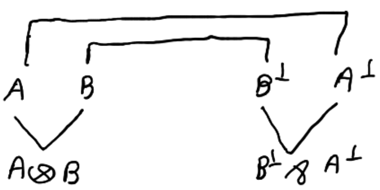
A⅋B conclusion

- each occurrence of formula is <u>conclusion</u> of <u>exactly</u> one link
- each occurrence of formula is <u>premisse</u> of <u>at most</u> one link

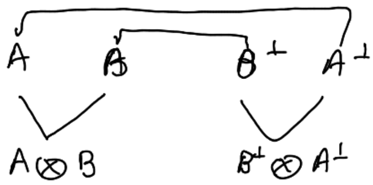
- by removing one edge in each (⅋)-link

"switch"

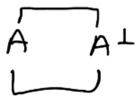
the graph becomes
- connected
- acyclic



is a proof-net



is not a proof-net



is not a proof net

a purely "geometrical" definition

Theorem: From sequent calculus to proof nets

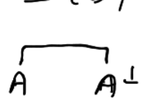
$\mathcal{D} : \vdash \Gamma$

derivation in sequent calculus

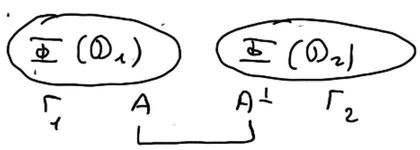
$\Phi(\mathcal{D})$ proof-net with conclusions Γ (formulas Γ are not premises of a link, are terminal nodes)

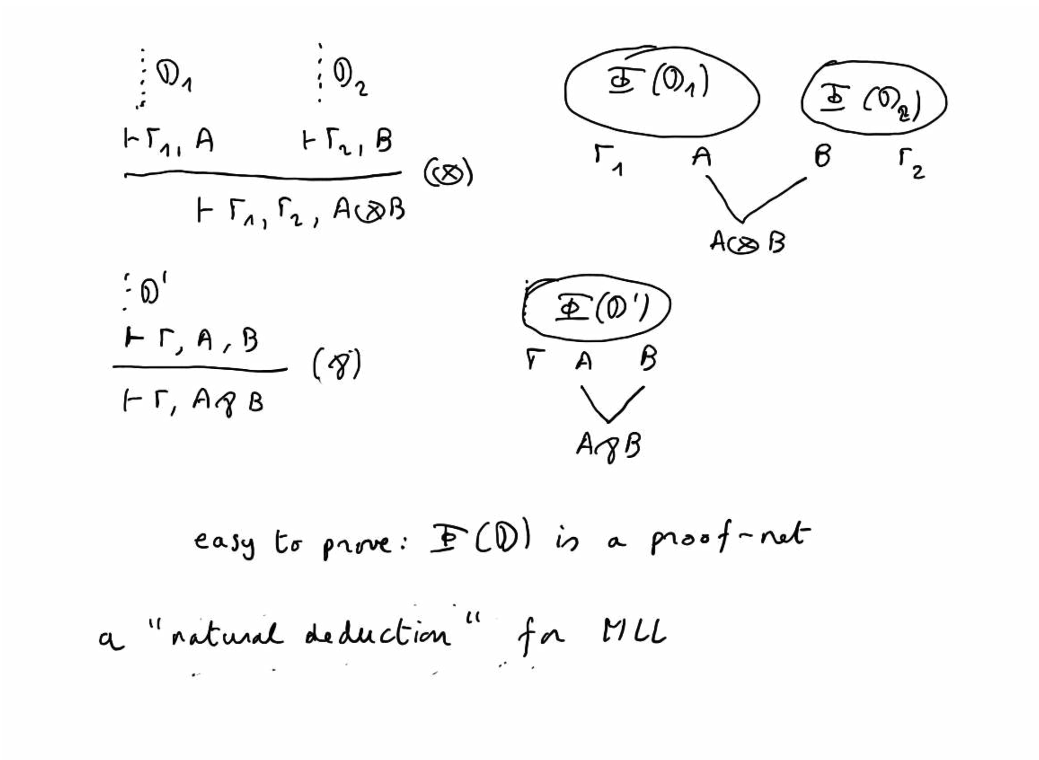
$\mathcal{D}$ $\Phi(\mathcal{D})$

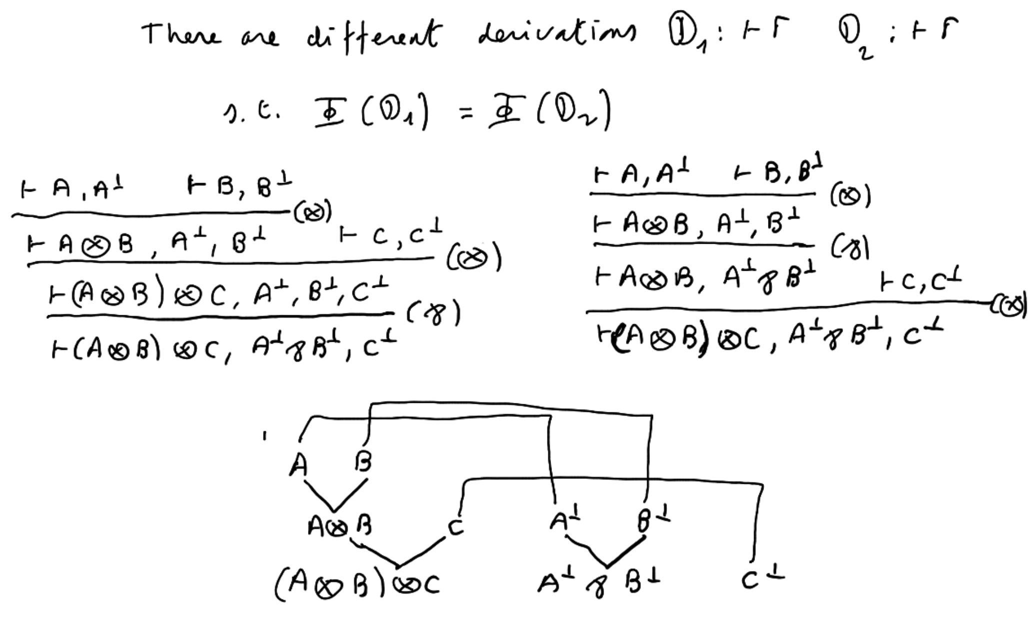
$\vdash A, A^{\perp}$



$$\frac{\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_2 \\ \vdash \Gamma_1, A & & \vdash \Gamma_2, A^{\perp}\end{matrix}}{\vdash \Gamma_1, \Gamma_2}\ (cut)$$

