Spektralzerlegung und Reskalierung für den Bolthausen-Sznitman-Coalescent

Dissertation
der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
Jonas Valentin Kukla
aus Rosenheim

Tübingen
2018

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation:	30.07.2018
Dekan:	Prof. Dr. Wolfgang Rosenstiel
1. Berichterstatter:	Prof. Dr. Martin Möhle
2. Berichterstatter:	Prof. Dr. Martin Zerner

Inhaltsverzeichnis

Einleitung und Zusammenfassung

Populationsgenetische Motivation

In der Mathematischen Populationsgenetik werden Modelle für das Verhalten biologischer Populationen betrachtet. Dabei werden meistens vereinfachende Annahmen getroffen, wie im vorliegenden Fall eine konstante, aber sehr große Populationsgröße N. Außerdem werden in der vorliegenden Arbeit ausschließlich haploide Populationen betrachtet, das heißt jedes Individuum hat immer nur ein Elterindividuum. Coalescent-Prozesse sind partitionswertige Markovprozesse, die verwendet werden können, um die Genealogie einer Stichprobe von n Indiviuen zu modellieren. Dabei läuft die Zeit rückwärts. In der Gegenwart, also zum Zeitpunkt 0, hat jedes Individuum seinen eigenen Block. Wenn man in die Vergangenheit zurückschaut, wächst der Zeitparameter t und sobald ein gemeinsamer Vorfahre von zwei oder mehr Individuen aus der Stichprobe auftritt, verschmelzen die Blöcke dieser Individuen. Da das Modell haploid ist, können, wenn t wächst, immer nur Blöcke verschmelzen. Dabei hängen die Raten für die Verschmelzungen nur von den Gruppengrößen der verschmelzenden Blöcke ab, aber nicht etwa von der Größe der verschmelzenden Blöcke. Außerdem gibt es keine systematischen Unterschiede im Verhalten der einzelnen Individuen. Das erste Beispiel eines Coalescent-Prozesses wurde von Kingman untersucht [21], spätere Verallgemeinerungen erfolgten zum Λ-Coalescent durch Pitman [36] und Sagitov [41] sowie weiter zum Ξ-Coalescent durch Schweinsberg [42] sowie Möhle und Sagitov [31].

Austauschbare Coalescents

Im folgenden wird für jede Menge B die Anzahl ihrer Elemente als *Mächtigkeit* $\#B$ oder $|B|$ bezeichnet. Eine *Partition* π einer Menge A, beispielsweise $A = [n] := \{1, \dots, n\}$, $n \in \mathbb{N}$ ist eine Menge nicht-leerer, paarweise disjunkter Teilmengen $B \subseteq A$, auch *Blöcke* genannt, deren Vereinigung A ergibt. Eine Partition kann auch als Äquivalenzrelation $\sim_\pi$ auf A aufgefasst werden, wobei für $i, j \in A$ gilt, dass $i \sim_\pi j$ genau dann, wenn es ein $B \in \pi$ gibt mit $i, j \in B$. Die Menge aller Partitionen von A wird im Folgenden mit $\mathcal{P}_A$ bezeichnet. Auf $\mathcal{P}_A$ lässt sich eine Ordnung $\leq$ definieren, indem $\pi \leq \rho$ genau dann, wenn jeder Block von π Teilmenge eines Blocks von ρ ist. Beachte, dass aus $\pi \leq \rho$ folgt $\#\pi \geq \#\rho$. Für eine Partition π von A und eine Permutation $\sigma : A \to A$ wird durch $i \sim_{\pi\circ\sigma} j \Leftrightarrow \sigma(i) \sim_\pi \sigma(j)$ eine Partition $\pi \circ \sigma$ auf A definiert. Für $C \subseteq A$ wird $\pi|_C := \{B \cap C : B \in \pi, B \cap C \neq \emptyset\}$ als *Einschränkung* von π auf C bezeichnet. Ein *austauschbarer n-Coalescent* auf einem Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, P)$ ist ein $\mathcal{P}_{[n]}$-wertiger homogener Markovprozess $(\Pi_t^{(n)})_{t\in[0,\infty)}$ mit den folgenden Eigenschaften:

- Für jedes $\omega \in \Omega$ ist $t \mapsto \Pi_t^{(n)}(\omega)$ in der diskreten Topologie auf $\mathcal{P}_{[n]}$ càdlàg, das heißt rechsstetig mit linksseitigen Grenzwerten.
- Es gilt $\Pi_0^{[n]} = \Delta_{[n]} := \{\{i\} : i \in [n]\}$.
- Für $0 \leq t_1 \leq t_2 < \infty$ gilt $\Pi_{t_1} \leq \Pi_{t_2}$.
- $(\Pi_t^{(n)})_{t\in[0,\infty)}$ ist *austauschbar*, das heißt für jede Permutation $\sigma : [n] \to [n]$ gilt

$$(\Pi_t^{(n)} \circ \sigma)_{t\in[0,\infty)} \stackrel{d}{=} (\Pi_t^{(n)})_{t\in[0,\infty)}.$$

Ist $P^{(n)}$ für jedes $n \in \mathbb{N}$ die Verteilung eines n-Coalescent $\Pi^{(n)}$ auf $\mathcal{P}_{[n]}$, dann heißt $(P^{(n)})_{n\in\mathbb{N}}$ eine *konsistente* Familie, wenn für $m \leq n$ gilt $\Pi^{(n)}|_{[m]} \stackrel{d}{=} P^{(m)}$. Für jedes $n \in \mathbb{N}$ und $\omega \in \Omega$ ist $\Pi^{(n)}(\omega)$ càdlàg und monoton wachsend auf dem endlichen Zustandsraum $\mathcal{P}_{[n]}$ und kann deshalb durch die Werte $\inf\{t \in [0,\infty) : \Pi_t^{(n)}(\omega) \geq \pi\}$, $\pi \in \mathcal{P}_{[n]}$ eindeutig beschrieben werden. Also ist der Zustandsraum von $\Pi^{(n)}$ in $[0,\infty]^{\#\mathcal{P}_{[n]}}$ einbettbar und damit ein Borelscher Raum. Der Erweiterungssatz von Kolmogorov garantiert die Existenz eines Prozesses $\Pi^{(\infty)}$ mit Werten in $\mathcal{P}_{\mathbb{N}}$ und $\Pi^{(\infty)}|_{[n]} \stackrel{d}{=} P^{(n)}$. Durch $\Pi^{(n)} := \Pi^{(\infty)}|_{[n]}$, $n \in \mathbb{N}$ erhält man eine Familie $(\Pi^{(n)})_{n\in\mathbb{N}}$ von auf natürlich Weise gekoppelten n-Coalescents.

Unter den austauschbaren Coalescents auf $\mathbb{N}$ sind die Λ-Coalescents hervorzuheben, bei denen im Gegensatz zu den Ξ-Coalescents fast sicher zu jedem Zeitpunkt höchstens eine Verschmelzung von mehreren Blöcken zu einem Block auftreten kann. Pitman [36] und Sagitov [41] haben jeweils gezeigt, dass sich diese wie folgt durch ein endliches Maß Λ auf $[0,1]$ charakterisieren lassen. Für jedes $n \in \mathbb{N}$ ist der auf $[n]$ eingeschränkte Coalescent, auch als Λ-n-Coalescent bezeichnet, ein homogener Markovprozess mit infinitesimalen Übergangsraten $(q_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$,

$$q_{\pi\rho} = \begin{cases} \int_0^1 x^{\#\pi-\#\rho-1}(1-x)^{\#\rho-1}\Lambda(\mathrm{d}x) & \text{wenn } \pi \prec \rho, \\ -\sum_{k=1}^{\#\pi-1} \int_0^1 k(1-x)^{k-1}\Lambda(\mathrm{d}x) & \text{wenn } \pi = \rho, \\ 0 & \text{sonst,} \end{cases}$$

wobei $\pi \prec \rho$ genau dann, wenn es genau einen Block in ρ gibt, der die Verschmelzung von mehreren Blöcken aus π ist und alle anderen Blöcke von ρ Blöcke von π sind. Die Formel für die totalen Raten ergibt sich aus $\sum_{\sigma\in\mathcal{P}_{[n]}:\pi\prec\sigma} x^{\#\pi-\#\sigma-1}(1-x)^{\#\sigma-1} = \sum_{k=1}^{\#\pi-1} \binom{\#\pi}{k-1} x^{\#\pi-k-1}(1-x)^{k-1} = \sum_{k=1}^{\#\pi-1} k(1-x)^{k-1}$.

Die *Blockzählprozesse* $N^{(n)} = (N_t^{(n)})_{t\in[0,\infty)}$ sind durch $N_t^{(n)} := \#\Pi_t^{(n)}$ definiert und es kann gezeigt werden, dass sie ebenfalls zeithomogene Markovprozesse sind mit Übergangsraten

$$q'_{ij} = \begin{cases} \binom{i}{j-1} \int_0^1 x^{i-j-1}(1-x)^{j-1}\Lambda(\mathrm{d}x) & \text{wenn } i > j, \\ -\sum_{k=1}^{i-1} \int_0^1 k(1-x)^{k-1}\Lambda(\mathrm{d}x) & \text{wenn } i = j, \\ 0 & \text{sonst.} \end{cases}$$

Die Fixation Line und Siegmund-Dualität

Die *Fixation Line* eines Coalescent wurde zuerst von Pfaffelhuber und Wakolbinger [35] eingeführt, um die Zeit zurück zum Urahnen im Kingman-Coalescent als Prozess betrachtet zu untersuchen. Die auf dem Look-Down-Prozess von Donnelly und Kurtz [8, 9] aufbauende Konstruktion der Fixation Line lässt sich für alle austauschbaren Coalescents durchführen. Die Fixation Line von Λ-Coalescents wurde von Labbé [24] verwendet und von Hénard [18] eingehend analysiert, die Fixation Line allgemeiner austauschbarer Coalescents durch Gaiser und Möhle [13]. Für die vorliegende Arbeit genügt es, die Fixation Line ausgehend von der natürlich gekoppelten Familie der Blockzählprozesse $(N^{(n)})_{n\in\mathbb{N}\cup\{\infty\}} = ((N_t^{(n)})_{t\geq 0})_{n\in\mathbb{N}\cup\{\infty\}}$ eines austauschbaren Coalescent über folgenden Zugang einzuführen. Zwei Familien I-wertiger Zufallsvariabler $(X^{(i)})_{i\in I}$ und $(Y^{(i)})_{i\in I}$ heißen *pfadweise dual* bezüglich eines messbaren, beschränkten Dualitätskerns $H : I^2 \to K \subseteq \mathbb{R}$, wenn für alle $(i,j) \in I^2$ gilt $H(X^{(i)}, j) = H(i, Y^{(j)})$. Ist eine Familie von Blockzählprozessen $(N^{(n)})_{n\in\mathbb{N}\cup\{\infty\}}$ gegeben, so definieren wir eine Familie $(L^{(n)})_{n\in\mathbb{N}\cup\{\infty\}}$ von Prozessen $L^{(n)} := (L_t^{(n)})_{t\geq 0}$ mit Zustandsraum $\mathbb{N}\cup\{\infty\}$, indem für jedes $t \geq 0$ gilt

$$L_t^{(n)} := \sup\{k \in \mathbb{N} : N_t^{(k)} \leq n\}.$$

Andersherum gilt auch $N_t^{(n)} = \inf\{k \in \mathbb{N} : L_t^{(k)} \geq n\}$. Dann heißt $L^{(n)}$ die n-Fixation Line des Coalescent. Gilt $N_t^{(n)} \to \infty$, $n \to \infty$, dann nimmt $L_t^{(n)}$, $n \in \mathbb{N}$ nur Werte in $\mathbb{N}$ an. Die Familien $(N^{(n)})_{n\in\mathbb{N}\cup\{\infty\}}$ und $(L^{(n)})_{n\in\mathbb{N}\cup\{\infty\}}$ sind pfadweise dual bezüglich des Dualitätskerns $H : \mathbb{N}^2 \to \{0,1\}$, wobei $H(i,j) = 1$ für $i \leq j$ und $H(i,j) = 0$ sonst. Dieser Dualitätskern heißt auch *Siegmund-Dualitätskern* [44]. Zwei Familien I-wertiger Zufallsvariablen $(X^{(i)})_{i\in I}$ und $(Y^{(i)})_{i\in I}$ heißen *dual* bezüglich eines Dualitätskerns $H : I^2 \to K$, wenn für alle $(i,j) \in I^2$ gilt $\mathbb{E}\left(H(X^{(i)}, j)\right) = \mathbb{E}\left(H(i, Y^{(j)})\right)$. Sind sie dual bezüglich des Siegmund-Dualitätskerns, heißen sie Siegmund-dual. Im Fall der Blockzählprozesse und der Fixation Line folgt die Dualität direkt aus der pfadweisen Dualität:

$$\mathbb{P}\left(N_t^{(n)} \leq m\right) = \mathbb{E}\left(H(N_t^{(n)}, m)\right) = \mathbb{E}\left(H(n, L_t^{(m)})\right) = \mathbb{P}\left(L_t^{(m)} \geq n\right).$$

Die Fixation Line ist ein zeithomogener Markovprozess. Wird der Dualitätskern H als Matrix $H = (h_{ij})_{i,j\in\mathbb{N}}$ mit $h_{ij} := H(i,j)$ aufgefasst und der infinitesimale Generator des Blockzählprozesses mit $Q' = (q'_{ij})_{i,j\in\mathbb{N}}$ sowie der infinitesimale Generator der Fixation Line mit $\Gamma = (\gamma_{ij})_{i,j\in\mathbb{N}}$ bezeichnet, so erfüllt Γ die Dualitätsgleichung $H\Gamma^\top = Q'H$ und daher gilt

$$\gamma_{ij} = \begin{cases} \binom{j}{i-1} \int_0^1 x^{j-i-1}(1-x)^i \Lambda(\mathrm{d}x) & \text{wenn } i < j, \\ -\sum_{k=1}^{i} \int_0^1 k(1-x)^{k-1}\Lambda(\mathrm{d}x) & \text{wenn } i = j, \\ 0 & \text{sonst.} \end{cases}$$

Durch die pfadweise Definition sind die $L^{(n)}$, $n \in \mathbb{N}\cup\{\infty\}$, auf natürliche Weise gekoppelt.

Der Bolthausen-Sznitman-Coalescent

Der Bolthausen-Sznitman-Coalescent ist ein Λ-Coalescent, bei dem Λ durch die uniforme Verteilung auf $[0,1]$ gegeben ist. Die Übergangsraten sind also durch

$$q_{\pi\rho} = \begin{cases} \frac{(\#\rho-1)!(\#\pi-\#\rho-1)!}{(\#\pi-1)!} & \text{wenn } \pi \prec \rho, \\ -(\#\pi - 1) & \text{wenn } \pi = \rho, \\ 0 & \text{sonst,} \end{cases}$$

gegeben. Die Übergangsraten des Blockzählprozesses sind durch

$$q'_{ij} = \begin{cases} \frac{i}{(i-j)(i-j+1)} & \text{wenn } i > j, \\ -(i-1) & \text{wenn } i = j, \\ 0 & \text{sonst,} \end{cases}$$

und die Übergangsraten der Fixation Line sind durch

$$\gamma_{ij} = \begin{cases} \frac{i}{(j-i)(j-i+1)} & \text{wenn } i > j, \\ -i & \text{wenn } i = j, \\ 0 & \text{sonst,} \end{cases}$$

gegeben. Der Bolthausen-Sznitman-Coalescent wurde von Bolthausen und Sznitman im Zusammenhang mit dem aus der theoretischen Physik stammenden Sherrington-Kirkpatrick-Modell für Spingläser eingeführt [6]. Durch Bertoin und Le Gall wurde gezeigt, dass er die Genealogie von Neveu's Verzweigungsprozess ist [4]. Für $n \in \mathbb{N}$ kann der Bolthausen-Sznitman-n-Coalescent auch durch schrittweises Zurückstutzen eines zufälligen rekursiven Baumes der Größe n konstruiert werden. Diese Konstruktion geht auf Goldschmidt und Martin zurück [15] und wird im ersten Artikel nochmals beschrieben. Im selben Artikel wird auch eine Rechtseigenvektormatrix für die Übergangsraten und die Übergangswahrscheinlichkeiten des n-Bolthausen-Sznitman-Coalescent hergeleitet und die Einträge werden im Kontext der Konstruktion von Goldschmidt und Martin wie folgt als Wahrscheinlichkeiten interpretiert. Ein *rekursiver Baum* auf einer aufsteigenden Menge von Elementen, genannt *Labelmenge* ist ein Baum mit Wurzel, der als Wurzel das niedrigste Element hat und bei dem die Elemente der Pfade von der Wurzel weg aufsteigend sind. Im Folgenden wird als Labelmenge eine Partition einer geordneten Menge verwendet, die Labels sind also die Blöcke der Partition und werden nach ihrem niedrigsten Element geordnet. Auf einer Menge A von n Labels gibt es $n!$ verschiedene rekursive Bäume. Ein zufällig uniform unter diesen ausgewählter Baum wird *zufälliger rekursiver Baum auf der Labelmenge A* genannt. Ein rekursiver Baum kann an einer seiner Kanten gestutzt werden, indem alle Knoten, die durch diese Kante von der Wurzel getrennt werden, entfernt werden. Der letzte Knoten vor dieser Kante erhält dann als neues Label die Vereinigung seines Labels und aller abgetrennten Label. Der neue kleinere Baum ist wieder ein rekursiver Baum auf einer neuen Labelmenge, nämlich einer gröberen Partition [15, Proposition 2.1]. Sind nun π und ρ Partitionen auf einer aufsteigenden endlichen Menge und $\mathcal{T}$ ein rekursiver Baum auf π, so kann man fragen, ob $\mathcal{T}$ mit beliebig vielen frei wählbaren Stutzungen zu einem Baum auf ρ gemacht werden kann. Wenn $\mathcal{T}$ ein zufälliger rekursiver Baum ist, kann die Wahrscheinlichkeit hierfür in Abhängigkeit von π und ρ explizit berechnet werden und es wird gezeigt, dass die Matrix mit von π und ρ abhängigen Einträgen eine Rechtseigenvektormatrix des Generators des Bolthausen-Sznitman-Coalescent ist.

Spektralzerlegungen

Eine Spektralzerlegung einer Matrix Q ist eine Darstellung $Q = RDL$, wobei D eine Diagonalmatrix ist und R und L invers zueinander sind. Spektralzerlegungen für bestimmte populationsgenetische Modelle kommen bereits bei Gladstien [14, p. 635] für ein bestimmtes Moranmodell und bei Tavaré [46, Appendix I] für den Kingman-Coalescent mit Mutation vor. Eine Spektralzerlegung für den Blockzählprozess des Kingman-Coalescent, das heißt des Λ-Coalescent mit $\Lambda = \delta_{\{0\}}$ wurde von Möhle und Pitters hergeleitet [30], ebenso in einem weiteren Artikel [29] eine Spektralzerlegung für den Blockzählprozess des Bolthausen-Sznitman-Coalescent. Anwendungen finden sich beispielsweise in der Berechnung der Verteilung der Absorptionszeit, also des Zeitpunkts, an dem der n-Coalescent den Zustand $\{[n]\}$ erreicht. Spektralzerlegungen für den partitionswertigen Bolthausen-Sznitman-n-Coalescent und den partitionswertigen Kingman-n-Coalescent selbst wurden zuerst im ersten Artikel dieser Dissertation hergeleitet.

Theorem (Theorem 1.1 in Kukla, Pitters, 2015 [23])**.** Sei $Q = (q_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ die Matrix der infinitesimalen Übergangsraten des Bolthausen-Sznitman-n-Coalescent und seien die Matrizen $L = (l_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ und $R = (r_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ definiert durch

$$l_{\pi\rho} := \begin{cases} (-1)^{\#\pi-\#\rho}\frac{(\#\rho-1)!}{(\#\pi-1)!} & \text{wenn } \pi \leq \rho, \\ 0 & \text{sonst,} \end{cases}$$

und

$$r_{\pi\rho} := \begin{cases} \frac{(\#\rho-1)!}{(\#\pi-1)!}\prod_{B\in\rho}(\#\pi|_B - 1)! & \text{wenn } \pi \leq \rho, \\ 0 & \text{sonst.} \end{cases}$$

Dann ist eine Spektralzerlegung von Q durch $Q = RDL$ gegeben, wobei $D = (d_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ durch $d_{\pi\pi} = -(\#\pi - 1)$ und $d_{\pi\rho} = 0$ wenn $\pi \neq \rho$ gegeben ist.

Daraus lassen sich die bereits bekannten Übergangswahrscheinlichkeiten des partitionswertigen Bolthausen-Sznitman-Coalescent herleiten, sowie Green's Matrix und eine Spektralzerlegung des Blockzählprozesses. Ebenso wird eine Spektralzerlegung für Kingmans Coalescent hergeleitet. Wie bereits im vorhergehenden Abschnitt angedeutet, lassen sich die $r_{\pi\rho}$ als Wahrscheinlichkeit interpretieren, dass ein zufälliger rekursiver Baum auf π sich mit beliebig vielen frei wählbaren Stutzungen zu einem Baum auf ρ machen lässt. Diese Interpretation ermöglicht einen zweiten Beweis für die Rechtseigenvektoreigenschaft.

Im zweiten Artikel wird eine Spektralzerlegung des Generators der Fixation Line hergeleitet.

Theorem (Theorem 2.2 in Kukla, Möhle, 2018 [22])**.** Der Generator $\Gamma = (\gamma_{ij})_{i,j\in\mathbb{N}}$ der Fixation Line $(L_t)_{t\geq 0}$ des Bolthausen–Sznitman–Coalescent hat Spektralzerlegung $\Gamma = RDL$, wobei $D = (d_{ij})_{i,j\in\mathbb{N}}$ die Diagonalmatrix ist mit $d_{ij} = -i$ für $i = j$ und $d_{ij} = 0$ für $i \neq j$, und $R = (r_{ij})_{i,j\in\mathbb{N}}$ und $L = (l_{ij})_{i,j\in\mathbb{N}}$ obere rechte Dreieckmatrizen sind mit Einträgen

$$r_{ij} = \frac{i!}{j!}(-1)^{i+j}S(j,i) \quad \text{und} \quad l_{ij} = \frac{i!}{j!}(-1)^{i+j}s(j,i), \qquad i,j \in \mathbb{N},$$

wobei $s(i,j) = (-1)^{i-j}\left[{i \atop j}\right]$ die vorzeichenbehafteten Stirlingzahlen erster Art sind und $S(i,j) = \left\{{i \atop j}\right\}$ die Stirlingzahlen zweiter Art. Diese Spektralzerlegung lässt sich sowohl eigenständig über Erzeugendenfunktionen herleiten, ähnlich wie in [29] die Spektralzerlegung des Blockzählprozesses, als auch unter Verwendung dieses Ergebnisses über die Dualität. Die Fixation Line hat die Erzeugendenfunktion $\mathbb{E}\left(z^{L_t^{(i)}}\right) = (1-(1-z)^\alpha)^i$, $\alpha := e^{-t}$ und somit die Verzweigungseigenschaft. Das ist eine Ausnahme, denn im Allgemeinen ist die Fixation Line eines austauschbaren Coalescent kein Verzweigungsprozess.

Die in 1 gestartete Fixation Line ist Sibuya-verteilt mit Parameter α, das heißt $\mathbb{P}\left(L_t^{(1)} = j\right) = (-1)^{j+1}\binom{\alpha}{j}$ und $L_t^{(n)}$ ist verteilt wie die Summe von n unabhängigen Zufallsvariablen mit derselben Verteilung wie $L_t^{(1)}$. Zu beachten ist jedoch, dass die $L_t^{(k)} - L_t^{(k-1)}$, $k \in \mathbb{N}$ ($L_t^{(0)} := 0$) nicht unabhängig und identisch verteilt sind, denn es gilt zum Beispiel

$$\begin{aligned}\mathbb{P}\left(L_t^{(1)} = 2, L_t^{(2)} - L_t^{(1)} = 1\right) = \mathbb{P}\left(N_t^{(2)} = 1, N_t^{(3)} = 2, N_t^{(4)} = 3\right) = \frac{\alpha^2 - \alpha^3}{3} \\ \neq \frac{\alpha^2 - \alpha^3}{2} = \mathbb{P}\left(L_t^{(1)} = 1\right)\mathbb{P}\left(L_t^{(1)} = 2\right).\end{aligned}$$

Der Mittag-Leffler-Prozess und Neveu's Verzweigungsprozess

Die (einseitige) α-stabile Verteilung L_α mit $\alpha \in (0,1]$ ist eine Verteilung auf $[0,\infty)$ und durch die Laplacetransformierten $\mathbb{E}\left(e^{-\lambda Y}\right) = e^{-\lambda^\alpha}$, $\lambda \in [0,\infty)$ für α-stabile (das heißt L_α-verteilte) Zufallsvariable Y eindeutig bestimmt.

Die (Typ-2-) Mittag-Leffler-Verteilung ML_α mit Parameter $\alpha \in [0,1]$ ist eine Verteilung auf $[0,\infty)$. Für $\alpha > 0$ und α-stabiles Y ist $X := Y^{-\alpha}$ Mittag-Leffler-verteilt mit Parameter α. Aus der Definition der Gammafunktion $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}\mathrm{d}s = y^z\int_0^\infty s^{z-1}e^{-sy}\mathrm{d}s$, $y > 0$, folgt $y^{-m\alpha} = \Gamma(m\alpha)^{-1}\int_0^\infty s^{m\alpha-1}e^{-sy}\mathrm{d}s$ und mit dem Satz von Tonelli für $m \in [0,\infty)$

$$\mathbb{E}\left(X^m\right) = \mathbb{E}\left(Y^{-m\alpha}\right) = \mathbb{E}\left(\int_0^\infty \frac{s^{m\alpha-1}e^{-sY}}{\Gamma(m\alpha)}\,\mathrm{d}s\right) = \int_0^\infty \frac{s^{m\alpha-1}e^{-s^\alpha}}{\Gamma(m\alpha)}\,\mathrm{d}s = \frac{\Gamma(1+m)}{\Gamma(1+m\alpha)}.$$

(Vergleiche hierzu [37, S. 12].) Als Verteilung auf $[0,\infty)$ ist ML_α durch diese Momente ebenfalls eindeutig bestimmt. Für $\alpha = 0$ gilt $\frac{\Gamma(1+m)}{\Gamma(1+m\alpha)} = m! = \mathbb{E}\left(E^m\right)$ für standard-exponentialverteiltes $E \overset{d}{=} \mathrm{Exp}(1)$ und daher wird $\mathrm{ML}_0 := \mathrm{Exp}(1)$ definiert. Die Mittag-Leffler-Verteilung erfüllt die Eigenschaft, dass für $\alpha, \beta \in [0,1]$ und unabhängige $X_1 \overset{d}{=} \mathrm{ML}_\alpha$ und $X_2 \overset{d}{=} \mathrm{ML}_\beta$ gilt $X_1^\beta X_2 \overset{d}{=} \mathrm{ML}_{\alpha\beta}$. Der Beweis erfolgt durch Vergleich der Momente. Insbesondere folgt für $\alpha = 0$ und jedes von X unabhängige $E \overset{d}{=} \mathrm{Exp}(1)$, dass $E^\beta X \overset{d}{=} \mathrm{Exp}(1)$. Für $\alpha, \beta \in (0,1]$ und unabhängige $Y_1 \overset{d}{=} L_\alpha$ und $Y_2 \overset{d}{=} L_\beta$ folgt $Y_1^{1/\beta} Y_2 \overset{d}{=} L_{\alpha\beta}$.

