Trapping of light in stationary spacetimes

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1 Summary

1.1 Summary in English

In this thesis, we study phenomena related to trapping of light in stationary spacetimes.

After a general introduction to Mathematical General Relativity, we first prove a uniqueness result for "quasilocal photon spheres" and static horizons in asymptotically flat so-called pseudo-electrostatic systems. Our result implies that an asymptotically Reissner–Nordström electrostatic system of arbitrary dimension which contains a "subextremal" photon sphere is a Reissner–Nordström manifold.

We define pseudo-electrostatic systems as a generalization of electrostatic systems by replacing one of the dimensionally reduced Einstein equations for electrovacuum with an inequality for the scalar curvature. Furthermore, we define "quasilocal photon spheres" in (pseudo-)electrostatic systems by equalities relating their intrinsic and extrinsic geometry; this notion generalizes (as we will show) photon spheres in electrostatic systems. We need to postulate a subextremality condition (an inequality relating mean curvature and scalar curvature of quasilocal photon spheres) as an assumption of our theorem, since the conclusion of the theorem does not hold, for example, for superextremal ($|q| > m$) Reissner–Nordström manifolds; and these manifolds may contain superextremal (in the sense of the quasilocal inequality) photon spheres.

The methods used in the proof of this theorem go back to the classical black hole uniqueness proofs of Bunting and Masood-ul-Alam [6] and Ruback [55], which rely on an application of the rigidity case of the positive mass theorem. These techniques are combined with newer ideas developed by Cederbaum–Galloway [12] and Cederbaum [8], which allow (by gluing suitably constructed "necks" to the photon sphere inner boundary of the (pseudo-)electrostatic system up to a static horizon) to reduce the (quasilocal) photon sphere case to the static horizon case. Our theorem extends to the much weaker case of pseudo-electrostatic systems due to the realization that the equation for the Ricci tensor is not necessary for this type of proof but can be replaced by an inequality for the scalar curvature. Generalizing from the higher-dimensional vacuum case treated in [8] and the $3 + 1$-dimensional electrovacuum case from [12] (which are both covered by our result) to the higher-dimensional electrovacuum case also requires a different strategy for the calculations that prove quasilocal properties of photon spheres, a new choice of the "mass" and "charge" of the glued-in necks, the usage of a different partial differential equation in the last step of the proof, and adjustment of many calculations (for example to prove regularity statements) along the way.

In the second part of this thesis, we leave the static setting and investigate trapped light in the Kerr spacetime. Studying the photon region in this paradigmatic example of a stationary but not static spacetime is an important step towards a better under-

standing of the structure of trapped light in stationary spacetimes. We give a new and (compared to [20]) more direct proof that the photon region in the Kerr spacetime can be naturally understood as a submanifold of the phase space and has topology $SO(3) \times \mathbb{R}^2$. We first prove rigorously that the photons of constant trapped coordinate radius, which are explicitly given in [64], are the only trapped photons in the Kerr spacetime in the sense that they stay away both from the horizon and from spatial infinity. We then proceed to describe the photon region as a zero set of a smooth map from the Kerr phase space to $\mathbb{R}^3$, using the characterization of photons of constant radius via their constants of motion from [64], and use the implicit function theorem to show that this set is a submanifold of the phase space. Finally, we show that this submanifold is $\mathbb{R}^2$ times a closed 3-dimensional manifold. By explicitly calculating the first fundamental group of this 3-manifold (using the Seifert–van Kampen theorem) as $\mathbb{Z}_2$, and by then appealing to the elliptization conjecture, we conclude that the manifold in question is $SO(3)$, so that the Kerr photon region in the phase space is topologically $SO(3) \times \mathbb{R}^2$.

1.2 Zusammenfassung auf Deutsch (Summary in German)

In dieser Arbeit betrachten wir Phänomene in Zusammenhang mit gefangenem Licht in stationären Raumzeiten.

Nach einer allgemeinen Einführung in die Mathematische Allgemeine Relativitätstheorie zeigen wir zunächst ein Eindeutigkeitsergebnis für „quasilokale Photonensphären“ und statische Horizonte in asymptotisch flachen, sogenannten pseudo-elektrostatischen Systemen. Unser Ergebnis impliziert, dass ein elektrostatisches System, das asymptotisch Reissner–Nordströmsch ist und eine „subextremale“ Photonensphäre besitzt, eine Reissner–Nordström-Mannigfaltigkeit ist.

Wir definieren pseudo-elektrostatische Systeme als eine Verallgemeinerung elektrostatischer Systeme, indem wir eine der Einsteingleichungen für zeitsymmetrische Raumschnitte elektrostatischer Raumzeiten durch eine Ungleichung für ihre Skalarkrümmung ersetzen. Des weiteren definieren wir „quasilokale Photonensphären“ in (pseudo-) elektrostatischen Systemen durch Gleichungen, die Größen ihrer intrinsischen und extrinsischen Geometrie zueinander in Beziehung setzen; wie wir zeigen werden, verallgemeinert in elektrostatischen Systemen dieser Begriff der quasilokalen Photonensphäre denjenigen der Photonensphäre.

In unserem Theorem kann nicht darauf verzichtet werden, Subextremalität (eine Ungleichung, die die mittlere Krümmung mit der Skalarkrümmung quasilokaler Photonensphären in Beziehung setzt) vorauszusetzen, da die Konklusion des Theorems beispielsweise nicht für superextremale ($|q| < m$) Reissner–Nordström-Mannigfaltigkeiten gilt, und diese können superextremale (im Sinne der quasilokalen Ungleichung) Photonensphären enthalten.

Die Beweismethoden gehen zurück auf die klassischen Beweise für die Eindeutigkeit schwarzer Löcher von Bunting und Masood-ul-Alam [6] und Ruback [55], die ihrerseits den Rigiditätsfall des Positive-Masse-Theorems verwenden. Die darin verwendeten Techniken kombinieren wir mit neueren Ideen, die von Cederbaum–Galloway [12] und Cederbaum [8] entwickelt wurden und es erlauben, durch Einkleben geeigneter „Hälse“ an den inneren Rand (an die (quasilokale) Photonensphäre) des gegebenen elektrostatischen Systems den Fall (quasilokaler) Photonensphären auf den Fall statischer Horizonte zurückzuführen.

Unser Theorem gilt auch für pseudo-elektrostatische Systeme, also unter einer Voraussetzung, die deutlich schwächer als die elektrostatischen Gleichungen ist; dies beruht darauf, dass die Gleichung für den Ricci-Tensor für die verwendeten Beweistechniken nicht nötig ist, sondern eine Unleichung an die Skalarkrümmung genügt. Beim Verallgemeinern von der Situation im höherdimensionalen Vakuum (in [8]) und dem $3 + 1$-dimensionalen Elektrovakuum-Fall in [12] auf den von uns behandelten Fall des höherdimensionalen Elektrovakuums wurden neue Strategien nötig für die Berechnungen, die die quasilokalen Eigenschaften von Photonensphären zeigen, ebenso wie neue Wahlen für die “Masse“ und die „Ladung“ der eingeklebten Hälse. Desweiteren unterscheidet sich die im letzten Beweisschritt verwendete partielle Differentialgleichung von denjenigen in [8, 12], und viele Rechungen (beispielsweise in den Beweisen von Regularitätsaussagen) mussten unserem Fall angepasst werden.

Im zweiten Teil der vorliegenden Arbeit verlassen wir das statische Setting und untersuchen gefangenes Licht in der Kerr-Raumzeit. Die Photonenregion in diesem paradigmatischen Beispiel einer stationären, nicht statischen Raumzeit zu untersuchen, ist ein wichtiger Schritt in Richtung eines besseren Verständnisses der Struktur gefangenen Lichts in stationären Raumzeiten. Wir präsentieren einen neuen und (verglichen mit [20]) direkteren Beweis dafür, dass die Photonenregion in der Kerr-Raumzeit auf natürliche Weise als Untermannigfaltigkeit des Phasenraums aufgefasst werden kann und Topologie $SO(3) \times \mathbb{R}^2$ hat. Zunächst beweisen wir rigoros, dass die Photonen mit konstanter radialer Koordinate (die in [64] explizit angegeben werden) die einzigen Photonen in der Kerr-Raumzeit sind, die „gefangen“ sind in dem Sinne, dass sie sich weder dem Horizont noch der raumartigen Unendlichkeit nähern. Anschließend beschreiben wir die Photonenregion als Nullstellengebilde einer glatten Abbildung vom Phasenraum der Kerr-Raumzeit nach $\mathbb{R}^3$, wobei wir uns die Charakterisierung von Photonen mit konstantem Koordinatenradius durch ihre Erhaltungsgrößen aus [64] zunutze machen. Mithilfe des impliziten Funktionensatzes zeigen wir, dass diese Nullstellenmenge eine Untermannigfaltigkeit des Phasenraums ist. Schließlich bestimmen wir die Topologie dieser Untermannigfaltigkeit als das Produkt von $\mathbb{R}^2$ mit einer geschlossenen 3-Mannigfaltigkeit. Nachdem wir die erste Fundamentalgruppe dieser 3-Mannigfaltigkeit mithilfe des Satzes von Seifert und van Kampen als $\mathbb{Z}_2$ berechnen, schließen wir mithilfe der Elliptisierungsvermutung, dass die fragliche 3-Mannigfaltigkeit homömorph zu $SO(3)$ ist, so dass die Photonenregion im Phasenraum der Kerr-Raumzeit Topologie $L(2;1) \times \mathbb{R}^2 \approx SO(3) \times \mathbb{R}^2$ besitzt.

2 Overview and contributions

Chapter 3 contains an introduction to General Relativity, focussing on concepts that are relevant for the research in this work. All of the material presented herein is well-known and can be found in the standard literature, e.g. [66, 45, 16, 41].

In Chapter 4, one of our results (a static photon sphere uniqueness theorem for higher-dimensional electrostatic spacetimes) is presented and proven. The content of this section was submitted to Classical and Quantum Gravity on July 2nd, 2019 [33]. All parts of the work are my own, but I am indebted to Carla Cederbaum for proposing this problem and for helpful discussions.

Chapter 5 covers our results for trapping of light in the subcritical Kerr spacetime. The content of this section was accepted for publication by General Relativity and Gravitation [13], co-authored with Carla Cederbaum. As it is usual in mathematics, the authors of the manuscript [13] are listed alphabetically.

Carla Cederbaum and I jointly discussed all of the statements and proof methods in the first part of the publication [13] (where we prove that the set of photons in the Kerr spacetime can be viewed as a submanifold of the phase space). The idea to make use of the specific topological methods in the second part (wherein the topology of the photon region in the phase space is determined) as well as the details of the proof are due to me. Literature research and the technical calculations were done by me; the paper-writing is estimated to be 90% by me and 10% by Carla Cederbaum. The categories "data generation" and "analysis and interpretation" do not apply. The images in [13] (and in Chapter 5) were generated by Oliver Schön.

3 An introduction to General Relativity

3.1 An introduction to spacetimes

General Relativity is a theory of gravitation, describing the structure of the universe on a large scale. Mathematically, it is a geometric theory: its objects are $n+1$-dimensional smooth manifolds $\mathfrak{M}^{n+1}$ carrying a smooth Lorentzian metric $\mathfrak{g}$. We choose the convention that the metric be of signature $(-,+,\dots,+)$, and its components are indexed with latin letters running from 0 to n. Furthermore, in this introduction we will denote the Levi-Civita connection of $\mathfrak{g}$ by ${}^{\mathfrak{g}}\nabla$. The case $n=3$ is the most relevant one, since the world typically presents itself to us as possessing 3 spatial dimensions and 1 temporal dimension. However, there has also been a lot of interest in the higher-dimensional case $n>3$, mainly sparked by the rise of string theory (see e.g. [43]).

The special case $(\mathfrak{M}^{n+1}=\mathbb{R}^{n+1}, \mathfrak{g}=\eta)$ with

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{E}_n \end{pmatrix}$$

(called the $n+1$-*dimensional Minkowski space*) is the setting of Special Relativity, where no gravitational forces curve the universe.

For any $x \in \mathfrak{M}^{n+1}$, a vector $v \in T_x\mathfrak{M}^{n+1}$ is called *timelike*(*lightlike, spacelike, causal*) if $\mathfrak{g}(v,v)<0$ ($\mathfrak{g}(v,v)=0$, $\mathfrak{g}(v,v)>0$, $\mathfrak{g}(v,v)\geq 0$); and this terminology extends in a straightforward way to vector fields. A global timelike vector field X on $\mathfrak{M}^{n+1}$ can be thought of as indicating the direction of time at any point in $\mathfrak{M}^{n+1}$ and is called a *time orientation* of $(\mathfrak{M}^{n+1}, \mathfrak{g})$; and a Lorentzian manifold $(\mathfrak{M}^{n+1}, \mathfrak{g})$ with a time orientation is called time-oriented. The choice of a time orientation W determines the notions of past and future in $(\mathfrak{M}^{n+1}, \mathfrak{g})$: a causal vector $v \in T_x\mathfrak{M}^{n+1}$ is called *future-directed* if $\mathfrak{g}(v, W_x)<0$ (and *past-directed* otherwise).

The movement of light and of matter particles in a time-oriented Lorentzian manifold are modelled in the following way: Matter particles move on *timelike curves*, that is, on smooth paths $\gamma : \mathbb{R} \supseteq I \to \mathfrak{M}^{n+1}$ whose every tangential vector is timelike. The particle is unaccelerated if the timelike curve is a geodesic.

Light, on the other hand, moves along *lightlike* curves, that is, along smooth paths $\gamma : \mathbb{R} \supseteq I \to \mathfrak{M}^{n+1}$ whose every tangential vector is lightlike. Lightlike geodesics will also be called *photons.* (Note that this terminology is used with no quantization in mind.) Curves with timelike or lightlike tangential vectors are called *causal.* It is usual to restrict attention to future-directed causal curves, that is, to those causal curves γ for which $\dot{\gamma}(s)$ is future-directed for one (hence, every) $s \in I$.

A submanifold of $\mathfrak{M}^{n+1}$ is called *timelike* (resp. *spacelike*) if its bilinear form induced by $\mathfrak{g}$ has signature $(-,+\dots,+)$ (resp. $(+\dots,+)$. It is called *lightlike* if the induced bilinear form is degenerate. The spacelike submanifolds of codimension 1 play an important role: they model the space and time "at a given point in time" and are called *spatial slices* of $\mathfrak{M}^{n+1}$. We will also use the term *null* interchangably with "lightlike".

General Relativity relates curvature quantities of a Lorentzian manifold to the matter that may be present in it: the Einstein equations (without a cosmological constant) give this relationship as

$$\mathfrak{Ric} - \frac{1}{2}\mathfrak{R}\mathfrak{g} = 8\pi\mathfrak{T}. \tag{1}$$

Here, $\mathfrak{Ric}$ is the *Ricci tensor* of $(\mathfrak{M}^{n+1}, \mathfrak{g})$, where we use the sign convention that $\mathfrak{Ric}$ is the trace of the Riemannian curvature tensor in the contravariant and the third covariant component, with the *Riemannian curvature tensor* $\mathfrak{Rm}$ defined as the $(3,1)$-tensor

$$\mathfrak{Rm}(Z, X, Y) = {}^{\mathfrak{g}}\nabla^2(X, Y)Z = {}^{\mathfrak{g}}\nabla_X {}^{\mathfrak{g}}\nabla_Y Z - {}^{\mathfrak{g}}\nabla_Y {}^{\mathfrak{g}}\nabla_X Z - {}^{\mathfrak{g}}\nabla_{[X,Y]} Z.$$

The trace $\mathfrak{R}$ of the Ricci tensor is the *scalar curvature* of $(\mathfrak{M}^{n+1}, \mathfrak{g})$.

The right-hand side $\mathfrak{T}$ of Equation (1) is the *energy-momentum tensor*; it describes the matter that is present in the model. In stating the above Einstein equations (1), we chose geometric units by setting the speed of light and the gravitational constant to 1.

The set of partial differential equations (1) are derived by applying the principle of least action to the so-called Einstein–Hilbert action (see e.g. [66, 41]).

We can now define a *spacetime* as a smooth, time-oriented Lorentzian manifold which fulfills the Einstein equations (1). Some examples of spacetimes will be given below. While some spacetimes try to model the universe as a whole and are therefore called *cosmological spacetimes*, others attempt to describe objects such as stars, binary systems of stars, or black holes. In the present work, we will only study member of this latter class of spacetimes (see Section 3.4). One may also classify spacetimes according to the matter they contain, see the following Section 3.2.

It is possible to include a *cosmological constant* $\Lambda \in \mathbb{R}$ in the Einstein equations, which represents the vacuum energy of the spacetime; with a cosmological constant, the Einstein equations take the more general form

$$\mathfrak{Ric} - \frac{1}{2}\mathfrak{R}\mathfrak{g} + \Lambda\mathfrak{g} = 8\pi\mathfrak{T}. \tag{2}$$

In this work, we will only be in the setting of vanishing cosmological constant $\Lambda = 0$, with the exception of the remarks at the end of Chapter 5.

3.2 Some matter models and important examples

The energy-momentum tensor $\mathfrak{T}$ on the right-hand side of the Einstein equations (1) is a priori not further detailed; different forms of $\mathfrak{T}$ model different kinds of matter that may be present in a spacetime. It is usual to require at least some "energy conditions" on $\mathfrak{T}$, most prominently the *dominant energy condition* (DEC), which requires that for every timelike or lightlike future-pointing vector field V, the vector field corresponding to the 1-form $-\mathfrak{T}(V, \cdot)$ must be future-pointing timelike or lightlike. The DEC implies the *null energy condition* (NEC), which stipulates that for every future-pointing null vector field V, always $\mathfrak{T}(V, V) \geq 0$.

The simplest matter model is *vacuum*, that is, $\mathfrak{T} = 0$. A part of the present work studies properties of a certain vacuum spacetime family, the *Kerr spacetimes*: the metric of the ($3 + 1$-dimensional) Kerr spacetime of mass m and angular momentum a is given in *Boyer–Lindquist coordinates* $(t, r, \vartheta, \varphi)$ on a patch of $\mathbb{R}^4$ with suitably large radial coordinate r as

$$-\left(1 - \frac{2mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{r^2 - 2mr + a^2} dr^2 + \rho^2 d\vartheta^2 - \frac{4mra \sin^2 \vartheta^2}{\rho^2} dt d\varphi + \frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2 \vartheta}{\rho^2} S^2 d\varphi^2$$

with $\rho^2 := r^2 + a^2 \cos^2 \vartheta$.

If the modulus of the angular momentum parameter a is smaller than the mass parameter m, the Kerr spacetime is called *subcritical*; if $|a| = m$, the Kerr spacetime is called *critical*. The *supercritical* case $|a| > m$ is thought to be physically less relevant since it violates cosmic censorship: the ring-shaped singularity at $\rho^2 = 0$ (where curvature blows up) is not hidden behind any horizon in this case. (For a discussion of horizons, see Section 3.7). By a coordinate change, one may always assume that the rotational parameter a be nonnegative.

The special case of vanishing rotational parameter $a = 0$ is known as the family of *Schwarzschild spacetimes* (and the coordinates in which the metric is given above are then called *Schwarzschild coordinates*); if moreover the mass parameter m vanishes, we recover the Minkowski spacetime. There are higher-dimensinal equivalents of Kerr spacetimes, the so-called Myers–Perry solutions [44, 43], but we will only mention them in passing in this work.

For more details about the Kerr family, see Chapter 5 or the standard reference [46].

Another important matter model is *electrovacuum* (see e.g. [66]), where the energy-momentum tensor takes the form

$$\mathfrak{T}_{ab} = \frac{1}{4\pi} \left(F_{aj} F_b{}^j - \frac{1}{4} \mathfrak{g}_{ab} F^{kl} F_{kl} \right). \tag{3}$$

Here, F is the electromagnetic field tensor which must satisfy Maxwell's equations. In the presence of an electric potential Ψ, it can be expressed as

$$F_{ab} = {}^{\mathfrak{g}}\nabla_a d\Psi_b - {}^{\mathfrak{g}}\nabla_b d\Psi_a.$$

An example of an electrovacuum space that will play a prominent role in the present work are the Reissner–Nordström spacetimes: the $n+1$-dimensional *Reissner–Nordström spacetime* of mass m and charge q is the manifold $\mathbb{R} \times \mathbb{R}^n \setminus \{0\}$, with the metric

$$-(1 - \tfrac{2m}{r^{n-2}} + \tfrac{q^2}{r^{2(n-2)}})dt^2 + (1 - \tfrac{2m}{r^{n-2}} + \tfrac{q^2}{r^{2(n-2)}})^{-1}dr^2 + r^2\Omega_{n-1}, \tag{4}$$

where Ω_{n-1} denotes the standard metric on $\mathbb{S}^{n-1}$. For vanishing charge parameter q, the Reissner–Nordström family reduces to the Schwarzschild family. If the modulus of the charge parameter q is smaller than the mass parameter m, the Reissner–Nordström spacetime is called *subextremal*; the *superextremal* case $|q| > m$ is, again, physically less relevant by violation of cosmic censorship.

For more details about the Reissner–Nordström family, see e.g. Chapter 4.

3.3 $n+1$ decomposition

It can be useful to leave the spacetime picture and to switch to a Riemannian framework. That is, one considers a spacelike submanifold (or *spatial slice*) (M^n, g) of a spacetime $(\mathfrak{M}^{n+1}, \mathfrak{g})$ and studies its geometry. A standard formula for submanifold geometry known as the Gauss–Codazzi–Mainardi–equation (see e.g. [36, 45]) states that for a frame $\{E_I\}$ of M^n,

$$\begin{aligned}\mathfrak{Rm}^\flat(E_I, E_J, E_K, E_L) = \mathrm{Rm}^\flat(E_I, E_J, E_K, E_L) &+ \mathrm{II}(E_I, E_L)\mathrm{II}(E_J, E_L)\\ &- \mathrm{II}(E_I, E_K)\mathrm{II}(E_J, E_L),\end{aligned}$$

where II is the mean curvature of M^n in $\mathfrak{M}^{n+1}$, Rm is the Riemannian curvature tensor of M^n, and the superscript $\flat$ denotes a contravariant-to-covariant type change. Applied to the Einstein equations (1) and writing R for the scalar curvature of M^n and H for its mean curvature in $\mathfrak{M}^{n+1}$, this yields the *constraint equations*

$$\mathrm{R} + H^2 - |\mathrm{II}|_g^2 = 16\mu \qquad \text{and} \tag{5}$$

$$\mathrm{div}_{M^n}(\mathrm{II} - Hg) = -8\pi J, \tag{6}$$

where the *enery density* μ and the *momentum density* J are defined for a unit normal

η to M^n as

$$\mu := \mathfrak{T}(\eta, \eta),$$
$$J := -\mathfrak{T}(\eta, \cdot\,|_{TM^n}).$$

It does indeed make sense to study General Relativity in terms of such spatial hypersurfaces, since—given a Riemannian manifold (M^n, g) that fulfills the constraint equations—a theorem by Choquet-Bruhat ensures the existence (and, under suitable additional requirements, uniqueness) of a spacetime of which M^n is a spacial slice with geometric quantities as in the Equations (5)–(6) (for a more precise formulation, see [24]).

If M^n is totally geodesic in $\mathfrak{M}^{n+1}$, it is called *time symmetric*. In that case, the dominant energy condition can be reformulated as

$$R \geq 0.$$

3.4 Isolated systems

While cosmological models are designed to model the universe as a whole, other spacetimes are models for objects on a smaller scale. They describe phenomena such as stars or black holes. To study such objects, it is convenient to consider them as isolated systems; that is, one models them by spacetimes which resemble the empty, flat Minkowski space $(\mathbb{R}^{n+1}, \eta)$ in a region "far out", but might possess curvature and nontrivial topology close to the "center". Mathematically, this intuition is captured by the definition of an asymptotically flat spacetime: an $n+1$-dimensional spacetime $(\mathfrak{M}^{n+1}, \mathfrak{g})$ is called *asymptotically flat* if it contains a spacelike hypersurface (M^n, g, II) (g the induced metric and II the second fundamental form) which "closely resembles Euclidean space" outside a compact set K. More precisely, it is usual to require that there is an open ball $B^n_S(0)$ and a diffeomorphism

$$\Phi = (x^i) : M^n \setminus K \to \mathbb{R}^n \setminus \overline{B^n_S(0)}$$

such that

1. on $M^n \setminus B^n_S(0)$ one has $\Phi_* g_{ij} = \delta_{ij} + a_{ij}$, where where δ_{ij} are the components of the Euclidean metric and a_{ij} fulfill suitable fall-off conditions, and

2. the components of the second fundamental form of fulfill suitable fall-off conditions on $M^n \setminus B^n_S(0)$,

see e.g. [16]. The region $M^n \setminus K$ is called an *asymptotic end.* Note also that sometimes it is required that in addition the scalar curvature be integrable on M^n.

