Totally geodesic periods over hyperbolic manifolds

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Abstract

Let X be a compact hyperbolic manifold with hyperbolic measure dx, $\{\phi_i\}$ be an orthonormal basis of $L^2(X,\, dx)$ such that ϕ_i's are Laplace eigenfunctions. Let Y be a totally geodesic compact submanifold of X with the induced measure dy. In this work we shall investigate some properties of the period integral $P_Y(\phi_i) = \int_Y \phi_i(y)dy$. We get an upper bound of $|P_Y(\phi_i)|$ for ϕ_i with large eigenvalue. Based on this bound, we use trace formula to derive the asymptotic of sums of all $|P_Y(\phi_i)|^2$.

Zusammenfassung

Sei X eine kompakte hyperbolische Mannigfaltigkeit mit Volumenform dx und sei $\{\phi_i\}$ eine Orthonormalbasis von $L^2(X, dx)$ bestehend aus Laplace-Eigenfunktionen. Sei Y eine totalgeodätische Untermannigfaltigkeit von X mit induzierter Volumenform dy. In dieser Arbeit werden einige Eigenschaften der Periodenintegrale $P_Y(\phi_i) = \int_Y \phi_i(y)\, dy$ untersucht. Wir erhalten eine obere Schranke für $|P_Y(\phi_i)|$ für große Eigenwerte. Wir benutzen dann diese Abschätzung und die relative Spurformel, um eine asymptotische Formel für die Summe aller $|P_Y(\phi_i)|^2$ herzuleiten.

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Contents

Introduction

The notions of *periods* spread in various areas of mathematics. Here, roughly speaking, periods are integrals of certain differentials over some (sub-)geometric objects. For specific problems, both differentials and (sub-)geometric objects need to be clearly described. Periods have been playing important roles in algebraic geometry, automorphic forms and number theory, often as bridges between other interesting and important things. There are a great deal of splendid results and conjectures about them. In what follows, we shall illustrate the notions of periods, as well as their close relations with other things, by some (among so many) examples.

In number theory, according to [**KZ**], we define the period to be a complex number whose real and imaginary parts are both expressed as convergent integrals of rational functions with coefficients in $\mathbb{Q}$ over domains in $\mathbb{R}^n$ where the domain is given by polynomial inequalities with coefficients in $\mathbb{Q}$. In general, these rational functions can also be algebraic functions with coefficients being algebraic numbers. Clearly, the collection $\mathcal{P}$ of all periods is countable. Some interesting irrational numbers, even transcendental numbers, are periods:

$$\sqrt{2} = \int\limits_{2x^2 \leqslant 1} dx, \qquad \pi = \iint\limits_{x^2+y^2 \leqslant 1} dxdy = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

We see that the period is not unique with respect to the integration expression. A more interesting example is:

$$\zeta(3) = \iiint\limits_{0<x<y<z<1} \frac{dxdydz}{(1-x)yz}$$

where $\zeta(s)$ is the Riemann zeta-function. A famous result by Apéry is that $\zeta(3)$ is irrational. In [**Za**], it is shown that all values of Riemann zeta-function at positive integers $n \geqslant 2$ are periods. At an advanced level, there is a conjecture by Deligne, Beilinson an Scholl which asserts that, if the motivic L-functions has vanishing order r at the integer m, then $L^{(r)}(m) \in \hat{\mathcal{P}}$ where $\hat{\mathcal{P}} = \mathcal{P}[1/\pi]$ ($1/\pi$ is conjectured not a period).

Let E be an elliptic curve defined over $\mathbb{Q}$. The Mordell-Weil group

$$E(\mathbb{Q}) = \{\, \text{rational point on } E \,\}$$

is (algebraically) decomposed into two parts: $E(\mathbb{Q}) \approx \mathbb{Z}^r \otimes T$ where T is a finite group. Call the natural number r the *algebraic rank* of E, denoted by E_{alg}. Exactly for those primes p which do not divide the discriminant Δ of the elliptic curve E, E_p (E modulo p) defines an elliptic curve over the finite field $\mathbb{F}_p$. Let $E(\mathbb{F}_p)$ be the Mordel-Weil group of E_p and $a_p = p + 1 - \#E(\mathbb{F}_p)$. Define

$$\widetilde{L}(E,\, s) = \prod_p L_p(E,\, s)$$

where

$$L_p(E,\, s) = \begin{cases} \frac{1}{1-a_p p^{-s}+p^{1-2s}}, & \text{if } p \nmid \Delta \\ \frac{1}{1-a_p p^{-s}}, & \text{if } p \| \Delta \\ 1, & \text{if } p^2 \nmid \Delta \end{cases}$$

This function can be analytically extended to all $z \in \mathbb{C}$. The *analytic rank* E_{anl} of E is defined to be the vanishing order of $\widetilde{L}(E,\, s)$ at $s = 1$: $E_{\text{anl}} = \text{ord}_{s=1} L(E,\, s)$. The Birch-Swinnerton-Dyer conjecture predicts that $E_{\text{alg}} = E_{\text{anl}}$, moreover

$$\frac{L^{(r)}(E,\, 1)}{r!} = \frac{\Omega_E \cdot \text{Reg}(E) \cdot \# ⧢ (E/\mathbb{Q}) \cdot \prod_p c_p}{(\#E(\mathbb{Q})_{\text{tor}})^2}$$

where $\Omega_E = \int_{E(\mathbb{R})} \omega$ is just a period (ω is some differential), $⧢(E/\mathbb{Q})$ is the Shafarevich-Tate group of E, c_p are some Tamagawa numbers of E (equal to 1 for all $p \nmid \Delta$) and $\text{Reg}(E)$ is the regulator of E (basically it is the absolute value of the determinant of the matrix (x_{ij}) where $x_{ij} = \langle P_i,\, P_j \rangle$ for P_i being the basis of $E(\mathbb{Q})/E_{\text{tor}}(\mathbb{Q})$ and $\langle\, ,\rangle$ being the Néron-Tate canonical height pairing).

In the theory of automorphic forms, periods are indispensable for various formulas which express the special values of L-functions or encode the important information on Fourier coefficients of automorphic forms. Let G be a reductive group over a number field F. Let H be a subgroup of G, usually coming as the set of fixed points of some (anti-)automorphisim on G. Then define the period over H, in the simplest way, to be

$$P_H(\phi) = \int\limits_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(x)dx$$

where $\mathbb{A}$ denotes the adele ring of F, Z_H is the split center of H and ϕ is an automorphic form (i.e., a cusp form or Eisenstein series) of $G(\mathbb{A})$. Naturally, we can replace ϕ with other things, e.g., the product of cusp forms, Eisenstein series or some other functions (e.g., characters of $H(\mathbb{A})$ trivial on $Z_H(\mathbb{A})H(F)$). Likely, with proper integral functions, the integration domain can also switch to other domains. For example, we can integrate over $T(F)\backslash T(\mathbb{A}) \times T(F)\backslash T(\mathbb{A})$ where T is the maximal split torus of G, or $N(F)\backslash N(\mathbb{A}) \times N(F)\backslash N(\mathbb{A})$ where N denotes the unipotent radical of the standard Borel subgroup, or $Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A}) \times N(F)\backslash N(\mathbb{A})$, even $Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A}) \times N(E)\backslash N(\mathbb{A}_E)$ for E

being an algebraic extension of F. All these examples turn to be useful. In the classical non-adelic case, i.e., $G = PSL_2(\mathbb{R})$, $\Gamma = PSL_2(\mathbb{Z})$, the Kuznetsov trace formula can be obtained via the integration of the automorphic kernel $K_f(x, y)$ over $(\Gamma \cap N)\backslash N \times (\Gamma \cap N)\backslash N$. Here f is a proper test function and $K_f(x, y)$ has two types of expansions:

$$K_f(x, y) = \sum_{\gamma\in\Gamma} K\left(x^{-1}\gamma y\right) = \sum_{\phi_i:\,\text{cusp forms}} K_{\phi_i}(x, y) + \sum_{E_i:\,\text{Eisenstein series}} K_{E_i}(x, y),$$

geometric and spectral expansions respectively. Another example is Waldspurger's formula. Let E be a quadratic extension of F, π be a cuspidal representation of $GL_2(\mathbb{A}_F)$, χ be a unitary character of $\mathbb{A}_E^\times$ trivial on $E^\times\mathbb{A}_F^\times$. Let π_χ be the induced representation $\mathrm{Ind}_{\mathbb{A}_E^\times}^{GL_2(\mathbb{A}_F)}(\chi)$ and π_E be the base change of π to $GL_2(\mathbb{A}_E)$. By Jacquet-Langlands correspondence, there is a quaternion algebra D over F such that $E \subset D$ and π_D corresponds to π. For T as above, Waldspurger showed in [**Wa**] that, for any $\phi \in \pi_D$,

$$L\left(\frac{1}{2}, \pi\times\chi\right)\cdot P = \frac{\left|\int_{Z(\mathbb{A}_E T(\mathbb{A}_F\backslash T(\mathbb{A}_E)))}\phi(x)\overline{\chi(x)}dx\right|^2}{\|\phi\|^2}$$

where χ, $\|\phi\|^2 = \int_{Z(\mathbb{A}_F)D^\times(F)\backslash D^\times(\mathbb{A}_F)}\|\phi(x)\|^2dx$ and P is a number dependent on ϕ, π. In practice, especially in the setting of the applications of trace formulas, one has to refine (e.g., by use of truncations) the integral function to deal with the convergence problem.

Main Results

Let X be a d-dimensional connected compact hyperbolic manifold, Y be a compact totally geodesic submanifold (or *cycle*) of X. Let $\{\phi_i\}_{i=0}^\infty$ be a family of orthonormal basis of $L^2(X, dx)$ where ϕ_i's are eigenfunctions of the Laplace operator Δ with eigenvalues $\lambda_i = \left(\frac{d-1}{2}\right)^2 - \nu_i^2$, $\nu_i \in \left[-\frac{d-1}{2}, \frac{d-1}{2}\right] \cup \mathbf{i}\,\mathbb{R}$, and dx denotes the hyperbolic measure of X. Define the *period* of ϕ_i over Y as follows:

$$P_Y(\phi_i) = \int_Y \phi_i(y)dy$$

where dy is the hyperbolic measure of Y induced from dx. In the present work we shall investigate some properties of periods. Our results are two-fold, namely, on one hand we study a single period to get its unform upper bound in terms of eigenvalues, on the other hand we study the family of periods and get the asymptotic of the sum of them. The latter achievement depends partly on the previous one, partly on a formula that explicitly expresses the volume of Y in terms of the periods. The central tool to derive such formula is the trace formula. We shall first think about the most simple case, i.e., when Y is one-dimensional, or equivalently Y is a closed geodesic. Afterwards, we try to think about higher-dimensional case. As more parameters occur in this situation, various results are needed in their *uniform* versions, although the strategy we shall follow stays unchanged. This makes our work for the higher-dimensional cases more complicated than the geodesic case. Note that we do not require that Y, after the embedding into X, is still smooth, i.e., there might be self-intersections on Y. For example, the geodesic C might be not *simple* in which case we call Y a "cycle". However, the self-intersections have no impact on our conclusion since they are of lower dimension than Y.

At the moment we shall point out that the paper [**MW**] is a key inspiration for our work. Actually the readers can find that we have followed the general philosophy of it.

In what follows, we list our main results according to the order of the presentation.

Let C be a closed geodesic over compact hyperbolic manifold X, then we have:

Theorem 0.1.

$$\lim_{\mu\to\infty} \frac{2^{d-1}\, e^{\mu}}{\sqrt{2(d-1)}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{i=0}^{\infty} K_{\nu_i}(\mu)\ |P_C(\phi_i)|^2 = \mathrm{len}(C) \tag{1}$$

where $K_z(x)$ is the K-Bessel function.

This is a generalization of the formula (22) of [**MW**] where the argument is done for compact Riemann surfaces (with genus $g \geqslant 2$ so that these surfaces are hyperbolic). As a consequence, we have:

Corollary 0.2. *There are infinitely many ϕ_i's such that $P_C(\phi_i) \neq 0$.*

More can be done in this situation:

- We can twist a unitary character χ along C (see Sect. 2.6 for its definition) to ϕ_i to get the "weighted period":

  $$P_C(\phi_i,\, \chi) = \int_C \phi_i(x)\chi(x)dx.$$

  Then we have:

  Theorem 0.3.

  $$\lim_{\mu\to\infty} \frac{2^{d-1}\, e^{\mu}}{\sqrt{2(d-1)}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{i=0}^{\infty} K_{\nu_i}(\mu)\left|P_C(\phi_i,\, \chi)\right|^2 = \mathrm{len}(C). \tag{2}$$

  Corollary 0.4. *There are infinitely many ϕ_i's such that $P_C(\phi_i,\, \chi) \neq 0$.*

  For two *distinct* unitary characters χ_1 and χ_2 along the geodesic C, we have:

  Theorem 0.5.

  $$\lim_{\mu\to\infty} \frac{2^{d-1}\, e^{\mu}}{\sqrt{2(d-1)}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{i=0}^{\infty} K_{\nu_i}(\mu)\, P_C(\phi,\, \chi_1)\, \overline{P_C(\phi,\, \chi_2)} = 0.$$

- Consider two *distinct* geodesics C_1, C_2 and the periods along them: $P_{C_1}(\phi_i)$, $P_{C_2}(\phi_i)$. We have the following formula on mixed periods $P_{C_1}(\phi_i)\overline{P_{C_2}(\phi_i)}$:

  Theorem 0.6.

$$\lim_{\mu\to\infty} e^{\mu}\,\mu^{-\frac{d}{2}+\frac{3}{2}-\epsilon}\sum_{i=0}^{\infty} K_{\nu_i}(\mu)P_{C_1}(\phi_i)\overline{P_{C_2}(\phi_i)} = 0,\ \epsilon > 0.$$

  Corollary 0.7. *Let X be a compact hyperbolic manifold with dimension $d \geqslant 3$. Suppose that $C_1 \cap C_2 \neq \varnothing$, then there are infinitely many ϕ_i's such that $P_{C_1}(\phi_i)$ and $P_{C_2}(\phi_i)$ are nonvanishing at the same time.*

- Consider the (squared) L^2-norms of ϕ_is along C. We have:

  Theorem 0.8. $\lim\limits_{\mu\to\infty} e^{\mu}\sum\limits_{i=0}^{\infty} 2^d \cdot \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu)\int_C |\phi_i|^2 = \mathrm{len}(C).$

  Modifying this formula and applying Tauberian Theorem, one can derive the asymptotics of the L^2-norms for surfaces:

  Corollary 0.9. *When $d = 2$, i.e., X is a compact Riemann surface with genus $g \geqslant 2$, the following asymptotic holds:*

$$\sum_{\lambda_n \leqslant x}\int_C |\phi_n|^2 \sim \frac{\mathrm{len}(C)}{4\pi}x \quad \text{as} \quad x\to\infty.$$

Motivated by the above results for closed geodesics, we try to consider higher dimensional compact submanifold $Y \subset X \cong \Gamma\backslash G/K$ on which periods are defined. Here $G = SO_0(1, d)$, Γ is a lattice in G, K is a maximal compact subgroup of G. It suffices to focus on a special case: $Y \cong \Gamma_0\backslash G^*/K^* \hookrightarrow X$ where

$$G^* = \{\tau = \mathrm{diag}(\tau_1, \tau_2)\,|\,\tau_1 \in O(1, n),\ \tau_2 \in O(d-n)\} \cap G,$$

$\Gamma_0 = \Gamma \cap G^*$, $K^* = K \cap G^*$, X is compact and $d \geqslant n \geqslant 2$ ($n = 1$ has been treated in the previous chapter). It is reasonable to choose Y formulated in such a way since one can conjugate G^* to get all possible totally geodesic submanifolds. We follow the strategy for the geodesic case which, with some extra technical arguments, still works!

Theorem 0.10. *For any n-dimensional totally geodesic compact submanifold (or cycle) Y on X, we have:*

$$\lim_{\mu\to\infty}\sum_{i=0}^{\infty} 2^{d-n}e^{\mu}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1-n} K_{\nu_i}(\mu)\,|P_Y(\phi_i)|^2 = \mathrm{vol}(Y). \tag{3}$$

Corollary 0.11. *There are infinitely many ϕ_i's with nonvanishing periods over Y.*

All the above conclusions result from the treatment of the geometric side of the trace formula. Now we turn to the spectral side. In particular, we shall refine the left hand side of these formulas so that they are in the form suitable for the application (of Tauberian Theorem). The first thing in demand is to bound a (single) period uniformly.

Proposition 0.12. *Let $\nu_j = ir_j$ where $r_j \in \mathbb{R}_{\geqslant 0}$.*

- *If $n = 1$, that is, Y is a closed geodesic, then for any fixed unitary character χ along Y and $\epsilon > 0$,*
$$\int_Y \phi_j(z)\chi(z)dz \ll r_j^{-\frac{1}{2}+\epsilon}, \quad \text{as } r_j \to \infty$$
where the implied $\mathcal{O}$-constant depends on χ.
- *If $n \geqslant 2$, then for any fixed $\epsilon > 0$,*
$$\int_Y \phi_j(z)dz \ll r_j^{-\frac{n}{2}+\epsilon}, \quad \text{as } r_j \to \infty$$
where the implied $\mathcal{O}$-constant depends on n.

Based on this proposition we can refine the above formulas (1), (2) and (3) as:

$$\lim_{\mu\to\infty} 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} \sum_{j=0}^{\infty} e^{-\frac{r_j^2}{2\mu}} \left|P_C(\phi_j)\right|^2 = 2\|E\| \operatorname{len}(C).$$

$$\lim_{\mu\to\infty} 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} \sum_{j=0}^{\infty} e^{-\frac{r_j^2}{2\mu}} \left|P_C(\phi_j,\, \chi)\right|^2 = 2\|E\| \operatorname{len}(C).$$

$$\lim_{\mu\to\infty} 2^{d-n} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-n} \sum_{j=0}^{\infty} e^{-\frac{r_j^2}{2\mu}} \left|P_Y(\phi_j)\right|^2 = \operatorname{vol}(Y).$$

By Tauberian Theorem, we get:

Theorem 0.13.
$$\sum_{\lambda_j \leqslant x} |P_C(\phi_j)|^2 \sim \frac{\|E\| \operatorname{len}(C)}{(d-1)!!\, \pi^{\frac{d}{2}} 2^{\frac{d-2}{2}}} \cdot x^{\frac{d-1}{2}}, \quad \text{as } x \to \infty.$$

Theorem 0.14.
$$\sum_{\lambda_j \leqslant x} |P_C(\phi_j,\, \chi)|^2 \sim \frac{\|E\| \operatorname{len}(C)}{(d-1)!!\, \pi^{\frac{d}{2}} 2^{\frac{d-2}{2}}} \cdot x^{\frac{d-1}{2}}, \quad \text{as } x \to \infty.$$

Let Y be a n-dimensional totally geodesic compact submanifold in X where $2 \leqslant n \leqslant d-1$, then

Theorem 0.15.
$$\sum_{\lambda_j \leqslant x} |P_Y(\phi_j)|^2 \sim \frac{\operatorname{vol}(Y)}{(2\pi)^{\frac{d-n-1}{2}}\, (d-n)!!} \cdot x^{\frac{d-n}{2}}, \quad \text{as } x \to \infty.$$

For a very general result on the asymptotic of periods on any submanifold of any compact Riemann manifold, see [**Ze**], especially the formula (3.4) there.

Organization of the thesis

This thesis is organized as follows. In Chapter 1, we present the necessary background knowledge to be used in Chapter 2. Using representation theory, harmonic analysis and the structure theory on the Lorentz group, we give a detailed argument on the trace formula for compact manifolds and express the Harish-Chandra-Selberg transform with explicit terms in the Lie algebra of G. In Chapter 2, we choose a test function which exponentially decays with respect to the slight variation of the hyperbolic distance. Then we compute the geometric side of the trace formula under this test function. Indexed by the double coset classes in $\Gamma_0\backslash\Gamma/\Gamma_0$ where Γ_0 denotes the stabilizer of the regular geodesics, the geometric side splits into two parts: the main and error terms. The main term comes from the trivial class $\tilde{1}$, while the error term comes from all other classes. This type of phenomenon is quite popular in the application of trace formulas. For example, in the representation-theoretic setting, usually the trivial representation contributes most for the spectral side. But one has to strictly realize this for a specific problem. Actually it is the most tricky part to deal with the error term. In Chapter 3, we focus on the higher dimensional compact submanifolds (or cycles) $Y \subset X$ where X is still compact. The convergence problem for the application of the trace formula has been solved in Chapter 2. The main difference with the geodesic case is that there are extra terms to deal with as Y is of higher dimension, although only part of these terms really matters. Hence we have to get the *unform* results (with respect to Γ) on the necessary terms which are parallel to those occurring in the geodesic case. Chapter 2 guides our work here, namely, the strategy is close to that of Chapter 2, only with some techniques to be overcome. In Chapter 4 we focus on the spectral side and refine it, based on the work on the bound of a single period, to be in the form availible for applying Tauberian Theorem. Then we get the asymptotics of periods. In the last chapter we discuss the noncompact case. Due to the lack of the deeper understanding of Eisenstein series, we can not show any essential results there. The main content is to give a connection between our work and the important but still open Selberg-Roelcke conjecture.

The outlook

1. As Waldspurger's fromula shows, central values of automorphic L-functions for GL_2 is related with the torus periods of cusp forms. In the future we would like to consider the counterpart for real case and adelic case of the Lorentz group, with the aid of trace formula.
2. It is a hard and vital task to improve the upper bound on the average growth order of the Eisenstein series over the critical line $\mathrm{Re}(s) = \frac{d-1}{2}$. In fact, this is where the *possible* resolution of the Selberg-Roelcke conjecture most hopefully lies.

Chapter 1

A Relative Trace Formula

1.1 Hyperbolic manifolds as symmetric spaces

Let X be an orientable connected hyperbolic manifold of finite volume, i.e., a complete Riemannian manifold with constant sectional curvature -1 which is orientable and has finite volume. Then the universal cover $\widetilde{X}$ of X is isomorphic to the hyperboloid model $\mathcal{H}^d$ where d is the dimension of X. Recall that

$$\mathcal{H}^d = \left\{ \xi = (\xi_0, \dots, \xi_d) \in \mathbb{R}^{d+1} \,\middle|\, \xi_0^2 - \sum_{i=1}^{d} \xi_i^2 = 1,\, \xi_0 > 0 \right\}.$$

For $\xi = (\xi_i)_{i=0}^d$ and $\eta = (\eta_i)_{i=0}^d \in \mathcal{H}^d$, define the *pseudo-metric* to be $\langle \xi, \eta \rangle = \xi_0\eta_0 - \sum_{i=1}^d \xi_1\eta_i$. Over $T_{\xi^0}\mathcal{H}^d$, the tangent space of $\mathcal{H}^d$ at the point $\xi^0 = (1,\, 0,\, \dots,\, 0)$, there is a positive definite inner product : $\langle\!\langle \alpha,\, \beta \rangle\!\rangle = \sum\limits_{i=1}^{d} \alpha_i\beta_i - \alpha_0\beta_0$ for $\alpha = (\alpha_i)_{i=0}^d$, $\beta = (\beta_i)_{i=0}^d \in T_{\xi^0}\mathcal{H}^d$, with which $\mathcal{H}^d$ is a hyperbolic manifold. Let $O(1,\, d)$ be the linear transformation group of $\mathcal{H}^d$ that preserves the pseudo-metric $\langle\, ,\, \rangle$. Denote by $G = SO_0(1,\, d)$ the connected component of $O(1,\, d)$ which contains the identity element. The maximal compact subgroup K of G is chosen to be the isotropic subgroup of the point ξ^0 in G. Then K is connected and isomorphic to $SO(d)$. The group G acts transitively and properly on $\mathcal{H}^d$. Let Γ be the fundamental group $\pi_1(X)$ of X. It is known that Γ is torsion-free and can be identified with a subgroup of G. Hence $\mathcal{H}^d \cong G/K$ and $X \cong \Gamma\backslash\widetilde{X} \cong \Gamma\backslash G/K$. In this paper we mainly work on compact hyperbolic manifolds. This means that $\Gamma\backslash G/K$ is compact, i.e., Γ is a uniform lattice in G. Also we shall discuss the noncompact hyperbolic manifolds. In that case Γ is not uniform anymore, but still torsion-free. When X is of higher dimension or noncompact, we have to use the structure and representation theory of G together with its harmonic analysis to further our work.

1.2 The representation-theoretic formulation

Let G be a real connected semisimple Lie group endowed with the Cartan involution Θ, K be its maximal compact subgroup which is the set of elements fixed by Θ in G. Assume that the symmetric pair (G, K, Θ) is of noncompact type. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. The Cartan involution θ on Lie algebra level gives rise to the Cartan decomposition: $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ where $\mathfrak{p} = \{X \in \mathfrak{g} | \, \theta X = -X\}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. For each linear functional λ on $\mathfrak{a}$, define$\mathfrak{n}$

$$\mathfrak{g}_\lambda = \big\{X \in \mathfrak{g} \,\big|\, [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\big\}.$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$, λ is called a *restricted root* of $\mathfrak{g}$. The set of restricted roots is denoted by Σ and can be shown to be a root system. Given an order on the dual space $\mathfrak{a}^*$, we can single out a subset Σ^+ of positive restricted roots in Σ. Define

$$\mathfrak{n} = \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

It is known that $\mathfrak{n}$ is nilpotent and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$. G acts on itself by conjugation. Under this action, we have two subgroups in G: $N_G(A)$, the normalizer of A in G, and $C_G(A)$, the centralizer of A in G. Define *Weyl group* $W(G, A)$ to be the quotient of these two subgroups:

$$W(G, A) := N_G(A)/C_G(A).$$

The Weyl group acts on A, thus on $\mathfrak{a}$ linearly. The following two decompositions are well known:

Theorem 1.2.1. (Iwasawa Decomposition) *Any $g \in G$ can be written as $g = nak$ for some unique $a \in A$, $n \in N$ and $k \in K$.*

Theorem 1.2.2. (KAK Decomposition) *Any $g \in G$ can be written as $g = k_1 a(g) k_2$ for some k_1, $k_2 \in K$ and $a(g) \in A$ where $a(g)$ is uniquely determined up to the action of the Weyl group.*

Denote $\exp \overline{\mathfrak{a}^+}$ by A^+, where $\overline{\mathfrak{a}^+}$ stands for the closure of the subset $\{X \in \mathfrak{a} \,|\, \alpha(X) > 0, \ \forall\, \alpha \in \Sigma^+\} \subset \mathfrak{a}$. If we require $a(g)$ in the KAK-decomposition to lie in A^+ (this is always possible), then $a(g)$ is unique, denoted by $a(g)$ and called the A^+-part of g. Note that k_1, k_2 in the KAK decomposition are not uniquely determined. The subgroups A, N and $NA = AN$ are all simply connected and closed. These facts are standard, see Ch. 5 of [**Kn**].

Under the assumptions on G and K, the Killing form

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$$

on $\mathfrak{g}$, when restricted to $\mathfrak{p}$ (written as $B|_{\mathfrak{p}}$), is positive-definite. Let $g \cdot o$ denote the image of g in G/K under the natural projection $G \to G/K$. By identifying $\mathfrak{p}$ with the tangent space $T_{e \cdot o}(G/K)$ of G/K at $e \cdot o \in G/K$, the $\mathrm{Ad}(K)$-invariance of the Killing form indicates that $B|_{\mathfrak{p}}$ induces a G-invariant Riemannian metric η on the manifold G/K. With this metric, the A^+-part of g determines the distance between the two points $g \cdot o$ and $e \cdot o$ on G/K. More precisely,

$$\mathrm{dist}_{G/K}(g \cdot o,\, e \cdot o) = B\big(\log a(g),\, \log a(g)\big)^{1/2} =: \|\log a(g)\|.$$

Let Γ be a torsion-free lattice in G such that $\Gamma\backslash G/K$ is a closed smooth manifold. By invariance, η induces a metric η' on $\Gamma\backslash G/K$ which defines a Laplace operator Δ on $\Gamma\backslash G/K$. The Laplace operator Δ is self-adjoint with respect to the volume form defined by η'. The volume form is equal (up to a positive scalar) to the Radon measure μ' in the following lemma since both of them are induced by η'. Let ϕ be a function on $\Gamma\backslash G/K$. We can lift ϕ to $\Gamma\backslash G$. This is nothing but the pull-back of ϕ according to the principal bundle $\Gamma\backslash G \to \Gamma\backslash G/K$ with the structure group K. In this respect, the lift of ϕ is an eigenfunction of $\Box$ over $\Gamma\backslash G$ with the eigenvalue unchanged, where $\Box$ is the Laplace operator defined by the G-invariant Riemannian metric over $\Gamma\backslash G$ induced by

$$\langle X,\, Y\rangle = -B(X,\, \theta Y)$$

for $X, Y \in \mathfrak{g}$. Note that $\langle\, ,\, \rangle$ is positive definite on $\mathfrak{g}$ and, when restricted to $\mathfrak{p}$, we have $\langle\, ,\, \rangle|_{\mathfrak{p}} = B(\, ,\,)|_{\mathfrak{p}}$. The tangent space of $\Gamma\backslash G$ at the point $\Gamma \cdot e$ is equal to $\mathfrak{g}$.

Any locally compact group which admits a lattice must be unimodular (see Theorem 9.1.6 of [**DE**]), so it is reasonable to equip $\Gamma\backslash G$ with a right G-invariant Radon measure.

Lemma 1.2.3. *Let dk be a Haar measure on K, μ be a right G-invariant Radon measure on $\Gamma\backslash G$, then there exists a unique Radon measure μ' on $\Gamma\backslash G/K$ such that, for any continuous function $f \in C_c(\Gamma\backslash G)$, we have*

$$\int\limits_{\Gamma\backslash G} f(x)d\mu(x) = \int\limits_{\Gamma\backslash G/K} \int\limits_{K} f(yk)dkd\mu'(y).$$

Proof. The lattice Γ is a closed subgroup in G. We equip it with the counting-measure, i.e., each point $\gamma \in \Gamma$ possesses the mass one. Let Δ_H denote the modular function of H. A semisimple Lie group is always unimodular, hence $\Delta_G|_\Gamma = \Delta_\Gamma \equiv 1$. It follows that there is a unique Haar measure on G such that, for any $h \in C_c(G)$ one has

$$\int_G h(g)dg = \sum_{\gamma\in\Gamma} \int_{\Gamma\backslash G} h(\gamma x)d\mu(x). \tag{1.1}$$

This is an application of the quotient integral formula (see Theorem 1.5.2 of [**DE**]). Conversely, given any Haar measure dg on G, there exists a unique right G-invariant

Radon measure μ on $\Gamma\backslash G$ such that the above formula holds. The ensuing map is surjective (see Lemma 1.5.1 of [**DE**]):

$$C_c(G) \to C(\Gamma\backslash G), \quad h \mapsto h^\Gamma : x \mapsto \sum_{\gamma\in\Gamma} h(\gamma x).$$

So, for a given $f \in C_c(\Gamma\backslash G)$ we may assume $f(x) = \sum_{\gamma\in\Gamma} F(\gamma x)$ for some $F \in C_c(G)$. Then

$$\begin{aligned} \int_{\Gamma\backslash G} f(x) d\mu(x) &= \sum_{\gamma\in\Gamma} \int_{\Gamma\backslash G} F(\gamma x) d\mu(x) \\ &\overset{(a)}{=} \int_G F(g) dg \\ &\overset{(b)}{=} \int_{G/K} \int_K F(xk) dk dx \\ &\overset{(c)}{=} \sum_{\gamma\in\Gamma} \int_{\Gamma\backslash G/K} \int_K F(\gamma xk) dk dx \\ &\overset{(d)}{=} \int_{\Gamma\backslash G/K} \int_K f(xk) dk dx \end{aligned}$$

The equality (a) follows from (1.1). For (b), we use the quotient integral formula again, noting that $\Delta_G|_K = \Delta_K \equiv 1$ since K is compact. Here dx is a left G-invariant Radon measure on G/K. Since G/K can be obtained by the left translations of Γ applying to $\Gamma\backslash G/K$, we get (c). In this step, thanks to the left G-invariance of dx, we keep using it to denote the measure on $\Gamma\backslash G/K$. The last step follows from the definition of f. It is clear that dx is identical to the expected measure $\mu(x)$ in the lemma. The uniqueness of $\mu(x)$ is a consequence of the quotient integral formula, implied in the step (b). $\square$

We normalize the Haar measure dk on K such that $\mathrm{vol}(K) = 1$. There are two spaces $L^2(\Gamma\backslash G, \mu)$ and $L^2(\Gamma\backslash G/K, \mu')$. The former space is a representation space of G under the right regular action R:

$$(R(g)f)(x) = f(xg)$$

for $f \in L^2(\Gamma\backslash G)$, $x \in \Gamma\backslash G$. The Laplace operator $\Box$ acts on the dense subset of smooth functions of $L^2(\Gamma\backslash G, \mu)$ as a symmetric operator, and it has a unique self-adjoint extension to $L^2(\Gamma\backslash G, \mu)$; the similar conclusion holds for Δ and $L^2(\Gamma\backslash G/K, \mu')$ (see [**Ch**]). Since $\Gamma\backslash G/K$ is compact, there is a family $\{\phi_i\}_{i=0}^\infty$ of countably many analytic functions over $\Gamma\backslash G/K$ such that they are eigenfunctions of Δ: $\Delta\phi_i = \lambda_i \cdot \phi_i$, meanwhile they constitute an orthonormal basis of $L^2(\Gamma\backslash G/K)$.

Remark 1.2.4. *Here we summarize the process of choosing various measures such that the quotient integral formulas and the lemma hold. First we fix three measures: the* *point-counting measure on the lattice* Γ*, the Haar measure* dk *on* K *such that* $\mathrm{vol}(K) = 1$ *and the Haar measure* dg *on* G*, then we get a right* G*-invariant Radon measure on* $\Gamma\backslash G$ *and a left* G*-invariant Radon measure on* G/K *which lead to the* μ' *on* $\Gamma\backslash G/K$*. Later we shall use the Haar measures* da *on* A*,* dn *on* N *and* dk *on* K *to give* dg*.*

In view of the above lemma, one has:

$$L^2(\Gamma\backslash G/K) \cong L^2(\Gamma\backslash G)^K,$$

the subset of elements in $L^2(\Gamma\backslash G)$ fixed by K under the action R. When Γ is uniform, the representation R can be decomposed into irreducible classes (see Theorem 9.2.2 of [**DE**]):

$$R \cong \bigoplus_{\pi\in\widehat{G}} N(\pi)\,\pi \tag{1.2}$$

where $\widehat{G}$ denotes the unitary dual of G, i.e., the set of equivalent classes of unitary irreducible representations of G, $N_\Gamma(\pi)$ denotes the multiplicity of π which is always a finite number, i.e., each π occurs (as isomorphic copies) finitely many times in R. Hence

$$L^2(\Gamma\backslash G/K) \cong \bigoplus_{\pi\in\widehat{G}^{\,K}} N_\Gamma(\pi)\,V_\pi^K \tag{1.3}$$

where $\widehat{G}^{\,K}$ means the subset of $\widehat{G}$ whose elements π's satisfy the condition $V_\pi^K \neq \{0\}$. Here we use V_π to denote the representation space of π. Such π's are called *spherical representations*. Let ρ be the half sum of positive roots and $M = C_K(A)$, the centralizer of A in K. By the subrepresentation theorem (see Theorem 8.37, Ch. 8 of [**Kn**]), any $\pi \in \hat{G}$ can be realized as a subrepresentation of some induced representation $\mathrm{Ind}_{MAN}^G(\sigma\otimes e^\nu\otimes\mathbf{1})$ where $\nu\in\mathfrak{a}_{\mathbb{C}}^*$ and σ is some irreducible unitary representation of M. Let $L^2(K, V_\sigma)$ be the collection of V_σ-valued L^2-functions on K. Recall that

$$\mathrm{Ind}_{MAN}^G(\sigma\otimes e^\nu\otimes\mathbf{1}) = \left\{ h : G\to V_\sigma \;\middle|\; \begin{array}{c} h(gman) = e^{-(\nu+\rho)\log a}\sigma(m)^{-1}h(g) \text{ for} \\ man\in MAN,\ g\in G;\ h|_K \in L^2(K, V_\sigma) \end{array} \right\}$$

endowed with the left regular action L of G:

$$(L(g)h)(x) = h\left(g^{-1}x\right).$$

When π is spherical, σ is trivial. The reason is as follows. By restriction to K, we have a natural isomorphisim: $\mathrm{Ind}_{MAN}^G(\sigma\otimes e^\nu\otimes\mathbf{1})|_K \cong \mathrm{Ind}_M^K(\sigma)$. The Frobenius Reciprocity Theorem gives $\left[\mathrm{Ind}_{MAN}^G(\sigma\otimes e^\nu\otimes\mathbf{1})|_K : \tau\right] = \left[\mathrm{Ind}_M^K(\sigma) : \tau\right] = [\tau|_M : \sigma]$ for any unitary irreducible representation τ of K. Let τ be trivial, then $\left[\mathrm{Ind}_{MAN}^G(\sigma\otimes e^\nu\otimes\mathbf{1})|_K : \mathrm{triv}\right] \geqslant 1$ since π is spherical. Thus we have $[\mathrm{triv}|_M : \sigma] \geqslant 1$ which immediately implies that σ is trivial.

For $\phi \in L^2(\Gamma\backslash G/K)$, denote by $\widetilde{\phi}$ the lift of ϕ to $\Gamma\backslash G$. Let $V_{\widetilde{\phi}}$ be the closed subspace in $L^2(\Gamma\backslash G)$ generated by $\widetilde{\phi}$ under the right regular action R of G. Then $V_{\widetilde{\phi}}$ is a representation space of G with the action R. Let $V(\lambda) = \bigoplus V_{\widetilde{\phi}}$ where ϕ runs through an orthonormal basis of $C^\infty(\Gamma\backslash G/K)_\lambda$, the space of smooth functions over $\Gamma\backslash G/K$ with Laplace eigenvalue λ. Clearly $V(\lambda)$ is a representation space of G as well. The decomposition (1.2) implies that, as a subrepresentation of $(L^2(\Gamma\backslash G), R)$, $V(\lambda)$ is decomposed into irreducibles:

$$V(\lambda) \cong \bigoplus_j m_j V_j(\lambda)$$

where $V_j(\lambda)$'s are among the irreducible unitary representation classes of G. For any representation (π, V_π) of G, let V_π^∞ denote the subset of smooth functions in V_π. By (1.3), one has:

$$C^\infty(\Gamma\backslash G/K)_\lambda \cong V_\lambda^{\infty, K} \cong \bigoplus_j m_j V_j(\lambda)^{\infty, K}.$$

The Duality Theorem in [**GS**] says that, each class $V_j(\lambda)$ occurs with multiplicity $m_j = \dim C^\infty(\Gamma\backslash G/K)_\lambda$. This implies that $V_j(\lambda)^{\infty, K}$ can not be $\{0\}$ for all i. Assume that $V_{j_0}(\lambda)^{\infty, K} \neq \{0\}$, then $\dim\left(m_{j_0} V_{j_0}(\lambda)^{\infty, K}\right) \geqslant m_{j_0} = \dim C^\infty(\Gamma\backslash G/K)_\lambda$. Hence only $V_{j_0}(\lambda)^{\infty, K}$ occurs in the decomposition of $V(\lambda)^{\infty, K}$:

$$V(\lambda)^{\infty, K} \cong m_{j_0} V_{j_0}(\lambda)^{\infty, K}.$$

Moreover $\dim V_{j_0}(\lambda)^{\infty, K} = 1$, i.e., $V_{j_0}(\lambda)$ is an irreducible unitary spherical representation of G. From now on till the end of this thesis, we shall always focus on the Lorentz group $G = SO_0(1, d)$. Notations are consistent with before. Irreducible unitary spherical representations of G are realized as induced representations (see [**Do**] or [**Th**]):

$$V_{j_0}(\lambda) \cong \mathrm{Ind}_{MAN}^G(\mathbf{1} \otimes e^\nu \otimes \mathbf{1}) \tag{1.4}$$

for some $\nu \in \mathfrak{a}_\mathbb{C}^*$. We use $I(\nu)$ to denote the subset of smooth elements in $\mathrm{Ind}_{MAN}^G(\mathbf{1} \otimes e^\nu \otimes \mathbf{1})$, and V_ν to denote the subset of smooth elements in $V_{j_0}(\lambda)$. Both I_ν and V_ν are representation spaces of G.