Der Mittag-Leffler-Prozess X und Neveu's Verzweigungsprozess Y sind zwei zeithomogene Markovprozesse in stetiger Zeit mit Zustandsraum $E := [0,\infty)$, die durch ihre Semigruppe und ihren Anfangszustand charakterisiert sind. Abhängig vom Anfangszustand wird der in x gestartete Mittag-Leffler-Prozess mit $X^{(x)}$ und Neveu's Verzweigungsprozess gestartet in y mit $Y^{(y)}$ bezeichnet, wobei $X := X^{(1)}$ und $Y := Y^{(1)}$.

Neveu's Verzweigungsprozess ist ein *Verzweigungsprozess mit stetigem Zustandsraum* und in stetiger Zeit, auf englisch *continuous state branching process (CSBP)*. Ein CSBP ist ein Markovprozess Y mit rechtsstetigen Pfaden, sodass $Y^{(y+z)}$ für alle $y, z \in E := [0,\infty)$ verteilt ist wie die Summe zweier unabhängiger, wie $Y^{(y)}$ und wie $Y^{(z)}$ verteilter Prozesse.

Für seine Laplacetransformierten gilt $\mathbb{E}\left(e^{-\lambda Y_t^{(y)}}\right) = \mathbb{E}\left(e^{-\lambda Y_t}\right)^y$. Er lässt sich durch seinen *Verzweigungsmechanismus* wie folgt charakterisieren. Jeder CSBP hat Laplacetransformierte der Form $\mathbb{E}\left(e^{-\lambda Y_t}\right) = e^{-u_t(\lambda)}$ für Funktionen $u_t : [0,\infty) \to [0,\infty]$, wobei u_t die Lösungen der Integralgleichung

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))\,\mathrm{d}s = \lambda$$

sind für den Laplaceexponenten ψ eines Lévy-Prozesses, das heißt, für eine konvexe Funktion $\psi : [0,\infty) \to \mathbb{R}$ von der Form

$$\psi(u) = au + bu^2 + \int_{(0,\infty)} \left(e^{-xu} - 1 + xu\mathbf{1}_{(0,1]}(x)\right)\mu(\mathrm{d}x).$$

Dabei ist $a \in \mathbb{R}$, $b \geq 0$ und μ das Sprungmaß eines Lévy-Prozesses, das heißt es genügt der Bedingung $\int_{(0,\infty)}(1 \wedge x^2)\,\mu(\mathrm{d}x) < \infty$. Diese Funktion ψ heißt *Verzweigungsmechanismus* und bestimmt den Verzweigungsprozess eindeutig. Neveu's Verzweigungsprozess hat den Verzweigungsmechanismus $\psi(u) = u\log u$, das heißt $b = 0$ und $\mu(\mathrm{d}x) = x^{-2}\mathrm{d}x$. Somit ist $u_t(\lambda) = \lambda^\alpha$, das heißt Y_t ist α-stabil. Also gilt für alle $t \geq 0$, dass $\mathbb{E}\left(e^{-\lambda Y_t^{(y)}}\right) = \mathbb{E}\left(e^{-\lambda Y_t}\right)^y = e^{-\lambda^\alpha y} = \mathbb{E}\left(e^{-\lambda y^{1/\alpha} Y_t}\right)$ und somit $y^{-1/\alpha}Y_t^{(y)} \stackrel{d}{=} L_\alpha$, $\alpha := e^{-t}$. Daher gilt für die Semigruppe, dass

$$T_t^Y g(y) = \mathbb{E}\left(g(y^{1/\alpha}Y_t)\right), \qquad t, y \geq 0$$

für alle beschränkten messbaren Funktionen $g : E \to \mathbb{R}$.

Die zum Zeitpunkt 0 in den verschiedenen $y \in [0,\infty)$ gestarteten CSBPs können durch $Y_t^{(y)} := y^{1/\alpha}Y_t$ auf natürliche Weise gekoppelt werden. Der Mittag-Leffler-Prozess lässt sich dann pfadweise durch $X_t^{(x)} := x^\alpha Y_t^{-\alpha}$ definieren. Aus dieser Definition folgt die pfadweise Dualität zu Neveu's Verzweigungsprozess bezüglich des Dualitätskerns $H : [0,\infty)^2 \to \{0,1\}$ mit $H(x,y) = 1$ für $x \leq y$ und $H(x,y) = 0$ für $x > y$, das heißt $X_t^{(x)} \leq y$ genau dann, wenn $Y^{(y)} \geq x$. Aus der pfadweisen Dualität folgt die Siegmund-Dualität. Der Mittag-Leffler-Prozess heißt so, da seine eindimensionalen Randverteilungen Mittag-Leffler-verteilt sind. Genauer gilt für alle $t \geq 0$, dass $x^{-\alpha}X_t^{(x)}$ Mittag-Leffler-verteilt zum Parameter α ist. Somit erfüllt die Semigruppe $(T_t^X)_{t\geq 0}$ des Mittag-Leffler-Prozesses X die Eigenschaft, dass

$$T_t^X f(x) = \mathbb{E}\left(f(x^\alpha X_t)\right), \qquad t, x \geq 0$$

für alle beschränkten messbaren Funktionen $f : E \to \mathbb{R}$.

Für die folgenden Betrachtungen ist es hilfreich, die logarithmierten Prozesse $\tilde{X}^{(x)}$, $\tilde{X}_t^{(x)} := \log X_t^{(e^x)}$, $x \in \mathbb{R}$ und $\tilde{Y}^{(y)}$, $\tilde{Y}_t^{(y)} := \log Y_t^{(e^y)}$, $y \in \mathbb{R}$ zu betrachten mit $\tilde{X} := \tilde{X}^{(0)}$ und $\tilde{Y} := \tilde{Y}^{(0)}$. Für die logarithmierten Prozesse ergibt sich $\tilde{X}_t^{(x)} - xe^{-t} \stackrel{d}{=} \mathrm{ML}_\alpha \circ \log^{-1}$ und $\tilde{Y}_t^{(y)} - ye^t \stackrel{d}{=} L_\alpha \circ \log^{-1}$, beziehungsweise für die Semigruppen

$$T_t^{\tilde{X}} f(x) = \mathbb{E}\left(f(xe^{-t} + \tilde{X}_t)\right), \qquad t \geq 0, x \in \mathbb{R}$$

und

$$T_t^{\tilde{Y}} g(y) = \mathbb{E}\left(g(ye^t + \tilde{Y}_t)\right), \qquad t \geq 0, y \in \mathbb{R}$$

für alle stetigen beschränkten Funktionen $f, g : \mathbb{R} \to \mathbb{R}$. Würde man in den Gleichungen für $\tilde{X}$ stattdessen annehmen, dass $\tilde{X}_t$ normalverteilt mit Erwartungswert 0 und Varianz $t\sigma^2$ sei, so wäre $\tilde{X}$ ein Ornstein-Uhlenbeck-Prozess. Die Prozesse $\tilde{X}$ und $\tilde{Y}$ gehören also zu einer Klasse verallgemeinerter Ornstein-Uhlenbeck-Prozesse. Die zugehörigen Semigruppen werden als *verallgemeinerte Mehler-Semigruppen* bezeichnet und wurden unter anderem von Bogachev, Röckner und Schmuland [5] eingehend untersucht, wobei der Zustandsraum ein Banachraum ist, auf dem eine Semigruppe $(U_t)_{t\geq 0}$ wirkt, in unserem Fall $U_t x = xe^{-t}$, $x \in \mathbb{R}$ beziehungsweise $U_t y = ye^t$, $x \in \mathbb{R}$. Nach Proposition 2.2 in [5] definiert

$$T_t^Z h(z) = \mathbb{E}\left(h(U_t z + Z_t)\right), \qquad t, z \geq 0$$

für eine Familie von Wahrscheinlichkeitsmaßen $(P_t)_{t\geq 0}$ auf E mit $P_{Z_t} = P_t$ genau dann eine Semigruppe, wenn

$$P_{Z_{s+t}} = P_{U_t Z_s} * P_{Z_t} \qquad \text{für alle } s, t \geq 0.$$

Dies ist für $Z = \tilde{X}$ beziehungsweise $Z = \tilde{Y}$ der Fall. Auf $\mathbb{R}$ haben alle Semigruppen die Gestalt $U_t x = xe^{\lambda t}$ für ein $\lambda \in \mathbb{R}$. Für $\lambda < 0$ kann eine invariante Verteilung existieren. Und in der Tat gilt für jede von X unabhängige Zufallsvariable $E \overset{d}{=} \mathrm{Exp}(1) = \mathrm{ML}_0$ und für jedes $t \geq 0$, dass $X_t^{(E)} \overset{d}{=} E^\alpha X_t \overset{d}{=} \mathrm{Exp}(1)$. Da $G := -\log E$ standard-Gumbel-verteilt ist, folgt die Selbstähnlichkeitsgleichung der Gumbelverteilung

$$-G \overset{d}{=} \tilde{X}_t^{(-G)} \overset{d}{=} -e^{-t}G + \tilde{X}_t.$$

Analog dazu folgt aus $Y_t^{-\alpha} \overset{d}{=} X_t$ für $\alpha \in (0,1]$ als weitere Selbstähnlichkeitsgleichung der Gumbelverteilung mit anderem Vorzeichen $G \overset{d}{=} e^{-t}G + e^{-t}\tilde{Y}_t$.

Konvergenzresultate

Betrachte nun den reskalierten Blockzählprozess $X^{(n)}$, $X_t^{(n)} := n^{-e^{-t}} N_t^{(n)}$ und die reskalierte Fixation Line $Y^{(n)}$, $Y_t^{(n)} := n^{-e^t} L_t^{(n)}$. Da die Pfade von $N^{(n)}$ und $L^{(n)}$ càdlàg sind, sind auch die Pfade von $X^{(n)}$ und $Y^{(n)}$ càdlàg. Der Raum der càdlàg-Funktionen von $[0,\infty)$ nach E wird mit $D_E[0,\infty)$ bezeichnet und mit der Skorohodschen J_1-Topologie versehen. Diese kann dadurch charakterisiert werden, dass $f_n \to f$ in $(D_E[0,\infty), J_1)$ für $f_n \in D_E[0,\infty), n \in \mathbb{N}$ und $f \in D_E[0,\infty)$ genau dann wenn es zeitdeformierende Homöomorphismen $\lambda_n : [0,\infty) \to [0,\infty)$ gibt, sodass für alle $t > 0$ gilt

$$\sup_{s\leq t} |\lambda_n(s) - s| + \sup_{s\leq t} |f(\lambda_n(s)) - f(s)| \longrightarrow 0, \qquad n \to \infty.$$

Das Hauptresultat des zweiten Artikels betrifft die Asymptotik des Blockzählprozesses und der Fixation Line:

Theorem (Theorem 2.1 in Kukla, Möhle, 2018 [22])**.** Für den Bolthausen–Sznitman–Coalescent gilt:

(a) Für $n \to \infty$ konvergiert der reskalierte Blockzählprozess $X^{(n)}$, $X_t^{(n)} := n^{-e^{-t}} N_t^{(n)}$ in Verteilung in $(D_E[0,\infty), J_1)$ gegen den Mittag–Leffler–Prozess $X = (X_t)_{t\geq 0}$.

(b) Für $n \to \infty$ konvergiert die reskalierte Fixation Line $Y^{(n)}$, $Y_t^{(n)} := n^{-e^t} L_t^{(n)}$ in Verteilung in $(D_E[0,\infty), J_1)$ gegen Neveu's Verzweigungsprozess $Y = (Y_t)_{t\geq 0}$.

Die Konvergenz der $(X_{t_1}, \ldots, X_{t_k})$ beziehungsweise $(Y_{t_1}, \ldots, Y_{t_k})$, $k \in \mathbb{N}$ ist bekannt und gilt sogar fast sicher. Für jedes $t \geq 0$ ist die Zufallspartition $\Pi_t^{(n)}$ eine $(e^{-t}, 0)$-Zufallspartition im Sinne von Pitman [37] und daher konvergiert nach Theorem 3.8 aus [37] $n^{-e^{-t}} N_t^{(n)}$ fast sicher gegen eine $\mathrm{ML}_{e^{-t}}$-verteilte Zufallsvariable. Daraus lässt sich die fast sichere Konvergenz von $n^{-e^t} L_t^{(n)}$ gegen eine α-stabile Zufallsvariable folgern. Der Beweis für die Konvergenz in Verteilung der Prozesse besteht darin, die Konvergenz auf die Konvergenz in Verteilung der eindimensionalen Randverteilungen zurückzuführen.

Danksagung

Ich möchte mich sehr herzlich bei Martin Möhle bedanken, der mich in allen Phasen meiner Doktorarbeit voll und ganz unterstützt hat und von dem ich insbesondere beim gemeinsamen Verfassen des zweiten Artikels sehr viel gelernt habe. Desweiteren möchte ich mich sehr herzlich bei Helmut Pitters bedanken, der genau im richtigen Moment mit einem Forschungsprojekt auf mich zukam, dessen gemeinsame Bearbeitung mir große Freude bereitet hat und mir beim Eintauchen in die Welt des Bolthausen-Sznitman-Coalescent sehr geholfen hat. Mein Dank gilt dem ganzen Arbeitsbereich Stochastik, insbesondere Elmar Teufl, der mir bei einigen TEX-nischen Fragen mit Rat und Tat zur Seite stand. Schließlich gilt mein Dank meiner Familie, auf die ich mich in jeder Situation verlassen kann.

Artikel 1

A spectral decomposition for the Bolthausen-Sznitman coalescent and the Kingman coalescent

Jonas Kukla   Helmut H. Pitters

Abstract. We consider both the Bolthausen-Sznitman and the Kingman coalescent restricted to the partitions of $\{1, \ldots, n\}$. Spectral decompositions of the corresponding generators are derived. As an application we obtain a formula for the Green's functions and a short derivation of the well-known formula for the transition probabilities of the Bolthausen-Sznitman coalescent.

Electron. Commun. Probab. **20** (2015), no. 87, 13 pages.

Keywords. Bolthausen-Sznitman coalescent; Kingman coalescent; spectral decomposition; Green's matrix; random recursive tree.

2010 *Mathematics Subject Classification.* 60J27; 60C05; 92D15.

1.1 Introduction

An exchangeable coalescent process is a discrete or continuous-time Markov chain that encodes the dynamics of particles grouped into so-called blocks. The jumps of the process consist of mergers of two or more blocks, and the rate at which a merger happens only depends on the current number of blocks, but not, for instance, on their sizes or the specific particles they contain. The theory of exchangeable coalescent processes has its origins in the study of genealogies in population genetics, culminating in the seminal work of Kingman [21].

Among exchangeable coalescent processes the so-called Λ-coalescents have received increasing attention in recent years. The latter were introduced independently by Donnelly and Kurtz [9], Pitman [36] and Sagitov [41]. A Λ-coalescent $\{\Pi(t), t \geq 0\}$ is a time-homogeneous exchangeable coalescent process in continuous time with state space $\mathcal{P}_{\mathbb{N}}$, the set of partitions of the non-negative integers $\mathbb{N} := \{1, 2, \ldots\}$, that only allows for one merger of blocks at any jump. It can be characterized via its restrictions $\{\Pi^n(t), t \geq 0\}$ to $[n] := \{1, \ldots, n\}$ as follows. If at any given time $\Pi^n(t)$ contains $b \geq 2$ blocks, then any $2 \leq k \leq b$ of these blocks merge at rate $\lambda_{b,k} := \int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx)$, where Λ denotes a finite measure on the unit interval. This measure Λ together with the initial state $\Pi(0)$ uniquely determines Π, hence the name Λ-coalescent.

In this note we consider both the Kingman coalescent $\Pi^K = \{\Pi^K(t), t \geq 0\}$ and the Bolthausen-Sznitman coalescent $\Pi^{BS} = \{\Pi^{BS}(t), t \geq 0\}$, which are both Λ-coalescents. For convenience we drop the superscripts K and BS when there is no risk of ambiguity.

The article is organized as follows. Our main results are spectral decompositions of the generator of the Bolthausen-Sznitman n-coalescent, Theorem 1.1, respectively of the generator of Kingman's n-coalescent, Theorem 1.5. As Corollaries we obtain for the Bolthausen-Sznitman coalescent a derivation of the formula for the transition probabilites that goes back to [6], and a formula for its Green's matrix. As a further application we obtain a spectral decomposition of the generator of the block counting process of the Bolthausen-Sznitman, respectively Kingman's coalescent.

1.2 Results

Let us introduce some notation. A partition of a set A is a set, π say, of nonempty pairwise disjoint subsets of A whose union is A. The members of π are called the blocks of π. Let $\#A$ denote the cardinality of A and let $\mathcal{P}_A$ denote the set of partitions of A.

1.2.1 Bolthausen-Sznitman n-coalescent

The Bolthausen-Sznitman n-coalescent $\Pi^{n,BS} = \{\Pi^{n,BS}(t), t \geq 0\}$ is obtained by choosing Λ to be the uniform measure on $[0,1]$. From the definition of the rates in the introduction it follows that the corresponding Q-matrix $Q := Q^{n,BS} = (q_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ is given by

$$q_{\pi\rho} = \begin{cases} \frac{(\#\rho-1)!(\#\pi-\#\rho-1)!}{(\#\pi-1)!} & \text{if } \pi \prec \rho, \\ -(\#\pi - 1) & \text{if } \pi = \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where $\pi \prec \rho$ if and only if ρ is obtained by exactly one merger of blocks of π. In this section we only consider the Bolthausen-Sznitman n-coalescent and therefore write Π instead of $\Pi^{n,BS}$.

For any two sets A and $B \subseteq A$ and a partition $\pi \in \mathcal{P}_A$ we call $\pi|_B := \{C \cap B \colon C \in \pi, C \cap B \neq \emptyset\} \in \mathcal{P}_B$ the restriction of π to B.

Theorem 1.1 (Spectral decomposition of the Bolthausen-Sznitman coalescent)**.** *Let $L = (l_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ and $R = (r_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ be matrices defined by*

$$l_{\pi\rho} := \begin{cases} (-1)^{\#\pi-\#\rho}\frac{(\#\rho-1)!}{(\#\pi-1)!} & \textit{if } \pi \leq \rho, \\ 0 & \textit{otherwise,} \end{cases} \tag{1.2}$$

and

$$r_{\pi\rho} := \begin{cases} \frac{(\#\rho-1)!}{(\#\pi-1)!} \prod_{B\in\rho}(\#\pi|_B - 1)! & \text{if } \pi \le \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

where $\pi \le \rho$ if and only if each block of π is contained in a block of ρ. Then a spectral decomposition of Q is given by $Q = RDL$, where $D = (d_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ is defined by $d_{\pi\pi} = -(\#\pi - 1)$ and $d_{\pi\rho} = 0$ if $\pi \ne \rho$. In particular, $l_{\pi\pi} = r_{\pi\pi} = 1$ for any $\pi \in \mathcal{P}_{[n]}$.

The right eigenvectors $r_{\pi\rho}$ may be interpreted as the probability that a random recursive tree on the label set π can be cut down to a tree on the label set ρ, see the paragraph "A connection with random recursive trees" in Section 1.3.1 below. As an application of this spectral decomposition we derive the well-known formula for the transition probabilities of Π given by Bolthausen and Sznitman in [6], Proposition 1.4.

Corollary 1.2 (Transition probabilities of the Bolthausen-Sznitman coalescent)**.** *For any two partitions $\pi, \rho \in \mathcal{P}_{[n]}$ and any time $t \ge 0$ the transition probabilities $p_{\pi\rho}(t) := \mathbb{P}\{\Pi(t) = \rho | \Pi(0) = \pi\}$ of the Bolthausen-Sznitman n-coalescent are given by*

$$p_{\pi\rho}(t) = (-1)^{\#\rho} e^t \frac{(\#\rho-1)!}{(\#\pi-1)!} \prod_{B\in\rho} (-e^{-t})^{\overline{\#\pi|_B}} = (e^{-t})^{\#\rho-1} \frac{(\#\rho-1)!}{(\#\pi-1)!} \prod_{B\in\rho} (1-e^{-t})^{\overline{\#\pi|_{B-1}}},$$

where for $x \in \mathbb{R}$, $k \in \mathbb{N}$ we denote by $x^{\overline{k}} := x(x+1)\cdots(x+k-1)$ the ascending factorial power with the convention $x^{\overline{0}} := 1$.

Thanks to the spectral decomposition, Theorem 1.1, we obtain a formula for the Green's matrix $G = (g_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ of the Bolthausen-Sznitman n-coalescent defined by $g_{\pi\rho} := \int_0^\infty p_{\pi\rho}(t)dt$. Recall that $g_{\pi\rho} = \mathbb{E}[\int_0^\infty 1_{\{\Pi(t)=\rho\}} dt | \Pi(0) = \pi]$ is the expected total time that Π spends in ρ starting from π. We denote by $\left[{i \atop j}\right]$ the Stirling permutation numbers, which count the number of permutations of a set of i elements with j cycles.

Corollary 1.3 (Green's matrix of the Bolthausen-Sznitman coalescent)**.** *The Green's matrix $G = (g_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ is given by*

$$g_{\pi\rho} = \begin{cases} (-1)^{\#\rho} \frac{(\#\rho-1)!}{(\#\pi-1)!} \sum_{(k_B)_{B\in\rho}\in\mathbb{N}^{\#\rho}} \frac{(-1)^{|k|}}{|k|-1} \prod_{B\in\rho} \left[{\#\pi|_B \atop k_B}\right] & \text{if } \pi \le \rho \ne \{[n]\}, \\ \infty & \text{if } \pi \le \rho = \{[n]\}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

where $|k| := \sum_{B\in\rho} k_B$.

Remark. Notice that since the return probability to any state $\pi \in \mathcal{P}_{[n]}$, $\pi \ne \{[n]\}$, equals 0 for any Λ-coalescent Π there is a close connection between the Green's matrix G of Π and its hitting probabilities defined by $h(\pi, \rho) := \mathbb{P}\{\Pi \text{ hits } \rho \text{ when started from } \pi\}$, namely via $g_{\pi\rho} = h(\pi,\rho)/q_\rho = h(\pi,\rho)/(1-\#\rho)$, cf. [33], p. 146, where $q_\rho := \sum_{\sigma\in\mathcal{P}_{[n]},\sigma\ne\rho} q_{\rho\sigma}$ is the total rate in ρ.

For a set A and $j \in \mathbb{N}$ let $\mathcal{P}_{A,j}$ denote the set of partitions of A into j blocks. Moreover, $\left\{{i \atop j}\right\}$ denotes the Stirling partition numbers, which count the number of partitions into j blocks of a set of i elements.

Remark. To the best of the authors' knowledge, Corollary 1.3 is the first result on the Green's matrix, respectively hitting probabilities, of the (partition-valued) Bolthausen-Sznitman coalescent. However, the hitting probabilities of the corresponding block counting process have been considered before in [29, Corollary 1.5] and the references given in Remark 1.6 therein. From Corollary 1.3 it follows by a technical but straightforward computation that for any $\pi \in \mathcal{P}_{[n],i}$ and $j \leq i$

$$\sum_{\rho \in \mathcal{P}_{[n],j}} g_{\pi\rho} = (-1)^j \frac{(j-1)!}{(i-1)!} \sum_{k=j}^{i} \frac{(-1)^k}{k-1} \begin{bmatrix} i \\ k \end{bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix}$$

in agreement with the entries of the Green's matrix of the block counting process of the Bolthausen-Sznitman coalescent provided in the proof of Corollary 1.5 in [29].

As a further application of the spectral decomposition of Q, Theorem 1.1, we derive a spectral decomposition of the generator $Q' = (q'_{ij})_{i,j \in [n]}$ of the block counting process $\{N(t), t \geq 0\}$ of the Bolthausen-Sznitman n-coalescent defined by $N(t) := \#\Pi(t)$. It is well-known that the matrix Q' is given by $q'_{ij} = i/((i-j)(i-j+1))$ if $i > j$, $q'_{ij} = 1 - i$ if $i = j$, and $q'_{ij} = 0$ otherwise.

In [29] this spectral decomposition of Q' was derived by means of generating functions without recourse to the partition-valued process Π.

Corollary 1.4. *Let* $L' = (l'_{ij})_{i,j \in [n]}$, $R' = (r'_{ij})_{i,j \in [n]}$, *and* $D' = (d'_{ij})_{i,j \in [n]}$ *be matrices given by*

$$l'_{ij} := (-1)^{i-j} \frac{(j-1)!}{(i-1)!} \begin{Bmatrix} i \\ j \end{Bmatrix}, \quad r'_{ij} := \frac{(j-1)!}{(i-1)!} \begin{bmatrix} i \\ j \end{bmatrix}, \quad d'_{ij} := (1-i) 1_{\{i=j\}}. \tag{1.5}$$

Then a spectral decomposition of Q' *is given by* $Q' = R'D'L'$.

1.2.2 Kingman's n-coalescent

Kingman's n-coalescent $\Pi^{n,K} = \{\Pi^{n,K}(t), t \geq 0\}$ is obtained by choosing Λ to be δ_0, the Dirac measure in 0, that is the corresponding Q-matrix $Q := Q^{n,K} = (q_{\pi\rho})_{\pi,\rho \in \mathcal{P}_{[n]}}$ is given by

$$q_{\pi\rho} = \begin{cases} 1 & \text{if } \pi \lessdot \rho, \\ -\binom{\#\pi}{2} & \text{if } \pi = \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{1.6}$$

where $\pi \lessdot \rho$ if and only if $\pi \leq \rho$ and $\#\pi - \#\rho = 1$. In other words, the jump chain of Kingman's n-coalescent is the directed simple random walk on the partition lattice $\mathcal{P}_{[n]}$, where at each step the chain jumps into a coarser partition. From now on we only consider Kingman's n-coalescent and therefore write Π instead of $\Pi^{n,K}$.