We do not go into detail here since we will, in fact, use the exact asymtotic behavior only in the case that (M^n, g) is *asymptotically Reissner–Nordström* (see Definition 9); and these conditions entail all usual notions of asymptotic flatness. For a precise definition of asymptotic flatness, see e.g. [16].

It is also usual to call a Riemannian manifold (M^n, g) asymptotically flat if it fulfills the conditions that we required from the spatial slice above, ignoring of course the condition on the second fundamental form.

3.5 ADM mass and the positive mass theorem

There are several notions of the mass of a system in General Relativity, some of them quasilocal, others global. We introduce a global concept of mass that will be used in Chapter 4; it is known as the ADM mass (after Arnowitt, Deser, and Misner) [2]. It is defined for asymptotically flat Riemannian manifolds (M^n, g) (with integrable scalar curvature) as

$$\frac{1}{16\pi} \lim_{r\to\infty} \sum_{i,j=1}^{n} \int_{\mathbb{S}_r} (g_{ij,j} - g_{jj,i})\nu^i \, d\sigma_r,$$

where the metric components are the ones induces by the chart Φ from the definition of asymptotic flatnass, ν is the unit normal to the sphere $\mathbb{S}_r$ of coordinate radius r pointing towards the asymptotic end, and $d\sigma_r$ is the induced volume element on $\mathbb{S}_r$.

In the context of General Relativity, one may think of the Riemannian manifold (M^n, g) in the definition of the ADM mass as of a spatial slice in a spacetime; and the concept of the ADM mass is meant to capture the mass that is contained in the universe. To justify that the ADM mass is a physically meaningful quantity which deserves the name mass, some sanity checks are required:

It can be verified by standard calculations that the ADM mass is finite. It was moreover shown in [5] and in [14] that the ADM mass does not depend on the choice of the diffeomeorphism Φ, as long as Φ is as required in the definition of asymptotic flatness. For manifolds which are asymptotically Reissner–Nordström (see Definition 9) and in particular for Reissner–Nordström manifolds themselves, the ADM mass coincides with the mass parameter m from the asymptotics, as can be checked by direct computations.

One wishes moreover for nonnegativity of the ADM mass in physically relevant settings. A physically relevant setting is that (M^n, g) is an embedded, totally geodesic submanifold of a spacetime (that is, a time symmetric spatial slice), where the spacetime fulfills the dominant energy condition. As mentioned in Section 3.3, the dominant energy condition translates to nonnegativity of the scalar curvature of the time symmetric slice.

Therefore, the following theorem known as the *positive mass theorem* is of utmost importance:

If (M^n, g) is an asymptotically flat, geodesically complete Riemannian manifold with nonnegative scalar curvature, then its ADM mass is nonnegative, and it is zero if and only if (M^n, g) is Euclidean space.

This was first shown by Schoen and Yau for the 3-dimensional case in [56] and extended to dimensions up to 7 in [57]; Witten generalized it to arbitrary dimensions under an additional spin assumption ([67], see also [48]). The theorem was established in its full generality only recently in [58]. Recently, also low regularity versions of the theorem were shown (e.g. [35, 40]), see also Section 4.5.3.

For an accessible sketch of the proof of the positive mass theorem in the 3-dimensional case, see the presentation in [7], where one may also find an explanation of the heuristics behind the definition of the ADM mass via analogy to the Newtonian setting.

3.6 Stationary and static spacetimes

Stationarity of a spacetimes expresses the idea that the spacetime does not change in time. More precisely, an asymptotically flat spacetime $(\mathfrak{M}^{n+1}, \mathfrak{g})$ with a spatial submanifold M^n as in the definition of asymptotically flat spacetimes is called *stationary* if there exists a nowhere vanishing, complete Killing vector field on $\mathfrak{M}^{n+1}$ (often called the *stationary Killing vector field* of $\mathfrak{M}^{n+1}$) which is timelike in the asymptotic end (see [16]); it is usually (and often tacitly) even assumed that the norm of the Killing vector field is bounded away from zero in the asymptotic end.

In the presence of matter (if the energy-stress tensor is not identically zero), one requires additionally that the matter be compatible with the stationary symmetry; in the case of electrovacuum, this means (more precisely) that the electromagnetic field tensor F must be invariant under the flow of the stationary Killing vector field.

The Kerr family and the Reissner–Nordström family are examples of stationary spacetimes, and the stationary Killing vector field is in both cases the coordinate vector field ∂_t, as can be seen using the following local characterization:

A stationary spacetime $(\mathfrak{M}^{n+1}, \mathfrak{g})$ may be decomposed as $\mathbb{R} \times M^n$; and one chooses a coordinates t on $\mathbb{R}$ such that the stationary Killing vector fields is given as the coordinate vector field ∂_t. The spacetime $(\mathfrak{M}^{n+1}, \mathfrak{g})$ is thus foliated by a family of spatial hypersurfaces $\{M^n_t\}$, which are isometric to one another by the family of isometries generated by the stationary Killing vector field. Denoting by η a unit normal to the hypersurfaces M^n_t, one may define the *lapse* or *lapse function* by $\mathfrak{g}(\eta, \partial_t)$, and the *shift* by $\mathfrak{g}(\partial_t, \cdot)$. Wherever the stationary Killing field is timelike, the metric takes with respect to local coordinates (x_i) $(1 \leq i \leq n)$ on M^n the form

$$\mathfrak{g} = -N^2 \left(dt + h_i dx^i\right)^2 + g,$$

where the smooth functions N and $h_i = \mathfrak{g}(\partial_t, \partial_i)$ $(1 \leq i \leq n)$ do not depend on t, and g is a (spacelike) metric on M^n. (Conversely, a spacetime with a metric of such a form is always stationary, provided neither N nor g depend on t.)

Such a decomposition is always possible in the *exterior region* (the development of the asymptotic end $M^n \setminus K$ under the flow of the stationary Killing vector field).

If in a stationary spacetime $(\mathfrak{M}^{n+1}, \mathfrak{g})$ additionally the stationary Killing vector field is hypersurface orthogonal (or, equivalently, integrable in the Frobenius sense), $(\mathfrak{M}^{n+1}, \mathfrak{g})$ is called *static*, which expresses the idea that there is no rotation present in the spacetime. Some examples of static spacetimes are the members of the Reissner–Nordström family, including, of course, the Schwarzschild family, as well as in particular the Minkowski spacetime. The members of the Kerr family, on the other hand, are not static (provided that $a \neq 0$).

Locally (and where the stationary Killing vector field is timelike), one may express staticity of the metric $\mathfrak{g}$ by requiring that the shift vanishes, that is, the metric $\mathfrak{g}$ is locally of the form

$$\mathfrak{g} = -N^2 dt^2 + g,$$

and in that case the lapse function coincides with the function N. (Conversely, a spacetime with a metric of such a form is always static.)

For a static spacetime, one may express the Einstein equations with a electrovacuum tensor energy-stress tensor (that is, with an energy-stress tensor of the form given in Equation (3)) in terms of quantities on M^n: if there is an potential $\Psi : M^n \to \mathbb{R}$, the Einstein equations take the form

$$\begin{aligned} \Delta N &= 2\frac{n-2}{n-1}\frac{1}{N}|d\Psi|^2, \\ 0 &= \operatorname{div}\left(\frac{\operatorname{grad}\Psi}{N}\right), \\ N\operatorname{Ric} &= \nabla^2 N - 2\frac{d\Psi \otimes d\Psi}{N} + \frac{2}{(n-1)N}|d\Psi|^2 g. \end{aligned}$$

Here, Ric and R denote the Ricci tensor and the scalar curvature of (M^n, g). Such a quadruple (M^n, g, N, Ψ) is called an *electrostatic system*. Conversely, if the above equations are fulfilled on $M^n \setminus K$, then

$$\left(\mathbb{R} \times (M^n \setminus K), -N^2 dt^2 + g\right)$$

fulfills the Einstein equations with an electromagnetic energy-stress tensor as the right-hand side.

3.7 Black holes

Roughly speaking, a black hole is a spacetime region out of which no object, be it a matter particle or light, can escape. While the black hole definition for general spacetimes is rather involved and can, for example, be phrased with the vocabulary of the conformal compatification of a spacetime (see [66]), the definition of black holes in stationary spacetimes is easier to formulate and to handle: Let $(\mathfrak{M}^{n+1}, \mathfrak{g})$ be a stationary spacetime, and let M_{ext} denote its exterior region (see Section 3.6). The *domain of outer communication* (abbreviated as DOC) of $\mathfrak{M}^{n+1}$ is the spacetime region

$$\langle\langle M_{\text{ext}} \rangle\rangle$$

that can be reached from the exterior region M_{ext} by a future-directed timelike curve as well as by a past-directed timelike curve; in other words, a point $x \in \mathfrak{M}^{n+1}$ is in the DOC if and only if there is a future-directed timelike curve

$$\gamma_1 : [0,1] \to \mathfrak{M}^{n+1}$$

and a past-directed timelike curve

$$\gamma_2 : [0,1] \to \mathfrak{M}^{n+1}$$

with $\gamma_i(0) \in M_{\text{ext}}$ and $\gamma_i(1) = x$ for $i = 1, 2$. The spacetime region

$$\mathfrak{M}^{n+1} \setminus \overline{\langle\langle M_{\text{ext}} \rangle\rangle}$$

is called the *black hole* or *black hole region*, and its boundary $\partial\langle\langle M_{\text{ext}} \rangle\rangle$ the *event horizon*.

In a Kerr spacetime of mass m and angular momentum a with $|a| < m$, the hypersurface $\{r^2 - 2mr + a^2 = 0\}$ is an event horizon; it is semi-permeable in the sense that there is no future-directed causal curve from the region $\{r^2 - 2mr + a^2 < 0\}$ to the region $\{r^2 - 2mr + a^2 > 0\}$.

In a Reissner–Nordström spacetime of mass m and charge q with $|q| < m$, the null hypersurface

$$\left\{ r = \left(m + \sqrt{m^2 - q^2} \right)^{\frac{1}{n-2}} \right\}$$

(given in the coordinates of (4)) is an event horizon, the *outer horizon*.

The notion of an event horizon is a global one; it is hard to check since its verification requires knowledge of the whole spacetime. There are other concepts of horizons that are easier to handle because they rely on quasilocal properties; one of them, which is useful in the static setting, is the concept of a static horizon.

For a static spacetime $(\mathbb{R} \times M^n, -N^2dt^2 + g)$, an $n-1$-dimensional submanifold $\Sigma^{n-1} \subseteq M^n$ is called a *static horizon* if the lapse N vanishes on Σ^{n-1}, and it is called *nondegenerate* if moreover the outer normal derivative of the lapse is positive.

In the above example of the subextremal and extremal Reissner–Nordström spacetimes, the event horizon is a static horizon. One can verify by a straightforward calculation (after extending the lapse to the horizon) that it is degenerate if the Reissner–Nordström spacetime is extremal, and nondegenerate otherwise.

Static horizons are interesting and useful by virtue of their quasilocal properties; for instance, it is standard knowledge that they are minimal surfaces. This is true without any assumptions on the energy-stress tensor and can, for instance, be checked by computing the Kretschmann scalar of the spacetime (see e.g. [34]).

3.8 Trapped light

In the vicinity of heavy objects like black holes or neutron stars, the curvature of the spacetime can be so large that some light rays orbit the central object forever at a fixed distance, without escaping towards infinity nor falling towards the central object. Since it is very imprecise to speak of a "fixed distance" (which is at the outset only defined in a possibly not very meaningful coordinate sense), one needs a more precise definition for such phenomena: in a static spacetime $(\mathfrak{M}^{n+1}, \mathfrak{g})$ with lapse function N, a photon $\gamma : \mathbb{R} \to \mathfrak{M}^{n+1}$ is called *trapped* if $N \circ \gamma =$ *const.*. In the case of radially symmetric static spacetimes like the Schwarzschild and Reissner–Nordström families, constancy of the lapse along γ is, of course, equivalent to constancy of the radial coordinate along γ.

The easiest case of trapped light occurs in the Schwarzschild spacetime (of mass m), where every photon γ with $\gamma(0) = 3m$ and $\dot{\gamma}(0) = 0$ has constant radial coordinate $r = 3m$ throughout (in the coordinates of (4)). This means that the cylinder with spherical base $\mathbb{R} \times \{r = 3m\}$ has the property that every photon that is once tangent to is remains tangent for as long as it exists. Remarkably, the set $\mathbb{R} \times \{r = 3m\}$, where trapped photons may be found, is a smooth submanifold of the spacetime.

For the case of stationary, non-static spacetimes, constancy of the lapse function does not seem to be the right way to define trapping: in the subcritical Kerr spacetimes with nonvanishing rotational parameter a, there are no photons of constant lapse function (as can be seen, for example, from Proposition 27), but instead there are photons of constant Boyer–Lindquist coordinate r (see Chapter 5). In view of this, we will define trapped light for stationary spacetimes in terms of the spacetime symmetry and a topological condition (see Definition 26), and our definition entails that the lapse is bounded along any trapped photon.

Other than being interesting in its own right, understanding trapping of light is also relevant in a variety of contexts: this phenomenon plays a role in the analysis of stability of black holes (see e.g. the lecture notes [18]), and since our Definition 26 entails that the lapse is bounded along any trapped photon, it is workable for the stability context, where scattering of the photon's energy is relevant.

Of current interest is also the role that trapped light plays in gravitational lensing (see e.g. the review [52]) and in the understanding of black hole shadows (see e.g. [27]), two concepts that are closely related to possible observations indicating black holes in our universe; see, for example, the pictures recently taken by the Event Horizon Telescope [23].

4 Photon sphere uniqueness in higher-dimensional electrovacuum spacetimes

Recall that the $n+1$-dimensional Reissner–Nordström spacetimes are a 2-parameter family (labelled with a *mass* m and a *charge* q) of static, spherically symmetric, electrically charged, asymptotically flat solutions to the Einstein equations, and a Reissner–Nordström spacetime is called *subextremal* (*extremal, superextremal*) if $m^2 > q^2$ (if $m^2 = q^2$, $m^2 < q^2$).

A subextremal or extremal Reissner–Nordström spacetime contains a unique photon sphere, on which light can get trapped (for a precise definition of photon spheres, see Definition 13), while a superextremal Reissner–Nordström spacetime may contain two photon spheres or none, depending on the mass-charge ratio.

Subextremality in the Reissner–Nordström family is equivalent to a quasilocal subextremality condition on the photon sphere (see Definition 19).

We establish the following uniqueness result for subextremal Reissner–Nordström spacetimes:

Theorem 1. *Let (M^n, g, N, Ψ) be an asymptotically Reissner–Nordström electrostatic system of mass m and charge q and with $n \geq 3$, such that M^n has a (possibly disconnected) compact photon sphere as an inner boundary with only subextremal connected components Σ_i^{n-1}, $1 \leq i \leq l$, $l \in \mathbb{N}$. Assume moreover that*

1. *$N \restriction_{\partial M^n} > 0$, and*

2. *R_{σ_i} (the scalar curvature of Σ_i^{n-1} with respect to the induced metric) is constant.*

Then (M^n, g) is isometric to the Reissner–Nordström manifold of mass m and charge q, and $m > |q|$. In particular, ∂M^n is the photon sphere in the Reissner–Nordström manifold of mass m and charge q, it has only one connected component and is a topological sphere.

The above theorem is implied by the more general Theorem 3 below. First, we will comment on some of the assumptions of Theorem 1:

Remark 2. It is natural to require that $N \restriction_{\partial M^n} \geq 0$, meaning that none of the photon sphere component are inside a black hole. Moreover, we will see in Proposition 17 how the scalar curvature of Σ_i^{n-1} relates to the normal derivative of Ψ on Σ_i^{n-1}, and in view of this relation, it is possible to replace the constant scalar curvature condition by the requirement that $|d\Psi|$ be constant on each Σ_i^{n-1}. If ∂M^n has only one connected component, the condition $R_{\sigma_i} = const.$ does not need to be assumed but is fulfilled automatically, as has been argued in [31], see also the exposition in [30]: in this case, one can show how Ψ can be written as a function of N. Since we show in Proposition 17 that $|dN|$ is constant on every connected component of the photon sphere, this would imply that $|d\Psi|$ and hence R_{σ_i} are also constant. However, this is only possible in case the photon sphere has only one connected component. Lastly, it is not possible to do away with the quasilocal subextremality condition on the photon sphere; since a superextremal Reissner–Nordström manifold may contain a photon sphere, it may fulfill the assumptions of Theorem 1 except for the subextremality condition, but the proof of Theorem 1 must break down in this case due to the absence of a horizon.

The assumptions of Theorem 1 may be considerably weakened: First, it is possible to allow for static horizon components as inner boundary components. Second, one may wish to replace the electrostatic equation for the Ricci tensor (Equation (12)) with the much weaker inequality $N^2 R \geq 2|d\Psi|^2$. We will refer to objects fulfilling this inequality and the other two electrostatic equations (10) and (11) as *pseudo-electrostatic*, see Section 4.1.2.

Since photon spheres in electrostatic settings are characterized by quasilocal geometry, it is useful to define in the pseudo-electrostatic setting the notion of a *quasilocal photon sphere* as a hypersurface characterized by certain quasilocal properties. We will see in Propositions 16, 18, and 17 that photon spheres in electrostatic systems are always quasilocal photon spheres; hence, the following is a generalization of Theorem 1:

Theorem 3. *Let (M^n, g, N, Ψ) be a pseudo-electrostatic system which is asymptotically Reissner–Nordström of mass m and charge q, and $n \geq 3$.*

Assume M^n has an orientable, compact inner boundary whose connected components are either nondegenerate static horizons or subextremal quasilocal photon spheres.

Then (M^n, g) is isometric to a piece of the Reissner–Nordström manifold of mass m and charge q, and $m > |q|$.

From Theorem 3, we also immediately get black hole uniqueness in higher-dimensional static asymptotically Reissner–Nordström electrovacuum:

Corollary 4. *Let (M^n, g, N, Ψ) be (pseudo-)electrostatic and asymptotically Reissner–Nordström (with mass m and charge q). Assume that ∂M^n is a (possibly disconnected) nondegenerate static horizon. Then (M^n, g) is isometric to the region of Reissner–Nordström manifold of mass m and charge q which is outside the horizon, and $m > |q|$.*

This corollary is also a result of [34], where it was proven under the additional assumption that $m > |q|$.

The above black hole uniqueness result actually does not require the full electrostatic equations; the pseudo-electrostatic conditions are sufficient (in contrast to [34] and most other black hole uniqueness proofs, where the full (electro-)static equations are explicitly required, with the exception of [8]).

The proof of Theorem 3 uses seminal ideas from the classical black hole uniqueness proofs by Bunting and Masood-ul Alam [6] and by Ruback [55], which were generalized in [12] and [11] to prove photon sphere uniqueness results, as well as on the techniques that were developed in [8] (see also [9]) to treat higher-dimensional cases. Its main part breaks down into the following steps: in the first step, we glue to each boundary component an explicitly constructed Riemannian manifold resembling a suitable piece of a Reissner–Nordström manifold up to a static horizon. The horizon allows to reflect the manifold in a second step along its boundary, obtaining an "upper" and a "lower" half. Both the gluing and the doubling can be done with $C^{1,1}$-regularity.

In a third step, we perform a conformal change of the doubled manifold such that the conformally transformed upper half has vanishing ADM mass, and the conformally transformed lower half can be one-point compactified (with $C^{1,1}$-regularity). This will allow to apply a low regularity version of the rigidity case of the positive mass theorem to conclude that the conformally transformed manifold is the Euclidean space. In a fourth and last step, uniqueness will be established through some topological arguments and by recovering the conformal factor applying a maximum principle to an elliptic PDE.

The chapter is organized as follows: In Section 4.1, we recall some definitions and known facts about the $n+1$-dimensional Reissner–Nordström spacetime and about asymptotically flat, static electrovacuum spacetimes in general and introduce our asymptotic assumptions. In Section 4.2, we give a definition of photon spheres that is adjusted to our setting and prove some statements about their geometry, which are interesting in their own right. Section 4.3 provides the prerequisites for the conformal change we need to perform in the third step. In Section 4.4, we prove the assertions in Theorem 3 about mass and charge. Section 4.5 presents the above sketched four steps of the proof of Theorem 1.

4.1 Setup and definitions

4.1.1 The $n+1$-dimensional Reissner–Nordström spacetime

The $n+1$-dimensional Reissner–Nordström spacetime of mass m and charge q is the manifold $(\mathbb{R}\times\mathbb{R}^n\setminus\{0\},\mathfrak{g}_{m,q})$, where the metric $\mathfrak{g}_{m,q}$ is given by

$$\mathfrak{g}_{m,q} = -(1-\tfrac{2m}{r^{n-2}}+\tfrac{q^2}{r^{2(n-2)}})dt^2+(1-\tfrac{2m}{r^{n-2}}+\tfrac{q^2}{r^{2(n-2)}})^{-1}dr^2+r^2\Omega_{n-1}, \tag{7}$$

and Ω_{n-1} denotes the standard metric on $\mathbb{S}^{n-1}$. The n-dimensional (spatial) Reissner–Nordström manifold is a canonical spatial slice of the Reissner–Nordström spacetime, that is, the manifold $\mathbb{R}^n\setminus\{0\}$ with the metric

$$g_{m,q} = (1 - \tfrac{2m}{r^{n-2}} + \tfrac{q^2}{r^{2(n-2)}})^{-1} dr^2 + r^2 \Omega_{n-1}.$$

The lapse $N_{m,q}$ and the *potential* Ψ_q of the n-dimensional Reissner–Nordström manifold of mass m and charge q are the functions

$$\begin{aligned} N_{m,q}(r) &:= (1 - \tfrac{2m}{r^{n-2}} + \tfrac{q^2}{r^{2(n-2)}})^{1/2}, \\ \Psi_q(r) &:= \frac{q}{\widehat{C} r^{n-2}} \end{aligned}$$

with

$$\widehat{C} := \sqrt{2\frac{n-2}{n-1}}.$$

Remark 5. In isotropic coordinates, $g_{m,q}$ can be written as

$$g_{m,q} = \left(1 + \frac{m+q}{2s^{n-2}}\right)^{\frac{2}{n-2}} \left(1 + \frac{m-q}{2s^{n-2}}\right)^{\frac{2}{n-2}} \delta =: \varphi_{m,q}^{\frac{2}{n-2}} \delta, \tag{8}$$

where the radial coordinates s and r transform by the rule

$$r = s \left(1 + \frac{m+q}{2s^{n-2}}\right)^{\frac{1}{n-2}} \left(1 + \frac{m-q}{2s^{n-2}}\right)^{\frac{1}{n-2}}.$$

We can rewrite the lapse and the potential as

$$\begin{aligned} N_{m,q}(s) &= \frac{\left(1 - \frac{m^2-q^2}{4s^{2(n-2)}}\right)}{\left(1 + \frac{m+q}{2s^{n-2}}\right)\left(1 + \frac{m-q}{2s^{n-2}}\right)}, \\ \Psi_q(s) &= \frac{q}{\widehat{C} s^{n-2} \left(1 + \frac{m+q}{2s^{n-2}}\right)\left(1 + \frac{m-q}{2s^{n-2}}\right)}. \end{aligned}$$

A straightforward computation allows to express $\varphi_{m,q}$ in terms of $N_{m,q}$ and Ψ_q as

$$\varphi_{m,q} = \left(\frac{(N_{m,q}+1)^2 - \widehat{C}^2 \Psi_q^2}{4}\right)^{-1}. \tag{9}$$

In the coordinates of (7), the outer horizon of the Reissner-Nordström black hole of mass $m > 0$ and charge q with $m^2 > q^2$ is located at $\left(m + \sqrt{m^2 - q^2}\right)^{\frac{1}{n-2}}$. In isotropic coordinates, the location of the outer horizon is at $s_{m,q} := \left(\frac{m^2-q^2}{4}\right)^{\frac{1}{2(n-2)}}$.

4.1.2 (Pseudo-)electrostatic spacetimes

The above introduced $n+1$-dimensional Reissner–Nordström spacetime is the paradigmatic example of a static electrovacuum spacetime.