Let U be a compact neighborhood of the unity e in G, f be a continuous function on G. Define

$$f_U : G \to \mathbb{R}_{\geqslant 0}, \quad g \mapsto \sup_{x, y \in U} \left|f(xgy)\right|.$$

We say f is *uniformly integrable* if there exists some U such that f_U lies in $L^1(G)$. Let $C_{\mathrm{unif}}(G)$ be the set of all continuous uniformly integrable functions over G, then $C_{\mathrm{unif}}(G)$ is a convolution algebra. Note that $C_{\mathrm{unif}}(G) \subset L^1(G)$ since $\left|f\right| \leqslant f_U$. Let $f \in C_{\mathrm{unif}}(G)$, define

$$(R(f)\phi)(x) = \int_G f(g) R(g)\phi(x) dg$$

for $\phi \in L^2(\Gamma\backslash G)$. Then $R(f)$ is an integral operator by the following lemma:

Lemma 1.2.5.

$$(R(f)\phi)(x) = \int\limits_{\Gamma\backslash G} K_f(x,\, y)\phi(y)d\mu(y),$$

where $K_f(x,\, y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$ *is continuous on* $\Gamma\backslash G \times \Gamma\backslash G$.

For details about $C_{\mathrm{unif}}(G)$ and the proof of this lemma, see Sect. 9.2 of [**DE**]. The assumption in the reference, that H is uniform, is necessary for the decomposition (1.2), but not for this lemma.

Let f be a bi-K-invariant function in $C_{\mathrm{unif}}(G)$. Then $R(f)$ acts on $V_\lambda^K \subset L^2(\Gamma\backslash G)^K$ with the integral kernel K_f since $R(f)\psi$ is still K-invariant for any $\psi \in V_\lambda^K$. The space $I(\nu)^K$ is one-dimensional: any K-fixed function in $I(\nu)$ is determined by its values at the points in $P = MAN$ thanks to the Langlands decomposition $G = KMAN$ and the the transformation law in $I(\nu)$. Consequently there exists a scalar $h_f(\lambda)$ such that $R(f)\psi = h_f(\lambda)\psi$. In view of the bi-$K$-invariance of f and the definition of K_f, we may regard $K_f(x,\, y)$ as a function over $\Gamma\backslash G/K \times \Gamma\backslash G/K$. By Lemma 1.2.3, the action of $R(f)$ over $\psi \in V_\lambda^K$ is identified with an integral operator (denoted by $R'(f)$) acting on $\hat{\psi}$ where $\hat{\psi}$ means the restriction of ψ over $\Gamma\backslash G/K$:

$$\begin{aligned}(R'(f)\hat{\psi})(x) := (R(f)\psi)(x) &= \int\limits_{\Gamma\backslash G} K_f(x,\, y)\psi(y)d\mu(y)\\ &= \int\limits_{\Gamma\backslash G/K}\int\limits_{K} K_f(x,\, yk)\psi(yk)dkd\mu'(y)\\ &= \int\limits_{\Gamma\backslash G/K} K_f(x,\, y)\hat{\psi}(y)d\mu'(y)\end{aligned}$$

The last step follows from the K-invariance of ψ and we regard K_f as a function over $\Gamma\backslash G/K$ in this step. Thus the kernel of $R'(f)$ is still K_f. In this way, we get: $R'(f)\phi = h_f(\lambda)\phi$ for any $\phi \in C^\infty(\Gamma\backslash G/K)_\lambda$. Likewise, $L(f)$ acts on $I(\nu)$:

$$(L(f)h)(x) = \int_G f(g)L(g)h(x)dg, \quad h \in I(\nu)$$

with integral kernel K_f and $L(f)\eta = h_f(\lambda)\eta$ for any nontrivial element η in $I(\nu)^K$.

To compute $h_f(\nu)$, we just pick a nontrivial element in $I(\nu)$ and apply it to $L(f)$. In what follows, the function η defined over G such that $\eta(kman) = e^{-(\nu+\rho)\log a}$ is a natural choice. Since $\eta(1) = 1$, it follows that

$$(L(f)\eta)(1) = h_f(\nu)\,\eta(1) = h_f(\nu).$$

By definition,

$$\begin{aligned}
(L(f)\eta)(1) &= \int_G f(g)\eta\left(g^{-1}\right) dg \\
&\overset{(a)}{=} \int_G f\left(g^{-1}\right)\eta(g)\, dg \\
&\overset{(b)}{=} \int_N \int_A \int_K f\left(n^{-1}a^{-1}k^{-1}\right)\eta(kan)e^{2\rho(\log a)} dk da dn \\
&\overset{(c)}{=} \int_N \int_A f\left(n^{-1}a^{-1}\right) e^{-(\nu+\rho)\log a} e^{2\rho(\log a)} da dn \\
&= \int_N \int_A f\left(n^{-1}a^{-1}\right) e^{-(\nu-\rho)\log a} da dn \qquad (1.5)
\end{aligned}$$

We have made the variable exchange $g \to g^{-1}$ in (a). Note that $dg = d\left(g^{-1}\right)$ since G is semisimple. For (b), we use an integral formula for functions on G with the variable written in the KAN-order (see Proposition 5.1, Ch. I of [**He**]). For (c), note that f is bi-K-invariant and the measure dk on K has been normalized such that $\mathrm{vol}(K) = 1$. Now we choose the Haar measures on A and N. Let $a = e^X$, $n = e^Y$ for $X \in \mathfrak{a}$, $Y \in \mathfrak{n}$. Since A is abelian, $da := dX$ is a Haar measure on A, where dX is a Lebesgue measure on the Euclidean space $\mathfrak{a}$.

Lemma 1.2.6. *Let dY be a Lebesgue measure on the Euclidean space $\mathfrak{n}$, then the measure dn on N such that*

$$\int_N f(n)dn = \int_{\mathfrak{n}} f(\exp Y)dY, \quad \forall f \in L^1(N)$$

is a Haar measure on N.

Proof. For nilpotent groups, the push-forwards of the Lebesgue measures on their Lie algebras are just Haar measures on them. See Theorem 2.1 of [**CG**] for this fact. The formula in the lemma reflects the nature of the measure obtained in this way. □

With the above measures da and dn, the formula (1.5) implies

$$(L(f)\eta)(1) = \int_{Y\in\mathfrak{n}} \int_{X\in\mathfrak{a}} f\left(e^{-Y}\cdot e^{-X}\right) e^{-\nu(X)+\rho(X)} dX dY.$$

Hence

$$h_f(\nu) = \int_{Y\in\mathfrak{n}} \int_{X\in\mathfrak{a}} f\left(e^{-Y}\cdot e^{-X}\right) e^{-\nu(X)+\rho(X)} dX dY. \qquad (1.6)$$

Remark 1.2.7. *One can also use Harish-Chandra's theory on spherical functions* [**HC**] *to describe* h_f*. This is roughly the idea of A. Selberg in his seminal paper* [**Se**].

For $\phi_i \in \{\phi_i\}_{i=0}^{\infty}$ with Laplace eigenvalue λ_i, let $\nu_i \in \mathfrak{a}_{\mathbb{C}}^*$ be such that $V_{j_0}(\lambda_i) \cong I(\nu_i)$. From now on, we shall also use $h_f(\lambda_i)$ instead of $h_f(\nu_i)$. If we choose a bi-K-invariant *test function* $f \in C_{\text{unif}}(G)$ such that the series

$$k_f(z, w) := \sum_{i=0}^{\infty} h_f(\lambda_i)\phi_i(z)\overline{\phi_i(w)}, \quad z, w \in \Gamma\backslash G/K$$

converges locally uniformly everywhere, then

Proposition 1.2.8. K_f *being viewed as a function over* $\Gamma\backslash G/K$*, we have:* $K_f = k_f$.

Proof. We already know that $R'(f)$ is an integral operator with continuous integral kernel K_f. Meanwhile $R'(f)\phi_i = h_f(\lambda_i)\phi_i$. Define

$$T_k : L^2(\Gamma\backslash G/K) \to L^2(\Gamma\backslash G/K), \quad \phi \mapsto \int\limits_{\Gamma\backslash G/K} k_f(z, w)\phi(w)d\mu'(w).$$

Then, by the definition of k_f and the assumption and k_f is locally uniformly convergent, T_k is an integral operator such that $T_k(\phi_i) = h_f(\lambda_i)\,\phi_i$ $(i \geqslant 0)$ as ϕ_i's are orthonormal to each other. So T_k and $R'(f)$ are identical to each other as operators and their integral kernels are equal to each other except a possible subset of measure zero. The locally uniform convergence of k_f implies that k_f is a continuous function as all ϕ_i's are analytic over $\Gamma\backslash G/K$. Hence $K_f = k_f$. □

Remark 1.2.9. *In literature,* K_f *is called "automorphic kernel". Later we shall choose the test function* f *to be of the form:* $f(g) = \Phi_\mu\big(\text{dist}_{G/K}(e \cdot o, g \cdot o)\big)$ *where* Φ_μ *is a smooth function (with* μ *as a parameter) on* $\mathbb{R}_{\geqslant 0}$ *with rapid decay at* ∞ *but not compactly supported. The absolute and locally uniform convergence of* k_f *needs to be checked when* f *is chosen.*

1.3 Two decompositions

The group $G = SO_0(1, d)$ is the connected component (containing 1) of the subset of $g \in SL_{n+1}(\mathbb{R})$ such that $gJg^T = J$ where

$$J = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

So its Lie algebra $\mathfrak{g}$ is the set of all matrices $X \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that $JXJ = -X^T$. Every $X \in \mathfrak{g}$ can be written as

$$X = \begin{pmatrix} 0 & a^T \\ a & B \end{pmatrix}$$

where $a \in \mathbb{R}^d$ and $B^T = -B \in \mathrm{Mat}_d(\mathbb{R})$. The group K, as a subgroup of G via the map $k \mapsto \begin{pmatrix} 1 & \\ & k \end{pmatrix}$, is the set of fixed points of the Cartan involution $\Theta(g) = g^{-T}$ on G. Define

$$E = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \in \mathrm{Mat}_{d+1}(\mathbb{R})$$

and

$$E_i = \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{d+1}(\mathbb{R})$$

for $1 \leqslant i \leqslant d-1$. Here ± 1 appear in the (i+2)-*th* row or column in E_i. Write $u = (u_1, u_2, \ldots, u_{d-1}) \in \mathbb{R}^{d-1}$ for short. Define

$$\omega_r^+ = \exp(rE), \quad \theta_u = \exp\left(\sum_{i=1}^{d-1} u_i E_i\right)$$

for $(r, u) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Then it is easy to verify that

$$\omega_r^+ = \begin{pmatrix} \cosh r & \sinh r & 0 & 0 & \cdots & 0 \\ \sinh r & \cosh r & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$\theta_u = \begin{pmatrix} 1 + \frac{|u|^2}{2} & -\frac{|u|^2}{2} & u_1 & u_2 & \cdots & u_{d-1} \\ \frac{|u|^2}{2} & 1 - \frac{|u|^2}{2} & u_1 & u_2 & \cdots & u_{d-1} \\ u_1 & -u_1 & 1 & 0 & \cdots & 0 \\ u_2 & -u_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{d-1} & -u_{d-1} & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where $|u|^2 = \sum_{i=1}^{d-1} u_i^2$. With these terms we have the following descriptions for Iwasawa and KAK decompositions of $G = SO_0(1, d)$:

Theorem 1.3.1. (Iwasawa Decomposition) *Any $g \in G$ can be written as $g = \theta_u \cdot \omega_r^+ \cdot \rho$ for some unique $\rho \in K$ and $(u, r) \in \mathbb{R}^{d-1} \times \mathbb{R}$.*

Theorem 1.3.2. (KAK Decomposition) *Any $g \in G$ can be written as $g = \rho_1 \cdot \omega_r^+ \cdot \rho_2$ for some $\rho_1, \rho_2 \in K$ and unique $r \in \mathbb{R}_{\geqslant 0}$.*

For proofs of these two theorems, see I.7 of [**FJ**]. The name "Iwasawa Decomposition" in Theorem 1.3.1 is valid in view of the fact $[E, E_i] = E_i$ for any $1 \leqslant i \leqslant d-1$. That is to say, if we define $\mathfrak{a} = \{tE \,|\, t \in \mathbb{R}\}$, $\mathfrak{n} = \left\{\sum_{i=1}^{d-1} u_i E_i \,|\, u_i \in \mathbb{R}\right\}$, then the linear functional $\alpha_0 \in \mathfrak{a}^*$ such that $\alpha_0(E) = 1$ is just a positive restricted root such that $\mathfrak{n} = \mathfrak{g}_{\alpha_0}$. Let $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}$. Note that G is of rank one, i.e., the maximal split torus A is of dimension one, so there is only one positive root. The uniqueness of $r \geqslant 0$ in Theorem 1.3.2 is clear since $r \geqslant 0$ uniquely determines an element rE in the closed positive Weyl chamber $\overline{\mathfrak{a}^+}$. It is easy to see that both A and N are abelian groups. Moreover N is unipotent: $(n-1)^3 = 0$ for any $n \in N$. The groups A, N, $NA = AN$ are all simply connected closed subgroups of G.

We have the following property of the Killing form

Lemma 1.3.3. *$B(E, E) = 2(d-1)$, $B(E, E_i) = 0$, $B(E_i, E_j) = 0$ for any $1 \leqslant i, j \leqslant d-1$.*

Proof. This follows easily from a combination of Proposition I.3.1 and formula I.13 of [**FJ**]. Note that here E is the E_1 there, E_i is the $\widetilde{E}_{i+1}$ there. □

By Theorem 1.3.1, the subgroup NA is topologically isomorphic to $\mathcal{H}^d$. The isomorphisim is realized by the map

$$S : N \times A \to \mathcal{H}^d, \quad (n, a) \mapsto S(na) = na \cdot \xi^0.$$

Any element $p \in NA$ is uniquely determined by some parameter $(u, r) \in \mathbb{R}^{d-1} \times \mathbb{R}$:

$$T : \mathbb{R}^{d-1} \times \mathbb{R} \to NA, \ (u, r) \mapsto p = \theta_u \, \omega_r^+.$$

There is a one-to-one correspondence between $\mathbb{R}^{d-1} \times \mathbb{R}$ and $\mathbb{R}^{d-1} \times \mathbb{R}_{>0}$ via:

$$H : \mathbb{R}^{d-1} \times \mathbb{R}_{>0} \to \mathbb{R}^{d-1} \times \mathbb{R}, \quad (u, r) \mapsto (u, \log r).$$

So we can and will use $\mathbb{R}^{d-1} \times \mathbb{R}_{>0}$ to characterize $\mathcal{H}^d$. The model $\mathcal{P}^d = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$ is called *Poincaré upper half space*. Those elements $(u, r) \in \mathcal{P}^d$, when used to represent the points on $\mathcal{H}^d$ via the map $\sigma = S \circ T \circ H$, are called *Poincaré coordinates*. One can use Poincaré coordinates to define a *hyperbolic distance* over $\mathcal{H}^d$:

$$\mathrm{dist}_{\mathcal{H}^d}(a, b) = \operatorname{arccosh}\left[\frac{|u-v|^2 + t^2 + s^2}{2ts}\right] \tag{1.7}$$

for $a = \sigma(x)$, $b = \sigma(y) \in \mathcal{H}^d$ where $x = (u, t)$, $y = (v, s) \in \mathcal{P}^d$. Here we require the value of the function arccosh to be non-negative. One should be warned that the Poincaré model is different from the usual upper half space model when they are used to parameterize $\mathcal{H}^d$. It is convenient to denote $\omega^+_{\log r}$ by ω_r ($r > 0$), or equivalently $\omega_{e^r} = \omega^+_r$ ($r \in \mathbb{R}$). Be careful that ω^+ is additive while ω is multiplicative with respect to their variables: $\omega^+_{\ell_1+\ell_2} = \omega^+_{\ell_1}\omega^+_{\ell_2}$, $\omega_{r_1 r_2} = \omega_{r_1}\omega_{r_2}$. For the KAK-decomposition of g: $g = \rho_1 \cdot \omega^+_r \cdot \rho_2$ where $r \geqslant 0$, define $\log \|g\|$ to be r. Then $\log \|g^{-1}\| = \log \|g\|$ and

$$\mathrm{dist}_{\mathcal{H}^d}(a, b) = \log \left\|P(x)^{-1} P(y)\right\|$$

where $P = T \circ H$. For more details, see Proposition I.7.3 and I.7.5 of [**FJ**]. As a consequence, $\log \|g\|$ defines a *hyperbolic metric* on $\mathcal{H}^d$. It is well-known that (G, K) is a symmetric pair of noncompact type, so $B|_{\mathfrak{p}}$ induces a Riemannian metric on the manifold G/K. Under this metric, the distance between the two points $P(x) \cdot o$, $P(y) \cdot o$ on G/K is

$$\mathrm{dist}_{G/K}(P(x) \cdot o,\, P(y) \cdot o) = \left\|\log a \left(P(x)^{-1} P(y)\right)\right\| = \|E\|\, r$$

where $\|E\| = \sqrt{B(E, E)}$, $P(x)^{-1} P(y) = \rho_1 \cdot \omega^+_r \cdot \rho_2$ for some $\rho_1, \rho_2 \in K$, $r \geqslant 0$. As a result we have the following connection between the two distances on G/K and $\mathcal{H}^d$:

$$\mathrm{dist}_{G/K}(S^{-1}(a) \cdot o,\, S^{-1}(b) \cdot o) = \|E\| \,\mathrm{dist}_{\mathcal{H}^d}(a, b). \tag{1.8}$$

1.4 A relative trace formula

As G is of split rank one, we can identify $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$ as follows:

$$\tau : \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}, \quad \alpha \mapsto (d-1)\alpha(E).$$

Then $\tau(\rho) = \frac{d-1}{2}\,\alpha_0(E) = \frac{d-1}{2}$. With such identification, it is known that $\mathrm{Ind}^G_{MAN}(\mathbf{1} \otimes e^\nu \otimes \mathbf{1})$ is irreducible and unitarizable if and only if $\tau(\nu)$ lies in $i\,\mathbb{R}$ (unitary principal series) or $(-\tau(\rho), \tau(\rho))$ (complementary series) (see [**Do**] or [**Th**]). Moreover $\mathrm{Ind}^G_{MAN}(\mathbf{1} \otimes e^\nu \otimes \mathbf{1}) \cong \mathrm{Ind}^G_{MAN}(\mathbf{1} \otimes e^{-\nu} \otimes \mathbf{1})$. The Casmir operator Ω (see § 3, Ch. 8 of [**Kn**] for the definition) acts on $I(\nu)$ as a scalar (see Lemma 12.28 of [**Kn**])

$$\chi_\nu(\Omega) = \tau(\rho)^2 - \tau(\nu)^2$$

and this action is equivalent to the action of the Laplacian $\square$. Hence the Laplace eigenvalue of ϕ_i is

$$\lambda_i = \chi_{\nu_i}(\Omega) = \tau(\rho)^2 - \tau(\nu_i)^2.$$

To simplify notations, we shall use ρ and ν_i to denote $\tau(\rho)$ and $\tau(\nu_i)$ respectively, and even call them roots. Note that there might be other Laplace eigenfunctions (over $\Gamma\backslash G/K$) which share the same eigenvalue but are linearly independent from ϕ_i.

Let $a = \omega_x^+ = \exp(xE)$, $n = \theta_u = \exp\left(\sum_{i=1}^{d-1} u_i E_i\right)$ $(x,\ u_i \in \mathbb{R})$. Define $da = dx$, $dn = du = du_1 \cdots du_{d-1}$ which are Lebesgue measures of the Euclidean spaces $\mathfrak{a}$ and $\mathfrak{n}$ respectively. By the formula (1.6), we have

$$h_f(\lambda_i) = \int\limits_{u\in\mathbb{R}^{d-1}} \int\limits_{x\in\mathbb{R}} f\left(\theta_{-u}\omega_{-x}^+\right) e^{-x\cdot\nu_i + x\cdot\rho} dx du. \tag{1.9}$$

For a given eigenvalue λ_i, the number ν_i (thus ν_i) is unique up to ± 1, while $I(\nu_i) \cong I(-\nu_i)$, so $h_f(\nu_i) = h_f(-\nu_i)$. Hence it is reasonable to parameterize h_f by λ_i.

The ensuing formula follows from the equality of two expressions of the automorphic kernel K_f, called "*pre-trace formula*":

$$\sum_{\gamma\in\Gamma} f(z^{-1}\gamma w) = \sum_{i=0}^{\infty} h_f(\lambda_i)\phi_i(z)\overline{\phi_i(w)}, \quad z,\ w \in \Gamma\backslash G/K. \tag{1.10}$$

As remarked before, the left hand side of the above formula is well-defined over $\Gamma\backslash G/K \times \Gamma\backslash G/K$ with respect to the variable $(z,\ w)$. We integrate K_f over two closed geodesics C_1 and C_2 on $\Gamma\backslash G/K$. The absolute and locally uniform convergence of the series k_f, which is necessary to justify Proposition 1.2.8, will be checked later (see Sect. 2.4).

The integration of the right hand side of (1.10), called "spectral side", is

$$\sum_{i=0}^{\infty} h_f(\lambda_i) \int_{z\in C_1} \phi_i(z) d\eta'(z) \cdot \overline{\int_{w\in C_2} \phi_i(w)\, d\eta'(w)}.$$

Recall that η' denotes the Radon metric on $\Gamma\backslash G/K$ induced from $B|_{\mathfrak{p}}$. The integration of the left hand side of (1.10), called "geometric side", is

$$\int_{w\in C_2} \int_{z\in C_1} \sum_{\gamma\in\Gamma} f\left(z^{-1}\gamma w\right) d\eta'(z) d\eta'(w).$$

Denote by $P_C(\phi_i)$ the period integral $\int_C \phi_i(z) d\eta'(z)$. We have:

$$\int_{w\in C_2} \int_{z\in C_1} \sum_{\gamma\in\Gamma} f(z^{-1}\gamma w) d\eta'(z) d\eta'(w) = \sum_{i=0}^{\infty} h(\lambda_i)\, P_{C_1}(\phi_i)\, \overline{P_{C_2}(\phi_i)}. \tag{1.11}$$

This is the *relative trace formula* to be used for compact hyperbolic manifolds. For both (1.10) and (1.11) to hold, the test function f should satisfy: (1) $f \in C_{\text{unif}}(G)$; (2) f is bi-K-invariant; (3) k_f is locally uniformly convergent.

In Chapter 4 we shall use another model of the unitary spherical irreducible representation of G, namely, the *noncompact picture* $J(\nu)$. Here, for the convenience of the reader, we include some details on this picture, which is merely a copy of Sect. 2.3 of [**MØ**]. Remember that, in the Bruhat decomposition

$$G = MAN \cup \overline{N}MAN,$$

$\overline{N}MAN$ is open and dense in G, so $f \in I(\nu)$ is completely decided by its restriction to $\overline{N}$ in view of the definition of $I(\nu)$. For $u = (u_1, \cdots, u_{d-1}) \in \mathbb{R}^{n-1} \cong \overline{\mathfrak{n}} = \mathrm{Lie}(\overline{N})$, let $\overline{n}_u = \exp\left(\sum_{i=1}^{d-1} u_i E_i^T\right)$ where $E_i^T \in \overline{\mathfrak{n}}$ denotes the transpose of E_i. For every $f \in I(\nu)$, define $\mathcal{R}f \in C^\infty(\mathbb{R}^{n-1})$ by

$$(\mathcal{R}f)(u) := f(u) = f(\overline{n}_u), \quad \overline{n}_u \in \mathbb{R}^{n-1}.$$

Denote by $J(\nu)$ the image of $I(\nu)$ under the map $\mathcal{R}$. Then $C_c^\infty(\mathbb{R}^{d-1}) \subset J(\nu)$. The action of G on $J(\nu)$ is given by

$$g.(\mathcal{R}(f)) = \mathcal{R}(L(g).f).$$

For $\nu \in i\,\mathbb{R}$, the case with which we shall concern ourselves later, the invariant Hermitian form on $J(\nu)$ is

$$\|h\|_\nu^2 := \frac{\Gamma(2\rho)}{\pi^\rho \Gamma(\rho)} \int_{\mathbb{R}^{d-1}} |h(u)|^2 du, \quad h \in J(\nu).$$

1.5 The primitive closed geodesics

In this section we consider the case $C_1 = C_2$ over $\Gamma\backslash G/K$ and denote this closed geodesic by C. By "geodesic" over a Riemannian manifold M, we mean a smooth map $c : \mathbb{R} \to M$ of constant speed $\|\dot{c}(t)\|$ for all $t \in \mathbb{R}$ (here $\|\ \|$ means the product over the tangent space of M which defines the Riemann metric), such that the following condition hold: $\nabla_{\dot{c}}(\dot{c}) = 0$ for the metric connection ∇ over M, or equivalently, c is locally distance minimizing, i.e., for any $t_0 \in \mathbb{R}$ there exists $\epsilon > 0$ such that c is the shortest curve connecting $c(s)$ and $c(t)$ for all $s, t \in (t_0 - \epsilon, t_0 + \epsilon)$. A *closed* geodesic is a pair (c, t) where c is a geodesic and t is a positive number, such that $c(x+t) = c(x)$ for all $x \in \mathbb{R}$. When t is minimal and positive, the closed geodesic (c, h) is said to be *primitive*. Note that any closed geodesic is a unique power of the primitive one: $(c, t) = (c, h)^n := (c, nh)$ where $n \in \mathbb{Z}$ is unique and (c, h) is primitive. We shall always focus on closed primitive geodesics. This means that only the parameter t in the period domain, e.g., the segment $[0, h]$, is under consideration. It is possible that the geodesic is not *simple*, i.e., c might be not injective over $[0, h)$. For a geodesic $c(t)$, we frequently mix the map with its image. Likewise, for a closed primitive geodesic, we just identify the map $c(t)$ with its image $\{c(t) \mid t \in [0, h]\}$. In this way, any geodesic D over G/K is of the form:

$$D = \left\{c(t) = ge^{tX} \cdot o \,\middle|\, X \in \mathfrak{p},\, t \in \mathbb{R}\right\}$$

for some $g \in G$. Remember that we have identified $\mathfrak{p}$ with the tangent space $T_{e\cdot o}(G/K)$. Since G/K is homogeneous and the metric η is left G-invariant, the left action of g^{-1}

translates D to be the geodesic D' over G/K which originates from $e \cdot o$ with the direction $X \in \mathfrak{p}$. By the following fact (see Proposition 5.13 of [**Kn**]):

$$\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}, \tag{1.12}$$

there exist $Y \in \mathfrak{a}$ and $k \in K$ such that $X = kYk^{-1}$. Hence the left action of kg^{-1} translates D to a new geodesic $D'' = kg^{-1}D$ over G/K which originates from $e \cdot o$ with direction $Y \in \mathfrak{a}$. A normalization on the parameter t allows us to make the assumption that $Y = E$, i.e., $D'' = A \cdot o := \left\{ e^{tE} \cdot o \,\middle|\, t \in \mathbb{R} \right\}$. Denote by Γ' the lattice $kg^{-1}\Gamma(kg^{-1})^{-1}$. If D is among the fibres of C according to the principal bundle $G/K \to \Gamma\backslash G/K$, then there is a closed geodesic C' over the new quotient $\Gamma'\backslash G/K$ whose fibre over G/K is $A \cdot o$. From now on, we call $A \cdot o$ the *regular geodesic* over G/K and C' the *closed regular geodesic* over $\Gamma'\backslash G/K$. By abuse of notation, we use $kg^{-1}C$ to denote C' although G does not act on $\Gamma\backslash G/K$.

As remarked before, we shall choose a test function f with the distance $\mathrm{dist}_{G/K}(e \cdot o,\, g \cdot o)$ as its variable. Let η'' denote the metric on $\Gamma'\backslash G/K$ which is induced from the left G-invariant metric η on G/K, we have: $\eta'(gk^{-1}z) = \eta''(z)$. The following two simple observations

$$\mathrm{dist}_{G/K}(\gamma g k^{-1} z,\, g k^{-1} w) = \mathrm{dist}_{G/K}(k g^{-1} \gamma g k^{-1} z,\, w), \quad z,\, w \in C'$$

and

$$\int_C \phi_i(w) d\eta'(w) = \int_{C'} \phi_i(gk^{-1}z) d\eta'(gk^{-1}z) = \int_{C'} \phi_i(gk^{-1}z) d\eta''(z)$$

show respectively that the automorphic kernel K_f, as a sum over the lattice Γ, is reduced (or, equal) to a sum over the new lattice Γ' and the periods of ϕ_i's along the geodesic C are reduced (or, equal) to the periods of $L_{kg^{-1}}(\phi_i)$'s along the new geodesic C'. Here L is the left regular action. The new family $\{L_{kg^{-1}}(\phi_i)\}_{i=0}^{\infty}$ constitutes an orthonormal basis of $L^2(\Gamma'\backslash G/K,\, \mu'')$ where μ'' is the Radon measure over $\Gamma'\backslash G/K$ satisfying Lemma 1.2.3. Thus it is reasonable for us to assume, for the rest of the paper, the existence of the closed regular geodesic (still denoted by C) on $\Gamma\backslash G/K$ and concentrate on such geodesic. One should distinguish $e \cdot o$ from $e^Z \cdot o$, the former being the initial point on G/K while the latter being the point on a geodesic with direction Z.

Let $\widetilde{C} \subset G/K$ be a lift of C according to the principal bundle $G/K \to \Gamma\backslash G/K$ and denote by $\mathrm{Stab}_\Gamma(\widetilde{C})$ the stabilizer of $\widetilde{C}$ in Γ:

$$\mathrm{Stab}_\Gamma(\widetilde{C}) = \left\{ \gamma \in \Gamma \,\middle|\, \gamma \widetilde{C} = \widetilde{C} \right\}.$$

For any continuous function ϕ over $\Gamma\backslash G/K$, the integration of ϕ over C is equal to the integration of its lift $\widetilde{\phi}$ (to G/K) over a fundamental domain C_0 of $\mathrm{Stab}_\Gamma(\widetilde{C})$ in $\widetilde{C}$. For this reason, we shall not distinguish C and C_0, as well as ϕ and $\widetilde{\phi}$. By assumption, we choose $\widetilde{C} = A \cdot o$, the regular geodesic over G/K. All other lifts of C are the translations

$\gamma\widetilde{C}$ on G/K where $\gamma \in \Gamma \setminus \mathrm{Stab}_\Gamma(\widetilde{C})$. As C is primitive and closed, there is a positive number T such that C_0 can be chosen to be

$$C_0 = \left\{ e^{tE} \cdot o \,\middle|\, 0 \leqslant t \leqslant T \right\}.$$

There exists $\gamma_0 \in \Gamma$ such that $\gamma_0 \cdot o = e^{TE} \cdot o$, so $\gamma_0 = e^{TE} k_0$ for some $k_0 \in K$. Let Γ_0 be the subgroup of Γ generated by γ_0: $\Gamma_0 = \langle \gamma_0 \rangle$.

Lemma 1.5.1. $C_0 \approx \Gamma_0 \backslash \widetilde{C}$*, or equivalently,* $\mathrm{Stab}_\Gamma(\widetilde{C}) = \Gamma_0$.

Here, by "$\approx$" we mean that $\Gamma_0 \backslash \widetilde{C}$ can be viewed as the fundamental domain C_0 of $\mathrm{Stab}_\Gamma(\widetilde{C})$ in $\widetilde{C}$.

Proof. Assume that $\gamma_0^2 \cdot o \notin \widetilde{C}$ and let $\eta \cdot o$ ($\eta \in \Gamma$) be the closest point on $\widetilde{C}$ which is Γ-equivalent to $\gamma_0 \cdot o$ and lies in the opposite direction to $e \cdot o$, i.e., the direction from $\gamma_0 \cdot o$ to $\eta \cdot o$ is compatible with the direction from $e \cdot o$ to $\gamma_0 \cdot o$. Then the segment D_1 (on $\widetilde{C}$) between $\gamma_0 \cdot o$ and $\eta \cdot o$ is isomorphic to C and the geodesic segment D_2 (over G/K) between $\gamma_0 \cdot o$ and $\gamma_0^2 \cdot o$ is also isomorphic to C since $D_2 = \gamma_0 D_0$ where $D_0 = \left\{ e^{tE} \cdot o \,\middle|\, 0 \leqslant t \leqslant T \right\}$. So there exists some $\delta \in \Gamma$ such that $\delta D_1 = D_2$, i.e., $\delta\gamma_0 \cdot o = \gamma_0 \cdot o$, $\delta\eta \cdot o = \gamma_0^2 \cdot o$ or $\delta\gamma_o \cdot o = \gamma_0^2 \cdot o$, $\delta\eta \cdot o = \gamma_0 \cdot o$. The former case implies that $\gamma_0^{-1}\delta\gamma_0$ lies in K, hence $\delta = 1$ (the intersection of Γ with any compact subgroup is trivial, otherwise there will be torsion element in Γ) and $\eta \cdot o = \gamma_0^2 \cdot o$ lies on $\widetilde{C}$, contrary to the hypothesis. By the similar reason, the latter case shows: $\gamma_0^{-2}\delta\gamma_0 = 1$, i.e., $\delta = \gamma_0$, which implies that $\delta\eta \cdot o = \gamma_0\eta \cdot o = \gamma_0 \cdot o$, so η lies in $K \cap \Gamma = \{1\}$, a contradiction. Thus $\gamma_0^2 \cdot o$ lies on $\widetilde{C}$. More generally, $\gamma_0^n \cdot o$ lies on $\widetilde{C}$ for all $n \in \mathbb{Z}$. These points are the Γ-periodic points on $\widetilde{C}$, so the lemma follows. □

We know that SO_d is isomorphic to K via the map $\mu : SO_d \xrightarrow{\cong} K$, $k \mapsto \mathrm{diag}(1, k)$. The group SO_{d-1} embeds into SO_d via the map $\nu : SO_{d-1} \hookrightarrow SO_d$, $k \mapsto \mathrm{diag}(1, k)$. Let M be the image of SO_{d-1} in K under the embedding $\mu \circ \nu$, i.e.,

$$M = \{\mathrm{diag}(1, 1, k) \,|\, k \in SO_{d-1}\} \subset K.$$

Note that M is just the centralizer of A in K.

Lemma 1.5.2. $k_0 \in M$.

Proof. As γ_0 preserves the geodesic $\widetilde{C}$ and acts on it in the same way as e^{TE} does, the element $k_0 = e^{-TE}\gamma_0 \in K$ fixes $\widetilde{C}$ pointwise, one has $k_0 \exp(X) \cdot o = \exp(k_0 X k_0^{-1}) \cdot o = \exp(X) \cdot o$. In view of Theorem 1.3.1 and (1.12), k_0 fixes $\mathfrak{a}$ pointwise, i.e., $k_0 X k_0^{-1} = X$ for any $X \in \mathfrak{a}$, so k_0 lies in M. □

An immediate consequence is:

Lemma 1.5.3. $AM \cap \Gamma = \Gamma_0$.

Proof. By Lemma 1.5.2, it is clear that Γ_0 lies in AM, thus in $AM \cap \Gamma$. Let $\gamma = ak \in AM \cap \Gamma$. The action of ak on $\widetilde{C}$ is equivalent to the action of a on $\widetilde{C}$ since A commutes with M, so $a = e^{nTE}$ for some $n \in \mathbb{Z}$. Then $\gamma_0^{-n}\gamma = k_0^{-n}k \in K \cap \Gamma = \{1\}$. This implies that $k = k_0^n$. So $\gamma = e^{nTE}k_0^n = \gamma_0^n \in \Gamma_0$. The proof is complete. □

To divide the summation over $\gamma \in \Gamma$ on the geometric side of the formula (1.11) into the summation over double coset classes in $\Gamma_0\backslash\Gamma/\Gamma_0$, we check the uniqueness of expressing elements in Γ through double cosets:

Proposition 1.5.4. *Let $\gamma \in \Gamma$ be such that $\widetilde{\gamma} \in \Gamma_0\backslash\Gamma/\Gamma_0 \smallsetminus \{\tilde{1}\}$, or equivalently $\gamma \notin \Gamma_0$. Then any element $\eta \in \Gamma$ in the same double coset class with γ can be written as $\eta = \gamma_1\gamma\gamma_2$ for unique γ_1, $\gamma_2 \in \Gamma_0$.*

For $g \in G$, let

$$\ell(g) = \inf\big\{\mathrm{dist}_{G/K}(gx,\, x) \,|\, x \in G/K\big\}.$$

Then g is called *hyperbolic* if $\ell(g) > 0$. Let

$$M(g) = \big\{x \in G/K \,|\, \mathrm{dist}_{G/K}(gx,\, x) = \ell(g)\big\}.$$

It is known from hyperbolic geometry that $M(g)$ is a geodesic and g translates along this geodesic.

Proof of the Proposition. Assume that $\gamma_1\gamma\gamma_2 = \gamma_3\gamma\gamma_4$ for some $\gamma \in \Gamma \smallsetminus \Gamma_0$ and $\gamma_i \in \Gamma_0$ $(1 \leqslant i \leqslant 4)$, then γ can be written as $\gamma = \gamma'\gamma\gamma''$ for γ', $\gamma'' \in \Gamma_0$. If we can show that $\gamma \in \Gamma_0$, then a contradiction arises and the proposition is proved.

Claim 1.5.5. *If $g \in G$ is hyperbolic, then hgh^{-1} is hyperbolic for any $h \in G$, moreover, $\ell(hgh^{-1}) = \ell(g)$ and*

$$M(hgh^{-1}) = hM(g).$$

Proof of the claim. The first two conclusions are clear in view of the G-invariance of the distance function. For the last conclusion, let $x \in M(hgh^{-1})$, then

$$\mathrm{dist}_{G/K}(hgh^{-1}x,\, x) = \mathrm{dist}_{G/K}(gh^{-1}x,\, h^{-1}x)$$

is minimal, which means that $h^{-1}x \in M(g)$, i.e., $x \in hM(g)$. Conversely, from $x \in hM(g)$, one easily gets $x \in M(hgh^{-1})$. □

Since $\gamma = \gamma'\gamma\gamma''$, one has $\gamma'' = \gamma^{-1}\gamma'^{-1}\gamma$. If $\gamma'' = 1$, then $\gamma' = 1$ and we are done. Now assume that $\gamma'' \neq 1$, then γ'' is hyperbolic. By the above claim,

$$\widetilde{C} = M(\gamma'') = M(\gamma^{-1}\gamma'^{-1}\gamma) = \gamma^{-1}M(\gamma') = \gamma^{-1}\widetilde{C}.$$

Therefore, $\gamma \in \Gamma_0$, a contradiction. □

Remark 1.5.6. *In the above proof, we do not assume Γ is uniform, so this proposition holds for non-uniform lattices as well.*

Remark 1.5.7. *In each nontrivial double coset class in $\Gamma_0\backslash\Gamma/\Gamma_0$, we choose one representative element γ and use it to achieve, in a unique way, all elements lying in this class (denoted by $\widetilde{\gamma}$) by two-sided multiplication of elements in Γ_0.*

Let Φ be a smooth function on $[0, +\infty)$. Define $f(g) = \Phi\big(\mathrm{dist}_{G/K}(e\cdot o,\, g\cdot o)\big)$. Then

$$K_f(z, w) = \sum_{\gamma\in\Gamma} f\left(z^{-1}\gamma w\right) = \sum_{\gamma\in\Gamma} \Phi\big(\mathrm{dist}_{G/K}(\gamma z,\, w)\big).$$

For simplicity, from now on we shall use dz, dw and $d(\ ,\ )$ to denote $d\eta'(z)$, $d\eta'(w)$ and $\mathrm{dist}_{G/K}(\ ,\ )$ respectively. By the above remark we have:

$$\begin{aligned}
\int_C\int_C \sum_{\gamma\in\Gamma} \Phi\big(d(\gamma z,\, w)\big) dz dw &= \int_C\int_C \sum_{\gamma_1,\,\gamma_2\in\Gamma_0}\ \sum_{\widetilde{\gamma}\in\Gamma_0\backslash\Gamma/\Gamma_0} \Phi\left(d(\gamma_2^{-1}\gamma\gamma_1 z,\, w)\right) dz dw \\
&= \int_C\int_C \sum_{\gamma\in\Gamma_0} \Phi\left(d(\gamma z,\, w)\right) dz dw \\
&\quad + \int_C\int_C \sum_{\gamma_1,\,\gamma_2\in\Gamma_0}\ \sum_{\widetilde{\gamma}\in\Gamma_0\backslash\Gamma/\Gamma_0\smallsetminus\{\tilde{1}\}} \Phi\left(d(\gamma\gamma_1 z,\, \gamma_2 w)\right) dz dw \\
&= \int_C\int_{\widetilde{C}} \Phi(d(z,\, w)) dz dw \\
&\quad + \int_{\widetilde{C}}\int_{\widetilde{C}} \sum_{\widetilde{\gamma}\in\Gamma_0\backslash\Gamma/\Gamma_0\smallsetminus\{\tilde{1}\}} \Phi\left(d(\gamma z,\, w)\right) dz dw
\end{aligned}$$

Let Σ_0 denote the term

$$\int_C\int_{\widetilde{C}} \Phi(d(z,\, w)) dz dw,$$

and Σ_1 denote $\sum_{\widetilde{\gamma}\neq\tilde{1}} I_\gamma$ where

$$I_\sigma = \int_{\widetilde{C}}\int_{\widetilde{C}} \Phi\left(d(\gamma z,\, w)\right) dz dw.$$

Remark 1.5.8. *A remarkable class of uniform lattices in the group $SO_0(1, d)$, for almost all d (namely except $d = 3$ and 7), arises from the totally real algebraic extensions of $\mathbb{Q}$. Lattices of this type are arithmetic and they even exhaust all unform arithmetic lattices (up to commensurability and conjugates) when d is an even integer. For more precise accounts, see 6.C of* [**Mo**]*. The lattice Γ is uniform if and only if each nontrivial element in Γ is semisimple, i.e., conjugate to a diagonal element within $GL_{d+1}(\mathbb{C})$. For this fact, see Theorem 9.21 of* [**Mo**].