Theorem 1.5 (Spectral decomposition of Kingman's coalescent)**.** *Define the matrices* $L = (l_{\pi\rho})_{\pi,\rho \in \mathcal{P}_{[n]}}$ *and* $R = (r_{\pi\rho})_{\pi,\rho \in \mathcal{P}_{[n]}}$ *by*

$$l_{\pi\rho} := \begin{cases} (-1)^{\#\pi - \#\rho} \frac{(\#\pi + \#\rho - 2)!}{(2\#\pi - 2)!} \prod_{B \in \rho} \#\pi|_B! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

and

$$r_{\pi\rho} := \begin{cases} \frac{(2\#\rho-1)!}{(\#\pi+\#\rho-1)!} \prod_{B\in\rho} \#\pi|_B! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

Then a spectral decomposition of Q *is given by* $Q = RDL$, *where* $D = (d_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ *is defined by* $d_{\pi\pi} := -\binom{\#\pi}{2}$ *and* $d_{\pi\rho} = 0$ *if* $\pi \neq \rho$. *In particular,* $l_{\pi\pi} = r_{\pi\pi} = 1$ *for any* $\pi \in \mathcal{P}_{[n]}$.

Remark. i) Although the spectral decomposition of Q, Theorem 1.5, directly yields a spectral decomposition of the matrix of transition probabilites $p_{\pi\rho}(t) := \mathbb{P}\{\Pi(t) = \rho|\Pi(0) = \pi\}$, analoguously to the one given in (1.19) for the Bolthausen-Sznitman n-coalescent, this expression is not particularly handy. In fact, a better approach to calculating $p_{\pi\rho}(t)$ is the one given by Kingman in [21] Equation (2.5).

ii) A combinatorial interpretation of the right eigenvectors $r_{\pi\rho}$ is given in the proofs. Unlike in the Bolthausen-Sznitman case, the authors are not aware of a probabilistic interpretation of the $r_{\pi\rho}$.

As in the case of the Bolthausen-Sznitman coalescent, consider the block counting process $\{N(t), t \geq 0\}$ of Kingman's n-coalescent defined by $N(t) := \#\Pi(t)$ and let $Q' = (q'_{ij})_{i,j\in[n]}$ denote the generator of $N(t)$. From Theorem 1.5 we obtain a spectral decomposition of Q'.

Corollary 1.6. *Let* $L' = (l'_{ij})_{i,j\in[n]}$, $R' = (r'_{ij})_{i,j\in[n]}$ *be matrices defined by*

$$l'_{ij} := (-1)^{i+j}\frac{(i+j-2)!}{(2i-2)!}\left\lfloor {i \atop j} \right\rfloor, \qquad r'_{ij} := \frac{(2j-1)!}{(i+j-1)!}\left\lfloor {i \atop j} \right\rfloor, \tag{1.9}$$

where $\left\lfloor {i \atop j} \right\rfloor := \binom{i-1}{j-1}\frac{i!}{j!}$ *denotes the unsigned Lah numbers which count the number of partitions into* j *blocks of a set of* i *elements, where the elements in each block are ordered. Then* $Q' = R'D'L'$ *is a spectral decomposition of* Q', *where* $D' = (d'_{ij})_{i,j\in[n]}$ *is defined by* $d'_{ii} = -\binom{i}{2}$ *and* $d'_{ij} = 0$ *if* $i \neq j$.

1.3 Proofs

In order to prove our results, we need some notions and facts from the theory of lattices that we collect from [45]. Recall that a partially ordered set (poset for short) $(P, \leq)$ is a set P together with a binary relation $\leq$ satisfying:

1. For all $p \in P$, $p \leq p$ (reflexivity).

2. If $p \leq q$ and $q \leq p$, then $p = q$ (antisymmetry).

3. If $p \leq q$ and $q \leq r$, then $p \leq r$ (transitivity).

The binary relation $\leq$ is called the order and the elements of P are said to be orderd with respect to $\leq$. Recall that two posets P, Q are called isomorphic, in which case we write $P \cong Q$, if there exists an order-preserving bijection $\phi\colon P \to Q$ whose inverse is order-preserving. The cartesian product $P \times Q$ of two posets P, Q is defined on the set $\{(p, q)\colon p \in P, q \in Q\}$ by letting $(p, q) \leq (p', q')$ in $P \times Q$ if and only if $p \leq p'$ in P and $q \leq q'$ in Q. For $p, q \in P$ an upper bound of p and q is an element $r \in P$ such that $p, q \leq r$. A least upper bound of p and q is an upper bound $s \in P$ of p and q such that for any upper bound r of p and q one has $s \leq r$. Clearly, if a least upper bound of two elements p and q exists, it is unique. A greatest lower bound is defined in complete analogy. A lattice is a poset with the property that any two of its elements have a least upper bound and a greatest lower bound. It is well-known, that $\mathcal{P}_{[n]}$ together with the relation $\leq$ as defined in Theorem 1.1 is a lattice, the so-called partition lattice. For $\pi, \rho \in \mathcal{P}_{[n]}$ with $\pi \leq \rho$ we call the set $[\pi, \rho] := \{\sigma \in \mathcal{P}_{[n]}\colon \pi \leq \sigma \leq \rho\}$ an interval. We will make repeated use of the isomorphism

$$[\pi, \rho] \cong \bigtimes_{B \in \rho} \mathcal{P}_{\pi|_B}, \tag{1.10}$$

cf. Example 3.10.4 in [45]. For more information on posets in general and the partition lattice in particular the reader is referred to [45]. Evidently, we have $\mathcal{P}_{[n]} = [\varDelta_{[n]}, \{[n]\}]$, where for any set A we let $\varDelta_A := \{\{a\}\colon a \in A\}$ be the partition of A into singletons. Occasionally, we write $\varDelta_n$ instead of $\varDelta_{[n]}$.

Using the notation we just introduced, the n-$\varLambda$-coalescent $\varPi^n$ is a Markov chain with state space $\mathcal{P}_{[n]}$, initial state $\varDelta_{[n]}$ and Q-matrix $Q^n = (q^n_{\pi\rho})_{\pi,\rho \in \mathcal{P}_{[n]}}$ given by

$$q^n_{\pi\rho} := \begin{cases} \lambda_{\#\pi, \#\pi - \#\rho + 1} & \text{if } \pi \prec \rho, \\ -\lambda_{\#\pi} & \text{if } \pi = \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{1.11}$$

where $\lambda_b := \sum_{k=2}^{b} \binom{b}{k} \lambda_{b,k}$, $b \geq 2$, is the infinitesimal rate.

There are several extensions of $\leq$ to a linear order on $\mathcal{P}_{[n]}$. Let us fix such an extension $\leq_{\text{ex}}$ and notice that the following quantities of Q^n that we are interested in do not depend on the specific extension chosen. The linear order $\leq_{\text{ex}}$ induces a natural bijection ψ from $\mathcal{P}_{[n]}$ to $[B_n]$, where B_n denotes the nth Bell number, which is the number $\#\mathcal{P}_{[n]}$ of partitions of $[n]$, defined inductively by letting $\psi(\varDelta_{[n]}) = 1$ and $\psi(\pi) \leq \psi(\rho)$ iff $\pi \leq_{\text{ex}} \rho$. Then Q^n can be seen as an upper right triangular matrix with entries ordered according to $\leq_{\text{ex}}$, if we define row/column π to be lower than row/column ρ iff $\pi \leq_{\text{ex}} \rho$. The determinant of Q^n is therefore given by the product of its diagonal entries. Hence, the characteristic polynomial of Q^n is given by $\chi_{Q^n}(x) = \det(Q^n - x\mathbb{I}_n) = (-1)^{B_n} \prod_{\pi \in \mathcal{P}_{[n]}} (\lambda_{\#\pi} + x) = (-1)^{B_n} \prod_{i=1}^{n} (\lambda_i + x)^{\left\{ n \atop i \right\}}$, where $\mathbb{I}_n$ is the identity matrix on $\mathcal{P}_{[n]}$. Hence, for each $i \in [n]$, $-\lambda_i$ is an eigenvalue of Q^n with algebraic multiplicity $\#\mathcal{P}_{[n],i} = \left\{ n \atop i \right\}$.

From now on we fix an $n \in \mathbb{N}$, $n \geq 2$, and drop this index in the notation, if there is no risk of confusion.

1.3.1 Bolthausen-Sznitman n-coalescent

In order to prepare the proof of the spectral decomposition of $Q = Q^{n,BS}$, Theorem 1.1, we calculate the left and right eigenvectors of Q. In the sequel we give two proofs for the right eigenvectors of Q that are of rather different flavours. The first proof is completely self-contained and only makes use of the partition lattice $\mathcal{P}_{[n]}$. Together with the proof of Lemma 1.11 it might serve as a starting point to find a spectral decomposition for more general coalescents, e.g. beta coalescents. There is a probabilistic interpretation of the right eigenvector of Q in terms of random recursive trees which then motivates our second proof that heavily draws on random recursive trees and their connection to the Bolthausen-Sznitman coalescent as explored in Goldschmidt and Martin [15].

Lemma 1.7. *For $\rho \in \mathcal{P}_{[n]}$ the vector $(r_{\pi\rho})_{\pi\in\mathcal{P}_{[n]}}$ defined by*

$$r_{\pi\rho} := \begin{cases} \frac{(\#\rho-1)!}{(\#\pi-1)!}\prod_{B\in\rho}(\#\pi|_B - 1)! & \text{if } \pi \le \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{1.12}$$

is a right eigenvector of Q with corresponding eigenvalue $1 - \#\rho$.

Proof. (first proof of Lemma 1.7) In order to carry out the following calculations, for any two partitions $\pi, \rho \in \mathcal{P}_{[n]}$ we need to parameterize the set $\{\sigma\colon \pi \prec \sigma \le \rho\}$. To construct an arbitrary partition σ such that $\pi \prec \sigma$, we could choose a subset $C \subseteq \pi$ of at least two blocks of π and merge them in order to obtain σ. In this case $\#\sigma = \#\pi - \#C + 1$. If, additionally, we require $\sigma \le \rho$, we certainly cannot choose any collection C of blocks in π. Instead, all blocks chosen have to be in $\pi|_B$ for some block $B \in \rho$, in which case $\#\sigma|_B = \#\pi|_B - \#C + 1$. To summarize, we have $\{\sigma\colon \pi \prec \sigma \le \rho\} = \left\{\left\{\bigcup_{D\in C} D\right\} \cup (\pi \setminus C) : B \in \rho, C \subseteq \pi|_B, \#C \ge 2\right\}$. Using this parametrization, we obtain

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{P}_{[n]}} q_{\pi\sigma} r_{\sigma\rho} = \sum_{\sigma\colon \pi\prec\sigma\le\rho} q_{\pi\sigma} r_{\sigma\rho} - (\#\pi - 1) r_{\pi\rho} \\
&= \sum_{B\in\rho} \sum_{\substack{C\subseteq\pi|_B \\ \#C\ge 2}} \frac{(\#\pi - \#C)!(\#C - 2)!}{(\#\pi - 1)!} \frac{(\#\rho - 1)!}{(\#\pi - \#C)!} \\
&\qquad \times (\#\pi|_B - \#C)! \prod_{D\in\rho\setminus\{B\}} (\#\pi|_D - 1)! - (\#\pi - 1) r_{\pi\rho} \\
&= \sum_{B\in\rho} \sum_{c=2}^{\#\pi|_B} \binom{\#\pi|_B}{c} \frac{(c-2)!}{(\#\pi - 1)!} (\#\rho - 1)! \frac{(\#\pi|_B - c)!}{(\#\pi|_B - 1)!} \prod_{D\in\rho} (\#\pi|_D - 1)! - (\#\pi - 1) r_{\pi\rho} \\
&= \left(\sum_{B\in\rho} \sum_{c=2}^{\#\pi|_B} \binom{\#\pi|_B}{c} (c-2)! \frac{(\#\pi|_B - c)!}{(\#\pi|_B - 1)!} - (\#\pi - 1) \right) r_{\pi\rho} \\
&= \left(\sum_{B\in\rho} \#\pi|_B \sum_{c=2}^{\#\pi|_B} \frac{1}{c(c-1)} - (\#\pi - 1) \right) r_{\pi\rho} \\
&= \left(\sum_{B\in\rho} (\#\pi|_B - 1) - (\#\pi - 1) \right) r_{\pi\rho} = (1 - \#\rho) r_{\pi\rho},
\end{aligned}$$

and the claim follows. □

A connection with random recursive trees. Let us now recall the notion of a random recursive tree in order to prepare our second proof of Lemma 1.7, where we closely follow Goldschmidt and Martin [15]. We call a tree on n nodes labelled by $1, 2, \dots, n$ an increasing tree if the root has label 1 and the labels in any path from the root to another node are increasing.

If we have an increasing tree on $n-1$ nodes, we obtain an increasing tree on n nodes by adding a node labelled n and attaching it by an edge to one of the nodes in the given tree. Starting from the tree that only consists of the root node 1 this gives an explicit construction of all increasing trees on n nodes. Consequently, there are $(n-1)!$ increasing trees on n nodes. A random recursive tree is a tree chosen uniformly at random from all increasing trees on n nodes. An explicit construction of a random recursive tree on n nodes is the following. Start with the root node labelled 1. If the tree has k nodes, choose one of these nodes uniformly at random and attach to it node $k+1$ by an edge. Stop after attaching node n.

For any partition $\pi \in \mathcal{P}_{[n]}$ an increasing tree on π is a tree with $\#\pi$ nodes that are labelled by the blocks in π such that the labels in any path from the root to another node are increasing with respect to their least element. We denote a random recursive tree on π by $\mathfrak{T}_\pi$.

Crucial to the construction of Π^n via random recursive trees is the following cutting procedure. When given a tree $\mathcal{T}$ on π, a cut is performed by picking an edge, removing the subtree above this edge (here we picture trees as they grow in nature: from the root at the bottom to the leaves at the top) and adding the labels of this subtree to the labels of the node below the edge. For a simple tree the cutting procedure is depicted in Figure 1.1. We denote by $c\mathcal{T}$ the tree obtained by cutting $\mathcal{T}$ at an edge chosen uniformly at random.

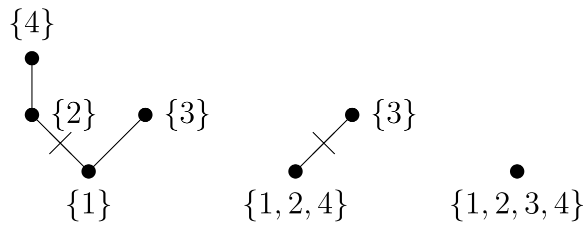

Figure 1.1: An increasing tree on $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ cut down successively to a tree on $\{\{1, 2, 3, 4\}\}$.

For any tree $\mathcal{T}$ on the label set π let $p(\mathcal{T}) = \pi$. A striking result is [15, Proposition 2.1], which states that after cutting a random recursive tree $\mathfrak{T}_\pi$ at an edge chosen uniformly at random, the new tree $c\mathfrak{T}_\pi$ is a random recursive tree on the new label set (which is again a partition of $[n]$), more formally: $c\mathfrak{T}_\pi =_d \mathfrak{T}_{p(c\mathfrak{T}_\pi)}$.

For $\pi \in \mathcal{P}_{[n]}$ define a time-homogeneous continuous-time Markov chain $\mathcal{R}_\pi := \{\mathcal{R}_\pi(t), t \geq 0\}$ with values in the set of random recursive trees as follows. The initial state $\mathcal{R}_\pi(0)$ is the random recursive tree $\mathfrak{T}_\pi$ on π. The process $\mathcal{R}_\pi$ evolves according to the following dynamics. Suppose $\mathcal{R}_\pi$ is in state $\mathcal{T}$. If $\mathcal{T}$ has only one vertex, do nothing. Otherwise, attach to each edge in $\mathcal{T}$ an exponential 1 clock, all clocks being independent. When the first clock rings, cut $\mathcal{T}$ at the associated edge to obtain the next state of $\mathcal{R}_\pi$. Proposition 2.2 in [15] then establishes that Π^n is equal in distribution to $\{p(\mathcal{R}_{\Delta_n}(t)), t \geq 0\}$.

Let $\{R_\pi(k), k \geq 0\}$ denote the jump chain of $\mathcal{R}_\pi$. From the definition of $\mathcal{R}$ it follows that a process that is equal in distribution to the jump chain R can be constructed recursively by letting

$$R_\pi(0) := \mathfrak{T}_\pi, \quad R_\pi(k+1) := cR_\pi(k), \quad k \geq 0.$$

Fix two partitions $\pi, \rho \in \mathcal{P}_{[n]}$ such that $\pi \leq \rho$. We say that an increasing tree $\mathcal{T}$ on π contains ρ (in symbols: $\rho \sqsubset \mathcal{T}$) iff one can obtain an increasing tree on ρ by successively cutting $\mathcal{T}$. A simple question then is: what is the probability $\mathbb{P}\{\rho \sqsubset \mathfrak{T}_\pi\}$ that a random recursive tree on π contains ρ? Notice first, that an increasing tree $\mathcal{T}$ on π contains ρ if for each block $B \in \rho$ we can find a node $v = v(B)$ in $\mathcal{T}$ such that the labels in the subtree above v coincide with the elements in $\pi|_B$. In other words, we can construct all increasing trees on π that contain ρ by first constructing an increasing tree $\mathcal{T}$ on ρ. Then each node $v \in \mathcal{T}$ is labelled by some block $B \in \rho$. Now if $\#\pi|_B > 1$, build an increasing tree on $\pi|_B$ and replace v by this tree. This procedure is done for all nodes $v \in \mathcal{T}$ to obtain an increasing tree on π that contains ρ. Therefore, the number of increasing trees on π containing ρ is $\#\{\mathcal{T} \colon \mathcal{T} \text{ increasing tree on } \pi, \rho \sqsubset \mathcal{T}\} = (\#\rho - 1)! \prod_{B \in \rho} (\#\pi|_B - 1)!$. On the other hand, the total number of increasing trees on π is $(\#\pi - 1)!$. Since a random recursive tree on π is a tree chosen uniformly at random from all increasing trees on π, we obtain

$$\mathbb{P}\{\rho \sqsubset \mathfrak{T}_\pi\} = \begin{cases} (\#\rho - 1)! \prod_{B \in \rho} (\#\pi|_B - 1)!/(\#\pi - 1)! & \text{if } \pi \leq \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{1.13}$$

which is just $r_{\pi\rho}$. We can now turn to our second proof of Lemma 1.7 in terms of random recursive trees.

Proof. (second proof of Lemma 1) We rewrite the statement as $\sum_{\sigma \colon \pi \prec \sigma} q_{\pi\sigma} r_{\sigma\rho} = (\#\pi - \#\rho) r_{\pi\rho}$. Let $J = \{J(k), k \geq 0\}$ denote the jump chain of $\Pi = \Pi^{BS,n}$. Since $\mathbb{P}\{J(1) = \sigma | J(0) = \pi\} = q_{\pi\sigma}/q_\pi$ is the probability that J jumps from π to σ, after dividing the rewritten statement by $q_\pi = \#\pi - 1$ we obtain for the left hand side

$$\begin{aligned} \sum_{\sigma \colon \pi \prec \sigma} \frac{q_{\pi\sigma}}{q_\pi} r_{\sigma\rho} &= \sum_{\sigma \colon \pi \prec \sigma} \mathbb{P}\{J(1) = \sigma | J(0) = \pi\} \mathbb{P}\{\rho \sqsubset \mathfrak{T}_\sigma\} \\ &= \sum_{\sigma \colon \pi \prec \sigma} \mathbb{P}\{p(c\mathfrak{T}_\pi) = \sigma\} \mathbb{P}\{\rho \sqsubset \mathfrak{T}_\sigma\} = \mathbb{P}\{\rho \sqsubset \mathfrak{T}_{p(c\mathfrak{T}_\pi)}\} \\ &= \mathbb{P}\{\rho \sqsubset c\mathfrak{T}_\pi\} = \mathbb{P}\{\rho \sqsubset c\mathfrak{T}_\pi | \rho \sqsubset \mathfrak{T}_\pi\} \mathbb{P}\{\rho \sqsubset \mathfrak{T}_\pi\}, \end{aligned}$$

where we used Propositions 2.2 and 2.1 of [15]. Conditional on $\rho \sqsubset \mathfrak{T}_\pi$, there are precisely $\#\pi - \#\rho$ among the $\#\pi - 1$ edges in $\mathfrak{T}_\pi$ that we may cut in order to obtain a tree $c\mathfrak{T}_\pi$ that contains ρ. Since, by definition, $c\mathfrak{T}_\pi$ is obtained by cutting $\mathfrak{T}_\pi$ at an edge chosen uniformly at random, we have $\mathbb{P}\{\rho \sqsubset c\mathfrak{T}_\pi | \rho \sqsubset \mathfrak{T}_\pi\} = (\#\pi - \#\rho)/(\#\pi - 1)$. The claim follows. ☐

Lemma 1.8. *For $x \in \mathbb{R} \setminus \{0\}$ we have*

$$\sum_{\sigma \in \mathcal{P}_{[n]}} r_{\pi\sigma} x^{\#\sigma - 1} l_{\sigma\rho} = (-1)^{\#\rho} x^{-1} \frac{(\#\rho - 1)!}{(\#\pi - 1)!} \prod_{B \in \rho} (-x)^{\overline{\#\pi|_B}}. \tag{1.14}$$

Proof. Notice that

$$\sum_{\sigma\in\mathcal{P}_{[n]}} r_{\pi\sigma}x^{\#\sigma-1}l_{\sigma\rho} = \frac{(\#\rho-1)!}{(\#\pi-1)!}\sum_{\sigma\in[\pi,\rho]}(-1)^{\#\sigma-\#\rho}x^{\#\sigma-1}\prod_{B\in\sigma}(\#\pi|_B-1)!$$
$$= (-1)^{\#\rho}x^{-1}\frac{(\#\rho-1)!}{(\#\pi-1)!}\sum_{\sigma\in[\pi,\rho]}\prod_{B\in\sigma}-x(\#\pi|_B-1)!. \tag{1.15}$$

Since $\left[{n \atop i}\right]$ counts the number of permutations of $[n]$ with i cycles it is clear that $\left[{n \atop i}\right] = \sum_{\pi\in\mathcal{P}_{[n],i}}\prod_{B\in\pi}(\#B-1)!$, cf. equation (1.15) in [37]. Using the isomorphism $[\pi,\rho] \cong \times_{B\in\rho}\mathcal{P}_{\pi|_B}$, we calculate

$$\sum_{\sigma\in[\pi,\rho]}\prod_{B\in\sigma}-x(\#\pi|_B-1)! = \sum_{\tau'\in\times_{B\in\rho}\mathcal{P}_{\pi|_B}}\prod_{B\in\rho}\prod_{C\in\tau'_B}-x(\#C-1)!$$
$$= \prod_{B\in\rho}\sum_{\tau\in\mathcal{P}_{\pi|_B}}\prod_{C\in\tau}-x(\#C-1)! = \prod_{B\in\rho}\left(\sum_{k=1}^{\#\pi|_B}(-x)^k\sum_{\tau\in\mathcal{P}_{\pi|_B,k}}\prod_{C\in\tau}(\#C-1)!\right)$$
$$= \prod_{B\in\rho}\sum_{k=1}^{\#\pi|_B}(-x)^k\left[{\#\pi|_B \atop k}\right] \tag{1.16}$$
$$= \prod_{B\in\rho}(-x)^{\overline{\#\pi|_B}}, \tag{1.17}$$

where in the last step we used $x^{\overline{n}} = \sum_{k=1}^n\left[{n \atop k}\right]x^k$, cf. equation (1.16) in [37]. □

Lemma 1.9. *The matrix $L = (l_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ defined by*

$$l_{\pi\rho} := \begin{cases}(-1)^{\#\pi-\#\rho}\frac{(\#\rho-1)!}{(\#\pi-1)!} & \text{if } \pi\le\rho,\\ 0 & \text{otherwise,}\end{cases} \tag{1.18}$$

is the inverse matrix of R, i.e. $\sum_{\sigma\in\mathcal{P}_{[n]}} r_{\pi\sigma}l_{\sigma\rho} = \delta_{\pi\rho}$.