Since we will be working in n-dimensional spatial slices, it is more convenient to use the dimensionally reduced Einstein–Maxwell equations. We recall from Chapter 3 the following definition:

Definition 6. Let (M^n, g) be a Riemannian manifold and let $N : M^n \to \mathbb{R}_{>0}$, $\Psi : M^n \to \mathbb{R}$ be smooth functions such that

$$\Delta N = \frac{\widehat{C}^2}{N}|d\Psi|^2, \tag{10}$$

$$0 = \operatorname{div}\left(\frac{\operatorname{grad}\Psi}{N}\right), \tag{11}$$

$$N\operatorname{Ric} = \nabla^2 N - 2\frac{d\Psi \otimes d\Psi}{N} + \frac{2}{(n-1)N}|d\Psi|^2 g. \tag{12}$$

Then (M^n, g, N, Ψ) is called an *electrostatic system.*

Here and onwards, Ric and R denote the Ricci tensor and the scalar curvature of (M^n, g).

Taking the trace of Equation (12) and plugging in Equation (10), one obtains

$$N^2 R = 2|d\Psi|^2. \tag{13}$$

Definition 7. If (M^n, g, N, Ψ) fulfills Equations (10), (11), and (13), but not necessarily (12), it is called an *traced-electrostatic system.* If Equations (10), (11) and the inequality

$$N^2 R \geq 2|d\Psi|^2$$

are fulfilled, we say that the system is *pseudo-electrostatic.*

4.1.3 Asymptotic considerations

Remark 8 (Weighted norms)**.** We will use weighted norms defined as follows:

$$\|f\|_{C^2_{-k}(U)} := \sup_{x\in U} ||x|^k \cdot |f(x)| + |x|^{k+1} \cdot |Df(x)| + |x|^{k+2} \cdot |D^2 f(x)||$$

for a twice differentiable function f on an open domain $U \subseteq \mathbb{R}^n$.

We will use the following definition of asymptotically Reissner–Nordström manifolds:

Definition 9. A smooth Riemannian manifold (M^n, g) of dimension $n \geq 3$ is called *asymptotically Reissner-Nordström* of mass m and charge q if

1. M^n is diffeomorphic to $K \sqcup E$, where K is a compact set, and E is an *asymptotic end* which is diffeomorphic to $\mathbb{R}^n \setminus \overline{B^n_S(0)}$ for some $S > s_{m,q}$,

2. for the diffeomorphism $\Phi = (x^i) : E^n \to \mathbb{R}^n \setminus \overline{B^n_S(0)}$ and the metric g there is a constant $C > 0$ such that

$$\|(\Phi_* g)_{ij} - (g_{m,q})_{ij}\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B^n_S(0)})} \leq C, \; i, j = 1, \ldots, n,$$

3. $\Phi_* g$ is uniformly positive definite and uniformly continuous on $\mathbb{R}^n \setminus \overline{B^n_S(0)}$.

Here, $(g_{m,q})_{ij}$ are the components of the Reissner-Nordström metric in isotropic coordinates, see Equation (8).

Notation 10. We will often notationally omit Φ and Φ_* whenever this does not lead to ambiguity. Moreover, we use the coordinates (x^i) defined by Φ to define a radial coordinate $s := \sqrt{\sum_{i=1}^{n} |x^i|^2}$ on the asymptotic end.

Definition 11. Let (M^n, g) be an asymptotically Reissner–Nordström manifold of mass m and charge q with an asymptotic end E^n, and $\Phi = (x^i) : E^n \to \mathbb{R}^n \setminus \overline{B^n_S(0)}$ a diffeomeorphism as in Definition 9. A smooth function $N : M^n \to \mathbb{R}$ is called an *asymptotic Reissner–Nordström lapse (of mass m)* if there is a constant C such that

$$\|\Phi_* N - N_{m,q}\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B^n_S(0)})} \leq C$$

for some (hence, all) $q \in \mathbb{R}$.

A smooth function $\Psi : M^n \to \mathbb{R}$ is called an *asymptotic Reissner–Nordström potential (of charge q)* if there is a constant C such that

$$\|\Phi_* \Psi - \Psi_q\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B^n_S(0)})} \leq C.$$

A quadruple (M^n, g, N, Ψ) is called an *asymptotically Reissner–Nordström system* if (M^n, g) an asymptotically Reissner–Nordström manifold of mass m and charge q, $N : M^n \to \mathbb{R}$ is an asymptotic Reissner–Nordström lapse of the same mass m, and $\Psi : M^n \to \mathbb{R}$ is an asymptotic Reissner–Nordström potential of the same charge q.

4.2 Quasilocal geometry

We cite the following fundamental theorem:

Theorem 12 ([17, 53])**.** *A timelike hypersurface P in a Lorentzian manifold is totally umbilical if and only if every lightlike geodesic that is initially tangent to P stays tangent to P for as long as it exists.*

We remind the reader that a submanifold is called totally umbilical if the trace-free part of its second fundamental form vanishes.

The above theorem motivates the following definition, following [17, 68, 10]:

Definition 13. Let (M^n, g, N, Ψ) be a pseudo-electrostatic system (see Definition 7).

A timelike embedded orientable hypersurface P^n in $(\mathbb{R} \times M^n, -N^2dt^2 + g)$ is called a *photon sphere* if it is totally umbilical and N and Ψ are constant on every connected component of P^n.

It is immediate that $P^n \cap M^n$ is totally umbilical in M^n. If P^n is a photon sphere, we will occasionally also refer to $P^n \cap M^n$ as a photon sphere.

In the $n+1$-dimensional Reissner–Nordström spacetime with mass parameter $m > 0$, there is a photon sphere located at the radius

$$\left(\frac{1}{2}mn + \frac{1}{2}\sqrt{m^2n^2 - 4(n-1)q^2}\right)^{\frac{1}{n-2}}$$

(in the coordinates of (7)), provided that the mass-charge ratio is such that $m^2n^2 - 4(n-1)q^2$ is nonnegative (which is always the case in subextremal Reissner–Nordström spacetime), while a Reissner–Nordström spacetime of negative or vanishing mass does not possess a photon sphere.

For the rest of this chapter, we fix the following notation:

Notation 14. For a photon sphere $(P^n, p) \hookrightarrow (\mathbb{R} \times M^n, -N^2dt^2 + g)$ of a pseudo-electrostatic system (M^n, g, N, Ψ), we write

$$(P^n, p) = \bigcup_{i=1}^{l} (\mathbb{R} \times \Sigma_i^{n-1}, -N_i^2 dt^2 + \sigma_i),$$

where each $\mathbb{R} \times \Sigma_i^{n-1}$ is a connected component of P^n.

We define

$$N_i := N \restriction_{\Sigma_i^{n-1}},$$
$$\Psi_i := \Psi \restriction_{\Sigma_i^{n-1}}.$$

Moreover, $\mathfrak{H}$ denotes the mean curvature of $\bigcup_{i=1}^{l} \left(\mathbb{R} \times \Sigma_i^{n-1}\right)$, while H denotes the mean curvature of $\bigcup_{i=1}^{l} \Sigma_i^{n-1}$ in M^n, and we set

$$\mathfrak{H}_i := \mathfrak{H} \restriction_{\mathbb{R} \times \Sigma_i^{n-1}},$$
$$H_i := H \restriction_{\Sigma_i^{n-1}}.$$

A choice of unit normal to $\bigcup_{i=1}^{l} \Sigma_i^{n-1}$ in M^n (pointing towards the asymptotic end if (M^n, g) is asymptotically flat) will be denoted by ν, and we set

$$\nu(N)_i := \nu(N) \restriction_{\Sigma_i^{n-1}}$$
$$\nu(\Psi)_i := \nu(\Psi) \restriction_{\Sigma_i^{n-1}} .$$

Photon spheres in the electrostatic setting are characterized by quasilocal properties which make them a *quasilocal photon sphere* defined as follows:

Definition 15. A totally umbilical hypersurface in a (pseudo-)electrostatic system (M^n, g, N, Ψ) that fulfills $\nu(N)_i > 0$, $N_i = const.$, $R_{\sigma_i} = const. > 0$, $H_i = const.$ is called a *quasilocal photon sphere component* if the equations

$$R_{\sigma_i} = \frac{n}{n-1} H_i^2 + \frac{2}{N_i^2} \nu(\Psi)_i^2. \tag{14}$$

and

$$\frac{H_i}{N_i} \nu(N)_i = \frac{H_i^2}{n-1} \tag{15}$$

are fulfilled.

The following three propositions serve to show that a photon sphere component in an electrostatic system is a quasilocal photon sphere component. While Propositions 16 and 18 are straightforward generalizations from the $3+1$-dimensional setting (see [68] and [10] for Proposition 16 and [12] and [11] for Proposition 18), we will prove Proposition 17 for our setting. If $\Psi = 0$, the equations reduce to the ones given for the curvature quantities of photon spheres in [8]. For a similar proof in dimension 3, see also [68].

Proposition 16 ([68], [10])**.** *Let (M^n, g, N, Ψ) be an electrostatic system and P^n a photon sphere in $(\mathbb{R} \times M^n, \mathfrak{g} := -N^2 dt^2 + g)$ with induced metric p. Then for every $1 \leq i \leq l$, $\mathfrak{H}_i$ and H_i are constant.*

Proposition 17. *Let (M^n, g, N, Ψ) be an electrostatic system and let (P^n, p) be a photon sphere in $(\mathbb{R} \times M^n, -N^2 dt^2 + g)$.*

Then Equation (14) *holds. In particular, R_{σ_i} is nonnegative, and it is positive provided that $H_i \neq 0$.*

Moreover, Equation (15) *holds, and H_i and $\nu(N)_i$ are constant in this case. Moreover, in this case R_{σ_i} is constant if and only if $\nu(\Psi)_i$ is.*

Proof. First we show Formula (14). We write $\mathfrak{Ric}$ and $\mathfrak{R}$ for the Ricci tensor and the scalar curvature of $(\mathbb{R} \times M^n, -N^2 dt^2 + g)$ and choose $\eta := \frac{1}{N}\partial_t$ as a unit normal to

$\bigcup_{i=1}^{l} \Sigma_i^{n-1}$ in P^n. One calculates (applying a general formula for the curvature of warped products to $(\mathbb{R} \times M^n, -N^2 dt^2 + g)$, see e.g. [45], and using Equation (10)), that

$$\mathfrak{Ric}(\eta, \eta) = \frac{\Delta N}{N} = \frac{\widehat{C}^2}{N^2} |d\Psi|^2.$$

The traced Gauss equation and the fact that M^n is totally geodesic in $\mathbb{R} \times M^n$ give

$$\mathfrak{R} = R - 2\mathfrak{Ric}(\eta, \eta),$$

so that we arrive at

$$\mathfrak{R} = \frac{2}{N^2} |d\Psi|^2 \left(1 - \widehat{C}^2\right) \tag{16}$$

$$= \frac{2\nu(\Psi)^2}{N^2} \left(1 - \widehat{C}^2\right), \tag{17}$$

where we also used Equation (13) and the fact that Ψ is constant on P^n by definition of photon spheres.

The traced Gauss equation applied to $P^n \hookrightarrow \mathbb{R} \times M^n$ simplifies by umbilicity to

$$\mathfrak{R} - 2\mathfrak{Ric}(\nu, \nu) = R_P - \frac{n-1}{n} \mathfrak{H}^2, \tag{18}$$

where R_P denotes the scalar curvature of P^n, and we abused notation by denoting the unit normal to P^n in $\mathbb{R} \times M^n$ by ν (like the unit normal to $\bigcup_{i=1}^{l} \Sigma_i^{n-1}$ in M^n).

Again by a standard warped product formula,

$$\mathfrak{Ric}(\nu, \nu) = \mathrm{Ric}(\nu, \nu) - \frac{1}{N} \nabla^2 N(\nu, \nu),$$

so that by Equation (12) and the fact that Ψ is constant on P^n,

$$\mathfrak{Ric}(\nu, \nu) = -\widehat{C}^2 \frac{\nu(\Psi)^2}{N^2}. \tag{19}$$

Combining Equations (17), (18), and (19) allows to express R_P as

$$R_P = \frac{n-1}{n} \mathfrak{H}^2 + \frac{2\nu(\Psi)^2}{N^2} \left(1 - \widehat{C}^2 + \widehat{C}^2\right)$$

$$= \frac{n}{n-1} H^2 + \frac{2\nu(\Psi)^2}{N^2},$$

and we have shown Equation(14) (where we also used that $R_P = R_{\sigma_i}$ and $H = \frac{n-1}{n} \mathfrak{H}$ since M^n is totally geodesic in $\mathfrak{M}^{n+1}$).

We now prove Formula (15).

By the traced Gauss equation and by umbilicity of Σ_i^{n-1} in M^n,

$$\mathrm{R} - 2\,\mathrm{Ric}(\nu,\nu) = R_{\sigma_i} - \frac{n-2}{n-1} H_i^2. \tag{20}$$

Plugging ν into both slots of Equation (12) and the fact that Ψ is constant on Σ_i^{n-1} give

$$\mathrm{Ric}(\nu,\nu) = \frac{\nabla^2 N}{N}(\nu,\nu) - 2\frac{n-2}{n-1}\frac{\nu(\Psi)^2}{N^2}. \tag{21}$$

Recall that in general for a smooth isometric embedding of manifolds $(M_1^{n-1}, g_1) \hookrightarrow (M_2^n, g_2)$ with a spacelike unit normal ν and a smooth function $f : M_2^n \to \mathbb{R}$, the formula

$$^{g_2}\Delta f = {}^{g_1}\Delta f + {}^{g_2}\nabla^2 f(\nu,\nu) + H_{M_1}\nu(f) \tag{22}$$

holds, where H_{M_1} denotes the mean curvature of M_1 in M_2.

Using this and Equation (10), we get

$$\begin{aligned}\nabla^2 N(\nu,\nu) &= \Delta N - \Delta_{\sigma_i} N - H_i \nu(N)\\ &= \Delta N - H_i\nu(N)\\ &= 2\frac{n-2}{n-1}\frac{\nu(\Psi)^2}{N} - H_i\nu(N).\end{aligned}$$

This gives

$$\mathrm{Ric}(\nu,\nu) = -\frac{H_i\nu(N)}{N}. \tag{23}$$

Likewise, Equation (13) reads in our case

$$R = 2\frac{\nu(\Psi)^2}{N^2}. \tag{24}$$

Plugging these expressions for $\mathrm{Ric}(\nu,\nu)$ and R into Equation (20), we get

$$2\frac{\nu(\Psi)_i^2}{N_i^2} + 2\frac{H_i\nu(N)_i}{N_i} = R_{\sigma_i} - \frac{n-2}{n-1}H_i^2. \tag{25}$$

Equation (15) now follows immediately from Equations (14) and (25).

We note that

$$nH_i = (n-1)\mathfrak{H}_i,$$

and since $\mathfrak{H}_i$ is constant (by Proposition 16), so is H_i. The assertions about constancy of $\nu(N)_i$, $\nu(\Psi)_i$, and R_{σ_i} follow directly from Equations (14) and (15).

□

Proposition 18 ([25, 12, 11]). *Let P^n be a quasilocalphoton sphere in* $(\mathbb{R} \times M^n, \mathfrak{g} :=$

$-N^2dt^2 + g)$, *and assume that* $(\mathbb{R} \times M^n, \mathfrak{g})$ *fulfills the null energy condition. Then* $H_i > 0$.

Moreover, we define a notion of subextremality for photon spheres and quasilocal photon spheres, in agreement with the definition in [11]:

Definition 19. A (quasilocal) photon sphere component Σ_i^{n-1} in a (pseudo-)electrostatic system is called *subextremal* if

$$\frac{H_i^2}{R_{\sigma_i}} > \frac{n-2}{n-1}.$$

If "$<$" ("$=$") holds, it is called *superextremal (extremal).*

We recall some well-known facts about static horizons in the electrostatic setting that carry over to the pseudo-electrostatic case:

Lemma 20. *Let* (M^n, g, N, Ψ) *be a (pseudo-)electrostatic system and* $\Sigma^{n-1} \subseteq M^n$ *a static horizon. Then*

1. $\Psi \restriction_{\Sigma^{n-1}} = const.$, *and*
2. $d\Psi \restriction_{\Sigma^{n-1}} = 0$.

For a proof, we refer the reader to the derivation of their Equations (11) and (13) in [34], where these statements were deduced in an electrostatic context, but without appealing to the electrostatic equation for the Ricci tensor (12). Recall also that any static horizon Σ^{n-1} has vanishing mean curvature.

4.3 Zero mass and one-point insertion

In this section, we prove two propositions about the asymptotic behavior of Reissner–Nordström manifolds after a specific conformal change which will be used in the proof of the main results Theorem 1 and Theorem 3.

Proposition 21 (Zero mass of an asymptotic end after a conformal change)**.**
Let (M^n, g, N, Ψ) *an asymptotically Reissner–Nordström system of mass* m *and charge* q.

Assume that $\Omega_+ := \left(\frac{(1+N)^2 - \widehat{C}^2\Psi^2}{4}\right)^{1/(n-2)} > 0$ *on all of* M^n.

Then the metric $\Omega_+^2 g$ *is asymptotically Reissner–Nordström with mass* 0 *and charge* 0.

Proof. We write $\Phi : E \to \mathbb{R}^n$ for the diffeomorphism that makes M^n asymptotically Reissner–NOrdström as in Definition 9 and recall that we required $S > s_{m,q}$ (see Definition 9).

Since $N_{m,q}$ and Ψ_q are given explicitly, we may check that

$$\left\| (\varphi_{m,q})^{-\frac{2}{n-2}} - \left(\frac{(N_{m,q}+1)^2 - \widehat{C}^2 \Psi_q^2}{4} \right)^{\frac{2}{n-2}} \right\|_{C_0^2(\mathbb{R}^n \setminus \overline{B_S^n(0)})} < C_1 \tag{26}$$

(see Equation (9)) on $\mathbb{R}^n \setminus \overline{B_S^n(0)}$ for some $C_1 = C_1(n, S)$.

The asymptotic behaviors

$$\|\Phi_* N - N_{m,q}\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B_S^n(0)})} \leq C_2$$

and

$$\|\Phi_* \Psi - \Psi_q\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B_S^n(0)})} \leq C_2$$

give that

$$\|(\Phi_* \Omega_+^2) - \varphi_{m,q}^{-\frac{2}{n-2}}\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B_S^n(0)})} < C_3 \tag{27}$$

for some $C_3 = C_3(C_2, n, S)$.

From these facts combined we conclude that

$$\left\| \Phi_* \Omega_+^2 \right\|_{C_0^2(\mathbb{R}^n \setminus \overline{B_S^n(0)})} < C_4$$

for some $C_4 = C_4(C_1, C_3, n, S)$.

Using the assumption that $\|(\Phi_* g)_{ij} - (g_{m,q})_{ij}\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B_S^n(0)})} \leq C_5$ for some C_5 and all $i, j = 1, \ldots, n$, we now get

$$\|(\Phi_* \Omega_+^2)(\Phi_* g)_{ij} - (\Phi_* \Omega_+^2)(g_{m,q})_{ij}\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B_S^n(0)})} < C_6$$

for some $C_6 = C_6(C_4, C_5, n, S)$ and all $i, j = 1, \ldots, n$.

On the other hand, the inequalities (26) and (27) also imply that there is a $C_7 = C_7(C_1, C_3, n, S)$ such that

$$\begin{aligned} &\|(\Phi_* \Omega_+^2)(g_{m,q})_{ij} - \varphi_{m,q}^{-\frac{2}{n-2}} (g_{m,q})_{ij}\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B_S^n(0)})} \\ &= \|\varphi_{m,q}^{-\frac{2}{n-2}} \left(\Phi_* \Omega_+^2 \delta_{ij} - (g_{m,q})_{ij} \right) \|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B_S^n(0)})} < C_7 \end{aligned}$$

for all $i, j = 1, \ldots, n$.

We can now compute

$$
\begin{aligned}
&\|(\Phi_*\Omega_+^2 g)_{ij} - (g_{m=0,q=0})_{ij}\|_{C^2_{-(n-1)}(\mathbb{R}^n\setminus\overline{B^n_S(0)})}\\
=&\|(\Phi_*\Omega_+^2 g)_{ij} - \delta_{ij}\|_{C^2_{-(n-1)}(\mathbb{R}^n\setminus\overline{B^n_S(0)})}\\
=&\|(\Phi_*\Omega_+^2)(\Phi_* g)_{ij} - \varphi_{m,q}^{-\frac{2}{n-2}}(g_{m,q})_{ij}\|_{C^2_{-(n-1)}(\mathbb{R}^n\setminus\overline{B^n_S(0)})}\\
\leq&\|(\Phi_*\Omega_+^2)(\Phi_* g)_{ij} - (\Phi_*\Omega_+^2)(g_{m,q})_{ij}\|_{C^2_{-(n-1)}(\mathbb{R}^n\setminus\overline{B^n_S(0)})}\\
&+\|(\Phi_*\Omega_+^2)(g_{m,q})_{ij} - \varphi_{m,q}^{-\frac{2}{n-2}}(g_{m,q})_{ij}\|_{C^2_{-(n-1)}(\mathbb{R}^n\setminus\overline{B^n_S(0)})}\\
<&C_6 + C_7,
\end{aligned}
$$

which proves that $(M^n, \Omega_+^2 g)$ is asymptotically Reissner-Nordström with mass 0 and charge 0. □

Proposition 22 (One-point insertion)**.** *Let (M^n, g, N, Ψ) an asymptotically Reissner–Nordström system of mass m and charge q.*

Assume that $\Omega_- := \left(\frac{(1-N)^2-\widehat{C}^2\Psi^2}{4}\right)^{1/(n-2)} > 0$ on all of M^n.

Then one can insert a point p_∞ into $(M^n, \Omega_-^2 g)$ to obtain a compact Riemannian manifold $(M^n_\infty := M^n \cup \{p_\infty\}, g_\infty)$ which is $C^{1,1}$-regular at p_∞ and with boundary $\partial M^n_\infty = \partial M^n$.

Proof. Again, we write $\Phi : E \to \mathbb{R}^n$ for the diffeomorphism that makes M^n asymptotically flat as in Definition 9 and recall that we required $S > s_{m,q}$.

We note that

$$(1-N_{m,q})^2 - \widehat{C^2}\Psi_q^2 = \frac{m^2-q^2}{s^{2(n-2)}\left(1+\frac{m+q}{2s^{n-2}}\right)\left(1+\frac{m-q}{2s^{n-2}}\right)}$$

and hence

$$\frac{(1-N_{m,q})^2 - \widehat{C^2}\Psi_q^2}{4}\cdot\varphi_{m,q} = \frac{m^2-q^2}{4s^{2(n-2)}} = \left(\frac{s_{m,q}}{s}\right)^{2(n-2)}.$$

With similar arguments as in the proof of Proposition 21 and following closely [8], we can use this and the asymptotic behavior of $N_{m,q}$ and Ψ_q to estimate

$$
\begin{aligned}
&\left\|\Phi_*(\Omega_-^2 g)_{ij} - \left(\tfrac{s_{m,q}}{s}\right)^4\delta_{ij}\right\|_{C^2_{-(n+3)}(\mathbb{R}^n\setminus\overline{B^n_S(0)})}\\
=&\left\|\Phi_*\left(\left(\tfrac{(1-N)^2-\widehat{C}^2\Psi^2}{4}\right)^{2/(n-2)} g_{ij}\right) - \left(\tfrac{(1-N_{m,q})^2-\widehat{C}^2\Psi_q^2}{4}\right)^{\frac{2}{n-2}}(g_{m,q})_{ij}\right\|_{C^2_{-(n+3)}(\mathbb{R}^n\setminus\overline{B^n_S(0)})}\\
\leq&C_1
\end{aligned}
$$

for some $C_1 = C_1(m, q, n, S)$ and all $i, j = 1, \dots, n$, where we used additionally that

$$\left\| \left(\tfrac{(1-N)^2 - \widehat{C}^2 \Psi^2}{4} \right)^{\frac{2}{n-2}} \right\|_{C^2_{-4}(\mathbb{R}^n \setminus \overline{B^n_S(0)})} \leq C_0$$

for some $C_0 = C_0(m, q, n, S)$.