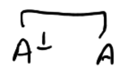



⊢ Γ1, A   ⊢ Γ2, B
⊢ Γ1, Γ2, A⊗B
(⊗)
Φ(D1)
Φ(D2)
A⊗B
⊢ Γ, A, B
⊢ Γ, A⅋B
(⅋)
Φ(D')
A⅋B
easy to prove: Φ(D) is a proof-net
a "natural deduction" for MLL


There are different derivations D1 : ⊢ Γ D2 : ⊢ Γ
s.t. Φ(D1) = Φ(D2)
⊢ A, A⊥   ⊢ B, B⊥
⊢ A⊗B, A⊥, B⊥   ⊢ C, C⊥
⊢ (A⊗B)⊗C, A⊥, B⊥, C⊥
⊢ (A⊗B)⊗C, A⊥⅋B⊥, C⊥
⊢ A⊗B, A⊥⅋B⊥   ⊢ C, C⊥
(A⊗B)⊗C
A⊥⅋B⊥

Moreover

$A \vdash A$ becomes $\vdash A^\perp, A$

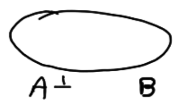


$A \vdash B$ becomes $\vdash A^\perp, B$

$B^\perp \vdash A^\perp$ " $\vdash A^\perp, B$

$$\begin{array}{c} A \\ \vdots \\ B \end{array}$$

becomes a proof-net π



and π corresponds also to

$$\begin{array}{c} B^\perp \\ \vdots \\ A^\perp \end{array}$$

Theorem: from proof-nets to sequent calculus

If π is a proof-net with conclusions Γ,

then $\pi = \Phi(\mathcal{D})$ for some derivation $\mathcal{D} : \vdash \Gamma$ in sequent calculus

Proof: not easy! it depends on the following lemma

Lemma (Splitting): if π is a proof-net with conclusions Γ, and no formula in Γ is conclusion of a $⅋$-link, then there is a <u>splitting</u> cut-link or a <u>splitting</u> $\otimes$-link,

PART II. SOME MOTIVATIONS OF PROOF NETS,
interesting for Proof Theoretic Semantics

A) Study of equivalence between proofs and identity of proofs

When $\Phi(\mathcal{D}) = \Phi(\mathcal{D}')$ $\quad \mathcal{D} : \vdash \Gamma \quad \mathcal{D}' : \vdash \Gamma \quad \mathcal{D} \neq \mathcal{D}'$

then $\mathcal{D}$ and $\mathcal{D}'$ are "equivalent"

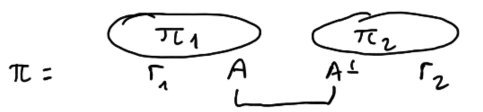
represent the "same" proof

the proof-net $\Phi(\mathcal{D}) = \Phi(\mathcal{D}')$

are two different ways to construct the same proof-net

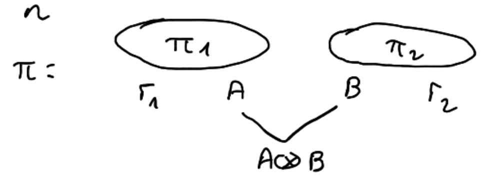
$\mathcal{D}$ and $\mathcal{D}'$ differ for some "inessential" order of rules

i.e.



$\Gamma_1, \Gamma_2 = \Gamma$

or



$\Gamma_1, A \otimes B, \Gamma_2 = \Gamma$

with π_1, π_2 distinct graphs (proof-nets)

Remark that linear logic starts in the field of
"semantics of proofs"
("denotational semantics")

another way to consider "identity" of proofs:

$\mathcal{D}$ and $\mathcal{D}'$ are equivalent (correspond to the <u>same</u> proof)
when $\mathcal{D} \rightsquigarrow \mathcal{D}^*$ and $\mathcal{D}' \rightsquigarrow \mathcal{D}^*$

$\rightsquigarrow$: cut-reduction, $\mathcal{D}^*$ cut-free

(the search of what is preserved under cut-reduction)

B) The meaning of a logical operator is given by introduction (and elimination) rules

This approach is at the basis of linear logic,
and at the basis of Proof Theoretic Semantics.

Novelty of linear logic:

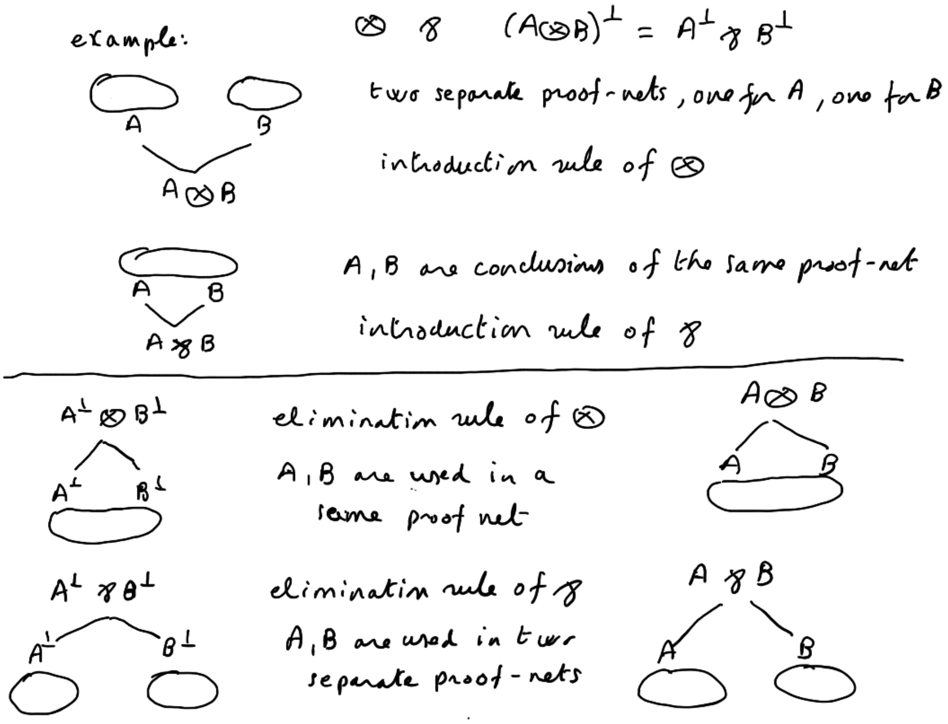
- each operator c is given together its dual $c^\perp$