Chapter 2

Periods along Closed Geodesics over Compact Hyperbolic Manifolds

In this chapter we apply the relative trace formula obtained in last chapter to get identities between the length of the closed geodesic and the periods along the geodesic. These two terms come from the geometric and spectral sides respectively. The main conclusions are placed at the end of each section.

2.1 Inserting a test function

In this section we shall choose a test function f for the application of the trace formula and give the very preliminary formula for h_f. A direct computation (or see Proposition I.4.2 of [**FJ**]) gives the following frequently used commutativity property on ω_r and θ_u:

$$\omega_r\,\theta_u = \theta_{ru}\,\omega_r$$

based on which, together with the formula (1.9), we get

$$\begin{aligned}
h(\lambda_i) &= \int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}} \Phi(d(\theta_{-u}\omega_{-x}\cdot o,\, e\cdot o))e^{-x\cdot\nu_i+x\cdot\rho}dxdu \\
&= \int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}} \Phi(d(e\cdot o,\, \omega_x\,\theta_u\cdot o))e^{-x\cdot\nu_i+x\cdot\rho}dxdu \\
&= \int\limits_{\mathcal{P}^d} \Phi(d(e\cdot o,\, \omega_r\theta_u\cdot o))r^{-\nu_i+\rho}\frac{dr}{r}du \\
&= \int\limits_{\mathcal{P}^d} \Phi(d(e\cdot o,\, \theta_{ru}\omega_r\cdot o))r^{C_i-1}drdu
\end{aligned}$$

where $(u, r) = (y, e^x) \in \mathcal{P}^d$, $C_i = -\nu_i + \rho$. Substituting r and u into the equations (1.7) and (1.8), we get

$$d(e \cdot o, \theta_{ru}\omega_r o) = \|E\| \left[\operatorname{arccosh}\left(\frac{|ru|^2 + 1 + r^2}{2r}\right)\right]$$

noting that $e = \theta_0 \omega_1$. For x, $\mu \geqslant 0$, define

$$\Phi_\mu(x) = \exp\left[-\mu \cdot \cosh\left(\frac{x}{\|E\|}\right)\right].$$

Here $\|E\|$ is explicitly known by Lemma 1.3.3. Originally we would like to insert the heat kernel (which is explicitly known, see [**GN**]), but then it is difficult to deal with the geometric side. Let $\Phi = \Phi_\mu$. Then

$$h(\lambda_i) = \int\limits_{\mathcal{P}^d} \exp\left[-\mu\left(\frac{|u|^2+1}{2}r + \frac{\frac{1}{2}}{r}\right)\right] r^{C_i - 1} dr du \tag{2.1}$$

2.2 Analysis on the spectral side

Now we compute the spectral side of the trace formula (in particular, h_f), under the test function $f = \Phi_\mu$. The following two formulas on K-Bessel functions are useful to us:

$$\int\limits_0^\infty x^{\nu-1} \exp\left(-\frac{\alpha}{x} - \beta x\right) dx = 2\left(\frac{\alpha}{\beta}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\alpha\beta}\right), \quad \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0. \tag{2.2}$$

$$\int\limits_0^\infty \left(x^2+b^2\right)^{-\frac{\nu}{2}} K_\nu\left(a\sqrt{x^2+b^2}\right) \cos(cx) dx = \sqrt{\frac{\pi}{2}} a^{-\nu} b^{\frac{1}{2}-\nu} \left(a^2+c^2\right)^{\frac{\nu}{2}-\frac{1}{4}} K_{\nu-\frac{1}{2}}\left(b\sqrt{a^2+c^2}\right) \tag{2.3}$$

where $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$, c is a real number. These are the formulas 3.471.9 and 6.726.4 of [**GR**] respectively.

Let $\alpha = \frac{\mu}{2}$, $\beta = \mu \cdot \frac{|u|^2+1}{2}$, $\nu = C_i$ in the formula (2.2), then by (2.1) we have

$$\begin{aligned} h(\lambda_i) &= \int\limits_{\mathbb{R}^{d-1}} \int\limits_0^\infty \exp\left[-\mu\left(\frac{|u|^2+1}{2}r + \frac{\frac{1}{2}}{r}\right)\right] r^{C_i-1} dr du \\ &= \int\limits_{\mathbb{R}^{d-1}} 2\left(|u|^2+1\right)^{-\frac{C_i}{2}} K_{C_i}\left(\mu\sqrt{|u|^2+1}\right) du \\ &= 2^d \cdot \int\limits_0^\infty \cdots \int\limits_0^\infty \left(|u|^2+1\right)^{-\frac{C_i}{2}} K_{C_i}\left(\mu\sqrt{|u|^2+1}\right) du_1 \cdots du_{d-1} \end{aligned} \tag{2.4}$$

Let $x = u_1$, $b^2 = u_2^2 + \cdots + u_{d-1}^2 + 1$, $a = \mu$, $c = 0$, $\nu = C_i$ in the formula (2.3), then

$$(2.4) = 2^d \cdot \int_0^\infty \cdots \int_0^\infty \sqrt{\frac{\pi}{2}}\, \mu^{-\frac{1}{2}} \left(\sqrt{u_2^2 + \cdots + u_{d-1}^2 + 1}\right)^{\frac{1}{2} - C_i} \times K_{C_i - \frac{1}{2}}\left(\mu\sqrt{u_2^2 + \cdots + u_{d-1}^2 + 1}\right) du_2 \cdots du_{d-1} \quad (2.5)$$

Let $x = u_2$, $b^2 = u_3^2 + \cdots + u_{d-1}^2 + 1$, $a = \mu$, $c = 0$, $\nu = C_i - \frac{1}{2}$ in the formula (2.3), then

$$(2.5) = 2^d \cdot \left(\sqrt{\frac{\pi}{2}}\, \mu^{-\frac{1}{2}}\right)^2 \int_0^\infty \cdots \int_0^\infty \left(\sqrt{u_3^2 + \cdots + u_{d-1}^2 + 1}\right)^{1 - C_i} \times K_{C_i - 1}\left(\mu\sqrt{u_3^2 + \cdots + u_{d-1}^2 + 1}\right) du_3 \cdots du_{d-1}$$

Repeating the above process, i.e., doing integrations along u_3, u_4, $\ldots$, u_{d-1} step by step in use of (2.3), we finally get

$$\begin{aligned} h_f(\lambda_i) = 2^d \cdot \left(\sqrt{\frac{\pi}{2}}\, \mu^{-\frac{1}{2}}\right)^{d-1} K_{C_i - \frac{d-1}{2}}(\mu) &= 2^d \cdot \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{-\nu_i}(\mu) \\ &= 2^d \cdot \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu) \end{aligned}$$

Now the spectral side of (1.11) is:

$$\sum_{i=0}^{\infty} 2^d \cdot \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu)\ |P_C(\phi_i)|^2 .$$

2.3 Analysis on the geometric side

In this section we focus on the geometric side of (1.11). We shall compute Σ_0 and Σ_1 separately. It turns out that, when μ tends to infinity, Σ_0 is the main term, while Σ_1 is the error term. For later applications, more information on the error term needs to be known: we shall get its order (with respect to μ). This requires more effort to put into Σ_1 than Σ_0.

2.3.1 The term Σ_0

Let $z = e^{tE} \cdot o \in \widetilde{C}$, $w = e^{sE} \cdot o \in C_0$ where $t \in (-\infty, +\infty)$, $s \in [0, T]$. As remarked in Sect. 1.5, instead of C, we work on C_0, the fundamental domain of Γ_0 in $\widetilde{C}$. The distance between z and w is:

$$d(z, w) = \|E\| \cdot |t - s|.$$

Applying to Φ_μ, we have

$$\Phi_\mu\left(d\left(e^{tE}\cdot o,\, e^{sE}\cdot o\right)\right) = \exp\left(-\mu\cdot\cosh(t-s)\right).$$

Note that $dz = \|E\|\,dt$ at the point $z = e^{tE}\cdot o$. Thus,

$$\Sigma_0 \;=\; \int\limits_{s=0}^{T}\int\limits_{t=-\infty}^{+\infty} \exp\left(-\mu\cdot\cosh(t-s)\right)\, B(E,\,E)\,dtds.$$

Let $L = t-s$, $S = s$, then

$$\Sigma_0 = B(E,\,E)\int\limits_{0}^{T}\int\limits_{-\infty}^{+\infty}\exp\left(-\mu\cdot\cosh\,L\right)dLdS = B(E,\,E)\,T\int\limits_{-\infty}^{+\infty}\exp\left(-\mu\cdot\cosh\,L\right)dL.$$

The following formula is useful to us at places (see 3.337.1 of [**GR**]):

$$\int_{-\infty}^{+\infty}\exp(-\alpha x-\beta\cosh x)\,dx = 2K_\alpha(\beta),\quad |\arg\beta| < \frac{\pi}{2}. \tag{2.6}$$

Let $\alpha = 0$, $\beta = \mu$ in (2.6), then we get

$$\Sigma_0 = 2B(E,\,E)\,T\cdot K_0(\mu) = 2\|E\|\,\mathrm{len}(C)\,K_0(\mu).$$

2.3.2 The term Σ_1

Let $\gamma = a_\gamma n_\gamma k_\gamma = \omega_{r_0}\,\theta_{w_0}\begin{pmatrix}1 & 0\\ 0 & u_0\end{pmatrix}$ for some $r_0 > 0$, $w_0 = \sum_{i=1}^{d-1} w_{0\,i}E_i \in \mathfrak{n}$ ($w_{0\,i}\in\mathbb{R}$) and $u_0 = (u_{ij}) \in SO_d$. Let $z = \omega_r\cdot o$, $w = \omega_{r'}\cdot o \in \widetilde{C}$ where $r,\,r' > 0$. Then

$$k_\gamma\,\omega_r = \begin{pmatrix}1 & 0\\ 0 & u_0\end{pmatrix}\begin{pmatrix}\frac{r+r^{-1}}{2} & \frac{r-r^{-1}}{2} & 0\\ \frac{r-r^{-1}}{2} & \frac{r+r^{-1}}{2} & 0\\ 0 & 0 & 1_{d-2}\end{pmatrix} = \begin{pmatrix}\frac{r+r^{-1}}{2} & \frac{r-r^{-1}}{2} & 0 & \cdots & 0\\ u_{11}\frac{r-r^{-1}}{2} & u_{11}\frac{r+r^{-1}}{2} & u_{12} & \cdots & u_{1d}\\ u_{21}\frac{r-r^{-1}}{2} & u_{21}\frac{r+r^{-1}}{2} & u_{22} & \cdots & u_{2d}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ u_{d1}\frac{r-r^{-1}}{2} & u_{d1}\frac{r+r^{-1}}{2} & u_{d2} & \cdots & u_{dd}\end{pmatrix} \tag{2.7}$$

Assume that $k_\gamma\,\omega_r = \theta_v\,\omega_s\,k$ for some $s > 0$, $v = (v_1,\,\cdots,\,v_{d-1}) \in \mathbb{R}^{d-1}$ and $k = \begin{pmatrix}1 & 0\\ 0 & k_1\end{pmatrix} \in K$ where $k_1 \in SO_d$.

$$\theta_v\,\omega_s\,k \;=\; \begin{pmatrix}1+\frac{|v|^2}{2} & -\frac{|v|^2}{2} & v_1 & \cdots & v_{d-1}\\ \frac{|v|^2}{2} & 1-\frac{|v|^2}{2} & v_1 & \cdots & v_{d-1}\\ v_1 & -v_1 & & & \\ \vdots & \vdots & & 1_{d-2} & \\ v_{d-1} & -v_{d-1} & & & \end{pmatrix}\begin{pmatrix}\frac{s+s^{-1}}{2} & \frac{s-s^{-1}}{2} & 0\\ \frac{s-s^{-1}}{2} & \frac{s+s^{-1}}{2} & 0\\ 0 & 0 & 1_{d-2}\end{pmatrix}k$$

$$= \begin{pmatrix} \left(1+\frac{|v|^2}{2}\right)\frac{s+s^{-1}}{2} - \frac{|v|^2}{2}\frac{s-s^{-1}}{2} & \cdots & \cdots \\ \frac{|v|^2}{2}\frac{s+s^{-1}}{2} + \left(1-\frac{|v|^2}{2}\right)\frac{s-s^{-1}}{2} & \cdots & \cdots \\ v_1\, s^{-1} & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ v_{d-1}\, s^{-1} & \cdots & \cdots \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & k_1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s+s^{-1}}{2} + \frac{s^{-1}}{2}|v|^2 & \cdots & \cdots \\ \frac{s-s^{-1}}{2} + \frac{s^{-1}}{2}|v|^2 & \cdots & \cdots \\ v_1\, s^{-1} & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ v_{d-1}\, s^{-1} & \cdots & \cdots \end{pmatrix} \tag{2.8}$$

The comparison of (2.7) and (2.8) gives:

$$\frac{r+r^{-1}}{2} = \frac{s+s^{-1}}{2} + \frac{s^{-1}}{2}|v|^2, \tag{2.9}$$

$$u_{11}\,\frac{r-r^{-1}}{2} = \frac{s-s^{-1}}{2} + \frac{s^{-1}}{2}|v|^2, \tag{2.10}$$

$$u_{i+1,\,1}\,\frac{r-r^{-1}}{2} = v_i \cdot s^{-1},\ 1 \leqslant i \leqslant d-1. \tag{2.11}$$

Combining (2.9) and (2.10), we have

$$s^{-1} = \frac{r+r^{-1}}{2} - u_{11}\frac{r-r^{-1}}{2}. \tag{2.12}$$

Note that

$$\begin{aligned} d(\gamma z,\, w) &= d(\omega_{r_0}\theta_{w_0}\cdot\theta_v\omega_s\cdot o,\ \omega_{r'}\cdot o) \\ &= d(\theta_{w_0+v}\,\omega_s\cdot o,\ \omega_{r_0^{-1}r'}\cdot o) \\ &= \|E\|\operatorname{arccosh}\left(\frac{|w_0+v|^2+s^2+r_0^{-2}r'^2}{2sr_0^{-1}r'}\right) \end{aligned} \tag{2.13}$$

At the point $z=\omega_r\cdot o$, $dz = \|E\|\,d(\log r) = \|E\|\,\frac{dr}{r}$. By definition,

$$\begin{aligned} I_\gamma &= \int_0^\infty\!\!\int_0^\infty \Phi_\mu\big(d(\gamma\omega_r\cdot o,\ \omega_{r'}\cdot o)\big)\frac{B(E,\,E)}{rr'}dr'dr \\ &= \int_0^\infty\!\!\int_0^\infty \exp\left(-\mu\,\frac{|w_0+v|^2+s^2+r_0^{-2}r'^2}{2sr_0^{-1}r'}\right)\frac{B(E,\,E)}{rr'}dr'dr \end{aligned}$$

Let $\nu = 0$, $\alpha = \mu \frac{|w_0+v|^2+s^2}{2sr_0^{-1}}$, $\beta = \mu \frac{r_0^{-1}}{2s}$ in the formula (2.2), then

$$\int_0^\infty \exp\left(-\mu \frac{|w_0+v|^2+s^2+r_0^{-2}r'^2}{2sr_0^{-1}r'}\right)\frac{dr'}{r'} = 2K_0\left(\mu\sqrt{\left|\frac{w_0+v}{s}\right|^2+1}\right).$$

As a result,

$$I_\gamma = 2B(E,\,E)\int_0^\infty K_0\left(\mu\sqrt{\left|\frac{w_0+v}{s}\right|^2+1}\right)\frac{dr}{r}.$$

Substituting (2.11) and (2.12) into the right hand side of the above formula, we have

$$I_\gamma = 2B(E,\,E)\int_0^\infty K_0\left(\mu\sqrt{f_\gamma(r)}\right)\frac{dr}{r}$$

where

$$\begin{aligned} f_\gamma(r) &= \left|\frac{w_0+v}{s}\right|^2+1 = \sum_{i=1}^{d-1}\left(\frac{v_i}{s}\right)^2+\left|\frac{w_0}{s}\right|^2+2\sum_{i=1}^{d-1}w_{0\,i}\,v_i\,s^{-2}+1 \\ &= \sum_{i=1}^{d-1}\left(\frac{v_i}{s}\right)^2+\left|\frac{w_0}{s}\right|^2+2\sum_{i=1}^{d-1}w_{0\,i}\,u_{i+1,\,1}\,\frac{r-r^{-1}}{2}\,s^{-1}+1 \\ &= M(\gamma)r^2+N(\gamma)r^{-2}+Q(\gamma) \end{aligned}$$

Here

$$M(\gamma) = \sum_{i=1}^{d}\left(w_{0\,i}\,\frac{1-u_{11}}{2}+\frac{u_{i+1,\,1}}{2}\right)^2, \tag{2.14}$$

$$N(\gamma) = \sum_{i=1}^{d}\left(w_{0\,i}\,\frac{1+u_{11}}{2}-\frac{u_{i+1,\,1}}{2}\right)^2, \tag{2.15}$$

$$Q(\gamma) = 2\sum_{i=1}^{d}\left(w_{0\,i}\,\frac{1-u_{11}}{2}+\frac{u_{i+1,\,1}}{2}\right)\left(w_{0\,i}\,\frac{1+u_{11}}{2}-\frac{u_{i+1,\,1}}{2}\right)+1 \tag{2.16}$$

The simple property $\sum_{i=1}^d u_{i,\,1}^2 = 1$ is used in the above computation. Define

$$\delta(\gamma) := 2\sqrt{M(\gamma)N(\gamma)}+Q(\gamma).$$

When $M(\gamma),\,N(\gamma) > 0$, we have $f_\gamma(r) \geqslant \delta(\gamma)$ where "=" can be achieved since r ranges over all positive numbers. When $M(\gamma) = 0$, $\delta(\gamma) = Q(\gamma) = \lim_{r\to\infty} f_\gamma(r)$. When $N(\gamma) = 0$, $\delta(\gamma) = Q(\gamma) = \lim_{r\to 0} f(r)$. The number $\delta(\gamma)$ has remarkable geometric meaning. Writing $\frac{|w_0+v|^2+s^2+r_0^{-2}r'^2}{2sr_0^{-1}r'}$ as $\frac{B}{r'}+Dr'$ where

$$B = \frac{|w_0+v|^2+s^2}{2sr_0^{-1}}, \quad D = \frac{r_0^{-1}}{2s}.$$

Then

$$\frac{B}{r'} + Dr' \geqslant 2\sqrt{BD} = \sqrt{\left|\frac{w_0+v}{s}\right|^2 + 1} = \sqrt{f(r)}. \tag{2.17}$$

Since $f_\gamma(r) = \left(\sqrt{M(\gamma)}\, r - \frac{\sqrt{N(\gamma)}}{r}\right)^2 + 2\sqrt{M(\gamma)N(\gamma)} + Q(\gamma) = \left(\sqrt{M(\gamma)}\, r - \frac{\sqrt{N(\gamma)}}{r}\right)^2 + \delta(\gamma)$, so

$$\inf_{r,\, r'>0} \frac{|w_0+v|^2 + s^2 + r_0^{-2} r'^2}{2 s r_0^{-1} r'} = \inf_{r>0} \sqrt{f_\gamma(r)} = \sqrt{\delta(\gamma)}.$$

By (2.13), we have

$$\sqrt{\delta(\gamma)} = \cosh\left(\|E\|^{-1} \inf_{z,\, w\in\widetilde{C}} d(\gamma z,\, w)\right). \tag{2.18}$$

Hence the number $\delta(\gamma)$ measures the minimal distance between the points on the geodesics $\widetilde{C}$ and $\gamma\,\widetilde{C}$. It is clear that δ is a well-defined function on the double coset classes $\Gamma_0\backslash\Gamma/\Gamma_0$, i.e., $\delta(\gamma) = \delta(\gamma')$ for γ and γ' in the same class: geometrically, we have: $\eta_1\widetilde{C} = \widetilde{C}$ and $\gamma\widetilde{C} = \gamma\eta_2\widetilde{C}$ for any $\eta_1,\, \eta_2 \in \Gamma_0$, so the minimal distance between $\eta_1\widetilde{C}$ and $\gamma\eta_2\widetilde{C}$ is identical to the minimal distance between $\widetilde{C}$ and $\gamma\widetilde{C}$ which means that $\delta\left(\eta_1^{-1}\gamma\eta_2\right) = \delta(\gamma)$. Define

$$\pi(x) = \#\left\{\widetilde{\gamma} \in \Gamma_0\backslash\Gamma/\Gamma_0 \,\middle|\, \delta(\gamma) \leqslant x\right\}.$$

Our conclusion is

Theorem 2.3.1. *$\pi(x) = \mathcal{O}\left(x^{\frac{d-1}{2}}\right)$, as $x \to \infty$.*

Proof. To count the classes $\widetilde{\gamma}$, it suffices to choose one representative element in each class and then count these representatives. Write $\gamma = \omega_{r_0}\,\theta_{w_0}\,k_\gamma$ as before and let $\eta = \omega_r\, k \in AM$ where $k = \mathrm{diag}(1,\, 1,\, \rho)$ for some $\rho \in SO_{d-1}$. Note that $ak = ka$ for $a \in A$, $k \in M$ and $k\theta_u = \theta_{u\rho^T} k$ where ρ^T is the transpose of ρ and $u\rho^T$ is the usual matrix multiplication (see Proposition I.4.2 of [**FJ**]). Then the left action of η on γ is as follows:

$$\begin{aligned} \eta\cdot\gamma &= \omega_r\, k \cdot \omega_{r_0}\,\theta_{w_0}\,k_\gamma \\ &= \omega_r\,\omega_{r_0}\, k\cdot\theta_{w_0}\,k_\gamma \\ &= \omega_{rr_0}\,\theta_{w_0\rho^T}\, kk_\gamma \end{aligned}$$

Clearly $|w_0| = |w_0\rho^T|$ and we can choose some $\eta_1 \in \Gamma_0$ such that $\eta_1\cdot\gamma = \omega_{\ell_0}\,\theta_{u_0}\,k$ where ℓ_0 lies in $[1,\, e^T]$. Now consider the right action of η on γ:

$$\begin{aligned} \gamma\cdot\eta &= \omega_{r_0}\,\theta_{w_0}\,k_\gamma\,\omega_r\,k \\ &= \omega_{r_0}\,\theta_{w_0}\cdot\theta_v\,\omega_s\cdot k' \end{aligned}$$

$$= \omega_{r_0 s}\, \theta_{(w_0+v)s^{-1}}\, k'$$

Recall that

$$\left|\frac{w_0+v}{s}\right|^2 + 1 = \left(\sqrt{M(\gamma)}\, r - \frac{\sqrt{N(\gamma)}}{r}\right)^2 + \delta(\gamma).$$

Later we shall show that neither $M(\gamma)$ nor $N(\gamma)$ can be zero for $\gamma \notin \Gamma_0$(see Lemma 2.3.6). Since $\gamma \in \Gamma_0$ (i.e., $\widetilde{\gamma} = \tilde{1}$) only contributes to $\pi(x)$ by 1, we shall assume that $M(\gamma)N(\gamma) \neq 0$ in the following. Let x_0 be the positive root of $\sqrt{M(\gamma)}\, x - \frac{\sqrt{N(\gamma)}}{x} = 0$, then there exist $r \in \{\exp(nT) \,|\, n \in \mathbb{Z}\}$ and $t \in [1,\, e^T]$ such that $x_0 = rt$. Let $\eta_2 = \omega_r\, k \in \Gamma_0$ and $b = \omega_t \in A$, then $\gamma \cdot \eta_2 \cdot b = \omega_{r_0 s'}\, \theta_{(w_0+v')s'^{-1}}\, k''$ where $s'^{-1} = \frac{rt+(rt)^{-1}}{2} - u_{11}\frac{rt-(rt)^{-1}}{2}$. Here $u(x_0) := \frac{w_0+v'}{s'}$ characterizes the number $\delta(\gamma)$: $|u_0| = (\delta(\gamma)-1)^{1/2} \leqslant (x-1)^{1/2}$. The above discussion on the two-sided action of Γ_0 shows that, for any $\gamma \in \Gamma \setminus \Gamma_0$, we can pick up some $\gamma_1,\, \gamma_2 \in \Gamma_0$ and $b_\gamma = \omega_t \in A$ such that $t \in [1,\, e^T]$ and $\gamma^* := \gamma_1 \gamma \gamma_2 b_\gamma \in \Omega_x$ where

$$\delta(\gamma) = |u(\gamma^*)|^2 + 1$$

and

$$\Omega_x := \left\{ g = \omega_r\, \theta_u\, k \in G \,\middle|\, r \in [1,\, e^T],\, |u| \leqslant (x-1)^{1/2},\, k \in K \right\}.$$

Such element $\gamma^* \in \Omega_x$ is unique: there is only one positive root x_0 for the equation $\sqrt{M(\gamma)}x - \frac{\sqrt{N(\gamma)}}{x} = 0$ and t is obtained from x_0 modulo (multiplicatively) proper e^{nT}. For two given representatives γ_1 and γ_2 of different classes in $\Gamma_0 \backslash \Gamma / \Gamma_0 \setminus \{\tilde{1}\}$, we have $\gamma_1^* \neq \gamma_2^*$: if $\gamma_1^* = \gamma_2^*$, then $\gamma_1 \cdot b_{\gamma_1} = \gamma_2 \cdot b_{\gamma_2}$, thus $\gamma_1^{-1}\gamma_2 = b_{\gamma_1} b_{\gamma_2}^{-1}$ lies in $A \cap \Gamma \subset \Gamma_0$, i.e., $\widetilde{\gamma}_1 = \widetilde{\gamma}_2$, a contradiction. Thus, counting $\pi(x)$ for large x is equivalent to counting $\pi'(x) := \#\{\gamma^* \in \Omega_x \,|\, \gamma \in \Gamma \setminus \Gamma_0\}$. For the latter, we have to know the distribution property of those γ^*s. This is stated in the following lemma. With this lemma we know that $\pi'(x)$ is bounded by the Euclidean volume of Ω_x. So $\pi(x) = \mathcal{O}\left(x^{\frac{d-1}{2}}\right)$ where the implied $\mathcal{O}$-constant is unconditional.

Lemma 2.3.2. *For any sequence of pairs*

$$\left\{(\gamma_{i1}^*,\, \gamma_{i2}^*) \,\middle|\, \gamma_{i1},\, \gamma_{i2} \in \Gamma,\ \widetilde{\gamma}_{i1} \neq \widetilde{\gamma}_{i2},\ \gamma_{i1}^*,\, \gamma_{i1}^* \in \Omega_\infty \right\}_{i=1}^{\infty},$$

$\gamma_{i1}^*,\, \gamma_{i2}^*$ *can not be close enough (as $i \to \infty$) with respect to the topology of G. Here* $\Omega_\infty = \left\{ g = \omega_r\, \theta_u\, k \in G \,\middle|\, r \in [1,\, e^T],\, k \in K \right\}$.

Proof. If the conclusion does not hold, then we get a sequence of pairs $\{(\gamma_{i1}^*,\, \gamma_{i2}^*)\}_{i=1}^{\infty}$ such that ${\gamma_{i1}^*}^{-1}\gamma_{i2}^* \to 1$, i.e., $b_{\gamma_{i1}}^{-1}\gamma_{i1}^{-1}\gamma_{i2}b_{\gamma_{i2}} \to 1$ as $i \to \infty$. Then $\gamma_{i1}^{-1}\gamma_{i2}$ lies in $b_{\gamma_{i1}} U_i\, b_{\gamma_{i2}}^{-1}$ where U_i is an open neighborhood of the identity. As $i \to \infty$, U_i can be small enough. Remember that b_γ lies in $\left\{\omega_r \,\middle|\, r \in [1,\, e^T]\right\}$. Since Γ is discrete, we can choose proper U_i such that those (finitely many) elements in $\Gamma \cap b_{\gamma_{i1}} U_i b_{\gamma_{i2}}^{-1}$ all lie in $\Gamma \cap A \subset \Gamma_0$. So $\gamma_{i1}^{-1}\gamma_{i2} \in \Gamma_0$ for large i, a contradiction as γ_{i1} and γ_{i2} are of different classes. $\square$

This completes the proof of the theorem. □

The following corollary is clear from the proof of the above theorem:

Corollary 2.3.3. *If there are infinitely many classes in* $\Gamma_0\backslash\Gamma/\Gamma_0$*, then the unique accumulation point of* $\{\delta(\gamma)\,|\,\gamma\in\Gamma\}$ *is* ∞*.*

One has the stronger information about $M(\gamma)N(\gamma)$ via the following lemmas:

Lemma 2.3.4. *If there are infinitely many classes in* $\Gamma_0\backslash\Gamma/\Gamma_0$*, then the unique accumulation point of* $\{M(\gamma)N(\gamma)\,|\,\gamma\in\Gamma\}$ *is* ∞*.*

Proof. By Cauchy inequality and (2.14), (2.15) and (2.16), we see

$$M(\gamma)N(\gamma)\geqslant Q(\gamma)-1.$$

If there exists a sequence $\{\gamma_i\}\subset\Gamma$ such that $M(\gamma_i)N(\gamma_i)\to y_0$ as $i\to\infty$, then $\delta(\gamma_i)=2\sqrt{M(\gamma_i)N(\gamma_i)}+Q(\gamma_i)$ is bounded. Thus $\{\delta(\gamma_i)\}$ has a convergent subsequence which contradicts Corollary 2.3.3. □

Lemma 2.3.5. *If* $M(\gamma)N(\gamma)=0$*, then* $Q(\gamma)=1$*.*

Proof. Clear from (2.14), (2.15) and (2.16). □

Lemma 2.3.6. *For* $\gamma\notin\Gamma_0$*,* $M(\gamma)$ *and* $N(\gamma)$ *can not be zero simultaneously.*

Proof. Assume that $M(\gamma)=N(\gamma)=0$. If $u_{11}=1$, by (2.15) we have $|w_0|=0$, then $\gamma\in\Gamma_0$, a contradiction. If $u_{11}=-1$, by (2.14) we have $|w_0|=0$, then $\gamma^2=\omega_{r_0}k_\gamma\,\omega_{r_0}k_\gamma=\omega_{r_0}\omega_{r_0^{-1}}k_\gamma k_\gamma=k_\gamma^2\in K\cap\Gamma=\{1\}$, which means that $\gamma=1$ since Γ is torsion-free. This is impossible as $\gamma\notin\Gamma_0$. Now assume that $u_{11}\neq\pm1$. By (2.14) and (2.15), $w_{0\,i}=\frac{u_{i+1,\,1}}{1+u_{11}}=\frac{-u_{i+1,\,1}}{1-u_{11}}$ which implies that $u_{i+1,1}=0$ for any $1\leqslant i\leqslant d$. Then $u_{11}=\pm1$, a contradiction shown as above. The proof is complete. □

Lemma 2.3.7. *For each class* $\widetilde{\gamma}\neq\tilde{1}$ *and any representative element* γ *in the class* $\widetilde{\gamma}$*,* $M(\gamma)N(\gamma)\neq0$*.*

Proof. Assume that $N(\gamma)=0$ for some γ in the class $\widetilde{\gamma}$, then $M(\gamma)\neq0$ by Lemma 2.3.6. As before, write $\gamma=\omega_{r_0}\,\theta_{w_0}\,k_\gamma$. Let $\gamma_2=\omega_r\cdot o$, then $\gamma\gamma_2=\omega_s\theta_w k$ where $|w|^2+1=M(\gamma)r^2+Q(\gamma)=M(\gamma)r^2+1$ (see Lemma 2.3.5). With $r\in\left\{e^{nT}\,\middle|\,n<0,\,|n|\text{ large enough}\right\}$, we see that there are infinitely many distinct γ's lying in Ω_x for any fixed number $x>1$. However, $\Gamma\cap\Omega_x$ is a finite set as Γ is discrete and Ω is compact. Up to now we have shown that $N(\gamma)\neq0$. The similar argument shows that $M(\gamma)\neq0$. We omit the details. □

By Lemma 2.3.4 and Lemma 2.3.7,

Corollary 2.3.8. *For any family of representatives* $\Lambda = \{\gamma\}$ *for all classes* $\widetilde{\gamma} \in \Gamma_0\backslash G/\Gamma_0 \smallsetminus \{\tilde{1}\}$*, we have*

$$\inf\{M(\gamma)N(\gamma) \mid \gamma \in \Lambda\} \geqslant \alpha$$

for some $\alpha > 0$*.*

Remark 2.3.9. *When* $|\,\Gamma_0\backslash\Gamma/\Gamma_0| < \infty$*, this corollary is trivial by Lemma 2.3.7.*

Remark 2.3.10. *We point out that the numbers* $M(\gamma)$*,* $N(\gamma)$*,* $Q(\gamma)$ *are all well-defined on* $\Gamma_0\backslash\Gamma$ *but not on* Γ/Γ_0*. One can check that the left action of* Γ_0 *on* γ *does not change the parameters* u_{11}*,* $|w_0|$ *and* $A(\gamma) = \sum_{i=1}^{d-1} w_{0i}u_{i+1,1}$ *which are the ingredients of* $M(\gamma)$*,* $N(\gamma)$*,* $Q(\gamma)$*, while the right action of* Γ_0 *does change some of these parameters.*

We reorder the those $\delta(\gamma)$'s ($\gamma \in \Lambda$) to get a sequence $\{\delta_n\}_{n=1}^{\infty}$ such that δ_n increases. Denote by I_n the corresponding n-th I_γ. If $|\,\Gamma_0\backslash\Gamma/\Gamma_0| < \infty$, there are only finitely many terms I_γ in Σ_1. We can estimate Σ_1 in the same way with the estimate for $\sum\limits_{n=1}^{N} I_n$ in the case $|\,\Gamma_0\backslash\Gamma/\Gamma_0| = \infty$ (see below). So we might as well assume that $|\,\Gamma_0\backslash\Gamma/\Gamma_0| = \infty$. By Theorem 2.3.1, we have

$$\delta_n \gg n^{\frac{1}{\frac{d-1}{2}+\epsilon}}, \quad \epsilon > 0.$$

In the following we choose $\epsilon = \frac{1}{2}$, then $\delta_n \gg n^{\frac{2}{d}}$. Let $x = \sqrt{M(\gamma)}\,r - \frac{\sqrt{N(\gamma)}}{r}$, then $r = \frac{x+\sqrt{x^2+4\sqrt{M(\gamma)N(\gamma)}}}{2\sqrt{M(\gamma)}}$, noting that $M(\gamma)N(\gamma) \neq 0$. Hence

$$\begin{aligned} I_\gamma &= 2B(E,\,E)\int_0^\infty K_0\left(\mu\sqrt{f_\gamma(r)}\right)\frac{dr}{r} \\ &= 2B(E,\,E)\int_{-\infty}^{+\infty}\frac{K_0\left(\mu\sqrt{x^2+\delta(\gamma)}\right)}{\sqrt{x^2+4\sqrt{M(\gamma)N(\gamma)}}}dx \\ &= 4B(E,\,E)\int_0^\infty\frac{K_0\left(\mu\sqrt{x^2+\delta(\gamma)}\right)}{\sqrt{x^2+4\sqrt{M(\gamma)N(\gamma)}}}dx \end{aligned}$$

When x is very large, the following inequality holds

$$K_0(x) \leqslant \sqrt{\frac{\pi}{2x}}e^{-x}\left(1+\frac{1}{8x}\right). \tag{2.19}$$

As μ tends to ∞, we have: $\mu\sqrt{x^2+\delta(\gamma)}$ tends to ∞ (note that $\delta(\gamma) \geqslant 1$) and

$$\begin{aligned} \frac{K_0\left(\mu\sqrt{x^2+\delta_n}\right)}{\sqrt{x^2+4\sqrt{M(\gamma)N(\gamma)}}} &\leqslant \frac{1}{\sqrt{x^2+4\sqrt{M(\gamma)N(\gamma)}}}\sqrt{\frac{\pi}{2\mu\sqrt{x^2+\delta_n}}}\,e^{-\mu\sqrt{x^2+\delta_n}}\left(1+\frac{1}{8\mu\sqrt{x^2+\delta_n}}\right) \\ &\leqslant \frac{1}{\sqrt{x^2+4\sqrt{\alpha}}}\sqrt{\frac{\pi}{2\mu\sqrt{x^2+1}}}\,e^{-\mu\sqrt{\delta_n}}\left(1+\frac{1}{8\mu}\right) \end{aligned}$$

$$< \quad 2\sqrt{\frac{\pi}{2\mu}}\,\frac{1}{\sqrt{x^2+4\sqrt{\alpha}}}\,\frac{1}{\sqrt[4]{x^2+1}}\,e^{-\mu\sqrt{\delta_n}}$$

Substituting this inequality into I_γ, we get a uniform upper bound for any n:

$$\begin{aligned} I_n \quad &\ll \quad \sqrt{\frac{2\pi}{\mu}}\int_0^\infty \frac{e^{-\mu\sqrt{\delta_n}}}{\sqrt{x^2+4\sqrt{\alpha}}\,\sqrt[4]{x^2+1}}dx \\ &= \quad \sqrt{\frac{2\pi}{\mu}}e^{-\mu\sqrt{\delta_n}}\int_0^\infty \frac{dx}{\sqrt{x^2+4\sqrt{\alpha}}\,\sqrt[4]{x^2+1}}, \quad \text{as } \mu\to\infty. \end{aligned} \tag{2.20}$$

It is clear that the integral

$$\int_0^\infty \frac{dx}{\sqrt{x^2+4\sqrt{\alpha}}\,\sqrt[4]{x^2+1}}$$

converges. Since $\delta_n \gg n^{\frac{2}{d}}$, there exists $N\in\mathbb{N}$ sush that $\delta_n \geqslant n^{\frac{2}{d}}$ for $n\geqslant N$. Thus by (2.20),

$$\sum_{n=N}^\infty I_n < \sqrt{\frac{2\pi}{\mu}}\sum_{n=N}^\infty e^{-\mu\sqrt{\delta_n}} \ll \frac{1}{\sqrt{\mu}}\sum_{n=N}^\infty e^{-\mu\, n^{1/d}}.$$

The term $\sum_n e^{-\mu\, n^{1/d}}$ is bounded by the integral

$$\int_1^\infty e^{-\mu\, x^{1/d}}dx = d\int_1^\infty e^{-\mu\, y}y^{d-1}dy \quad (\text{letting}\, y = x^{1/d}).$$

An elementary calculus shows that this integral is bounded by $e^{-\mu}\,\mu^{-1}$. Consequently we get

$$\sum_{n=N}^\infty I_n = \mathcal{O}\left(e^{-\mu}\,\mu^{-\frac{3}{2}}\right).$$

Now let's consider the terms I_n for $1\leqslant n\leqslant N$. The following argument also applies to the case when $|\,\Gamma_0\backslash\Gamma/\Gamma_0| < \infty$.

$$\begin{aligned} I_n = 4\|E\|^2\int_0^\infty \frac{K_0\left(\mu\sqrt{x^2+\delta_n}\right)}{\sqrt{x^2+4\sqrt{M(\gamma_n)N(\gamma_n)}}}dx \quad &\leqslant \quad 4\|E\|^2\int_0^\infty \frac{K_0\left(\mu\sqrt{x^2+1}\right)}{\sqrt{x^2+4\sqrt{\alpha}}}dx \\ &\leqslant \quad \frac{2\|E\|^2}{\sqrt[4]{\alpha}}\int_0^\infty K_0\left(\mu\sqrt{x^2+1}\right)dx \end{aligned}$$

Let $x^2+1=y$, then

$$\int_0^\infty K_0\left(\mu\sqrt{x^2+1}\right)dx = \frac{1}{2}\int_1^\infty \frac{K_0\left(\mu\sqrt{y}\right)}{\sqrt{y-1}}dy = \frac{1}{\sqrt{2}}\Gamma\left(\frac{1}{2}\right)\mu^{-\frac{1}{2}}K_{-\frac{1}{2}}(\mu).$$

The last step follows from the formula 6.592.12 of [**GR**]:

$$\int_1^\infty x^{-\frac{\nu}{2}}(x-1)^{\mu-1}K_\nu\left(a\sqrt{x}\right)dx = \Gamma(\mu)2^\mu a^{-\mu}K_{\nu-\mu}(a), \quad \mathrm{Re}(a)>0,\ \mathrm{Re}(\mu)>0. \tag{2.21}$$

The Bessel function has the well-known asymptotic (where $\nu \in \mathbb{C}$ and $x \in \mathbb{R}$):

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x}, \quad \text{as } x \to \infty. \tag{2.22}$$

By this asymptotic, we immediately get: $I_n = \mathcal{O}\left(e^{-\mu}\,\mu^{-1}\right)$. Hence

$$\sum_{n=1}^{N} I_n = \mathcal{O}\left(e^{-\mu}\,\mu^{-1}\right).$$

So far, we have obtained:

$$\Sigma_1 = \mathcal{O}\left(e^{-\mu}\,\mu^{-1}\right) + \mathcal{O}\left(e^{-\mu}\,\mu^{-\frac{3}{2}}\right).$$

When $\delta_1 = 1$, the term $\mathcal{O}\left(e^{-\mu}\,\mu^{-1}\right)$ does exist since I_1 just contributes with it. We explain in more details. Let $\beta = 4\sqrt{M(\gamma_1)N(\gamma_1)}$ where γ_1 is such that $\delta(\gamma_1) = \delta_1 = 1$. Let $y = \sqrt{x^2+1}$, then

$$I_1 = \int_1^\infty \frac{y}{\sqrt{y^2-1+\beta}} \frac{K_0(\mu\, y)}{\sqrt{y^2-1}} dy.$$

For $y \geqslant 1$, one easily checks the following: if $\beta \leqslant 1$, then $\frac{y}{\sqrt{y^2-1+\beta}} \geqslant 1$; if $\beta > 1$, then $\frac{y}{\sqrt{y^2-1+\beta}} \geqslant \frac{1}{\sqrt{\beta}}$. In summary, $\frac{y}{\sqrt{y^2-1+\beta}} \geqslant c := \min\left\{\frac{1}{\sqrt{\beta}},\, 1\right\} > 0$. Hence

$$I_1 \geqslant c\int_1^\infty \frac{K_0(\mu\, y)}{\sqrt{y^2-1}} dy = \frac{c}{2}\left[K_0\left(\frac{\mu}{2}\right)\right]^2.$$

The last step follows from the formula 6.567.15 of [**GR**]:

$$\int_1^\infty x^\nu \left(x^2-1\right)^{\nu-\frac{1}{2}} K_\nu(bx) = \frac{2^{\nu-1}}{\sqrt{\pi}} b^{-\nu}\Gamma\left(\nu+\frac{1}{2}\right)\left[K_\nu\left(\frac{b}{2}\right)\right]^2, \quad \mathrm{Re}(b) > 0,\ \mathrm{Re}(\nu) > -\frac{1}{2}$$

and the well-known formula $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. By (2.22), we see $I_1 \geqslant \frac{\pi c}{2} e^{-\mu}\mu^{-1}$ as $\mu \to \infty$.