Proof. Choosing $x = 1$ in Lemma 1.8 we have that $\sum_{\sigma\in\mathcal{P}_{[n]}} r_{\pi\sigma}l_{\sigma\rho} = \delta_{\pi\rho}$. □

Proof. (of Theorem 1.1) The claim follows by Lemmata 1.7 and 1.9. □

Proof. (of Corollary 1.2) Since Π is a continuous-time Markov chain with finite state space, we have for $P(t) = (p_{\pi\rho}(t))_{\pi,\rho\in\mathcal{P}_{[n]}}$ the identity $P(t) = \exp(tRDL) = R\exp(tD)L$. In the last step we made use of the spectral decomposition, Theorem 1.1. In particular, this yields

$$p_{\pi\rho}(t) = \sum_{\sigma\in\mathcal{P}_{[n]}} r_{\pi\sigma}e^{-t(\#\sigma-1)}l_{\sigma\rho} = \sum_{\sigma\in[\pi,\rho]} r_{\pi\sigma}e^{-t(\#\sigma-1)}l_{\sigma\rho}. \tag{1.19}$$

Letting $x = e^{-t}$ in Lemma 1.8 proves the claim. □

Proof. (of Corollary 1.3) From (1.16) we have

$$x^{-1}\sum_{\sigma\in[\pi,\rho]}\prod_{B\in\sigma}-x(\#\pi|_B-1)! = x^{-1}\prod_{B\in\rho}\sum_{k=1}^{\#\pi|_B}(-x)^k\begin{bmatrix}\#\pi|_B\\ k\end{bmatrix}$$
$$= \sum_{(k_B)_{B\in\rho}\in\mathbb{N}^{\#\rho}}(-1)^{|k|}x^{|k|-1}\prod_{B\in\rho}\begin{bmatrix}\#\pi|_B\\ k_B\end{bmatrix},$$

where $|k| := \sum_{B\in\rho} k_B$. Letting $x = e^{-t}$ and integrating out with respect to t we see for $\rho \neq \{[n]\}$

$$\int_0^\infty e^t\sum_{\sigma\in[\pi,\rho]}\prod_{B\in\sigma}-e^{-t}(\#\pi|_B-1)!\,dt = \sum_{(k_B)_{B\in\rho}\in\mathbb{N}^{\#\rho}}\frac{(-1)^{|k|}}{|k|-1}\prod_{B\in\rho}\begin{bmatrix}\#\pi|_B\\ k_B\end{bmatrix},$$

where for $\rho = \{[n]\}$

$$\int_0^\infty e^t\sum_{\sigma\in[\pi,\rho]}\prod_{B\in\sigma}-e^{-t}(\#\pi|_B-1)!\,dt = \sum_{k=1}^{\#\pi}(-1)^k\int_0^\infty e^{-t(k-1)}\,dt\begin{bmatrix}\#\pi\\ k\end{bmatrix} = -\infty.$$

The claim follows from (1.15) and the definition of $g_{\pi\rho}$. □

Proof. (of Corollary 1.4) Evidently, the rate at which a jump from i to j blocks occurs equals the sum of all rates at which a jump from a partition $\pi\in\mathcal{P}_{[n],i}$ to a partition $\rho\in\mathcal{P}_{[n],j}$ occurs, i.e. $q'_{ij} = \sum_{\rho\in\mathcal{P}_{[n],j}} q_{\pi\rho}$, where $\pi\in\mathcal{P}_{[n],i}$ is fixed arbitrarily. The quantity $\sum_{\rho\in\mathcal{P}_{[n],j}} q_{\pi\rho}$ does not depend on the choice of $\pi\in\mathcal{P}_{[n],i}$, as the following calculation shows. By the spectral decomposition of Q, Theorem 1.1, we obtain

$$q'_{ij} = \sum_{\rho\in\mathcal{P}_{[n],j}}\sum_{\sigma\in[\pi,\rho]} r_{\pi\sigma}d_{\sigma\sigma}l_{\sigma\rho}$$
$$= \sum_{\substack{\sigma:\,\pi\le\sigma\\ \#\sigma\ge j}}\sum_{\substack{\rho:\,\sigma\le\rho\\ \#\rho=j}}\frac{(\#\sigma-1)!}{(\#\pi-1)!}\prod_{B\in\sigma}(\#B-1)!(1-\#\sigma)(-1)^{\#\sigma-j}\frac{(j-1)!}{(\#\sigma-1)!}$$
$$= \frac{(j-1)!}{(i-1)!}(-1)^j\sum_{\substack{\sigma:\,\pi\le\sigma\\ \#\sigma\ge j}}(\#\sigma-1)!\prod_{B\in\sigma}(\#B-1)!(1-\#\sigma)\frac{(-1)^{\#\sigma}}{(\#\sigma-1)!}\begin{Bmatrix}\#\sigma\\ j\end{Bmatrix}$$
$$= \frac{(j-1)!}{(i-1)!}(-1)^j\sum_{k=j}^{i}(k-1)!\begin{bmatrix}i\\ k\end{bmatrix}(1-k)\frac{(-1)^k}{(k-1)!}\begin{Bmatrix}k\\ j\end{Bmatrix} = \sum_{k=j}^{i} r'_{ik}d'_{kk}l'_{kj},$$

hence, $Q' = R'D'L'$. □

1.3.2 Kingman's n-coalescent

For any two partitions $\pi,\rho\in\mathcal{P}_{[n]}$ such that $\pi\le\rho$ one may ask in how many different ways the jump chain of Kingman's coalescent may reach ρ when started in π. Since at each step only one merger of a pair of blocks occurs, there are $\#\pi - \#\rho + 1$ steps to be taken, and so the set of different ways is $C(\pi, \rho) := \{(\pi_1, \dots, \pi_m)\colon \pi = \pi_1 \lessdot \dots \lessdot \pi_m = \rho, m = \#\pi - \#\rho + 1\}$, where we defined $\lessdot$ in the paragraph preceding Theorem 1.5. We call each element in $C(\pi, \rho)$ a maximal chain in $[\pi, \rho]$ and denote by $m(\pi, \rho) := \#C(\pi, \rho)$ the number of maximal chains in $[\pi, \rho]$. Before we turn to the proof of the spectral decomposition of Q, Theorem 1.5, we count the number of maximal chains $m(\pi, \rho)$ in $[\pi, \rho]$ in the next Lemma.

Lemma 1.10 (Number of maximal chains). *For $\pi, \rho \in \mathcal{P}_{[n]}$ with $\pi \le \rho$ we have that*

$$m(\pi, \rho) = 2^{\#\rho - \#\pi}(\#\pi - \#\rho)! \prod_{B \in \rho} \#\pi|_B!. \tag{1.20}$$

Proof. Notice that any maximal chain $(\pi_1, \dots, \pi_n)$ in $[\Delta_{[n]}, \{[n]\}]$ can be constructed as follows. Let $\pi_1 := \Delta_{[n]}$, and if π_i with $i < n$ is constructed, π_{i+1} is obtained by merging two blocks in π_i, which can be done in $\binom{\#\pi_i}{2}$ ways. When $\pi_n = \{[n]\}$ is reached, the construction is finished. This construction shows that there are $m(\Delta_{[n]}, \{[n]\}) = \binom{n}{2}\binom{n-1}{2} \cdots \binom{2}{2} = 2^{1-n} n!(n-1)!$ maximal chains in $[\Delta_{[n]}, \{[n]\}]$. Hence (1.20) holds in the case $(\pi, \rho) = (\Delta_{[n]}, \{[n]\})$.

For the general case, recall the isomorphism $[\pi, \rho] \cong \bigtimes_{B \in \rho} \mathcal{P}_{\pi|_B}$. As a consequence, any maximal chain in $[\pi, \rho]$ can be built by choosing a maximal chain in each factor $\mathcal{P}_{\pi|_B}$ and then 'intertwining' these chains, i.e. ordering their elements (excluding the first element — the partition into singletons — in each chain) in any order subject to the restriction that the order of elements of the same chain is preserved. Consequently, we have

$$\begin{aligned} m(\pi, \rho) &= (\#\pi - \#\rho)! \prod_{B \in \rho} [(\#\pi|_B - 1)!]^{-1} \prod_{B \in \rho} m(\Delta_{\pi|_B}, \{\pi|_B\}) \\ &= (\#\pi - \#\rho)! \prod_{B \in \rho} 2^{1 - \#\pi|_B} \#\pi|_B! = 2^{\#\rho - \#\pi}(\#\pi - \#\rho)! \prod_{B \in \rho} \#\pi|_B!. \end{aligned}$$

□

Lemma 1.11. *For any $\rho \in \mathcal{P}_{[n]}$ the vector $(r_{\pi\rho})_{\pi \in \mathcal{P}_{[n]}}$ defined by*

$$r_{\pi\rho} := \begin{cases} \frac{2^{\#\pi - \#\rho}(2\#\rho - 1)!}{(\#\pi - \#\rho)!(\#\pi + \#\rho - 1)!} m(\pi, \rho) & \text{if } \pi \le \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{1.21}$$

$$= \begin{cases} \frac{(2\#\rho - 1)!}{(\#\pi + \#\rho - 1)!} \prod_{B \in \rho} \#\pi|_B! & \text{if } \pi \le \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{1.22}$$

is a right eigenvector of Q with corresponding eigenvalue $-\binom{\#\rho}{2}$.

Proof. Fix $\pi, \rho \in \mathcal{P}_{[n]}$. If $\pi < \rho$, we have

$$\begin{aligned} \sum_{\sigma \in \mathcal{P}_{[n]}} q_{\pi\sigma} r_{\sigma\rho} &= \sum_{\sigma\colon \pi \lessdot \sigma} r_{\sigma\rho} - \binom{\#\pi}{2} r_{\pi\rho} \\ &= \frac{2^{\#\pi - 1 - \#\rho}(2\#\rho - 1)!}{(\#\pi - 1 - \#\rho)!(\#\pi + \#\rho - 2)!} \sum_{\sigma\colon \pi \lessdot \sigma} m(\sigma, \rho) - \binom{\#\pi}{2} r_{\pi\rho} \\ &= \left(\frac{(\#\pi - \#\rho)(\#\pi + \#\rho - 1)}{2} - \binom{\#\pi}{2} \right) r_{\pi\rho} = -\binom{\#\rho}{2} r_{\pi\rho}, \end{aligned}$$

where we used $\sum_{\sigma:\,\pi\lessdot\sigma} m(\sigma,\rho) = m(\pi,\rho)$. If $\pi=\rho$, we have $\sum_{\sigma\in\mathcal{P}_{[n]}} q_{\pi\sigma}r_{\sigma\rho} = q_{\pi\pi} = -\binom{\#\rho}{2}r_{\pi\rho}$, since $m(\pi,\pi)=1$, hence $r_{\pi\pi}=1$. Finally, if $\pi\le\rho$ does not hold, thus $r_{\pi\rho}=0$, we cannot have $\pi\le\sigma\le\rho$ for any $\sigma\in\mathcal{P}_{[n]}$ and therefore $\sum_{\sigma\in\mathcal{P}_{[n]}} q_{\pi\sigma}r_{\sigma\rho}=0$. This shows (1.21). Now (1.22) follows from Lemma 1.10 on the number of maximal chains. □

Evidently, the B_n eigenvectors of Q defined by (1.21) are linearly independent. We are now interested in the inverse matrix of $R=(r_{\pi\rho})_{\pi\rho\in\mathcal{P}_{[n]}}$, that is the matrix $L=(l_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$ such that $\delta_{\pi\rho}=\sum_{\sigma\in\mathcal{P}_{[n]}} r_{\pi\sigma}l_{\sigma\rho}$ for all $\pi,\rho\in\mathcal{P}_{[n]}$.

Lemma 1.12. *For any $\pi\in\mathcal{P}_{[n]}$ the vector $(l_{\pi\rho})_{\rho\in\mathcal{P}_{[n]}}$ given by*

$$l_{\pi\rho} := \begin{cases} (-1)^{\#\pi-\#\rho}\frac{2^{\#\pi-\#\rho}(\#\pi+\#\rho-2)!}{(2\#\pi-2)!(\#\pi-\#\rho)!}m(\pi,\rho) & \text{if } \pi\le\rho,\\ 0 & \text{otherwise,}\end{cases} \tag{1.23}$$

$$= \begin{cases} (-1)^{\#\pi-\#\rho}\frac{(\#\pi+\#\rho-2)!}{(2\#\pi-2)!}\prod_{B\in\rho}\#\pi|_B! & \text{if } \pi\le\rho,\\ 0 & \text{otherwise,}\end{cases} \tag{1.24}$$

is a left eigenvector of Q with corresponding eigenvalue $-\binom{\#\pi}{2}$.

Proof. Use $\sum_{\sigma:\,\sigma\lessdot\rho} m(\pi,\sigma)=m(\pi,\rho)$ to obtain

$$\begin{aligned}\sum_{\sigma\in\mathcal{P}_{[n]}} l_{\pi\sigma}q_{\sigma\rho} &= \sum_{\sigma:\,\sigma\lessdot\rho} l_{\pi\sigma} - \binom{\#\rho}{2}l_{\pi\rho}\\ &= (-1)^{\#\pi-\#\rho-1}\frac{2^{\#\pi-\#\rho-1}(\#\pi+\#\rho-1)!}{(2\#\pi-2)!(\#\pi-\#\rho-1)!}\sum_{\sigma:\,\sigma\lessdot\rho} m(\pi,\sigma) - \binom{\#\rho}{2}l_{\pi\rho}\\ &= \left(-\frac{(\#\pi+\#\rho-1)(\#\pi-\#\rho)}{2} - \binom{\#\rho}{2}\right)l_{\pi\rho} = -\binom{\#\pi}{2}l_{\pi\rho},\end{aligned}$$

thus (1.23) holds. Equation (1.24) follows from the Lemma on the number of maximal chains, Lemma 1.10. □

Proof. (of Theorem 1.5) The inverse matrix of R, let us call it $U=(u_{\pi\rho})_{\pi,\rho\in\mathcal{P}_{[n]}}$, is uniquely determined, and is a matrix of left eigenvectors of Q, i.e. $UQ=DU$. Moreover, for any $\pi\in\mathcal{P}_{[n]}$ we have by assumption $u_{\pi\pi}=\sum_{\sigma:\,\pi\le\sigma\le\rho} u_{\pi\sigma}r_{\sigma\rho}=\delta_{\pi\pi}=1$. This uniquely determines a matrix of left eigenvectors of Q, since for any $\pi,\rho\in\mathcal{P}_{[n]}$ with $\pi<\rho$ we have $-\binom{\#\rho}{2}u_{\pi\rho}=\sum_{\sigma\in[\pi,\rho]} u_{\pi\sigma}q_{\sigma\rho} = 1_{\{\pi\lessdot\rho\}}+\sum_{\sigma:\,\pi<\sigma\le\rho} u_{\pi\sigma}q_{\sigma\rho}$, and $u_{\pi\rho}=0$ for $\pi\not\le\rho$. Since $QR=RD$ by Lemma 1.12, and evidently $l_{\pi\pi}=1$ for any $\pi\in\mathcal{P}_{[n]}$, we have $U=L$ and the claim follows. □

Remark. Instead of calculating the hitting probabilites $h(\pi,\rho)$ of Kingman's n-coalescent via the spectral decomposition, Theorem 1.5, we use the observation from section 1.2 that the jump chain of Π can be interpreted as the directed simple random walk on $\mathcal{P}_{[n]}$. This implies that the jump chain of Π (when started from π) traces out any maximal chain in $[\pi,\{[n]\}]$ with equal probability $m(\pi,\{[n]\})^{-1}$. Clearly, the total number of maximal chains in $[\pi,\{[n]\}]$ that contain ρ is $m(\pi,\rho)m(\rho,\{[n]\})$, and thus

$$h(\pi,\rho)=\frac{m(\pi,\rho)m(\rho,\{[n]\})}{m(\pi,\{[n]\})}=\binom{\#\pi-1}{\#\rho-1}^{-1}\frac{\#\rho!}{\#\pi!}\prod_{B\in\rho}\#\pi|_B! = \left\lfloor{\#\pi \atop \#\rho}\right\rfloor^{-1}\prod_{B\in\rho}\#\pi|_B!,$$

where we used the Lemma on the number of maximal chains, Lemma 1.10, in the second step. In the special case $\pi = \Delta_{[n]}$ this formula was given by Kingman in [21], equation (2.3).

Proof. (of Corollary 1.6) In complete analogy to the argument in Corollary 1.4, we have $q'_{ij} = \sum_{\rho \in \mathcal{P}_{[n],j}} q_{\pi\rho}$ independent of the particular partition $\pi \in \mathcal{P}_{[n],i}$, as the following calculation shows. Using the spectral decomposition of Q, Theorem 1.5, we obtain

$$\begin{aligned} q'_{ij} &= \sum_{\rho \in \mathcal{P}_{[n],j}} \sum_{\sigma \in [\pi,\rho]} r_{\pi\sigma} d_{\sigma\sigma} l_{\sigma\rho} \\ &= -\sum_{\substack{\sigma:\ \pi \leq \sigma \\ \#\sigma \geq j}} \frac{(2\#\sigma - 1)!}{(i + \#\sigma - 1)!} \left(\prod_{B \in \sigma} \#\pi|_B! \right) \binom{\#\sigma}{2} \sum_{\substack{\rho:\ \sigma \leq \rho \\ \#\rho = j}} (-1)^{\#\sigma + j} \frac{(\#\sigma + j - 2)!}{(2\#\sigma - 2)!} \prod_{B \in \rho} \#\sigma|_B! \\ &= -\sum_{k=j}^{i} \frac{(2k-1)!}{(i+k-1)!} \left\lfloor {i \atop k} \right\rfloor \binom{k}{2} (-1)^{k+j} \frac{(k+j-2)!}{(2k-2)!} \left\lfloor {k \atop j} \right\rfloor = \sum_{k=j}^{i} r'_{ik} d'_{kk} l'_{kj}, \end{aligned}$$

where in the third step we used the identity $\left\lfloor {i \atop k} \right\rfloor = \sum_{\sigma \in \mathcal{P}_{[i],k}} \prod_{B \in \sigma} \#B!$ twice. This identity is obvious from the interpretation of $\left\lfloor {i \atop k} \right\rfloor$ as the number of partitions into k ordered blocks of a set of i elements, where the elements in each block are ordered. □

Acknowledgements. This is part of the authors' PhD theses. They would like to thank their respective supervisors Martin Möhle and Alison Etheridge for their support and numerous discussions. They also thank an anonymous referee for his constructive comments. H. P. thankfully acknowledges financial support by the foundation "Private Stiftung Ewald Marquardt für Wissenschaft und Technik, Kunst und Kultur."

Artikel 2

On the block counting process and the fixation line of the Bolthausen-Sznitman coalescent

Jonas Kukla Martin Möhle

Abstract. The block counting process and the fixation line of the Bolthausen–Sznitman coalescent are analyzed. It is shown that these processes, properly scaled, converge in the Skorohod topology to the Mittag–Leffler process and to Neveu's continuous-state branching process respectively as the initial state tends to infinity. Strong relations to Siegmund duality, Mehler semigroups and self-decomposability are pointed out. Furthermore, spectral decompositions for the generators and transition probabilities of the block counting process and the fixation line of the Bolthausen–Sznitman coalescent are provided leading to explicit expressions for functionals such as hitting probabilities and absorption times.

Stoch. Process. Appl. **128** (2018), 939-962.

Keywords. absorption time; block counting process; Bolthausen–Sznitman coalescent; fixation line; hitting probabilities; Mehler semigroup; Mittag–Leffler process; Neveu's continuous-state branching process; self-decomposability; Siegmund duality; spectral decomposition.

2010 *Mathematics Subject Classification.* Primary 60F05; 60J27; Secondary 92D15; 97K60.

2.1 Introduction

Exchangeable coalescents are Markovian processes $(\Pi_t)_{t\geq 0}$ with state space $\mathcal{P}$, the set of partitions of $\mathbb{N} := \{1, 2, \ldots\}$. These processes have attracted the interest of several researchers, mainly in biology, mathematics and physics, during the last decades. The full family of exchangeable coalescents (with simultaneous multiple collisions) is a class of partition valued Markovian processes with a rich probabilistic structure and hence important for mathematical studies. Moreover, coalescents are useful in mathematical population genetics to model the ancestry of a sample of individuals or genes and therefore important for biological applications.

Exchangeable coalescents with multiple collisions but without simultaneous multiple collisions are characterized by a measure Λ on the unit interval $[0,1]$ and therefore called Λ-coalescents. For further information on these processes we refer the reader to the independent works of Pitman [36] and Sagitov [41]. The most important coalescent is probably the Kingman coalescent [21], which allows only for binary mergers of ancestral lineages. In this case the measure Λ is the Dirac measure at 0.

For $t \geq 0$ let N_t denote the number of blocks of Π_t and let $N_t^{(n)}$ denote the number of blocks of $\Pi_t^{(n)}$, where $\Pi_t^{(n)}$ denotes the partition of Π_t restricted to a sample of size $n \in \mathbb{N}$. The processes $(N_t)_{t\geq 0}$ and $(N_t^{(n)})_{t\geq 0}$ are called the block counting processes of $(\Pi_t)_{t\geq 0}$ and $(\Pi_t^{(n)})_{t\geq 0}$ respectively.

Hénard [18] introduced for $n \in \mathbb{N}$ the so-called fixation line $(L_t^{(n)})_{t\geq 0}$ of a Λ-coalescent. Recently [13] the fixation line was extended to arbitrary exchangeable coalescents. One possible definition of the fixation line is based on the lookdown construction going back to Donnelly and Kurtz [8, 9]. A precise pathwise construction of the Markovian processes $(L_t^{(n)})_{t\geq 0}$, $n \in \mathbb{N}$, is provided in [18, p. 3010] for the Λ-coalescent and in [13, Section 1] for general exchangeable coalescents. By this construction, $(L_t^{(n)})_{t\geq 0}$ has state space $\{n, n+1, \ldots\} \cup \{\infty\}$, initial state $L_0^{(n)} = n$ and non-decreasing paths. Moreover, $L_t^{(n)} \leq L_t^{(n+1)}$ for all $n \in \mathbb{N}$ and $t \geq 0$. The infinitesimal rates of the process $(L_t)_{t\geq 0} := (L_t^{(1)})_{t\geq 0}$ are provided in [18, Lemma 2.3] for the Λ-coalescent and in [13, Proposition 2.2] for arbitrary exchangeable coalescents.

The fixation line can be traced back to Pfaffelhuber and Wakolbinger [35] for the Kingman coalescent. For the Λ-coalescent the fixation line appears in Labbé [24] and was further studied by Hénard [17, 18].

Note that we omit the pathwise definition of the fixation line via the lookdown construction here because it is provided in detail in [13] and [18]. In fact, our proofs concerning the fixation line are mainly based on the infinitesimal rates and do not rely on the pathwise construction except for the fact that $L_t^{(n)}$ is non-decreasing in n.

The fact that the block counting process $(N_t)_{t\geq 0}$ of a coalescent with multiple collisions is Siegmund dual to the fixation line $(L_t)_{t\geq 0}$ is explicitly mentioned in [2, Remark 3.6] and already contained in Hénard [18, Lemma 2.4] even though the name Siegmund dual is not mentioned there. For the full class of coalescents with simultaneous multiple collisions this Siegmund duality is provided in [13, Theorem 2.9] and may also be derived from the pathwise relations $L_t^{(n)} = \sup\{k \in \mathbb{N} : N_t^{(k)} \leq n\}$ and $N_t^{(n)} = \inf\{k \in \mathbb{N} : L_t^{(k)} \geq n\}$, $t \geq 0$, $n \in \mathbb{N}$.

In this article we focus on the Bolthausen–Sznitman coalescent [6], which is the particular Λ-coalescent with Λ being the uniform distribution on the unit interval. The generator $Q = (q_{ij})_{i,j\in\mathbb{N}}$ of the block counting process and the generator $\Gamma = (\gamma_{ij})_{i,j\in\mathbb{N}}$ of the fixation line of the Bolthausen–Sznitman coalescent have entries (see, for example, [29, Eq. (1.1)] and [18, p. 3015, Eq. (2.8) with $\alpha = 1$])

$$q_{ij} = \begin{cases} \frac{i}{(i-j)(i-j+1)} & \text{for } j < i, \\ 1-i & \text{for } j = i, \\ 0 & \text{for } j > i, \end{cases} \quad \text{and} \quad \gamma_{ij} = \begin{cases} \frac{i}{(j-i)(j-i+1)} & \text{for } j > i, \\ -i & \text{for } j = i, \\ 0 & \text{for } j < i. \end{cases}$$

The block counting process and the corresponding generator Q have been studied extensively in the literature. In this article we focus on both processes $(N_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ with an emphasis on the fixation line $(L_t)_{t\geq 0}$, which has been studied less extensively so far. As already observed by Hénard [18], it follows from $\gamma_{i,i+j} = i/(j(j+1))$, $i, j \in \mathbb{N}$, that $(L_t)_{t\geq 0}$ is a continuous-time branching process with offspring law $p_k := 1/(k(k-1))$, $k \in \{2, 3, \ldots\}$, and probability generating function (pgf) $\mathbb{E}\left(s^{L_t^{(n)}}\right) = (1-(1-s)^{e^{-t}})^n$, $s \in [0, 1]$, $t \geq 0$, $n \in \mathbb{N}$. These properties of the fixation line turn out to be fundamental and simplify several calculations. We furthermore stress the duality relation between the block counting process and the fixation line.

Section 2.2 deals with the behavior of the block counting process $(N_t^{(n)})_{t\geq 0}$ and the fixation line $(L_t^{(n)})_{t\geq 0}$ as the initial state n tends to infinity. The main convergence result (Theorem 2.1) states that both processes, properly scaled, converge in the Skorohod sense as $n \to \infty$ to the Mittag–Leffler process and to Neveu's continuous-state branching process respectively.

In Section 2.3 the processes $(N_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ are analyzed with an emphasis on spectral decompositions. These spectral decompositions lead to explicit expressions for several functionals of these processes such as hitting probabilities and absorption times.

The proofs provided in Section 2.4 rely on both analytic and probabilistic arguments which demonstrates the interplay between analysis and probability. The proofs concerning the asymptotic results in Section 2.2 do not depend on the proofs of the results concerning the spectral decomposition in Section 2.3 and vice versa. A short appendix collects some results of independent interest used in the proofs.

2.2 Asymptotics

We are interested in the behavior of the block counting process $(N_t^{(n)})_{t\geq 0}$ and the fixation line $(L_t^{(n)})_{t\geq 0}$ of the Bolthausen–Sznitman n-coalescent as the sample size n tends to infinity. In order to state the main convergence result (see Theorem 2.1 below) let us recall some properties of the Mittag–Leffler process $X = (X_t)_{t\geq 0}$ and Neveu's [32] continuous-state branching process $Y = (Y_t)_{t\geq 0}$.

The Mittag–Leffler process X is a Markovian process in continuous time with initial state $X_0 = 1$ and state space $E := [0, \infty)$. The name of this process comes from the fact that for every $t \geq 0$ the marginal random variable X_t is Mittag–Leffler distributed with parameter e^{-t}. Note that X_t has moments $\mathbb{E}\,(X_t^m) = \Gamma(1+m)/\Gamma(1+me^{-t})$, $m \in [0, \infty)$. The semigroup $(T_t^X)_{t\geq 0}$ of the Mittag–Leffler process X is given by

$$T_t^X f(x) = \mathbb{E}\left(f(x^{e^{-t}} X_t)\right), \qquad t, x \geq 0, f \in B(E), \tag{2.1}$$

where $B(E)$ denotes the set of all bounded measurable functions $f : E \to \mathbb{R}$. Conditional on $X_s = x$ the random variable $x^{-e^{-t}} X_{s+t}$ is Mittag–Leffler distributed with parameter e^{-t}. Some further information on the process X can be found in [3] and [28].

Neveu's [32] continuous-state branching process Y is as well a Markovian process in continuous time with initial state $Y_0 = 1$ and state space E. For every $t \geq 0$ the marginal random variable Y_t is α-stable with Laplace transform $\mathbb{E}\left(e^{-\lambda Y_t}\right) = e^{-\lambda^\alpha}$, $\lambda \geq 0$, where $\alpha := e^{-t}$. The semigroup $(T_t^Y)_{t\geq 0}$ of Neveu's continuous-state branching process Y is given by

$$T_t^Y g(y) = \mathbb{E}\left(g(y^{e^t} Y_t)\right), \qquad t, y \geq 0, g \in B(E). \tag{2.2}$$

Conditional on $Y_s = y$ the random variable $y^{-e^t} Y_{s+t}$ has the same distribution as Y_t. Note that (see, for example, [28]) the Mittag–Leffler process X is Siegmund dual to Neveu's continuous state branching process Y, i.e. $\mathbb{P}(X_t \leq y | X_0 = x) = \mathbb{P}(Y_t \geq x | Y_0 = y)$ for all $t, x, y \geq 0$.