$(c^{\perp\perp} = c)$

$$(c(A_1, \dots, A_n))^\perp = c^\perp(A_1^\perp, \dots, A_n^\perp)$$

$\begin{cases} \text{- introduction rule(s) of } c = \text{elimination rule(s) of } c^\perp \\ \text{- elimination rule(s) of } c = \text{introduction rule(s) of } c^\perp \end{cases}$

- it is sufficient give introduction rule(s) of c and $c^\perp$



C) An internal notion of "valid proof"

The notion of "valid proof" is not established by means of references to "external" reality.

"Valid proof" is a proof net.

The definition of "proof net" states a "correctness criterion" ("for every switch, the graph becomes acyclic and connected") purely internal to the proof.

Indeed, given a graph π with conclusion A, each "switch" of π correspond to a possible refutation of A

and the fact that under a given switch the graph π becomes acyclic and connected corresponds to the "victory against refutation"

so: a proof net with conclusion A is "able to win against every possible refutation"

possible refutations are expressed by modifications of the graph ("switches").

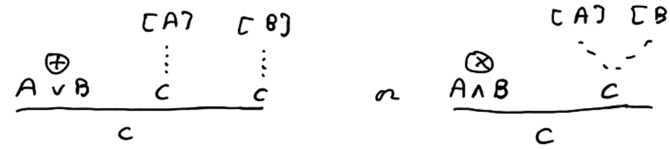
Each derivation $\mathcal{D}$ in sequent calculus of a sequent $\vdash \Gamma$ is a way to build a proof-net π with conclusions Γ, by preserving in each step the "correctness" of π.

But the "correctness" of a proof-net π may be established as a "global" property, without the reference to a given way to build the graph.

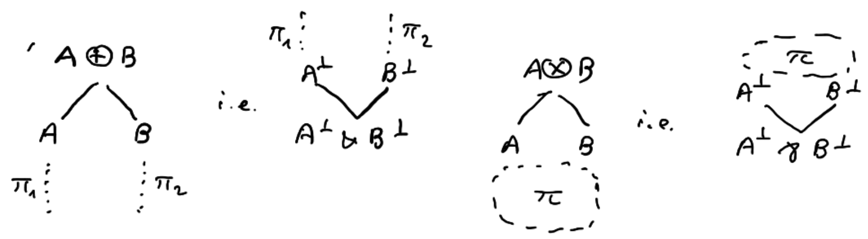
D) "Impredicative" elimination rules

Proof nets avoid "impredicative" elimination rules,

e.g.



(impredicative: C is arbitrary, may be $C = A \vee B$, $C = A \wedge B$)



PART III - RESULTS ON PROOF NETS
useful for Proof Theoretic Semantics

A) Positive / Negative logical constants

A logical constant (logical unit, connective, quantifier, ...) is

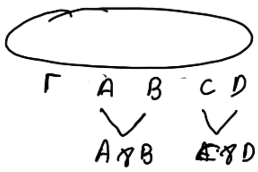
- "negative" (only one introduction rule, reversible) $\top \ \bot \ ⅋ \ \& \ ? \ \forall$
- "positive" (otherwise) $0 \ 1 \ \otimes \ \oplus \ ! \ \exists$

c, c^* dual : c positive c^* negative

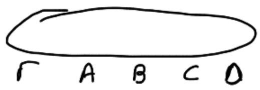
c negative c^* positive

Proof-nets : negative , positive

⅋ negative   a proof net π



is essentially ("geometrically") the same as



<u>reversibility</u>

each proof of $\vdash \Gamma, A \parr B$ may be transformed

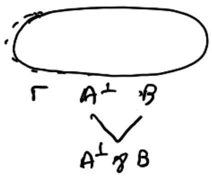
into a proof

$$\frac{\begin{matrix}\vdots \\ \vdash \Gamma, A, B\end{matrix}}{\vdash \Gamma, A \parr B}(\parr)$$

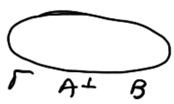
"canonical proof"

in particular a proof-net with conclusions $\Gamma, A \multimap B$

$\Gamma, A^\perp \parr B$



is the same as



each proof of $\vdash \Gamma, A \multimap B$ may be transformed into

a proof of

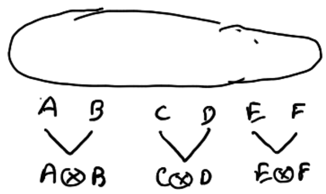
$$\frac{\begin{matrix}\vdots \\ \vdash \Gamma, A^{\perp}, B\end{matrix}}{\vdash \Gamma, A \multimap B}(\wp) \quad \text{"a canonical proof of } A \multimap B\text{"}$$

In general, this holds for every negative logical constant:

- terminal links may be removed
- existence of a canonical proof.