Remark 2.3.11. *By the formula (2.18), there is an equivalent condition for $\delta_1 = 1$:*

$$\inf_{z,\, w \in \widetilde{C}} d(\gamma z,\, w) = 0$$

for some $\gamma \in \Gamma \setminus \Gamma_0$. A sufficient condition is, $\widetilde{C} \cap \gamma\,\widetilde{C} \neq \varnothing$, i.e., $\widetilde{C}$ intersects its translation $\gamma\,\widetilde{C}$ where $\gamma\,\widetilde{C} \neq \widetilde{C}$. Actually this is also a necessary condition for $\delta_1 = 1$. Since M, N are nonzero, the minimal distance between $\widetilde{C}$ and $\gamma\,\widetilde{C}$ is achieved at some finite r where r is such that $\sqrt{M}\,r - \frac{\sqrt{N}}{r} = 0$. Such r induces a finite r' such that $\frac{B}{r'} = Dr'$ (see (2.17)). The two points $\omega_r \cdot o$ and $\omega_{r'} \cdot o$ are both regular points, i.e., they are not at infinity. So $\widetilde{C}$ and $\gamma\,\widetilde{C}$ intersect at these two points.

2.4 f and k_f

We examine the properties on f and k_f to show that our argument in above is valid. Remember that $f(g) = \Phi_\mu\left(d(e\cdot o,\, g\cdot o)\right)$ for $g\in G$. It is clear that f is bi-K-invariant, so $f = f_K$ (see Sect. 1.2 for the definition of f_K). To show that $f\in C_{\rm unif}(G)$, it suffices to show that $f_{A_0N_0}\in L^1(G)$ for some compact neighborhood A_0N_0 of $e\in G$. Since KA_0U_0 is compact, by the integral formula, that $f_{A_0N_0}$ is integrable is equivalent to $\int_{AN} f_{A_0N_0}(ank)d(an) < \infty$ for any $k\in K$ (note that $f_{A_0N_0}$ is continuous, see Lemma 9.2.3 of [**DE**]). Write $ka_2n_2 = a'n'k'$ for $a_2\in A_0$, $n_2\in N_0$. Then a' and n' are contained in compact subsets of A_0 and N_0 respectively. In view of the commutativity relation between a and n, the hyperbolic distance in terms of a, n, and the test function we have chosen (see Sect. 2.1), the left multiplications of $a_1n_1\in A_0N_0$ to ank and the right multiplications of $a_2n_2\in A_0N_0$ to ank do not cause convergence problem to f, i.e., $f_{A_0N_0}\in L^1(G)$ is equivalent to $f = f_K\in L^1(G)$. We have the following computation:

$$\begin{aligned}\int_G f_K(g)dg = \int_G f(g)dg &= \int_N\int_A \Phi_\mu\left(d(an\cdot o,\, e\cdot o)\right)e^{2\rho\log(a)}dadn\\ &= 2^d\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\frac{d-1}{2}}(\mu) < \infty\end{aligned}$$

This is a copy of that of $h_f(\phi_i)$ by dropping the term η_i there (see formula (1.5) and Sect. 2.2). Hence $f\in C_{\rm unif}(G)$. The supremum norm of ϕ_n satisfies the classical Hörmander's bound (see [**Ho**], [**So**]): $\sup|\phi_n| \leqslant A\,\lambda_n^{\frac{d-1}{4}}\|\phi_n\|_{L^2(X)}$ where λ_n is the Laplace eigenvalue of ϕ_n and A is uniform for all n. Since ϕ_n's are orthonormal basis of $L^2(X)$, we have: $\sup|\phi_n|\leqslant A\,\lambda_n^{\frac{d-1}{4}}$. When the eigenvalue $\lambda_n = \rho^2 - \nu_n^2\in\mathbb{R}$ is large, i.e., $\lambda_n > \rho^2 = \left(\frac{d-1}{2}\right)^2$, it is clear that ν_n lies in $i\,\mathbb{R}$. This means that there are only finitely many ϕ_n's such that ν_n is real. For the convergence problem of k_f, it suffices to consider those ϕ_n's with large eigenvalues. Thus we may write $\nu_n = i\,r_n$ for $r_n\in\mathbb{R}_{>0}$. Then $\lambda_n = \left(\frac{d-1}{2}\right)^2 + r_n^2$. By the following formula (see 8.432.5 of [**GR**])

$$K_\nu(xz) = \frac{\Gamma\left(\nu+\frac12\right)(2z)^\nu}{x^\nu\Gamma\left(\frac12\right)}\int_0^\infty \frac{\cos xt\,dt}{(t^2+z^2)^{\nu+\frac12}},\quad \mathrm{Re}\left(\nu+\frac12\right)\geqslant 0,\ x>0\ |\arg z| < \frac{\pi}{2},$$

we have:

$$K_{ir_n}(x) = \frac{\Gamma\left(1/2+ir_n\right)}{\Gamma(1/2)}(2x)^{ir_n}\int_0^\infty \frac{\cos t\,dt}{(t^2+x^2)^{1/2+ir_n}},\quad x>0.$$

The integration by parts shows that

$$\int_0^\infty \frac{\cos t\,dt}{(t^2+x^2)^{1/2+ir_n}} = (1+2ir_n)\int_0^\infty \frac{t\,\sin t\,dt}{(t^2+x^2)^{3/2+ir_n}}.$$

The integral on the right hand side of the above equality clearly exists. Thus $K_{ir_n}(x)$ is bounded by $\left|\Gamma\left(\frac12+ir_n\right)\right| r_n$ for fixed x. By the following standard formula on Gamma function (where $a,\, b \in \mathbb{R}$):

$$|\Gamma(a+ib)| = \sqrt{2\pi}\, |b|^{a-1/2} e^{-a-|b|\,\pi/2} \left[1 + \mathcal{O}\left(\frac{1}{|b|}\right)\right], \quad \text{as } |b| \to \infty,$$

we get a bound: $K_{ir_n}(x) = \mathcal{O}\left(r_n e^{-\frac{\pi}{2} r_n}\right)$. Combining this bound with Hörmander's bound, we have:

$$K_{ir_n}(x)\phi_n(z)\overline{\phi_n(w)} = \mathcal{O}\left(r_n e^{-\frac{\pi}{2} r_n} \left[\left(\frac{d-1}{2}\right)^2 + r_n^2 \right]^{\frac{d-1}{2}} \right) = \mathcal{O}\left(r_n^d e^{-\frac{\pi}{2} r_n}\right) \quad \text{as} \quad n \to \infty.$$

The spectrum $\{\lambda_n\}$ of the Laplacian is discrete with ∞ as the unique accumulation point and each eigenvalue λ_n occurs with finite multiplicity, so is $\{r_n \in \mathbb{R}\}$. Let $N(x)$ be the counting function of Laplace eigenvalues with multiplicities over any smooth compact Riemannian manifold X:

$$N(x) := \sum_{\lambda_n \leqslant x} 1.$$

Assume that X is of dimension d. Weyl's law gives the asymptotic of $N(x)$ for large x (see [**MP**]).

$$N(x) = \frac{\mathrm{vol}(X)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}\, x^{\frac{d}{2}} + o\left(x^{\frac{d}{2}}\right), \quad \text{as } x \to \infty.$$

Since $\lambda_n = \left(\frac{d-1}{2}\right)^2 + r_n^2$, we have: $r_n = \sqrt{\lambda_n} - A_n$ where $A_n = \sqrt{\frac{(d-1)^2}{4} + r_n^2} - r_n > 0$. Clearly $A_n = o(1)$ as $n \to \infty$. With the bound on $K_{ir_n}(\mu)\phi_n(z)\overline{\phi_n(w)}$ obtained in above and the formula $h_f(\lambda_n) = 2^d \cdot \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_n}(\mu)$, we have:

$$k_f \ll \sum_n r_n^d e^{-\frac{\pi}{2} r_n} < \sum_n \lambda_n^{\frac{d}{2}} e^{-\frac{\pi}{2}\left(\sqrt{\lambda_n} - A_n\right)} \asymp \sum_n \lambda_n^{\frac{d}{2}} e^{-\frac{\pi}{2}\sqrt{\lambda_n}} = \int_{\frac{(d-1)^2}{4}}^{\infty} x^{\frac{d}{2}} e^{-\frac{\pi}{2}\sqrt{x}} dN(x).$$

Here $dN(x)$ means the measure on $\mathbb{R}_{>0}$ with mass 1 at Laplace eigenvalues $x = \lambda_n$ (with multiplicities), otherwise 0. Partial integration shows that

$$\int_{\frac{(d-1)^2}{4}}^{\infty} x^{\frac{d}{2}} e^{-\frac{\pi}{2}\sqrt{x}} dN(x) = x^{\frac{d}{2}} e^{-\frac{\pi}{2}\sqrt{x}} N(x)\Big|_{\frac{(d-1)^2}{4}}^{\infty} - \int_{\frac{(d-1)^2}{4}}^{\infty} e^{-\frac{\pi}{2}\sqrt{x}} \left(\frac{d}{2} x^{\frac{d}{2}-1} - \frac{\pi}{4} x^{\frac{d-1}{2}}\right) N(x) dx.$$

Applying Weyl's law on $N(x)$ to the right hand side of the above formula, we know this integral exists. The absolute and locally uniform convergence of k_f then follows.

2.5 The comparison

Now we put the data on the two sides of the trace formula together:

$$\sum_{i=0}^{\infty} 2^d \cdot \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu)\, |P_C(\phi_i)|^2 = 2\|E\|\, \mathrm{len}(C) \cdot K_0(\mu) + \mathcal{O}\left(e^{-\mu}\, \mu^{-1}\right) + \mathcal{O}\left(e^{-\mu}\, \mu^{-\frac{3}{2}}\right). \tag{2.23}$$

Multiplying $\sqrt{\frac{2\mu}{\pi}}\, e^{\mu}$ on both sides of (2.23) and taking the limitation $\mu \to \infty$, by the asymptotic formula (2.22) we easily get:

$$\lim_{\mu\to\infty} 2^d \cdot e^{\mu} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{i=0}^{\infty} K_{\nu_i}(\mu)\ |P_C(\phi_i)|^2 = 2\|E\|\,\mathrm{len}(C). \tag{2.24}$$

Substituting the data on Killing form in Lemma (1.3.3) into this formula, we have:

Theorem 2.5.1. *For any compact hyperbolic manifold and primitive closed geodesic C over it, the following holds:*

$$\lim_{\mu\to\infty} \frac{2^{d-1}\, e^{\mu}}{\sqrt{2(d-1)}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{i=0}^{\infty} K_{\nu_i}(\mu)\ |P_C(\phi_i)|^2 = \mathrm{len}(C).$$

An immediate consequence is

Corollary 2.5.2. *There are infinitely many ϕ_i's such that $P_C(\phi_i) \neq 0$.*

Proof. Assume that there exists a finite subset I of $\mathbb{N}$ such that $P_C(\phi_i) \neq 0$ for $i \in I$, $P_C(\phi_i) = 0$ for $i \notin I$. Then

$$\lim_{\mu\to\infty} \frac{2^{d-1}\, e^{\mu}}{\sqrt{2(d-1)}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{i\in I} K_{\nu_i}(\mu)\ |P_C(\phi_i)|^2 = \mathrm{len}(C).$$

Applying (2.22), we have:

$$\text{L.H.S.} = \lim_{\mu\to\infty} \frac{2^{d-1}}{\sqrt{2(d-1)}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} \sum_{i\in I} |P_C(\phi_i)|^2 = 0$$

This is a contradiction as $\mathrm{len}(C) \neq 0$. □

There is a representation-theoretic formulation for this corollary. Let G be a reductive group defined over the number field F. Let H be a subgroup of G (usually obtained as the set of fixed points of some involution on G). An automorphic (cuspidal) representation $(\pi, V_\pi) \hookrightarrow L^2\left(Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)\right)$ is called *H-distinguished* if the period

$$\int_{H(F)\backslash H(\mathbb{A}_F)^1} \phi(z)dz \neq 0$$

for some $\phi \in V_\pi$. At the moment, we are dealing with real groups. It is reasonable to call the irreducible representation π occurring in $L^2_0(\Gamma\backslash G)$ "*real automorphic representation*". Here $L^2_0(\Gamma\backslash G) = L^2(\Gamma\backslash G)$ if $\Gamma\backslash G$ is compact, $L^2_0(\Gamma\backslash G) = \overline{\langle\phi_i\rangle}$ if $\Gamma\backslash G$ is noncompact, the closure of the subspace of $L^2(\Gamma\backslash G)$ spanned by cusp forms. Denote $\Gamma \cap H$ by Γ_H. In our setting, the *H-period* is defined to be

$$\int_{\Gamma_H\backslash H} \phi(z)dz$$

for any $\phi \in V_\pi$. Any split torus $H \subset G$ is of dimension 1, and it gives rise to closed geodesic over X provided $\Gamma_H \backslash H$ is compact. In view of this, Corollary 2.5.2 reads as follows:

Theorem 2.5.3. *If $\Gamma \backslash G$ is compact, there are infinitely many real spherical automorphic representations occurring in $L^2(\Gamma \backslash G)$ which are H-distinguished for any split torus $H \subset G$ such that $\Gamma_H \backslash H$ is compact.*

Remark 2.5.4. *The Gan-Gross-Prasad conjecture* [**GP**] *asserts that (in the adelic setting, for unitary groups) the distinguishment of π is equivalent to the non-vanishing of the central critical value of certain Rankin-Selberg L-function associated with π (here we are satisfied with this very rough description, not mentioning the field extensions, base change etc.). Ichino and Ikeda re-formulated this conjecture for orthogonal groups (see* [**II**]*). The unitary case of this conjecture has been verified by W. Zhang in* [**Zh**] *under some local assumptions. It is desirable to consider the conjecture in our setting.*

2.6 Weighted periods

Assume that $\Gamma_A \backslash A$ is compact. We can use $\omega_r^+ \cdot o$ to parameter the points on $\Gamma_A \backslash A$. Let χ_1 and χ_2 be two unitary characters on $\Gamma_A \backslash A$ defined as

$$\chi_1 : z = \omega_r^+ \cdot o \mapsto e^{\frac{2\pi i m r}{T}}, \quad \chi_2 : z = \omega_r^+ \cdot o \mapsto e^{\frac{2\pi i n r}{T}}$$

for some m, $n \in \mathbb{Z}$ and $T > 0$. These two characters are well-defined over $\Gamma_0 \backslash \widetilde{C}$ if C is simple, but not so if C is a cycle. Nevertheless we call such characters as characters along the geodesic C. Define the *weighted period with character* χ to be

$$P_C(\phi, \chi) := \int_C \phi(z)\, \chi(z) dz.$$

We can view the weighted period as the period integral with respect to the complex measure $\chi(z)dz$ on C. Hence

$$\sum_{i=0}^{\infty} h_f(\lambda_i)\, P_C(\phi, \chi_1)\, \overline{P_C(\phi, \chi_2)} = \sum_{\gamma \in \Gamma} \int_C \int_C \Phi\left(d(\gamma z, w)\right) \chi_1(z)\, \chi_2^{-1}(w) dz dw.$$

As before, we divide the summation on the geometric side (i.e., the right hand side) of the above equality into $\Sigma_0^{\chi_1, \chi_2}$ and $\Sigma_1^{\chi_1, \chi_2}$ both of which have obvious meanings (similar to Σ_0 and Σ_1). Since χ_1, χ_2 are unitary, it is clear that $|\Sigma_1^{\chi_1, \chi_2}| \leqslant \Sigma_1$. With the test function $f = \Phi_\mu$ inserted, we have: $\Sigma_1^{\chi_1, \chi_2} = \mathcal{O}\left(e^{-\mu}\, \mu^{-1}\right)$ as $\mu \to \infty$. Let $z = e^{tE} \cdot o$ where $t \in (-\infty, +\infty)$, $w = e^{sE} \cdot o$ where $s \in [0, T]$, then

$$\Sigma_0^{\chi_1, \chi_2} = \int_{w \in C} \int_{z \in \widetilde{C}} \Phi_\mu\left(d(z, w)\right) \chi_1(z)\, \chi_2^{-1}(w) dz dw$$

$$= \|E\|^2 \int\limits_{s=0}^{T} \int\limits_{t=-\infty}^{+\infty} \exp\left(-\mu \cdot \frac{e^{t-s}+e^{s-t}}{2}\right) e^{\frac{2\pi imt}{T}} e^{-\frac{2\pi ins}{T}} dtds$$

$$= \|E\|^2 \int\limits_{s=0}^{T} \int\limits_{t=-\infty}^{+\infty} \exp\left(-\mu \cdot \frac{e^{t-s}+e^{s-t}}{2}\right) e^{\frac{2\pi im(t-s)}{T}} e^{\frac{2\pi i(m-n)s}{T}} dtds \quad (2.25)$$

Let $L = t - s$, $S = s$. The integration along S gives:

$$\begin{aligned}
(2.25) \quad &= \|E\|^2 \int\limits_{S=0}^{T} \int\limits_{L=-\infty}^{+\infty} \exp\left(-\mu \cdot \frac{e^{L}+e^{-L}}{2}\right) e^{\frac{2\pi imL}{T}} e^{\frac{2\pi i(m-n)S}{T}} dLdS \\
&= \begin{cases} \|E\|^2\, T \int\limits_{-\infty}^{\infty} \exp\left(-\mu \cdot \frac{e^{L}+e^{-L}}{2}\right) e^{\frac{2\pi imL}{T}} dL, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases} \\
&= \delta_{m\,n}\, \|E\|^2\, T \int_{-\infty}^{+\infty} \exp\left(-\mu \cdot \cosh L + \frac{2\pi im}{T} L\right) dL \\
&= \delta_{m\,n}\, 2\|E\|^2\, T\, K_{\frac{2\pi im}{T}}(\mu)
\end{aligned}$$

where $\delta_{m\,n}$ denotes the Kronecker symbol. The last step results from the formula (2.6). The spectral side is:

$$\sum_{i=0}^{\infty} h_f(\lambda_i)\, P_C(\phi_i,\, \chi_1)\, \overline{P_C(\phi_i,\, \chi_2)} = \sum_{i=0}^{\infty} 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu)\, P_C(\phi_i,\, \chi_1)\, \overline{P_C(\phi_i,\, \chi_2)}.$$

Putting the data on the two sides together, we have:

$$\sum_{i=0}^{\infty} 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu)\, P_C(\phi_i,\, \chi_1)\, \overline{P_C(\phi_i,\, \chi_2)} = \delta_{m\,n}\, 2B(E,\, E)\, T\, K_{\frac{2\pi im}{T}}(\mu) + \mathcal{O}\left(e^{-\mu}\, \mu^{-1}\right).$$

Multiplying $\sqrt{\frac{2\mu}{\pi}}\, e^{\mu}$ on both sides and applying (2.22), we have:

Theorem 2.6.1. *For any compact hyperbolic manifold and unitary character χ along the geodesic C,*

$$\lim_{\mu\to\infty} 2^d\, e^{\mu} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{i=0}^{\infty} K_{\nu_i}(\mu)\, |P_C(\phi,\, \chi)|^2 = 2\|E\|\mathrm{len}(C). \quad (2.26)$$

Corollary 2.6.2. *There are infinitely many i such that $P_C(\phi_i,\, \chi) \neq 0$.*

The two characters χ_1 and χ_2 are different if and only if $m \neq n$.

Theorem 2.6.3. *For any compact hyperbolic manifold and two distinct unitary characters χ_1 and χ_2 along the geodesic C,*

$$\lim_{\mu\to\infty} \frac{2^{d-1}\, e^{\mu}}{\sqrt{2(d-1)}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{i=0}^{\infty} K_{\nu_i}(\mu)\, P_C(\phi,\, \chi_1)\, \overline{P_C(\phi,\, \chi_2)} = 0.$$

Like Corollary 2.5.2, Corollary 2.6.2 imples

Theorem 2.6.4. *Let χ be a continuous unitary character of the split torus $H \in G$ such that χ is trivial on Γ_H. Suppose that $\Gamma_H\backslash H$ and $\Gamma\backslash G$ are both compact, then there are infinitely many real spherical automorphic representations π occurring in $L^2(\Gamma\backslash G)$ such that $\pi \otimes \chi$'s are H-distinguished.*

2.7 Twisted periods on two geodesics

Consider two distinct closed geodesics C_1 and C_2 on $\Gamma\backslash G/K$. Without loss of generality, we assume that C_1 is regular and $\widetilde{C}_2 = g\,\widetilde{C}_1$ for some $g \in G$. There exist $T,\, S > 0$ such that $C_1 \approx \{\exp(tE)\cdot o \,|\, 0 \leqslant t \leqslant T\}$ and $C_2 \approx \{g\exp(tE)\cdot o \,|\, 0 \leqslant t \leqslant S\}$. We have: $C_1 \approx \Gamma_1\backslash\widetilde{C}_1$ where $\Gamma_1 = \langle\gamma_1\rangle \subset \Gamma$, $\gamma_1 = e^{TE}k_0$ and $C_2 \approx \Gamma_2\backslash\widetilde{C}_2$ where $\Gamma_2 = \langle\gamma_2\rangle \subset \Gamma$ such that $\gamma_2\, g\cdot o = ge^{SE}\cdot o$. It follows that

$$g^{-1}\gamma_2 g\cdot o = e^{SE}\cdot o.$$

So there exists $k_1 \in K$ such that $g^{-1}\gamma_2 g = e^{SE}\cdot k_1$. The left action of γ_2^n transforms $g\cdot o$ to $ge^{nSE}\cdot o$. This implies that $g^{-1}\gamma_2^n g = \left(e^{SE}\cdot k_1\right)^n = e^{nSE}k_n$ for some $k_n \in K$. An argument analogous to Lemma 1.5.2 shows that $k_1 \in M$. Thus $\gamma_2 = ge^{SE}k_1 g^{-1}$ and $\Gamma_2 \subset \Gamma \cap gAMg^{-1}$. Conversely, for any $\gamma = gakg^{-1} \in \Gamma \cap gAMg^{-1}$ where $k \in M$, one has: $g^{-1}\gamma g = ak$, then $g^{-1}\gamma g\cdot o = a\cdot o$, hence $\gamma g\cdot o = ga\cdot o = ge^{yE}\cdot o \in \widetilde{C}_2$ for some $y \in \mathbb{R}$. This means that $\gamma \in \Gamma_2$. So we have shown that

Lemma 2.7.1. $\Gamma_2 = \Gamma \cap gAMg^{-1}$.

The relative trace formula with the test function Φ_μ is

$$\sum_{i=0}^{\infty} 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu) P_{C_1}(\phi_i)\overline{P_{C_2}(\phi_i)} = \|E\|^2 \sum_{\gamma\in\Gamma} \int_{s=0}^{S}\int_{t=0}^{T} \Phi_\mu\left(d\left(\gamma e^{tE}\cdot o,\, e^{sE}\cdot o\right)\right) dtds.$$

Let $\gamma \in \Gamma$ be such that $g^{-1}\gamma \in AM$, then η which is of the same class $\left(\text{in } \Gamma_2\backslash\Gamma/\Gamma_1\right)$ with γ also satisfies the condition $g^{-1}\eta \in AM$: $\eta = \gamma_2^m\gamma\gamma_1^n$ for some $m,\, n \in \mathbb{Z}$, then $g^{-1}\eta = g^{-1}\cdot ge^{mSE}k_1^m g^{-1}\cdot\gamma\gamma_1^n = e^{mSE}k_1^m\cdot g^{-1}\gamma\cdot\gamma_1^n \in AM$ since $g^{-1}\gamma,\, \gamma_1 \in AM$. If $g^{-1}\gamma$ does not lie in AM, then the above argument shows that $\eta \notin AM$ for any η of the same class with γ. To divide the geometric side of the trace formula into a summation with respect to double coset classes, we check the uniqueness of expressing an element, say η, as the form $\eta = \gamma'\gamma\gamma''$ where $\gamma' \in \Gamma_2$, $\gamma'' \in \Gamma_1$.

Proposition 2.7.2. *Any element η in the class $\widetilde{\gamma} \in \Gamma_2\backslash\Gamma/\Gamma_1$ can be written as $\eta = \gamma'\gamma\gamma''$ for unique $\gamma' \in \Gamma_2$, $\gamma'' \in \Gamma_1$.*

This proposition results from a weaker statement:

Lemma 2.7.3. *Any element η in the class $\widetilde{\gamma} \in \Gamma_2\backslash\Gamma/\Gamma_1$ such that $g^{-1}\gamma \notin AM$ can be written as $\eta = \gamma'\gamma\gamma''$ for unique $\gamma' \in \Gamma_2$, $\gamma'' \in \Gamma_1$.*

Proof of the Lemma. It suffices to show that if $\gamma = \gamma_2^m\gamma\gamma_1^n$ then $m = n = 0$. The assumption $\gamma = \gamma_2^m\gamma\gamma_1^n$, i.e., $\gamma = ge^{mSE}k_1^m g^{-1}\cdot\gamma\cdot e^{nTE}k_0^n$ indicates that $g^{-1}\gamma = e^{mSE}k_1^m\cdot g^{-1}\gamma\cdot e^{nTE}k_0^n$. As $e^{mSE}k_1^m$ and $e^{nTE}k_0^n$ lie in AM, there exists $\delta = \mathrm{diag}(k, h)$ for some $k \in O_2(\mathbb{R})$, $h \in GL_{d-1}(\mathbb{C})$ such that $\delta\gamma_1^n\delta^{-1} = \delta e^{nTE}k_0^n\delta^{-1} = \mathrm{diag}(\epsilon, \epsilon^{-1}, u_1, \cdots, u_{d-1})$ where $\epsilon > 0$ and $|u_i| = 1$. Denote $\mathrm{diag}(u_1, \cdots, u_{d-1})$ by u and let $\tau = \delta g^{-1}\gamma\delta^{-1}$ be of the form

$$\tau = \begin{pmatrix} a & b & x \\ c & d & y \\ z^T & w^T & \kappa \end{pmatrix}$$

where $x, y, z, w \in \mathbb{C}$ and z^T is the transpose of z. Then there exists $\alpha \in \mathbb{R}$ such that $ke^{mSE}k^{-1} = \mathrm{diag}(\epsilon^\alpha, \epsilon^{-\alpha})$. The conjugacy of δ applying to $g^{-1}\gamma$ shows that

$$\begin{pmatrix} a & b & x \\ c & d & y \\ z^T & w^T & \kappa \end{pmatrix} = \begin{pmatrix} \epsilon^\alpha & & \\ & \epsilon^{-\alpha} & \\ & & hk_1^m h^{-1} \end{pmatrix} \begin{pmatrix} a & b & x \\ c & d & y \\ z^T & w^T & \kappa \end{pmatrix} \begin{pmatrix} \epsilon & & \\ & \epsilon^{-1} & \\ & & u \end{pmatrix}.$$

By comparison, we have

$$x_i = \epsilon^\alpha x_i u_i \tag{2.27}$$

$$y_i = \epsilon^{-\alpha} y_i u_i \tag{2.28}$$

$$z^T = hk_1^m h^{-1} z^T \epsilon \tag{2.29}$$

$$w^T = hk_1^m h^{-1} w^T \epsilon^{-1} \tag{2.30}$$

Clearly $m = 0$ is equivalent to $n = 0$. Let us assume that neither m nor n is zero. It follows that $\alpha \neq 0$, otherwise $e^{mSE} = 1$ which implies that $m = 0$ (since $S \neq 0$), a contradiction. If $x \neq 0$, say $x_i \neq 0$, then $\epsilon^\alpha u_i = 1$ by (2.27). Since $\alpha \neq 0$, $\epsilon > 0$ and $|u_i| = 1$, we get: $\epsilon = 1$. This means that $e^{nTE} = 1$ which implies that $n = 0$, a contradiction. Thus $x = 0$. Similarly, $y = 0$ by (2.28). As for z, consider the Euclidean norm $\left|h^{-1}z^T\right|$. From (2.29), we have: $h^{-1}z^T = k_1^m \cdot h^{-1}z^T \cdot \epsilon$. The factor k_1^m lies in SO_{d-1}, so

$$\left|h^{-1}z^T\right| = \left|k_1^m \cdot h^{-1}z^T \cdot \epsilon\right| = \left|h^{-1}z^T\right| \cdot \epsilon.$$

If $z \neq 0$, then $h^{-1}z^T \neq 0$ and $\epsilon = 1$ from the above equality. This is a contradiction as already discussed. Hence $z = 0$. Similarly $w = 0$ by (2.30). Now it is clear that $g^{-1}\gamma$

lies in $\{\mathrm{diag}(\beta,\, \eta) \,|\, \beta \in GL_2,\, \eta \in GL_{d-1}\} = AM$, a contradiction as we have assumed that $g^{-1}\gamma \notin AM$. Consequently, $m = n = 0$. □

Proof of the Proposition. In view of the Lemma 2.7.3, we just have to show that the class $\widetilde{\gamma}$ such that $g^{-1}\gamma \in AM$ does not exist. This is trivial: if $g^{-1}\gamma \in AM$, then the commutativity between A and M shows that, for any $a \in A$, there exists a unique $b \in A$ such that $g^{-1}\gamma a \cdot o = b \cdot o$, i.e., $\gamma a \cdot o = gb \cdot o$; conversely, for any $b \in A$, there exists a unique $a \in A$ such that $\gamma a \cdot o = gb \cdot o$, which implies that $C_1 = C_2$, a contradiction as we assume that C_1 and C_2 are two distinct geodesics in this section. □

By the proposition, the geometric side ("G. S.") is:

$$\begin{aligned}
\text{G. S.} &= B(E,\, E) \sum_{\widetilde{\gamma} \in \Gamma_2 \backslash \Gamma / \Gamma_1} \sum_m \sum_n \int_{s=0}^{S} \int_{t=0}^{T} \Phi_\mu \left(d\left(\gamma \gamma_1^m e^{tE} \cdot o,\, \gamma_2^n g e^{sE} \cdot o \right) \right) dtds \\
&= B(E,\, E) \sum_{\widetilde{\gamma} \in \Gamma_2 \backslash \Gamma / \Gamma_1} \sum_n \int_{s=0}^{S} \int_{t=0}^{\infty} \Phi_\mu \left(d\left(\gamma e^{tE} \cdot o,\, g e^{nSE} k_1^n g^{-1} g e^{sE} \cdot o \right) \right) dtds \\
&= B(E,\, E) \sum_{\widetilde{\gamma} \in \Gamma_2 \backslash \Gamma / \Gamma_1} \sum_n \int_{s=0}^{S} \int_{t=-\infty}^{\infty} \Phi_\mu \left(d\left(\gamma e^{tE} \cdot o,\, g e^{nSE} k_1^n e^{sE} \cdot o \right) \right) dtds \\
&= B(E,\, E) \sum_{\widetilde{\gamma} \in \Gamma_2 \backslash \Gamma / \Gamma_1} \sum_n \int_{s=0}^{S} \int_{t=-\infty}^{\infty} \Phi_\mu \left(d\left(\gamma e^{tE} \cdot o,\, g e^{nSE} e^{sE} k_1^n \cdot o \right) \right) dtds \\
&= B(E,\, E) \sum_{\widetilde{\gamma} \in \Gamma_2 \backslash \Gamma / \Gamma_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_\mu \left(d\left(g^{-1} \gamma e^{tE} \cdot o,\, e^{sE} \cdot o \right) \right) dtds
\end{aligned}$$

Now $g^{-1}\gamma$ plays the same role with γ in Sect. 2.3.2, so we have to know the information on $M(g^{-1}\gamma)\, N(g^{-1}\gamma)$ and $\delta(g^{-1}\gamma)$. The left and right actions of Γ_2 and Γ_1 (resp.) on γ is equivalent to the left action of $\{\exp(mSE) \,|\, m \in \mathbb{Z}\}$ and right action of Γ_1 on $g^{-1}\gamma$. Such an observation and the trivial fact $\{g^{-1}\gamma \,|\, \gamma \in \Gamma\}$ is discrete enable us to show that the conclusions before Lemma 2.3.7 still hold with γ replaced by $g^{-1}\gamma$. The Lemma 2.3.7 also holds when we replace γ with $g^{-1}\gamma$, once noticing the following lemma and that AM plays the same role with $\tilde{1}$ there. Define

$$M' = \left\{ \begin{pmatrix} 1 & & \\ & -1 & \\ & & \rho \end{pmatrix} :\, \rho \in O_{d-1}(\mathbb{R}),\, \det(\rho) = -1 \right\}.$$

Lemma 2.7.4. *$g^{-1}\gamma \notin AM'$ for any $\gamma \in \Gamma$.*

Proof. Similar to the case AM, only noting that $kak^{-1} = a^{-1}$ for $a \in A$, $k \in M'$. □

As in the Sect. 2.3.2, we have:

$$\text{G. S.} = \mathcal{O}\left(e^{-\mu}\,\mu^{-1}\right) + \mathcal{O}\left(e^{-\mu}\,\mu^{-\frac{3}{2}}\right), \text{ as } \mu \to \infty.$$

Note that the main term $2\|E\|\text{len}(C)K_0(\mu)$ in (2.23) does not appear here. The reason is that, in the present setting, there is no term corresponding to Σ_0 on the geometric side. Multiplying $e^{\mu}\mu^{1-\epsilon}$ ($\forall\epsilon > 0$) on both sides of the trace formula and taking the limitation $\mu \to \infty$, we have:

Theorem 2.7.5. *For any compact hyperbolic manifold, the following holds:*

$$\lim_{\mu\to\infty} e^{\mu}\,\mu^{-\frac{d}{2}+\frac{3}{2}-\epsilon} \sum_{i=0}^{\infty} K_{\nu_i}(\mu) P_{C_1}(\phi_i)\overline{P_{C_2}(\phi_i)} = 0, \quad \epsilon > 0.$$

Proposition 2.7.6. *Let X be a compact hyperbolic manifold with dimension $d \geqslant 3$. Suppose that $C_1 \cap C_2 \neq \varnothing$, then there are infinitely many ϕ_i's such that $P_{C_1}(\phi_i)$ and $P_{C_2}(\phi_i)$ are nonvanishing at the same time.*

Proof. The condition $C_1 \cap C_2 \neq \varnothing$ implies that there exists $\gamma \in \Gamma$ such that $\delta\left(g^{-1}\gamma\right) = 1$. This means that the term $\mathcal{O}\left(e^{-\mu}\,\mu^{-1}\right)$ does exist on the geometric side (see Remark 2.3.11), i.e., the order of the geometric side, when multiplied by e^{μ}, is $\frac{1}{\mu}$. Assume that there is a finite subset $I \subset \mathbb{N}$ such that $P_{C_1}(\phi_i) \neq 0$, $P_{C_2}(\phi_i) \neq 0$ for $i \in I$ and $P_{C_1}(\phi_i) = 0$ or $P_{C_2}(\phi_i) = 0$ for $i \notin I$. Clearly $I \neq \varnothing$ as $\phi_0 \in I$. Then by (2.22),

$$\lim_{\mu\to\infty} e^{\mu}\,\mu^{-\frac{d-1}{2}} \sum_{i\in I} K_{\nu_i}(\mu) P_{C_1}(\phi_i)\overline{P_{C_2}(\phi_i)} = \lim_{\mu\to\infty} \mu^{-\frac{d}{2}} \sum_{i\in I} P_{C_1}(\phi_i)\overline{P_{C_2}(\phi_i)}.$$