Analogously to the proof of Proposition 2.8 in [8], let now (y^i) denote coordinates in $\mathbb{R}^n \setminus \overline{B^n_S(0)}$ so that $s = |y|_\delta$, and perform an inversion at the sphere of radius $s_{m,q}$ to new coordinates $\eta^i := \left(\frac{s_{m,q}}{s}\right)^2 y^i$. Then

$$(\Phi_* \Omega_-^2 g)(\partial_{\eta^k}, \partial_{\eta^l}) = (\Phi_* \Omega_-^2 g)(\partial_{y^i}, \partial_{y^j}) \left(\frac{s}{s_{m,q}} \right)^4 \left(\delta^i_k - 2\frac{y^i y_k}{s^2} \right) \left(\delta^j_l - 2\frac{y^j y_l}{s^2} \right),$$

$$\delta(\partial_{\eta^k}, \partial_{\eta^l}) = \delta(\partial_{y^i}, \partial_{y^j}) \left(\frac{s}{s_{m,q}} \right)^4 \left(\delta^i_k - 2\frac{y^i y_k}{s^2} \right) \left(\delta^j_l - 2\frac{y^j y_l}{s^2} \right) = \left(\frac{s}{s_{m,q}} \right)^4 \delta_{kl},$$

where the indices are lowered and raised with the flat metric δ.

Together with the above estimate, it follows that

$$\begin{aligned} &\left\| (\Phi_* \Omega_-^2 g)(\partial_{\eta^k}, \partial_{\eta^l}) - \delta_{kl} \right\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B^n_S(0)})} \\ =&\left\| \left[\Phi_* (\Omega_-^2 g)_{ij} - \left(\tfrac{s_{m,q}}{s}\right)^4 \delta_{ij} \right] \left(\tfrac{s}{s_{m,q}}\right)^4 \left(\delta^i_k - 2\tfrac{y^i y_k}{s^2} \right) \left(\delta^j_l - 2\tfrac{y^j y_l}{s^2} \right) \right\|_{C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B^n_S(0)})} \\ \leq& C_2 \end{aligned}$$

for some $C_2 = C_2(C_0, C_1, m, q, n, S)$, where the subscript $C^2_{-(n-1)}(\mathbb{R}^n \setminus \overline{B^n_S(0)})$ is to be interpreted in (y^i)-coordinates.

In terms of the new coordinates (η^i) and writing $S' := \frac{s^2_{m,q}}{S}$, this (along with the assumption that $n \geq 3$) allows to conclude that

$$\left\| (\Phi_* \Omega_-^2 g)(\partial_{\eta^k}, \partial_{\eta^l}) - \delta_{kl} \right\|_{C^2_2(B^n_{S'}(0))} < C_3$$

for some $C_3 = C_3(C_0, C_1, C_2, m, q, n, S)$.

We can thus insert a point p_∞ (with $\eta^i(p_\infty) = 0$ for all $i = 1, \dots, n$) into M^n and extend g to a metric g_∞ on $M^n_\infty = M^n \cup \{p_\infty\}$ by letting

$$g_\infty(x) := \begin{cases} \Omega_-^2 g(x) & \text{for } x \neq p_\infty, \\ \delta & \text{for } x = p_\infty, \end{cases}$$

and g_∞ has $C^{1,1}$-regularity at p_∞. □

4.4 Mass and charge

We now prove two lemmata which together show $m > |q|$ under assumptions that are weaker than those of Theorem 3.

Lemma 23. *Let* (M^n, g, N, Ψ) *be asymptotically Reissner–Nordström and let Equation* (10) *be fulfilled. If* N *is constant on each component of* ∂M^n *and* $\nu(N) > 0$ *on the inner boundary* ∂M^n*, then* $m > 0$.

Proof. Let us first assume that $N > 0$ on all of ∂M^n. By Stokes' theorem and Equation (10),

$$\begin{aligned} 0 < \int_{\partial M^n} \nu(N) &= -\int_{M^n} \Delta N + \int_{\mathbb{S}^{n-1}_\infty} \nu(N) \\ &= -\int_{M^n} \frac{\widehat{C}^2}{N} |d\Psi|^2 + \int_{\mathbb{S}^{n-1}_\infty} \nu(N) \\ &\leq \int_{\mathbb{S}^{n-1}_\infty} \nu(N), \end{aligned}$$

where $\mathbb{S}^{n-1}_\infty$ is a sphere at infinity. By the asymptotic behavior of N,

$$\int_{\mathbb{S}^{n-1}_\infty} \nu(N) = (n-2)\,\mathrm{vol}\left(\mathbb{S}^{n-1}_1\right) m$$

so m is positive.

If ∂M^n has components where N vanishes, we pass from these components to a close-by level surface of N where $N > 0$ and $\nu(N) > 0$. □

For the remainder of the present Chapter 4, (M^n, g, N, Ψ) will be as in the assumptionts of Theorem 3. We will continue to denote those boundary components of (M^n, g, N, Ψ) which are quasilocal photon spheres by Σ^{n-1}_i $(1 \leq i \leq l)$ (in agreement with the notation fixed in Section 4.2), while the horizon components will be denoted by $\widehat{\Sigma}^{n-1}_i$ $(l+1 \leq i \leq L)$.

Lemma 24. *For* (M^n, g, N, Ψ) *consider the following conditions:*

1. *each quasilocal photon sphere component is subextremal and each static horizon component is nondegenerate,*
2. $F_\pm := N - 1 \pm \widehat{C}\Psi < 0$ *on* M^n*,*
3. $m^2 > q^2$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. "(1) $\Rightarrow$ (2)": Due to the electrostatic equations (10) and (11), $F_\pm$ fulfills

$$\Delta F_\pm \mp \frac{\widehat{C} d\Psi(\operatorname{grad} F_\pm)}{N} = 0, \tag{28}$$

which we will use to apply a maximum principle to $F_\pm$. On the other hand, the asymptotic behavior of N and Ψ gives $F_\pm \to 0$ as $r \to \infty$. If $F_\pm$ was positive at some point in M^n, then $F_\pm$ had a positive maximum on $M^n \cup \partial M^n$, hence (by the maximum principle) on ∂M^n, which means that at least on one boundary component Σ_i^{n-1}, the normal derivative $\nu(F_\pm)$ could not be positive. Hence, $F_\pm$ is negative on M^n provided that $\nu(F_\pm)$ is positive on ∂M^n; it thus remains to be shown that $\nu(F_\pm) \restriction_{\partial M^n} > 0$:

We fix $1 \leq i \leq l$. If Σ_i^{n-1} is a quasilocal photon sphere component, then $\nu(F_\pm)$ is positive on Σ_i^{n-1} if and only if

$$\begin{aligned} & \nu(N)^2 - \widehat{C}^2 \nu(\Psi)^2 > 0 \\ \overset{(14),(15)}{\Longleftrightarrow} & \frac{H_i^2}{(n-1)^2} - \frac{n-2}{n-1}\left(R_{\sigma_i} - \frac{n}{n-1} H_i^2\right) > 0 \\ \Leftrightarrow & \frac{H_i^2}{R_{\sigma_i}} > \frac{n-2}{n-1}, \end{aligned}$$

and the last inequality is the subextremality condition.

For static horizon components, $\nu(F_\pm) \restriction_{\Sigma_i^{n-1}} > 0$ was shown in [34] as a consequence of Equation (10) and the nondegeneracy condition $\nu(N) > 0$, by analyzing the near-horizon asymptotics.

We have shown that $F_\pm$ is negative on M^n provided that every quasilocal photon sphere component is subextremal and every static horizon component is nondegenerate.

"(2) $\Rightarrow$ (3)":

For the Reissner–Nordström lapse and potential, we note the asymptotic behavior

$$N_{m,q} - 1 \pm \widehat{C}\Psi_q = (-m \pm q) r^{-n+2} + \mathcal{O}(r^{-n+1})$$

for $r \to \infty$.

By the asymptotic conditions for N and Ψ, we deduce that also

$$F_\pm = (-m \pm q) r^{-n+2} + \mathcal{O}(r^{-n+1})$$

for $r \to \infty$.

By the assumption, $F_\pm$ is negative in the asymptotic region, and hence $m > \pm q$. $\square$

Note that the implication "(2) ⇒ (3)" was already shown in [11] for the case that the boundary is a photon sphere and $n = 3$; our proof here is very similar.

4.5 Main part of the proof of the main results

In the remainder of the present Chapter 4, we will prove Theorem 3. Since we have shown in Section 4.2 that a photon sphere in an electrostatic system is a quasilocal photon sphere, this will imply Theorem 1.

The proof follows the steps of constructing suitable pseudo-electrostatic fill-ins and attaching them to the boundary, doubling this new manifold along its new boundary, conformally compactifying it and applying the positive mass theorem, and determining the conformal factor to show that the original manifold was a piece of a Reissner–Nordström manifold.

4.5.1 Gluing in pseudo-Reissner–Nordström necks

In this section, we glue suitable pieces of pseudo-electrostatic spacetimes to quasilocal photon sphere components of the inner boundary of the (pseudo-)electrostatic system (M^n, g, N, Ψ), thereby getting a horizon as a new inner boundary.

We fix a quasilocal photon sphere Σ_i^{n-1} and define its scalar curvature radius

$$r_i := \sqrt{\frac{(n-1)(n-2)}{R_{\sigma_i}}},$$

keeping in mind that the scalar curvature R_{σ_i} is strictly positive.

Now we define a "charge"

$$q_i := \sqrt{\frac{2}{(n-1)(n-2)}} \frac{\nu(\Psi)_i}{N_i} r_i^{n-1} \tag{29}$$

as well as a "mass"

$$m_i := \frac{r_i^{n-2}}{n} + \frac{(n-1)q_i^2}{n r_i^{n-2}}. \tag{30}$$

We need to show for later use that $m_i^2 > q_i^2$. To this end, we calculate (plugging in Definition 30 for m_i)

$$r_i^{2(n-2)} n^2 \left(m_i^2 - q_i^2\right) = r_i^{4(n-2)} - (n^2 - 2n + 2) q_i^2 r_i^{2(n-2)} + (n-1)^2 q_i^4,$$

and this quantity is positive provided that

$$r_i^{2(n-2)} > (n-1)^2 q_i^2.$$

Plugging in Definition 29 for q_i and then using Equation (14) to substitute for $\frac{\nu(\Psi)^2}{N_i^2}$, this is seen to be equivalent to

$$\frac{H_i^2}{R_{\sigma_i}} > \frac{n-2}{n-1},$$

which is exactly the subextremality condition. This shows that $m_i^2 > q_i^2$.

We may now define

$$I_i := [a_i, b_i] := \left[\left(m_i + \sqrt{m_i^2 - q_i^2}\right)^{\frac{1}{n-2}}, r_i = \left(\frac{m_i n}{2} + \frac{1}{2}\sqrt{m_i^2 n^2 - 4(n-1)q_i^2}\right)^{\frac{1}{n-2}}\right]$$

and set

$$\gamma_i := \frac{1}{N_{m_i,q_i}(r)^2} dr^2 + \frac{r^2}{r_i^2}\sigma_i,$$

and recall $N_{m_i,q_i}(r) = \sqrt{1 - \frac{2m_i}{r^{n-2}} + \frac{q_i^2}{r^{2(n-2)}}}$ and $\Psi_{q_i}(r) = \frac{q_i}{\widehat{C} r^{n-2}}$.

Because of its importance for the subsequent arguments, we state the following fact as a lemma:

Lemma 25. *Let $\alpha_i, \beta_i > 0$ be constants. The system $(I_i \times \Sigma_i^{n-1}, \gamma_i, \alpha_i N_{m_i,q_i}, \alpha_i \Psi_{q_i} + \beta_i)$ is a traced-electrostatic system. Furthermore, $\mathbb{R} \times \{b_i\} \times \Sigma_i^{n-1}$ is a photon sphere in $\left(\mathbb{R} \times I_i \times \Sigma_i^{n-1}, -(\alpha_i N_{m_i,q_i})^2 dt^2 + \gamma_i\right)$ and $\{a_i\} \times \Sigma_i^{n-1}$ is a nondegenerate static horizon.*

Proof. For the system system $(I_i \times \Sigma_i^{n-1}, \gamma_i, N_{m_i,q_i}, \Psi_{q_i})$, Equations (10), (11), and (13) can be verified by a straightforward computation involving the Christoffel symbols of the metric γ_i and a comparison with those of the Reissner–Nordström manifold of mass m_i and charge q_i. Since Equations (10), (11) , and (13) are invariant under scaling $(N, \Psi) \mapsto (\alpha N, \alpha\Psi + \beta)$ of a lapse N and a potential Ψ by positive constants α, β, the system $(I_i \times \Sigma_i^{n-1}, \gamma_i, \alpha_i N_{m_i,q_i}, \alpha_i \Psi_{q_i} + \beta_i)$ is also a traced-electrostatic system.

One verifies by direct computations that $\{b_i\} \times \Sigma_i^{n-1}$ is a photon sphere and $\{a_i\} \times \Sigma_i^{n-1}$ a nondegenerate static horizon. □

We now choose

$$\alpha_i := \frac{N_i}{N_{m_i,q_i}(r_i)} > 0,$$
$$\beta_i := \Psi_i - \alpha_i \frac{q_i}{\widehat{C} r_i^{n-2}}.$$

Now we combine (M^n, g, N, Ψ) with $(I_i \times \Sigma_i^{n-1}, \gamma_i, \alpha_i N_{m_i,q_i}, \alpha_i \Psi_{q_i} + \beta_i)$ to a new system $(\widetilde{M}^n, \widetilde{g}, \widetilde{N}, \widetilde{\Psi})$ by gluing along the boundary components Σ_i^{n-1} and setting

$$\widetilde{g} := \begin{cases} g & \text{on } M^n, \\ \gamma_i & \text{on } \Sigma_i^{n-1}, \end{cases}$$

$$\widetilde{N} := \begin{cases} N & \text{on } M^n, \\ \alpha_i N_{m_i,q_i} & \text{on } \Sigma_i^{n-1}, \end{cases}$$

$$\widetilde{\Psi} := \begin{cases} \Psi & \text{on } M^n, \\ \alpha_i \Psi_{q_i} + \beta_i & \text{on } \Sigma_i^{n-1}. \end{cases}$$

We proceed to show that $\widetilde{g}, \widetilde{N}$, and $\widetilde{\Psi}$ are well-defined and $C^{1,1}$ across all gluing surfaces Σ_i^{n-1}.

We intend to use $\widetilde{N}$ as a smooth collar function across the gluing surface Σ_i^{n-1}; to this end, we first collect some facts about $\widetilde{N}$ and $\widetilde{\Psi}$.

By the choice of the scaling constants α_i and β_i, both $\widetilde{N}$ and $\widetilde{\Psi}$ are well-defined and continuous across Σ_i^{n-1}.

On the side of the glued-in necks $I_i \times \Sigma_i^{n-1}$, the unit normal to Σ_i^{n-1} is given as $\widetilde{\nu} = N_{m_i,q_i}(r_i)\partial_r$.

We use the explicit form of $\widetilde{\Psi}$ on the necks and the definition of q_i to calculate that

$$\widetilde{\nu}(\widetilde{\Psi}) = \alpha_i N_{m_i,q_i}(r_i)\partial_r(\Psi_{q_i}) = -\alpha_i N_{m_i,q_i}(r_i)\frac{q_i}{\widehat{C}}(n-2)r_i^{-n+1} = \nu(\Psi)_i$$

on $I_i \times \Sigma_i^{n-1}$; therefore, the normal derivative of $\widetilde{\Psi}$ is the same on both sides. Note that R_{σ_i} agrees with the scalar curvature of the photon sphere in the Reissner–Nordström manifold of mass m_i and charge q_i. This can be seen by solving the definition of m_i for $r_i^{n-2} = \frac{1}{2}m_i n + \frac{1}{2}\sqrt{m_i^2 n^2 - 4(n-1)q_i^2}$, where one also needs to use subextremality $r_i^{2(n-2)} > (n-1)^2 q_i^2$ to rule out the possibility $r_i^{n-2} = \frac{1}{2}m_i n - \frac{1}{2}\sqrt{m_i^2 n^2 - 4(n-1)q_i^2}$. Plugging this into the definition of r_i and solving for R_{σ_i} then gives the term for R_{σ_i} which is exactly the scalar curvature of the induced metric of the Reissner–Nordström photon sphere of mass m_i and q_i.

Since R_{σ_i} agrees with the respective Reissner–Nordström term, one sees from by definition of q_i that $\nu(\widetilde{\Psi})$ (which we already showed to agree from both sides) also agrees with the respective value for Reissner–Nordström.

Now, by Equation (14), the square of the mean curvature of Σ_i^{n-1} on the original side is the same as the square of the mean curvature of the Reissner–Nordström photon sphere with mass m_i and q_i. But since the sign of the mean curvature on the original side is positive by Proposition 18), the mean curvature agrees with the mean curvature of the photon sphere in the Reissner–Nordström manifold of mass m_i and charge q_i (both with respect to the outward pointing unit normal), which is positive.

On the newly glued-in side, it can be verified by a direct comparison with the Reissner–Nordström manifold of mass m_i and charge q_i that the mean curvature of Σ_i^{n-1} agrees with the one from Reissner–Nordström. Therefore, the mean curvature of Σ_i^{n-1} agrees from both sides.

We proceed to show that the normal derivative of $\widetilde{N}$ agrees from both sides.

On the original side M^n, it can be expressed via Equation (15) as

$$\nu(N)_i = (n-1)H_i N_i.$$

On the side of the glued-in necks, we compare $\widetilde{\nu}(\widetilde{N})_i$ with the normal derivative of the lapse in the Reissner–Nordström manifold of mass m_i and charge q_i at the photon sphere. Denoting the outward pointing unit normal to the photon sphere in this Reissner–Nordström manifold by ν_{m_i,q_i}, we explicitly calculate for the neck that

$$\widetilde{\nu}(\widetilde{N}) = \alpha_i \nu_{m_i,q_i}\left(N_{m_i,q_i}\right).$$

Since for the photon sphere of the Reissner–Nordström manifold Equation (14) holds and we already know that H_i agrees with the the mean curvature H_{m_i,q_i} of the photon sphere in the Reissner–Nordström manifold of mass m_i and charge q_i, we get on the glued-in side that

$$\widetilde{\nu}(\widetilde{N}) = \alpha_i \nu_{m_i,q_i}\left(N_{m_i,q_i}\right) = \alpha_i (n-1) H_{m_i,q_i} N_{m_i,q_i} = (n-1)H_i \widetilde{N}_i$$

agrees with $\widetilde{\nu}(\widetilde{N})$ on the original side.

Now, as $\widetilde{N}$ is well-defined, constant on Σ_i^{n-1}, and its normal derivatives do not vanish and agree from both sides, we can use $\widetilde{N}$ as a smooth collar function in a neighborhood of Σ_i^{n-1}. This finally shows that $\widetilde{M^n}$ is a smooth manifold.

Since we also showed along the way that $\widetilde{\Psi}$ is well-defined and its normal derivatives agree from both sides (and since it is smooth away from the gluing surfaces), $\widetilde{\Psi}$ is indeed $C^{1,1}$ across Σ_i^{n-1}.

Only the regularity of the metric $\widetilde{g}$ remains to be proven. To this effect, let $\{y^A\}$ be local coordinates on Σ_i^{n-1} and flow them to a neighborhood of Σ_i^{n-1} in $\widetilde{M^n}$ along the level set flow of $\widetilde{N}$. We will show that the components $\widetilde{g}_{\widetilde{N}\widetilde{N}}$, $\widetilde{g}_{\widetilde{N}A}$, and $\widetilde{g}_{AB}$ are $C^{1,1}$ across Σ_i^{n-1} with respect to the coordinates $\left(\widetilde{N}, y^A\right)$ for all $A, B = 1, \ldots, n-1$. This is done exactly as in [11], so we will be brief:

Continuity and smoothness in tangential directions of $\widetilde{g}$ in the chosen coordinates are immediate by construction of $\widetilde{g}$. The components $\widetilde{g}_{\widetilde{N}A}$ vanish in a neighborhood of Σ_i^{n-1} (for each $A = 1, \ldots, n-1$).

Hence, we only need to consider the normal derivatives of $\widetilde{g}_{\widetilde{N}\widetilde{N}}$ and $\widetilde{g}_{AB}$ for $A, B = 1, \ldots, n-1$.

Now

$$\widetilde{g}_{AB,\widetilde{N}} = \frac{2}{\widetilde{\nu}(\widetilde{N})}\widetilde{h}_{AB}$$

holds on Σ_i^{n-1}, where $\widetilde{h}_{AB}$ denotes the second fundamental form of Σ_i^{n-1} in $(\widetilde{M}^n, \widetilde{g})$. By umbilicity and the fact that the mean curvature agrees from both sides, $\partial_{\widetilde{N}}(\widetilde{g}_{AB})$ is the same from both sides.

Also,

$$\widetilde{g}_{\widetilde{N}\widetilde{N},\widetilde{N}} = -2\widetilde{\nu}(\widetilde{N})^2\nabla^2\widetilde{N}(\widetilde{\nu}, \widetilde{\nu}),$$

and by Equation (22), constancy of $\widetilde{N}$ and $\widetilde{\Psi}$ on Σ_i^{n-1}, and Equation (10), one gets

$$\nabla^2\widetilde{N}(\widetilde{\nu}, \widetilde{\nu}) = \frac{\widehat{C}^2}{\widetilde{N}}\widetilde{\nu}(\widetilde{\Psi})^2 - H_i\widetilde{\nu}(\widetilde{N}),$$

so that $\widetilde{g}_{\widetilde{N}\widetilde{N},\widetilde{N}}$ agrees from both sides.

Summing up, the system $\left(\widetilde{M}^n, \widetilde{g}, \widetilde{N}, \widetilde{\Psi}\right)$ we constructed is $C^{1,1}$ on a finite set of hypersurfaces and smooth elsewhere and is at least pseudo-electrostatic. Furthermore, its boundary consists of nondegenerate static horizons.

4.5.2 Doubling

Like the authors of [8, 11] and following the original models for this procedure in [6, 55], in this section we double the Riemannian manifold $\widetilde{M}^n$ that we constructed in the previous section and glue the two copies along their shared boundary. The metric, the lapse, and the potential will be extended to all of $\widehat{M}^n$.

As the arguments mirror those of [8] and others, we will just briefly sketch them and show that they carry over to our situation with only slight modifications.

First, we rename $\left(\widetilde{M}^n, \widetilde{g}, \widetilde{N}, \widetilde{\Psi}\right)$ to $\left(\widetilde{M}^n_+, \widetilde{g}_+, \widetilde{N}_+, \widetilde{\Psi}_+\right)$, reflect $\widetilde{M}^n$ as well as $\widetilde{g}$, $\widetilde{N}$ and $\widetilde{\Psi}$ through the boundary $\partial\widetilde{M}^n$ to obtain a new system that we call $\left(\widetilde{M}^n_-, \widetilde{g}_-, \widetilde{N}_-, \widetilde{\Psi}_-\right)$. Then we glue $\widetilde{M}^n_+$ and $\widetilde{M}^n_-$ along their shared boundary and name the resulting manifold $\widehat{M}^n$. We also set

$$\widehat{g} := \begin{cases} \widetilde{g}_+ & \text{on } \widetilde{M}^n_+, \\ \widetilde{g}_- & \text{on } \widetilde{M}^n_-, \end{cases}$$

$$\widehat{N} := \begin{cases} \widetilde{N}_+ & \text{on } \widetilde{M}^n_+, \\ -\widetilde{N}_- & \text{on } \widetilde{M}^n_-, \end{cases}$$

$$\widehat{\Psi} := \begin{cases} \widetilde{\Psi}_+ & \text{on } \widetilde{M}^n_+, \\ \widetilde{\Psi}_- & \text{on } \widetilde{M}^n_-. \end{cases}$$

We will denote the connected components of the gluing surface $\partial\widetilde{M}^n \subseteq \widehat{M}^n$ by $\widehat{\Sigma}_i^{n-1}$, that is,

$$\partial\widetilde{M}^n = \bigcup_{i=1}^{l} \widehat{\Sigma}_i^{n-1}.$$

Each boundary component $\widehat{\Sigma}_i^{n-1}$ $(1 \leq i \leq L)$ is a nondegenerate static horizon, either by the construction in the previous subsection (for $1 \leq i \leq l$), or by the assumptions of Theorem 3 (for $l+1 \leq i \leq L$). We fix an $1 \leq i \leq L$ to show $C^{1,1}$-regularity of $\left(\widehat{M}^n, \widehat{g}, \widehat{N}, \widehat{\Psi}\right)$ across $\partial\widetilde{M}^n$.

By its construction as an odd function, $\widehat{N}$ is smooth across $\partial\widetilde{M}^n$. This allows us to use $\widehat{N}$ as a smooth collar function across $\partial\widetilde{M}^n$, showing that $\widehat{M}^n$ is a smooth manifold.

The fact that $d\Psi \restriction_{\widehat{\Sigma}_i^{n-1}} = 0$ from Lemma 20 gives at once that $\widehat{\Psi}$ is $C^{1,1}$ across $\partial\widetilde{M}^n$.