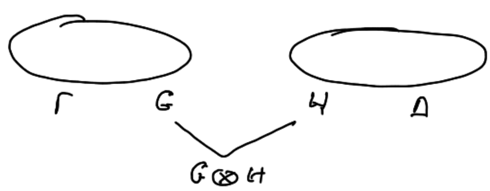
ex. of $\vdash \Gamma, A \,\&\, B$

⊗ positive given a proof net

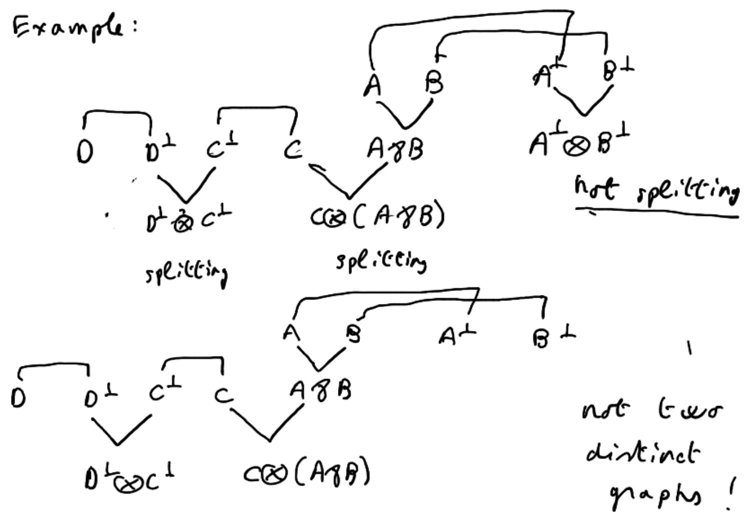


one of these ⊗-links is SPLITTING

but in general not every terminal ⊗-link is SPLITTING



Example:



So, $\vdash \Gamma, A \otimes B$ is provable,

does not imply that

there is a proof

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ \vdash \Gamma_1, A & \vdash \Gamma_2, B \end{array}}{\vdash \Gamma_1, A \otimes B}\ (\otimes)$$

i.e. a "canonical" proof

The existence of canonical proofs is assured only for negative logical operators.

But, theorems on proof-nets and on sequent calculus for Linear Logic, (in particular, Focalization Theorem proved by Andreoli and Pareschi) allow to say that

if $\vdash \Gamma$ is provable

then there is a "canonical" proof of $\vdash \Gamma$

⋮

positive rules

negative rules

positive rules — a block of positive rules

negative rules — a block of negative rules "in parallel"

$\vdash \Gamma$

i.e. a proof that is (bottom up)

an alternation of blocks of negative rules and positive rules

and in each block the order of rules is not relevant

i.e. the rules of each block may be performed "at the same time"

B) Relationships between a logical constant and its dual

given a logical constant (e.g. a n-ary connective)
c and its dual $c^{\perp}$,
<u>why</u> $c^{\perp}$ is dual of c (c is dual of $c^{\perp}$)?

Study the relationships between c and $c^{\perp}$,
i.e. between introduction rule(s) of c
and elimination rule(s) of c
(introduction rule(s) of $c^{\perp}$),
the "harmony" between c and $c^{\perp}$
between introduction / elimination rules

in a geometrical way,
in order to satisfy the requirements of
cut-reduction procedure.

("Multiplicative connectives, generalised")

A n-ary multiplicative connective **c** is
a <u>set</u> of permutations of $\{1, \ldots, n\}$

c and $c^{\perp}$ must satisfy the following requirement:
$\forall \sigma \in c \ \forall \tau \in c^{\perp}$ <u>$\sigma\tau$ is cyclic</u>
(and so $c^{\perp\perp} = c$) ("harmony"?)

e.g. $\otimes = \{\{\langle 1,2\rangle, \langle 2,1\rangle\}\} = \{p_{1,2}\}$

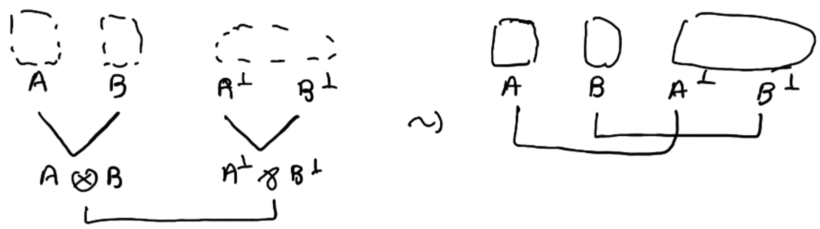
$⅋ = \{\{\langle 1,1\rangle, \langle 1,1\rangle\}\} = \{id\}$

clearly $p_{1,2} \circ id = id \circ p_{1,2}$ is cyclic

(in this way, Girard discovered also pairs of connectives which are **not sequentializable**)

$p_{1,2}$ correspond to the fact that the premisses of $\otimes$-introduction must be conclusions of distinct proofs

id correspond to the fact that the premisse of $⅋$-introduction must be conclusions of a same proof



acyclic connected for every switch

acyclic connected for every switch

My discovery (with P. Ruet) of non-commutative multiplicative conjunction $\odot$ and non-commutative multiplicative disjunction ∇ correspond to consider also

partial permutations

$A \odot B \neq B \odot A$
$A \nabla B \neq B \nabla A$

and so

$\odot$ and ∇ are sets of partial permutations s.t. $\forall \sigma \in \odot$ $\forall \tau \in \nabla$ $\sigma\tau$ has exactly one cycle

C) Generalized positive connectives , generalized negative connectives

Theorems on proof-nets, and focalisation theorem on sequent calculus, allow to define

n-ary positive connectives

n-ary negative connectives

in such a way that the dual of a positive connective is a negative connective (and viceversa).