This formula shows that the spectral side, when multiplied by e^{μ}, has order $\mu^{-\frac{d}{2}}$. So we have $\frac{d}{2} = 1$, a contradiction as $d \geqslant 3$. □

Like Proposition 2.5.2, Proposition 2.7.6 implies

Theorem 2.7.7. *For $d \geqslant 3$, let H_1, H_2 be two distinct split tori in G such that $\Gamma_{H_1}\backslash H_1$, $\Gamma_{H_2}\backslash H_2$ are compact and $H_1 = g^{-1}H_2 g$ for some $g \in G$. Assume that $\Gamma\backslash G$ is compact and $H_1 \cap g\gamma H_2 k \neq \emptyset$ for some $\gamma \in \Gamma$, $k \in K$, then there are infinitely many real spherical automorphic representations π's occurring in $L^2(\Gamma\backslash G)$ such that $\pi \otimes \pi^{\vee}$ are $H_1 \times H_2$-distinguished.*

2.8 The L^2-norm

The integration of $\phi_i(z)\overline{\phi_i(w)}$ over the diagonal subset $\{(z,\, z)\,|\, z \in C\}$ of $C \times C$ gives the (square of) L^2-norm of ϕ_i over C. Starting from the pre-trace formula, we have:

$$\sum_{\gamma\in\Gamma} \int_C f(z^{-1}\gamma z)dz = \sum_{i=0}^{\infty} h_f(\lambda_i) \int_C |\phi_i(z)|^2 dz.$$

The test function f is as before: bi-K-invariant, uniformly continuous such that the kernel K_f converges everywhere. In particular, we still use Φ_μ. The geometric side is again divided into two parts Σ_0 and Σ_1 based on double cosets in $\Gamma_0\backslash G/\Gamma_0$ where

$$\Sigma_0 = \sum_{\gamma\in\Gamma_0} \int_C \Phi_\mu\left(d(\gamma z,\, z)\right) dz$$

and

$$\Sigma_1 = \sum_{\tilde\gamma\neq\tilde 1} \sum_{\gamma_1\in\Gamma_0} \sum_{\gamma_2\in\Gamma_0} \int_C \Phi_\mu\left(d(\gamma\gamma_1 z,\, \gamma_2 z)\right) dz.$$

We still choose the representative elements γ's in the set Λ. Let $\gamma = \gamma_0^n$ and $z = e^{xE}\cdot o$ for $x \in [0,\, T]$. Then $\gamma z = e^{(nT+x)E}\cdot o$ and

$$\begin{aligned}
\Sigma_0 &= \sum_{n=-\infty}^{\infty} \int_0^T \Phi_\mu(d(e^{(nT+x)E}\cdot o,\, e^{xE}\cdot o))\|E\| dx \\
&= T\|E\| \sum_{n=-\infty}^{\infty} \exp\left(-\mu\cdot\cosh nT\right) \\
&= T\|E\| \left(2\sum_{n=1}^{\infty} \exp\left(-\mu\cdot\cosh nT\right) + \exp(-\mu)\right)
\end{aligned}$$

For fixed μ and positive x, the function $\exp\left(-\mu\cdot\cosh xT\right)$ decreases as x increases. So

$$\int_1^\infty \exp(-\mu\cdot\cosh xT)dx < \sum_{n=1}^{\infty} \exp\left(-\mu\cdot\cosh nT\right) < \int_0^\infty \exp(-\mu\cdot\cosh xT)dx.$$

By (2.6), the right hand side of the above inequality is equal to $\frac{K_0(\mu)}{T}$, the left hand side is equal to $\frac{K_0(\mu)}{T} - \frac{1}{T}\int_0^T \exp(-\mu\cosh x)dx$. Hence

$$T\|E\|\left(\frac{2K_0(\mu)}{T} + e^{-\mu}\right) - 2\|E\|\int_0^T \exp(-\mu\cosh x)dx < \Sigma_0 < T\|E\|\left(\frac{2K_0(\mu)}{T} + e^{-\mu}\right).$$

Multiplying e^μ to this inequality, we have:

$$2\|E\|K_0(\mu)e^\mu + T\|E\| - 2\|E\|\int_0^T \exp(-\mu(\cosh x - 1))dx < e^\mu\,\Sigma_0 < 2\|E\|K_0(\mu)e^\mu + T\|E\|. \tag{2.31}$$

It is easy to see that $\lim\limits_{\mu\to\infty} K_0(\mu)e^\mu = 0$ by (2.22) and $\lim\limits_{\mu\to\infty}\int_0^T \exp(-\mu(\cosh x - 1))dx = 0$. So both sides of (2.31) tend to $T\|E\|$ as $\mu\to\infty$, which implies that:

$$\lim_{\mu\to\infty} e^\mu\,\Sigma_0 = T\|E\| = \text{len}(C).$$

Let $\gamma = \omega_{r_0}\theta_{w_0}k_\gamma \in \Gamma\smallsetminus\Gamma_0$ as before. Recall that

$$d\left(\gamma\gamma_0^m e^{xE}\cdot o,\, \gamma_0^n e^{xE}\cdot o\right) \;=\; d\left(\gamma\omega_{e^{mT+x}}\cdot o,\, \omega_{e^{mT+x}}\cdot o\right)$$

$$= d\left(\theta_{w_0+v}\,\omega_s\cdot o,\ \omega_{r_0^{-1}e^{nT+x}}\cdot o\right)$$

$$= \|E\|\operatorname{arccosh}\left(\frac{\left|\frac{w_0+v}{s}\right|^2+1+\frac{e^{2(nT+x)}}{(r_0 s)^2}}{\frac{2e^{nT+x}}{r_0 s}}\right)$$

For the meanings of v and s, see Sect. 2.3.2. Note that e^{mT+x} is the r there and e^{nT+x} is the r' there. Clearly there exists $x_0 \in [0,\, T]$ such that

$$\int_0^T \sum_{\widetilde{\gamma}\neq\widetilde{1}}\sum_{m,\,n}\Phi_\mu\left(d(\gamma\gamma_0^m e^{xE}\cdot o,\ \gamma_0^n e^{xE}\cdot o)\right)dx = T\cdot\sum_{\widetilde{\gamma}\neq\widetilde{1}}\sum_{m,\,n}\Phi_\mu\left(d(\gamma\gamma_0^m e^{x_0E}\cdot o,\ \gamma_0^n e^{x_0E}\cdot o)\right).$$

Let $s_m^{-1} = \frac{1-u_{11}}{2}\,e^{mT+x_0} + \frac{1+u_{11}}{2}\,e^{-(mT+x_0)}$ and $\delta_m(\gamma) = \left(\left|\frac{w_0+v}{s}\right|^2+1\right)_{x=x_0} = f\left(e^{mT+x_0}\right)$ where $f(r) = M(\gamma)r^2 + N(\gamma)r^{-2} + Q(\gamma)$. For the meaning of the terms $M(\gamma)$, $N(\gamma)$, $Q(\gamma)$, see Sect. 2.3.2. Denote

$$X_1 = T\|E\|\sum_{\widetilde{\gamma}\neq\widetilde{1}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\mu\cdot\frac{\delta_m(\gamma)+\left(\frac{e^x}{r_0 s_m}\right)^2}{\frac{2e^x}{r_0 s_m}}\right)dx$$

and

$$X_2 = T\|E\|\sum_{\widetilde{\gamma}\neq\widetilde{1}}\sum_{m=-\infty}^{\infty}\max_{x\in\mathbb{R}}\exp\left(-\mu\cdot\frac{\delta_m(\gamma)+\left(\frac{e^x}{r_0 s_m}\right)^2}{\frac{2e^x}{r_0 s_m}}\right).$$

The function $\psi(x) = \exp\left(-\mu\cdot\frac{\delta_m(\gamma)+\left(\frac{e^x}{r_0 s_m}\right)^2}{\frac{2e^x}{r_0 s_m}}\right)$ increases monotonously at first, then decreases for $x\in\mathbb{R}$. Note that $\delta_m(\gamma)$, s_m are independent from n, x. So we have:

$$X_1 - X_2 \leqslant \Sigma_1 = T\|E\|\sum_{\widetilde{\gamma}\neq\widetilde{1}}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\left(-\mu\cdot\frac{\delta_m(\gamma)+\frac{e^{2(nT+x_0)}}{(r_0 s_m)^2}}{\frac{2e^{nT+x_0}}{r_0 s_m}}\right) \leqslant X_1 + X_2.$$

Let $L = e^x$, then

$$X_1 = T\|E\|\sum_{\widetilde{\gamma}\neq\widetilde{1}}\sum_{m=-\infty}^{\infty}\int_0^{\infty}\exp\left(-\mu\cdot\frac{\delta_m(\gamma)+\left(\frac{L}{r_0 s_m}\right)^2}{\frac{2L}{r_0 s_m}}\right)\frac{dL}{L}$$

$$= T\|E\|\sum_{\widetilde{\gamma}\neq\widetilde{1}}\sum_{m=-\infty}^{\infty}2K_0\left(\mu\sqrt{\delta_m(\gamma)}\right)$$

by formula (2.2). Denote

$$X_3 = 2T\|E\|\sum_{\widetilde{\gamma}\neq\widetilde{1}}\int_{-\infty}^{\infty}K_0\left(-\mu\cdot\sqrt{M(\gamma)e^{2x}+N(\gamma)e^{-2x}+Q(\gamma)}\right)dx$$

and

$$X_4 = T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \max_{x\in\mathbb{R}} K_0\left(\mu\cdot\sqrt{M(\gamma)e^{2x}+N(\gamma)e^{-2x}+Q(\gamma)}\right)$$

For the similar reason, we have: $X_3 - X_4 \leqslant X_1 \leqslant X_3 + X_4$.

It is clear that $X_2 = T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \sum_{m=-\infty}^{\infty} \exp\left(-\mu\sqrt{\delta_m(\gamma)}\right)$. Denote

$$X_5 = T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \int_{-\infty}^{\infty} \exp\left(-\mu\cdot\sqrt{M(\gamma)e^{2x}+N(\gamma)e^{-2x}+Q(\gamma)}\right) dx$$

and

$$X_6 = T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \max_{x\in\mathbb{R}} \exp\left(-\mu\cdot\sqrt{M(\gamma)e^{2x}+N(\gamma)e^{-2x}+Q(\gamma)}\right).$$

As before, $X_5 + X_6 \leqslant X_2 \leqslant X_5 + X_6$. In conclusion, we have:

$$X_3 - X_4 + X_5 - X_6 \leqslant \Sigma_1 \leqslant X_3 + X_4 + X_5 + X_6.$$

Noting that $M(\gamma)e^{2x}+N(\gamma)e^{-2x}+Q(\gamma) \geqslant \left(\sqrt{M(\gamma)}e^{x} - \sqrt{N(\gamma)}e^{-x}\right)^2 + \delta(\gamma)$, the term X_6 is easy to be dealt with:

$$X_6 = T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \exp\left(-\mu\cdot\sqrt{\delta(\gamma)}\right) = \mathcal{O}\left(e^{-\mu}\mu^{-1}\right) + \mathcal{O}\left(e^{-\mu}\,\mu^{-\frac{3}{2}}\right). \tag{2.32}$$

For the second step, see the ending part of Sect. 2.3.2. It is clear that

$$X_4 = T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} K_0\left(\mu\sqrt{\delta(\gamma)}\right).$$

When μ is large, there is a uniform (for all $\gamma\in\Lambda$) number C_μ such that $K_0\left(\mu\sqrt{\delta(\gamma)}\right) \leqslant C_\mu\, e^{-\mu\sqrt{\delta(\gamma)}}$. Hence

$$X_4 = \mathcal{O}\left(\sum_{\tilde{\gamma}\neq\tilde{1}} \exp\left(-\mu\cdot\sqrt{\delta(\gamma)}\right)\right) = \mathcal{O}\left(e^{-\mu}\mu^{-1}\right).$$

Actually we can replace the symbol $\mathcal{O}$ with o, i.e., $C_\mu \to 0$ as $\mu\to\infty$, while such a change makes no difference to our conclusion as we shall estimate $\lim_{\mu\to\infty} e^{\mu}\,\Sigma_1$.

Now we estimate X_5, based on which X_3 is estimated like X_4. Let $L = e^{2x}$, then

$$\begin{aligned} X_5 &= T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \int_0^{\infty} \exp\left(-\mu\cdot\sqrt{M(\gamma)L+N(\gamma)L^{-1}+Q(\gamma)}\right)\frac{dL}{L} \\ &= T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \int_0^{\infty} \exp\left(-\mu\cdot\sqrt{\left(\sqrt{M(\gamma)L}-\sqrt{N(\gamma)L^{-1}}\right)^2+\delta(\gamma)}\right)\frac{dL}{L} \end{aligned} \tag{2.33}$$

Let $S = \sqrt{M(\gamma)L} - \sqrt{N(\gamma)L^{-1}}$, then

$$\begin{aligned}
(2.33) \quad &= 4T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \int_0^\infty \exp\left(-\mu\sqrt{S^2+\delta(\gamma)}\right) \frac{dS}{\sqrt{S^2+4\sqrt{M(\gamma)N(\gamma)}}} \\
&\leqslant T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \frac{2}{\sqrt[4]{M(\gamma)N(\gamma)}} \int_0^\infty \exp\left(-\mu\sqrt{S^2+\delta(\gamma)}\right) dS \\
&= T\|E\| \sum_{\tilde{\gamma}\neq\tilde{1}} \frac{2\sqrt{\delta(\gamma)}}{\sqrt[4]{M(\gamma)N(\gamma)}} K_1\left(\mu\sqrt{\delta(\gamma)}\right) \qquad (2.34)
\end{aligned}$$

The last step is an application of the formula 3.461.6 of [**GR**]:

$$\int_0^\infty \exp\left(-a\sqrt{x^2+b^2}\right) dx = bK_1(ab), \quad \operatorname{Re} a > 0,\ \operatorname{Re} b > 0. \qquad (2.35)$$

When μ is large, we have: $\sqrt{\delta(\gamma)}K_1\left(\mu\sqrt{\delta(\gamma)}\right) \leqslant C_\mu \sqrt{\delta(\gamma)}\, e^{-\mu\sqrt{\delta(\gamma)}} = \mathcal{O}\left(e^{-(\mu-\epsilon)\sqrt{\delta(\gamma)}}\right)$ where the $\mathcal{O}$-constant is uniform for all $\gamma \in \Lambda$. Thus

$$X_5 = \mathcal{O}\left(\sum_{\tilde{\gamma}\neq\tilde{1}} e^{-(\mu-\epsilon)\sqrt{\delta(\gamma)}}\right) = \mathcal{O}\left(e^{-(\mu-\epsilon)}(\mu-\epsilon)^{-1}\right) + \mathcal{O}\left(e^{-(\mu-\epsilon)}\,(\mu-\epsilon)^{-\frac{3}{2}}\right), \quad \epsilon > 0.$$

The same bound holds for X_3. In view of what have been obtained now, the following hold:

$$\lim_{\mu\to\infty} e^\mu X_i = 0, \quad i = 3,\ 4,\ 5,\ 6.$$

Consequently, we have: $\lim_{\mu\to\infty} e^\mu \Sigma_1 = 0$. The main conclusion is summarized as

Theorem 2.8.1.

$$\lim_{\mu\to\infty} e^\mu \sum_{i=0}^\infty 2^d \cdot \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu) \int_C |\phi_i|^2 = \operatorname{len}(C). \qquad (2.36)$$

Corollary 2.8.2. *When $d = 2$, i.e., X is a compact Riemann surface with genus $g \geqslant 2$, the following asymptotic holds:*

$$\sum_{\lambda_n \leqslant x} \int_C |\phi_n|^2 \sim \frac{\operatorname{len}(C)}{4\pi} x \quad \text{as} \quad x \to \infty.$$

Proof. Such asymptotic is derived in an analogous way with the Theorem 2 of [**MW**]. We omit the detailed discussions since it is almost trivial once familiarizing with the argument in [**MW**]. Nevertheless we give an outline. First we have a refined version of (2.36):

$$\lim_{\mu\to\infty} \frac{\pi}{2\mu} \sum_{n=0}^\infty e^{-\frac{r_n^2}{2\mu}} \int_C |\phi_n|^2 = \frac{\operatorname{len}(C)}{4} \qquad (2.37)$$

where $r_n = \nu_n$. In [**MW**], there is a refined formula (see Theorem 1 there) which is derived from a formula (see formula (22) in Proposition 3 there) of the same type with (2.36) here by using the uniform estimates of K-Bessel function and Reznikov's nontrivial bound on L^2-norm (for compact hyperbolic surfaces): $\int_C |\phi_n|^2 = \mathcal{O}\left(\lambda_n^{\frac{1}{4}}\right)$. Note that there are two variables involved in $K_z(\mu)$, so the *uniform* estimate on $K_z(\mu)$ is needed when we take the limitation $\mu \to \infty$. To obtain the above formula, we just do a similar argument. Actually we have a term $\sqrt{\frac{\pi}{2\mu}}$ on the left hand side of (2.36) which does not appear in the formula (22) of [**MW**] (this is not surprising since $\int_C |\phi_n|^2 \geqslant \left|\int_C \phi_n\right|^2$). So all those separate steps for building Theorem 1 of [**MW**] clearly hold for our situation because we have lower μ-order in (2.36) compared to formula (22) of [**MW**]: the term $\sqrt{\frac{\pi}{2\mu}}$ can lower the order of μ. In the present case, $\lambda_n = \frac{1}{4} + r_n^2$ where $r_n \in \mathbb{R}$: single out finitely many terms on the left hand side of the above formula for which λ_n's are small and take the limitation $\mu \to \infty$, then these terms vanish in view of (2.22), so it is without loss for us to consider only those ϕ_n's with large eigenvalues, i.e., r_n's are large enough real numbers. A slight modification of this refined formula gives

$$\sum_{n=0}^{\infty} e^{-\frac{\lambda_n}{2\mu}} \int_C |\phi_n|^2 \sim \frac{\text{len}(C)}{4 \cdot \frac{\pi}{2\mu}} \cdot e^{-\frac{1}{4}\frac{1}{2\mu}}, \quad \text{as} \quad \mu \to \infty.$$

Let $\rho = 1$ and $L(x) = \frac{\text{len}(C)}{4\pi} e^{-\frac{1}{4x}}$. If we define the probability measure at λ_n to be $\int_C |\phi_n|^2$ and denote this measure by $U\{d\lambda\}$, then the above asymptotic reads

$$\int_0^{\infty} e^{-y\lambda}\, U\{d\lambda\} \sim \frac{\text{len}(C)}{4\pi y} \cdot e^{-\frac{1}{4}y} = y^{-\rho} L\left(\frac{1}{y}\right) \quad \text{as} \quad y \to 0$$

where $y = \frac{1}{2\mu}$. The Tauberian Theorem (see Theorem 2 on p. 445 of [**Fe**]) implies that

$$\sum_{\lambda_n \leqslant x} \int_C |\phi_n|^2 = \int_0^x U\{d\lambda\} \sim \frac{x^{\rho}}{\Gamma(\rho+1)} L(x) = \frac{x}{\Gamma(2)} \frac{\text{len}(C)}{4\pi} e^{-\frac{1}{4x}} \sim \frac{\text{len}(C)}{4\pi} x \quad \text{as} \quad x \to \infty.$$

□

Remark 2.8.3. *In* [**Ze**], *S. Zelditch obtained the following general result* (*see formula (3.4) there*)*: let X be a compact manifold of dimension n and $Y \subset X$ be a submanifold of dimension d, then*

$$\sum_{\sqrt{\lambda_i} \leqslant T} \left| \int_Y \phi_i \right|^2 \sim C_{n,Y} \cdot T^{n-d} \quad \text{as} \quad T \to \infty$$

where $\{\phi_i\}$ is a family of Laplacian eigenfunctions which are orthonormal over X, $C_{n,Y}$ is a constant dependent on n and Y. Note that the author used the operator $\sqrt{\Delta}$ there, so the respective eigenvalues μ_i's are just $\sqrt{\lambda_i}$'s. The result in [**MW**] *gives the explicit term $C_{n,Y}$ for compact Riemann surfaces with genus $g \geqslant 2$, while our Corollary 2.8.2 gives the asymptotic of the squared L^2 norms for compact Riemann surfaces.*

Remark 2.8.4. *To get an asymptotic of L^2-norms or periods for higher-dimensional compact hyperbolic manifolds, one has to use good sup-norm estimate of any single eigenfunction on the manifold or rather when restricted to geodesics (like Reznikov's bound). The classical Hörmander's bound is not enough even for surfaces.*

Remark 2.8.5. *In* [**MP**]*, the authors obtained an asymptotic for $\sum_{\lambda_n \leqslant x} |\phi_n(z)|^2$ where z is any point on a compact smooth Riemannian manifold. Such an asymptotic is derived from a Wiener-Ikehara Theorem while the proof of Wiener-Ikehara's theorem under use does not indicate the reliability (i.e., the locally continuous dependence) of the asymptotic on the point z.*

Chapter 3

Periods over Totally Geodesic Submanifolds — Compact Case

In light of what we have done up to now, it is natural to consider the more general sub-objects of the hyperbolic manifolds on which the integration of ϕ_i is done. Those immediately coming into mind are compact totally geodesic submanifolds (or *cycles* if they have self-intersections in X) Y which are realized by the embedding: $Y \hookrightarrow X := \Gamma\backslash G/K$. Closed geodesics are special examples: $C \approx \Gamma_0\backslash\widetilde{C} \hookrightarrow X$ where $\widetilde{C} = SO(1,\,1)\cdot o \hookrightarrow G/K$. This motivates us to focus on Y of the form: $Y \approx \Gamma_0\backslash G^*/K^* \hookrightarrow X$ where

$$G^* = \{\tau = \mathrm{diag}(\tau_1,\,\tau_2) \in G \,|\, \tau_1 \in O(1,\,n),\ \tau_2 \in O(d-n)\}\,,$$

$$K^* = K \cap G^* = \{\mathrm{diag}(\rho_1,\,\rho_2) \in K \,|\, \rho_1 \in O(n),\ \rho_2 \in O(d-n)\}$$

is the maximal compact subgroup of G^* and $\Gamma_0 = \Gamma \cap G^*$ is a torsion-free uniform lattice in G^* (the fundamental group of Y). Actually, by a proper conjugacy in G, any totally geodesic n-dimensional submanifold can be realized as Y of the above form. When $n = 1$, $G^*/K^* \approx A \cdot o$: if $\det(\tau_1) = -1$ for $\tau \in G^*$, then as a point in G^*/K^*, $\tau K^* = \tau\rho K^* \in SO(1,\,1)K^*$ where ρ is such that $\det(\rho_1) = -1$. Actually G^* can arise in the following (more often used) way. Let V be the $(d+1)$-dimensional vector space over real numbers equipped with the pseudo-metric $q_V = \langle\,,\,\rangle_V$ (see 1.1). For a given subspace $W \subset V$ of dimension $(n+1)$ (where $1 \leqslant n \leqslant d$) and its orthogonal supplement U (with respect to q_V), let $q_V|_W$, $q_V|_U$ denote the restrictions of q_V on W and U respectively. Then we have a subgroup $H := (GL\,(W,\,q_V|_W) \cap GL\,(U,\,q_V|_U))^0 \subset G$ where $GL\,(W,\,q_V|_W)$ stands for the group of $q_V|_W$-preserving linear transforms of W, $(\ )^0$ means the connected component of the corresponding group which contains the identity element. The aforementioned group G^* is obtained by choosing a special subspace W: for $0 \leqslant i \leqslant d$, let $W = \left\{x = (x_i) \in \mathbb{R}^{d+1} \,\middle|\, x_i = 0,\ \forall\ i \geqslant n+1\right\}$, then $U = \left\{x = (x_i) \in \mathbb{R}^{d+1} \,\middle|\, x_i = 0,\ \forall\ i \leqslant n\right\}$ and $H = G^*$. One should be careful that those points on Y, when pulled back to G according to the principal bundle $G \to G/K \to X \hookleftarrow Y$, do not necessarily lie in G^* although Y is characterized in terms of G^*. Throughout this chapter we assume that $n \geqslant 2$ and X is compact, i.e., Γ is uniform and torsion-free in G. Our main conclusion is Theorem 3.3.6.

3.1 The geometric side

The results in Sect. 2.4 hold for any X which is compact, so we have the absolute and locally uniform convergence for the spectral expansion. With the test function Φ_μ as before, integrating both sides of the pre-trace formula over $Y \times Y$ gives:

$$\sum_{i=0}^{\infty} 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu) \left| \int_Y \phi_i(z) dz \right|^2 = \sum_{\gamma \in \Gamma} \int_Y \int_Y \Phi_\mu \left(\gamma z_1,\, z_2\right) dz_1 dz_2$$

where dz is the hyperbolic measure of Y. One can use Poincaré coordinates $(u,\, r) \in \mathcal{P}^n$ to characterize those points in Y by identifying the fundamental domain of G^*/K^* (for the lattice Γ_0) with Y. More precisely, $z \in Y$ can be written as $z = \theta_u \omega_r \cdot o \in Y$ where $(u,\, r)$ lies in a (bounded) domain $\Omega \subset \mathcal{P}^n$ since Y is compact. Such a parametrization is of course not unique. But within G^*/K^*, it is unique up to the left action of Γ_0. Under this coordinate, the hyperbolic measure dz is:

$$dz = \frac{dr du}{r^n}.$$

It is known that dz is a left G^*-invariant Radon measure on G^*/K^*. As before, the geometric side is divided into two parts indexed by double coset classes. But, in the present case, we do not have the uniqueness result (to use γ to express, via two sided action of Γ_0, any η in the class $\widetilde{\gamma}$) like Proposition 1.5.4, hence we could only say that the geometric side is bounded by $\Sigma_0 + \sum_{\widetilde{\gamma} \in \Gamma_0 \backslash \Gamma / \Gamma_0 \smallsetminus \{\tilde{1}\}} I_\gamma$ where

$$\Sigma_0 = \sum_{\gamma \in \Gamma_0} \int_{Y \times Y} \Phi_\mu(\gamma z_1,\, z_2) dz_1 dz_2 \quad \text{and} \quad I_\gamma = \sum_{\gamma_1 \in \Gamma_0} \sum_{\gamma_2 \in \Gamma_0} \int_{Y \times Y} \Phi_\mu(\gamma \gamma_1 z_1,\, \gamma_2 z_2) dz_1 dz_2.$$

In the following two sections, we investigate these two parts separately. It is a general philosophy that the term Σ_0 should be the main term while the other part is the error term, i.e., the trivial element $\tilde{1} \in \Gamma_0 \backslash \Gamma / \Gamma_0$ contributes most. However we need to know the detailed information on the order of the error term. When $n = d$, we have $G^* = G$ and $\Gamma = \Gamma_0$. In this case, only Σ_0 occurs on the geometric side. Without the rest terms I_γ, we shall see that it is much easier to derive our conclusions. Besides, the case $n = 1$ has been considered in previous chapters. Hence we assume $d > n \geqslant 2$ till the end. We shall see that the restriction $n \geqslant 2$, unlike $d > n$, is essential.

3.1.1 The term Σ_0

By definition, we have:

$$\Sigma_0 = \int_{z_2\in Y} \int_{z_1\in\widetilde{Y}} \Phi_\mu(d(z_1,\, z_2))dz_1dz_2$$

where $\widetilde{Y} \cong G^*/K^*$. Let $z_1 = \theta_v\,\omega_s \cdot o$ denote the point in $\widetilde{Y}$. Let $z_2 = \theta_w\,\omega_t \cdot o$ denote the point in Y where $(u,\,r)$ lies in Ω, a Γ_0-fundamental domain in $\mathcal{P}^n$ which is isomorphic to Y. Then

$$\begin{aligned}\Sigma_0 &= \int_{(w,t)\in\Omega}\int_{(v,s)\in\mathcal{P}^n} \exp\left(-\mu\,\frac{|w-v|^2+s^2+t^2}{2st}\right)\frac{dsdv}{s^n}\frac{dtdw}{t^n}\\ &= \int_{\Omega}\int_{\mathbb{R}^{n-1}\times\mathbb{R}_{>0}} \exp\left[-\frac{\mu}{2}\left(\frac{s}{t}+\frac{1+\left|\frac{v-w}{t}\right|^2}{\frac{s}{t}}\right)\right]\frac{dsdv}{s^n}\frac{dtdw}{t^n}\end{aligned}$$

Let $v' = \frac{v-w}{t}$ and $s' = \frac{s}{t}$, then $dv = dv_1 \cdots dv_{n-1} = t^{n-1}dv_1' \cdots dv_{n-1}' = t^{n-1}dv'$. By (2.2), the integration along s' gives:

$$\begin{aligned}\Sigma_0 &= \int_{\Omega}\int_{(v',s')\in\mathbb{R}^{n-1}\times\mathbb{R}_{>0}} \exp\left[-\frac{\mu}{2}\left(s'+\frac{1+|v'|^2}{s'}\right)\right]\frac{ds'dv'}{s'^n}\frac{dtdw}{t^n}\\ &= \int_{\Omega}\int_{\mathbb{R}^{n-1}} \left(1+|v'|^2\right)^{-\frac{n-1}{2}} K_{n-1}\left(\mu\sqrt{1+|v'|^2}\right)dv'\frac{dtdw}{t^n}\end{aligned}$$

As for the integration over along v', we copy the procedure of the computation for $h_f(\lambda_i)$ (dropping the term η_i there). See Sect. 2.2 for details. It turns out that

$$\begin{aligned}\Sigma_0 &= \int_{\Omega} 2^n\left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1} K_{\frac{n-1}{2}}(\mu)\frac{dtdw}{t^n}\\ &= 2^n\left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1} K_{\frac{n-1}{2}}(\mu)\cdot \mathrm{vol}(Y)\end{aligned}$$

3.1.2 The term I_γ

By definition, we have:

$$I_\gamma = \int_{z_1\in\widetilde{Y}}\int_{z_2\in\widetilde{Y}} \Phi_\mu(\gamma z_1,\, z_2)dz_1dz_2.$$

Let $z_1 = \theta_u\,\omega_r \cdot o$, $z_2 = \theta_w\,\omega_t \cdot o$ and assume that $\gamma z_1 = \theta_{v_1}\,\omega_{s_1} \cdot o$ where $\gamma = \omega_{r_0}\theta_{w_0}k_\gamma$. Then

$$I_\gamma = \int_{(u,r)\in\mathcal{P}^n}\int_{(w,t)\in\mathcal{P}^n} \exp\left(-\mu\,\frac{|w-v_1|^2+s_1^2+t^2}{2s_1t}\right)\frac{dtdw}{t^n}\frac{drdu}{r^n}.$$

The integration along t gives (again, by formula (2.2)):

$$
\begin{aligned}
I_\gamma &= 2\int_{\mathcal{P}^n}\int_{\mathbb{R}^{n-1}} \left(|w-v_1|^2+s_1^2\right)^{-\frac{n-1}{2}} K_{n-1}\left(\mu\sqrt{\left|\frac{w-v_1}{s_1}\right|^2+1}\right) dw\frac{dr du}{r^n} \\
&= 2\int_{\mathcal{P}^n}\int_{\mathbb{R}^{n-1}} \left(\left|\frac{w-v_1}{s_1}\right|^2+1\right)^{-\frac{n-1}{2}} K_{n-1}\left(\mu\sqrt{\left|\frac{w-v_1}{s_1}\right|^2+1}\right) d\left(\frac{w}{s_1}\right)\frac{dr du}{r^n} \\
&= 2\int_{\mathcal{P}^n}\int_{\mathbb{R}^{n-1}} \left(\left|w'-\frac{v_1}{s_1}\right|^2+1\right)^{-\frac{n-1}{2}} K_{n-1}\left(\mu\sqrt{\left|w'-\frac{v_1}{s_1}\right|^2+1}\right) dw'\frac{dr du}{r^n}
\end{aligned}
$$

where we have put $w' = \frac{w}{s_1}$. For any $\theta_w\omega_t\cdot o \in G^*/K^*$, w lies in

$$\mathbb{R}^{n-1}_{d-1} := \left\{x=(x_i)_{i=1}^{d-1}\in\mathbb{R}^{d-1}\,\middle|\, x_i = 0\ \forall i\geqslant n\right\}.$$

Thus, when we do the integration along w', those first $(n-1)$ components of $\frac{v_1}{s_1}$ can be absorbed into w', meanwhile those last $(d-n)$ components of $\frac{v_1}{s_1}$ remain after the integration, i.e., if we denote $x_n^2+\cdots+x_{d-1}^2$ by $|x|^2_{\geqslant n}$ for $x=(x_1,\cdots,x_{d-1})\in\mathbb{R}^{d-1}$, then

$$I_\gamma = 2\int_{\mathcal{P}^n}\int_{\mathbb{R}^{n-1}} \left(|w'|^2+\left|\frac{v_1}{s_1}\right|^2_{\geqslant n}+1\right)^{-\frac{n-1}{2}} K_{n-1}\left(\mu\sqrt{|w'|^2+\left|\frac{v_1}{s_1}\right|^2_{\geqslant n}+1}\right) dw'\frac{dr du}{r^n}.$$

As before, a copy of the computation on $h_f(\lambda_i)$ (dropping η_i) gives:

$$I_\gamma = 2^n\left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1}\int_{\mathcal{P}^n}\left(\sqrt{\left|\frac{v_1}{s_1}\right|^2_{\geqslant n}+1}\right)^{-\frac{n-1}{2}} K_{\frac{n-1}{2}}\left(\mu\sqrt{\left|\frac{v_1}{s_1}\right|^2_{\geqslant n}+1}\right)\frac{dr du}{r^n}.$$

Assume that $k_\gamma = \mathrm{diag}(1,\, u)$ where $u=(u_{ij})\in SO_d(\mathbb{R})$. Let $k_\gamma\theta_u\,\omega_r = \theta_v\,\omega_s k$ for some $k\in K$, then the computation shows that

$$
k_\gamma\theta_u\,\omega_r = \begin{pmatrix}
\left(1+\frac{|u|^2}{2}\right)\frac{r+r^{-1}}{2} - \frac{|u|^2}{2}\frac{r-r^{-1}}{2}, & \cdots \\
\left(u_{11}\frac{|u|^2}{2}+\sum\limits_{i=2}^n u_{1i}u_{i-1}\right)\frac{r+r^{-1}}{2} + \left[u_{11}\left(1-\frac{|u|^2}{2}\right)-\sum\limits_{i=2}^n u_{1i}u_{i-1}\right]\frac{r-r^{-1}}{2}, & \cdots \\
\left(u_{21}\frac{|u|^2}{2}+\sum\limits_{i=2}^n u_{2i}u_{i-1}\right)\frac{r+r^{-1}}{2} + \left[u_{21}\left(1-\frac{|u|^2}{2}\right)-\sum\limits_{i=2}^n u_{2i}u_{i-1}\right]\frac{r-r^{-1}}{2}, & \cdots \\
\vdots & \vdots \\
\left(u_{d1}\frac{|u|^2}{2}+\sum\limits_{i=2}^n u_{di}u_{i-1}\right)\frac{r+r^{-1}}{2} + \left[u_{d1}\left(1-\frac{|u|^2}{2}\right)-\sum\limits_{i=2}^n u_{di}u_{i-1}\right]\frac{r-r^{-1}}{2}, & \cdots
\end{pmatrix}.
$$

The term $\theta_v\,\omega_s k$ has been (partly) computed in Sect. 2.3.2, see (2.8) there. By comparison of the first columns of these two matrices, we have:

$$\frac{s+s^{-1}}{2}+\frac{s^{-1}}{2}|v|^2=\left(1+\frac{|u|^2}{2}\right)\frac{r+r^{-1}}{2}-\frac{|u|^2}{2}\frac{r-r^{-1}}{2} \tag{3.1}$$

$$\frac{s-s^{-1}}{2}+\frac{s^{-1}}{2}|v|^2=\left(u_{11}\,\frac{|u|^2}{2}+\sum_{i=2}^{n}u_{1\,i}u_{i-1}\right)\frac{r+r^{-1}}{2}+\left[u_{11}\left(1-\frac{|u|^2}{2}\right)-\sum_{i=2}^{n}u_{1\,i}u_{i-1}\right]\frac{r-r^{-1}}{2} \tag{3.2}$$

$$v_i\,s^{-1}=\left(u_{i+1,\,1}\,\frac{|u|^2}{2}+\sum_{j=2}^{n}u_{i+1,\,j}u_{j-1}\right)\frac{r+r^{-1}}{2}+\left[u_{i+1,\,1}\left(1-\frac{|u|^2}{2}\right)-\sum_{j=2}^{n}u_{i+1,\,j}u_{j-1}\right]\frac{r-r^{-1}}{2} \tag{3.3}$$

The equalities (3.1) and (3.2) imply

$$s^{-1}=\left(1+(1-u_{11})\,\frac{|u|^2}{2}-\sum_{i=2}^{n}u_{1\,i}u_{i-1}\right)\frac{r+r^{-1}}{2}-\left(u_{11}+(1-u_{11})\,\frac{|u|^2}{2}-\sum_{i=2}^{n}u_{1\,i}u_{i-1}\right)\frac{r-r^{-1}}{2}.$$

Let $\beta=(1-u_{11})\,\frac{|u|^2}{2}-\sum\limits_{i=2}^{n}u_{1\,i}u_{i-1}$, then

$$s^{-1}=\frac{1-u_{11}}{2}\,r+\left(\frac{1+u_{11}}{2}+\beta\right)r^{-1}. \tag{3.4}$$

Let $\alpha_i=u_{i+1,\,1}\,\frac{|u|^2}{2}+\sum\limits_{j=2}^{n}u_{i+1,\,j}u_{j-1}$, then

$$v_i\,s^{-1}=\frac{u_{i+1,\,1}}{2}\,r+\left(\alpha_i-\frac{u_{i+1,\,1}}{2}\right)r^{-1},\quad 1\leqslant i\leqslant d-1. \tag{3.5}$$

Clearly $v_1=(w_0+v)r_0$, $s_1=r_0 s$. A computation with those terms in above shows that

$$\left|\frac{v_1}{s_1}\right|^2_{\geqslant n}+1=\left|\frac{w_0+v}{s}\right|^2_{\geqslant n}+1=M(\gamma)r^2+N_u(\gamma)r^{-2}+Q_u(\gamma)=:f_\gamma(u,\,r)$$

where

$$M(\gamma)=\sum_{i=n}^{d-1}\left(w_{0\,i}\,\frac{1-u_{11}}{2}+\frac{u_{i+1,1}}{2}\right)^2=:\sum_{i=n}^{d-1}m_i^2 \tag{3.6}$$

$$N_u(\gamma)=\sum_{i=n}^{d-1}\left[w_{0\,i}\left(\frac{1+u_{11}}{2}+\beta\right)+\left(\alpha_i-\frac{u_{i+1,1}}{2}\right)\right]^2=:\sum_{i=n}^{d-1}n_i^2 \tag{3.7}$$

$$Q_u(\gamma)=1+2\sum_{i=n}^{d-1}m_i n_i \tag{3.8}$$

Here $w_{0\,i}$ is the i-th component of w_0: $w_0 = (w_{0\,1},\, \cdots, w_{0\,d-1})$. Now we have:

$$I_\gamma = 2^n \left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1} \int\limits_{\mathcal{P}^n} \left(\sqrt{f_\gamma(u,\,r)}\right)^{-\frac{n-1}{2}} K_{\frac{n-1}{2}}\left(\mu\sqrt{f_\gamma(u,\,r)}\right)\frac{dr du}{r^n}.$$

Write $f_\gamma(u,\,r)$ as $f_\gamma(u,\,r) = \left(\sqrt{M(\gamma)}\,r - \frac{\sqrt{N_u(\gamma)}}{r}\right)^2 + 2\sqrt{M(\gamma)N_u(\gamma)} + Q_u(\gamma)$. Define

$$\delta_u(\gamma) = 2\sqrt{M(\gamma)N_u(\gamma)} + Q_u(\gamma).$$

The parameter u is a token that N, Q and δ depend on it as well as γ. Note that M depends only on γ and this M is slightly different from the M in Ch. 2: here M is the summation of parts of m_i^2's while the M in Ch. 2 is the summation of all m_i^2's. We still use M by abuse of notations. The number $\delta_u(\gamma)$ has remarkable geometric meaning. Recall that

$$\delta(\gamma z_1,\, z_2) = \frac{|w-v_1|^2+s_1^2+t^2}{2s_1t} \geqslant 2\sqrt{\frac{|w-v_1|^2+s_1^2}{2s_1}\cdot\frac{1}{2s_1}} = \sqrt{\left|\frac{w-v_1}{s_1}\right|^2+1} \geqslant \sqrt{\left|\frac{v_1}{s_1}\right|^2_{\geqslant n}+1}$$

where $d(\gamma z_1,\, z_2) = \|E\| \operatorname{arccosh}\delta(\gamma z_1,\, z_2)$ is the hyperbolic distance between z_1 and z_2. The "=" at the first inequality can be achieved as t ranges among all positive numbers. The last step follows from the fact that w is a vector in $\mathbb{R}^{n-1}$ whose last $(d-n)$ components vanish. Here the "=" can be achieved for w such that $w_{0\,i} = v_{1\,i}$ $(1 \leqslant i \leqslant n-1)$. Since r ranges over all positive numbers, $\left|\frac{v_1}{s_1}\right|^2_{\geqslant n} + 1 = f_\gamma(u,\,r) \geqslant \delta_u(\gamma)$ where "=" can be obtained when $\sqrt{M(\gamma)}\,r - \frac{\sqrt{N_u(\gamma)}}{r} = 0$, i.e., $r = \sqrt{\frac{N_u(\gamma)}{M(\gamma)}}$ (if $M(\gamma) \neq 0$) or $r = \infty$ (if $M(\gamma) = 0$). So $\delta_u(\gamma)$ measures the minimal distance between the geodesic $\gamma\,\theta_u\,A\cdot o$ and the submanifold $\widetilde{Y}$:

$$\sqrt{\delta_u(\gamma)} = \cosh\left(\|E\|^{-1}\cdot \inf_{z_1\in A\cdot o,\, z_2\in\widetilde{Y}} d(\gamma\,\theta_u z_1,\, z_2)\right)$$

where A is the maximal split torus of G (as before) and $A\cdot o$ is the regular geodesic over G/K. By this formula we know that $\delta_u(\,\cdot\,)$ is well-defined over $\Gamma_0\backslash\Gamma$, but not on Γ/Γ_0. For any fixed $u \in \mathbb{R}^{n-1}_{d-1}$, define

$$\pi_u^*(x) := \#\left\{\gamma \in \Lambda \,|\, \delta_u(\gamma) \leqslant x\right\}.$$

It is clear from the above discussion that the number $f_\gamma(u,\,r)$ also has remarkable geometric meaning: it measures the (hyperbolic) distance between the point $\gamma\theta_u\omega_r\cdot o$ and the submanifold $\widetilde{Y}$. More precisely,

$$\sqrt{f_\gamma(u,\,r)} = \cosh\left(\|E\|^{-1}\cdot\inf_{z\in\widetilde{Y}} d(\gamma\,\theta_u\omega_r\cdot o,\, z)\right).$$

Let $x = \sqrt{M(\gamma)}\, r - \frac{\sqrt{N_u(\gamma)}}{r}$, then $r = \frac{x+\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}}{2\sqrt{M(\gamma)}}$ and

$$I_\gamma = 2^n \left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1} \int\limits_{\mathbb{R}^{n-1}_{d-1}} \int\limits_{-\infty}^{\infty} F_\gamma(u,\, x) dx du \tag{3.9}$$

where

$$F_\gamma(u,\, x) = \frac{\left(2\sqrt{M(\gamma)}\right)^{n-1}}{\left(\sqrt{x^2+\delta_u(\gamma)}\right)^{\frac{n-1}{2}}} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+\delta_u(\gamma)}\right)}{\left(x+\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}}.$$

Denote by $d(\gamma)$ the (minimal) distance between the manifold $\widetilde{Y}$ and its translation $\gamma\widetilde{Y}$:

$$d(\gamma) := \inf_{z,\, w \in \widetilde{Y}} d\left(\gamma z,\, w\right).$$

Let $\delta(\gamma) = \|E\| \cosh d(\gamma)$. Clearly $d(\,\cdot\,)$ (thus $\delta(\,\cdot\,)$) is well-defined on $\Gamma_0 \backslash \Gamma / \Gamma_0$. Define

$$\pi^*(x) = \#\left\{\widetilde{\gamma} \in \Gamma_0\backslash\Gamma/\Gamma_0 \,|\, \delta(\gamma) \leqslant x\right\}.$$

In the next section, we shall prove the following two results:

Proposition 3.1.1.

$$\pi^*(x) = \mathcal{O}\left(x^{\frac{d-n}{2}}\right), \quad \text{as } x \to \infty.$$

Here the implied $\mathcal{O}$-constant depends only on Γ.