We imitate closely the argumentation of [8] to show that the metric is of regularity $C^{1,1}$ across the gluing surfaces. To this end, we switch to adapted coordinates $(\widehat{N}, y^A)$ in a neighborhood of $\widehat{\Sigma}_i^{n-1}$. It is immediate that

$$\partial_{\widehat{N}}\left(g_{A\widehat{N}}\right) = 0$$

for all $A, B = 2, \ldots, n$.

Denoting by $\widehat{\nu}_+$ the unit normal to $\partial\widetilde{M}^n$ pointing into $\widetilde{M}^n_+$, the level set flow equations give

$$\partial_{\widehat{N}}\left(g_{\widehat{N}\widehat{N}}\right) = -2\left(\widehat{\nu}_+(\widehat{N})\right)^2 \widehat{\nabla}^2\widehat{N}(\widehat{\nu}_+, \widehat{\nu}_+)$$

on $\partial\widetilde{M}^n$.

By Formula (22), this reduces to

$$\partial_{\widehat{N}}\left(g_{\widehat{N}\widehat{N}}\right) =) = -2\left(\widehat{\nu}_+(\widehat{N})\right)^2 \Delta\widehat{N},$$

where we also made use of the facts that $\widehat{N} \restriction_{\widehat{\Sigma}_i^{n-1}} = 0$ and that $\widehat{\Sigma}_i^{n-1}$ has vanishing mean curvature.

Jointly with Equation (10) and Lemma 20, this gives

$$\partial_{\widehat{N}}\left(g_{\widehat{N}\widehat{N}}\right)=0$$

on $\widehat{\Sigma}_i^{n-1}$. Since the same holds on the other side of Σ_i^{n-1}, this shows that $g_{\widehat{N}\widehat{N}}$ is $C^{1,1}$ across Σ_i^{n-1}.

To see that g_{AB} is $C^{1,1}$, one calculates that

$$\partial_{\widehat{N}}\left(g_{AB}\right)=\frac{2}{\nu_+(\widehat{N})}h_{AB}$$

for all $A,B=2,\ldots,n$, where h is the second fundamental form of $\widehat{\Sigma}_i^{n-1}$, which vanishes since $\widehat{\Sigma}_i^{n-1}$ is a static horizon from both sides; so that

$$\partial_{\widehat{N}}\left(g_{AB}\right)=0$$

for all $A,B=2$.

Summing up the results of the last two sections, we have constructed a system $\left(\widehat{M^n},\widehat{g},\widehat{N},\widehat{\Psi}\right)$ with an "upper half" $\widetilde{M_+^n}$ and a "lower half" $\widetilde{M_-^n}$, which is smooth except possibly on a finite collection of hypersurfaces where it is $C^{1,1}$, and such that (M^n,g) embeds isometrically into the upper half of $\left(\widehat{M^n},\widehat{g}\right)$, and $\widehat{N}\restriction_{M^n}=N$, $\widehat{\Psi}\restriction_{M^n}=\Psi$.

Furthermore, $\widehat{g}$, $\pm\widehat{N}$, and $\widehat{\Psi}$ fulfill Equations (10)–(11) and the inequality (7) on $\widetilde{M_\pm^n}$ (except possibly on the gluing surfaces, where second derivatives might not exist), and $\left(\widetilde{M_\pm^n},\widehat{g},\pm\widehat{N},\widehat{\Psi}\right)$ are asymptotically Reissner–Nordström of mass m and charge q.

4.5.3 Conformal transformation and applying the positive mass theorem

In this step, the Riemannian manifold $\left(\widehat{M^n},\widehat{g}\right)$ that was constructed in the previous step will turn out to be conformally equivalent to $(\mathbb{R}^n,\delta)$ by a conformal factor that is constructed from the functions $\widehat{N}$ and $\widehat{\Psi}$.

We define

$$\Omega:=\left(\frac{(1+\widehat{N})^2-\widehat{C}^2\widehat{\Psi}^2}{4}\right)^{1/(n-2)}.$$

Note that Ω is smooth everywhere on $\widehat{M^n}$, except possibly on a finite collection of hypersurfaces, where it is $C^{1,1}$.

We need to show that Ω is positive everywhere.

Defining

$$F_\pm:=\widehat{N}-1\pm\widehat{C}\widehat{\Psi}$$

on $\widehat{M^n}$, we know from Lemma 24 that $F_\pm<0$ on the original manifold M^n. Hence,

$$0 < F_+F_- = (1-N)^2 - \widehat{C}^2\Psi^2 < (1+N)^2 - \widehat{C}^2\Psi^2 = 4\Omega^{n-2} \text{ on } M^n.$$

On the mirrored image of M^n in the lower half $\widetilde{M}^n_-$, we can now conclude that

$$0 < (1+\widehat{N})^2 - \widehat{C}^2\widehat{\Psi}^2 = 4\Omega^{n-2},$$

using that in this region $\widehat{N} = -N$ and that we just showed that $(1-N)^2 - \widehat{C}^2\Psi^2 > 0$.

On the glued-in necks, we can apply a similar trick; since we already know that $F_\pm < 0$ on Σ_i^{n-1}, it suffices to check (using the explicit form of $F_\pm$ on the necks) that $F_\pm < 0$ on $\widehat{\Sigma}_i^{n-1}$ and again apply a maximum principle to $F_\pm$ (recalling the PDE for $F_\pm$ given in Equation (28)). Summing up, $\Omega > 0$ on all of $\widehat{M}^n$.

We immediately see that the assumptions of Propositions 21 and 22 are met and may thus conclude that by Proposition 21, the upper half $\widetilde{M}^n$ with the metric $\Omega^2\widehat{g}$ is asymptotically Reissner–Nordström with mass 0 and charge 0.

To the lower half $\widetilde{M}^n_- \subseteq \widehat{M}^n$, we apply Proposition 22 to insert a point p_∞ into $(\widehat{M}^n, \Omega^2\widehat{g})$ such that the resulting manifold $(\widehat{M}^n_\infty, g_\infty)$ has $C^{1,1}$-regularity at p_∞.

The scalar curvature of a traced-electrostatic system after the conformal transformation we performed was calculated in [34] (using Equations (10), (11), and (13)) as

$$\frac{1}{8\widehat{N}^2\Omega^{2(n-3)}}\left|2\widehat{N}\widehat{\Psi}\nabla\widehat{N} - \left(\widehat{N}^2 - 1 + \widehat{C}^2\widehat{\Psi}^2\right)\nabla\widehat{\Psi}\right|^2 \geq 0. \tag{31}$$

It is easy to check (using the standard formula for the conformally transformed scalar curvature and the just mentioned calculation in [34]) that for the pseudo-electrostatic system $(\widehat{M}^n, \Omega^2\widehat{g})$ the transformed scalar curvature is bounded from below by the left-hand side expression in (31) and therefore also nonnegative.

To sum up, we have constructed a geodesically complete manifold $(\widehat{M}^n_\infty, g_\infty)$ with nonnegative scalar curvature and vanishing mass. The positive mass theorem for smooth manifolds of arbitrary dimensions was proven in [58]. A Ricci flow argument (which is independent of the dimension) in [39] shows that if the rigidity case of the positive mass theorem holds in a certain class of smooth manifolds, then it also holds for lower regularity "manifolds with corners" in that same class (and the isomorphism to the Euclidean space is smooth wherever the metric is smooth). In particular, the authors of [39] cover the case that the metric is only $C^{1,1}$-regular on a finite collection of hypersurfaces. Summing up, we have the rigidity statement of the positive mass theorem in arbitrary dimensions for smooth manifolds with $C^{1,1}$-regular hypersurfaces at our disposition. We thereby get that $(\widehat{M}^n_\infty, g_\infty)$ is isometric to the Euclidean space $(\mathbb{R}^n, \delta)$, and the isometry is smooth except possibly on the lower regularity submanifolds (see also [8] for the application of the positive mass theorem to an analogous situation).

4.5.4 Recovering the Reissner–Nordström manifold

In a last step, we show that the original Riemannian manifold (M^n, g) must have been a piece of the n-dimensional spatial Reissner–Nordström manifold. We will not explicitly denote the isometry $(\widehat{M}^n_\infty, g_\infty) \approx (\mathbb{R}^n, \delta)$ in what follows.

Just like the author of [8], we first recall that each boundary component Σ_i^{n-1} is a closed, totally umbilical hypersurface of (M^n, g); and hence (umbilicity being invariant under the conformal transformation $g \mapsto \Omega^2 g = \delta$) a closed, totally umbilical hypersurface of the Euclidean space. As a consequence, each Σ_i^{n-1} is a round sphere in $(\mathbb{R}^n, \delta)$, and thus—the conformal factor being constant on each boundary component—each $\left(\Sigma_i^{n-1}, g \restriction_{\Sigma_i^{n-1}}\right)$ is an intrinsically round sphere.

Second (and again as in [8]), since $\widehat{M}^n_\infty$ is homeomorphic to $\mathbb{R}^n$, the doubled manifold $\widehat{M}^n$ (without the inserted point) must be homeomorphic to $\mathbb{R}^n \setminus \{0\}$. In particular, the $(n-1)$-th fundamental group of $\widehat{M}^n$ is

$$\pi_{n-1}\left(\widehat{M}^n\right) = \pi_{n-1}\left(\mathbb{R}^n \setminus \{0\}\right) = \mathbb{Z}.$$

Recalling that each boundary component Σ_i^{n-1} of M^n is a topological sphere, we may now conclude that ∂M^n has only one component. This allows us to drop from now on the index i.

In the last step, we will determine the conformal factor Ω^2, applying a maximum principle. Since we know that the conformally transformed manifold $(\widehat{M}^n_\infty, g_\infty)$ is flat and in particular has vanishing scalar curvature, Inequality (31) (for the conformally transformed scalar curvature) reduces to the equality

$$2\widehat{N}\widehat{\Psi}\nabla\widehat{N} = \left(\widehat{N}^2 - 1 + \widehat{C}^2\widehat{\Psi}^2\right)\nabla\widehat{\Psi}. \tag{32}$$

It was computed in [34] as a consequence of Equations (32), (10), and (11) that the functions

$$v_\pm := (1 + \widetilde{N} \pm \widehat{C}\widetilde{\Psi})^{-1} : \widetilde{M}^n_+ \to \mathbb{R} \tag{33}$$

are harmonic with respect to the conformally changed metric $\Omega^2 g = \delta$ on $\widetilde{M}^n_+ \subseteq \mathbb{R}^n$.

The asymptotic conditions from Definition 11 imply that $v_\pm(y) \to \frac{1}{2}$ as $|y| \to \infty$.

By the same arguments as above for Σ^{n-1}, the horizon $\widehat{\Sigma}^{n-1}$ is a round sphere $\mathbb{S}_T(a) \subseteq \mathbb{R}^n$ with radius T and center $a \in \mathbb{R}^n$, and $v_\pm \restriction_{\widehat{\Sigma}^{n-1}} = v_\pm \restriction_{\mathbb{S}_T(a)}$ are constants. We can exclude $v_\pm \restriction_{\mathbb{S}_T(a)} \in \{0, \pm\infty\}$ because $v_+ v_- = \frac{1}{4}\Omega^{-(n-2)}$ on all of $\widetilde{M}^n$ and $\Omega^{-(n-2)}$ vanishes nowhere. Thus, by a maximum principle for elliptic PDEs, the functions $v_\pm$ are uniquely determined as the guessed solutions that are known from Reissner–Nordström, namely

$$v_{\pm}(y) = \left(1 + \left(1 - \frac{T^{2(n-2)} + c_{\pm}^2}{T^{n-2}|y-a|^{n-2}} + \frac{c_{\pm}^2}{|y-a|^{2(n-2)}}\right)^{\frac{1}{2}} \pm \frac{c_{\pm}}{|y-a|^{n-2}}\right)^{-1}, \tag{34}$$

where $c_{\pm}$ are constants that are determined by the constants $v_{\pm}\restriction_{\mathbb{S}_T(a)}$ via $v_{\pm}\restriction_{\mathbb{S}_T(a)}= \left(1 \pm \frac{c_{\pm}}{T^{n-2}}\right)^{-1}$.

Now, adding the two equations

$$1 \pm \frac{c_{\pm}}{T^{n-2}} = \frac{1}{v_{\pm}\restriction_{\mathbb{S}_T(a)}} = 1 \pm \widehat{C}\Psi \restriction_{\mathbb{S}_T(a)},$$

leads to $c_+ = c_- =: c$.

To determine the constant c, we compare the asymptotic behavior of $v_{\pm}$ that we get from Equations (33) and (34) and conclude that

$$m \pm q = \frac{T^{n-2}}{2} + \frac{c^2}{2T^{n-2}} \pm c.$$

This is equivalent to

$$\begin{aligned} 2m &= T^{n-2} + \frac{c^2}{T^{n-2}}, \\ q &= c, \end{aligned}$$

so that

$$v_{\pm} = \left(1 - N_{m,q} \pm \widehat{C}\Psi_q\right)^{-1}.$$

We have now determined $v_{\pm}$ and hence also N, Ψ, and the conformal factor Ω^2 as the respective functions known from the Reissner–Nordström manifold with mass m and charge q, and hence we know that (M^n, g) is exactly the Reissner–Nordström manifold of mass m and charge q. The inequality $m > |q|$ was already proven in Lemma 24, so that now the proof of Theorem 3 and thereby also of Theorem 1 is complete.

4.6 Discussion of the uniqueness result in a historical context

Our results in the present Chapter 4 stand in a tradition of uniqueness results for black holes and/or photon spheres in static, asymptotically flat spacetimes. In 1967, Israel gave a first proof that the Schwarzschild solution is unique among all 3+1 static vacuum systems with suitable asymptotic behavior (and under certain topological and regularity assumptions) [31]. His method relies on a geometric investigation of level surfaces of the lapse in a time symmetric spatial slice and on integrating certain inequalities (derived from the Einstein vacuum equations), taking into account the asymptotic and the near-horizon behavior of the quantities that are involved. A main drawback in Israel's proof is the assumption that the lapse regularly foliate the spacetime; that is, that $dN \neq 0$ everywhere. As one might want to a priori admit multiple connected horizon components, this condition seems rather restrictive, as it would already by itself heuristically preclude the existence of multiple heavy objects in the spacetime. Müller zum Hagen was able to remove it in [42]; other significant simplifications were made by Robinson in [54]. Israel extended his method in [32] to the electrovacuum case in dimension $3+1$, proving uniqueness of the Reissner–Nordström manifolds.

These early methods do not seem naturally prone to generalizations to a higher-dimensional setting since they make crucial use of the Gauss–Bonnet theorem. Also, they assume a priori spherical topology of the level sets of the lapse (and hence also of the static horizons), which would be a rather restrictive condition in the higher-dimensional setting, considering the possibility of non-spherical horizon topologies for $n \geq 4$ [21], even though the only known nonspherical black holes so far are not static.

A new approach to static black hole uniqueness was introduced in [6] by Bunting and Masood-ul-Alam. Their proof relies on conformal flatness of the spatial slice: after doubling the static vacuum system along its boundary (the horizon) and thereby getting a new system with two asymptotic ends, one finds a conformal factor (built out of the static lapse) which changes the ADM mass in one end to 0, and allows to insert a point into the other end to obtain a geodesically complete manifold, which is also scalar flat. The rigidity case of the positive mass theorem yields that the conformally transformed manifold is flat, and computations involving the vanishing Bach tensor are used to establish spherical symmetry, so that the original manifold can be recovered as being Schwarzschild.

The approach to black hole uniqueness via the positive mass theorem was later used to prove black hole uniquenes in the electrostatic setting [55, 37]. In the higher-dimensional setting, the computation of the Bach tensor has to be replaced by different methods, since the Bach tensor is no longer a conformal invariant in higher dimensions. But this last step can be replaced by applying a maximum principle to the conformal factor, which fulfills a partial differential equations that is derived from the dimensionally reduced static (electro-)vacuum Einstein equations [26]; the higher-dimensional electrovacuum case was treated in [34].

As a side note, we remark that a new strategy for proving the classical $3+1$ vacuum black hole uniqueness (under the assumption that the boundary is connected) was developed in [1], where the authors use monotonicity of certain quantities along the level set flow of the static potential.

For more details about black hole uniqueness theorems and their history, the reader may consult e.g. [30], [16], or [38].

It is much later that photon sphere uniqueness theorems enter the stage. In 2015, Cederbaum proved photon sphere uniqueness in the vacuum $3+1$ case [10], adapting Israel's black hole uniqueness proof. Yazadjiev and Lazov's proof of photon sphere uniqueness in $3+1$ electrovacuum [68] also follows Israel's ideas. Like in Israel's original proof, these two results assume a priori connectedness of the inner boundary (the photon sphere), as well as a lapse which regularly foliates the manifold. By changing the proof stategy to one adapting the Bunting–Masood-ul-Alam methods, it is possible to do away with these assumptions. The first photon sphere uniqueness proof in a Bunting–Masood-ul-Alam spirit is for the $3+1$ vacuum case [12]; later, the $3+1$ electrovacuum case [11] and the higher-dimensional vacuum case [8] followed, all of them also covering the case that some boundary components are static horizons. In one specific case, even perturbative uniqueness was established [69], see also [60, 61].

It was a new discovery of [8] that not the full vacuum equations are needed for a uniqueness proof via positive mass rigidity. In our Theorem 3, we likewise do not use the full electrostatic equations, but replace the equation

$$N\,\mathrm{Ric} = \nabla^2 N - 2\frac{d\Psi \otimes d\Psi}{N} + \frac{2}{(n-1)N}|d\Psi|^2 g$$

with the inequality

$$N^2 R \geq 2|d\Psi|^2,$$

which turns out to be sufficient to ensure that the conformally transformed manifold $(\widehat{M}^n_\infty, g_\infty)$ is of non-negative scalar curvature. To apply the Result 3 to photon spheres, however, one still needs to assume the full electrostatic equations near the photon sphere, since they are needed to prove that photon spheres are quasilocal photon spheres.

Just like its predecessor results in [8, 11, 12] which it generalizes, our Theorem 1 does not a priori assume connectedness of the boundary, thereby rigorously proving the physically reasonable intuition that two or more bodies that are dense enough to form a photon sphere cannot be in static equilibrium in the absence of other forces that could pull them apart. It is also important to notice that we do not assume that the inner boundary has spherical topology, but rather it turns out to be spherical as a result of the theorem. This implies that there are no electrovacuum, static objects with the specified decay conditions which possess a non-spherical horizon or a non-spherical photon sphere; so that our result can also be considered as a photon sphere or horizon topology statement.

In contrast to the result in [34], we do not assume a priori that the mass m and the charge q from the asymptotic assumptions on the lapse and the electric potential fulfill $m > |q|$. Instead, the inequality $m^2 > q^2$ follows from the subextremality of the photon sphere components and the non-degeneracy of the horizon components which together constitute the inner boundary (Lemma 24), and positivity of the mass follows from constancy of the lapse on the boundary components (Lemma 23). (Note, however, that it is known at least since [15], where the $3+1$ case was treated, that the mass-charge inequality can be dropped from the assumptions of static electrovacuum uniqueness results.)

Interest in higher dimensional gravity was first sparked by string theory, which lead to the discovery of the Myers–Perry solution ([44], see also [43]). More recently, TeV-scale black holes, which can be approximated by asymptotically flat, higher dimensional black holes [26], have renewed interest in higher dimensional gravity, specifically for the electrically charged case: since the electric force is several orders of magnitude larger than the gravitational force, one does not expect black holes with a significant electric charge as the result of stellar collapse [4]. As to the case of black holes created by high energy collisions, since the minimum energy required to create a black hole is currently estimated to be far beyond the range of even the Large Hadron Collider, we cannot (thinking in a classical framework) expect to observe such black holes [4]. However, in higher dimensions the effect of gravity decreases not as rapidly in distance as in a $3+1$-dimensional setting, so that some models (for example large extra dimensions) may allow for the creation of electrically charged black holes via collision [19, 3].

Gravity is much richer in higher dimensions than in the classical case of 3 spatial dimensions and 1 temporal dimension. This is famously highlighted by the discovery of a black hole with ringlike ($\mathbb{S}^2 \times \mathbb{S}^1$) horizon topology [21]. This black ring is, however, rotating; and so far no static example with a similar behavior is known. Since one reason for unexpected properties of exact solutions in higher dimensions is the possibility of multible independent planes of rotation (see [22]), it might be that static solutions are still relatively well-behaved in higher dimensions. Our Result 1 points in this direction, since it precludes the existence of objects with nonspherical horizon or photon sphere topology in a certain electrovacuum, asymptotically flat setting.

5 Geometry and topology of the Kerr photon region in the phase space

As mentioned in the Introduction (Chapter 3), in the exterior of the subcritical Kerr spacetime there are photons which stay at fixed coordinate radius; and they form a 2-parameter family [64]. We give a thorough, explicit proof that these are the only trapped photons in this spacetime family (Proposition 27). This fact is often tacitly assumed to be well-known and indeed follows from Dyatlov's rigorous analysis [20]; whereas our proof explicitly demonstrates that the conventional computations (see e.g. [64]) and heuristic arguments indeed do not overlook any trapped photons.

Unlike in the Schwarzschild situation, the spacetime region in a subcritical Kerr spacetime where trapped photons are found (the *region accessible to trapping* or the *photon region*) is not a submanifold (with or without boundary) of the spacetime. Therefore, the spacetime itself seems not the right setting to investigate geometric properties of photon regions; instead, the (co-)tangent bundle, where the Kerr photon region will be seen to be a submanifold, is a much better setting for further investigation of its geometry. Of these two bundles, the cotangent bundle seems a more appropriate structure to understand photon regions, since the natural symplectic form it carries allows to understand constants of motion as symmetries of the phase space (see e.g. [62]); and the analysis of the constants of motion plays an important role in understanding trapped light.

For the Schwarzschild spacetime of positive mass, a direct computation shows that the set of trapped photons in the cotangent bundle is a submanifold of topology $SO(3)\times\mathbb{R}^2$. The fact that the photon region in the Kerr phase space is a manifold of topology $SO(3)\times\mathbb{R}^2$ as well was shown earlier as a consequence of more general results by Dyatlov in [20], where the implicit function theorem was used on a family of slowly rotating Kerr spacetimes considering them as perturbations of the Schwarzschild spacetime. Our proof uses a different, more direct approach, which does not rely on knowledge about the Schwarzschild case and might help gain better insights into why the set of trapped photons in the Kerr phase space exhibits the properties in question. Knowing this alternative way to determine the topology might also be useful in possibly proving a uniqueness theorem for asymptotically flat, stationary, rotating, vacuum spacetimes with a photon region, in the spirit of e.g. Chapter 4 of the present thesis or its predecessor results in e.g. [10, 12].

The present chapter is organized as follows: In Section 5.1, we recall some well-known facts about the Kerr family and introduce some notation. In Section 5.2, we give a precise definition of trapping of light for stationary spacetimes and show that only photons of constant Boyer–Lindquist radius can be trapped in a subcritical Kerr spacetime. As a consequence of this, there are no null geodesically complete timelike umbilical hypersurfaces in the domain of outer communication in subcritical Kerr (Corollary 28).

The subsequent Section 5.3 explains how the set of trapped light can be understood as a subset of the (co-)tangent bundle; we prove that the photon region in the Kerr (co-)tangent bundle is a submanifold (Theorem 34). To this end, we make use of the characterization of trapped photons in terms of constants of motion and apply the implicit function theorem twice. Theorem 35 shows that the photon region in the phase space of a subcritical Kerr spacetime has topology $SO(3) \times \mathbb{R}^2$. This is done by calculating its fundamental group via the Seifert–van Kampen theorem and using the classification of 3-manifolds.

5.1 Basic facts and notation

We describe the Kerr spacetime of mass m and angular momentum a in Boyer–Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $t \in \mathbb{R}$, radius $r > 0$ suitably large, latitude $0 \leq \vartheta \leq \pi$, and longitude $0 \leq \varphi \leq 2\pi$. We use the following abbreviations:

$$\begin{aligned} S &:= \sin\vartheta \\ C &:= \cos\vartheta \\ \rho^2 &:= r^2 + a^2\cos^2\vartheta \\ \Delta &:= r^2 - 2mr + a^2 \\ \mathscr{A} &:= \left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\vartheta. \end{aligned}$$

The metric of the Kerr spacetime in Boyer–Lindquist coordinates is given by

$$-\left(1 - \frac{2mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\vartheta^2 - \frac{4mraS^2}{\rho^2} dt d\varphi + \frac{\mathscr{A}}{\rho^2} S^2 d\varphi^2.$$

The domain of outer communication (DOC) is the spacetime patch where $r > m + \sqrt{m^2 - a^2}$. The Boyer–Lindquist coordinates cover the entire DOC, except the *axis* $\{S = 0\}$. The breakdown of the metric in this form is a mere coordinate artefact; it can be extended smoothly over the axis, and so can the Killing vector fields ∂_t and ∂_φ (see [46]). We treat the subcritical case $|a| < m$ and assume that $a \geq 0$. Recall that in the subcritical case, the horizon $\{\Delta = 0\}$ has two connected components, and its outer component $\{r = m + \sqrt{m^2 - a^2}\}$ is the boundary of the DOC.