Define the scaled block counting process $X^{(n)} = (X_t^{(n)})_{t\geq 0}$ and the scaled fixation line $Y^{(n)} := (Y_t^{(n)})_{t\geq 0}$ of the Bolthausen–Sznitman n-coalescent via

$$X_t^{(n)} := \frac{N_t^{(n)}}{n^{e^{-t}}} \quad \text{and} \quad Y_t^{(n)} := \frac{L_t^{(n)}}{n^{e^t}}, \qquad t \geq 0, n \in \mathbb{N}. \tag{2.3}$$

Note that, for $n \geq 2$, the processes $X^{(n)}$ and $Y^{(n)}$ are time-inhomogeneous because of the time-dependent scalings $n^{e^{-t}}$ and n^{e^t}. Let us consider the one-dimensional distributions of $X^{(n)}$ and $Y^{(n)}$ respectively. We first turn to $Y^{(n)}$. As already mentioned in the introduction, $(L_t)_{t\geq 0}$ is a continuous-time branching process with offspring law $p_k := 1/(k(k-1))$, $k \in \{2, 3, \ldots\}$, and $\mathbb{E}\left(s^{L_t^{(n)}}\right) = (1 - (1-s)^{e^{-t}})^n$, $s \in [0,1]$, $t \geq 0$, $n \in \mathbb{N}$. See also Eq. (2.7) in Corollary 2.3 in the following Section 2.3. Thus, for all $t, \lambda \geq 0$,

$$\mathbb{E}\left(e^{-\lambda Y_t^{(n)}}\right) = (1 - (1 - e^{-\lambda/n^{e^t}})^{e^{-t}})^n \longrightarrow e^{-\lambda^{e^{-t}}} = \mathbb{E}\left(e^{-\lambda Y_t}\right), \qquad n \to \infty.$$

Hence, $Y_t^{(n)} \to Y_t$ in distribution as $n \to \infty$. The convergence $X_t^{(n)} \to X_t$ in distribution as $n \to \infty$ is now obtained via duality as follows. For $n \in \mathbb{N}$, $t \geq 0$ and $x > 0$ define $m := \lfloor x n^{e^{-t}} \rfloor$ for convenience. Since $(N_t)_{t\geq 0}$ is Siegmund dual to $(L_t)_{t\geq 0}$ we conclude that

$$\begin{aligned}\mathbb{P}\left(X_t^{(n)} \leq x\right) &= \mathbb{P}\left(N_t^{(n)} \leq m\right) = \mathbb{P}\left(L_t^{(m)} \geq n\right) = \mathbb{P}\left(Y_t^{(m)} > (n-1)/m^{e^t}\right) \\ &\longrightarrow \mathbb{P}\left(Y_t > x^{-e^t}\right) = \mathbb{P}\left(Y_t^{-e^{-t}} < x\right) = \mathbb{P}\left(Y_t^{-e^{-t}} \leq x\right), \qquad n \to \infty,\end{aligned}$$

since $(n-1)/m^{e^t} \to x^{-e^t}$ as $n \to \infty$. It is well known that $Y_t^{-e^{-t}}$ is Mittag–Leffler distributed with parameter e^{-t}. Thus, $X_t^{(n)} \to X_t$ in distribution as $n \to \infty$. An alternative proof (avoiding duality) of the latter convergence based on moment calculations is provided in [28, p. 46, Step 1]. The convergence of the one-dimensional distributions motivates the following convergence result.

Theorem 2.1 (Asymptotics of the block counting process and the fixation line). *For the Bolthausen–Sznitman coalescent the following two assertions hold.*

(a) As $n \to \infty$ the scaled block counting process $X^{(n)}$, defined in (2.3), converges in $D_E[0, \infty)$ to the Mittag–Leffler process $X = (X_t)_{t\geq 0}$.

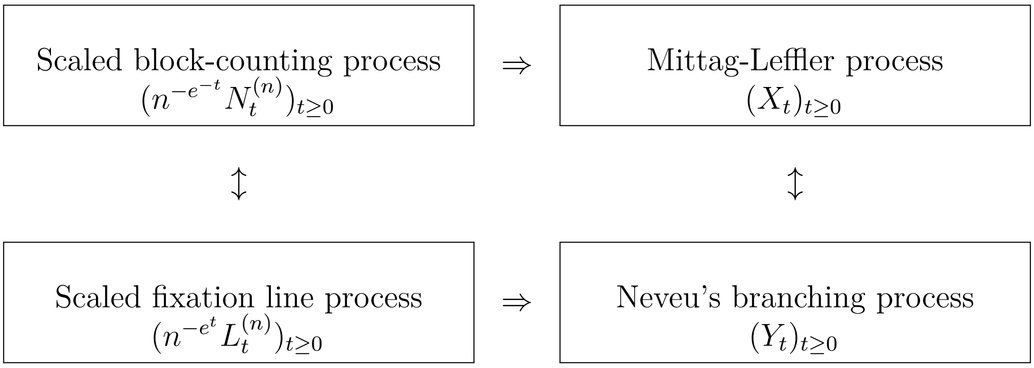

Figure 2.1: Commutative diagram for the block counting process $(N_t^{(n)})_{t\geq 0}$ and the fixation line $(L_t^{(n)})_{t\geq 0}$ of the Bolthausen–Sznitman coalescent. The right-arrows '$\Rightarrow$' stand for 'convergence in $D_E[0,\infty)$ as $n\to\infty$'. The vertical updown-arrows '$\updownarrow$' stand for 'duality', on the left hand side the duality of the block counting process $(N_t)_{t\geq 0}$ and the fixation line $(L_t)_{t\geq 0}$ with respect to the Siegmund duality kernel $H:\mathbb{N}^2\to\{0,1\}$ defined via $H(i,j):=1$ for $i\leq j$ and $H(i,j):=0$ otherwise, on the right hand side the duality of $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ with respect to the Siegmund duality kernel $H:[0,\infty)^2\to\{0,1\}$ defined via $H(x,y):=1$ for $x\leq y$ and $H(x,y):=0$ otherwise.

(b) As $n\to\infty$ the scaled fixation line $Y^{(n)}$, defined in (2.3), converges in $D_E[0,\infty)$ to Neveu's continuous-state branching process $Y=(Y_t)_{t\geq 0}$.

The proof of Theorem 2.1 provided in Section 2.4 indeed shows that it suffices to verify the convergence of the one-dimensional distributions. Theorem 2.1 demonstrates the intimate relation between the Bolthausen–Sznitman coalescent, the Mittag–Leffler process and Neveu's continuous state branching process. We refer the reader to Bertoin and Le Gall [4] for further insights concerning these relations.

Theorem 2.1 (a) is known from the literature [28, Theorem 1.1] and provided here for completeness. Our proof of Theorem 2.1 (a) is significantly shorter than the proof provided in [28] and gives further insights into the structure of the scaled block counting process $X^{(n)}$.

Part (b) of Theorem 2.1 is likely to be known from branching process theory, however the authors have not been able to trace this result in the literature. Note that the offspring distribution of the branching process $(L_t^{(n)})_{t\geq 0}$ has pgf $f(s)=s+(1-s)\log(1-s)$ and, hence, infinite mean. For related convergence results for the critical case when the offspring distribution has mean 1 we refer the reader to Sagitov [40] and the references therein. Note that in Theorem 2.1 of [40] the space-scaling is n and an additional time-scaling occurs. Theorem 2.1 (b) may be viewed as a kind of boundary case of Theorem 2.1 of [40] for $\alpha\to 1$. Similar convergence results for sequences of discrete-time branching processes can be traced back to Lamperti [25, 26].

In summary the commutative diagram in Fig. 1 holds.

Let us point out that Theorem 2.1 is strongly related to Mehler semigroups, to self-decomposability and to the Gumbel distribution. Clearly, Theorem 2.1 can be stated logarithmically as follows. The process $(\log N_t^{(n)} - e^{-t}\log n)_{t\geq 0}$ converges in $D_{\mathbb{R}}[0,\infty)$ to

$\tilde{X} := (\tilde{X}_t)_{t\geq 0} := (\log X_t)_{t\geq 0}$ and the process $(\log L_t^{(n)} - e^t \log n)_{t\geq 0}$ converges in $D_{\mathbb{R}}[0,\infty)$ to $\tilde{Y} := (\tilde{Y}_t)_{t\geq 0} := (\log Y_t)_{t\geq 0}$ as $n \to \infty$. Note that the semigroup $(T_t^{\tilde{X}})_{t\geq 0}$ of $\tilde{X}$ is given by

$$T_t^{\tilde{X}} f(x) = \mathbb{E}\left(f(xe^{-t} + \tilde{X}_t)\right), \qquad t \geq 0, f \in B(\mathbb{R}), x \in \mathbb{R}, \tag{2.4}$$

whereas the semigroup $(T_t^{\tilde{Y}})_{t\geq 0}$ of $\tilde{Y}$ is given by

$$T_t^{\tilde{Y}} g(y) = \mathbb{E}\left(g(ye^{t} + \tilde{Y}_t)\right), \qquad t \geq 0, g \in B(\mathbb{R}), y \in \mathbb{R}. \tag{2.5}$$

Semigroups of this form belong to the class of so called generalized Mehler semigroups [5] corresponding to generalized Ornstein–Uhlenbeck type processes. Note that (2.4) and (2.5) define the semigroups of $\tilde{X}$ and $\tilde{Y}$ completely, since for every $t \geq 0$ the distributions of the marginals $\tilde{X}_t = \log X_t$ and $\tilde{Y}_t = \log Y_t$ can be characterized as follows. Let E be standard exponentially distributed and independent of X and Y. Note that $G := -\log E$ is standard Gumbel distributed. From $E \stackrel{d}{=} (E/Y_t)^{e^{-t}}$ (see, for example, [43]) we conclude by an application of the transformation $x \mapsto -\log x$ that the distribution of $\tilde{Y}_t$ is characterized via the self-decomposable distributional equation

$$G \stackrel{d}{=} e^{-t}G + e^{-t}\tilde{Y}_t.$$

Thus, $\tilde{Y}_t$ has characteristic function $u \mapsto \Gamma(1 - iue^t)/\Gamma(1 - iu)$, $u \in \mathbb{R}$, and cumulants $\kappa_j(\tilde{Y}_t) = (e^{jt} - 1)\kappa_j(G)$, $j \in \mathbb{N}$, $t \geq 0$, where $\kappa_j(G)$ are the cumulants of the Gumbel distribution, i.e. $\kappa_1(G) = \gamma$ (Euler–Mascheroni constant) and $\kappa_j(G) = (-1)^j \Psi^{(j-1)}(1) = (j-1)!\zeta(j)$ for $j \in \mathbb{N} \setminus \{1\}$, where Ψ and ζ denote the digamma function (logarithmic derivative of the gamma function) and the Riemann zeta function respectively.

Similarly, the distribution of $\tilde{X}_t$ is characterized via the self-decomposable distributional equation

$$S \stackrel{d}{=} e^{-t}S + \tilde{X}_t,$$

where $S := -G$. Therefore, $\tilde{X}_t$ has characteristic function $u \mapsto \Gamma(1 + iu)/\Gamma(1 + iue^{-t})$, $u \in \mathbb{R}$, and cumulants $\kappa_j(\tilde{X}_t) = (-1)^j(1 - e^{-jt})\kappa_j(G)$, $j \in \mathbb{N}$, $t \geq 0$.

2.3 Spectral decompositions and applications

Spectral decompositions are of fundamental interest since they lead to diagonal representations of the corresponding operators or matrices which simplify many mathematical calculations and numerical computations significantly. Explicit spectral decompositions for (the block counting process of) the Kingman coalescent and the Bolthausen–Sznitman coalescent are provided in [23] and [29]. We are interested in analog spectral decompositions for the fixation line. A spectral decomposition of the generator Γ of the fixation line of the Kingman coalescent is provided in the appendix (Lemma 2.10) for completeness. Our first result in this section (Theorem 2.2) provides an explicit spectral decomposition for the generator Γ of the fixation line of the Bolthausen–Sznitman coalescent. This spectral decomposition is for example used afterwards to derive exact expressions and sharp approximations for the absorption time of the Bolthausen–Sznitman coalescent (see Corollaries 2.5 and 2.6). It turns out to be convenient to express the spectral decomposition in terms of the Stirling numbers $s(i,j)$ and $S(i,j)$ of the first and second kind respectively. Note that $(-1)^{i-j}s(i,j)$ is the number of permutations of i elements having j cycles whereas $S(i,j)$ counts the number of ways to partition a set of i elements into j nonempty subsets. For more insights why the Stirling numbers occur naturally in this context we refer the reader to [23], where a spectral decomposition of the generator of the full (partition valued) Bolthausen–Sznitman coalescent is provided.

Theorem 2.2 (Spectral decomposition of the generator of the fixation line)**.** *The generator $\varGamma = (\gamma_{ij})_{i,j\in\mathbb{N}}$ of the fixation line $(L_t)_{t\geq 0}$ of the Bolthausen–Sznitman coalescent has spectral decomposition $\varGamma = RDL$, where $D = (d_{ij})_{i,j\in\mathbb{N}}$ is the diagonal matrix with entries $d_{ij} = -i$ for $i = j$ and $d_{ij} = 0$ for $i \neq j$, and $R = (r_{ij})_{i,j\in\mathbb{N}}$ and $L = (l_{ij})_{i,j\in\mathbb{N}}$ are upper right triangular matrices with entries*

$$r_{ij} = \frac{i!}{j!}(-1)^{i+j}S(j,i) \quad \textit{and} \quad l_{ij} = \frac{i!}{j!}(-1)^{i+j}s(j,i), \qquad i,j\in\mathbb{N}. \tag{2.6}$$

As already explained in the introduction, the fixation line $(L_t)_{t\geq 0}$ of the Bolthausen–Sznitman coalescent has the branching property. Alternatively, one may derive this branching property from the spectral decomposition of the generator (Theorem 2.2). We state the following corollary.

Corollary 2.3 (Branching property/transition probabilities of the fixation line)**.** *For the Bolthausen–Sznitman coalescent, the random variable $L_t^{(i)}$ has probability generating function (pgf)*

$$\mathbb{E}\left(z^{L_t^{(i)}}\right) = (1-(1-z)^{e^{-t}})^i, \qquad |z|<1, t\geq 0, i\in\mathbb{N}. \tag{2.7}$$

Thus, $(L_t)_{t\geq 0}$ is a Markovian continuous-time branching process with state space $\mathbb{N}$ and offspring distribution $p_k = 1/(k(k-1))$, $k\in\{2,3,\ldots\}$, having infinite mean. Moreover, the transition probabilities $p_{ij}(t) := \mathbb{P}(L_t = j|L_0 = i) = \mathbb{P}\left(L_t^{(i)} = j\right)$ are given by

$$p_{ij}(t) = (-1)^{i+j}\frac{i!}{j!}\sum_{k=i}^{j} S(k,i)e^{-tk}s(j,k) = (-1)^j\sum_{k=1}^{i}(-1)^k\binom{i}{k}\binom{e^{-t}k}{j}, \qquad i,j\in\mathbb{N}. \tag{2.8}$$

Remarks. 1. For $i = 1$ it follows that $L_t = L_t^{(1)}$ has pgf $\mathbb{E}\left(z^{L_t}\right) = 1-(1-z)^\alpha = -\sum_{j=1}^\infty \binom{\alpha}{j}(-z)^j$ and Sibuya distribution

$$\mathbb{P}(L_t = j) = p_{1j}(t) = (-1)^{j+1}\binom{\alpha}{j} = \frac{\alpha\varGamma(j-\alpha)}{\varGamma(1-\alpha)\varGamma(j+1)}, \qquad j\in\mathbb{N}, \tag{2.9}$$

where $\alpha := e^{-t}$. Note that $\mathbb{P}(L_t = j) \sim \alpha/(\varGamma(1-\alpha)j^{\alpha+1})$ as $j\to\infty$ and that L_t has a Pareto like tail $\mathbb{P}(L_t\geq j) = \varGamma(j-\alpha)/(\varGamma(1-\alpha)\varGamma(j)) \sim 1/(\varGamma(1-\alpha)j^\alpha)$ as $j\to\infty$. Thus, $\mathbb{E}(L_t^q) = \sum_{j=1}^\infty j^q\mathbb{P}(L_t = j) < \infty$ if and only if $q<\alpha$. Particular reciprocal factorial moments of L_t are known explicitly. For example,

$$\mathbb{E}\left(\frac{1}{(L_t+1)(L_t+2)\cdots(L_t+k)}\right) = \frac{\alpha}{\varGamma(1-\alpha)}\sum_{j=1}^\infty\frac{\varGamma(j-\alpha)}{\varGamma(j+k+1)} = \frac{\alpha}{k!(\alpha+k)}, \quad k\in\mathbb{N}.$$

The Sibuya distribution (2.9) and similar distributions occur in [7, Eq. (2)], [19, p. 9], [20, p. 225] and [37, p. 70, Eq. (3.38)].

2. The pgf $f(s) := \sum_{k=2}^{\infty} p_k s^k = s + (1-s)\log(1-s)$ of the offspring distribution satisfies
$$\int_{(1-\varepsilon,1)} \frac{\lambda(\mathrm{d}s)}{f(s)-s} = \int_{(1-\varepsilon,1)} \frac{\lambda(\mathrm{d}s)}{(1-s)\log(1-s)} = \int_{(0,\varepsilon)} \frac{\lambda(\mathrm{d}x)}{x\log x} = -\infty$$
for all $\varepsilon \in (0,1)$, where λ denotes Lebesgue measure on $(0,1)$. This implies (Harris [16, p. 107]) that the fixation line $(L_t)_{t\geq 0}$ does not explode, in agreement (see [13]) with the fact that the Bolthausen–Sznitman coalescent stays infinite.

As a second application we study the probability $h(i,j) = \mathbb{P}\left(L_t^{(i)} = j \text{ for some } t \geq 0\right)$ that the fixation line hits state $j \in \mathbb{N}$ started from state $i \in \mathbb{N}$.

Corollary 2.4 (Hitting probabilities)**.** *The hitting probabilities $h(i,j)$ have horizontal generating function*
$$\sum_{j=i}^{\infty} h(i,j)z^{j-1} = \frac{z^i}{(1-z)(-\log(1-z))}, \qquad i \in \mathbb{N}, |z| < 1. \tag{2.10}$$
Moreover, for all $i \in \mathbb{N}$,
$$h(i,j) = \frac{1}{\log j} - \frac{\gamma}{\log^2 j} + O\left(\frac{1}{\log^3 j}\right), \qquad j \to \infty, \tag{2.11}$$
where $\gamma := -\Gamma'(1) \approx 0.577216$ denotes the Euler–Mascheroni constant. The hitting probability $h(i,j)$ can be computed via
$$h(i,j) = \sum_{k=1}^{j-i} \mathbb{P}\left(\eta_1 + \cdots + \eta_k = j-i\right), \qquad 1 \leq i < j, \tag{2.12}$$
where $\eta_1, \eta_2, \ldots$ are iid random variables with distribution $\mathbb{P}(\eta_1 = n) := u_n := 1/(n(n+1))$, $n \in \mathbb{N}$. The hitting probabilities can be also expressed in terms of the Stirling numbers $s(.,.)$ and $S(.,.)$ of the first and second kind as
$$h(i,j) = (-1)^{i+j}\frac{i!}{(j-1)!}\sum_{k=i}^{j} \frac{s(j,k)S(k,i)}{k} \tag{2.13}$$
$$= (-1)^{j-i}\frac{1}{(j-i)!}\sum_{k=1}^{j-i+1} \frac{s(j-i+1,k)}{k}, \qquad 1 \leq i \leq j. \tag{2.14}$$
Moreover, $h(i,j)$ has representations
$$h(i,j) = \frac{1}{(j-i)!}\int_0^1 \frac{\Gamma(j-i+x)}{\Gamma(x)}\,\mathrm{d}x = \frac{1}{(j-i)!}\sum_{k=0}^{j-i} \frac{|s(j-i,k)|}{k+1}, \qquad 1 \leq i \leq j. \tag{2.15}$$

Remark. Concrete values of the hitting probabilities $h(i,j)$ for $i = 1$ and $j \in \{1,\ldots,7\}$ are provided in the remark after the proof of Corollary 2.4.

We now turn to the block counting process $(N_t^{(n)})_{t\geq 0}$ of the Bolthausen–Sznitman n-coalescent. For $n \in \mathbb{N}$ and $i \in \{1,\ldots,n\}$ let $\tau_{ni} := \inf\{t > 0 : N_t^{(n)} \leq i\}$ denote the first time the block counting process $(N_t^{(n)})_{t\geq 0}$ jumps to a state smaller than or equal to i. Note that $\tau_n := \tau_{n1}$ is the absorption time of $(N_t^{(n)})_{t\geq 0}$.

Corollary 2.5 (Distribution function and asymptotics of τ_{ni})**.** *For all $n \in \mathbb{N}$ and $i \in \{1,\dots,n\}$, τ_{ni} has distribution function*

$$\mathbb{P}(\tau_{ni} \le t) = \sum_{j=1}^{i} (-1)^{n+j} \binom{i}{j} \binom{je^{-t}-1}{n-1}, \qquad t \in (0,\infty). \tag{2.16}$$

In particular, for every $i \in \mathbb{N}$, $\tau_{ni} - \log\log n \to \min(G_1,\dots,G_i)$ in distribution as $n \to \infty$, where $G_1, G_2, \dots$ are independent standard Gumbel distributed random variables.

Remark. Note that $\min(G_1,\dots,G_i)$ has distribution function $F_i(x) := 1-(1-F(x))^i$, where $F(x) := e^{-e^{-x}}$, $x \in \mathbb{R}$. For $i = 1$ we recover the well known convergence result (see Goldschmidt and Martin [15, Proposition 3.4], Freund and Möhle [12, Corollary 1.2] or Hénard [18, Theorem 3.9]) that the scaled absorption time $\tau_n - \log\log n$ is asymptotically standard Gumbel distributed.

The fact that the distribution function (2.16) of τ_{ni} is known explicitly can be further exploited. For example, the following Edgeworth expansion holds.

Corollary 2.6 (Edgeworth expansion)**.** *For every $i \in \mathbb{N}$ and $x \in \mathbb{R}$ the following Edgeworth expansion of order $K \in \mathbb{N}_0$ holds.*

$$\mathbb{P}(\tau_{ni} - \log\log n \le x) = \sum_{k=0}^{K} c_k d_{ki}(x) \frac{e^{-kx}}{\log^k n} + O\left(\frac{1}{\log^{K+1} n}\right), \qquad n \to \infty, \tag{2.17}$$

where $c_0, c_1, \dots$ are the coefficients in the series expansion $1/\Gamma(1-x) = \sum_{k=0}^\infty c_k x^k$, $|x| < 1$, and

$$d_{ki}(x) := \left(e^x \frac{\mathrm{d}}{\mathrm{d}x}\right)^k F_i(x) = \sum_{j=1}^{i} (F(x))^j (-1)^{j-1} \binom{ij}{j} j^k, \qquad k \in \mathbb{N}_0, i \in \mathbb{N}, x \in \mathbb{R}, \tag{2.18}$$

with F_i and F as defined in the previous remark. Alternatively, $d_{0i}(x) = F_i(x)$ and

$$d_{ki}(x) = \sum_{j=1}^{k} S(k,j)(-1)^{j-1}(i)_j (F(x))^j (1-F(x))^{i-j}, \qquad k, i \in \mathbb{N}, x \in \mathbb{R}, \tag{2.19}$$

where the $S(k,j)$ are the Stirling numbers of the second kind and $(i)_j := i(i-1)\cdots(i-j+1)$.

Remarks. 1. The coefficients c_k, $k \in \mathbb{N}_0$, are related to the moments of the Gumbel distribution (see Lemma 2.8). The concrete values c_k for $k \le 3$ are provided in the remark after the proof of Lemma 2.8.

2. For $K = 1$ Corollary 2.6 reads $\mathbb{P}(\tau_{ni} - \log\log n \le x) = F_i(x) - \gamma F_i'(x)/\log n + O(1/\log^2 n)$. In particular, for every $x \in \mathbb{R}$, $\mathbb{P}(\tau_{ni} - \log\log n \le x) - F_i(x) \sim -\gamma F_i'(x)/\log n$ as $n \to \infty$. Thus, the speed of the convergence of $\tau_{ni} - \log\log n$ to G_i is of order $1/\log n$.