Proposition 3.1.2. *For any $\gamma \notin \Gamma_0$ and $u \in \mathbb{R}^{n-1}_{d-1}$, there exists a positive number c such that*

$$M(\gamma)N_u(\gamma) \geqslant c.$$

An immediate implication is:

Corollary 3.1.3. $\pi^*_u(x) = \mathcal{O}\left(x^{\frac{d-n}{2}}\right)$ *as $x \to \infty$. Here the $\mathcal{O}$-constant does not depend on u.*

Proof. Since $\delta_u(\gamma) \geqslant \delta(\gamma)$, we have: $\pi^*_u(x) \subset \pi^*(x)$. The corollary then follows from Corollary 3.1.1. □

When μ is very large, $\mu\sqrt{x^2+\delta_u(\gamma)} \geqslant \mu$ is also very large (for any x, u and γ) as $\delta_u(\gamma) \geqslant 1$, hence by (2.19)

$$K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+\delta_u(\gamma)}\right) \leqslant \frac{C}{\sqrt{\mu\sqrt{x^2+\delta_u(\gamma)}}} e^{-\mu\sqrt{x^2+\delta_u(\gamma)}} \leqslant \frac{C}{\sqrt{\mu}\,\sqrt[4]{x^2+1}} e^{-\mu\sqrt{x^2+\delta_u(\gamma)}}$$

where C is a fixed number which is independent from x and u. Since $\sqrt{x^2+\delta_u(\gamma)} \geqslant \frac{\sqrt{2}}{2}\left(|x|+\sqrt{\delta_u(\gamma)}\right)$ (see the end part of Sect. ??), we have:

$$K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+\delta_u(\gamma)}\right) \leqslant \frac{C}{\sqrt{\mu}\,\sqrt[4]{x^2+1}}\, e^{-\frac{\mu\sqrt{2}}{2}|x|}\cdot e^{-\frac{\mu\sqrt{2}}{2}\sqrt{\delta_u(\gamma)}} \tag{3.10}$$

The function

$$G_\gamma(u,\,x) := x+\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}$$

is monotonously increasing as x increases: the derivative

$$\frac{\partial G_\gamma(u,\,x)}{\partial x} = 1+\frac{x}{\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}}$$

is positive for all $x\in\mathbb{R}$ as $\left|\frac{x}{\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}}\right| < 1$. Note that $G_\gamma(u,\,x)$ is positive for all $x\in\mathbb{R}$. Thus $G_\gamma(u,\,x) > \alpha_\gamma(u)$ for $x > -1$. Here

$$\alpha_\gamma(u) = G_\gamma(u,\,-1) = \sqrt{1+4\sqrt{M(\gamma)N_u(\gamma)}}-1 \geqslant \sqrt{1+4\sqrt{c}}-1.$$

Let $\alpha = \sqrt{1+4\sqrt{c}}-1$, then $G_\gamma(u,\,x) > \alpha$ for $x > -1$.

We analyze the terms which occur in $F_\gamma(u,\,x)$ for μ very large:

- If $x > -1$, by Proposition 3.1.2 and the above argument, the following holds

$$F_\gamma(u,\,x) < \frac{C}{\sqrt{\mu}}\frac{\left(2\sqrt{M(\gamma)}\right)^{n-1}}{\left(\sqrt{x^2+1}\right)^{\frac{n}{2}-\frac{1}{4}}}\frac{e^{-\frac{\mu\sqrt{2}}{2}|x|}\cdot e^{-\frac{\mu\sqrt{2}}{2}\sqrt{\delta_u(\gamma)}}}{\alpha^{n-1}\sqrt{x^2+4\sqrt{c}}}$$

  So the integral $\int_{-1}^{\infty} F_\gamma(u,\,x)dx$ is (upper) bounded by

$$\frac{A}{\sqrt{\mu}}\left(2\sqrt{M(\gamma)}\right)^{n-1} e^{-\frac{\mu\sqrt{2}}{2}\sqrt{\delta_u(\gamma)}}$$

  where A is a constant:

$$A = \frac{C}{\alpha^{n-1}}\int_{-1}^{\infty}\frac{e^{-\frac{\mu\sqrt{2}}{2}|x|}\,dx}{\left(\sqrt{x^2+1}\right)^{\frac{n}{2}-\frac{1}{4}}\sqrt{x^2+4\sqrt{c}}}.$$

  This integral clearly converges.

- If $x \leqslant -1$, then

$$\frac{1}{\left(\sqrt{x^2+\delta_u(\gamma)}\right)^{\frac{n-1}{2}}\left(x+\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}}$$

$$= \frac{\left(\sqrt{x^2 + 4\sqrt{M(\gamma)N_u(\gamma)}} - x\right)^{n-1}}{\left(4\sqrt{M(\gamma)N_u(\gamma)}\right)^{n-1}\left(\sqrt{x^2+\delta_u(\gamma)}\right)^{\frac{n-1}{2}}}$$

$$= \frac{\left(\sqrt{\frac{1+\frac{4\sqrt{M(\gamma)N_u(\gamma)}}{x^2}}{16M(\gamma)N_u(\gamma)}} + \frac{1}{4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}(-x)^{n-1}}{\left(\sqrt{x^2+\delta_u(\gamma)}\right)^{\frac{n-1}{2}}}$$

$$\leqslant \frac{\left(\sqrt{\frac{1+4\sqrt{c}}{16c}} + \frac{1}{4\sqrt{c}}\right)^{n-1}(-x)^{n-1}}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}}$$

The last step follows from Proposition 3.1.2 and our assumption on x. By (3.10) we have:

$$F_\gamma(u,\, x) \leqslant C \cdot \frac{\left(\sqrt{\frac{1+4\sqrt{c}}{16c}} + \frac{1}{4\sqrt{c}}\right)^{n-1}(-x)^{n-1}}{\left(\sqrt{x^2+1}\right)^{\frac{n}{2}-\frac{1}{4}}} \frac{\left(2\sqrt{M(\gamma)}\right)^{n-1} e^{\frac{\mu\sqrt{2}}{2}x} \cdot e^{-\frac{\mu\sqrt{2}}{2}\sqrt{\delta_u(\gamma)}}}{\sqrt{\mu}\,\sqrt{x^2+4\sqrt{c}}}.$$

So the integral $\int_{-\infty}^{-1} F_\gamma(u,\, x)dx$ is (upper) bounded by

$$\frac{B}{\sqrt{\mu}}\left(2\sqrt{M(\gamma)}\right)^{n-1} e^{-\frac{\mu\sqrt{2}}{2}\sqrt{\delta_u(\gamma)}}$$

where B is a constant:

$$B = \int_{-\infty}^{-1} \frac{C'(-x)^{n-1}\, e^{\frac{\mu\sqrt{2}}{2}x}}{\left(\sqrt{x^2+1}\right)^{\frac{n}{2}-\frac{1}{4}}\sqrt{x^2+4\sqrt{c}}}\, dx.$$

Here $C' = C\left(\sqrt{\frac{1+4\sqrt{c}}{16c}} + \frac{1}{4\sqrt{c}}\right)^{n-1}$. Clearly this integral converges.

Remark 3.1.4. *$G_\gamma(u,\, x)$ occurs in the denominator of $F_\gamma(u,\, x)$. As $x \to -\infty$, $G_\gamma(u,\, x) \to 0$. This is the motivation for the above argument.*

Our conclusion is summarized as: there exists a positive number C (universal for all γ and u) such that

$$\int_{-\infty}^{\infty} F_\gamma(u,\, x)dx \leqslant \frac{C}{\sqrt{\mu}}\left(2\sqrt{M(\gamma)}\right)^{n-1} e^{-\frac{\mu\sqrt{2}}{2}\sqrt{\delta_u(\gamma)}}.$$

This implies that

$$I_\gamma \leqslant 2^n\left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1}\frac{C}{\sqrt{\mu}}\int_{u\in\mathbb{R}^{n-1}}\left(2\sqrt{M(\gamma)}\right)^{n-1} e^{-\frac{\mu\sqrt{2}}{2}\sqrt{\delta_u(\gamma)}}du. \tag{3.11}$$

Expanding n_i, we get:

$$n_i = |u|^2 \, \frac{w_{0i}(1-u_{11})+u_{i+1,1}}{2} + \sum_{j=2}^{n} \left(u_{i+1,j} - w_{0i}u_{1j}\right) u_{j-1} + w_{0i}\frac{1+u_{11}}{2} - \frac{u_{i+1,1}}{2}.$$

Note that $\frac{w_{0i}(1-u_{11})+u_{i+1,1}}{2} = m_i$ and $M(\gamma) \neq 0$ for $\gamma \notin \Gamma_0$ (see Proposition 3.1.2). So

$$\begin{aligned}
\sum_{i=n}^{d-1} m_i n_i &= |u|^2 \underbrace{\sum_{i=n}^{d-1} m_i^2}_{=M(\gamma)} + \sum_{i=n}^{d-1}\sum_{j=2}^{n} m_i \left(u_{i+1,j} - w_{0i}u_{1j}\right) u_{j-1} + \sum_{i=n}^{d-1} m_i \left(w_{0i}\frac{1+u_{11}}{2} - \frac{u_{i+1,1}}{2}\right) \\
&= \sum_{j=2}^{n} \left(\sqrt{M(\gamma)}\, u_{j-1} + \frac{\sum_{i=n}^{d-1} m_i \left(u_{i+1,j} - w_{0i}u_{1j}\right)}{2\sqrt{M(\gamma)}} \right)^2 + \sum_{i=n}^{d-1} m_i \left(w_{0i}\frac{1+u_{11}}{2} - \frac{u_{i+1,1}}{2}\right) \\
&\qquad - \sum_{j=2}^{n} \frac{\left(\sum_{i=n}^{d-1} m_i \left(u_{i+1,j} - w_{0i}u_{1j}\right)\right)^2}{4M(\gamma)} \\
&= \frac{1}{4}\sum_{j=2}^{n} u_{j-1}'^2 + H_\gamma
\end{aligned}$$

Here we denote

$$u'_{j-1} = 2\sqrt{M(\gamma)}\, u_{j-1} + \frac{\sum_{i=n}^{d-1} m_i \left(u_{i+1,j} - w_{0i}u_{1j}\right)}{\sqrt{M(\gamma)}} \tag{3.12}$$

and

$$H_\gamma = \sum_{i=n}^{d-1} m_i \left(w_{0i}\frac{1+u_{11}}{2} - \frac{u_{i+1,1}}{2}\right) - \sum_{j=2}^{n} \frac{\left(\sum_{i=n}^{d-1} m_i \left(u_{i+1,j} - w_{0i}u_{1j}\right)\right)^2}{4M(\gamma)}. \tag{3.13}$$

The number H_γ depends only on γ. Now we have:

$$\begin{aligned}
\delta_u(\gamma) &= 2\sqrt{M(\gamma)N_u(\gamma)} + Q_u(\gamma) \\
&= 2\sqrt{\left(\sum_{i=n}^{d-1} m_i^2\right)\left(\sum_{i=n}^{d-1} n_i^2\right)} + 2\sum_{i=n}^{d-1} m_i n_i + 1 \\
&\geqslant 4\sum_{i=n}^{d-1} m_i n_i + 1 \\
&= \sum_{j=2}^{n} u_{j-1}'^2 + 4H_\gamma + 1
\end{aligned} \tag{3.14}$$

3.2 Proofs of Proposition 3.1.1 and 3.1.2

Assume that C is a closed geodesic over Y and $\Gamma_{00} \subset \Gamma_0$ is the stabilizer of C, or equivalently $C \approx \Gamma_{00}\backslash\widetilde{C}$ where $\widetilde{C}$ is one of those lifts of C in G^*/K^*. Then there exists some $g \in G^*$ such that $g\widetilde{C}$ is the regular geodesic $\widetilde{C}' = A \cdot o \subset G/K$. Let $\Gamma_0' = g\Gamma_0 g^{-1}$ and $\Gamma_{00}' = g\Gamma_{00}g^{-1}$, then over the quotient manifold $\Gamma_0'\backslash G^*/K^*$, the regular geodesic is closed (denoted by C'). If we identify C and C' with some fundamental domains for $\Gamma_{00}\backslash\widetilde{C}$ and $\Gamma_{00}'\backslash\widetilde{C}'$ respectively, then we may recognize C' as gC, or C as $g^{-1}C'$. Accordingly we identify Y' with gY, or Y with $g^{-1}Y'$ where Y' is the fundamental domain for $\Gamma_0'\backslash G^*/K^*$ which contains the above mentioned fundamental domain for C'. From Ch. 2 we have: $\Gamma_{00}' = \langle\gamma_0\rangle$ where $\gamma_0 = \omega_{e^T}k_0$ for some $T > 0$ and $k_0 \in K^*$. Here $T = \frac{\mathrm{len}(C')}{\|E\|}$ (see the remark at the end of this chapter) and k_0 is of the form: $k_0 = \mathrm{diag}(1, 1, \eta, \rho)$ for some $(\eta, \rho) \in O_{n-1} \times O_{d-n}$. With $\{\phi_i\}$ being replaced by $\{L_g(\phi_i)\}$ where L is the left regular action of g, the period $\int_Y \phi_i$ is equal to $\int_{Y'} L_g(\phi)$. The only condition we have posed on $\{\phi_i\}$ is that they are the eigenfunctions of the integral operator T_f (see Ch. 1) which are orthonormal. Such a property is preserved for the family $\{L_g(\phi_i)\}$. Hence, by passing to Γ', Γ_0' (especially Γ_{00}'), Y' and $\{L_g(\phi_i)\}$ if necessary (a normalization process), we may assume that the regular geodesic over Y is closed. The main implication is that there exists γ_0 (which is of the form $\gamma_0 = \omega_{e^T}k_0$ as above) such that $\langle\gamma_0\rangle \subset \Gamma_0$.

The idea for the proof of Proposition 3.1.1 is, in principle, similar to that of Theorem 2.3.1: find elements $\gamma' \in \Gamma$ which share the same δ, i.e., $\delta_u(\gamma) = \delta_v(\gamma')$ for some $v \in \mathbb{R}^{n-1}_{d-1}$, then count these representatives. The point is to realize these elements in some special domain $\Omega \in \mathcal{P}^d$ so that the counting is reduced to computing the volume of Ω. The difference from the proof of Theorem 2.3.1 is that, here the number $\delta_u(\gamma)$ involves in partial terms of u. We make use of the right action of Γ_{00} to reduce these terms to be in a bounded domain (in $\mathbb{R}^{d-n}$). Meanwhile we make use of the left action of Γ_0 to reduce the rest terms of u, as well as the term ω, to be in a bounded domain (in $\mathbb{R}_{>0} \times \mathbb{R}^{n-1}$), without essentially changing the previous ones.

Lemma 3.2.1. *The natural map $\Gamma \to G/K$ is injective.*

Proof. Assume $\gamma_1 \cdot o = \gamma_2 \cdot o$ for some $\gamma_1, \gamma_2 \in \Gamma$, then $\gamma_1^{-1}\gamma_2 \in K$. This implies that $\gamma_1^{-1}\gamma_2 = 1$ (note that Γ is torsion-free), i.e., $\gamma_1 = \gamma_2$, so the map is injective. □

Lemma 3.2.2. *The image of Γ in G/K is discrete and has no accumulation point in G/K.*

Proof. Assume that $\{\gamma_i \cdot o \,|\, \gamma_i \in \Gamma,\ \gamma_i \cdot o \neq \gamma_j \cdot o \text{ for } i \neq j\}$ is a convergent sequence, then $\gamma_i^{-1}\gamma_j \to K$ as $i, j \to \infty$. Meanwhile, by Lemma 3.2.1, $\gamma_i \neq \gamma_j$ for $i \neq j$. So $\{\gamma_i^{-1}\gamma_j \,|\, i \neq j\}$ lies in a compact neighborhood of K. By passing to a subsequence if necessary, we know that $\gamma_i^{-1}\gamma_j$ converges to some $\gamma \in \Gamma$, noting that Γ is closed (since it is discrete). Hence the sequence $\{\gamma_i^{-1}\gamma_j \mid i \neq j\}$ is stationary for large i, j, i.e., there exists $\delta \in \Gamma$ such that $\gamma_j = \gamma_i\delta$. Thus $\gamma_{j+\ell} = \gamma_{j+\ell-1}\delta = \cdots = \gamma_j\delta^\ell = \gamma_j\delta$ which means that $\delta^\ell = \delta$. As Γ is torsion-free, we have: $\delta = 1$. So $\gamma_i = \gamma_j$ $(i \neq j)$, a contradiction. The first part of the lemma is proved. Assume that there exist a sequence $\{\gamma_i \cdot o\}$ and some $g \in G$ such that $\gamma_i \cdot o \to g \cdot o \in G/K$ as $i \to \infty$. Here $\gamma_i \cdot o \neq \gamma_j \cdot o$ for $i \neq j$. Then $\{g^{-1}\gamma_i\}$ lies in a compact neighborhood of K. By passing to a subsequence if necessary, we know that $g^{-1}\gamma_j$ converges to some $h \in G$, noting that $g^{-1}\Gamma$ is discrete (hence closed). The sequence $\{g^{-1}\gamma_i\}$, thus $\{\gamma_i\}$ as well, is stationary for large i. But we have assumed that $\gamma_i \neq \gamma_j$ for $i \neq j$. □

Lemma 3.2.3. *The image of* $\Gamma_0\backslash\Gamma$ *for the map* $\Gamma_0\backslash\Gamma \to \Gamma_0\backslash G/K$ *is discrete and has no accumulation point in* $\Gamma_0\backslash G/K$.

Proof. For $g \in G$, denote by $\widetilde{g}$ the image of g in G/K, by $\bar{g}$ the image of g in $\Gamma_0\backslash G/K$. Assume that the sequence $\{\bar{\gamma}_i\}$ converges to $\bar{\gamma}$ where $\bar{\gamma}_i \neq \bar{\gamma}_j$ for $i \neq j$. Then there exist a sequence $\{\eta_i\} \subset \Gamma_0$ and a compact neighborhood W of K such that $\gamma^{-1}\eta_i\gamma_i \in W \cap \Gamma$. By passing to a subsequence if necessary, let's assume that $\gamma^{-1}\eta_i\gamma_i \to \gamma' \in \Gamma$. In view of the discreteness of Γ, $\eta_i\gamma_i \equiv \gamma\gamma'$ for large i. This means that $\bar{\gamma}_i \equiv \bar{\gamma}_j$ for large i and j, a contradiction. This proves the first part of the lemma. The argument for the second part is similar to that of the Lemma 3.2.2. We omit the details. □

Proof of Proposition 3.1.1. First we assume that $M(\gamma)N_u(\gamma) \neq 0$. There exist unique $r \in [1, e^T]$ and $r_1 \in \left\{e^{\mathbb{Z}T}\right\}$ such that $\gamma\theta_u\omega_{rr_1} \cdot o = \omega_{r_0 s}\theta_{v'} \cdot o$ where $|v'|^2_{\geqslant n} + 1 = \left|\frac{w_0+v}{s}\right|^2_{\geqslant n} + 1 = \delta_u(\gamma)$: rr_1 is the unique positive solution of the equation $\sqrt{M(\gamma)}x - \frac{\sqrt{N_u(\gamma)}}{x} = 0$, thus modulo $\left\{e^{\mathbb{Z}T}\right\}$ (multiplicatively), r is unique. Clearly we have: $\gamma\theta_u\omega_{rr_1}\cdot o = \gamma\gamma_1\omega_r\theta_{(rr_1)^{-1}u\left(\rho_1^{-1}\right)^T}\cdot o$ where $\gamma_1 = \omega_{r_1}k_1 \in \Gamma_{00}$ for some $k_1 = \mathrm{diag}(1,\, 1,\, \rho_1) \in K^*$. For any $v = (v_i)_{i=1}^{d-1} \in \mathbb{R}^{d-1}$, let $v_{<n}$ and $v_{\geqslant n}$ denote $(v_1,\, \cdots,\, v_{n-1},\, 0,\, \cdots,\, 0)$ and $(0,\, \cdots,\, 0,\, v_n,\, \cdots,\, v_{d-1})$ respectively. As before, let $\Omega \approx Y$ denote the fundamental domain of Γ_0 in G^*/K^*. Then there exists a unique $\gamma_2 \in \Gamma_0$ such that $\gamma_2\omega_{r_0 s}\theta_{v'_{<n}} = \omega_t\theta_w k \in G^*$ such that $(t,\, w)$ lies in Ω. Assume that $k = \mathrm{diag}(1,\, \rho_2,\, \rho_3) \in K^*$. Define

$$\mathbb{R}_{d-1,\,n} := \left\{x = (x_i)_{i=1}^{d-1} \in \mathbb{R}^{d-1} \,\middle|\, x_i = 0 \;\forall\; i \leqslant n-1\right\}$$

and

$$G_0^* := \{\omega_r\theta_u k \,|\, (r,\, u) \in \Omega,\;\, k \in K^*\} \subset G^*.$$

Denote $\mathring{1}_{\ell,\,i} = \mathrm{diag}(1, \cdots, -1, \cdots, 1) \in \mathrm{Mat}_{\ell\times\ell}(\mathbb{R})$ where -1 is the i-th entry of the diagonal. Then

$$\gamma_2\omega_{r_0 s}\theta_{v'}\cdot o = \gamma_2\omega_{r_0 s}\theta_{v'_{<n}}\theta_{v'_{\geqslant n}}\cdot o = \omega_t\theta_w \begin{pmatrix} 1 & & \\ & \rho_2 & \\ & & \rho_3 \end{pmatrix} \theta_{v'_{\geqslant n}}\cdot o = \underbrace{\omega_t\theta_w \begin{pmatrix} 1 & & \\ & \hat{\rho}_2 & \\ & & \hat{1}_{d-n} \end{pmatrix}}_{\in G_0^*} \theta_{v'_{\geqslant n}\rho_3^{*\,T}}\cdot o$$

where

- if $\det(\rho_2) = 1$, then $\hat{\rho}_2 = \rho_2$, $\hat{1}_{d-n} = 1_{d-n}$ and $\rho_3^* = \mathrm{diag}(1,\, 1_n,\, \rho_3)$.
- if $\det(\rho_2) = -1$, then $\hat{\rho}_2 = \rho_2 \cdot \mathring{1}_{n,\,i}$ where $2 \leqslant i \leqslant n$ (this is available since $n \geqslant 2$), $\hat{1}_{d-n} = \mathring{1}_{d-n,\,j}$ where $1 \leqslant j \leqslant d-n$, $\rho_3^* = \mathrm{diag}\big(1,\, \mathring{1}_{n,\,i},\, \mathring{1}_{d-n,\,j} \cdot \rho_3\big)$.

Note that $v'_{\geqslant n}\rho_3^{*T} \in \mathbb{R}_{d-1,\,n}$ and $\big|v'_{\geqslant n}\rho_3^{*T}\big|^2 + 1 = \big|v'_{\geqslant n}\big|^2 + 1 = \delta_u(\gamma)$. Define

$$\Omega_x^* := \big\{ g = g^*\theta_u k \in G \,\big|\, g^* \in G_0^*,\ u \in \mathbb{R}_{d-1,\,n},\ |u|^2 + 1 \leqslant x,\ k \in K \big\} \subset G.$$

Denote by $\gamma^*(u)$ the element $\gamma_2\gamma\gamma_1\omega_r\theta_{(rr_1)^{-1}u\left(\rho_1^{-1}\right)^T}$, then $\gamma^*(u) \in \Omega_x^*$ for some $x \geqslant 1$. When $M(\gamma)N_u(\gamma) = 0$, the existence of $\gamma^*(u)$ is clear in view of the above argument: if $M(\gamma) = 0$, one chooses $\gamma_1 = \gamma_0^\ell$ for ℓ large enough; if $N_u(\gamma) = 0$, one chooses $\gamma_1 = \gamma_0^\ell$ for $\ell < 0$ and $-\ell$ large enough.

Lemma 3.2.4. *$\gamma^*(u) \neq \eta^*(w)$ for any $u,\ w \in \mathbb{R}_{d-1}^{n-1}$ and $\gamma,\ \eta$ of different classes in $\Gamma_0\backslash\Gamma/\Gamma_0$.*

Proof. Likely, we have: $\eta^*(w) = \eta_2\eta\eta_1\omega_\ell\theta_{(\ell\ell_1)^{-1}w\left(\tau_1^{-1}\right)^T} \in \Omega_x^*$ for some $x \geqslant 1$, $\eta_2 \in \Gamma_0$ and $\eta_1 = \omega_{\ell_1}k_1' \in \Gamma_{00}$ where $k_1' = \mathrm{diag}(1,\, 1,\, \tau_1)$. Let us denote by $\hat{\gamma}$ and $\hat{\eta}$ the elements $\gamma_2\gamma\gamma_1$ and $\eta_2\eta\eta_1$ respectively. Then $\gamma^*(u) = \eta^*(w)$ implies that $\hat{\gamma}^{-1}\hat{\eta} = \omega_r\theta_{(rr_1)^{-1}u\left(\rho_1^{-1}\right)^T} \cdot \theta^{-1}_{(\ell\ell_1)^{-1}w\left(\tau_1^{-1}\right)^T}\,\omega_{\ell^{-1}} \in \Gamma \cap AN_{<n}$ where $N_{<n} = \{\theta_u \,|\, u \in \mathbb{R}_{d-1}^{n-1}\}$. It is clear that $AN_{<n} \subset G^*$, hence $\hat{\gamma}^{-1}\hat{\eta} \in \Gamma \cap G^* = \Gamma_0$ which means that γ and η are of the same class in $\Gamma_0\backslash\Gamma/\Gamma_0$. This contradicts our assumption. $\square$

This lemma tells us that, those representative elements $\gamma^*(u)$'s are distinguished in Ω_x^* with respect to γ's (of different classes). A further property is about the discreteness of these $\gamma^*(u)$'s:

Lemma 3.2.5. *For any sequence of pairs*

$$\big\{ \big(\gamma_{i1}^*(u_i), \gamma_{i2}^*(w_i)\big) \,\big|\, \gamma_{i1},\ \gamma_{i2} \in \Gamma \smallsetminus \Gamma_0,\ \widetilde{\gamma}_{i1} \neq \widetilde{\gamma}_{i2},\ \gamma_{i1}^*(u_i),\ \gamma_{i2}^*(w_i) \in \Omega_\infty^*,\ \forall\ u_i,\ w_i \in \mathbb{R}_{d-1}^{n-1} \big\}_{i=1}^{\infty},$$

$\gamma_{i1}^*(u_i)$ and $\gamma_{i2}^*(w_i)$ can not be close enough (as $i \to \infty$) with respect to the topology of G.

Proof. Let $\gamma_{i1}^*(u_i) = \gamma_{i1}'\gamma_{i1}\gamma_{i1}''\omega_{r_{i1}}\theta_{u_{i1}} \in \Omega_\infty^*$ and $\gamma_{i2}^*(w_i) = \gamma_{i2}'\gamma_{i2}\gamma_{i2}''\omega_{r_{i2}}\theta_{u_{i2}} \in \Omega_\infty^*$ for some $\gamma_{i1}',\ \gamma_{i2}' \in \Gamma_0$, $\gamma_{i1}'',\ \gamma_{i2}'' \in \Gamma_{00}$, $r_{i1},\ r_{i2} \in [1,\, e^T]$ and $u_{i1},\ u_{i2} \in \mathbb{R}_{d-1}^{n-1}$. Assume that $\gamma_{i1}^*(u_i)$ and $\gamma_{i2}^*(w_i)$ are close enough as $i \to \infty$, that is, $\left(\gamma_{i1}^*(u_i)\right)^{-1}\gamma_{i2}^*(w_i) \to 1$, then $\theta_{-u_{i1}}\omega_{r_{i1}^{-1}}\left(\gamma_{i1}'\gamma_{i1}\gamma_{i1}''\right)^{-1}\left(\gamma_{i2}'\gamma_{i2}\gamma_{i2}''\right)\omega_{r_{i2}}\theta_{u_{i2}} \in U_i$, i.e., $\eta_i := \left(\gamma_{i1}'\gamma_{i1}\gamma_{i1}''\right)^{-1}\left(\gamma_{i2}'\gamma_{i2}\gamma_{i2}''\right) \in V_i := \omega_{r_{i1}}\theta_{u_{i1}}\,U_i\,\theta_{-u_{i2}}\omega_{r_{i2}^{-1}}$ where U_i is a neighborhood of 1 that can be small enough for large i. For any i, V_i is a neighborhood of the element $\omega_{r_{i1}}\theta_{u_{i1}}\theta_{-u_{i2}}\omega_{r_{i2}^{-1}} \in AN_{<n}$. Here $N_{<n}$ has the same meaning with that in the proof of Lemma 3.2.4. As $i \to \infty$, $\eta_i \to$

$AN_{<n}$ by which we mean the following: there exists $\alpha_i \in AN_{<n}$ such that $\alpha_i^{-1}\eta_i \to 1$, i.e., η_i can be close enough to $AN_{<n}$. This implies that, over $\Gamma\backslash G/K$, there exists a compact neighborhood W of the compact submanifold $\Gamma_0\backslash G^*/K^*$ (more properly, W is the neighborhood of $(\Gamma_0 \cap AN_{<n})\backslash AN_{<n} \hookrightarrow \Gamma_0\backslash G^*/K^*$) such that all the points $\widetilde{\eta}_i \in \Gamma_0\backslash\Gamma/K$ lie in W. As $\Gamma \cap K = \{1\}$, we can identify the image $\widetilde{\eta}_i$ of η_i under $\Gamma \to \Gamma_0\backslash\Gamma/K$ and the image of η_i under $\Gamma \to \Gamma_0\backslash\Gamma$. Hence there is a subsequence $\{\widetilde{\eta}_{i_m}\}$ which converges. The subset $\Gamma_0\backslash\Gamma \subset W$ is closed has no accumulation point (see Lemma 3.2.3), so $\widetilde{\eta}_{i_m} \equiv \widetilde{\eta}$ for some $\eta \in \Gamma$ and any m larger than some N_0. As η_i can be close enough to $AN_{<n}$, we can find some $\widetilde{\eta}$ such that it lies in $(\Gamma_0 \cap AN_{<n})\backslash AN_{<n}$. It immediately follows that $\eta \in \Gamma \cap AN_{<n} \subset \Gamma_0$. Hence $\eta_{i_m} \in \Gamma_0$ which means that γ_{i_m1} and γ_{i_m2} are of the same class in $\Gamma_0\backslash\Gamma/\Gamma_0$ for $m \geqslant N_0$, a contradiction. □

Remark 3.2.6. *In the above proof, the neighborhood W plays a key role. By such W we get a convergent subsequence $\{\eta_{i_m}\}$ with the accumulation point in Γ_0. Meanwhile this subsequence is stationary for large m. Note that W does not depend on u_i or w_i. Actually it depends only on Γ as V_i is a neighborhood of $\eta_i \in \Gamma$. The property we have essentially used on u_i and w_i is that they lie in $\mathbb{R}^{n-1}_{d-1}$. So the above lemma is universal for all u_i, $w_i \in \mathbb{R}^{n-1}_{d-1}$.*

Clearly, $\delta(\gamma) = \min_{u\in\mathbb{R}^{n-1}_{d-1}} \delta_u(\gamma)$. To count $\pi^*(x)$ is to count the representative elements $\gamma_i^*(u)$'s such that γ_i's are of different classes in $\Gamma_0\backslash\Gamma/\Gamma_0 \setminus \{\tilde{1}\}$ and $\delta(\gamma_i) = \delta_u(\gamma_i)$. The special element $\tilde{1}$ does not influence the order of $\pi^*(x)$. Lemma 3.2.4 and Lemma 3.2.5 hold for any u, w and u_i, $w_i \in \mathbb{R}^{n-1}_{d-1}$ respectively. Hence the cardinality of $\pi^*(x)$ is (upper) bounded by the volume of Ω_x^*. Both G_0^* and K are compact, so the volume of Ω_x^* depends on the $(d-n)$ many free parameters u's. Consequently, we have: $\pi^*(x) = \mathcal{O}\left(\mathrm{vol}\left(\Omega_x^*\right)\right) = \mathcal{O}\left(x^{\frac{d-n}{2}}\right)$ as $x \to \infty$. Here the $\mathcal{O}$-constant is unconditional. This completes the proof of Proposition 3.1.1. □

The following conclusions is immediate:

Corollary 3.2.7. *If $|\,\Gamma_0\backslash\Gamma/\Gamma_0| = \infty$, then the unique accumulation point of $\{\delta(\gamma)\}$ is ∞.*

Likewise, by Corollary 3.1.3,

Corollary 3.2.8. *For any fixed $u \in \mathbb{R}^{n-1}_{d-1}$, if $|\,\Gamma_0\backslash\Gamma/\Gamma_0| = \infty$, then the unique accumulation point of $\{\delta_u(\gamma)\}$ is ∞.*

Proposition 3.2.9. *For any fixed $u \in \mathbb{R}^{n-1}_{d-1}$, the unique accumulation point of $\{M(\gamma)N_u(\gamma)\}$ is ∞.*

Proof. Assume that $M(\gamma_i)N_u(\gamma_i) \leqslant c$ for a sequence $\{\gamma_i\}$ and fixed number c. By (3.8),

$$Q_u(\gamma) \leqslant 1 + 2\sqrt{M(\gamma)N_u(\gamma)}.$$

Hence $\delta_u(\gamma_i) = 2\sqrt{M(\gamma_i)N_u(\gamma_i)} + Q_u(\gamma_i) \leqslant 4\sqrt{c}+1$ which implies that there exists a convergent subsequence $\{\delta_u(\gamma_{i_m})\}$ with finite accumulation value. However, this contradicts Corollary 3.2.8. □

Lemma 3.2.10. *If $M(\gamma)N_u(\gamma) = 0$, then $Q_u(\gamma) = 1$.*

Proof. This is clear in view of (3.6), (3.7) and (3.8). □

Lemma 3.2.11. *For any $g \in G$, $\Gamma_0 \cdot g \cdot \Gamma_0$ is discrete in G.*

Proof. It is well-known that the product of a finite number of discrete groups is discrete. So $\Gamma_0 \cdot g \cdot \Gamma_0 = g \cdot g^{-1}\Gamma_0 g \cdot \Gamma_0$ is discrete. □

Lemma 3.2.12. *For any $\gamma \notin \Gamma_0$ and $u \in \mathbb{R}^{n-1}_{d-1}$, $M(\gamma)$ and $N_u(\gamma)$ can not be zero simultaneously.*

Proof. Assume that $M(\gamma) = N_u(\gamma) = 0$ for some $\gamma \notin \Gamma_0$ and $u \in \mathbb{R}^{n-1}_{d-1}$. As before, let $\gamma\theta_u\omega_r = \omega_{r_0 s}\theta_{\frac{w_0+v}{s}} k$ for some $k \in K$. Then $\left|\frac{w_0+v}{s}\right|^2_{\geqslant n} + 1 = M(\gamma)r^2 + \frac{N_u(\gamma)}{r^2} + Q_u(\gamma) \equiv 1$ for any $r > 0$ (see Lemma 3.2.10). which shows that $(w_0+v)_i \equiv 0$, $i \geqslant n$. Thus, for any $\gamma_0^n = \omega_{e^{nT}} k_0^n \in \Gamma_{00} \subset \Gamma_0$ where $k_0 = \mathrm{diag}(1, 1, \rho)$, there exists $\gamma_{2\,n} \in \Gamma_0$ such that $\gamma^*(u) = \gamma_{2\,n}\gamma\theta_u\gamma_0^n \in \Omega_1^* \subset G$. One can write $\theta_u\gamma_0^n = \gamma_0^n\theta_{u_n}$ where $u_n = e^{-nT}u(\rho^T)^n$. Clearly, $|u_n| \to 0$ as $n \to \infty$.