The Kerr spacetime will be denoted by $(\mathfrak{K}, \mathfrak{g})$. It is well-known that the motion of regular (that is, not entirely contained in the axis) photons in the DOC of Kerr is governed by the following equations of motion (see e.g. [46]):

$$\Delta\rho^2\dot{t} = \mathscr{A}E - 2mraL, \tag{35}$$

$$\rho^4\dot{r}^2 = E^2r^4 + (a^2E^2 - L^2 - \mathfrak{Q})r^2 + 2m((aE-L)^2 + \mathfrak{Q})r - a^2\mathfrak{Q} =: R(r), \tag{36}$$

$$\rho^4\dot{\vartheta}^2 = \mathfrak{Q} - \left(\frac{L^2}{S^2} - E^2a^2\right)C^2 =: \Theta(\vartheta), \tag{37}$$

$$\Delta\rho^2\dot{\varphi} = 2mraE - \left(\rho^2 - 2mr\right)\frac{L}{S^2}. \tag{38}$$

The dot denotes the derivative with respect to the affine parameter.

The quantity $E = -\langle\dot{\gamma}, \partial_t\rangle$ is the *energy* of a geodesic γ, $L = \langle\dot{\gamma}, \partial_\varphi\rangle$ its *angular momentum*, and $\mathfrak{Q}$ its *Carter constant*, given as $T(\dot{\gamma}, \dot{\gamma})$, where T is the Killing tensor given by $T^{\mu\nu} = \frac{1}{r^2}\mathfrak{g}^{\mu\nu} + 2\rho^2 l^{(\mu}n^{\nu)}$ (where $l^\mu = \frac{1}{\Delta}(r^2+a^2, 1, 0, a)$ and $n^\nu = \frac{1}{2\rho^2}(r^2+a^2, -\Delta, 0, a)$ are the *principal null directions*).

For dealing with general geodesics, it is useful to further introduce the quantity $\mathrm{q} := g(\dot{\gamma}, \dot{\gamma})$, which equals zero for null geodesics (photons).

These so-called *constants of motion* $\mathrm{q}, E, L, \mathfrak{Q}$ are constant along each geodesic, since they are derived from Killing vectors and Killing tensors, respectively. Note that they extend over the axis, since the generating tensors do so (see [46]).

Wherever ∂_t is timelike, all photons have positive energy $E > 0$. In case $E \neq 0$, we can rewrite the so-called *R-equation* (36) and *Θ-equation* (37) as the *scaled R-equation* and the *scaled Θ-equation*, respectively:

$$\left(\frac{\rho^2}{E}\right)^2\dot{r}^2 = r^4 + (a^2 - \Phi^2 - Q)r^2 + 2m((a-\Phi)^2 + Q)r - a^2Q\left(= \tfrac{R}{E^2}\right), \tag{39}$$

$$\left(\frac{\rho^2}{E}\right)^2\dot{C}^2 = Q - (Q + \Phi^2 - a^2)C^2 - a^2C^4\left(= \tfrac{\Theta}{E^2}\right), \tag{40}$$

with the *conserved quotients* $\Phi := \frac{L}{E}$, and $Q := \frac{\mathfrak{Q}}{E^2}$ (see [64]).

It was shown in [64] that in the DOC of a subcritical Kerr spacetime with $a \neq 0$ all photons with constant r-coordinate (*spherical photon orbits*) belong to the one-parameter class of solutions of $R(r) = \frac{\partial R(r)}{\partial r} = 0$ given by

$$\Phi_{\text{trap}}(r) := -\frac{r^3 - 3mr^2 + a^2r + a^2m}{a(r-m)}, \tag{41}$$

$$Q_{\text{trap}}(r) := -\frac{r^3(r^3 - 6mr^2 + 9m^2r - 4a^2m)}{a^2(r-m)^2} \tag{42}$$

(see (36) and (39)), where r ranges from

$$\hat{r}_1 := 2m(1 + \cos(2/3\arccos(-a/m)))$$

to

$$\hat{r}_2 := 2m(1 + \cos(2/3 \arccos(a/m))).$$

The *corotating* (*counterrotating*) orbits (i.e., those with positive (negative) angular momentum L) are located inside (outside) the hypersurface $\{r = r_{\text{mid}}\}$, where

$$r_{\text{mid}} := m + 2\sqrt{m^2 - \frac{1}{3}a^2} \cos\left(\frac{1}{3} \arccos \frac{m(m^2 - a^2)}{(m^2 - \frac{1}{3}a^2)^{\frac{3}{2}}}\right), \tag{43}$$

while those at the intermediate radius r_{mid} have vanishing angular momentum L.

The function $\Phi_{\text{trap}} : [\hat{r}_1, \hat{r}_2] \to \mathbb{R}$ is strictly monotonically decreasing and has only one zero, located at r_{mid}, while $Q_{\text{trap}} : [\hat{r}_1, \hat{r}_2] \to \mathbb{R}_{\geq 0}$ has its zeros at $\hat{r}_1$ and $\hat{r}_2$, is strictly increasing on $[\hat{r}_1, 3m]$, and strictly decreasing on $[3m, \hat{r}_2]$. The quantity Φ_{trap} determines the maximal latitude for trapped photons. In particular, trapped photons with radius r can only reach the the axis $\{S^2 = 0\}$ if $\Phi_{\text{trap}}(r) = 0$.

For these facts and more information about *spherical photon orbits* (i.e., those of constant radius) in the Kerr spacetime, see [64].

5.2 Trapped photons in the Kerr spacetime

It is natural to ask whether there are more photons than those of constant radial coordinate which are “trapped” around the Kerr center, without ever falling through the horizon or escaping to spatial infinity. Before answering this question, we first need a more precise notion of what it means to be trapped.

We suggest the following definition for the rather general case of stationary spacetimes:

Definition 26. A photon in the DOC of a stationary spacetime is called *trapped* if its orbit in the quotient of the DOC under the action of the stationary Killing vector field ∂_t is contained in a compact set.

Thus, in the Kerr spacetime, a photon is trapped if and only if the range of its radial coordinate is a relatively compact set contained in $\left(m + \sqrt{m^2 - a^2}, \infty\right)$. In an arbitrary stationary spacetime, it is immediate (from the constancy of the metric components along the flow lines of the stationary symmetry) that the lapse along trapped photons is bounded.

Proposition 27. *The spherical photons are the only trapped photons in the DOC of a subcritical ($0 < a < m$) Kerr spacetime.*

Proof. Let $\gamma = (t, r, \vartheta, \varphi)$ be a trapped photon of non-constant radial coordinate in the DOC of a subcritical Kerr spacetime. We will first treat the **case that γ does not intersect the axis** $\{S^2 = 0\}$.

We focus on the **subcase that the photon has nonvanishing energy** $E \neq 0$ and investigate the right-hand side $R(r)$ of Equation (36). Since its left-hand side is manifestly non-negative, γ has to lie in a region of the Kerr space-time where $R(r)$ is non-negative.

Since the range of the trapped photon's radial coordinate is a relatively compact set contained in $\left(m + \sqrt{m^2 - a^2}, \infty\right)$, there must be two values $m + \sqrt{m^2 - a^2} < r_1 < r_2 < \infty$ such that the photon either turns around when reaching r_i (that is, the sign of $\dot{r}$ changes at some s_i with $r(s_i) = r_i$) or asymptotically approaches r_i (that is, $\lim_{s \to +\infty} r(s) = r_i$ or $\lim_{s \to -\infty} r(s) = r_i$).

As a consequence, for trapping with non-constant radial component, we need two zeros r_i $(i = 1, 2)$ of $R(r)$ that lie in the interval $\left(m + \sqrt{m^2 - a^2}, \infty\right)$, with $R(r) \geq 0$ between these zeros.

Thus, on the search for trapped photons of non-constant coordinate radius, analyzing the polynomial $R(r)$ of degree 4 with leading coefficient $E^2 > 0$, one would have to find constants of motion such that all roots of R are real, and (writing $r_0 \leq r_1 \leq r_2 \leq r_4$ for these roots) such that r_1, r_2, r_3 are outside the outer horizon component $\{r = m + \sqrt{m^2 - a^2}\}$, and r_1 and r_2 do not coincide, see Figure 1.

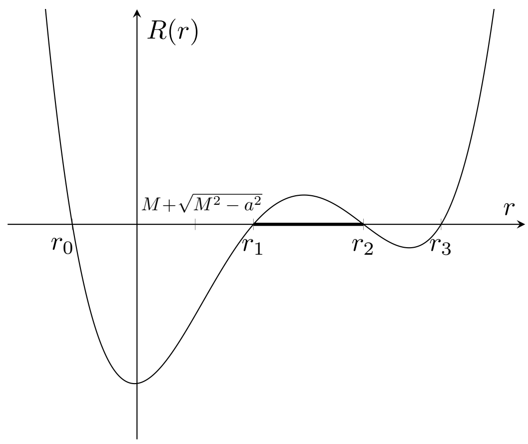

Figure 1: A qualitative plot of R against r for a hypothetical trapped photon of non-constant coordinate radius.

Since we assumed that $E \neq 0$, we may work with the scaled R-equation (39). We see immediately that the coefficient of r^2 has to be negative to allow for the constellation of roots we need (since otherwise $\frac{d}{dr}\left(\frac{R}{E^2}\right)$ would be monotonous and $\frac{R}{E^2}$ convex), that is,

$$a^2 - \Phi^2 - Q < 0. \tag{44}$$

On the other hand, since the left-hand side of the scaled Θ-equation is manifestly non-negative, choosing a non-positive $Q \leq 0$ forces $a^2 - \Phi^2 - Q \geq 0$, contradicting (44).

We have thus ruled out the case $Q \leq 0$ for trapped light and may focus on the case $Q > 0$.

Since $Q > 0$ implies $\frac{R(0)}{E^2} < 0$, a necessary condition for the second root r_1 of $\frac{R(r)}{E^2}$ to be outside the outer horizon component is $\frac{R(m)}{E^2} < 0$. But on the other hand, we can estimate (using $Q > 0$ and $a < m$):

$$\frac{R(m)}{E^2} = m^2((\Phi - 2a)^2 + m^2 - a^2) + m^2 Q - a^2 Q \geq m^2(\Phi - 2a)^2 \geq 0,$$

thus ruling out the existence of trapped light with $r \neq \mathit{const.}$ outside the axis with nonvanishing energy E.

In the **subcase of vanishing energy** $E = 0$, the R-equation reduces to

$$\rho^4 \dot{r}^2 = -(L^2 + \mathfrak{Q})r^2 + 2m(L^2 + \mathfrak{Q})r - a^2\mathfrak{Q}.$$

The roots of the right-hand side of this equation are given by

$$r_{1,2} = m \pm \sqrt{m^2 - \frac{a^2\mathfrak{Q}}{L^2 + \mathfrak{Q}}},$$

and at least one of them (if real) is smaller than the radius of the outer horizon component, $m + \sqrt{m^2 - a^2}$.

Summing up, so far we have shown that there are no trapped photons contained in $\mathfrak{K} \setminus \{S^2 = 0\}$ with nonconstant radial component.

We need to treat the **case of photons that are contained in the axis** $\{S^2 = 0\}$ separately. The axis is a 2-dimensional, totally geodesic submanifold with line element

$$-(1 - 2mr/\rho^2)dt^2 + (\rho^2/\Delta)dr^2,$$

so every photon contained in the axis has to fulfill

$$-(1 - 2mr/\rho^2)\dot{t}^2 + (\rho^2/\Delta)\dot{r}^2 = 0.$$

For trapped photons, $\dot{r}$ has to vanish at some parameter value (at least asymptotically), while $\dot{t} \neq 0$ cannot approach 0. This gives $(1 - 2mr/\rho^2) = 0$, that is, $\Delta = 0$, which means that the photon is not in the DOC.

The **case of photons that cross the axis but are not entirely contained in it** can be treated like the off-axis case: the R-equation is also valid on the axis, and the conditions on the conserved quotients that were derived from the scaled Θ-equation still would have to be fulfilled, since the conserved quotients can be calculated off-axis. $\square$

The statement of Proposition 27 was also discussed in [47], using different techniques, and follows from [20].

We can use Proposition 27 to show nonexistence of certain umbilical hypersurfaces in the DOC. We remind the reader that a semi-Riemannian manifold is called *null geodesically complete* if every inextendible null geodesic is defined on all of $\mathbb{R}$.

Corollary 28. *There are no null geodesically complete, timelike umbilical hypersurfaces in the DOC of the subcritical Kerr spacetime (with $a \neq 0$) whose quotient under the stationary Killing vector field ∂_t is contained in a compact set. In particular, none of the cylinders $\{r = const.\}$ are umbilical.*

Proof. Assume towards a contradiction that there is such a hypersurface $\mathscr{S}$. Recall that by Theorem 12 (proven in [53, 17]), a timelike hypersurface in a Lorentzian manifold is umbilical if and only if every null geodesic that is initially tangent to it remains tangent throughout. Since by the assumption the quotient of $\mathscr{S}$ after factoring out ∂_t is relatively compact in the quotient of the DOC, every photon that stays on $\mathscr{S}$ has the property that the range of its radial coordinate is a relatively compact set contained in $\big(m + \sqrt{m^2 - a^2}, \infty\big)$. These two facts combine to the conclusion that every photon that is initially tangent to $\mathscr{S}$ is trapped.

By Proposition 27, the only trapped photons are those of constant radial coordinate, hence for any given radius where trapped photons exist, there is only one choice for their conserved quotients Φ and Q (namely, the values for Φ_{trap} and Q_{trap} from Equations (41) and (42)). We are free to choose a positive value of E for a trapped photon; that is, there is one degree of freedom for choosing the constants of motion for a trapped photon at a given point. These constants of motion determine the off-axis trapped photons completely, possible up to a sign choice for $\dot{\vartheta}$. That is, at every point in the Kerr DOC, there is (at most) one degree of freedom for choosing an initial direction for a trapped photon.

On the other hand, at every point of $\mathscr{S}$, the cone of lightrays tangent to $\mathscr{S}$ is 2-dimensional, which means that there are two degrees of freedom for choosing trapped photons at a given point of $\mathscr{S}$. This gives a contradiction, proving that there is no such $\mathscr{S}$. [1] □

5.3 The Kerr photon region as a submanifold of the (co-)tangent bundle

We call the region in the Kerr spacetime where one can find trapped photons the *region accessible to trapping*. In contrast to the Schwarzschild case, in the subcritical

[1] In Schwarzschild, the statement is no longer true, as is demonstrated by the striking counterexample of the photon sphere $\{r = 3m\}$. The proof of Corollary 28 cannot be immitated in the $a = 0$ case since there we do not have Equations (41) and (42) at our disposition.

Kerr spacetime the region accessible to trapping is not a submanifold (with or without boundary) of the spacetime; it can be imagined as the region covered by a crescent moon that rotates about an axis through its pointy ends, see Figure 2.

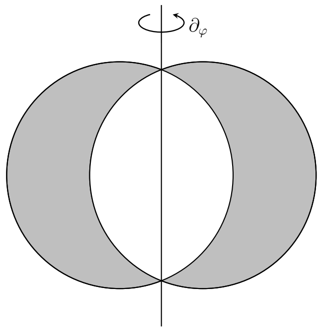

Figure 2: A $t = const.$ slice of the photon region in the Kerr spacetime.

We will identify the set of trapped photons in any spacetime with a subset of the (co-)tangent bundle. In the subcritical Kerr spacetime, it turns out that this subset of the (co-)tangent bundle is a much nicer geometrical object than the region accessible to trapping.

One can identify geodesics in any pseudo-Riemannian manifold $\left(\mathfrak{M}, {}^{\mathfrak{M}}\mathfrak{g}\right)$ with points in the tangent bundle $T\mathfrak{M}$ in the following way: The natural map

$$\{\text{geodesics in } \mathfrak{M}\} \to T\mathfrak{M}$$

given by

$$\gamma \mapsto (\gamma(0), \dot{\gamma}(0))$$

is a bijection by uniqueness of geodesics (where we distinguish between null geodesics with different affine parametrizations). The canonical isomorphism $T\mathfrak{M} \to T^*\mathfrak{M}$ now gives a similar identification of geodesics with points in $T^*\mathfrak{M}$.

We only consider future-directed trapped photons; by reversing the direction of time (using the symmetry $t \mapsto -t$) one can extend all statements to past directed photons.

The task in the present section is to show that the *photon region in the tangent bundle* of the subcritical Kerr spacetime $P' \subseteq T\mathfrak{K}$ and the *photon region in the cotangent bundle* $P \subseteq T^*\mathfrak{K}$ (that is, the image of P' under the canonical isomorphism $T\mathfrak{K} \to T^*\mathfrak{K}$) are smooth submanifolds of $T\mathfrak{K}$ and $T^*\mathfrak{K}$, respectively. Of course, statements about the submanifold structure and the topology of P' and P imply each other immediately, but it will turn out to be useful to work in both settings.

Remark 29. If $\left(\mathfrak{M}, {}^{\mathfrak{M}}\mathfrak{g}\right)$ is any spacetime, the above identification allows us to view the set of photons in $\mathfrak{M}$ as

$$\{(b, p) \in T\mathfrak{M} : {}^{\mathfrak{M}}\mathfrak{g}_b(p, p) = 0, \dot{t} > 0\}.$$

As the preimage of 0 under $(b,p) \mapsto {}^{\mathfrak{M}}\mathfrak{g}_b(p,p)$ (intersected with the open set $\{\dot{t} > 0\}$), this set is a submanifold of $T\mathfrak{M}$ (and as regular as the spacetime itself), since the differential of this map contains the components of $2 \cdot {}^{\mathfrak{M}}\mathfrak{g}_b(\cdot, p)$ as its last n matrix components, and this never vanishes by non-degeneracy of ${}^{\mathfrak{M}}\mathfrak{g}$.

We can naturally extend all functions that were defined on the DOC of Kerr to the tangent bundle (of the DOC) of the Kerr spacetime, so that now in particular the constants of motion and the conserved quotients, but also Φ_{trap} and Q_{trap} are functions on $T\mathfrak{K}$. Slightly abusing notation, we will use the same letters to denote these new functions. This remark applies—mutatis mutandis—also to $T^*\mathfrak{K}$.

The region accessible to trapping intersects the axis in only two points; justified by the equatorial symmetry $\vartheta \mapsto -\vartheta$, we will call one of them the North Pole and the other one the South Pole, and we will refer to the two connected components of the region accessible to trapping without the equatorial plane $\{S^2 = 1\}$ as the Northern and the Southern Hemisphere.

Remark 30. As it turns out, in order to obtain later the topology of P' and P we will need to construct smooth bundle charts for a spatial slice of the image of P under the canonical isomorphism $T\mathfrak{K} \to T^*\mathfrak{K}$. Nonetheless, this only proves that P is a manifold, not that it is a submanifold of $T^*\mathfrak{K}$. Even if one is only interested in the manifold structure of P' and P, not their submanifold structure, the proof of the submanifold property has advantages over the explicit construction of charts, since the latter will turn out to be quite technical.

On $T\mathfrak{K} = \mathfrak{K} \times \mathbb{R}^4$, we use coordinates $(t, r, \vartheta, \varphi, p^0, p^1, p^2, p^3)$, where (p^0, p^1, p^2, p^3) are the components of the contravariant tangent vector in the same coordinate basis.

Remark 31. In the Schwarzschild case $a = 0$, it is easy to see that the photon sphere lifts to a submanifold of the tangent bundle of Schwarzschild: this photon region consists precisely of all null geodesics of the form $(t, 3m, \vartheta, \varphi, p^0, 0, p^2, p^3)$ and is thus in particular a submanifold of the space of all photons.

We shall now consider the situation in the subcritical Kerr spacetime with $a \neq 0$.

Proposition 32. *The Kerr photon region in the (co-)tangent bundle $T\mathfrak{K}$ ($T^*\mathfrak{K}$) of a subcritical Kerr spacetime is a smooth submanifold when the axis $\{S^2 = 0\}$ is deleted; that is, $P' \setminus \{S^2 = 0\}$ is a smooth submanifold of $T\mathfrak{K}$ (and $P \setminus \{S^2 = 0\}$ is a smooth submanifold of $T^*\mathfrak{K}$).*

Proof. We will prove the statement for $P' \subseteq T\mathfrak{K}$.

By the equations of motion and the conditions for trapped photons (see [46] and [64]), a Kerr geodesic $\gamma = (t, r, \vartheta, \varphi, p^0, p^1, p^2, p^3)$ is a trapped photon if and only if

$$p^0 = \frac{E}{\Delta\rho^2}\left(\mathscr{A} - 2mra\Phi_{\text{trap}}\right), \tag{45}$$

$$p^1 = 0, \tag{46}$$

$$(p^2)^2 = \frac{E^2}{\rho^4}\left(Q_{\text{trap}} - \left(\frac{\Phi_{\text{trap}}^2}{S^2} - a^2\right)C^2\right), \tag{47}$$

$$p^3 = \frac{E}{\Delta\rho^2}\left(2mra + (\rho^2 - 2mr)\frac{\Phi_{\text{trap}}}{S^2}\right), \tag{48}$$

for some energy $E = -g_{00}p^0 - g_{03}p^3 > 0$, where Q_{trap} and Φ_{trap} are the functions of r which give the trapping conditions from [64] stated in Equations (41) and (42).

Note that these equation are invariant under the scaling

$$\left(E, p^0, p^1, p^2, p^3\right) \mapsto \lambda\left(E, p^0, p^1, p^2, p^3\right)$$

with $\lambda > 0$, which allows us to get rid of one of the above equations:

Solving Equation (45) for

$$e := E(r, \vartheta, p^0) = \frac{p^0\Delta\rho^2}{\mathscr{A} - 2mra\Phi_{\text{trap}}}$$

and plugging it into Equations (46)–(48) yields

$$p^1 = 0 \tag{49}$$

$$(p^2)^2 = \frac{e^2}{\rho^4}\left(Q_{\text{trap}} - \left(\frac{\Phi_{\text{trap}}^2}{S^2} - a^2\right)C^2\right), \tag{50}$$

$$p^3 = \frac{e}{\Delta\rho^2}\left(2mra + (\rho^2 - 2mr)\frac{\Phi_{\text{trap}}}{S^2}\right). \tag{51}$$

Hence, the off-axis photon region in the tangent bundle is the preimage of 0 under the following smooth function $f = (f_1, f_2, f_3) : T\mathfrak{K} \setminus \{S^2 = 0\} \to \mathbb{R}^3$,

$$f_1 := p^1, \tag{52}$$

$$f_2 := (p^2)^2 - \frac{e^2}{\rho^4}\left(Q_{\text{trap}} - \left(\frac{\Phi_{\text{trap}}^2}{S^2} - a^2\right)C^2\right), \tag{53}$$

$$f_3 := p^3 - \frac{e}{\Delta\rho^2}\left(2mra + (\rho^2 - 2mr)\frac{\Phi_{\text{trap}}}{S^2}\right). \tag{54}$$

We will show that the differential of f has full rank, so that we can use the submersion theorem.