2.4 Proofs

Proof of Theorem 2.1 (a). Let $Z^{(n)} := (X_t^{(n)}, t)_{t\geq 0}$ and $Z := (X_t, t)_{t\geq 0}$ denote the space-time processes of $X^{(n)} = (X_t^{(n)})_{t\geq 0}$ and $X = (X_t)_{t\geq 0}$ respectively. Note that $Z^{(n)}$ has state space $S_n := \{(j/n^{e^{-t}}, t) : j \in \{1,\ldots,n\}, t \geq 0\} = \bigcup_{t\geq 0}(E_{n,t} \times \{t\})$, where $E_{n,t} := \{j/n^{e^{-t}} : j \in \{1,\ldots,n\}\}$, and that Z has state space $S := E \times [0,\infty) = [0,\infty)^2$. The processes $Z^{(n)}$ and Z are time-homogeneous (see, for example, Revuz and Yor [39, p. 85, Exercise (1.10)]). In the following it is shown that $Z^{(n)}$ converges in $D_S[0,\infty)$ to Z as $n \to \infty$. Note that this convergence implies the desired convergence of $X^{(n)}$ in $D_E[0,\infty)$ to X as $n \to \infty$. Define $\pi_n : B(S) \to B(S_n)$ via $\pi_n f(x,s) := f(x,s)$ for all $f \in B(S)$ and $(x,s) \in S_n$. By Proposition 2.12 it suffices to verify that, for every $t \geq 0$ and $\lambda, \mu > 0$,

$$\lim_{n\to\infty} \sup_{s\geq 0} \sup_{x\in E_{n,s}} \left| T_t^{(n)} \pi_n f_{\lambda,\mu}(x,s) - \pi_n T_t f_{\lambda,\mu}(x,s) \right| = 0,$$

where $(T_t^{(n)})_{t\geq 0}$ and $(T_t)_{t\geq 0}$ denote the semigroups of the space-time processes $Z^{(n)}$ and Z respectively and the test functions $f_{\lambda,\mu} : S \to \mathbb{R}$ are defined via $f_{\lambda,\mu}(x,s) := e^{-\lambda x - \mu s}$ for all $(x,s) \in S$. Fix $t \geq 0$ and $\lambda, \mu > 0$. For convenience, define $\alpha := e^{-t}$ and $\beta := e^{-s}$. We have

$$\begin{aligned} T_t^{(n)} \pi_n f_{\lambda,\mu}(x,s) &= \mathbb{E}\left(f_{\lambda,\mu}(X_{s+t}^{(n)}, s+t) \middle| X_s^{(n)} = x \right) \\ &= (\alpha\beta)^\mu \mathbb{E}\left(\exp(-\lambda/n^{\alpha\beta} N_{s+t}^{(n)}) \middle| N_s^{(n)} = xn^\beta \right) \\ &= (\alpha\beta)^\mu \mathbb{E}\left(\exp(-\lambda/n^{\alpha\beta} N_t^{(xn^\beta)}) \right), \qquad (x,s) \in S_n, \end{aligned}$$

and

$$\begin{aligned} \pi_n T_t f_{\lambda,\mu}(x,s) &= \mathbb{E}\left(f_{\lambda,\mu}(X_{s+t}, s+t) | X_s = x \right) \\ &= (\alpha\beta)^\mu \mathbb{E}\left(\exp(-\lambda X_{s+t}) | X_s = x \right) \\ &= (\alpha\beta)^\mu \mathbb{E}\left(\exp(-\lambda x^\alpha X_t) \right), \qquad (x,s) \in S. \end{aligned}$$

Thus, we have to verify that

$$\lim_{n\to\infty} \sup_{s\geq 0} \sup_{x\in E_{n,s}} (\alpha\beta)^\mu \left| \mathbb{E}\left(\exp(-\lambda/n^{\alpha\beta} N_t^{(xn^\beta)}) \right) - \mathbb{E}\left(\exp(-\lambda x^\alpha X_t) \right) \right| = 0.$$

In the following it is shown that it suffices to verify the convergence of the one-dimensional distributions $X_t^{(k)} \to X_t$ in distribution as $k \to \infty$, $t \geq 0$. Since both expectations above are bounded between 0 and 1 and since $(\alpha\beta)^\mu = e^{-\mu(s+t)}$ tends to 0 as $s \to \infty$ it suffices to verify that, for every $s_0 > 0$,

$$\lim_{n\to\infty} \sup_{s\in[0,s_0]} \sup_{x\in E_{n,s}} \left| \mathbb{E}\left(\exp(-\lambda/n^{\alpha\beta} N_t^{(xn^\beta)}) \right) - \mathbb{E}\left(\exp(-\lambda x^\alpha X_t) \right) \right| = 0.$$

We will even verify that

$$\lim_{n\to\infty} \sup_{s\in[0,s_0]} \sup_{x>0} \left| \mathbb{E}\left(\exp(-\lambda/n^{\alpha\beta} N_t^{(\lfloor xn^\beta \rfloor)}) \right) - \mathbb{E}\left(\exp(-\lambda x^\alpha X_t) \right) \right| = 0.$$

The difference of the two expectations depends on n and s only via $n^\beta = n^{e^{-s}}$. Since the map $s \mapsto n^{e^{-s}}$ is non-increasing it follows that the convergence for fixed $s \in [0, s_0]$ is slower as s is larger. So the slowest convergence holds at the right end point $s = s_0$. Thus, it suffices to verify that, for every $s \geq 0$,

$$\lim_{n\to\infty} \sup_{x>0} \left| \mathbb{E}\left(\exp(-\lambda/n^{\alpha\beta} N_t^{(\lfloor xn^\beta \rfloor)})\right) - \mathbb{E}\left(\exp(-\lambda x^\alpha X_t)\right) \right| = 0.$$

The map $x \mapsto \mathbb{E}\left(\exp(-\lambda x^\alpha X_t)\right)$ is bounded, continuous, and non-increasing. Moreover, for every $n \in \mathbb{N}$ the map $x \mapsto \mathbb{E}\left(\exp(-\lambda/n^{\alpha\beta} N_t^{(\lfloor xn^\beta \rfloor)})\right)$ is bounded and non-increasing, since $N_t^{(1)} \leq N_t^{(2)} \leq \cdots$. Thus, by the theorem of Pólya [38, Satz I], it suffices to verify that, for every $s \geq 0$ and $x > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left(\exp(-\lambda/n^{\alpha\beta} N_t^{(\lfloor xn^\beta \rfloor)})\right) = \mathbb{E}\left(\exp(-\lambda x^\alpha X_t)\right).$$

Note that we have reduced the problem to verify the convergence uniformly for all $s \geq 0$ and $x \in E_{n,s}$ to the problem to verify the convergence pointwise for all points $(s, x) \in [0, \infty) \times (0, \infty)$.

Define $\tau := n^\beta$ and $k := \lfloor x\tau \rfloor$. Using this notation it remains to verify that

$$\lim_{\tau\to\infty} \mathbb{E}\left(\exp(-\lambda\tau^{-\alpha} N_t^{(\lfloor x\tau \rfloor)})\right) = \mathbb{E}\left(\exp(-\lambda x^\alpha X_t)\right)$$

or, equivalently, that

$$\lim_{k\to\infty} \mathbb{E}\left(\exp(-\lambda k^{-\alpha} N_t^{(k)})\right) = \mathbb{E}\left(\exp(-\lambda X_t)\right). \tag{2.20}$$

Thus, it suffices to verify the convergence of the one-dimensional distributions $X_t^{(k)} = k^{-\alpha} N_t^{(k)} \to X_t$ in distribution as $k \to \infty$. In the remaining part of the proof this convergence of the one-dimensional distributions is verified by the method of moments. We have

$$\mathbb{E}\left(\exp(-\lambda k^{-\alpha} N_t^{(k)})\right) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} \frac{\mathbb{E}\left((N_t^{(k)})^m\right)}{k^{\alpha m}}.$$

Note that the series on the right hand side is absolutely convergent, since $N_t^{(k)} \leq k$ and, hence, $\mathbb{E}\left((N_t^{(k)})^m\right) \leq k^m$. Applying the formula $z^m = \sum_{i=0}^m (-1)^{m-i} S(m,i)[z]_i$, $m \in \mathbb{N}_0$, $z > 0$, where $[z]_i := \Gamma(z+i)/\Gamma(z)$ for $z, i > 0$, it follows that

$$\frac{\mathbb{E}\left((N_t^{(k)})^m\right)}{k^{\alpha m}} = \sum_{i=0}^m (-1)^{m-i} S(m,i) \frac{\mathbb{E}\left([N_t^{(k)}]_i\right)}{k^{\alpha m}} = \sum_{i=0}^m (-1)^{m-i} S(m,i) \mathbb{E}\left(X_t^i\right) \frac{[k]_{\alpha i}}{k^{\alpha m}}$$

by Lemma 3.1 of [28]. From $[k]_{\alpha i} \sim k^{\alpha i}$ as $k \to \infty$ we conclude that only the summand $i = m$ yields asymptotically a non-zero contribution and it follows that

$$\lim_{k\to\infty} \frac{E(N_t^{(k)})^m}{k^{\alpha m}} = \mathbb{E}\left(X_t^m\right).$$

Moreover,

$$\frac{\mathbb{E}\left((N_t^{(k)})^m\right)}{k^{\alpha m}} \leq \frac{\mathbb{E}\left([N_t^{(k)}]_m\right)}{k^{\alpha m}} = \mathbb{E}\left(X_t^m\right) \frac{[k]_{\alpha m}}{k^{\alpha m}}.$$

It is readily checked that the map $k \mapsto [k]_{\alpha m}/k^{\alpha m}$ is non-increasing in k. Thus, we obtain the upper bound

$$\frac{\mathbb{E}\left((N_t^{(k)})^m\right)}{k^{\alpha m}} \le \mathbb{E}\left(X_t^m\right)\frac{[k_0]_{\alpha m}}{k_0^{\alpha m}} \quad \text{for all } k \ge k_0.$$

Note that

$$\frac{\lambda^m}{m!}\mathbb{E}\left(X_t^m\right)\frac{[k_0]_{\alpha m}}{k_0^{\alpha m}} = \frac{\lambda^m}{m!}\frac{m!}{\Gamma(1+\alpha m)}\frac{\Gamma(k_0+\alpha m)}{k_0^{\alpha m}\Gamma(k_0)} \sim \left(\frac{\lambda}{k_0^\alpha}\right)^m (\alpha m)^{k_0-1}$$

as $m \to \infty$. Thus, if we choose k_0 sufficiently large such that $\lambda/k_0^\alpha < 1$, for example $k_0 := (2\lambda)^{1/\alpha}$, then the dominating map $m \mapsto (\lambda^m/m!)\mathbb{E}\left(X_t^m\right)[k_0]_{\alpha m}/k_0^{\alpha m}$ is integrable with respect to the counting measure on $\mathbb{N}$. Thus, it is allowed to apply the dominated convergence theorem, which yields

$$\lim_{k\to\infty}\mathbb{E}\left(\exp(-\lambda k^{-\alpha}N_t^{(k)})\right) = \sum_{m=0}^\infty \frac{(-\lambda)^m}{m!}\mathbb{E}\left(X_t^m\right) = \mathbb{E}\left(\exp(-\lambda X_t)\right).$$

Thus, (2.20) is established. The proof is complete. □

Remark. The proof of Theorem 2.1 (a) shows (see (2.20)) that it suffices to verify the convergence of the one-dimensional distributions. The convergence of the one-dimensional distributions is then established by the method of moments. Alternatively, one may first prove Theorem 2.1 (b) and then, as already explained before Theorem 2.1, use the fact that the block counting process is Siegmund dual to the fixation line in order to verify the convergence of the one-dimensional distributions $X_t^{(k)} \to X_t$ in distribution as $k \to \infty$, $t \ge 0$.

Before we come to the proof of Theorem 2.1 (b), we provide a recursion for the Laplace transforms of the finite-dimensional distributions of Neveu's continuous-state branching process $Y = (Y_t)_{t\ge 0}$.

Lemma 2.7 (Recursion for the Laplace transforms of Y)**.** *Let $0 = t_0 \le t_1 < t_2 < \cdots$. For $k \in \mathbb{N}$ let $\psi_k : [0,\infty)^k \to [0,1]$, defined via $\psi_k(\lambda_1,\ldots,\lambda_k) := \mathbb{E}\left(e^{-\lambda_1 Y_{t_1}}\cdots e^{-\lambda_k Y_{t_k}}\right)$ for all $\lambda_1,\ldots,\lambda_k \ge 0$, denote the Laplace transform of $(Y_{t_1},\ldots,Y_{t_k})$. Then, ψ_k satisfies the recursion $\psi_1(\lambda_1) = e^{-\lambda_1^{\alpha_1}}$ for all $\lambda_1 \ge 0$ and*

$$\psi_k(\lambda_1,\ldots,\lambda_k) = \psi_{k-1}(\lambda_1,\ldots,\lambda_{k-2},\lambda_{k-1}+\lambda_k^{\alpha_k/\alpha_{k-1}}), \qquad k \in \mathbb{N}\setminus\{1\}, \lambda_1,\ldots,\lambda_k \ge 0,$$

where $\alpha_j := e^{-t_j}$, $1 \le j \le k$.

Proof of Lemma 2.7. Clearly, $\psi_1(\lambda_1) = \mathbb{E}\left(e^{-\lambda_1 Y_{t_1}}\right) = e^{-\lambda_1^{\alpha_1}}$ for all $\lambda_1 \ge 0$. Moreover, for all $\lambda_1,\ldots,\lambda_k \ge 0$,

$$\begin{aligned}\psi_k(\lambda_1,\ldots,\lambda_k) &= \mathbb{E}\left(e^{-\lambda_1 Y_{t_1}}\cdots e^{-\lambda_k Y_{t_k}}\right)\\ &= \mathbb{E}\left(\mathbb{E}\left(e^{-\lambda_1 Y_{t_1}}\cdots e^{-\lambda_k Y_{t_k}}\middle| Y_{t_1},\ldots,Y_{t_{k-1}}\right)\right)\\ &= \mathbb{E}\left(e^{-\lambda_1 Y_{t_1}}\cdots e^{-\lambda_{k-1} Y_{t_{k-1}}}\mathbb{E}\left(e^{-\lambda_k Y_{t_k}}\middle| Y_{t_{k-1}}\right)\right).\end{aligned}$$

Since $\mathbb{E}\left(e^{-\lambda_k Y_{t_k}}\middle| Y_{t_{k-1}}\right) = e^{-\lambda_k^{\alpha_k/\alpha_{k-1}}Y_{t_{k-1}}}$ almost surely it follows that

$$\begin{aligned}\psi_k(\lambda_1,\ldots,\lambda_k) &= \mathbb{E}\left(e^{\lambda_1 Y_{t_1}}\cdots e^{-\lambda_{k-2}Y_{t_{k-2}}}e^{-(\lambda_{k-1}+\lambda_k^{\alpha_k/\alpha_{k-1}})Y_{t_{k-1}}}\right)\\ &= \psi_{k-1}(\lambda_1,\ldots,\lambda_{k-2},\lambda_{k-1}+\lambda_k^{\alpha_k/\alpha_{k-1}}).\end{aligned}$$

□

We are now able to verify Theorem 2.1 (b).

Proof of Theorem 2.1 (b). The proof is divided into two parts. First the convergence of the finite-dimensional distributions is verified. Afterwards the convergence in $D_E[0,\infty)$ is considered. In fact Part 2 does not need results from Part 1, so one could omit Part 1. However, we think it is helpful for the reader to consider first the convergence of the finite-dimensional distributions.

Part 1. (Convergence of the finite-dimensional distributions) Fix $0 = t_0 \le t_1 < t_2 < \cdots$. For $k, n \in \mathbb{N}$ let $\psi_k^{(n)} : [0,\infty)^k \to [0,1]$ and $\psi_k : [0,\infty)^k \to [0,1]$ denote the Laplace transforms of $(Y_{t_1}^{(n)}, \ldots, Y_{t_k}^{(n)})$ and $(Y_{t_1}, \ldots, Y_{t_k})$ respectively. In the following the pointwise convergence $\psi_k^{(n)} \to \psi_k$ as $n \to \infty$ is verified by induction on $k \in \mathbb{N}$.

Obviously, $L_{t_1}^{(n)}$ has pgf $\mathbb{E}\left(z_1^{L_{t_1}^{(n)}}\right) = (1-(1-z_1)^{\alpha_1})^n$, $z_1 \in [0,1]$, where $\alpha_1 := e^{-t_1}$. Replacing z_1 by $e^{-\lambda_1/n^{1/\alpha_1}}$ with $\lambda_1 \ge 0$ it follows that

$$\psi_1^{(n)}(\lambda_1) = \mathbb{E}\left(e^{-\lambda_1 Y_{t_1}^{(n)}}\right) = (1-(1-e^{-\lambda_1/n^{1/\alpha_1}})^{\alpha_1})^n.$$

Clearly, $\psi_1(\lambda_1) = \mathbb{E}\left(e^{-\lambda_1 Y_{t_1}}\right) = e^{-\lambda_1^{\alpha_1}}$. Using the shorthand $x := \lambda_1/n^{1/\alpha_1}$ and the inequality $|a^n - b^n| \le n|a-b|$, $|a|, |b| \le 1$, it follows that

$$\begin{aligned}\left|\psi_1^{(n)}(\lambda_1) - \psi_1(\lambda_1)\right| &= \left|(1-(1-e^{-x})^{\alpha_1})^n - (e^{-x^{\alpha_1}})^n\right| \\ &\le n\left|1-(1-e^{-x})^{\alpha_1} - e^{-x^{\alpha_1}}\right| \\ &= n(e^{-x^{\alpha_1}} - 1 + (1-e^{-x})^{\alpha_1}),\end{aligned}$$

since $(1-e^{-x})^{\alpha_1} \ge 1-e^{-x^{\alpha_1}}$ by Lemma 2.9. From $1-e^{-x} \le x$, $x \in \mathbb{R}$, and $e^{-t}-1+t \le t^2/2$, $t \ge 0$, we conclude that

$$|\psi_1^{(n)}(\lambda_1) - \psi_1(\lambda_1)| \le n(e^{-x^{\alpha_1}} - 1 + x^{\alpha_1}) \le n\frac{(x^{\alpha_1})^2}{2} = \frac{\lambda_1^{2\alpha_1}}{2n} \longrightarrow 0, \qquad n \to \infty.$$

Thus, the pointwise convergence $\psi_1^{(n)} \to \psi_1$ as $n \to \infty$ is established.

Now fix $k \in \mathbb{N} \setminus \{1\}$. The induction step from $k-1$ to k works as follows. For convenience define $\alpha_j := e^{-t_j}$ for all $j \in \mathbb{N}$. For all $z_1, \ldots, z_k \in [0,1]$,

$$\begin{aligned}\mathbb{E}\left(z_1^{L_{t_1}^{(n)}} \cdots z_k^{L_{t_k}^{(n)}}\right) &= \mathbb{E}\left(\mathbb{E}\left(z_1^{L_{t_1}^{(n)}} \cdots z_k^{L_{t_k}^{(n)}} \,\middle|\, L_{t_1}^{(n)}, \ldots, L_{t_{k-1}}^{(n)}\right)\right) \\ &= \mathbb{E}\left(z_1^{L_{t_1}^{(n)}} \cdots z_{k-1}^{L_{t_{k-1}}^{(n)}} \mathbb{E}\left(z_k^{L_{t_k}^{(n)}} \,\middle|\, L_{t_{k-1}}^{(n)}\right)\right).\end{aligned}$$

Since $\mathbb{E}\left(z_k^{L_{t_k}^{(n)}} \,\middle|\, L_{t_{k-1}}^{(n)}\right) = (1-(1-z_k)^{\alpha_k/\alpha_{k-1}})^{L_{t_{k-1}}^{(n)}}$ almost surely it follows that

$$\mathbb{E}\left(z_1^{L_{t_1}^{(n)}} \cdots z_k^{L_{t_k}^{(n)}}\right) = \mathbb{E}\left(z_1^{L_{t_1}^{(n)}} \cdots z_{k-2}^{L_{t_{k-2}}^{(n)}} u_{k-1}^{L_{t_{k-1}}^{(n)}}\right),$$

where $u_{k-1} := z_{k-1}(1-(1-z_k)^{\alpha_k/\alpha_{k-1}})$. Replacing for each $j \in \{1,\ldots,k\}$ the variable z_j by $e^{-\lambda_j/n^{1/\alpha_j}}$ with $\lambda_j \geq 0$ it follows that

$$\begin{aligned}\psi_k^{(n)}(\lambda_1,\ldots,\lambda_k) &= \mathbb{E}\left(e^{-\lambda_1 Y_{t_1}^{(n)}}\cdots e^{-\lambda_k Y_{t_k}^{(n)}}\right)\\ &= \mathbb{E}\left(e^{-\lambda_1 Y_{t_1}^{(n)}}\cdots e^{-\lambda_{k-2} Y_{t_{k-2}}^{(n)}} e^{-\mu_{k-1}(n) Y_{t_{k-1}}^{(n)}}\right)\\ &= \psi_{k-1}^{(n)}(\lambda_1,\ldots,\lambda_{k-2},\mu_{k-1}(n)),\end{aligned} \tag{2.21}$$

where

$$\mu_{k-1}(n) := \lambda_{k-1} - n^{1/\alpha_{k-1}}\log(1-(1-e^{-\lambda_k/n^{1/\alpha_k}})^{\alpha_k/\alpha_{k-1}}).$$

A technical but straightforward calculation shows that $\mu_{k-1}(n) \to \lambda_{k-1} + \lambda_k^{\alpha_k/\alpha_{k-1}}$ as $n \to \infty$. Moreover, by induction, $\psi_{k-1}^{(n)}$ converges pointwise to ψ_{k-1} as $n \to \infty$. It is well known that the convergence $\psi_{k-1}^{(n)} \to \psi_{k-1}$ of Laplace transforms holds even uniformly on any compact subset of $[0,\infty)^{k-1}$. Taking these facts into account it follows from (2.21) that

$$\begin{aligned}\lim_{n\to\infty}\psi_k^{(n)}(\lambda_1,\ldots,\lambda_k) &= \lim_{n\to\infty}\psi_{k-1}^{(n)}(\lambda_1,\ldots,\lambda_{k-2},\mu_{k-1}(n))\\ &= \psi_{k-1}(\lambda_1,\ldots,\lambda_{k-2},\lambda_{k-1}+\lambda_k^{\alpha_k/\alpha_{k-1}}) = \psi_k(\lambda_1,\ldots,\lambda_k),\end{aligned}$$

where the last equality holds by Lemma 2.7. The induction is complete.

The pointwise convergence $\psi_k^{(n)} \to \psi_k$ of the Laplace transforms implies the convergence $(Y_{t_1}^{(n)},\ldots,Y_{t_k}^{(n)}) \to (Y_{t_1},\ldots,Y_{t_k})$ in distribution as $n \to \infty$.

Part 2. (Convergence in $D_E[0,\infty)$) We proceed essentially in the same way as in the proof of Theorem 2.1 (a), however the detail arguments differ slightly from those in the proof of part (a). Recall that $E := [0,\infty)$ is the state space of the limiting process Y. For $n \in \mathbb{N}$ and $t \geq 0$ define $E_{n,t} := \{j/n^{e^t} : j = n, n+1, \ldots\}$. Note that the processes $Y^{(n)}$ are time-inhomogeneous. In order to obtain time-homogeneous processes let $Z^{(n)} := (Y_t^{(n)}, t)_{t\geq 0}$ and $Z := (Y_t, t)_{t\geq 0}$ denote the space-time processes of $(Y_t^{(n)})_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ respectively. Note that $Z^{(n)}$ has state space $S_n := \{(j/n^{e^t}, t) : j = n, n+1, \ldots, t \geq 0\} = \bigcup_{t\geq 0}(E_{n,t} \times \{t\})$ and that Z has state space $S := E \times [0,\infty) = [0,\infty)^2$. According to Revuz and Yor [39, p. 85, Exercise (1.10)] the processes $Z^{(n)}$ and Z are time-homogeneous. Define $\pi_n : B(S) \to B(S_n)$ via $\pi_n g(y,s) := g(y,s)$ for all $g \in B(S)$ and $(y,s) \in S_n$. In the following it is shown that $Z^{(n)}$ converges in $D_S[0,\infty)$ to Z as $n \to \infty$. Note that this convergence implies the desired convergence of $Y^{(n)}$ in $D_E[0,\infty)$ to Y as $n \to \infty$. For $\lambda, \mu > 0$ define the test function $g_{\lambda,\mu} \in \widehat{C}(S)$ via $g_{\lambda,\mu}(y,s) := e^{-\lambda y - \mu s}$, $(y,s) \in S$. By Proposition 2.12 it suffices to verify that for every $t \geq 0$ and $\lambda, \mu > 0$,

$$\lim_{n\to\infty}\sup_{s\geq 0}\sup_{y\in E_{n,s}}\left|U_t^{(n)}\pi_n g_{\lambda,\mu}(y,s) - \pi_n U_t g_{\lambda,\mu}(y,s)\right| = 0, \tag{2.22}$$

where $U_t^{(n)} : B(S_n) \to B(S_n)$ is defined via $U_t^{(n)} g(y,s) := \mathbb{E}\left(g(Y_{s+t}^{(n)}, s+t)\middle| Y_s^{(n)} = y\right)$, $g \in B(S_n)$, $s \geq 0$, $y \in E_{n,s}$. Note that $(U_t^{(n)})_{t\geq 0}$ is the semigroup of $Z^{(n)}$.

Fix $t \ge 0$ and $\lambda, \mu > 0$. As before define $\alpha := e^{-t}$. For all $n \in \mathbb{N}$, $s \ge 0$ and $y \in E_{n,s}$ we have (with the notation $\beta := e^{-s}$)

$$\begin{aligned} U_t^{(n)} \pi_n g_{\lambda,\mu}(y,s) &= \mathbb{E}\left(\pi_n g_{\lambda,\mu}(Y_{s+t}^{(n)}, s+t) \middle| Y_s^{(n)} = y\right) \\ &= \mathbb{E}\left(\exp(-\lambda Y_{s+t}^{(n)} - \mu(s+t)) \middle| Y_s^{(n)} = y\right) \\ &= (\alpha\beta)^\mu \mathbb{E}\left(\exp(-\lambda/n^{1/(\alpha\beta)} L_{s+t}^{(n)}) \middle| L_s^{(n)} = yn^{1/\beta}\right) \\ &= (\alpha\beta)^\mu \mathbb{E}\left(\exp(-\lambda/n^{1/(\alpha\beta)} L_t^{(yn^{1/\beta})})\right) \end{aligned}$$

and

$$\begin{aligned} \pi_n U_t g_{\lambda,\mu}(y,s) = U_t g_{\lambda,\mu}(y,s) &= \mathbb{E}\left(\exp(-\lambda Y_{s+t} - \mu(s+t)) | Y_s = y\right) \\ &= (\alpha\beta)^\mu \mathbb{E}\left(\exp(-\lambda Y_{s+t}) | Y_s = y\right) = (\alpha\beta)^\mu \mathbb{E}\left(\exp(-\lambda y^{1/\alpha} Y_t)\right). \end{aligned}$$

Thus, one has to verify that

$$\lim_{n\to\infty} \sup_{s\ge 0} \sup_{y \in E_{n,s}} (\alpha\beta)^\mu \left| \mathbb{E}\left(\exp(-\lambda/n^{1/(\alpha\beta)} L_t^{(yn^{1/\beta})})\right) - \mathbb{E}\left(\exp(-\lambda y^{1/\alpha} Y_t)\right) \right| = 0. \qquad (2.23)$$

We will even verify that

$$\lim_{n\to\infty} \sup_{s\ge 0} \sup_{y>0} \left| \mathbb{E}\left(\exp(-\lambda/n^{1/(\alpha\beta)} L_t^{(\lfloor yn^{1/\beta} \rfloor)})\right) - \mathbb{E}\left(\exp(-\lambda y^{1/\alpha} Y_t)\right) \right| = 0.$$

The quantity inside the absolute values depends on n and s only via $n^{1/\beta} = n^{e^s}$. Since the map $s \mapsto n^{e^s}$ is non-decreasing it follows that the convergence for fixed $s \ge 0$ is slower as s is smaller. So the slowest convergence holds for $s = 0$ ($\Rightarrow \beta = 1$). Thus it suffices to verify that for every $t \ge 0$ and $\lambda > 0$

$$\lim_{n\to\infty} \sup_{y>0} \left| \mathbb{E}\left(\exp(-\lambda/n^{1/\alpha} L_t^{(\lfloor yn \rfloor)})\right) - \mathbb{E}\left(\exp(-\lambda y^{1/\alpha} Y_t)\right) \right| = 0.$$

The map $y \mapsto \mathbb{E}\left(\exp(-\lambda y^{1/\alpha} Y_t)\right)$ is bounded, continuous and non-increasing. Using that $L_t^{(1)} \le L_t^{(2)} \le \cdots$ it follows with the same argument as in the proof of Theorem 2.1 (a) (Pólya's theorem [38, Satz I]) that it suffices to verify the above convergence pointwise for every $y > 0$. Defining $k := \lfloor yn \rfloor$ it is readily seen that this is equivalent to the convergence of the one-dimensional distributions $Y_t^{(k)} = k^{-1/\alpha} L_t^{(k)} \to Y_t$ in distribution as $k \to \infty$, $t \ge 0$. But the convergence of the one-dimensional distributions holds as already shown before Theorem 2.1 (or by Part 1). The proof of part (b) of Theorem 2.1 is complete.