- If $u \neq 0$, then $u' \neq 0$. By Lemma 3.2.11, there are finitely many $\gamma^*(u)$'s in the domain Ω_1^*. Thus, $\gamma_{2\,n}\gamma\gamma_0^n\theta_{u_n} = \gamma_{2\,m}\gamma\gamma_0^m\theta_{u_m}$ for infinitely many m and n. This implies that $(\gamma_{2\,m}\gamma\gamma_0^m)^{-1}(\gamma_{2\,n}\gamma\gamma_0^n) = \theta_{u_m-u_n}$. Note that $\theta_{u_m-u_n} \in W_{mn}$ where W_{mn} is a neighborhood of 1 that can be small enough for large m and n. Hence $(\gamma_{2\,m}\gamma\gamma_0^m)^{-1}(\gamma_{2\,n}\gamma\gamma_0^n) = \theta_{u_m-u_n} \in \Gamma \cap W_{mn} = \{1\}$ (as Γ is discrete), i.e., $\gamma_{2\,n}\gamma\gamma_0^n = \gamma_{2\,m}\gamma\gamma_0^m$ for large m, n. However, $u_m \neq u_n$ (if $m \neq n$) since $u \neq 0$. So $\gamma_{2\,n}\gamma\gamma_0^n\theta_{u_n} \neq \gamma_{2\,m}\gamma\gamma_0^m\theta_{u_m}$ for any m, n large, a contradiction.

- If $u = 0$, then $M(\gamma) = \sum_{i=n}^{d-1}\left(w_{0\,i}\,\frac{1-u_{11}}{2} + \frac{u_{i+1,\,1}}{2}\right)^2$ and $N_u(\gamma) = \sum_{i=n}^{d-1}\left(w_{0\,i}\,\frac{1+u_{11}}{2} - \frac{u_{i+1,\,1}}{2}\right)^2$. Then the assumption that $M(\gamma) = N_u(\gamma) = 0$ immediately implies that $w_{0\,i} = u_{i+1,\,1} = 0$ for $n \leqslant i \leqslant d-1$, i.e., $\gamma \in AN_{<n}K$. There is a bijection between Γ_0 and the set of fibers of any point $z \in Y$ in G^*/K^* since Y is regarded as an embedded submanifold in X. The left translation of $e \cdot o$ by γ lies in G^*/K^*. So there exists $\eta \in \Gamma_0$ such that $\eta \cdot o = \gamma \cdot o$ from which it follows that $\gamma \in \Gamma_0$, but we have assumed that $\gamma \in \Gamma \setminus \Gamma_0$.

The proof of the lemma is complete. □

Lemma 3.2.13. *For any $\gamma \notin \Gamma_0$ and $u \in \mathbb{R}^{n-1}_{d-1}$, $M(\gamma)N_u(\gamma) \neq 0$.*

Proof. Assume that $M(\gamma) = 0$. By Lemma 3.2.12, $N_u(\gamma) \neq 0$ for any $u \in \mathbb{R}^{n-1}_{d-1}$. Thus, $\left|\frac{w_0+v}{s}\right|^2_{\geqslant n} + 1 = \frac{N_u(\gamma)}{r^2} + 1 > 1$ for any $r > 0$. Let $r = e^{nT}$. For n large enough, $\left|\frac{w_0+v}{s}\right|^2_{\geqslant n} + 1 \leqslant 1 + \epsilon$. There exist $\gamma_2 \in \Gamma_{00}$ and $\gamma_1 = \omega_{e^{nT}} k_0^n \in \Gamma_{00}$ such that $\gamma_r^* = \gamma_2\gamma\gamma_1\theta_u$ lies in $\Omega^*_{1+\epsilon}$, i.e., there are infinitely many (different) γ^*s in the domain $\Omega^*_{1+\epsilon}$, contradicting Lemma 3.2.11. The case for $N_u(\gamma) = 0$ can be disproved in the same way. We omit the details. □

Proof of Proposition 3.1.2. Assume that there exist a sequence $\{\gamma_i \in \Gamma \smallsetminus \Gamma_0 \,|\, \widetilde{\gamma}_i \neq \widetilde{\gamma}_j \text{ for } i \neq j\}$ and $u_i \in \mathbb{R}^{n-1}_{d-1}$ such that $M(\gamma_i)N_{u_i}(\gamma_i) \to 0$ as $i \to \infty$. By (3.6), (3.7) and (3.8), it follows that $\delta_{u_i}(\gamma_i) \to 1$ as $i \to \infty$. Since $\delta(\gamma_i) \leqslant \delta_{u_i}(\gamma_i)$, we get $\delta(\gamma_i) \to 1$ as $i \to \infty$ which means that there exists a subsequence $\{\delta(\gamma_{i_m})\}$ convergent to 1. This contradicts Corollary 3.2.7. The proposition follows in view of Lemma 3.2.13. □

3.3 The comparison

For $0 < \epsilon < 1$, define $\eta_{u,\epsilon}(\gamma) := \delta_u(\gamma) - \epsilon \cdot \delta(\gamma) > 0$.

Proposition 3.3.1. *For each class $\widetilde{\gamma} \in \Gamma_0 \backslash \Gamma / \Gamma_0 \smallsetminus \{\tilde{1}\}$, one can choose proper representative element γ such that*

$$\int\limits_{\mathbb{R}^{n-1}_{d-1}} \left(2\sqrt{M(\gamma)}\right)^{n-1} e^{-\frac{\mu}{2}\sqrt{\eta_{u,\epsilon}(\gamma)}} du \leqslant A$$

for some positive number A that is uniform for all γ chosen as above.

Remark 3.3.2. *Remember that $\delta(\,\cdot\,)$ is well-defined on $\Gamma_0\backslash\Gamma/\Gamma_0$, while $\delta_u(\,\cdot\,)$ is only well-defined over $\Gamma_0\backslash\Gamma$, not on Γ/Γ_0.*

Proof. By 3.14, $\eta_{u,\epsilon}(\gamma) = \delta_u(\gamma) - \epsilon \cdot \delta(\gamma) \geqslant \sum_{j=2}^{n} u'^2_{j-1} + 4H_\gamma + 1 - \epsilon \cdot \delta(\gamma)$. Remember that $\sum_{j=2}^{n} u'^2_{j-1}$ is a polynomial of degree 2 with respect to $2\sqrt{M(\gamma)}\,u$ (see formula 3.12). Besides, $\eta_{u,\epsilon}(\gamma)$ is positive with minimum value $\min_{u \in \mathbb{R}^{n-1}_{d-1}} \eta_{u,\epsilon}(\gamma) = (1-\epsilon)\cdot\delta(\gamma) \geqslant 1-\epsilon > 0$. Let $v = 2\sqrt{M(\gamma)}\,u$. Denote $4H_\gamma + 1 - \epsilon \cdot \delta(\gamma)$ by $\tau_\epsilon(\gamma)$. One has

- if $\tau_\epsilon(\gamma) \geqslant 0$, then

$$\int\limits_{\mathbb{R}^{n-1}_{d-1}} \left(2\sqrt{M(\gamma)}\right)^{n-1} e^{-\frac{\mu}{2}\sqrt{\eta_{u,\epsilon}(\gamma)}} du \leqslant \int\limits_{\mathbb{R}^{n-1}_{d-1}} e^{-\frac{\mu}{2}\cdot|v|} dv,$$

- if $\tau_\epsilon(\gamma) < 0$, then

$$\int\limits_{\mathbb{R}^{n-1}_{d-1}} \left(2\sqrt{M(\gamma)}\right)^{n-1} e^{-\frac{\mu}{2}\sqrt{\eta_{u,\epsilon}(\gamma)}} du \quad \leqslant \quad \int\limits_{\substack{v \in \mathbb{R}^{n-1}_{d-1} \\ |v|^2 \geqslant \tau_\epsilon(\gamma)}} e^{-\frac{\mu}{2}\sqrt{|v|^2 - |\tau_\epsilon(\gamma)|}} dv$$

$$+ \quad \left(2\sqrt{M(\gamma)}\right)^{n-1} e^{\sqrt{(1-\epsilon)\delta(\gamma)}} \cdot V_\epsilon(\gamma)$$

where $V_\epsilon(\gamma)$ is the standard volume of the ball $B_\epsilon(\gamma) = \left\{x \in \mathbb{R}^{n-1} \,\middle|\, |x| \leqslant \sqrt{|\tau_\epsilon(\gamma)|}\right\}$. Both integrals on the right hand side of the above inequalities converge. Up to now, we have shown that the integral in the proposition converges for each $\gamma \in \Gamma \smallsetminus \Gamma_0$. Next we show that the integral is uniformly upper bounded for properly chosen γ in each non-trivial class in $\Gamma_0\backslash\Gamma/\Gamma_0$. The key observation is that, with proper $\gamma \notin \Gamma_0$, the local minimal or maximal value of $\eta_{u,\epsilon}(\gamma)$ is obtained at u where u lies in a fixed bounded domain around zero in $\mathbb{R}^{n-1}_{d-1}$. Beyond this domain, there is a positive number b such that $\eta_{u,\epsilon}(\gamma)$ is larger than the polynomial $\sum_{j=2}^{n} u'^2_{j-1} + b$ when $\delta(\gamma)$ is large enough. Within this domain, $\eta_{u,\epsilon}(\gamma) \geqslant 1-\epsilon$. Thus the proposition follows. To verify the observation, we work on $2\sqrt{M(\gamma)N_u(\gamma)}$ and $2\sum_{i=n}^{d-1} m_i n_i$. With u as variable, n_i is a polynomial of degree at most 2 and $2\sum_{i=n}^{d-1} m_i n_i$ is a polynomial of exactly degree 2 (see Sect. 3.1.2). Hence we just have to choose $\gamma \notin \Gamma_0$ such that n_i^2 and $2\sum_{i=n}^{d-1} m_i n_i$ achieve their minimal or maximal values at u for u in a domain that is universal for all i and γ. Expand n_i as

$$n_i = \left(\frac{w_{0\,i}(1-u_{11}) + u_{i+1,1}}{2}\right)|u|^2 + \sum_{j=2}^{n} \left(u_{i+1,j} - w_{0\,i}u_{1\,j}\right) u_{j-1} + \left(\frac{w_{0\,i}(1+u_{11}) - u_{i+1,1}}{2}\right).$$

- if $m_i = \frac{w_{0\,i}(1-u_{11})+u_{i+1,1}}{2} \neq 0$, then by (3.7), $|n_i|$ achieves its minimal or maximal value at u where

$$u_{j-1} = -\frac{u_{i+1,j} - w_{0\,i}u_{1\,j}}{w_{0\,i}(1-u_{11}) + u_{i+1,1}}, \quad 1 \leqslant j \leqslant n-1 \tag{3.15}$$

or such that $n_i = 0$, i.e.,

$$m_i \sum_{j=2}^{n} \left(u_{j-1} + \frac{u_{i+1,j} - w_{0\,i}u_{1\,j}}{2m_i}\right)^2 = \sum_{j=2}^{n} \frac{\left(u_{i+1,j} - w_{0\,i}u_{1\,j}\right)^2}{4m_i} - \frac{w_{0\,i}(1+u_{11}) - u_{i+1,1}}{2}. \tag{3.16}$$

By (3.12), $2\sum_{i=n}^{d-1} m_i n_i$ achieves its minimal value at u where

$$u_{j-1} = -\frac{\sum_{i=n}^{d-1} m_i(u_{i+1,j} - w_{0\,i}u_{1\,j})}{2M(\gamma)}. \tag{3.17}$$

Denote $\gamma = \omega_{r_0}\theta_{w_0}k_\gamma$. Multiply $\gamma_0^\ell = e^{\ell TE}k_0^\ell \in \Gamma_{00}$ to γ from the right, and denote $\gamma\gamma_0^n = \omega_{r_0 s}\theta_{\frac{w_0+v}{s}}k$ as before. By (3.4) and (3.5) we see that

$$\left(\frac{w_0+v}{s}\right)_i = \frac{w_{0\,i}(1-u_{11}) + u_{i+1,1}}{2}e^{\ell T} + \frac{w_{0\,i}(1+u_{11}) - u_{i+1,1}}{2}e^{-\ell T}. \tag{3.18}$$

Since $w_{0\,i}(1-u_{11})+u_{i+1,\,1}\neq 0$, $\left(\frac{w_0+v}{s}\right)_i$ can be very large for ℓ large. The right multiplication by γ_0^ℓ does not change u_{11}, i.e., $u_{11}(\gamma)=u_{11}(\gamma\gamma_0^\ell)$. Thus $|w_{0\,i}|$ and $|m_i|$ can be large enough with γ replaced by proper $\gamma\gamma_0^\ell$. For γ in different classes, one can use different ℓ. The formulas (3.15), (3.16) and (3.17) show that it is $w_{0\,i}(1-u_{11})$ or $(1-u_{11})\sum_{i=n}^{d-1} w_{0\,i}^2$ that decides the order of the denominator of u_{j-1} for $|w_{0\,i}|$ and m_i large. The other terms in the denominator are from k_γ, thus bounded. The order of the numerator in these three formulas is no larger than that of the denominator (with respect to $w_{0\,i}$ or $\sum_{i=n}^{d-1} w_{0\,i}^2$). By the ensuing Lemma 3.3.5, we see that u lies in a bounded domain. Here we mention again that $u_{11}\neq 1$ for $\gamma\notin\Gamma_0$.

- if $m_i=\frac{w_{0\,i}(1-u_{11})+u_{i+1,\,1}}{2}=0$, then $n_i=\sum_{j=2}^{n}\left(u_{i+1,\,j}-w_{0\,i}u_{1\,j}\right)u_{j-1}+w_{0\,i}$.
  - if $w_{0\,i}=0$, then $n_i=0$ defines a hypersurface through the origin which means that $|n_i|$ achieves its minimal value at the origin.
  - if $w_{0\,i}\neq 0$, then $\frac{w_{0\,i}(1+u_{11})-u_{i+1,\,1}}{2}\neq 0$. By (3.18), $\left(\frac{w_0+v}{s}\right)_i\to 0$ as $\ell\to\infty$, meanwhile $\left(\frac{w_0+v}{s}\right)_i\neq 0$. This means that, we can get a new $w_{0\,i}$ with γ replaced by $\gamma\gamma_0^\ell$ such that this new $w_{0\,i}$ is small enough, but nonzero. In this process, $u_{i+1,\,1}$ maintains unchanged as the right multiplication of γ_0^ℓ does not change those entries lying in the first column of ρ_γ where $k_\gamma=\mathrm{diag}(1,\,\rho_\gamma)$. With the new terms, m_i does not vanish. Hence we might as well assume that $m_i\neq 0$. At this point, we are led back to the former cases.

The proof is complete. □

Lemma 3.3.3. *$ANM\cap\Gamma=\Gamma_{00}$, $ANM'\cap\Gamma=\{1\}$.*

Proof. Let $\gamma=\omega_{r_0}\theta_{w_0}k_\gamma$ for some $k\in M$, then $\gamma\gamma_0^\ell=\omega_{r_0e^{\ell T}}\theta_{w_0e^{-\ell T}}k_\gamma k_0^\ell$. Remember that $\gamma_0=\omega_{e^T}k_0$ where $k_0\in M$. Multiplying proper $\gamma_2\in\Gamma_{00}$ to the left side of $\gamma\gamma_0^\ell$, we get infinitely many distinct γ^*'s lying in Ω_x for some fixed $x>1$, except $w_0=0$. Here the notations γ^* and Ω_x are as those in the Chapter 2. By Lemma 2.3.1, there should be only finitely many γ^*'s in Ω_x. Hence $w_0=0$ and the first conclusion follows from Lemma 1.5.3. Note that the group Γ_0 in the first three chapters is denoted as Γ_{00} here. When k_γ lies in M', by the formula $\omega_r k_\gamma=k_\gamma\omega_{r^{-1}}$, the above argument still applies and we get $w_0=0$, i.e., $\gamma\in AM'\cap\Gamma=\{1\}$. □

The following lemma is an easy consequence of the proof of the above lemma:

Lemma 3.3.4. *$\Gamma_{00}\backslash ANM/\Gamma_{00}$ is compact.* □

Lemma 3.3.5. *For any subset* $\Lambda \subset \Gamma \setminus \Gamma_0$ *consisting of representative elements of classes in* $\Gamma_0\backslash\Gamma/\Gamma_0 \setminus \{\tilde{1}\}$ *such that each class has exactly one representative element in* Λ*, we have:*

$$\sup_{\gamma\in\Lambda} \left|u_{11}(\gamma)\right| < 1.$$

Proof. For $g \in G$, let g^* denote its image in $\Gamma_{00}\backslash G$ under the natural map $G \to \Gamma_{00}\backslash G$. Let $\{\gamma_i\} \subset \Lambda$ be a sequence such that $u_{11}(\gamma_i) \to 1$ as $i \to \infty$. Clearly, $\gamma_i^* \neq \gamma_j^*$ for $i \neq j$. For each i there exists $\eta_i \in ANM$ such that $\eta_i^{-1}\gamma_i \in W_i$ where W_i is a compact neighborhood of 1. We can and will assume that W_i is small enough for i large. By Lemma 3.3.4, there exist $\tau_i,\ \tau_i' \in \Gamma_{00}$ such that $\tau_i\eta_i^{-1}\tau_i'$ converges in (a fundamental domain of) $\Gamma_{00}\backslash ANM/\Gamma_{00}$. When W_i is small enough, $\Gamma_{00}\backslash W_i$ is isomorphic to W_i. The right action of Γ on $\bigcup\limits_{\gamma\in\Gamma} (\Gamma_{00}\backslash W_i)\,\gamma$ is thus discontinuous for i large enough. On one hand, $\left(\eta_i^{-1}\tau_i'\right)^*$ converges in a compact domain in $\Gamma_{00}\backslash ANM$, on the other hand $\left(\eta_i^{-1}\tau_i'\right)^*\tau_i'^{-1}\gamma_i = (\eta_i\gamma_i)^*$ converges in $\Gamma_{00}\backslash W_i$. Hence, by passing to a subsequence if necessary, we get: $\tau_i'^{-1}\gamma_i$ is stationary for large i. This implies that $\gamma_i^* = \gamma_j^*$ for large $i,\ j$. But we have assumed that $\gamma_i^* \neq \gamma_j^*$ for $i \neq j$. Thus $\sup\limits_{\gamma\in\Lambda} u_{11}(\gamma) < 1$. The case for $u_{11}(\gamma_i) \to -1$ is proved in a similar way. We omit the details. □

To estimate $\sum\limits_{\Gamma_0\backslash\Gamma/\Gamma_0\setminus\{\tilde{1}\}} I_\gamma$, let us assume that $|\,\Gamma_0\backslash\Gamma/\Gamma_0| = \infty$ because we can treat the case $|\,\Gamma_0\backslash\Gamma/\Gamma_0| < \infty$ in the same way with what we shall do for $\sum\limits_{m=1}^{N_{n/2}} I_m$ in the case $|\,\Gamma_0\backslash\Gamma/\Gamma_0| = \infty$ (see below).

By (3.11) and the above proposition,

$$\begin{aligned} I_\gamma &\leqslant 2^n\left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1}\frac{C}{\sqrt{\mu}}\int\limits_{\mathbb{R}^{n-1}_{d-1}}\left(2\sqrt{M(\gamma)}\right)^{n-1} e^{-\frac{\mu\sqrt{2}}{2}\left(\sqrt{\eta_{u,\,\epsilon}(\gamma)+\epsilon\cdot\delta(\gamma)}\right)}du \\ &\leqslant 2^n\left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1}\frac{C}{\sqrt{\mu}}\int\limits_{\mathbb{R}^{n-1}_{d-1}}\left(2\sqrt{M(\gamma)}\right)^{n-1} e^{-\frac{\mu}{2}\sqrt{\eta_{u,\,\epsilon}(\gamma)}}\cdot e^{-\frac{\mu}{2}\sqrt{\epsilon\cdot\delta(\gamma)}}du \\ &\leqslant A\cdot 2^n\left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1}\frac{C}{\sqrt{\mu}}\,e^{-\frac{\mu}{2}\sqrt{\epsilon\cdot\delta(\gamma)}} \end{aligned}$$

Thus, one has:

$$\sum_{\widetilde{\gamma}\in\Gamma_0\backslash\Gamma/\Gamma_0\setminus\{\tilde{1}\}} I_\gamma \ll \left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1}\frac{1}{\sqrt{\mu}}\sum_{\widetilde{\gamma}\in\Gamma_0\backslash\Gamma/\Gamma_0\setminus\{\tilde{1}\}} e^{-\frac{\mu}{2}\sqrt{\epsilon\cdot\delta(\gamma)}}.$$

We may arrange the order of elements in $\Gamma_0\backslash\Gamma/\Gamma_0$ to get the sequence $\{\widetilde{\gamma}_m\}_{m=1}^{\infty}$ where $\delta(\gamma_m)$ increases as m increases. For any $\alpha > 0$, by Proposition 3.1.1, there exists some natural number $N_\alpha > 4^d$ such that:

$$\delta(\gamma_m) \gg m^{\frac{1}{\frac{d-n}{2}+\alpha}}, \quad m \gg N_\alpha.$$

Let $\epsilon = \frac{1}{4}$ and $\alpha = \frac{n}{2}$, then

$$\sum_{m=N_{n/2}}^{\infty} I_m \leqslant \sum_{m=N_{n/2}}^{\infty} e^{-\frac{\mu}{4}\sqrt{\delta(\gamma_m)}} \ll \sum_{m=N_{n/2}}^{\infty} e^{-\frac{\mu}{4}m^{1/d}}.$$

The term $\sum\limits_{m=N_{n/2}}^{\infty} e^{-\frac{\mu}{4}m^{1/d}}$ is bounded by the integral

$$\int_{4^d}^{\infty} e^{-\frac{\mu}{4}x^{1/d}}dx = d\,4^d \int_1^{\infty} e^{-\mu y} y^{d-1} dy \ \ (\text{letting } \ y = \frac{x^{1/d}}{4})$$

which is bounded by $\mathcal{O}\left(e^{-\mu}\mu^{-1}\right)$ (see Sect. 2.3.2). Hence

$$\sum_{m=N_{n/2}}^{\infty} I_m = \mathcal{O}\left(e^{-\mu}\mu^{-\frac{n}{2}-1}\right).$$

Now let's check the terms $I(\gamma_m)$ for $1 \leqslant m \leqslant N_{n/2}$. The case $|\,\Gamma_0\backslash\Gamma/\Gamma_0| < \infty$ can be treated in the same approach used here. By (3.9), there exists some $L > 0$ such that

$$F_\gamma(u,\,x) \leqslant \frac{L}{\left(\sqrt{x^2+\delta_u(\gamma)}\right)^{\frac{n-1}{2}}} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+\delta_u(\gamma)}\right)}{\left(x+\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}}. \tag{3.19}$$

We have:

$$\frac{1}{\left(x+\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}} \geqslant \frac{1}{\left(2\sqrt[4]{M(\gamma)N_u(\gamma)}\right)^{n-1}}, \quad \text{for } x \geqslant 2 \tag{3.20}$$

$$\frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+\delta_u(\gamma)}\right)}{\left(\sqrt{x^2+\delta_u(\gamma)}\right)^{\frac{n-1}{2}}} \leqslant \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}} \quad (\text{as } \delta_u(\gamma) \geqslant 1). \tag{3.21}$$

As $M(\gamma)N_u(\gamma) \geqslant \left(\sum\limits_{i=n}^{d-1} m_i n_i\right)^2$ where $\sum\limits_{i=n}^{d-1} m_i n_i$ is a polynomial of degree 2 with respect to u (see the end part of Sect. 3.1.2). Since $M(\gamma)N_u(\gamma) \geqslant c > 0$, the integral

$$\int\limits_{\mathbb{R}^{n-1}_{d-1}} \frac{1}{\left(2\sqrt[4]{M(\gamma)N_u(\gamma)}\right)^{n}}\,du$$

converges for $n \geqslant 2$. Let $y = x^2+1$, then

$$\int_0^{\infty} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}}dx = \int_1^{\infty} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{y}\right)}{\left(\sqrt{y}\right)^{\frac{n-1}{2}}} \frac{dy}{2\sqrt{y-1}} = \frac{\sqrt{2}}{2}\Gamma\left(\frac{1}{2}\right)\mu^{-\frac{1}{2}}K_{\frac{n}{2}-1}(\mu)$$

The second step follows from the formula (2.21). By (3.19), (3.20) and (3.21),

$$\int\limits_{u\in\mathbb{R}^{n-1}_{d-1}} \int\limits_{x=0}^{\infty} F_{\gamma_m}(u,\,x)dxdu \;\leqslant\; L\cdot \int\limits_{0}^{\infty} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}}\,dx\cdot \int\limits_{\mathbb{R}^{n-1}_{d-1}} \frac{du}{\left(2\sqrt[4]{M(\gamma)N_u(\gamma)}\right)^n}$$

$$\leqslant\; L'\cdot\mu^{-\frac{1}{2}}K_{\frac{n}{2}-1}(\mu) \tag{3.22}$$

for some scalar L'. At the moment, there are finitely many $I(\gamma_m)$ under consideration, so we can assume that the scalars L' (and L'' in below) are uniform for all γ_m.

For $x\leqslant -\epsilon$ ($0<\epsilon<1$), we have:

$$\begin{aligned}
&\frac{1}{\left(x+\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}}\\
=\;& \left(\frac{\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}-x}{4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}\\
=\;& (-x)^{n-1}\left(\frac{\sqrt{1+\frac{4\sqrt{M(\gamma)N_u(\gamma)}}{x^2}}+1}{4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}\\
=\;& (-x)^{n-1}\left(\sqrt{\frac{1}{16M(\gamma)N_u(\gamma)}+\frac{1}{4x^2\sqrt{M(\gamma)N_u(\gamma)}}}+\frac{1}{4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}
\end{aligned} \tag{3.23}$$

For ϵ small enough, the following hold:

$$\frac{1}{16M(\gamma)N_u(\gamma)} < \frac{1}{12\epsilon^2\sqrt{M(\gamma)N_u(\gamma)}},$$

$$\frac{1}{4x^2\sqrt{M(\gamma)N_u(\gamma)}} < \frac{1}{4\epsilon^2\sqrt{M(\gamma)N_u(\gamma)}},$$

and

$$\frac{1}{4\sqrt{M(\gamma)N_u(\gamma)}} < \frac{1}{4\epsilon^2\sqrt{M(\gamma)N_u(\gamma)}}.$$

With these inequalities, (3.23) reads:

$$\frac{1}{\left(x+\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}} < (-x)^{n-1}\left(\frac{1}{8\epsilon\sqrt[4]{M(\gamma)N_u(\gamma)}}\right)^{n-1}.$$

Hence

$$\int\limits_{u\in\mathbb{R}^{n-1}_{d-1}} \frac{du}{\left(x+\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}\right)^{n-1}\sqrt{x^2+4\sqrt{M(\gamma)N_u(\gamma)}}}$$

$$< \frac{1}{2} \frac{(-x)^{n-1}}{(8\epsilon)^{n-1}} \int\limits_{u\in\mathbb{R}^{n-1}_{d-1}} \frac{du}{\left(\sqrt[4]{M(\gamma)N_u(\gamma)}\right)^n} \tag{3.24}$$

The integral on the right hand side of (3.24) converges for $n \geqslant 2$ (we have discussed this integral for $x > 0$). By (3.19), (3.21) and (3.24), we have:

$$\begin{aligned}
\int\limits_{u\in\mathbb{R}^{n-1}_{d-1}} \int\limits_{x=-\infty}^{-\epsilon} F_{\gamma_m}(u, x)dxdu \quad &< \quad \frac{L}{(8\epsilon)^{n-1}} \int\limits_{-\infty}^{-\epsilon} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}} (-x)^{n-1}dx \\
&\quad \times \int\limits_{\mathbb{R}^{n-1}_{d-1}} \frac{du}{\left(\sqrt[4]{M(\gamma)N_u(\gamma)}\right)^{n-1}} \\
&= \quad \frac{L'}{\epsilon^{n-1}} \int\limits_{\epsilon}^{\infty} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}} x^{n-1}dx
\end{aligned}$$

for some L''. Let $y = x^2 + 1$, then

$$\begin{aligned}
\int_{\epsilon}^{\infty} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}} x^{n-1}dx \quad &< \quad \int_{0}^{\infty} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}} x^{n-1}dx \\
&= \quad \frac{1}{2}\int_{1}^{\infty} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{y}\right)}{\left(\sqrt{y}\right)^{\frac{n-1}{2}}} (y-1)^{\frac{n}{2}-1}\, dy \\
&= \quad 2^{n/2-1}\Gamma\left(\frac{n}{2}\right)\mu^{-\frac{n}{2}}K_{-\frac{1}{2}}(\mu)
\end{aligned}$$

The last step follows from (2.21). Thus, there exists a positive number S such that

$$\int\limits_{\mathbb{R}^{n-1}_{d-1}} \int\limits_{-\infty}^{-\epsilon} F_{\gamma_m}(u, x)dxdu < \frac{S}{\epsilon^{n-1}\cdot\mu^{\frac{n}{2}}} K_{-\frac{1}{2}}(\mu). \tag{3.25}$$

However, it is $\int\limits_{\mathbb{R}^{n-1}_{d-1}} \int\limits_{-\infty}^{0} F_{\gamma_m}(u, x)dxdu$ which is to be estimated. For μ very large, by (2.22), $K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)$ decreases exponentially with maximum value $\sqrt{\frac{\pi}{2\mu}}\, e^{-\mu}$. Hence one has to choose proper ϵ for each μ such that the two integrals $\int\limits_{-\infty}^{-\epsilon} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}} x^{n-1}dx$ and $\int\limits_{-\infty}^{0} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}} x^{n-1}dx$ are bounded by each other, i.e.,

$$\int_{-\infty}^{-\epsilon} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}} x^{n-1}dx \asymp \int_{-\infty}^{0} \frac{K_{\frac{n-1}{2}}\left(\mu\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^{\frac{n-1}{2}}} x^{n-1}dx. \tag{3.26}$$

In the following, we will see that $\epsilon = \mu^{-\frac{1}{2}}$ is sufficient for establishing (3.26). Note that $\epsilon^{n-1}\mu^{\frac{n}{2}} = \mu^{1/2}$, so (3.25) reads

$$\int_{\mathbb{R}^{n-1}_{d-1}} \int_{-\infty}^{-\epsilon} F_{\gamma_m}(u,\, x)dxdu < \frac{S}{\mu^{1/2}} \cdot K_{-\frac{1}{2}}(\mu).$$

By computation,

$$\frac{e^{-\mu\sqrt{\epsilon^2+1}}}{e^{-\mu}} = e^{-\mu\frac{\epsilon^2}{1+\sqrt{\epsilon^2+1}}} = e^{-\frac{1}{1+\sqrt{\epsilon^2+1}}} =: \tau_\mu.$$

As $\mu \to \infty$, it is easy to see that $\tau_\mu \to e^{-1/2}$. Using the asymptotic (2.22), we see that $K_{\frac{n-1}{2}}(\mu) \asymp K_{\frac{n-1}{2}}\left(\mu\sqrt{\epsilon^2+1}\right)$. The property (3.26) then follows. Thus,

$$\int_{\mathbb{R}^{n-1}_{d-1}} \int_{-\infty}^{0} F_{\gamma_m}(u,\, x)dxdu = \mathcal{O}\left(\mu^{-1/2}K_{-\frac{1}{2}}(\mu)\right). \tag{3.27}$$

By (3.22) and (3.27), we have:

$$\begin{aligned}
\sum_{m=1}^{N_{n/2}} I_m &= \sum_{m=1}^{N_{n/2}} \int_{\mathbb{R}^{n-1}_{d-1}} \int_{-\infty}^{\infty} F_{\gamma_m}(u,\, x)dxdu \\
&= 2^n \left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1} \sum_{m=1}^{N_{n/2}} \left(\int_{\mathbb{R}^{n-1}_{d-1}} \int_{-\infty}^{0} F_{\gamma_m}(u,\, x)dxdu + \int_{\mathbb{R}^{n-1}_{d-1}} \int_{0}^{\infty} F_{\gamma_m}(u,\, x)dxdu \right) \\
&= \mathcal{O}\left(e^{-\mu}\mu^{-\frac{n+1}{2}}\right)
\end{aligned}$$

Putting the data on geometric side and spectral side together,

$$\begin{aligned}
\sum_{i=0}^{\infty} 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{\nu_i}(\mu) \left| \int_Y \phi_i(z)dz \right|^2 &= 2^n \left(\sqrt{\frac{\pi}{2\mu}}\right)^{n-1} K_{\frac{n-1}{2}}(\mu) \cdot \mathrm{vol}(Y) \\
&\quad + \mathcal{O}\left(e^{-\mu}\mu^{-\frac{n+1}{2}}\right) + \mathcal{O}\left(e^{-\mu}\mu^{-\frac{n}{2}-1}\right)
\end{aligned}$$

Multiplying $2^{-n}\left(\sqrt{\frac{2\mu}{\pi}}\right)^n e^\mu$ on both sides of this formula and taking the limitation $\mu \to \infty$, one gets:

Theorem 3.3.6. *Let X be a compact d-dimensional hyperbolic manifold, $Y \cong \Gamma\backslash G^*/K^*$ be a totally geodesic submanifold of X where Γ_0, G^* and K^* are defined at the beginning of this chapter, then the following holds*

$$\lim_{\mu\to\infty} \sum_{i=0}^{\infty} 2^{d-n} e^{\mu} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1-n} K_{\nu_i}(\mu) \left|P_Y(\phi_i)\right|^2 = \mathrm{vol}(Y). \tag{3.28}$$

Corollary 3.3.7. *There are infinitely many ϕ_i's with nonvanishing periods over Y: $P_Y(\phi_i) \neq 0$.*

Remark 3.3.8. *Comparing the formula in Theorem 3.3.6 with that in Theorem 2.5.1, one finds that there is a factor (involving Killing form) lost on the left hand side. The reason for this phenomenon is that in this chapter we fix the hyperbolic measure (of Y) without presuming the metric in advance, while in Ch. 2 we start from the hyperbolic metric. This choice will of course leads to new hyperbolic metric which differs from the old one only by a nonzero scalar. If we, by contrast, firstly fix the metric, then there will be a factor occurring on the left hand side of the formula in Theorem 3.3.6. Such difference is not essential as it is only a matter of modification.*

In representation-theoretic language, Corollary 3.3.7 reads:

Theorem 3.3.9. *Let $H = G^*$ or the conjuation of G^*. Assume that $\Gamma_H \backslash H$ is compact, $\Gamma \backslash G$ is also compact, then there are infinitely many real spherical automorphic representations which are H-distinguished.*

Chapter 4

Asymptotics of Periods

In this chapter we shall refine formulas (2.24), (2.26) and (3.28), based on which we derive the asymptotics of periods in use of the Tauberian theorem. This depends on a careful study of the spectral side. The nontrivial bound of periods is particularly important to our work. The main results in this chapter are Theorem 4.3.1, 4.3.2 and 4.3.3.

4.1 The refined formulas

It is clear from the proof of Corollary 2.5.2 that any finitely many terms on the left hand sides of (2.24), (2.26) and (3.28) are killed in the process of taking the limitation $\mu \to \infty$. So we may always focus on the eigenfunctions with large eigenvalues, i.e., we can and will assume that $\nu_j \in i\,\mathbb{R}$. Write $\nu_j = ir_j$ where $r_j \in \mathbb{R}_{\geqslant 0}$.

Theorem 4.1.1. *The formula (2.24) can be refined as*

$$\lim_{\mu\to\infty} 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} \sum_{j=0}^{\infty} e^{-\frac{r_j^2}{2\mu}} \, |P_C(\phi_j)|^2 = 2\|E\| \operatorname{len}(C). \tag{4.1}$$

Theorem 4.1.2. *The formula (2.26) can be refined as*

$$\lim_{\mu\to\infty} 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} \sum_{j=0}^{\infty} e^{-\frac{r_j^2}{2\mu}} \, |P_C(\phi_j,\, \chi)|^2 = 2\|E\| \operatorname{len}(C). \tag{4.2}$$

Theorem 4.1.3. *The formula (3.28) can be refined as*

$$\lim_{\mu\to\infty} 2^{d-n} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-n} \sum_{j=0}^{\infty} e^{-\frac{r_j^2}{2\mu}} \, |P_Y(\phi_j)|^2 = \operatorname{vol}(Y). \tag{4.3}$$

In this subsection we show a nontrivial bound on periods to be used in the proof of these theorems. Let Y be a closed geodesic or compact totally geodesic submanifold on the compact hyperbolic manifold X. Let $n = \dim Y$ and $\{\psi_\ell\}$ be an orthonormal basis of $L^2(Y, dz)$ such that ψ_ℓ's are eigenfunctions of the Laplace operator Δ_Y over Y (defined by the hyperbolic metric of Y): $\Delta_Y \psi_\ell = \lambda'_\ell \cdot \psi_\ell$. Like λ_j, we write $\lambda'_\ell = \rho'^2 - \nu'^2_\ell$ where $\rho' = \frac{n-1}{2}$ denotes the half sum of positive roots of G^*. The restriction of ϕ_j on Y can be expanded as the linear combination of ψ_i's:

$$\phi_j|_Y = \sum_\ell a_{j,\ell}\, \psi_\ell, \quad a_{j,\ell} \in \mathbb{C}. \tag{4.4}$$

By the assumption on ψ_ℓ, we have

$$a_{j,\ell} = \int_Y \phi_j(z) \overline{\psi_\ell}(z) dz. \tag{4.5}$$

So the periods in our context is nothing but the Fourier coefficients of $\phi_j|_Y$ in its expansion (4.4). In particular, the period $P_Y(\phi_j)$ is, up to some scalar, just the *zero*-th coefficient or the constant term.

Proposition 4.1.4.

- *Let $n = 1$, that is, Y is a closed geodesic. For any fixed ψ_ℓ and $\epsilon > 0$,*

$$\int_Y \phi_j(z) \overline{\psi_\ell}(z) dz \ll r_j^{-\frac{1}{2}+\epsilon}, \quad \text{as } r_j \to \infty \tag{4.6}$$

where the implied $\mathcal{O}$-constant depends on ψ_ℓ.

- *Let $2 \leqslant n \leqslant d-1$, then for any fixed $\epsilon > 0$,*

$$\int_Y \phi_j(z) dz \ll r_j^{-\frac{n}{2}+\epsilon}, \quad \text{as } r_j \to \infty \tag{4.7}$$

where the implied $\mathcal{O}$-constant depends on n.