Some partial derivatives of f are:

$$
\begin{aligned}
&\frac{\partial f_1}{\partial p^1} = 1 \qquad \frac{\partial f_1}{\partial p^3} = 0 \\
&\frac{\partial f_2}{\partial p^1} = 0 \qquad \frac{\partial f_2}{\partial p^2} = 2p^2 \qquad \frac{\partial f_2}{\partial p^3} = 0 \\
&\frac{\partial f_2}{\partial r} = \frac{\partial}{\partial r}\left(\frac{e^2}{\rho^4}\right)\left(Q_{\text{trap}} - \left(\frac{\Phi^2_{\text{trap}}}{S^2} - a^2\right)C^2\right) + \frac{e^2}{\rho^4}\left(\frac{\partial Q_{\text{trap}}}{\partial r} - \frac{\partial \Phi^2_{\text{trap}}}{\partial r}\frac{C^2}{S^2}\right) \\
&\frac{\partial f_2}{\partial \vartheta} = \frac{\partial}{\partial \vartheta}\left(\frac{e^2}{\rho^4}\right)\left(Q_{\text{trap}} - \left(\frac{\Phi^2_{\text{trap}}}{S^2} - a^2\right)C^2\right) - \frac{e^2}{\rho^4}2C\left(\frac{\Phi^2_{\text{trap}}}{S^3} + \left(\frac{\Phi^2_{\text{trap}}}{S^2} - a^2\right)S\right) \\
&\frac{\partial f_3}{\partial p^3} = 1
\end{aligned}
$$

By the above formulas for the partial derivatives of f_1 and f_3 we see that the differential of f has at least rank 2. In order to show that f is indeed submersive, we assume (towards a contradiction) that the differential of f has rank 2 at some point of the off-axis photon region $P' \setminus \{S^2 = 0\}$ in $T\mathfrak{K}$. Then $\frac{\partial f_2}{\partial p^2}$, $\frac{\partial f_2}{\partial r}$, and $\frac{\partial f_2}{\partial \vartheta}$ vanish at this point; and in particular

$$\frac{\partial f_2}{\partial p^2} = 2p^2 = 0$$

implies

$$\left(Q_{\text{trap}} - \left(\frac{\Phi^2_{\text{trap}}}{S^2} - a^2\right)C^2\right) = 0 \tag{55}$$

because of Equation (53) and the fact that e is non-zero at the photon region in $T\mathfrak{K}$. Hence, $\frac{\partial f_2}{\partial r} = 0$ and $\frac{\partial f_2}{\partial \vartheta} = 0$ reduce to

$$\frac{\partial Q_{\text{trap}}}{\partial r} - \frac{\partial \Phi^2_{\text{trap}}}{\partial r}\frac{C^2}{S^2} = 0, \tag{56}$$

and

$$2C\left(\frac{\Phi^2_{\text{trap}}}{S^3} + \left(\frac{\Phi^2_{\text{trap}}}{S^2} - a^2\right)S\right) = 0. \tag{57}$$

In the **case** $C \neq 0$, combining Equations (55) and (57) yields

$$\Phi^2_{\text{trap}} + Q_{\text{trap}}\frac{S^4}{C^2} = 0.$$

But since $Q_{\text{trap}} \geq 0$ in the region accessible to trapping, this implies that Φ^2_{trap} and Q_{trap} vanish for the same value of r, which cannot happen (see Section 5.1).

In the **case** $C = 0$, Equations (55) and (56) give $Q_{\text{trap}} = 0$ and $\frac{\partial Q_{\text{trap}}}{\partial r} = 0$, which also cannot happen for the same radius r, see Section 5.1.

We have thus derived a contradiction from the assumption that the differential of f does not have full rank somewhere on the off-axis photon region.

Thus, 0 is a regular value of f, and we may conclude by the submersion theorem that the photon region of the Kerr spacetime with deleted axis is a submanifold of the tangent bundle $T\mathfrak{K}$.

□

Due to the coordinate failure, we have to deal with the axis separately. To do this, we will once again use the submersion theorem, this time applying it to a different function $T\mathfrak{K} \to \mathbb{R}^3$.

Proposition 33. *For a subcritical Kerr spacetime $\mathfrak{K}$, there is a neighborhood of the North (resp. South) Pole of the photon region P' which is a smooth submanifold of $T\mathfrak{K}$ (or equivalently, a neighborhood of the North (resp. South) Pole of the photon region P which is a smooth submanifold of $T^*\mathfrak{K}$).*

Proof. We consider the function $h : T\mathfrak{K} \to \mathbb{R}^3$ given by

$$h = (h_1, h_2, h_3) := (\mathrm{q}, \mathfrak{Q} - E^2 \cdot Q_{\text{trap}}, L - E \cdot \Phi_{\text{trap}}),$$

where Q_{trap}, Φ_{trap} are again the functions given in Equations (41) and (42).

Clearly, the upper hemisphere of the photon region can be described as

$$\{(b, p) \in T\mathfrak{K} : p^0((b, p)) > 0, h((b, p)) = 0\}.$$

The task is to show that h is a submersion in a neighborhood of $P' \cap \{S^2 = 0\}$ in $T\mathfrak{K}$. For the calculations near the axis, we change coordinates

$$(t, r, \vartheta, \varphi) \mapsto (t, r, x := r \cos\varphi \sin\vartheta, y := r \sin\varphi \sin\vartheta).$$

We can work with the new coordinates on each hemisphere; for simplicity let us only consider the Northern Hemisphere, which suffices by the equatorial symmetry. Since also p^2, p^3 fail on the axis, we change the p^i accordingly $(p^0, p^1, p^2, p^3) \mapsto (\widetilde{p}^0, \widetilde{p}^1, \widetilde{p}^2, \widetilde{p}^3)$, where $\widetilde{p}^0, \widetilde{p}^1, \widetilde{p}^2, \widetilde{p}^3$ are the natural coordinates adapted to our new coordinate system on the Northern Hemisphere of $\mathfrak{K}$, i.e.,

$$
\begin{aligned}
p^0 &= \widetilde{p}^0, \qquad p^1 = \widetilde{p}^1, \\
p^2 &= -\frac{(x^2+y^2)^{\frac{1}{2}}}{zr}\widetilde{p}^1 + \frac{1}{z(x^2+y^2)^{\frac{1}{2}}}\left(x\widetilde{p}^2 + y\widetilde{p}^3\right), \\
p^3 &= \frac{1}{x^2+y^2}\left(-y\widetilde{p}^2 + x\widetilde{p}^3\right),
\end{aligned}
$$

using the abbreviation $z := \sqrt{r^2 - x^2 - y^2}$.

We note that

$$
\begin{aligned}
p^2 &= \mathcal{O}(1) & \frac{\partial p^2}{\partial \widetilde{p}^2} &= \mathcal{O}(1) & \frac{\partial p^2}{\partial \widetilde{p}^3} &= \mathcal{O}(1) \\
p^3 &= \mathcal{O}(S^{-1}) & \frac{\partial p^3}{\partial \widetilde{p}^2} &= \mathcal{O}(S^{-1}) & \frac{\partial p^3}{\partial \widetilde{p}^3} &= \mathcal{O}(S^{-1})
\end{aligned}
$$

as $S \to 0$.

Furthermore, the metric components with respect to the new coordinates are given on the axis as

$$
\widetilde{\mathfrak{g}}_{\alpha\beta} = diag\left(-\frac{\Delta}{r^2+a^2}, \frac{r^2+a^2}{\Delta}, \frac{r^2+a^2}{r^2}, \frac{r^2+a^2}{r^2}\right).
$$

We calculate that on the axis

$$
\begin{aligned}
\frac{\partial \mathrm{q}}{\partial \widetilde{p}^0} &= -2\frac{\Delta}{r^2+a^2}\widetilde{p}^0, \\
\frac{\partial \mathrm{q}}{\partial \widetilde{p}^2} &= 2\frac{r^2+a^2}{r^2}\widetilde{p}^2, \\
\frac{\partial \mathrm{q}}{\partial \widetilde{p}^3} &= 2\frac{r^2+a^2}{r^2}\widetilde{p}^3.
\end{aligned}
$$

For a function $G : T\mathfrak{K} \to \mathbb{R}$, we will use the notation $G = \mathcal{O}_{1,\widetilde{p}}(S)$ to subsume $G = \mathcal{O}(S)$ and $\frac{\partial G}{\partial \widetilde{p}^\alpha} = \mathcal{O}(S)$ for all $0 \leq \alpha \leq 3$.

To calculate some partial derivatives of the Carter constant

$$
\mathfrak{Q} = r^2\mathrm{q} + \frac{1}{\Delta}\left((r^2+a^2)^2(p_0)^2 - \Delta^2(p_1)^2 + a^2(p_3)^2 + 2(r^2+a^2)ap_0p_3\right)
$$

on the axis, we first note that $p_0 = \left(-1 + \frac{2mr}{\rho^2}\right)\widetilde{p}^0 + \mathcal{O}_{1,\widetilde{p}}(S)$, $p_1 = \frac{\rho^2}{\Delta}\widetilde{p}^1$, and $p_3 = \mathcal{O}_{1,\widetilde{p}}(S)$ as $S \to 0$.

This gives

$$
\mathfrak{Q} = r^2\mathrm{q} + \frac{1}{\Delta}\left(\left(r^2+a^2\right)^2\left(-1+\frac{2mr}{\rho^2}\right)^2\left(\widetilde{p}^0\right)^2 - \rho^4\left(\widetilde{p}^1\right)^2\right) + \mathcal{O}_{1,\widetilde{p}}(S)
$$

as $S \to 0$, and hence

$$\frac{\partial \mathfrak{Q}}{\partial \widetilde{p}^0}\bigg|_{S^2=0} = r^2 \frac{\partial \mathrm{q}}{\partial \widetilde{p}^0}\,|_{S^2=0} \; + 2\Delta \widetilde{p}^0,$$
$$\frac{\partial \mathfrak{Q}}{\partial \widetilde{p}^\alpha}\bigg|_{S^2=0} = r^2 \frac{\partial \mathrm{q}}{\partial \widetilde{p}^\alpha}\,|_{S^2=0} \text{ for } \alpha = 2, 3.$$

Similarly, we calculate

$$\begin{aligned} L = \langle \partial_\varphi, \dot\gamma \rangle &= -2\frac{mraS^2}{\rho^2} p^0 + \left(r^2 + a^2 + \frac{2mra^2 S^2}{\rho^2} \right) S^2 p^3 \\ &= -2\frac{mraS^2}{\rho^2} \widetilde{p}^0 + \frac{1}{r^2}\left(r^2 + a^2 + \frac{2mra^2 S^2}{\rho^2} \right) \left(-y\widetilde{p}^2 + x\widetilde{p}^3 \right), \end{aligned}$$

and get, using $\frac{\partial S^2}{\partial x} = \mathcal{O}(S)$ and $\frac{\partial S^2}{\partial y} = \mathcal{O}(S)$:

$$\frac{\partial L}{\partial \widetilde{p}^\alpha}\,|_{S^2=0} = 0 \;\; \forall\, 0 \le \alpha \le 3,$$
$$\frac{\partial L}{\partial x}\bigg|_{S^2=0} = \frac{r^2 + a^2}{r^2} \widetilde{p}^3 \;,$$
$$\frac{\partial L}{\partial y}\bigg|_{S^2=0} = -\frac{r^2 + a^2}{r^2} \widetilde{p}^2 \;.$$

Finally,

$$E = \langle \partial_t, \dot\gamma \rangle = \left(-1 + \frac{2mr}{\rho^2} \right) \widetilde{p}^0 + \frac{2ma}{r\rho^2} \left(-y\widetilde{p}^2 + x\widetilde{p}^3 \right),$$

so that

$$\frac{\partial E}{\partial \widetilde{p}^0}\bigg|_{S^2=0} = -\frac{\Delta}{r^2 + a^2} \;,$$
$$\frac{\partial E}{\partial \widetilde{p}^\alpha}\,|_{S^2=0} = 0 \qquad \forall\, 1 \le \alpha \le 3.$$

Making use of the previous calculations, we see that the differential of

$$h = (\mathrm{q}, \mathfrak{Q} - E^2 \cdot Q_{\mathrm{trap}}, L - E \cdot \Phi_{\mathrm{trap}})$$

in terms of coordinates $(r, x, y, \widetilde{p}^\alpha)$ at an axis point $\{S^2 = 0\}$ of the photon region contains the submatrix

$$\begin{pmatrix} \frac{\partial h_j}{\partial \widetilde{p}^0} & \frac{\partial h_j}{\partial \widetilde{p}^2} & \frac{\partial h_j}{\partial \widetilde{p}^3} & \frac{\partial h_j}{\partial x} & \frac{\partial h_j}{\partial y} \end{pmatrix} |_{S^2=0}$$

$$= \begin{pmatrix} -2\frac{\Delta}{r^2+a^2}\widetilde{p}^0 & 2\frac{r^2+a^2}{r^2}\widetilde{p}^2 & 2\frac{r^2+a^2}{r^2}\widetilde{p}^3 & * & * \\ -2\frac{\Delta}{r^2+a^2}r^2\widetilde{p}^0 + \heartsuit & 2\left(r^2+a^2\right)\widetilde{p}^2 & 2\left(r^2+a^2\right)\widetilde{p}^3 & * & * \\ 0 & 0 & 0 & \frac{r^2+a^2}{r^2}\widetilde{p}^3 & -\frac{r^2+a^2}{r^2}\widetilde{p}^2 \end{pmatrix}$$

with $\heartsuit := 2\Delta\widetilde{p}^0\left(1 - Q_{\text{trap}}(r)\frac{\Delta}{(r^2+a^2)^2}\right)$.

Here we have already made use of the fact that Φ_{trap} vanishes at axis points of the photon region (see Section 5.1).

First, note that at least one of the partial derivatives $\frac{\partial h_1}{\partial \widetilde{p}^2}|_{S^2=0} = 2\frac{r^2+a^2}{r^2}\widetilde{p}^2$ or $\frac{\partial h_1}{\partial \widetilde{p}^3}|_{S^2=0} = 2\frac{r^2+a^2}{r^2}\widetilde{p}^3$ is non-zero, since we can exclude $\widetilde{p}^2 = \widetilde{p}^3 = 0$. (Photons on the axis with $\widetilde{p}^2 = \widetilde{p}^3 = 0$ are radial in- or outgoers, hence not trapped.) Similarly, the last row of the matrix cannot vanish. To see that h is submersive where the photon region intersects the axis (in the tangent bundle), it is enough to show that $\heartsuit \neq 0$ for trapped photons on the axis, since this implies linear independence of all three rows. Since obviously $2\widetilde{p}^0\Delta \neq 0$, for photons in the DOC, one only needs to show that $\left(1 - Q_{\text{trap}}(r)\frac{\Delta}{r^2+a^2}\right) \neq 0$. Since $\Phi = 0$ for trapped photons on the axis, we plug $\Phi = 0$ into $\frac{R(r)}{E^2} = 0$ and solve for Q. This gives $Q = \frac{1}{\Delta}\left(r^4 + a^2r^2 + 2ma^2r\right)$ for every trapped photon on the axis. Hence, by $Q = Q_{\text{trap}}$, we get

$$1 - Q_{\text{trap}}(r)\frac{\Delta}{\left(r^2+a^2\right)^2} = \frac{\Delta a^2}{\left(r^2+a^2\right)^2} > 0,$$

which makes $\heartsuit$ non-zero for trapped photons on the axis.

We have thus seen that the differential of h is surjective at every axis point of the photon region P' in the tangent bundle. By the submersion theorem we may conclude that a neighborhood of the North (similarly: South) Pole of P' is a submanifold of $T\mathfrak{K}$.

□

From Propositions (33) and (32), we immediately get the following:

Theorem 34. *For a subcritical Kerr spacetime $\mathfrak{K}$, the photon region P' in $T\mathfrak{K}$ and the photon region P in $T^*\mathfrak{K}$ are smooth submanifolds of dimension* 5.

5.4 The topology of the Kerr photon region in the (co-)tangent bundle

We now turn our attention to the topology of P' and P. From now on, it will be more useful to work with $P \subseteq T^*\mathfrak{K}$. We will prove the following theorem towards the end of this section and first show important lemmata and propositions to be used in the proof. The claim of Theorem 35 follows from Dyatlov's implicit function theorem argument in Section 3 of [20].

Theorem 35. *The Kerr photon region and in particular the Schwarzschild photon sphere in the (co-)tangent bundle have topology $SO(3) \times \mathbb{R}^2$.*

Proof. Since the photon region in any member of the Kerr family is invariant under time translation ∂_t and under rescaling of energy $(E, p^0, p^1, p^2, p^3) \mapsto \lambda\,(E, p^0, p^1, p^2, p^3)$, $\lambda > 0$, we may therefore restrict our attention to a 6-dimensional slice

$$\{t = 0, p_0(= -E) = -1\}$$

in the cotangent bundle and show that the photon region in this slice,

$$P_0 := P \cap \{t = 0, p_0(= -E) = -1\},$$

has topology $SO(3)$.

In the Schwarzschild case, we see immediately that P_0 is the bundle of tangent 1-spheres to the 2-sphere $\{r = 3m\}$. Such a sphere bundle is, of course, topologically just $T^1\mathbb{S}^2$, and it is well-known that $T^1\mathbb{S}^2$ is homeomorphic to $SO(3)$.

The remainder of this section is dedicated to the proof of the rotating Kerr case. In order to construct explicit bundle charts of P_0, we first prove the following

Lemma 36. *Consider a Kerr spacetime with $0 < a < m$.*

1. *There are smooth functions $r_{min}, r_{max} : (0, \pi) \to \mathbb{R}$ that give, for every $\vartheta \in (0, \pi)$, the minimal and maximal Boyer–Lindquist radii where trapped photons with latitude ϑ may be located.*

2. *There is a smooth function $\overline{r} : (0, \pi) \times [-1, 1] \to \mathbb{R}$ with*

$$\begin{aligned} \overline{r}(\vartheta, -1) &= r_{min} \\ \overline{r}(\vartheta, 0) &= r_{mid}, \text{ and} \\ \overline{r}(\vartheta, 1) &= r_{max} \end{aligned}$$

for all $\vartheta \in (0, \pi)$, and with $\overline{r}\left(\frac{\pi}{2} + \vartheta, \cdot\right) = \overline{r}\left(\frac{\pi}{2} - \vartheta, \cdot\right)$ for all $\vartheta \in \left(0, \frac{\pi}{2}\right)$,

(where r_{mid} is given by Formula (43)), and with the additional property that $\overline{r}(\vartheta, \cdot)$ has a smooth inverse for every $\vartheta \in (0, \pi)$.

The function $\overline{r}(\vartheta, \cdot)$ parametrizes the radial width of the crescent moon in Figure 2.

Proof. 1. The boundary of the region accessible to trapping in the spacetime is exactly where trapped photons have vanishing ϑ-motion ($\dot{\vartheta} = 0$); hence, the wanted functions $r_{\min}, r_{\max}$ are given implicitly by the requirements that the right-hand side of the Θ-equation (37) with $Q = Q_{\text{trap}}(r)$ and $\Phi = \Phi_{\text{trap}}(r)$ plugged in vanishes:

$$0 = Q_{\text{trap}}(r_{\text{min}}(\vartheta)) - \left(\Phi_{\text{trap}}(r_{\text{min}}(\vartheta)) - a^2\right)\cos^2(\vartheta),$$
$$0 = Q_{\text{trap}}(r_{\text{max}}(\vartheta)) - \left(\Phi_{\text{trap}}(r_{\text{max}}(\vartheta)) - a^2\right)\cos^2(\vartheta)$$

for all $\vartheta \in (0, \pi)$, and the additional conditions that $r_{\text{min}} < r_{\text{mid}} < r_{\text{max}}$ and that $r_{\text{min}}(\vartheta), r_{\text{max}}(\vartheta)$ are the the zeros of $\Theta(\cdot, \vartheta) = 0$ that are closest to $r_{\text{mid}}(\vartheta)$ for each ϑ.

Thus, r_{min} and r_{max} are well-defined by elementary properties of the region accessible to trapping, and smoothness follows from the smooth dependence of Θ on r and ϑ.

2. This follows directly from the first part of the lemma, for example by explicitly defining

$$\overline{r}(\vartheta, s) := \begin{cases} \left(r_{\text{mid}}(\vartheta) - r_{\text{min}}(\vartheta)\right) s + r_{\text{mid}}(\vartheta) & \text{for } s < 0, \\ r_{\text{mid}}(\vartheta)\left(\frac{r_{\text{max}}(\vartheta) + r_{\text{min}}(\vartheta) - r_{\text{mid}}(\vartheta)}{r_{\text{mid}}(\vartheta)}\right)^{s^2} + \left(r_{\text{mid}}(\vartheta) - r_{\text{min}}(\vartheta)\right) s & \text{for } s \geq 0. \end{cases}$$

□

For the following lemma, we fix some notation and interpret the product $\mathbb{S}^1 \times \mathbb{S}^1$ as follows: let the first factor $\mathbb{S}^1$ be parametrized by $\varphi \in [-\pi, \pi)$, and the second factor $\mathbb{S}^1$ be viewed as $[-1, 1] \times \{-1, 1\}/\sim_{\mathbb{S}^1}$, where $\sim_{\mathbb{S}^1}$ is the equivalence relation on $[-1, 1] \times \{-1, 1\}$ generated by identifying $(\pm 1, 1)$ with $(\pm 1, -1)$. We thus denote elements of $\mathbb{S}^1 \times \mathbb{S}^1$ in the form (φ, s, ς).

Recall that we write $P_0 = P \cap \{t = 0, p_0 = -1\}$.

Lemma 37. *There are functions $p_2, p_3 : (0, \pi) \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ such that the map*

$$H : (0, \pi) \times \mathbb{S}^1 \times \mathbb{S}^1 \to P_0 \setminus \{S^2 = 0\},$$

given in Boyer–Lindquist coordinates in the form

$$(\vartheta, \varphi, s, \varsigma) \mapsto H(\vartheta, \varphi, s, \varsigma) = (\overline{r}(\vartheta, s), \vartheta, \varphi, 0, p_2(\vartheta, \varphi, s, \varsigma), p_3(\vartheta, \varphi, s, \varsigma))$$

is a diffeomorphism, where $\overline{r}$ is defined as in Lemma 36 (and hence only depends on $|\vartheta - \frac{\pi}{2}|$ and s).

Proof. We define functions $p^2, p^3 : (0, \pi) \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ as the solutions of Equations (50) and (51) with $\overline{r}(\vartheta, s)$ plugged in for r and $E = 1$ plugged in for e, and with the additional requirement that $\operatorname{sgn} p^2(\vartheta,\varphi,s,\varsigma) = \varsigma$. In other words,

$$p^2(\vartheta,\varphi,s,\varsigma) = \varsigma\left(\frac{1}{\rho^2(\overline{r},\vartheta)}\cdot\left(Q_{\text{trap}}(\overline{r}) - \left(\frac{\Phi^2_{\text{trap}}(\overline{r})}{S^2} - a^2\right)C^2\right)\right)^{\frac{1}{2}},$$
$$p^3(\vartheta,\varphi,s,\varsigma) = \frac{1}{\Delta(\overline{r})\rho^2(\overline{r},\vartheta)}\cdot\left(2m\overline{r}a + \left(\rho^2(\overline{r},\vartheta) - 2m\overline{r}\right)\frac{\Phi_{\text{trap}}(\overline{r})}{S^2}\right),$$

where we use the shorthand $\overline{r} = \overline{r}\,(\vartheta,s)$.

Both p^2 and p^3 are smooth on $(0,\pi)\times\mathbb{S}^1\times\mathbb{S}^1$, and they are of course, as the notation suggests, meant to be used as 7-th and 8-th coordinate in $T\mathfrak{K}$.

Then, a function p^0 is defined as the unique positive solution to

$$\mathfrak{g}\left((p^0,0,p^2,p^3),(p^0,0,p^2,p^3)\right) = 0;$$

it is smooth by smoothness of the metric $\mathfrak{g}$.

The functions p_2, p_3 from the present lemma are obtained by type-changing the vector $(p^0,0,p^2,p^3) \in T\mathfrak{K}$ into a 1-form.

Bijectivity of H is clear by the construction, and smoothness of the inverse map H^{-1} follows directly from the fact that for every $\vartheta \in (0,\pi)$, the radial width function $\overline{r}(\vartheta,\cdot)$ has a smooth inverse.

□

Lemma 38. *Let* $0 < \vartheta_0 < \pi$ *and* $I_{\vartheta_0} := (\Phi_{trap}(r_{min}(\vartheta_0)), \Phi_{trap}(r_{max}(\vartheta_0)))$. *There is a unique smooth function* $\overline{p}_2 : I_{\vartheta_0} \to \mathbb{R}_{\geq 0}, p_3 \mapsto \overline{p}_2(p_3)$ *such that there is an element of* P_0 *with latitude* ϑ_0*, angular momentum* p_3*, and third covariant Boyer–Lindquist coordinate* $p_2 = \overline{p}_2(p_3)$*.*

If $\sin^2\vartheta_0$ *is small enough,* $\overline{p}_2$ *is concave.*

Proof. Recall that the covariant Boyer–Lindquist coordinate p_3 coincides with the scaled angular momentum $\Phi = \Phi_{\text{trap}}$ for every photon in P_0. Since Φ_{trap} is strictly decreasing in r on the photon region, we may regard r as a function of p_3 and write $r(p_3)$ for the solution of $p_3 = \Phi_{\text{trap}}(r)$.