The following calculations even provide an explicit upper bound for the difference

$$\begin{aligned} d &:= \left| \mathbb{E}\left(\exp(-\lambda/n^{1/(\alpha\beta)} L_t^{(yn^{1/\beta})})\right) - \mathbb{E}\left(\exp(-\lambda y^{1/\alpha} Y_t)\right) \right| \\ &= \left| (1 - (1 - e^{-\lambda/n^{1/(\alpha\beta)}})^\alpha)^{yn^{1/\beta}} - e^{-y\lambda^\alpha} \right| \end{aligned}$$

occurring in (2.23) as well as an alternative proof of the convergence. Define $m := yn^{1/\beta} \in \{n, n+1, \ldots\}$ and $x := \lambda/n^{1/(\alpha\beta)}$. In the following it is assumed that $n \ge \lambda$ which implies that $x \le 1$. Using the inequality $|a^m - b^m| \le mr^{m-1}|a-b|$, $m \in \mathbb{N}$, where $r := \max(|a|,|b|)$, it follows that

$$\begin{aligned} d &= \left|(1-(1-e^{-\lambda/n^{1/(\alpha\beta)}})^\alpha)^{yn^{1/\beta}} - e^{-y\lambda^\alpha}\right| \\ &= \left|(1-(1-e^{-x})^\alpha)^m - (e^{-x^\alpha})^m\right| \\ &\le mr^{m-1}\left|1-(1-e^{-x})^\alpha - e^{-x^\alpha}\right|, \end{aligned}$$

where $r := \max(1-(1-e^{-x})^\alpha, e^{-x^\alpha}) = e^{-x^\alpha}$ by Lemma 2.9. Note that $r \in (0,1)$.

The map $z \mapsto zr^{z-1}$, $z \ge 0$ takes its maximum at the point $z = 1/(-\log r) = 1/x^\alpha$. Thus, $mr^{m-1} \le 1/x^\alpha r^{1/x^\alpha - 1} \le 1/x^\alpha$, since $r \le 1$ and $x \le 1$, i.e. $1/x^\alpha - 1 \ge 0$. Furthermore, $|1-(1-e^{-x})^\alpha - e^{-x^\alpha}| = e^{-x^\alpha} - 1 + (1-e^{-x})^\alpha \le e^{-x^\alpha} - 1 + x^\alpha \le (x^\alpha)^2/2$. Therefore, we obtain the upper bound

$$d \le \frac{1}{x^\alpha}\frac{(x^\alpha)^2}{2} = \frac{x^\alpha}{2} = \frac{\lambda^\alpha}{2n^{e^s}} \le \frac{\lambda^\alpha}{2n}.$$

Note that this upper bound does not depend on y and s. Thus, for all $t \ge 0$, $\lambda, \mu > 0$ and all $n \in \mathbb{N}$ with $n \ge \lambda$,

$$\begin{aligned} &\sup_{s\ge 0}\sup_{y \in E_{n,s}} \left|U_t^{(n)}\pi_n g_{\lambda,\mu}(y,s) - \pi_n U_t g_{\lambda,\mu}(y,s)\right| \\ &\quad = \sup_{s\ge 0}\sup_{y\in E_{n,s}} \underbrace{(\alpha\beta)^\mu}_{\le 1} \left|(1-(1-e^{-\lambda/n^{1/(\alpha\beta)}})^\alpha)^{yn^{1/\beta}} - e^{-y\lambda^\alpha}\right| \le \frac{\lambda^\alpha}{2n}. \end{aligned} \tag{2.24}$$

In particular, (2.22) holds for all $t \ge 0$ and all $\lambda, \mu > 0$. □

Remark. The presented proof of Theorem 2.1 (b) does not use any duality argument and shows that it suffices to verify the convergence of the one-dimensional distributions. The proof gives some more information than stated in Theorem 2.1 (b). Note that (2.24) provides the explicit upper bound $\lambda^\alpha/(2n)$, showing that the convergence of the semigroups is of order $1/n$, at least for test functions of the form $g_{\lambda,\mu}$, $\lambda, \mu > 0$.

To the best of the authors' knowledge the convergence result on the fixation line has no counterpart in the literature on branching processes and may hence trigger further research in the field of continuous-time branching processes (with infinite offspring mean).

We now turn to the proofs concerning the results in Section 2.3.

Proof of Theorem 2.2. Two proofs are provided. The first proof is self-contained and based on generating functions. The second proof uses duality and the spectral decomposition [29, Theorem 1.1] of the generator of the block counting process.

Proof 1 via generating functions. The proof is similar to that of Theorem 1.1 of [29]. Let $D = (d_{ij})_{i,j\in\mathbb{N}}$ be the diagonal matrix with entries $d_{ii} := -\gamma_i = \gamma_{ii}$, $i \in \mathbb{N}$. Furthermore, let $R = (r_{ij})_{i,j\in\mathbb{N}}$ be the upper right triangular matrix with entries defined for each $j \in \mathbb{N}$ recursively via $r_{jj} := 1$ and

$$r_{ij} := \frac{1}{\gamma_i - \gamma_j}\sum_{k=i+1}^{j} \gamma_{ik} r_{kj}, \qquad i \in \{j-1, j-2, \ldots, 1\}. \tag{2.25}$$

Since $\gamma_{ii} = -\gamma_i$, $i \in \mathbb{N}$, we conclude that $r_{ij}\gamma_{jj} = \sum_{k=i}^{j} \gamma_{ik} r_{kj}$. Thus, the entries of R are defined such that $RD = \Gamma R$. Define $L := R^{-1}$. Then, the spectral decomposition $\Gamma = RDL$ holds. Moreover, $DL = L\Gamma$ and, hence, $\gamma_{ii} l_{ij} = \sum_{k=i}^{j} l_{ik}\gamma_{kj}$, $i, j \in \mathbb{N}$. Since $\gamma_{ii} = -\gamma_i$, $i \in \mathbb{N}$, we obtain for each $i \in \mathbb{N}$ the recursion $l_{ii} = 1$ and

$$l_{ij} = \frac{1}{\gamma_j - \gamma_i} \sum_{k=i}^{j-1} l_{ik}\gamma_{kj}, \qquad j \in \{i+1, i+2, \ldots\}. \tag{2.26}$$

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc. For $i \in \mathbb{N}$ define the generating function $l_i : U \to \mathbb{C}$ via $l_i(z) := \sum_{j=i}^{\infty} l_{ij} z^j$, $z \in U$, and consider the modified function $f_i : U \to \mathbb{C}$ defined via $f_i(z) := \sum_{j=i}^{\infty} (j-i) l_{ij} z^j$, $z \in U$. We have

$$f_i(z) = \sum_{j=i}^{\infty} j l_{ij} z^j - i \sum_{j=i}^{\infty} l_{ij} z^j = z l_i'(z) - i l_i(z).$$

On the other hand, by the recursion (2.26), we obtain the factorization

$$\begin{aligned} f_i(z) &= \sum_{j=i+1}^{\infty} (j-i) l_{ij} z^j = \sum_{j=i+1}^{\infty} \sum_{k=i}^{j-1} l_{ik}\gamma_{kj} z^j \\ &= \sum_{k=i}^{\infty} l_{ik} \sum_{j=k+1}^{\infty} \gamma_{kj} z^j = \sum_{k=i}^{\infty} k l_{ik} z^k \sum_{j=k+1}^{\infty} \frac{z^{j-k}}{(j-k)(j-k+1)} \\ &= \sum_{k=i}^{\infty} k l_{ik} z^k \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} = z l_i'(z) a(z), \end{aligned}$$

where the auxiliary function $a : U \to \mathbb{C}$ is defined via $a(z) := \sum_{n=1}^{\infty} z^n/(n(n+1)) = 1 - (1-z)(-\log(1-z))/z$, $z \in U$. Thus, l_i satisfies the differential equation $z l_i'(z) a(z) = z l_i'(z) - i l_i(z)$ or, equivalently,

$$l_i'(z) = \frac{i l_i(z)}{(1-a(z))z} = \frac{i l_i(z)}{(1-z)(-\log(1-z))}.$$

The solution of this homogeneous differential equation with initial conditions $l_i(0) = \cdots = l_i^{(i-1)}(0) = 0$ and $l_i^{(i)}(0) = i!$ is $l_i(z) = (-\log(1-z))^i$, $i \in \mathbb{N}$, $z \in U$. Here $l_i^{(j)}$ denotes the jth derivative of l_i. For $f(z) = \sum_{j=0}^{\infty} a_j z^j$ let $[z^j]f(z) := a_j$ denote the coefficient in front of z^j in the series expansion of f. By [1, p. 824], $l_i(z) = (-\log(1-z))^i = i! \sum_{j=i}^{\infty} |s(j,i)| z^j/j!$ and, hence,

$$l_{ij} = [z^j] l_i(z) = \frac{i!}{j!} |s(j,i)| = \frac{i!}{j!} (-1)^{i+j} s(j,i),$$

which is the second formula in (2.6). Let us now turn to the inverse $R = L^{-1}$ of L. We have $L(z, z^2, \ldots)^\top = (l_1(z), l_2(z), \ldots)^\top$. Multiplying from the left with R it follows that $(z, z^2, \ldots)^\top = R(l_1(z), l_2(z), \ldots)^\top$. Thus, $z^i = \sum_{j=i}^{\infty} r_{ij} l_j(z) = \sum_{j=i}^{\infty} r_{ij}(-\log(1-z))^j$. Replacing z by $1 - e^{-z}$ leads to $(1-e^{-z})^i = \sum_{j=i}^{\infty} r_{ij} z^j =: r_i(z)$, $i \in \mathbb{N}$, $z \in U$. The calculations between Eq. (2.9) and Eq. (2.10) in [29] show that r_i has expansion

$$r_i(z) = (1-e^{-z})^i = \sum_{j=0}^{\infty} (-1)^{i+j} \frac{i!}{j!} S(j,i) z^j,$$

which yields the formula in (2.6) for the coefficient $r_{ij} = [z^j]r_i(z)$ in front of z^j. $\square$

Proof 2 via duality. The duality kernel H can be interpreted as a non-singular matrix $H = (h_{ij})_{i,j\in\mathbb{N}}$ with entries $h_{ij} = 1$ for $j \geq i$ and $h_{ij} = 0$ for $j < i$. The entries of its inverse $H^{-1} =: (g_{ij})_{i,j\in\mathbb{N}}$ are given by $g_{ij} = \delta_{i,j} - \delta_{i+1,j}$. It is known [29] that the generator matrix Q of the block counting process has spectral decomposition $Q = \tilde{R}\tilde{D}\tilde{L}$, where the matrices $\tilde{R} = (\tilde{r}_{ij})_{i,j\in\mathbb{N}}$, $\tilde{D} = (\tilde{d}_{ij})_{i,j\in\mathbb{N}}$ and $\tilde{L} = (\tilde{l}_{ij})_{i,j\in\mathbb{N}}$ are given by $\tilde{r}_{ij} = ((j-1)!/(i-1)!)|s(i,j)|$, $\tilde{d}_{ij} = (i-1)\delta_{i,j}$ and $\tilde{l}_{ij} = (-1)^{i+j}((j-1)!/(i-1)!)S(i,j)$ respectively. The entries of $D = (d_{ij})_{i,j\in\mathbb{N}}$ can be read off from the diagonal of Γ and are therefore given by $d_{ij} = i\delta_{i,j}$. Define the matrices $A = (a_{ij})_{i,j\in\mathbb{N}}$ and $B = (b_{ij})_{i,j\in\mathbb{N}}$ by $a_{ij} = \delta_{i+1,j}$ and $b_{ij} = \delta_{i-1,j}$. Clearly $\tilde{D} = BDA$. This together with the duality relation $H\Gamma^\top = QH$ and the spectral decomposition of the block counting process $Q = \tilde{R}\tilde{D}\tilde{L}$ yields

$$\Gamma^\top = H^{-1}\tilde{R}\tilde{D}\tilde{L}H = (-H^{-1}\tilde{R}B)D(-A\tilde{L}H).$$

Hence $\Gamma = RDL$ with $R := (-A\tilde{L}H)^\top$ and $L := (-H^{-1}\tilde{R}B)^\top$. It remains to calculate the entries of R and L. Using the recursion $S(i+1,j) = jS(i,j) + S(i,j-1)$ we obtain

$$\begin{aligned} r_{ji} = (-A\tilde{L}H)_{ij} &= -(\tilde{L}H)_{i+1,j} = -\sum_{k=1}^{j} \tilde{l}_{i+1,k} = \sum_{k=1}^{j} (-1)^{i+k}\frac{(k-1)!}{i!}S(i+1,k) \\ &= \sum_{k=1}^{j} (-1)^{i+k}\frac{k!}{i!}S(i,k) + \sum_{k=1}^{j} (-1)^{i+k}\frac{(k-1)!}{i!}S(i,k-1) \\ &= \sum_{k=1}^{j} (-1)^{i+k}\frac{k!}{i!}S(i,k) - \sum_{k=0}^{j-1} (-1)^{i+k}\frac{k!}{i!}S(i,k) = (-1)^{i+j}\frac{j!}{i!}S(i,j). \end{aligned}$$

Using the recursion $|s(i+1,j+1)| = |s(i,j)| + i|s(i,j+1)|$ we get

$$\begin{aligned} l_{ji} = (-H^{-1}\tilde{R}B)_{ij} &= -(H^{-1}\tilde{R})_{i,j+1} = \tilde{r}_{i+1,j+1} - \tilde{r}_{i,j+1} \\ &= \frac{j!}{i!}|s(i+1,j+1)| - \frac{j!}{(i-1)!}|s(i,j+1)| = \frac{j!}{i!}|s(i,j)|. \end{aligned}$$ $\square$

Proof of Corollary 2.3. By Theorem 2.2, $\Gamma = RDL$, where R and $L = R^{-1}$ have entries (2.6). Hence, the transition matrix $P(t) = e^{t\Gamma}$ has spectral decomposition $P(t) = e^{tRDL} = Re^{tD}L$. Thus, $p_{ij}(t) = \mathbb{P}(L_t = j | L_0 = i) = (Re^{tD}L)_{ij} = \sum_{k=i}^{j} r_{ik}e^{-\gamma_k t}l_{kj}$. The first formula in (2.8) for $p_{ij}(t)$ follows from $\gamma_k = k$ and from (2.6). Recall that $\alpha := e^{-t}$. Conditional on $L_0 = i$ the random variable L_t has probability generating function

$$\begin{aligned} \mathbb{E}\left(z^{L_t}\middle|L_0 = i\right) &= \sum_{j=i}^{\infty} z^j p_{ij}(t) = \sum_{j=i}^{\infty} z^j (-1)^{i+j}\frac{i!}{j!}\sum_{k=i}^{j} S(k,i)\alpha^k s(j,k) \\ &= (-1)^i i! \sum_{k=i}^{\infty} S(k,i)\alpha^k \sum_{j=k}^{\infty} \frac{(-z)^j}{j!}s(j,k) \\ &= (-1)^i i! \sum_{k=i}^{\infty} S(k,i)\alpha^k \frac{(\log(1-z))^k}{k!} \\ &= (-1)^i (e^{\alpha\log(1-z)} - 1)^i = (1-(1-z)^\alpha)^i, \quad |z| < 1, t \geq 0, i \in \mathbb{N}. \end{aligned}$$

Expansion leads to

$$\mathbb{E}\left(z^{L_t}\middle|L_0=i\right)=\sum_{k=0}^{i}\binom{i}{k}(-1)^k(1-z)^{\alpha k}=\sum_{k=0}^{i}\binom{i}{k}(-1)^k\sum_{j=0}^{\infty}\binom{\alpha k}{j}(-z)^j$$
$$=\sum_{j=0}^{\infty}(-z)^j\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{\alpha k}{j}.$$

The coefficient in front of z^j in this expansion yields the second formula for $p_{ij}(t)$. □

Proof of Corollary 2.4. The hitting probability $h(i,j)$ is related to the entry $g(i,j) := \int_0^\infty \mathbb{P}\left(L_t^{(i)}=j\right)\,\mathrm{d}t$ of the Green matrix via $h(i,j)=\gamma_j g(i,j)=jg(i,j)$ (see, for example, Norris [33, p. 146]). Thus, for all $i\in\mathbb{N}$ and $|z|<1$,

$$h_i(z):=\sum_{j=i}^{\infty}h(i,j)z^{j-1}=\int_0^\infty\sum_{j=i}^{\infty}j\mathbb{P}\left(L_t^{(i)}=j\right)z^{j-1}\,\mathrm{d}t=\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}z}\sum_{j=i}^{\infty}\mathbb{P}\left(L_t^{(i)}=j\right)z^j\,\mathrm{d}t.$$

Plugging in the formula (2.7) for the pgf of $L_t^{(i)}$ it follows that

$$h_i(z)=\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}z}(1-(1-z)^{e^{-t}})^i\,\mathrm{d}t=\int_0^\infty i(1-(1-z)^{e^{-t}})^{i-1}e^{-t}(1-z)^{e^{-t}-1}\,\mathrm{d}t.$$

Substituting $x:=e^{-t}$ and noting that $\mathrm{d}t/\mathrm{d}x=-1/x$ leads to $h_i(z)=(1-z)^{-1}\int_0^1 i(1-(1-z)^x)^{i-1}(1-z)^x\,\mathrm{d}x$. Substituting further $y:=1-(1-z)^x$ and noting that $\mathrm{d}x/\mathrm{d}y = 1/((1-y)(-\log(1-z)))$ we obtain

$$h_i(z)=\frac{1}{(1-z)(-\log(1-z))}\int_0^z iy^{i-1}\,\mathrm{d}y=\frac{z^i}{(1-z)(-\log(1-z))},\qquad i\in\mathbb{N},|z|<1.$$

In particular, $h(i,j)=h(1,j-i+1)$. The asymptotic expansion (2.11) follows from Panholzer [34, Eq. (19)]. Formula (2.12) is obtained as follows. Let $(J_k)_{k\in\mathbb{N}_0}$ denote the jump chain of the fixation line $(L_t)_{t\geq 0}$. Given this chain is in state i it jumps to state $i+j$ with probability $\gamma_{i,i+j}/\gamma_i=1/(j(j+1))=:u_j$, $j\in\mathbb{N}$. From this property it is easily seen that the jump chain has independent increments, i.e. $J_0=1$, $J_1=1+\eta_1$, $J_2=1+\eta_1+\eta_2$ and so on, where $\eta_1,\eta_2,\dots$ are iid random variables with distribution $\mathbb{P}(\eta_1=j)=u_j$, $j\in\mathbb{N}$. For $1\leq i<j$ it follows that $h(i,j)=h(1,j-i+1)=\sum_{k=1}^{j-i}\mathbb{P}(J_k=j-i+1)=\sum_{k=1}^{j-i}\mathbb{P}(\eta_1+\cdots+\eta_k=j-i)$. Formula (2.13) for $h(i,j)$ follows from $h(i,j)=jg(i,j)=j\int_0^\infty\mathbb{P}\left(L_t^{(i)}=j\right)\,\mathrm{d}t$ and

$$\int_0^\infty\mathbb{P}\left(L_t^{(i)}=j\right)\,\mathrm{d}t=\int_0^\infty(-1)^{i+j}\frac{i!}{j!}\sum_{k=i}^{j}S(k,i)e^{-tk}s(j,k)\,\mathrm{d}t$$
$$=(-1)^{i+j}\frac{i!}{j!}\sum_{k=i}^{j}\frac{S(k,i)s(j,k)}{k}.$$

Eq. (2.14) follows from $h(i,j)=h(1,j-i+1)$ and $S(k,1)=1$ for all $k\in\mathbb{N}$. Moreover, for $i=1$ we have $\mathbb{P}(L_t=j)=\alpha\Gamma(j-\alpha)/(j!\Gamma(1-\alpha))$ with $\alpha:=e^{-t}$. Thus,

$$g(1,j)=\int_0^\infty\mathbb{P}(L_t=j)\,\mathrm{d}t=\frac{1}{j!}\int_0^1\frac{\Gamma(j-\alpha)}{\Gamma(1-\alpha)}\,\mathrm{d}\alpha=\frac{1}{j!}\int_0^1\frac{\Gamma(j-1+x)}{\Gamma(x)}\,\mathrm{d}x$$

and, hence, we obtain the integral representation

$$h(i,j) = h(1,j-i+1) = (j-i+1)g(1,j-i+1) = \frac{1}{(j-i)!}\int_0^1 \frac{\Gamma(j-i+x)}{\Gamma(x)}\,\mathrm{d}x, \quad 1 \le i \le j.$$

The last formula for $h(i,j)$ in (2.15) follows from $\Gamma(n+x)/\Gamma(x) = \sum_{k=0}^n |s(n,k)|x^k$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}$. The proof of Corollary 2.4 is complete. □

Remark. Note that $\mathbb{P}(\eta_1 + \cdots + \eta_k = j-i) = \sum_{i_1,\ldots,i_k} u_{i_1}\cdots u_{i_k}$, where the sum extends over all $i_1,\ldots,i_k \in \mathbb{N}$ satisfying $i_1 + \cdots + i_k = j-i$. Hence, concrete values of the hitting probabilities are $h(1,1) = 1$, $h(1,2) = \mathbb{P}(\eta_1 = 1) = u_1 = 1/2$, $h(1,3) = \mathbb{P}(\eta_1 = 2) + \mathbb{P}(\eta_1 + \eta_2 = 2) = u_2 + u_1^2 = 1/6 + 1/4 = 5/12 \approx 0.41667$, $h(1,4) = \mathbb{P}(\eta_1 = 3) + \mathbb{P}(\eta_1 + \eta_2 = 3) + \mathbb{P}(\eta_1 + \eta_2 + \eta_3 = 3) = u_3 + 2u_1u_2 + u_1^3 = 1/12 + 1/6 + 1/8 = 3/8 = 0.375$, $h(1,5) = u_4 + (2u_1u_3 + u_2^2) + 3u_1^2u_2 = 1/20 + 1/9 + 1/8 = 251/720 \approx 0.34861$, $h(1,6) = 95/288 \approx 0.32986$, $h(1,7) = 19087/60480 \approx 0.31559$ and so on.

Proof of Corollary 2.5. By the definition of τ_{ni} and the duality of $(N_t)_{t\ge0}$ and $(L_t)_{t\ge0}$ we have $\mathbb{P}(\tau_{ni} \le t) = \mathbb{P}\left(N_t^{(n)} \le i\right) = \mathbb{P}\left(L_t^{(i)} \ge n\right) = \sum_{j=n}^\infty p_{ij}(t)$. Using the second formula for $p_{ij}(t)$ in (2.8) yields

$$\begin{aligned}
\mathbb{P}(\tau_{ni} \le t) &= \sum_{j=n}^\infty (-1)^j \sum_{k=1}^i (-1)^k \binom{i}{k}\binom{e^{-t}k}{j} \\
&= \sum_{k=1}^i (-1)^k \binom{i}{k} \sum_{j=n}^\infty (-1)^j \binom{e^{-t}k}{j} \\
&= \sum_{k=1}^i (-1)^k \binom{i}{k} (-1)^n \binom{e^{-t}k-1}{n-1},
\end{aligned}$$

where the last equality holds since $\sum_{j=n}^\infty (-1)^j\binom{z}{j} = (-1)^n\binom{z-1}{n-1}$ for all $n \in \mathbb{N}$ and all $z \in \mathbb{R}$.

Fix $x \in \mathbb{R}$ and define $F(x) := e^{-e^{-x}}$ for convenience. Assume that n is sufficiently large such that $x + \log\log n > 0$. Choosing $t := x + \log\log n$ and noting that for all sufficiently large n

$$\begin{aligned}
(-1)^{n-1}\binom{e^{-t}k-1}{n-1} &= \frac{\Gamma(n-ke^{-x}/\log n)}{\Gamma(n)\Gamma(1-ke^{-x}/\log n)} \\
&\sim \frac{\Gamma(n-ke^{-x}/\log n)}{\Gamma(n)} \;\to\; e^{-ke^{-x}} = (F(x))^k
\end{aligned}$$

as $n \to \infty$ by an application of Stirling's formula $\Gamma(n+1) \sim (n/e)^n\sqrt{2\pi n}$ as $n \to \infty$, it follows that

$$\begin{aligned}
\mathbb{P}(\tau_{ni} - \log\log n \le x) &= \mathbb{P}(\tau_{ni} \le x + \log\log n) \\
&\to \sum_{k=1}^i (-1)^{k-1}\binom{i}{k}(F(x))^k = 1 - (1-F(x))^i, \qquad n \to \infty.
\end{aligned}$$

It remains to note that $x \mapsto 1-(1-F(x))^i$, $x \in \mathbb{R}$, is the distribution function of the minimum of i independent standard Gumbel distributed random variables. □

Before we will prove Corollary 2.6 we provide the Taylor expansion of the map $x \mapsto 1/\Gamma(1-x)$.

Lemma 2.8. *The map* $x \mapsto 1/\Gamma(1-x)$ *has Taylor expansion* $1/\Gamma(1-x) = \sum_{k=0}^\infty c_k x^k$, $|x| < 1$, *where the coefficients* $c_0, c_1, \dots$ *are related to the moments* $m_k = (-1)^k \Gamma^{(k)}(1)$, $k \in \mathbb{N}_0$, *of the Gumbel distribution via* $c_0 = m_0 = 1$ *and*

$$c_k = \sum_{j=1}^{k} (-1)^j \sum_{\substack{k_1,\dots,k_j \in \mathbb{N} \\ k_1+\dots+k_j = k}} \frac{m_{k_1} \cdots m_{k_j}}{k_1! \cdots k_j!}, \qquad k \in \mathbb{N}. \tag{2.27}$$

Alternatively,

$$c_k = \frac{(-1)^k}{k!} \sum_{l=1}^{k} (-1)^l \binom{k+1}{l+1} (\Gamma^l)^{(k)}(1) \qquad k \in \mathbb{N}, \tag{2.28}$$

where $(\Gamma^l)^{(k)}$ *denotes the kth derivative of the lth power of* Γ.

Remark. Concrete values are $c_1 = -m_1 = -\gamma \approx -0.577216$, $c_2 = m_1^2 - m_2/2 = \gamma^2 - (\gamma^2 + \zeta(2))/2 = \gamma^2/2 - \pi^2/12 \approx -0.655878$, $c_3 = -m_3/6 + m_1 m_2 - m_1^3 = \gamma\zeta(2)/2 - \zeta(3)/3 - \gamma^3/6 = \pi^2\gamma/12 - \zeta(3)/3 - \gamma^3/6 \approx 0.042003$ and so on.