The definition is based on

- introduction rule(s) for the positive connective
- introduction rule for its dual, negative connective

Ex. $\Phi(A,B,C)$ positive

and its dual $\Psi(A,B,C)$ negative

Φ : two rules

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, \Phi(A,B,C)}(\Phi 1) \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, C}{\vdash \Gamma, \Delta, \Phi(A,B,C)}(\Phi 2)$$

Ψ : one rule

$$\frac{\vdash \Gamma, A, B \quad \vdash \Gamma, A, C}{\vdash \Gamma, \Psi(A,B,C)}(\Psi)$$

Ψ is the dual of Φ $\quad (\Phi(A,B,C))^\perp = \Psi(A^\perp, B^\perp, C^\perp)$

and we have

$$\frac{\dfrac{\vdash A^\perp, A \quad \vdash B^\perp, B}{\vdash \Phi(A,B,C), A^\perp, B^\perp}(\Phi 1) \quad \dfrac{\vdash A^\perp, A \quad \vdash C^\perp, C}{\vdash \Phi(A,B,C), A^\perp, C^\perp}(\Phi 2)}{\vdash \Phi(A,B,C), \Psi(A^\perp, B^\perp, C^\perp)}(\Psi)$$

Remark that

$$\Phi(A,B,C) = (A \otimes B) \oplus (A \otimes C)$$
$$= A \otimes (B \oplus C)$$

$$\Psi(A,B,C) = (A \parr B) \& (A \parr C)$$
$$= A \parr (B \& C)$$

since

from $\vdash \Gamma, A$ $\quad \vdash \Delta, B$ $\qquad$ one proves

and from $\vdash \Gamma, A$ $\quad \vdash \Delta, C$ $\qquad (A\otimes B)\oplus(A\otimes C)$

$\qquad A\otimes(B\oplus C)$

$$\dfrac{\dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A\otimes B}(\otimes)}{\vdash \Gamma, \Delta, (A\otimes B)\oplus(A\otimes C)}(\oplus 1) \qquad \dfrac{\dfrac{\vdash \Gamma, A \quad \vdash \Delta, C}{\vdash \Gamma, \Delta, A\otimes C}(\otimes)}{\vdash \Gamma, \Delta, (A\otimes B)\oplus(A\otimes C)}(\oplus 2)$$

$$\dfrac{\vdash \Gamma, A \quad \dfrac{\vdash \Delta, B}{\vdash \Delta, B\oplus C}(\oplus 1)}{\vdash \Gamma, \Delta, A\otimes(B\oplus C)}(\otimes) \qquad \dfrac{\vdash \Gamma, A \quad \dfrac{\vdash \Delta, C}{\vdash \Delta, B\oplus C}(\oplus 2)}{\vdash \Gamma, \Delta, A\otimes(B\oplus C)}$$

$\Phi(A,B,C)$, $A\otimes(B\oplus C)$, $(A\otimes B)\oplus(A\otimes C)$

are "the same" formula because are provable from the same premisses

from $\vdash \Gamma, A, B$ $\quad \vdash \Gamma, A, C$

one proves $\vdash \Gamma, (A ⅋ B) \,\&\, (A ⅋ C)$

$\vdash \Gamma, A ⅋ (B \,\&\, C)$

$$\dfrac{\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A ⅋ B}(⅋) \quad \dfrac{\vdash \Gamma, A, C}{\vdash \Gamma, A ⅋ C}(⅋)}{\vdash \Gamma, (A ⅋ B) \,\&\, (A ⅋ C)}(\&)$$

$$\dfrac{\dfrac{\vdash \Gamma, A, B \quad \vdash \Gamma, A, C}{\vdash \Gamma, A, B \,\&\, C}(\&)}{\vdash \Gamma, A ⅋ (B \,\&\, C)}(⅋)$$

$\Psi(A,B,C)$, $A ⅋ (B \,\&\, C)$, $(A ⅋ B) \,\&\, (A ⅋ C)$

are the "same" formula since they are provable from the same premisses

$$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$$

and not only equivalent

$$A \parr (B \& C) = (A \parr B) \& (A \parr C)$$

and not only equivalent

since they are always the "same" proofs.

Conclusion. — PERSPECTIVES

We need to continue the Interaction between the field of Proof Theoretic Semantics and the field of Linear Logic, to better understand the Logic and applications of Logic,

Under the metodological principles that:

- "semantics" is not "to give the meaning" but "to discover the meaning"

- "proofs" are the main objects of Logic, the main objects to understand in Logic.

Thank you

fr your attention!

Predicative Hierarchies

Proof-Theoretic Semantics: Assessment and Future perspective

Gerhard Jäger

University of Bern

Tübingen, March 2019

My personal relationship to proof-theoretic semantics

- Proof-theoretic strength of set existence assertions
- Ordinal analysis of formal systems

Outline of this talk

- The early days of predicativity
- Predicative reducibility
- Subsystems of set theory

Predicativty – impredicativity

Russell and Poincaré (around 1901 – 1906)

- The vicious circle principle (VCP): A definition of an object S is *impredicative* if it refers to a totality to which S belongs.
- VPC is the essential source of inconsistencies.
- The structure of the natural numbers and the principle of induction on the natural numbers (for arbitrary properties) do not require foundational justification; further sets have to be introduced by purely predicative means.

Typical impredicative definitions

- $S \;=\; \{\, n \in \mathbb{N} : (\forall X \subseteq \mathbb{N})\varphi[X, n] \,\}$

  $$? : \quad m \in S \;\rightsquigarrow\; (\forall X \subseteq \mathbb{N})\varphi[X, m] \;\rightsquigarrow\; \varphi[S, m] \;\rightsquigarrow\; m \in S.$$

- Well-orderings

  Let $\prec$ be a (primitive recursive) linear ordering on $\mathbb{N}$ and X a subset of $\mathbb{N}$.

  $$Prog[\prec, X] \;:\Leftrightarrow\; (\forall m \in \mathbb{N})((\forall n \prec m)(n \in X) \to (m \in X)),$$

  $$Acc[\prec] \;:=\; \bigcap \{X \subseteq \mathbb{N} : Prog[\prec, X]\},$$

  $$WO[\prec] \;:\Leftrightarrow\; \mathbb{N} \subseteq Acc[\prec],$$

  $$(WO[\prec] \;\Leftrightarrow\; \text{every nonempty } X \subseteq \mathbb{N} \text{ has a } \prec\text{-least element}).$$

Typical predicative definitions

Pick an arbitrary arithmetic formula $A[X, n]$ of second order arithmetic.