Similarly to the treatment of the functions p^2, p^3 in the proof of Lemma 37, we use Equation (40) to see that $\overline{p}_2 > 0$ is given by

$$(\overline{p}_2)^2 = Q_{\text{trap}}(r(p_3)) - \left((p_3)^2 - a^2\right)\frac{C^2}{S^2} - a^2\frac{C^4}{S^2}.$$

Therefore,

$$\frac{d^2}{d(p_3)^2}\left(\overline{p}_2(r)\right)^2 = \frac{d^2}{d(p_3)^2}Q_{\text{trap}}(r(p_3)) - 2\frac{C^2}{S^2}.$$

Since $\frac{d^2}{d(p_3)^2} Q_{\text{trap}}(p_3)$ is bounded in a neighborhood of the photon region and $\frac{C^2}{S^2}$ approaches infinity at the poles, $\frac{d^2}{d(p_3)^2} \left(\overline{p}_2(r)\right)^2$ is negative in a punctured neighborhood of any pole. Hence, $(\overline{p}_2)^2$ and also $\overline{p}_2$ are concave if $\sin^2 \vartheta_0$ is chosen sufficiently small. □

Lemma 39. *There is a $0 < \varepsilon < 1$ such that the neighborhood $P_0 \cap \{\sin^2 \vartheta < \varepsilon\}$ of the North and South Pole of the photon region in the phase space slice $\{t = 0, p_0 = -1\}$ of a subcritical Kerr spacetime is diffeomorphic to $\left(\mathbb{S}^2 \cap \{\sin^2 \vartheta < \varepsilon\}\right) \times \mathbb{S}^1$ via*

$$\Psi : P_0 \cap \{\sin^2 \vartheta < \varepsilon\} \to \left(\mathbb{S}^2 \cap \{\sin^2 \vartheta < \varepsilon\}\right) \times \mathbb{S}^1,$$

where Ψ is given in the coordinates of the proof of Proposition 32 by

$$\Psi(0, r, \vartheta, \varphi, -1, 0, \widetilde{p}_2, \widetilde{p}_3) := (\vartheta, \varphi, [\operatorname{atan2}(\widetilde{p}_2, \widetilde{p}_3)]) .$$

(A definition of atan2 *is given in the proof below.)*

Proof. We now view the sphere $\mathbb{S}^1$ as the quotient of the real line that is obtained by factoring out the equivalence relation generated by $u \sim u + 2\pi$ and denote its elements by $[u] := \{u + 2\pi k : k \in \mathbb{Z}\}$.

Note that using (ϑ, φ) also on the poles causes no problem, even though they are not coordinate functions at the poles.

As usual, $\operatorname{atan2} : \mathbb{R}^2 \setminus \{(0,0)\} \to (-\pi, \pi]$ gives the angle between the positive $\widetilde{p}_2$-axis and $(\widetilde{p}_2, \widetilde{p}_3)$ (keeping track of its sign) and is defined by

$$\operatorname{atan2}(v, u) = \begin{cases} \arctan\left(\frac{v}{u}\right) & \text{for } u > 0, \\ \arctan\left(\frac{v}{u}\right) + \operatorname{sgn} v \cdot \pi & \text{for } u < 0, \\ \operatorname{sgn} v \cdot \frac{\pi}{2} & \text{for } u = 0. \end{cases}$$

Let $\varepsilon > 0$ be such that the function $\overline{p}_2(\vartheta, \cdot)$ from Lemma 38 is concave for every ϑ with $\sin^2 \vartheta < \varepsilon$.

The map Ψ as given in the claim is well-defined and smooth. We write Ψ_{North} and Ψ_{South} for the restrictions of Ψ to the Northern and Southern pole caps $P_0 \cap \{\sin^2 \vartheta < \varepsilon, \vartheta < (>) \frac{\pi}{2}\}$. Since the union of the pole caps is disjoint, it suffices to show that Ψ_{North} and Ψ_{South} are diffeomorphisms.

Focusing on Ψ_{North}, we prove that Ψ_{North} is a bijection by arguing that for fixed (ϑ_0, φ_0), the map

$$\Psi_{\text{North}}(0, r, \vartheta_0, \varphi_0, -1, 0, \cdot, \cdot) : P_0 \cap \{\vartheta = \vartheta_0, \varphi = \varphi_0, \} \to \{\vartheta_0, \varphi_0\} \times \mathbb{S}^1$$

is bijective:

First we fix a $(\vartheta_0, \varphi_0) \in \mathbb{S}^2$ with $\vartheta_0 < \pi$, $\sin^2 \vartheta_0 < \varepsilon$.

We need to show that [atan2] maps the set

$$\{(\widetilde{p}_2, \widetilde{p}_3) \in \mathbb{R}^2 : \exists\, r \text{ such that } (0, r, \vartheta_0, \varphi_0, -1, 0, \widetilde{p}_2, \widetilde{p}_3) \text{ is a trapped photon}\}$$

bijectively to $\mathbb{S}^1$. To this end, we change from the covariant coordinates $\widetilde{p}_2, \widetilde{p}_3$ to the ones that belong to Boyer–Lindquist coordinates, p_2 and p_3. The set

$$\{(p_2, p_3) \in \mathbb{R}^2 : \exists\, r \text{ such that } (0, r, \vartheta_0, \varphi_0, -1, 0, p_2, p_3) \text{ is a trapped photon}\}$$

bounds a convex region around the origin in $\mathbb{R}^2$, since it can be piecewise parametrized by $p_3 \mapsto (\pm\overline{p}_2(p_3), p_3)$, where $\overline{p}_2$ is the concave function from Lemma 38. Its image under the linear bijection

$$\mathbb{R}^2 \ni (p_2, p_3) \mapsto (\tan\vartheta_0 \cdot (\cos\varphi_0 p_2 - \sin\varphi_0 p_3), -\sin\varphi_0 p_2 + \cos\varphi_0 p_3) \in \mathbb{R}^2$$

is also a convex set around the origin, which means that [atan2] maps it bijectively to $\mathbb{S}^1$.

On the other hand, one can easily check that

$$\operatorname{atan2}(\widetilde{p}_2, \widetilde{p}_3) = \operatorname{atan2}\left(\tan\vartheta_0 \cdot (\cos\varphi_0 p_2 - \sin\varphi_0 p_3), -\sin\varphi_0 p_2 + \cos\varphi_0 p_3\right),$$

since $\frac{\widetilde{p}_2}{\widetilde{p}_3}$ can be rewritten in terms of p_2 and p_3 as

$$\frac{\widetilde{p}_2}{\widetilde{p}_3} = \tan\vartheta_0 \cdot \frac{\cos\varphi_0 p_2 - \sin\varphi_0 p_3}{-\sin\varphi_0 p_2 + \cos\varphi_0 p_3}$$

if $\widetilde{p}_3 \neq 0$, and $\operatorname{sgn}\widetilde{p}_2 = \operatorname{sgn}(\cos\varphi_0 p_2 - \sin\varphi_0 p_3)$.

This proves that $\Psi_{\text{North}}(0, r, \vartheta_0, \varphi_0, -1, 0, \cdot, \cdot) : P_0 \cap \{\vartheta = \vartheta_0, \varphi = \varphi_0\} \to \{\vartheta_0, \varphi_0\} \times \mathbb{S}^1$ is bijective for every $(\vartheta_0, \varphi_0) \in \mathbb{S}^2$ with $\vartheta_0 < \pi$, $\sin^2\vartheta_0 < \varepsilon$.

Now we show that $\Psi_{\text{North}}|_{\{S^2=0\}}$ is a bijection onto $\{S^2 = 0\} \times \mathbb{S}^1$:

At the North Pole $\{S^2 = 0\}$ of the region accessible to trapping, every lightlike point in the phase space with $\widetilde{p}_1 = 0$ is a trapped photon, and it is obvious by the rotational symmetry of the Kerr spacetime that the set $\{(\widetilde{p}_2, \widetilde{p}_3) : (-1, 0, \widetilde{p}_2, \widetilde{p}_3) \text{ is a photon}\}$ is a circle around the origin in the $\widetilde{p}_2$-$\widetilde{p}_3$-plane. As before, it is mapped bijectively under [atan2] onto $\mathbb{S}^1$.

We have now seen that Ψ_{North} is bijective. A straightforward calculation yields that the differential of Ψ_{North} has full rank everywhere, so that by the implicit function theorem Ψ_{North} is a diffeomorphism, which proves the claim. □

In order to determine the topology of P_0 (and thereby the topology of P), we will calculate its first fundamental group.

Proposition 40. *The Kerr photon region slice P_0 has first fundamental group $\mathbb{Z}_2$.*

Proof. The diffeomorphisms H from Lemma 37 and Ψ from Lemma 39 will be used in what follows. Recall (from the proof of Lemma 33) the covariant coordinates $\widetilde{p_i}$ that are naturally associated with the coordinates $(t, r, x = r\cos\varphi\sin\vartheta, y = r\sin\varphi\sin\vartheta)$.

While $U_{\text{North}} \approx B_1(0) \times \mathbb{S}^1$ and $U_{\text{South}} \approx B_1(0) \times \mathbb{S}^1$ both have $\mathbb{Z}$ as their first fundamental group, $U_{\text{Eq}} \approx (-\pi, \pi) \times \mathbb{S}^1 \times \mathbb{S}^1$ has first fundamental group $\mathbb{Z} \times \mathbb{Z}$. Moreover, $\pi_1(U_{\text{Eq}} \cap U_{\text{North}}) = \pi_1(U_{\text{Eq}} \cap U_{\text{South}}) = \mathbb{Z} \times \mathbb{Z}$.

Let $\frac{\pi}{2} < \vartheta_0 < \pi$ be such that every point of P_0 with latitude ϑ_0 is in the domain of Ψ_{North}. We define homotopies of closed paths $\gamma_1, \gamma_2 : \mathbb{S}^1 \times I \to P_0$ by

$$\gamma_{1,\lambda}(\varphi) := H\left(\vartheta_0 - \lambda(2\vartheta_0 - \pi), \varphi, 0, -1\right),$$
$$\gamma_{2,\lambda}(s, \varsigma) := H\left(\vartheta_0 - \lambda(2\vartheta_0 - \pi), 0, s, \varsigma\right).$$

By construction, the paths $\gamma_{1,0}$ and $\gamma_{2,0}$ are representatives of a set of generators for $\pi_1(U_{\text{Eq}} \cap U_{\text{North}})$, and the analogous statement holds for $\gamma_{1,1}$ and $\gamma_{2,1}$ in $\pi_1(U_{\text{Eq}} \cap U_{\text{South}})$, as well as for $\gamma_{1,\lambda}$ and $\gamma_{2,\lambda}$ in $\pi_1(U_{\text{Eq}})$ (for every λ).

We will now determine what elements in $\pi_1(U_{\text{North}})$ the paths $\gamma_{i,0}$ represent and what elements in $\pi_1(U_{\text{South}})$ the paths $\gamma_{i,1}$ represent (for $i = 1, 2$).

Note that by construction of H, the image of each $\gamma_{1,\lambda}$ consist of points with $r = r_{\text{mid}}$ and hence with $p_3 = 0$.

This makes it easy to calculate—using the standard transformations for the coordinates of the phase part of $T\mathfrak{K}^*$—that for points in the image of a path $\gamma_{1,\lambda}$ with $\lambda \neq \frac{1}{2}$:

$$\widetilde{p}_2 \circ \gamma_{1,\lambda}(\varphi) = \frac{p_2}{r\cos\vartheta}\cos\varphi \tag{58}$$
$$\widetilde{p}_3 \circ \gamma_{1,\lambda}(\varphi) = -\frac{p_2}{r\sin\vartheta}\sin\varphi, \tag{59}$$

where we use ϑ as shorthand for $\vartheta(\lambda) = \vartheta_0 - \lambda(2\vartheta_0 - \pi)$ and p_2 as shorthand for $p_2 \circ \gamma_{1,\lambda}(\varphi)$.

Clearly, $|p_2| \circ \gamma_{1,\lambda}(\varphi)$ can be calculated from $p_3 \circ \gamma_{1,\lambda} = 0$, $r \circ \gamma_{1,\lambda} = r_{\text{mid}}$, and the modulus of the latitude, $|\vartheta - \frac{\pi}{2}|$. Since we have chosen negative sign of p_2 for all points in the range of $\gamma_{1,\lambda}$, even $p_2 \circ \gamma_{1,\lambda}(\varphi)$ only depends on $|\vartheta - \frac{\pi}{2}|$.

This allows us to simplify (58)–(59) to

$$\begin{aligned}\widetilde{p}_2 \circ \gamma_{1,0}(\varphi) &= A\cos\varphi,\\ \widetilde{p}_2 \circ \gamma_{1,1}(\varphi) &= -A\cos\varphi,\\ \widetilde{p}_3 \circ \gamma_{1,\lambda}(\varphi) &= B\sin\varphi \quad \text{for } \lambda = 0,1\end{aligned}$$

for positive constants A, B.

We see from these formulas that the map $P_3 \circ \Psi \circ \gamma_{1,0} : \mathbb{S}^1 \to \mathbb{S}^1$ (where P_3 is the projection onto the third component), given by $[\text{atan2}(\widetilde{p}_2 \circ \gamma_{1,0}, \widetilde{p}_3 \circ \gamma_{1,0})]$, has index 1 and is hence homotope to the identity map on $\mathbb{S}^1$.

Similarly, $P_3 \circ \Psi \circ \gamma_{1,1} : \mathbb{S}^1 \to \mathbb{S}^1$ is of index -1 and can be thought of as the map -id : $\mathbb{S}^1 \to \mathbb{S}^1$.

We may thus note for later use **Fact 1**: the path $\Psi\circ\gamma_{1,0}$ is equivalent to $\varphi \mapsto (\vartheta_0, 0, \varphi)$ in $\pi_1(\Psi(U_{\text{North}} \cap U_{\text{Eq}}))$, and the path $\Psi \circ \gamma_{1,1}$ is equivalent to $\varphi \mapsto (\pi - \vartheta_0, 0, -\varphi)$ in $\pi_1(\Psi(U_{\text{South}} \cap U_{\text{Eq}}))$.

Similarly as we just did for $\gamma_{1,\lambda}$, we now calculate the components $\widetilde{p}_2$ and $\widetilde{p}_2$ on the orbits of $\gamma_{2,\lambda}$ for $\lambda \neq \frac{1}{2}$:

$$\begin{aligned}\widetilde{p}_2 \circ \gamma_{2,\lambda}(s,\varsigma) &= \frac{1}{r\cos\vartheta} p_2\\ \widetilde{p}_3 \circ \gamma_{2,\lambda}(s,\varsigma) &= \frac{1}{r\sin\vartheta} p_3.\end{aligned}$$

Here, p_2 and p_3 depend on $\vartheta(\lambda) = \vartheta_0 - \lambda(2\vartheta_0 - \pi)$ and (s,ς). Actually, p_2 and p_3 are independent of the sign of $\vartheta - \frac{\pi}{2}$, since the function $\overline{r}$ used in the construction of H is.

This allows us to write

$$\widetilde{p}_2 \circ \gamma_{2,0}(s,\varsigma) = -\widetilde{A} p_2, \tag{60}$$

$$\widetilde{p}_2 \circ \gamma_{2,1}(s,\varsigma) = \widetilde{A} p_2, \tag{61}$$

$$\widetilde{p}_3 \circ \gamma_{2,\lambda}(s,\varsigma) = \widetilde{B} p_3 \quad \text{for } \lambda = 0,1 \tag{62}$$

for positive constants $\widetilde{A}, \widetilde{B}$.

We already know that $\gamma_{2,0}$ represents one of the generators of $\pi_1(U_{\text{Eq}} \cap U_{\text{North}})$, and that $\gamma_{2,1}$ represents one of the generators of $\pi_1(U_{\text{Eq}} \cap U_{\text{South}})$. Choosing an orientation for $\mathbb{S}^1 \equiv [-1,1] \times \{-1,1\}/\sim_{\mathbb{S}^1}$, we can decide that $P_3 \circ \Psi \circ \gamma_{2,i} : \mathbb{S}^1 \to \mathbb{S}^1$ has index 1. Then $P_3 \circ \Psi \circ \gamma_{2,i} : \mathbb{S}^1 \to \mathbb{S}^1$ has index -1. We can now state **Fact 2**: the path $\Psi \circ \gamma_{2,0}$ is equivalent to $\varphi \mapsto (\vartheta_0, 0, \varphi)$ in $\pi_1(\Psi(U_{\text{North}} \cap U_{\text{Eq}}))$, and the path $\Psi \circ \gamma_{1,1}$ is equivalent to $\varphi \mapsto (\pi - \vartheta_0, 0, -\varphi)$ in $\pi_1(\Psi(U_{\text{South}} \cap U_{\text{Eq}}))$.

In the last step of the proof, we finally calculate $\pi_1(P_0)$ using the Seifert–van Kampen theorem. Combining Fact 1 and Fact 2, we note that in $\pi_1 \left(U_{\text{North}} \cap U_{\text{Eq}}\right)$,

$$[\gamma_{1,0}]_{\pi_1\left(U_{\text{North}} \cap U_{\text{Eq}}\right)} = [\gamma_{2,0}]_{\pi_1\left(U_{\text{North}} \cap U_{\text{Eq}}\right)},$$

and in $\pi_1 \left(U_{\text{South}} \cap U_{\text{Eq}}\right)$, that

$$[\gamma_{1,1}]_{\pi_1\left(U_{\text{South}} \cap U_{\text{Eq}}\right)} = [\gamma_{2,1}]_{\pi_1\left(U_{\text{South}} \cap U_{\text{Eq}}\right)}.$$

We apply the Seifert–van Kampen theorem two times; first to U_{North} and U_{Eq}, then to $U_{\text{North}} \cup U_{\text{Eq}}$ and U_{South}).

For the sake of readability, the homomorphisms of the different fundamental groups induced by the various inclusion maps are notationally omitted.

Using the group presentations

$$\pi_1 \left(U_{\text{North}}\right) = \langle [\gamma_{1,0}]_{\pi_1(U_{\text{North}})}, [\gamma_{2,0}]_{\pi_1(U_{\text{North}})} \rangle$$

and

$$\pi_1 \left(U_{\text{Eq}}\right) = \langle [\gamma_{1,0}]_{\pi_1\left(U_{\text{Eq}}\right)}, [\gamma_{2,0}]_{\pi_1\left(U_{\text{Eq}}\right)} \rangle,$$

one obtains $\pi_1 \left(U_{\text{Eq}} \cup U_{\text{North}}\right)$ by factoring the identifications

$$[\gamma_{1,0}]_{\pi_1(U_{\text{North}})} = [\gamma_{2,0}]_{\pi_1\left(U_{\text{Eq}}\right)},$$
$$[\gamma_{1,0}]_{\pi_1(U_{\text{North}})} = [\gamma_{1,0}]_{\pi_1\left(U_{\text{Eq}}\right)}$$

out of the group

$$\langle [\gamma_{1,0}]_{\pi_1(U_{\text{North}})}, [\gamma_{2,0}]_{\pi_1(U_{\text{North}})}, [\gamma_{1,0}]_{\pi_1\left(U_{\text{Eq}}\right)}, [\gamma_{2,0}]_{\pi_1\left(U_{\text{Eq}}\right)} \rangle.$$

Now, $\pi_1(P_0) = \pi_1 \left(\left(U_{\text{Eq}} \cup U_{\text{North}}\right) \cup U_{\text{South}}\right)$ can be calculated by factoring the identifications

$$[\gamma_{1,1}]_{\pi_1(U_{\text{South}})} = [\gamma_{2,0}]_{\pi_1\left(U_{\text{Eq}}\right)},$$
$$[\gamma_{1,1}]_{\pi_1(U_{\text{South}})} = [\gamma_{1,0}]_{\pi_1\left(U_{\text{Eq}}\right)},$$

(which come from Facts 1 and 2) out of the free product of $\pi_1 \left(U_{\text{Eq}} \cup U_{\text{North}}\right)$ with $\pi_1 \left(U_{\text{South}}\right)$.

Hence, $\pi_1(P_0)$ is the quotient of $\langle [\gamma_{1,0}]_{\pi_1\left(U_{\text{Eq}}\right)}, [\gamma_{2,0}]_{\pi_1\left(U_{\text{Eq}}\right)}, [\gamma_{1,0}]_{\pi_1(U_{\text{North}})}, [\gamma_{1,1}]_{\pi_1(U_{\text{South}})} \rangle$

after factoring out the identifications

$$
\begin{aligned}
[\gamma_{1,0}]_{\pi_1(U_{\text{North}})} &= [\gamma_{2,0}]^{-1}_{\pi_1(U_{\text{Eq}})}\,, \\
[\gamma_{1,0}]_{\pi_1(U_{\text{North}})} &= [\gamma_{1,0}]_{\pi_1(U_{\text{Eq}})}\,, \\
[\gamma_{1,1}]_{\pi_1(U_{\text{South}})} &= [\gamma_{2,0}]_{\pi_1(U_{\text{Eq}})}\,, \\
[\gamma_{1,1}]_{\pi_1(U_{\text{South}})} &= [\gamma_{1,0}]_{\pi_1(U_{\text{Eq}})}\,.
\end{aligned}
$$

The quotient we are left with is the group $\mathbb{Z}_2$. □

We are now going to conclude in a very short argument that the photon region in the phase space of a subcritical Kerr spacetime has topology $SO(3) \times \mathbb{R}^2$:

Proof of Theorem 35. Since P_0 is a closed 3-dimensional manifold with $\pi_1(P_0) = \mathbb{Z}_2$, it is doubly covered by an $\mathbb{S}^3$ (by the Poincaré conjecture). By the elliptization conjecture, this $\mathbb{S}^3$ can be taken to be the standard 3-sphere and the group $\mathbb{Z}_2$ as a subgroup of $S = (3)$ acting on it. (For the statements of the Poicaré and the elliptization conjecture, see e.g. [65, 59]; for the proof covering these conjectures see [49, 51, 50].) Hence, P_0 is the quotient $\mathbb{S}^3/\mathbb{Z}_2 \approx \mathbb{R}P^3 \approx SO(0)$.

Recalling how P_0 was obtained as a slice $P \cap \{t = 0, p_0 = -1\}$ of the photon region in the phase space, this proves Theorem 35.

□

5.5 Outlook

We have described how the phenomenon of trapping of light in subcritical Kerr spacetimes can be better understood in the framework of the cotangent bundle and have characterized the set of (future) trapped photons as a submanifold of $T^*\mathfrak{K}$ with topology $SO(3) \times \mathbb{R}^2$. It is natural to ask if analogous results can be obtained with similar methods for other stationary spacetimes, possibly involving a cosmological constant. Note that in [20], the implicit function argument for the computation of the topology of the photon sphere in the phase space is also applied to the Kerr–de Sitter spacetime.

Null geodesics of constant coordinate radius in (various subfamilies of) the Plebański–Demiański class of metrics have been studied in [28, 29]. To our knowledge, it has not yet been rigorously investigated whether these photon orbits are the only trapped photons in the respective spacetimes, which would be a crucial first step for a geometric understanding of the photon region. Results for spherical photons in the Plebański–Demiański spacetime family show a very similar spacetime picture of trapping as in the Kerr family ([28, 29]); one might also hope to prove in this setting that the photon region in the phase space is a submanifold with topology $SO(3) \times \mathbb{R}^2$, but of course, the calculations would become much more involved in this generalized setting.

We can also ask similar questions about the photon region in stationary spacetimes of higher dimensions. By the symmetry of a Schwarzschild–Tangherlini black hole [63] of dimension $n + 1$ and mass m, it is known that trapping of light occurs at a fixed radius $r = nm$ for every photon that is initially tangent to the hypersurface $\{r = nm\}$. Hence (by arguments just like in the 4-dimensional Schwarzschild case), the photon region in the phase space can be expected to be a submanifold with the topology of a tangent unit sphere bundle $T^1\mathbb{S}^{n-1}$, times $\mathbb{R}^2$.

It seems reasonable to conjecture that for Myers–Perry spacetimes of dimension $n+1$, the photon region in the phase space has the same topology and bundle structure as the one of the Schwarzschild–Tangherlini solution of dimension $n + 1$ (whether the various rotation parameters coincide or not), since letting the rotational parameters tend to zero should not cause jumps in the topology, just as in the $3 + 1$-dimensional Kerr case. This can, however, not be proved by similar methods as the ones we used for the 4-dimensional case, since our proof relies on the classification of 3-manifolds via their fundamental groups, and there is no similar classification available for higher dimensions.

The presented way to determine the topology of the Kerr photon region in phase space might turn out to be useful in proving a uniqueness theorem for asymptotically flat, stationary, vacuum spacetimes outside a photon region, in the spirit of the static results in this direction mentioned in Section 4.6.

6 References

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