Proof. A Gumbel distributed random variable τ has moment generating function $\mathbb{E}\left(e^{x\tau}\right) = \Gamma(1-x)$, $x < 1$. Thus, the map $x \mapsto \Gamma(1-x)$ has Taylor expansion $\Gamma(1-x) = \sum_{k=0}^\infty a_k x^k$, $|x| < 1$,where $a_k := m_k/k!$ and $m_k = \mathbb{E}\left(\tau^k\right)$, $k \in \mathbb{N}_0$, are the moments of the Gumbel distribution. For the reciprocal map $1/\Gamma(1-x)$ it follows that

$$\begin{aligned}
\frac{1}{\Gamma(1-x)} &= \sum_{j=0}^\infty (1 - \Gamma(1-x))^j = \sum_{j=0}^\infty \left(\sum_{k=1}^\infty -a_k x^k \right)^j \\
&= 1 + \sum_{j=1}^\infty \sum_{k_1,\dots,k_j \in \mathbb{N}} (-a_{k_1}) \cdots (-a_{k_j}) x^{k_1+\dots+k_j} \\
&= 1 + \sum_{j=1}^\infty (-1)^j \sum_{k=1}^\infty x^k \sum_{\substack{k_1,\dots,k_j \in \mathbb{N} \\ k_1+\dots+k_j = k}} a_{k_1} \cdots a_{k_j} = \sum_{k=0}^\infty c_k x^k
\end{aligned}$$

with $c_0 := 1$ and c_k, $k \in \mathbb{N}$, as given in (2.27), since $a_k = m_k/k!$, $k \in \mathbb{N}_0$. Since $m_k = (-1)^k \Gamma^{(k)}(1)$, (2.27) can be rewritten as

$$\begin{aligned}
c_k &= \sum_{j=1}^{k} (-1)^{j+k} \sum_{\substack{k_1,\dots,k_j \in \mathbb{N} \\ k_1+\dots+k_j = k}} \frac{\Gamma^{(k_1)}(1) \cdots \Gamma^{(k_j)}(1)}{k_1! \cdots k_j!} \\
&= \sum_{j=1}^{k} \frac{(-1)^{j+k}}{k!} \sum_{l=1}^{j} (-1)^{j-l} \binom{j}{k} (\Gamma^l)^{(k)}(1), \qquad k \in \mathbb{N},
\end{aligned}$$

where the last equality holds by Lemma 1 in the appendix of [27]. Interchanging the sums and noting that $\sum_{j=l}^k \binom{j}{l} = \binom{k+1}{l+1}$ yields (2.28). □

Proof of Corollary 2.6. Fix $x \in \mathbb{R}$ and define $F(x) := e^{-e^{-x}}$. By Corollary 2.5, for all sufficiently large n,

$$\mathbb{P}\left(\tau_{ni} - \log\log n \le x\right) = \sum_{j=1}^{i} (-1)^{j-1} \binom{i}{j} \frac{\Gamma(n - je^{-x}/\log n)}{\Gamma(n)\Gamma(1 - je^{-x}/\log n)}. \tag{2.29}$$

For every $c \in \mathbb{R}$ it is easily checked that $\Gamma(n + c/\log n)/\Gamma(n) = e^c + O(1/(n\log n))$ as $n \to \infty$. For $c = -je^{-x}$ we obtain

$$\frac{\Gamma(n - je^{-x}/\log n)}{\Gamma(n)} = (F(x))^j + O\bigg(\frac{1}{n\log n}\bigg). \tag{2.30}$$

Moreover (see Lemma 2.8), from $1/\Gamma(1-x) = \sum_{k=0}^{\infty} c_k x^k$ we conclude that, for all $K \in \mathbb{N}_0$,

$$\frac{1}{\Gamma(1 - je^{-x}/\log n)} = \sum_{k=0}^{K} c_k \bigg(\frac{je^{-x}}{\log n}\bigg)^k + O\bigg(\frac{1}{(\log n)^{K+1}}\bigg). \tag{2.31}$$

Multiplying (2.30) with (2.31) yields

$$\frac{\Gamma(n - je^{-x}/\log n)}{\Gamma(n)\Gamma(1 - je^{-x}/\log n)} = (F(x))^j \sum_{k=0}^{K} c_k \bigg(\frac{je^{-x}}{\log n}\bigg)^k + O\bigg(\frac{1}{(\log n)^{K+1}}\bigg).$$

Plugging this expansion into (2.29) and exchanging the sums yields

$$\mathbb{P}\left(\tau_{ni} - \log\log n \le x\right) = \sum_{k=0}^{K} c_k \bigg(\frac{e^{-x}}{\log n}\bigg)^k \sum_{j=1}^{i} (F(x))^j (-1)^{j-1} \binom{i}{j} j^k + O\bigg(\frac{1}{(\log n)^{K+1}}\bigg),$$

which is the desired Edgeworth expansion with coefficients $d_{ki}(x)$ as defined in (2.18). It remains to verify the alternative representation (2.19) of the coefficients $d_{ki}(x)$. It is readily checked by induction on $k \in \mathbb{N}_0$ that $(t\frac{\mathrm{d}}{\mathrm{d}t})^k f(t) = \sum_{j=0}^{k} S(k,j) t^j f^{(j)}(t)$ for every k-times differentiable function $f : \mathbb{R} \to \mathbb{R}$, where the $S(k,j)$ denote the Stirling numbers of the second kind. Applying this formula to $f(t) := 1 - (1-t)^i$ with $i \in \mathbb{N}$ it follows for all $k \in \mathbb{N}_0$ and $t \in \mathbb{R}$ that

$$\begin{aligned}
\sum_{j=1}^{i} (-1)^{j-1} \binom{i}{j} j^k t^j &= \bigg(t\frac{\mathrm{d}}{\mathrm{d}t}\bigg)^k \sum_{j=1}^{i} (-1)^{j-1} \binom{i}{j} t^j = \bigg(t\frac{\mathrm{d}}{\mathrm{d}t}\bigg)^k (1 - (1-t)^i) \\
&= \sum_{j=0}^{k} S(k,j) t^j \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\bigg)^j (1 - (1-t)^i) \\
&= S(k,0)(1 - (1-t)^i) + \sum_{j=1}^{k} S(k,j) t^j (-1)^{j-1} (i)_j (1-t)^{i-j},
\end{aligned}$$

where $(i)_j := i(i-1)\cdots(i-j+1)$. Replacing t by $F(x)$ and noting that $S(k,0) = 0$ for $k \in \mathbb{N}$ shows that (2.18) coincides for $k \in \mathbb{N}$ with (2.19). □

2.5 Appendix

Lemma 2.9. *For all $x \geq 0$ and all $\alpha \in [0,1]$ we have $(1-e^{-x})^\alpha \geq 1-e^{-x^\alpha}$.*

Proof. Fix $\alpha \in [0,1]$. If $x \geq 1$ then $x^\alpha \leq x$ and, hence, $(1-e^{-x})^\alpha \geq 1-e^{-x} \geq 1-e^{-x^\alpha}$. Assume now that $x \in [0,1]$. Then $x^\alpha \geq x$. The function $f(x) := (1-e^{-x})^\alpha - 1 + e^{-x^\alpha}$ satisfies $f(0) = 0$ and has derivative $f'(x) = \alpha e^{-x}(1-e^{-x})^{\alpha-1} - \alpha x^{\alpha-1}e^{-x^\alpha}$, which is nonnegative on $[0,1]$, since $e^{-x} \geq e^{-x^\alpha}$ and $(1-e^{-x})^{\alpha-1} \geq x^{\alpha-1}$ for $x \in [0,1]$. From $f(0) = 0$ and $f'(x) \geq 0$ for $x \in [0,1]$ it follows that $f(x) \geq 0$ for $x \in [0,1]$, which is the desired inequality. □

Lemma 2.10 (Spectral decomposition of Γ for the Kingman coalescent)**.** *The generator $\Gamma = (\gamma_{ij})_{i,j\in\mathbb{N}}$ of the fixation line $(L_t)_{t\geq 0}$ of the Kingman coalescent has spectral decomposition $\Gamma = RDL$, where $D = (d_{ij})_{i,j\in\mathbb{N}}$ is the diagonal matrix with entries $d_{ij} = -i(i+1)/2$ for $i = j$ and $d_{ij} = 0$ for $i \neq j$, and $R = (r_{ij})_{i,j\in\mathbb{N}}$ and $L = (l_{ij})_{i,j\in\mathbb{N}}$ are upper right triangular matrices with entries*

$$r_{ij} = (-1)^{j-i}\frac{j!\,(j-1)!\,(i+j)!}{(j-i)!\,i!\,(i-1)!\,(2j)!}, \qquad i,j \in \mathbb{N}, i \leq j, \tag{2.32}$$

and

$$l_{ij} = \frac{j!\,(j-1)!\,(2i+1)!}{i!\,(i-1)!\,(j-i)!\,(i+j+1)!}, \qquad i,j \in \mathbb{N}, i \leq j. \tag{2.33}$$

Remark. Note that $l_i(z) := \sum_{j=i}^\infty l_{ij}z^{j+1}$ satisfies the differential equation $z^2(1-z)l_i''(z) = i(i+1)l_i(z)$, $i \in \mathbb{N}$, $|z| < 1$.

Proof. For a pure birth process the recursion (2.25) reduces to $r_{ij} = \gamma_i/(\gamma_i - \gamma_j)r_{i+1,j}$, $i \in \{j-1, j-2, \ldots, 1\}$, with solution $r_{ij} = \prod_{k=i}^{j-1}\gamma_k/(\gamma_k - \gamma_j)$, $i \leq j$. Thus, for the Kingman coalescent, for all $i,j \in \mathbb{N}$ with $i \leq j$,

$$r_{ij} = \prod_{k=i}^{j-1}\frac{k(k+1)}{k(k+1) - j(j+1)} = \prod_{k=i}^{j-1}\frac{k(k+1)}{(k-j)(k+j+1)} = (-1)^{j-i}\frac{j!\,(j-1)!\,(i+j)!}{(j-i)!\,i!\,(i-1)!\,(2j)!}.$$

Similarly, the recursion (2.26) reduces to $l_{ij} = \gamma_{j-1}/(\gamma_j - \gamma_i)l_{i,j-1}$, $j \in \{i+1, i+2, \ldots\}$, with solution $l_{ij} = \prod_{k=i+1}^{j}\gamma_{k-1}/(\gamma_k - \gamma_i)$, $i \leq j$. Thus, for the Kingman coalescent, for all $i,j \in \mathbb{N}$ with $i \leq j$,

$$l_{ij} = \prod_{k=i+1}^{j}\frac{k(k-1)}{k(k+1) - i(i+1)} = \prod_{k=i+1}^{j}\frac{k(k-1)}{(k-i)(k+i+1)} = \frac{j!\,(j-1)!\,(2i+1)!}{i!\,(i-1)!\,(j-i)!\,(i+j+1)!}.$$

□

Let E be locally compact, i.e. every point $x \in E$ has a compact neighborhood. A function $f : E \to \mathbb{R}$ vanishes at infinity, if for every $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $|f(x)| < \varepsilon$ for all $x \in E \setminus K$. In other words $\{x \in E : |f(x)| \geq \varepsilon\}$ is compact. In the following $\widehat{C}(E)$ denotes the set of all real-valued continuous functions on E vanishing at infinity.

Lemma 2.11. *Let $d \in \mathbb{N}$. The set D of all functions $g : [0,\infty)^d \to \mathbb{R}$ of the form $g(y) = \sum_{i_1,\ldots,i_d=1}^m a_{i_1,\ldots,i_d}e^{-(i_1y_1+\cdots+i_dy_d)}$ with $m \in \mathbb{N}$ and $a_{i_1,\ldots,i_d} \in \mathbb{R}$ is dense in $\widehat{C}([0,\infty)^d)$.*

Proof. Let $g \in \widehat{C}([0,\infty)^d)$. Define $f : [0,1]^d \to \mathbb{R}$ via $f(x) := g(-\log x_1, \ldots, -\log x_d)$ for $x \in (0,1]^d$ and $f(x) := 0$ if $x_j = 0$ for some $j \in \{1, \ldots, d\}$. Since g is continuous and vanishes at infinity it follows that f is continuous. For $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_d) \in [0,1]^d$ let

$$p_n(x) := \sum_{k_1,\ldots,k_d=1}^{n} f\Big(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\Big) \prod_{j=1}^{d} \binom{n}{k_j} x_j^{k_j}(1-x_j)^{n-k_j}$$

denote the nth multivariate Bernstein polynomial of f. Note that the sum runs only over $k = (k_1, \ldots, k_d) \in \{1, \ldots, n\}^d$ (not as usual over $k \in \{0, \ldots, n\}^d$) since $f(x) = 0$ if $x_j = 0$ for some $j \in \{1, \ldots, d\}$. By a d-dimensional version of Bernstein's approximation theorem (see, for example, [10, Theorem 8]), $p_n \to f$ as $n \to \infty$ uniformly on $[0,1]^d$. Replacing x_j by e^{-y_j} it follows that $g_n \to g$ as $n \to \infty$ uniformly on $[0,\infty)^d$, where $g_n(y) := p_n(e^{-y_1}, \ldots, e^{-y_d})$. It remains to note that $g_n \in D$. □

Proposition 2.12 (Convergence of Markov processes)**.** *Let $d \in \mathbb{N}$, $E := [0,\infty)^d$ and $X = (X_t)_{t\geq 0}$ be an E-valued time-homogeneous Markov process. Furthermore, for every $n \in \mathbb{N}$ let $X^{(n)} = (X_t^{(n)})_{t\geq 0}$ be an E_n-valued time-homogeneous Markov process with state space $E_n \subseteq E$. Let $(T_t)_{t\geq 0}$ and $(T_t^{(n)})_{t\geq 0}$ denote the corresponding semigroups. Define $\pi_n : B(E) \to B(E_n)$ via $\pi_n f(x) := f(x)$ for all $f \in B(E)$ and $x \in E_n$. If, for every $t \geq 0$ and $\lambda \in \mathbb{N}^d$,*

$$\lim_{n\to\infty} \Big\| T_t^{(n)} \pi_n f_\lambda - \pi_n T_t f_\lambda \Big\| := \lim_{n\to\infty} \sup_{x\in E_n} \Big| T_t^{(n)} \pi_n f_\lambda(x) - \pi_n T_t f_\lambda(x) \Big| = 0,$$

where $f_\lambda(x) := e^{-\langle \lambda, x\rangle} := e^{-(\lambda_1 x_1 + \cdots + \lambda_d x_d)}$ for all $\lambda \in \mathbb{N}^d$ and $x \in E$, then $X^{(n)}$ converges in $D_E[0,\infty)$ to X as $n \to \infty$.

Proof. By assumption, $\lim_{n\to\infty} \|T_t^{(n)}\pi_n f - \pi_n T_t f\| = 0$ for all $f \in D$, where $D := \{f : E \to \mathbb{R} \,:\, f(x) = \sum_{i=1}^m a_i e^{-\langle\lambda,x\rangle}, m \in \mathbb{N}, \lambda \in \mathbb{N}^d, a_i \in \mathbb{R}\}$ Let $f \in \widehat{C}(E)$ and fix $\varepsilon > 0$. Since D is dense in $\widehat{C}(E)$ by Lemma 2.11 there exists $h \in D$ such that $\|f - h\| < \varepsilon$. It follows that

$$\begin{aligned}
\Big\| T_t^{(n)}\pi_n f - \pi_n T_t f \Big\| &\leq \Big\| T_t^{(n)}\pi_n (f-h) \Big\| + \Big\| T_t^{(n)}\pi_n h - \pi_n T_t h \Big\| + \|\pi_n T_t (h-f)\| \\
&\leq \Big\| T_t^{(n)} \Big\| \, \|f-h\| + \Big\| T_t^{(n)}\pi_n h - \pi_n T_t h \Big\| + \|T_t\| \, \|h-f\| \\
&\leq 2\varepsilon + \Big\| T_t^{(n)}\pi_n h - \pi_n T_t h \Big\| \to 2\varepsilon, \qquad n \to \infty.
\end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily we conclude that $\lim_{n\to\infty} \Big\| T_t^{(n)}\pi_n f - \pi_n T_t f \Big\| = 0$ for all $f \in \widehat{C}(E)$. The result follows from [11, p. 172, Theorem 2.11]. □

Acknowledgement. The authors thank two anonymous referees for helpful comments and suggestions leading to a significant improvement of the structure of the article.

Literaturverzeichnis

[1] Milton Abramowitz und Irene A. Stegun (Hrsg.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York; John Wiley & Sons, Inc., New York, 1984, Reprint of the 1972 edition, Selected Government Publications, MR757537. 41

[2] Ellen Baake, Ute Lenz, und Anton Wakolbinger, *The common ancestor type distribution of a Λ-Wright-Fisher process with selection and mutation*, Electron. Commun. Probab. **21** (2016), Paper No. 59, 16, `doi:10.1214/16-ECP16`, MR3548771. 26

[3] Erich Baur und Jean Bertoin, *The fragmentation process of an infinite recursive tree and Ornstein-Uhlenbeck type processes*, Electron. J. Probab. **20** (2015), no. 98, 20 pages, `doi:10.1214/EJP.v20-3866`, MR3399834. 28

[4] Jean Bertoin und Jean-François Le Gall, *The Bolthausen-Sznitman coalescent and the genealogy of continuous-state branching processes*, Probab. Theory Related Fields **117** (2000), no. 2, 249–266, `doi:10.1007/s004400050006`, MR1771663. 4, 29

[5] Vladimir I. Bogachev, Michael Röckner, und Byron Schmuland, *Generalized Mehler semigroups and applications*, Probab. Theory Related Fields **105** (1996), no. 2, 193–225, `doi:10.1007/BF01203835`, MR1392452. 8, 30

[6] Erwin Bolthausen und Alain-Sol Sznitman, *On Ruelle's probability cascades and an abstract cavity method*, Comm. Math. Phys. **197** (1998), no. 2, 247–276, MR1652734, `doi:10.1007/s002200050450`, MR1652734. 4, 12, 13, 26

[7] Gerd Christoph und Karina Schreiber, *Discrete stable random variables*, Statist. Probab. Lett. **37** (1998), no. 3, 243–247, `doi:10.1016/S0167-7152(97)00123-5`, MR1614930. 31

[8] Peter Donnelly und Thomas G. Kurtz, *A countable representation of the Fleming-Viot measure-valued diffusion*, Ann. Probab. **24** (1996), no. 2, 698–742, `doi:10.1214/aop/1039639359`, MR1404525. 3, 26

[9] ———, *Particle representations for measure-valued population models*, Ann. Probab. **27** (1999), no. 1, 166–205, `doi:10.1214/aop/1022677258`, MR1681126. 3, 12, 26

[10] Miloslav Duchoň, *A generalized Bernstein approximation theorem*, Tatra Mt. Math. Publ. **49** (2011), 99–109, `doi:10.2478/v10127-011-0029-x`, MR2867253. 48

[11] Stewart N. Ethier und Thomas G. Kurtz, *Markov Processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley

& Sons, Inc., New York, 1986, Characterization and convergence, MR0838085, doi:10.1002/9780470316658, MR838085. 48

[12] Fabian Freund und Martin Möhle, *On the time back to the most recent common ancestor and the external branch length of the Bolthausen-Sznitman coalescent*, Markov Process. Related Fields **15** (2009), no. 3, 387–416, MR2554368. 33

[13] Florian Gaiser und Martin Möhle, *On the block counting process and the fixation line of exchangeable coalescents*, ALEA Lat. Am. J. Probab. Math. Stat. **13** (2016), no. 2, 809–833, MR3546382. 3, 26, 32

[14] Keith Gladstien, *The characteristic values and vectors for a class of stochastic matrices arising in genetics*, SIAM J. Appl. Math. **34** (1978), no. 4, 630–642, MR0475977, https://doi.org/10.1137/0134050. 4

[15] Christina Goldschmidt und James B. Martin, *Random recursive trees and the Bolthausen-Sznitman coalescent*, Electron. J. Probab. **10** (2005), no. 21, 718–745, doi:10.1214/EJP.v10-265, MR2164028. 4, 17, 18, 19, 33

[16] Theodore E. Harris, *The Theory of Branching Processes*, Die Grundlehren der Mathematischen Wissenschaften, Bd. 119, Springer-Verlag, Berlin; Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963, MR0163361. 32

[17] Olivier Hénard, *Change of measure in the lookdown particle system*, Stochastic Process. Appl. **123** (2013), no. 6, 2054–2083, doi:10.1016/j.spa.2013.01.015, MR3038498. 26

[18] ———, *The fixation line in the Λ-coalescent*, Ann. Appl. Probab. **25** (2015), no. 5, 3007–3032, doi:10.1214/14-AAP1077, MR3375893. 3, 26, 27, 33

[19] Thierry Huillet und Martin Möhle, *On the extended Moran model and its relation to coalescents with multiple collisions*, Theor. Popul. Biol. **87** (2013), 5–14, MR number not available, doi:10.1016/j.tpb.2011.09.004. 31

[20] Alexander Iksanov und Martin Möhle, *On the number of jumps of random walks with a barrier*, Adv. in Appl. Probab. **40** (2008), no. 1, 206–228, doi:10.1239/aap/1208358893, MR2411821. 31

[21] John Frank Charles Kingman, *The coalescent*, Stochastic Process. Appl. **13** (1982), no. 3, 235–248, doi:10.1016/0304-4149(82)90011-4, MR671034. 1, 11, 15, 24, 26

[22] Jonas Kukla und Martin Möhle, *On the block counting process and the fixation line of the Bolthausen-Sznitman coalescent*, Stochastic Process. Appl. **128** (2018), no. 3, 939–962, doi:10.1016/j.spa.2017.06.012, MR3758343. 5, 8

[23] Jonas Kukla und Helmut H. Pitters, *A spectral decomposition for the Bolthausen-Sznitman coalescent and the Kingman coalescent*, Electron. Commun. Probab. **20** (2015), no. 87, 13 pages, doi:10.1214/ECP.v20-4612, MR3434204. 5, 30, 31

[24] Cyril Labbé, *From flows of Λ-Fleming-Viot processes to lookdown processes via flows of partitions*, Electron. J. Probab. **19** (2014), no. 55, 49 pages, doi:10.1214/EJP.v19-3192, MR3227064. 3, 26

[25] John Lamperti, *Continuous state branching processes*, Bull. Amer. Math. Soc. **73** (1967), 382–386, `doi:10.1090/S0002-9904-1967-11762-2`, MR0208685. 29

[26] ______, *The limit of a sequence of branching processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **7** (1967), 271–288, `doi:10.1007/BF01844446`, MR0217893. 29

[27] Martin Möhle, *Looking forwards and backwards in the multi-allelic neutral Cannings population model*, J. Appl. Probab. **47** (2010), no. 3, 713–731, `doi:10.1239/jap/1285335405`, MR2731344. 45

[28] ______, *The Mittag-Leffler process and a scaling limit for the block counting process of the Bolthausen-Sznitman coalescent*, ALEA Lat. Am. J. Probab. Math. Stat. **12** (2015), no. 1, 35–53, MR3333734. 28, 29, 35

[29] Martin Möhle und Helmut Pitters, *A spectral decomposition for the block counting process of the Bolthausen-Sznitman coalescent*, Electron. Commun. Probab. **19** (2014), no. 47, 11 pages, `doi:10.1214/ECP.v19-3464`, MR3246966. 5, 14, 27, 30, 40, 41, 42

[30] ______, *Absorption time and tree length of the Kingman coalescent and the Gumbel distribution*, Markov Process. Related Fields **21** (2015), no. 2, 317–338, MR3442826. 5

[31] Martin Möhle und Serik Sagitov, *A classification of coalescent processes for haploid exchangeable population models*, Ann. Probab. **29** (2001), no. 4, 1547–1562, MR1880231, `https://doi.org/10.1214/aop/1015345761`. 1

[32] Jacques Neveu, *A continuous-state branching process in relation with the GREM model of spin glass theory*, Rapport interne no 267, École Polytechnique, 1992. 27, 28

[33] James R. Norris, *Markov Chains*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 2, Cambridge University Press, Cambridge, 1998, Reprint of 1997 original, MR1600720. 13, 43

[34] Alois Panholzer, *Destruction of recursive trees*, Mathematics and Computer Science. III, Trends Math., Birkhäuser, Basel, 2004, pp. 267–280, MR2090518. 43

[35] Peter Pfaffelhuber und Anton Wakolbinger, *The process of most recent common ancestors in an evolving coalescent*, Stochastic Process. Appl. **116** (2006), no. 12, 1836–1859, `doi:10.1016/j.spa.2006.04.015`, MR2307061. 2, 26

[36] Jim Pitman, *Coalescents with multiple collisions*, Ann. Probab. **27** (1999), no. 4, 1870–1902, `doi:10.1214/aop/1022677552`, MR1742892. 1, 2, 12, 26

[37] ______, *Combinatorial Stochastic Processes*, Lecture Notes in Mathematics, vol. 1875, Springer-Verlag, Berlin, 2006, Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard, MR2245368. 6, 9, 20, 31

[38] Georg Pólya, *Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung und das Momentenproblem*, Math. Z. **8** (1920), no. 3-4, 171–181, `doi:10.1007/BF01206525`, MR1544437. 35, 39

[39] Daniel Revuz und Marc Yor, *Continuous Martingales and Brownian Motion*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999, `doi:10.1007/978-3-662-06400-9`, MR1725357. 34, 38

[40] Serik Sagitov, *A key limit theorem for critical branching processes*, Stochastic Process. Appl. **56** (1995), no. 1, 87–100, `doi:10.1016/0304-4149(94)00046-V`, MR1324323. 29

[41] ———, *The general coalescent with asynchronous mergers of ancestral lines*, J. Appl. Probab. **36** (1999), no. 4, 1116–1125, MR1742154. 1, 2, 12, 26

[42] Jason Schweinsberg, *Coalescents with simultaneous multiple collisions*, Electron. J. Probab. **5** (2000), Paper no. 12, 50, MR1781024, `https://doi.org/10.1214/EJP.v5-68`. 1

[43] Damodar N. Shanbhag und Maddipatla Sreehari, *On certain self-decomposable distributions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **38** (1977), no. 3, 217–222, `doi:10.1007/BF00537265`, MR0436267. 30

[44] David Siegmund, *The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes*, Ann. Probab. **4** (1976), no. 6, 914–924, MR0431386. 3

[45] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012, MR2868112. 15, 16

[46] Simon Tavaré, *Line-of-descent and genealogical processes, and their applications in population genetics models*, Theoret. Population Biol. **26** (1984), no. 2, 119–164, MR770050, `https://doi.org/10.1016/0040-5809(84)90027-3`. 4