Arithmetic definitions. Consider the process

$$Pow(\mathbb{N}) \ni S \longmapsto \{n \in \mathbb{N} : \mathbb{N} \models A[S, n]\} \in Pow(\mathbb{N}).$$

Arithmetical hierarchies. Given a set $S \subseteq \mathbb{N}$ we write

$$m \in (S)_n \;:\Leftrightarrow\; \langle n, m \rangle \in S.$$

Now suppose that $\prec$ is a primitive recursive linear ordering such that 0 is its least element and

$$n \oplus 1 \text{ the successor of } n \text{ in } \prec.$$

We may also assume that the field of $\prec$ is $\mathbb{N}$.

Now suppose that

$$\begin{aligned} (S)_0 &= \varnothing, \\ (S)_{n\oplus 1} &= \{m \in \mathbb{N} : \mathbb{N} \models A[(S)_n, m]\}, \\ (S)_\ell &= \text{disjoint union of } (S)_n \text{ with } n \prec \ell \text{ if } \ell \text{ limit.} \end{aligned}$$

Then we write $\mathcal{H}_A[\prec, S]$ and call S an A-hierarchy.

Question

For which linear orderings $\prec$ does this definition make sense?

First answer: well-orderings.

But is this enough if one wants to build up sets from below?

Further locally predicative hierarchies

Ramified analytic hierarchy

$$R_0 := \emptyset, \qquad R_{\alpha+1} := Def^{(2)}(R_\alpha), \qquad R_\lambda := \bigcup_{\xi<\lambda} R_\xi \ \ (\lambda \text{ limit}).$$

Gödel's constructible hierarchy

$$L_0 := \emptyset, \qquad L_{\alpha+1} := Def(L_\alpha), \qquad L_\lambda := \bigcup_{\xi<\lambda} L_\xi \ \ (\lambda \text{ limit}).$$

Every step $R_\alpha \mapsto R_{\alpha+1}$ and $L_\alpha \mapsto L_{\alpha+1}$ is justified from a predicative perspective.

Central question in connection with all these hierarchies:

How far are we allowed to iterate?

A first (model-theoretic) approach

Kleene, Spector, Kreisel, Wang, et al.

- $HYP = \Delta^1_1 = R_{\omega_1^{CK}} = L_{\omega_1^{CK}} \cap Pow(\mathbb{N})$.
- Conjecture: Predicatively justifiable subsets of $\mathbb{N} = HYP$.

However, this approach of iterating predicative set formation involves in an essential way the impredicative notion of being a well-ordering relation, even if one restrictes oneself to recursive well-orderings.

A step away from the semantic notion of well-ordered relation to predicatively provable well-orderings.

The proof-theoretic shift

Solomon Feferman (1928 – 2016)

Kurt Schütte (1909 – 1998)

Feferman – Schütte and the ordinal Γ_0

A boot-strap method

(i) We start off from a predicatively accepted ground theory, say ACA_0.

(ii) Then we systematically extend our framework: Whenever we have proved that a primitive recursive linear ordering is a well-ordering, we are allowed to iterate arithmetic comprehension along this well-ordering and to carry through bar induction along this well-ordering.

Originally done by Feferman and Schütte in the context of systems of ramified analysis or/and progressions of theories.

More modern terminology: the theory $\mathsf{AUT}(\Pi^0_\infty)$

Recall that for any formula $B[n]$ of second order arithmetic,

$$TI[\prec, B] \;:\Leftrightarrow\; Prog[\prec, B] \;\rightarrow\; \forall n B[n].$$

$$\mathsf{AUT}(\Pi^0_\infty) \;:=\; \mathsf{ACA}_0 \;+\; \frac{WO[\prec]}{\exists X\, \mathcal{H}_A[\prec, X]} \;+\; (BR)\; \frac{WO[\prec]}{TI[\prec, B]},$$

where $\prec$ is a primitive recursive linear ordering, $A[X, n]$ an arithmetic formula, and $B[n]$ an arbitrary formula.

Theorem

The proof-theoretic ordinal of $\mathsf{AUT}(\Pi^0_\infty)$ *is the ordinal* Γ_0, *and* $L_{\Gamma_0} \cap Pow(\mathbb{N})$ *is its least standard model.*

Reverse Mathematics (Friedman, Simpson, et al.)

Five central subsystems of second order arithmetic – The Big Five

$$\mathsf{RCA}_0 \;-\; \mathsf{WKL}_0 \;-\; \mathsf{ACA}_0 \;-\; \mathsf{ATR}_0 \;-\; \Pi^1_1\text{-}\mathsf{CA}_0$$

The principle (ATR) of arithmetic transfinite recursion

$$\forall R(WO[R] \;\rightarrow\; \exists X\, \mathcal{H}_A[\prec, X]),$$

where $A[X, n]$ is an arbitrary arithmetic formula which may contain additional parameters.

$$\mathsf{ATR}_0 \;:=\; \mathsf{ACA}_0 + (\mathsf{ATR})$$

Predicative reducibility of ATR_0

Theorem (Friedman, McAloon, Simpson, J)

1. *The proof-theoretic ordinal of* ATR_0 *is the ordinal* Γ_0.
2. ATR_0 *does not have a minimum* ω*-model or* β*-modell, but HYP is the intersection of all* ω*-models of* ATR_0.
3. Γ_{ε_0} *is the proof-theoretic ordinal of*

$$\mathsf{ATR} \ := \ \mathsf{ATR}_0 + \textit{induction on } \mathbb{N} \textit{ for all } \mathcal{L}_2 \textit{ formulas}$$

First consequences:

(1) $\mathsf{AUT}(\Pi^0_\infty)$ and ATR_0 are proof-theoretically equivalent but conceptually very different.

(2) And is there a big conceptual difference between ATR_0 and ATR?

Equivalences

Fixed points of positive arithmetic clauses (AFP)

$$\exists X \forall n (n \in X \ \leftrightarrow \ A[X^+, n]),$$

where $A[X^+, n]$ is an arbitrary X-positive arithmetic formula which may contain additional parameters.

Comparability of well-orderings (CWO)

$$\forall X, Y\, (WO[X] \wedge WO[Y] \ \rightarrow \ (|X| \leq |Y| \vee |Y| \leq |X|))$$

Π^1_1 reduction (Π^1_1-Red)

$$\forall n (A[n] \rightarrow B[n]) \ \rightarrow \ \exists X(\{n : A[n]\} \subseteq X \subseteq \{n : B[n]\}),$$

where $A[n]$ and $B[n]$ are arbitrary Σ^1_1 and Π^1_1 formulas, respectively.

Theorem (Avigad, Friedman, Simpson)

(ATR), (AFP), (CWO), *and* (Π^1_1-Red) *are pairwise equivalent over* ACA_0.

(Δ^1_1-TR)

$$\forall X \forall n(A[X,n] \leftrightarrow B[X,n]) \wedge WO[R] \rightarrow \exists X\, \mathcal{H}_A[R,X]$$

where $A[X,n]$ and $B[X,n]$ are arbitrary Σ^1_1 and Π^1_1 formulas, respectively.

Theorem (Bärtschi, J)

(Δ^1_1-TR) *and* ($M\Delta^1_1$-FP) *are both equivalent to* (ATR) *over* ACA_0.

From second order arithmetic to set theory

A major difference in building up the universe

- Second order arithmetic: Start off from a fixed/completed ground structure,

$$\mathcal{N} = (\mathbb{N}, \text{ prim.rec. functions and relations}).$$

Subsets of $\mathbb{N}$ are then introduced in a controlled way (predicaively, constructively, ...).

- Set theory: Start off from some basic sets and (sometimes) urelements and build new sets accoring to specific rules. In general there is no a priori bound (super collection) to which all sets belong.

Predicativity in set theory

The Platonic approach

We assume that we have a clear understanding of what an ordinal is and that the constructible universe exists,

$$L = \bigcup_{\alpha \in On} L_\alpha .$$

Then – in the Feferman-Schütte style – those ordinals can be identified that are "predicatively accessible (justified)". It can be shown (with some effort) that

$$\text{Predicative part of } L = L_{\Gamma_0}.$$

"Building the universe from below" or "predicatively acceptable closure conditions"

For example:

- Closure under pair, union, product, difference, ...
- Fixed points of positive arithmetic operators with set parameters,

$$\Phi_{\mathfrak{A}} : Pow(\omega) \ni X \longmapsto \{n \in \omega : \mathfrak{A}[S, X^+, n]\} \in Pow(\omega).$$

  Then:

  – $\Phi_{\mathfrak{A}}$ has a clear predicative meaning.

  – Define a fixed point of $\Phi_{\mathfrak{A}}$ via a pseudo-hierarchy argument.

  – Stays a fixed point independent of possible new sets.

Basic set theory BS^0

Formulated in the usual language $\mathcal{L}_\in$ of set theory with ω as constant for the first infinite ordinal and relation constants for all primitive recursive relations on $\mathbb{N}$.

Set-theoretic axioms of BS^0

(1) Equality and extensionality,

(2) closure under the rudimentary operations,

(3) Δ_0-Separation: For any Δ_0 formula $A[x]$,

$$\exists y \forall x (x \in y \;\leftrightarrow\; x \in a \wedge A[x]),$$

(4) ω-induction: $(\forall x \subseteq \omega)(x \neq \varnothing \;\rightarrow\; (\exists m \in x)(\forall n \in x)(m \leq n))$,

(5) The defining axioms for all primitive recursive relations.

BS^0 is clearly justified on predicative grounds. However, situation becomes more complicated if we turn to extensions of BS^0.

Simpson's ATR_0^{set}

$$BS^0 + (Reg) + (Count) + (Beta),$$

where

- (Reg) $:\Leftrightarrow$ $\forall a(a \neq \varnothing \;\rightarrow\; (\exists x \in a)(\forall y \in a)(y \notin x))$.
- (Count) $:\Leftrightarrow$ $\forall a(a$ is hereditarily countable$)$.
- $Wf[a,r] \;\Leftrightarrow\; (\forall b \subseteq a)(b \neq \varnothing \;\rightarrow\; (\exists x \in b)(\forall y \in b)(\langle y,x\rangle \notin r))$,
- $Cp[a,r,f] \;:\Leftrightarrow \begin{cases} Dom[f] = a \;\wedge \\ (\forall x \in a)(f(x) = \{f(y) : y \in a \wedge \langle y,x\rangle \in r\}) \end{cases}$
- (Beta) $:\Leftrightarrow$ $Wf[a,r] \;\rightarrow\; \exists f\, Cp[a,r,f]$.

Theorem (Simpson)

Every axiom of ATR_0 *is a theorem of* ATR_0^{set} *modulo the natural translation of* $\mathcal{L}_2$ *into* $\mathcal{L}_\in$.

Theorem (Simpson)

If A *is an axiom of* ATR_0^{set}, *then* $|A|$ *is a theorem of* ATR_0.

$$|A| \ ::: \ \begin{cases} \text{translation of the } \mathcal{L}_\in \text{ formula } A \text{ into the language } \mathcal{L}_2; \\ \text{sets are represented as well-founded trees;} \\ S \in^* T \ :\Leftrightarrow \ \exists n(\langle n \rangle \in T \wedge S \simeq T^{\langle n \rangle}) \end{cases}$$

Some aspects of this translation:

- Closure of $\in^*$ under $\simeq$ is required because of extensionality.
- $\in^*$ has a Σ^1_1 definition; with some extra effort it can be made Δ^1_1 in ATR_0.
- The translation of (Beta) is (more or less) for free under this interpretation of $\mathcal{L}_\in$ into $\mathcal{L}_2$.

Question

Is there a natural translation of $\mathcal{L}_\in$ into $\mathcal{L}_2$ that avoids the use of well-founded trees or graphs with specific decorations?

For example, is there a natural interpretation of $\mathcal{L}_\in$ into $\mathcal{L}_2$ – respecting extensionality – that can be developed within Σ^1_1-AC?

Hierarchies, fixed points, and reductions

The obvious analogues of (ATR), (AFP), and (Π^1_1-Red)?

The principle (Δ_0-TR) of Δ_0 transfinite recursion

$$(\forall r \subseteq \omega)(WO[r] \;\rightarrow\; (\exists x \subset \omega)\mathcal{H}_A[r,x])$$

where $A[X,n]$ is an arbitrary Δ_0 which may contain additional parameters.

Fixed points of positive Δ_0 clauses (Δ_0-FP)

$$(\exists x \subseteq \omega)(\forall n \in \omega)(n \in x \;\leftrightarrow\; A[x^+,n]),$$

where $A[x^+,n]$ is an arbitrary x-positive arithmetic formula which may contain additional parameters.

Π reduction (Π-Red)

$$(\forall x \in a)(A[x] \rightarrow B[x]) \;\rightarrow\; \exists y(\{x \in a : A[x]\} \subseteq y \subseteq \{x \in a : B[x]\}),$$

where $A[x]$ and $B[x]$ are arbitrary Σ and Π formulas, respectively.

Theorem (Bärtschi, J)

1. $BS^0 + (\Delta_0\text{-FP}) \;\vdash\; (\Delta_0\text{-TR})$.
2. $BS^0 + (\Pi\text{-Red}) \;\vdash\; (\Delta_0\text{-TR})$.
3. $ATR_0 \;\subseteq\; BS^0 + (\Delta_0\text{-TR}) \;\subseteq\; \begin{cases} BS^0 + (\Delta_0\text{-FP}), \\ BS^0 + (\Pi\text{-Red}). \end{cases}$

Theorem

1. $\mathsf{BS}^0 + (\Pi\text{-Red}) \leq \mathsf{ATR}_0$.
2. *The proof-theoretic ordinal of* $\mathsf{BS}^0 + (\Delta_0\text{-FP})$ *is* Γ_0.

The first reduction is via a modified Simpson translation of $\mathcal{L}_\in$ into $\mathcal{L}_2$, the second via an embedding into KPi^0.

Question

What is the exact relationship – over BS^0 – between

$$(\Delta_0\text{-TR}), \quad (\Delta_0\text{-FP}), \quad (\Pi\text{-Red})?$$

Kripke-Platek set theory KP

KP := BS^0 plus the following two axiom schemes

(1) (Δ_0-*Collection*): For all Δ_0 formulas $A[x,y]$,

$$(\forall x \in a)\exists y A[x,y] \;\rightarrow\; \exists z(\forall x \in a)(\exists y \in z)A[x,y].$$

(2) ($\mathcal{L}_\in$-$\mathsf{I}_\in$): For all $\mathcal{L}_\in$ formulas$B[x]$,

$$\forall x((\forall y \in x)B[y] \rightarrow B[x]) \;\rightarrow\; \forall x B[x].$$

Relationship between ATR_0^{set} and KP

Theorem (J)

1. *The proof-theoretic ordinal of* KP *is the Bachmann-Howard ordinal;* KP *is proof-theoretically equivalent to the theory* ID_1.
2. KP + (Beta) *is proof-theoretically equivalent to* Δ^1_2-CA + (BI).

Immediate consequence

$$\mathsf{KP} \not\subseteq \mathsf{ATR}_0^{set} \quad \text{and} \quad \mathsf{ATR}_0^{set} \not\subseteq \mathsf{KP}.$$

Further:

- KP $\nvdash$ $(\mathsf{AFP})^-$ (Gregoriades for parameter-free).
- KP + (AFP) and KP have the same proof-theoretic strength (Sato).
- KP + (Beta) + (Π-Red) proves Π^1_2 comprehension on ω.
- But what about KP + (Π-Red)?

Kripke-Platek without foundation and extensions

$$\mathsf{KP}^0 \;:=\; \mathsf{BS}^0 + (\Delta_0\text{-}\mathit{Collection})$$

$\mathcal{L}_\in \;:=\; \mathcal{L}_\in(\mathsf{Ad})$ with Ad a unary relation symbol to express admissibility.

Ad axioms

(Ad.1) $\mathsf{Ad}(d) \;\to\; d$ transitive $\wedge\; \omega \in d$.

(Ad.2) $\mathsf{Ad}(d) \;\to\; A^d$ for every closed instance of an axiom of KP^0.

(Ad.3) $\mathsf{Ad}(d_1) \wedge \mathsf{Ad}(d_2) \;\to\; d_1 \in d_2 \vee d_1 = d_2 \vee d_2 \in d_1$.

$$\mathsf{KPi}^0 \;:=\; \mathsf{KP}^0 + \forall x \exists y (x \in y \wedge \mathsf{Ad}(y)),$$

$$\mathsf{KPi} \;:=\; \mathsf{KP} + \forall x \exists y (x \in y \wedge \mathsf{Ad}(y)).$$

Remark

1. The least α such that $L_\alpha \models \mathsf{KPi}$ is the first rec. inacc. ordinal.
2. KPi^0 poves (Beta). However, (Beta) is very weak in the context of KPi^0 since there is no induction on the ordinals.
3. On the other hand, it is strong in KP since then it makes the Π_1 predicate "r is well-founded on a" a Δ_1 predicate.

Theorem (J)

1. $\mathsf{ATR}_0 \subseteq \mathsf{KPi}^0$ *and the proof-theoretic ordinal of* KPi^0 *is* Γ_0.
2. KPi *is proof-theoretically equivalent to* KP + (Beta), *and thus also to* Δ^1_2-CA + (BI).

Outlook

- The relationship between subsystems of second order arithmetic and set theory is rather transparent *as soon as Axiom (Beta) is available*.
- However, what can we say if we do not have Axiom (Beta)? Is there a general picture?
- Is Axiom (Beta) a philosophically relevant principle?
- The *fat versus high* question.

Thank you for your attention!