Corollary 4.1.5. *For any closed totally geodesic submanifold Y of a compact hyperbolic manifold X, one has:*

$$\int_Y \phi_j(z) dz \to 0, \quad \text{as } \lambda_j \to \infty.$$

The rest of this subsection will be devoted to the proof of this proposition. We adopt the trick in [**Re**] and some results in [**MØ**]. Let $J'(\nu'_\ell)$ denote the noncompact picture of the representation $(G^*, I'(\nu'_\ell))$ where $I'(\nu'_\ell) = \mathrm{Ind}_{M^*A^*N^*}^{G^*} \left(\mathbf{1} \otimes e^{\nu'_\ell} \otimes \mathbf{1}\right)^\infty$, the subset of smooth elements in $\mathrm{Ind}_{M^*A^*N^*}^{G^*} \left(\mathbf{1} \otimes e^{\nu'_\ell} \otimes \mathbf{1}\right)$. That is, $J'(\nu'_\ell)$ is the image of $I'(\nu'_\ell)$ under the map $\mathcal{R}$ (see Sect. 1.4). Here $M^* = M \cap G^*$, $A^* = A \cap G^*$ and $N^* = N \cap G^*$. For $u \in \mathbb{R}^{d-1}$, let $u = (u', u'') \in \mathbb{R}^{n-1} \times \mathbb{R}^{d-n}$. Under our assumption that $\nu_j \in i\,\mathbb{R}_{\geqslant 0}$, the integral operator

$$L_{\nu_j, \nu'_\ell} : J(\nu_j) \times J'(\nu'_\ell) \to \mathbb{C},$$

$$(f_1, f_2) \mapsto \int_{u_1 \in \mathbb{R}^{d-1}} \int_{u_2 \in \mathbb{R}^{n-1}} \left(|u'_1 - u_2|^2 + |u''_1|^2\right)^{\nu'_\ell - \rho'} |u''_1|^{\nu_j - \rho - (\nu'_\ell - \rho')} f_1(u_1) f_2(u_2) du_2 du_1 \tag{4.8}$$

defines a $\Delta(G^*)$-invariant (i.e., invariant under the diagonal G^*-action) bilinear form on $J(\nu_j) \times J'(\nu'_\ell)$. The space of $\Delta(G^*)$-invariant bilinear forms on $J(\nu) \times J'(\nu')$ is at most one-dimensional ("Multiplicity One Theorem") provided that

$$\nu + \rho \pm \nu' - \rho' \notin -2\mathbb{N}_0.$$

See Theorem 4.1 of [**MØO**] for this fact. For fixed ψ_ℓ, this condition is satisfied for $\nu = ir$ such that r is large enough. In the representation space $L^2(\Gamma_0 \backslash G^*)$ of G^*, there is an isotypic subrepresentation (arising from ψ_ℓ) whose irreducibles are isomorphic to $I'(\nu'_\ell)$. Denote by $V'_{\nu'_\ell}$ the smooth part of one of these isomorphic irreducibles. The bilinear form

$$L^\bullet_{\nu_j, \nu'_\ell} : V_{\nu_j} \times \overline{V}_{\nu'_\ell} \to \mathbb{C}, \quad (g_1, g_2) \mapsto \int_{\Gamma_0 \backslash G^*} g_1(z) \overline{g}_2(z) dz$$

also defines a $\Delta(G^*)$-invariant bilinear forms on $V_{\nu_j} \times \overline{V}_{\nu'_\ell}$. In view of the isomorphisims $I(\nu) \cong V_\nu \cong J(\nu)$ and $I'(\nu') \cong V'_{\nu'} \cong J'(\nu')$, the investigation of the coefficients $a_{j,\ell}$ in (4.5) is identical with that of $L^\bullet_{\nu_j, \nu'_\ell}$. Since $\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta(G^*)}(J(\nu) \times J'(\nu'), \mathbb{C}) \leqslant 1$ for our choice of ν and ν', there exist scalars $b_{\nu, \nu'} \in \mathbb{C}$ such that

$$L^\bullet_{\nu_j, \nu'_\ell}(g_1, g_2) = b_{\nu, \nu'} \cdot L_{\nu_j, \nu'_\ell}(f_1, f_2) \tag{4.9}$$

where f_1, f_2 correspond to g_1, g_2, and g_1, g_2 correspond to ϕ_j, ψ_ℓ respectively. For r_j very large (this is the case where ψ_ℓ is fixed and λ_j is large), $b_{\nu, \nu'}$ depends on ψ_ℓ, but not on ϕ_j. In [**MØ**], $L_{\nu_j, \nu'_\ell}(f_1, f_2)$ is explicitly computed (see Proposition 3.1 there):

$$L_{\nu_j, \nu'_\ell}(f_1, f_2) = \frac{\pi^{\rho+\rho'} \Gamma(\rho') \Gamma\left(\frac{(\nu_j+\rho)+(\nu'_\ell-\rho')}{2}\right) \Gamma\left(\frac{(\nu_j+\rho)-(\nu'_\ell+\rho')}{2}\right)}{\Gamma(2\rho')\Gamma(\rho-\rho')\Gamma(\nu_j+\rho)}.$$

By Stirling's asymptotic

$$|\Gamma(x+iy)| = \sqrt{2\pi}\, |y|^{x-1/2}\, e^{-\frac{\pi}{2}|y|} \left(1 + \mathcal{O}\left(|y|^{-1}\right)\right), \quad \text{as } |y| \to \infty,$$

one can easily show that, for any fixed $\nu'_\ell \in (-\rho', \rho') \cup i\,\mathbb{R}_{\geqslant 0}$,

$$L_{\nu_j, \nu'_\ell}(f_1, f_2) \ll r_j^{-\frac{n}{2}}, \quad \text{as } r_j \to \infty. \tag{4.10}$$

To show Proposition 4.1.4, it suffices to show that, for any fixed ϕ with large Laplace eigenvalue $\lambda = \rho^2 - \nu^2$ where $\nu = ir$, the following holds:

$$\sum_{r'_\ell \leqslant r^\epsilon} |b_{\nu, \nu'_\ell}|^2 \leqslant c_1\, r^\epsilon, \quad \epsilon > 0, \quad \text{as } r \to \infty, \tag{4.11}$$

for some number $c_1 > 0$. Assuming (4.11), the proposition follows from (4.9) and (4.10):

$$\left| \int_Y \phi_j(z) \overline{\psi_\ell}(z) dz \right|^2 = \left| L^\bullet_{\nu_j, \nu'_\ell}(g_1, g_2) \right|^2 \leqslant \left| L_{\nu_j, \nu'_\ell}(f_1, f_2) \right|^2 \cdot \sum_{r'_\ell \leqslant r_j^\epsilon} |b_{\nu_j, \nu'_\ell}|^2 \leqslant c_2\, r_j^{-n+2\epsilon}$$

for some number $c_2 > 0$. Note that $|b_{\nu_j,\nu'_\ell}|^2$ is contained in $\sum_{|\nu'_\ell|\leqslant r_j^\epsilon} |b_{\nu_j,\nu'_\ell}|^2$ for fixed ψ_ℓ and any large r_j.

The remaining task is to show (4.11) for which we follow the idea of [**Re**]. The key is to find a smooth function $w_r \in J(\nu)$ (where $\nu = ir$) and a small subset $U \subset G$ which contains the identity, such that, for any fixed $\sigma > 0$,

A. $\left|L_{\nu,\nu'_\ell}(g.w_r, \eta_\ell)\right|^2 \geqslant \alpha r^{-\sigma}$ for some $\alpha > 0$, any $g \in U$, $|\nu'_\ell| \leqslant r^\epsilon$ and fixed r which is large. Here $g.w_r$ stands for the action of g on w_r, and $\eta_\ell \in J'(\nu'_\ell)$ is the vector corresponding to ψ_ℓ. See Sect. 2.3 of [**MØ**] for details on the action $g.w_r$.

B. $\int_{x\in L\cdot U} |\Phi_{w_r}(x)|^2 \xi(x)dx \leqslant \beta$ for some $\beta > 0$. Here, Φ_{w_r} is the element in V_ν corresponding to w_r under the isomorphisim between V_ν and $J(\nu)$, L is the subset of G^* which is isomorphic to $\Gamma_0\backslash G^*$, $\delta_L = \frac{\mathbf{1}_L}{\mathrm{vol}(L)}$ where $\mathbf{1}_L$ is the characteristic function for L, $\xi = \delta_L * \xi_U$ is just the convolution of the two functions δ_L and ξ_U where ξ_U is a smooth nonnegative function with its support in U such that $\int_U \xi(x)dx = 1$.

Since $n < d$, we can choose L and U such that $L \cap U \subset G^* \cap U = \{1\}$. Thus $dx = dtdg$ for $x = tg \in L \cdot U$. Moreover,

$$\xi(tg) = \int_L \delta_L(s)\xi_U\left(s^{-1}tg\right)ds = \int_L \delta_L(t)\xi_U(g)dt = \xi_U(g).$$

The reason for the second step is that, ξ_U vanishes unless $s^{-1}t = 1$ since $L \cap U = \{1\}$. Let $\widetilde{\psi}_\ell$ denote the lift of ψ_ℓ on $\Gamma_0\backslash G^*$ for the natural map $\Gamma_0\backslash G^* \to \Gamma_0\backslash G^*/K^* \cong Y$. These $\widetilde{\psi}_\ell$'s are still orthonormal over $\Gamma_0\backslash G^*$. Extend them to be an (complete) orthonormal basis of $L^2(\Gamma_0\backslash G^*)$. So $g.\Phi_{w_r}$ can be written as the linear combination

$$g.\Phi_{w_r} = \sum_\ell \left\langle g.\Phi_{w_r}\big|_{\Gamma_0\backslash G^*}, \widetilde{\psi}_\ell\right\rangle_{\Gamma_0\backslash G^*} \cdot \widetilde{\psi}_\ell + \text{other terms}$$

Assuming the above two conditions (A), (B) and the property of U, (4.11) is shown as follows:

$$\begin{aligned}
\beta &\geqslant \int_{L\cdot U} |\Phi_{w_r}(x)|^2 \xi(x)dx \\
&= \int_{g\in U}\int_{t\in L} |\Phi_{w_r}(tg)|^2 \xi(tg)dtdg \\
&= \int_{g\in U}\left(\int_{t\in L} |\Phi_{w_r}(tg)|^2 dt\right)\xi_U(g)dg \\
&= \int_{g\in U}\left(\int_{t\in L} |(g.\Phi_{w_r})(t)|^2 dt\right)\xi_U(g)dg \\
&\geqslant \int_{g\in U}\left(\int_{t\in L} \sum_\ell \left|\left\langle g.\Phi_{w_r}\big|_{\Gamma_0\backslash G^*}, \widetilde{\psi}_\ell\right\rangle_{\Gamma_0\backslash G^*} \cdot \widetilde{\psi}_\ell(t)\right|^2 dt\right)\xi_U(g)dg
\end{aligned}$$

$$\begin{aligned}
&\geqslant \int_{g\in U}\left(\sum_{|\nu'_\ell|\leqslant r^\epsilon}\left|\left\langle g.\Phi_{w_r}\big|_{\Gamma_0\backslash G^*},\ \widetilde{\psi}_\ell\right\rangle_{\Gamma_0\backslash G^*}\right|^2\right)\xi_U(g)dg\\
&= \int_{g\in U}\xi_U(g)dg\cdot\sum_{|\nu'_\ell|\leqslant r^\epsilon}\left|\left\langle g_0.\Phi_{w_r}\big|_{\Gamma_0\backslash G^*},\ \widetilde{\psi}_\ell\right\rangle_{\Gamma_0\backslash G^*}\right|^2 \qquad \text{for some } g_0\in U\\
&= \sum_{|\nu'_\ell|\leqslant r^\epsilon}\left|b_{\nu,\nu'_\ell}\right|\cdot\left|L_{\nu,\nu'_\ell}(g_0.w_r,\ \eta_\ell)\right|^2\\
&\geqslant \alpha\, r^{-\sigma}\sum_{|\nu'_\ell|\leqslant r^\epsilon}\left|b_{\nu,\nu'_\ell}\right|^2
\end{aligned}$$

Hence we get: $\sum_{|\nu'_\ell|\leqslant r^\epsilon}\left|b_{\nu,\nu'_\ell}\right|^2 < c_1\, r^\sigma$ (as $r\to\infty$) for some number $c_1>0$. Note that, in above we have used the Multiplicity One Theorem for which we have to make sure that $|\nu'_\ell|\ll r$. This is guaranteed by choosing those ℓ such that $|\nu'_\ell|<r^\epsilon$ where r is large enough.

Now we are in the position to find w_r and U such that the aforementioned two conditions (A) and (B) are satisfied. Let w be a smooth function on $\mathbb{R}^{d-1}$ which is nonnegative, compactly supported in the unit ball of 0 and $w|_{B(0,x)}=1$ for $x<1$ close enough to 1. Assume that $\int_{\mathbb{R}^{d-1}} w(x)dx = C>0$. For any $\eta>0$, define

$$w_r := r^{\frac{d-1}{2}\eta}w\left(r^{(d-1)\eta}\left(x-\vec{1}\right)\right)$$

where $\vec{1}$ is a special element in $\mathbb{R}^{d-1}$ to be decided later. View w_r as a function in $J(\nu)$, then $\|w_r\|^2_{L^2(J(\nu))}=\frac{\Gamma(2\rho)}{\pi^\rho\Gamma(\rho)}$ (see Sect. 1.4). The kernel of the operator $L_{\nu,\nu'}$ (see (4.8)) has particularly good properties when $n=1$ or ψ_ℓ is a constant (i.e., $\lambda'_\ell=0$, or equivalently $\nu'_\ell=\rho'$). These two cases just correspond to the two cases in Proposition 4.1.4.

- When $n=1$, the kernel of L_{ν,ν'_ℓ} is

$$|u_1|^{(\nu-\rho)+(\nu'_\ell-\rho')}.$$

Note that the w_r is supported in a small neighborhood N_r of $\vec{1}$ where N_r has size

$$\left[-r^{-(d-1)\eta},\, r^{-(d-1)\eta}\right]^{d-1}\subset\mathbb{R}^{d-1}.$$

Let $\vec{1}=(1,0,\cdots,0)\in\mathbb{R}^{d-1}$. As $r\to\infty$, it is those $u_1\in N_r$ that contribute to the above integral. Denote $x_r=r^{-(d-1)\eta}$. In N_r, one has

$$|u_1|^{(\nu-\rho)+(\nu'_\ell-\rho')} = e^{(\nu-\rho)+(\nu'_\ell-\rho')\log|u_1|}\asymp e^{(\nu-\rho)\log\sqrt{(1+x_r)^2+(d-2)x_r^2}}.$$

Since $x_r\to 0$ as $r\to\infty$, we have: $|u_1|^{(\nu-\rho)+(\nu'_\ell-\rho')}\asymp e^{ir\log\sqrt{(1+x_r)^2+(d-2)x_r^2}}$. Substituting $y=2x_r+(d-1)x_r^2$ into the Taylor expansion of $\frac{1}{2}\log(1+y)$, one can show that

$$r\log\sqrt{(1+x_r)^2+(d-2)x_r^2}\to 0,\quad \text{as } r\to\infty.$$

This implies that $e^{(\nu-\rho)\log\sqrt{(1+x_r)^2+(d-2)x_r^2}}$ tends to be 1, thus $|u_1|^{(\nu-\rho)+(\nu'_\ell-\rho')}$ tends to be a nonzero constant as $r\to\infty$. Now it is clear that, in the present case, $L_{\nu,\nu'}(w_r)$, η_ℓ is (up to a positive scalar) essentially given by $\int_{\mathbb{R}^{d-1}} w_r(u_1)du_1$. It follows that $L_{\nu,\nu'}(w_r,\,\eta_\ell)$ is of the order $r^{-\frac{d-1}{2}\eta}$.

- When $\lambda'_\ell = 0$, the K-invariant function in $J'(\nu')$ is just $(1+|u_2|^2)^{-2\rho'}$ where $u_2\in\mathbb{R}^{n-1}$ (see Sect. 2.3 of [**MØ**]), the kernel of L_{ν,ν'_ℓ} is

$$|u''_1|^{\nu-\rho}(1+|u_2|^2)^{-2\rho'}.$$

So the variables in the integral of $L_{\nu,\nu'}$ are separated. It is clear that the integral $\int_{\mathbb{R}^{n-1}}(1+|u_2|^2)^{-2\rho'}du_2$ converges for $\rho'\geqslant\frac{1}{2}$, i.e., $n\geqslant 2$. The case $n=1$ has been discussed in above. As for the term $|u''_1|^{\nu-\rho}$, we apply the argument for the case $n=1$ by letting $\vec{1}=(x_i)_{i=1}^{d-1}$ where

$$x_i=\begin{cases}1, & \text{if } i=n,\\ 0, & \text{otherwise,}\end{cases}$$

and show that

$$|u''_1|^{\nu-\rho}\to 1, \quad \text{as } r\to\infty.$$

Finally, we get: $L_{\nu,\nu'}(w_r,\,\eta_\ell)$ is essentially given by $\int_{\mathbb{R}^{d-n}} w_r(u_1)du_1$. So $L_{\nu,\nu'}(w_r,\,\eta_\ell)$ is of the order $r^{-\frac{d-1}{2}\eta}$.

For U small enough, gN_r is close to N_r for any $g\in U$. The order of $L_{\nu,\nu'}(w_r,\,\eta_\ell)$ in above holds for any $g\in U$. Up to now, we have shown that the condition (A) is satisfied for the chosen function w_r and subset U. Actually it is easy to see that condition (B) holds:

$$\int_{x\in L\cdot U}|\Phi_{w_r}(x)|^2\,\xi(x)dx\leqslant\sup_{x\in U}\xi(x)\cdot\|\Phi_{w_r}\|^2=\sup_{x\in U}\xi(x)\cdot\|w_r\|^2=:\beta.$$

As $\eta>0$ is arbitary, the proof of Proposition 4.1.4 is complete.

Remark 4.1.6. *In the paper* [**MØ**]*, the authors did not assume that the ambient manifold is compact. The method in* [**Re**] *does not rely on the assumption that the surface (the ambient manifold treated there) is compact since it only uses the model of the spherical representation, but not the geometry of the manifold. So our bound in Proposition 4.1.4 also holds for noncompact X when one replaces those ϕ_j's with cusp forms of X. By cusp forms we mean the L^2 Laplace eigenfunctions on X, i.e., the Laplace eigenfunctions with vanishing constant in its Fourier expansion around each cusp of X.*

Remark 4.1.7. *In* [**Re**]*, the author conjectured the following bound on geodesic periods over surfaces: $|P_C(\phi)|\ll\lambda^{-\frac{1}{4}+\epsilon}$ where $\Delta\phi=\lambda\,\phi$. Our proposition gives an affirmative answer to this problem, noting that $r\sim\lambda^{1/2}$ uniformly for λ large.*

4.2 The refinement of the spectral side

In this subsection we shall use Proposition 4.1.4 to prove Theorem 4.1.1, 4.1.2 and 4.1.3, the refined versions of formulas (2.24), (2.26) and (3.28) respectively. For the former two theorems, what we really use is weaker than the original conclusion, namely we only need the fact that periods are bounded, while for Theorem 4.1.3, we have to make full use of Proposition 4.1.4. The argument splits into several steps according to the intervals in which r_j lies. Firstly we show Theorem 4.1.1.

Lemma 4.2.1. • *For all* $0 < r \leqslant x$, *we have*

$$0 < K_{ir}(x) \leqslant e^{-(\pi/2)r-\sqrt{x^2-r^2}+r\arccos(r/x)} \min\left(\frac{\sqrt{\pi/2}}{\sqrt[4]{x^2-r^2}}, \frac{\Gamma\left(\frac{1}{3}\right)}{2^{\frac{2}{3}}3^{\frac{1}{6}}}r^{-\frac{1}{3}}\right).$$

• *For all* $r > x \geqslant 1$, *we have*

$$|K_{ir}(x)| < e^{-\frac{\pi r}{2}} \begin{cases} \frac{5}{\sqrt[4]{r^2-x^2}}, & x \leqslant r - \frac{1}{2}r^{1/3}, \\ 4r^{-1/3}, & x \geqslant r - \frac{1}{2}r^{1/3}. \end{cases}$$

$$\left|\frac{\partial}{\partial r}K_{ir}(x)\right| < e^{-\frac{\pi}{2}r} \begin{cases} \frac{17+5\log(r/x)}{\sqrt[4]{r^2-x^2}}, & x \leqslant r - \frac{1}{2}r^{1/3}, \\ 12r^{-1/3}, & x \geqslant r - \frac{1}{2}r^{1/3}. \end{cases}$$

Proof. See Proposition 2 of [**Bo**]. □

By Lemma 4.2.1, for fixed, large enough $x \geqslant 1$ and arbitary $r > x$, an elementary computation shows that:

$$|K_{ir}(x)| \ll e^{-\frac{\pi r}{2}}, \quad \left|\frac{\partial}{\partial r}K_{ir}(x)\right| \ll e^{-\frac{\pi}{2}r} \tag{4.12}$$

where the two implied $\mathcal{O}$-constants depend on x. More precisely, they tend to 0 as $x \to \infty$. In the following we shall use the uniform asymptotic: $r \sim \sqrt{\lambda}$ for λ large.

Lemma 4.2.2.

$$\lim_{\mu\to\infty} e^{\mu}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2}\sum_{r_j>\mu} K_{ir_j}(\mu)\,|P_C(\phi_j)|^2 = \lim_{\mu\to\infty}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1}\sum_{r_j>\mu} e^{-\frac{r_j^2}{2\mu}}\,|P_C(\phi_j)|^2 = 0.$$

Proof. By (4.12) and Proposition 4.1.4,

$$\text{L.H.S.} \ll \int_{\mu}^{\infty} e^{\mu-\frac{\pi}{2}r}\mu^{-\frac{d-2}{2}}dN(r) < \mu^{-\frac{d-2}{2}}\int_{\mu}^{\infty} e^{\left(1-\frac{\pi}{2}\right)r}dN(r)$$

where μ is large. Weyl's law shows that $N(r)$ grows polynomially (see Sect. 2.4). So the integral $\int_{\mu}^{\infty} e^{\left(1-\frac{\pi}{2}\right)r}dN(r)$ converges and tends to 0 as $\mu \to \infty$. As for R.H.S., noting that $e^{-\frac{r_j^2}{2\mu}} < e^{-\frac{r_j}{2}}$ for $r_j > \mu$, we have:

$$\text{R.H.S.} \ll \mu^{-\frac{d-1}{2}}\int_{\mu}^{\infty} e^{-\frac{r}{2}}\,dN(r).$$

Applying the argument for L.H.S., we see that R.H.S. tends to 0 as $\mu \to \infty$. □

Lemma 4.2.3.

$$\lim_{\mu\to\infty} e^{\mu}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2}\sum_{\mu^{\frac{1}{2}+\epsilon}\leqslant r_j\leqslant\mu} K_{ir_j}(\mu)\left|P_C(\phi_j)\right|^2 = \lim_{\mu\to\infty}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1}\sum_{\mu^{\frac{1}{2}+\epsilon}\leqslant r_j\leqslant\mu} e^{-\frac{r_j^2}{2\mu}}\left|P_C(\phi_j)\right|^2 = 0$$

for any fixed $0 < \epsilon < \frac{1}{2}$.

Proof. Let $f(r) = \mu - (\pi/2)r - \sqrt{\mu^2 - r^2} + r\arccos(r/\mu)$. Lemma 4.2.1 shows that $e^{\mu}K_{ir}(\mu) \ll e^{f(r)}$ for $r \leqslant \mu$. The partial derivative of $f(r)$:

$$\frac{\partial f}{\partial r} = -\frac{\pi}{2} + \arccos\left(\frac{r}{\mu}\right) < 0, \quad 0 < r \leqslant \mu$$

indicates that

$$\max_{\mu^{1/2+\epsilon}\leqslant r_j\leqslant\mu} e^{f(r)} = e^{f\left(\mu^{1/2+\epsilon}\right)}.$$

Using Taylor expansion of $\arccos x$, one has

$$\begin{aligned} f\left(\mu^{1/2+\epsilon}\right) &= \mu - \frac{\pi}{2}\mu^{\frac{1}{2}+\epsilon} - \sqrt{\mu^2 - \mu^{1+2\epsilon}} + \mu^{\frac{1}{2}+\epsilon}\left(\frac{\pi}{2} - \mu^{-\frac{1}{2}+\epsilon} + \text{ lower order terms }\right) \\ &\asymp \mu - \sqrt{\mu^2 - \mu^{1+2\epsilon}} - \mu^{2\epsilon} \\ &= \mu\frac{\mu^{2\epsilon-1}}{1+\sqrt{1-\mu^{2\epsilon-1}}} - \mu^{2\epsilon} \\ &\sim -\frac{\mu^{2\epsilon}}{2}, \quad \text{as } \mu \to \infty \quad (\text{note that } 2\epsilon - 1 < 0) \end{aligned}$$

Hence

$$\text{L.H.S.} \ll \mu^{-\frac{d-2}{2}} e^{-\frac{\mu^{2\epsilon}}{2}} \mu^d \to 0, \quad \text{as } \mu \to \infty.$$

As for R.H.S., we have: $e^{-\frac{r_j^2}{2\mu}} \leqslant e^{\frac{-\mu^{2\epsilon}}{2}}$. Thus,

$$\text{R.H.S.} \ll \mu^{-\frac{d-1}{2}} e^{\frac{-\mu^{2\epsilon}}{2}} \int_{\mu^{1/2+\epsilon}}^{\mu} dN(r) < \mu^{-\frac{d-1}{2}} e^{\frac{-\mu^{2\epsilon}}{2}} \mu^{-d} \to 0, \quad \text{as } \mu \to \infty.$$

□

Lemma 4.2.4. *For* $4 < r < \mu$, *we have*

$$K_{ir}(\mu) = \sqrt{\frac{\pi}{2\mu}}\, e^{-\left(\mu+\frac{r^2}{2\mu}\right)}\left[1 + \frac{r^2-\mu}{8\mu^2}\right] + \frac{e^{-\mu}}{\sqrt{\mu}}\mathcal{O}\left(\frac{r}{\mu\sqrt{\mu}} + \frac{\sqrt{r}}{\sqrt{\mu}}e^{-\frac{\mu}{r}}\right).$$

Proof. See Proposition 4 of [**MW**]. □

Lemma 4.2.5.

$$\lim_{\mu\to\infty} e^{\mu}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2}\sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} K_{ir_j}(\mu)\left|P_C(\phi_j)\right|^2 = \lim_{\mu\to\infty}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1}\sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} e^{-\frac{r_j^2}{2\mu}}\left|P_C(\phi_j)\right|^2.$$

Proof. Let ϵ be small enough such that $\epsilon(d+1) < \frac{1}{2}$. Noting that $|P_C(\phi_j)|$'s are uniformly bounded, by Lemma 4.2.4 we have:

$$\begin{aligned} & e^{\mu}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} K_{ir_j}(\mu)\,|P_C(\phi_j)|^2 - \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} e^{-\frac{r_j^2}{2\mu}}\,|P_C(\phi_j)|^2 \\ = & \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} |P_C(\phi_j)|^2\, e^{-\frac{r_j^2}{2\mu}}\, \frac{r_j^2-\mu}{8\mu^2} + \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} |P_C(\phi_j)|^2\, \mathcal{O}\left(\frac{r_j}{\mu^2}\right) \\ \ll & \ \mu^{-\frac{d-1}{2}}\mu^{-1}\left(\mu^{\frac{1}{2}+\epsilon}\right)^d + \mu^{-\frac{d-2}{2}}\mu^{-2+\frac{1}{2}+\epsilon}\left(\mu^{\frac{1}{2}+\epsilon}\right)^d \\ = & \ \mu^{-\frac{1}{2}+\epsilon d} + \mu^{-\frac{1}{2}+\epsilon(d+1)} \to 0, \quad \text{as } \mu\to\infty \end{aligned}$$

□

Up to now, we have shown Theorem 4.1.1. Theorem 4.1.2 can be proved in the same way. We omit the details. The above two lemmas 4.2.2 and 4.2.3 clearly hold for higher dimensional Y. To prove Theorem 4.1.3, it remains to show the following lemma which is parallel to Lemma 4.2.5 for $n \geqslant 2$.

Lemma 4.2.6.

$$\lim_{\mu\to\infty} e^{\mu}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1-n} \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} K_{ir_j}(\mu)\,|P_Y(\phi_j)|^2 = \lim_{\mu\to\infty}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-n} \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} e^{-\frac{r_j^2}{2\mu}}\,|P_Y(\phi_j)|^2 .$$

Proof. By Lemma 4.2.4,

$$\begin{aligned} & e^{\mu}\left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1-n} \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} K_{ir_j}(\mu)\,|P_Y(\phi_j)|^2 - \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-n} \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} e^{-\frac{r_j^2}{2\mu}}\,|P_Y(\phi_j)|^2 \\ = & \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-n} |P_Y(\phi_j)|^2\, e^{-\frac{r_j^2}{2\mu}}\, \frac{r_j^2-\mu}{8\mu^2} + \sum_{r_j<\mu^{\frac{1}{2}+\epsilon}} \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1-n} |P_Y(\phi_j)|^2\, \mathcal{O}\left(\frac{r_j}{\mu^2}\right) \end{aligned}$$

For $r_j < \mu^{\frac{1}{2}+\epsilon}$, we have: $\frac{r_j^2-\mu}{8\mu^2} \ll \mu^{-1+2\epsilon}$ and $\frac{r_j}{\mu^2} < \mu^{-\frac{3}{2}+\epsilon}$. For r_j sufficiently large (say, $r_j > A > 0$), Proposition 4.1.4 says that $|P_Y(\phi_j)|^2 \leqslant c\, r_j^{-n+2\epsilon}$ for some positive number c. So the two terms in the above formula are bounded by

$$\mu^{-\frac{d-n}{2}-1+2\epsilon}\int_A^{\mu^{\frac{1}{2}+\epsilon}} r^{-n+2\epsilon}dN(r) \quad \text{and} \quad \mu^{-\frac{d-1-n}{2}-\frac{3}{2}+\epsilon}\int_A^{\mu^{\frac{1}{2}+\epsilon}} r^{-n+2\epsilon}dN(r)$$

respectively. The integration by parts shows that

$$\mu^{-\frac{d-n}{2}-1+2\epsilon}\int_A^{\mu^{\frac{1}{2}+\epsilon}} r^{-n+2\epsilon}dN(r) \ll \mu^{-\frac{d-n}{2}-1+2\epsilon} r^{-n+2\epsilon}N(r)\Big|_{r=\mu^{\frac{1}{2}+\epsilon}} \sim \mu^{-1+\epsilon(3-n+2\epsilon+d)} \tag{4.13}$$

$$\mu^{-\frac{d-1-n}{2}-\frac{3}{2}+\epsilon}\int_A^{\mu^{\frac{1}{2}+\epsilon}} r^{-n+2\epsilon}dN(r) \ll \mu^{-\frac{d-1-n}{2}-\frac{3}{2}+\epsilon}r^{-n+2\epsilon}N(r)\big|_{r=\mu^{\frac{1}{2}+\epsilon}} \sim \mu^{-1+\epsilon(2-n+2\epsilon+d)} \tag{4.14}$$

Here we have used Weyl's law for $N(r)$. For ϵ small enough, $-1+\epsilon(3-n+2\epsilon+d) < 0$. Hence both (4.13) and (4.14) tends to 0 as μ tends to infinity. The lemma is proved and Theorem 4.1.3 follows. □

4.3 The asymptotics of periods

Theorem 4.3.1.

$$\sum_{\lambda_j \leqslant x} |P_C(\phi_j)|^2 \sim \frac{\|E\|\,\mathrm{len}(C)}{(d-1)!!\,\pi^{\frac{d}{2}}2^{\frac{d-2}{2}}} \cdot x^{\frac{d-1}{2}}, \quad \text{as } x \to \infty.$$

Proof. Define $L(x) = C_d \cdot e^{\frac{-\rho^2}{x}}$ where $C_d = \frac{\|E\|\,\mathrm{len}(C)}{2^{d-1}\pi^{\frac{d-1}{2}}}$. For any fixed $x > 0$,

$$\frac{L(tx)}{L(t)} \to 1, \quad \text{as } t \to \infty.$$

Define the probility measure $U\{d\lambda\}$ to be $|P_C(\phi_j)|^2$ at $\lambda = \lambda_j$. Then Theorem 4.1.1 says that

$$\sum_{j=0}^{\infty} e^{-\frac{\lambda_j}{2\mu}}|P_C(\phi_j)|^2 \sim C_d \cdot \mu^{\frac{d-1}{2}}e^{\frac{-\rho^2}{2\mu}}, \quad \text{as } \mu \to \infty.$$

Let $y = \frac{1}{2\mu}$. The above formula reads as

$$\int_0^{\infty} e^{-y\lambda}U\{d\lambda\} \sim L\left(\frac{1}{y}\right) \cdot y^{-\rho}, \quad \text{as } y \to 0.$$

By the Tauberian Theorem (see Theorem 2 on p. 445 of [**Fe**]), one derives:

$$\sum_{\lambda_j \leqslant x} |P_C(\phi_j)|^2 = \int_0^x U\{d\lambda\} \sim \frac{C_d \cdot x^{\frac{d-1}{2}}e^{-\frac{\rho^2}{x}}}{\Gamma(\rho+1)} \sim \frac{C_d}{\Gamma(\rho+1)} \cdot x^{\frac{d-1}{2}}, \quad \text{as } x \to \infty.$$

Substituting special values $\Gamma(\rho+1) = \frac{d-1}{2}\Gamma\left(\frac{d-1}{2}\right) = \frac{d-1}{2}\sqrt{\pi}\,\frac{(d-3)!!}{2^{\frac{d-2}{2}}}$, we get the theorem. □

Theorem 4.3.2.

$$\sum_{\lambda_j \leqslant x} |P_C(\phi_j,\,\chi)|^2 \sim \frac{\|E\|\,\mathrm{len}(C)}{(d-1)!!\,\pi^{\frac{d}{2}}2^{\frac{d-2}{2}}} \cdot x^{\frac{d-1}{2}}, \quad \text{as } x \to \infty.$$

Proof. The same with Theorem 4.3.1. □

Theorem 4.3.3.

$$\sum_{\lambda_j \leqslant x} |P_Y(\phi_j)|^2 \sim \frac{\mathrm{vol}(Y)}{(2\pi)^{\frac{d-n-1}{2}}(d-n)!!} \cdot x^{\frac{d-n}{2}}, \quad \text{as } x \to \infty.$$

Proof. The same with Theorem 4.3.1. One needs to replace C_d with $C_{d,n} = \frac{\mathrm{vol}(Y)}{2^{d-n}\pi^{\frac{d-n}{2}}}$, replace $y^{-\rho}$ with $y^{-\frac{d-n}{2}}$. □

Remark 4.3.4. *At this point we would like to remind the reader of Remark 3.3.8.*

Remark 4.3.5. *See* [**Ze**] *for the general conclusion on asymptotics of periods over any compact Riemann manifold.*

Chapter 5

Periods along Closed Geodesics over Non-compact Hyperbolic Manifolds

In this chapter we extend the argument that was carried out in previous chapters to noncompact hyperbolic manifolds. The key obstruction for getting the formula of the type (2.5.1) arises from the continuous contribution of the spectral resolution. In Sect. 5.2, formulas of the type (2.5.1) are derived, however with error terms. At the end we also discuss the relation between our work and the Selberg-Roelcke conjecture.

5.1 Some preparation

In this section we shall make some preparations for later use. The lattice $\Gamma \subset G = SO_0(1, d)$ is not uniform anymore, but is still torsion-free. Besides we assume that it is of cofinite volume: $\mathrm{vol}(X) < \infty$ where $X = \Gamma\backslash G/K$. A *ray* in G/K is a geodesic $r : [0, \infty) \to G/K$ which realizes the shortest distance between any two points on it. Two rays r_1 and r_2 are equivalent if $d(r_1(t), r_2(t))$ is bounded as $t \to \infty$. Define the *visibility boundary* of G/K as

$$\partial(G/K) := \{\text{rays in } G/K\}/\sim$$

where $\sim$ means the equivalence of rays defined in above. Since G acts on G/K by isometry, G acts on $\partial(G/K)$. For each $x \in \partial(G/K)$, the stabilizer G_x is a proper parabolic subgroup of G. This induces a bijection

$$\partial(G/K) \longleftrightarrow \{\text{proper parabolic subgroups of } G\}.$$

Let $G_x = M_x A_x N_x$ be the Langlands decomposition of G_x. We say that x is a *cusp* of Γ, or Γ-*cusp*, if $\Gamma \cap N_x$ is a lattice in N_x (as N_x is unipotent, this means that $\Gamma \cap N_x$ is cocompact in N_x). Let $(G/K)^* = G/K \cup \{\Gamma\text{- cusps}\} \subset \overline{G/K} := G/K \cup \partial(G/K)$. When

$\mathrm{vol}(X) < \infty$, the boundary of X consists of finitely many points. Denoted these points by p_0, $p_1, ..., p_k$ and call each of them a *cusp* of X. Indeed, one has

$$\overline{X} = \Gamma\backslash G/K \cup \{p_0, \cdots, p_k\} \cong \Gamma\backslash(G/K)^*.$$

The visibility boundary of G/K, in the upper space model $\mathcal{H}^d$ (see Sect. 1.1), is identified with the hyperplane defined by $\xi_0 = 0$ together with the point at infinity. Hence

$$\partial\mathcal{H}^d \cong \mathbb{R}^{d-1} \cup \{\infty\} \cong S^{d-1}.$$

Let $\widetilde{p}_i$ be the set of fibres of p_i for natural map $T : (G/K)^* \to \overline{\Gamma\backslash G/K}$. For any $p \in \widetilde{p}_i$, one can find some element in G which translates p to $\omega_\infty \cdot o$ where $\omega_\infty \cdot o$ is defined to be the end point $\lim\limits_{r\to\infty} \omega_r \cdot o \in \partial(G/K)$ at the infinity. Here we have identified $\omega_\infty \cdot o$ (corresponding to ∞ of the upper half space model) as a rays class. Thus, without loss of generality, we may assume that p_0 is the image of $\omega_\infty \cdot o$ under T. Let Γ_∞ be the stabilizer of $\omega_\infty \cdot o$.

Lemma 5.1.1. $\Gamma_\infty = NM \cap \Gamma$.

Proof. It suffices to show that $\Gamma_\infty \subset NM$. Let $\gamma = \theta_{w_0}\,\omega_{r_0} k_\gamma \in \Gamma_\infty$ and $\gamma\omega_r \cdot o = \theta_{w_0}\,\omega_{r_0} \cdot \theta_v\,\omega_s \cdot o = \theta_{w_0 + r_0 v}\,\omega_{r_0 s} \cdot o$ where the terms v, s here and $u_{i\,j}$ to appear in below have the same meanings with those in Sect. 2.3.2. Remember that $s^{-1} = \frac{1-u_{11}}{2} r + \frac{1+u_{11}}{2} r^{-1}$. If $u_{11} = -1$, then $s = r^{-1} \to 0$ as $r \to \infty$, meanwhile $v = 0$ since $v_i\, s^{-1} = u_{i+1,1} \frac{r-r^{-1}}{2} = 0$. This means $\gamma\omega_\infty \cdot o = \theta_{w_0}\omega_0 \cdot o$ where $\omega_0 = \lim\limits_{r\to 0^+} \omega_r$, a contradiction. If $u_{11} \neq \pm 1$, then $s \to 0$ as $r \to \infty$, meanwhile

$$v_i = s u_{i+1,1}\, \frac{r - r^{-1}}{2} = u_{i+1,1}\, \frac{\frac{r-r^{-1}}{2}}{\frac{1-u_{11}}{2} r + \frac{1+u_{11}}{2} r^{-1}} \to \frac{u_{i+1,1}}{1 - u_{11}}.$$

This means that $\gamma\omega_\infty \cdot o = \theta_{w_0 + r_0 v}\omega_0 \cdot o$ where $|w_0 + r_0 v|$ is bounded, a contradiction. Hence $u_{11} = 1$, i.e., $\gamma \in NAM$. If $r_0 \neq 1$, by taking the inverse if necessary (this is always available since $u_{11} = 1$), we assume that $r_0 < 1$, then $\gamma^2 = \theta_{w_0}\,\omega_{r_0} k_\gamma \cdot \theta_{w_0}\,\omega_{r_0} k_\gamma = \theta_{w_0 + r_0 w_0 \rho^T}\,\omega_{r_0^2} k^2$ where $k_\gamma = \mathrm{diag}(1,\,1,\,\rho)$, $\rho \in SO_{d-1}$. The computation by induction shows that $\gamma^{2^n} = \theta_{w_n}\,\omega_{r_n}\,k_n$ where

$$w_n = w_0 \left(1 + r_0 \rho^T + r_0^2 \left(\rho^T\right)^2 + \cdots + r_0^{2^n - 1} \left(\rho^T\right)^{2^n - 1}\right) \quad \text{and} \quad r_n = r_0^{2^n}.$$

Since $r_0 < 1$ and $|w_0 \left(\rho^T\right)^i| = |w_0|$ for any $i \geqslant 1$, the above formulas show that w_n converges (say, to w_∞) as $n \to \infty$, meanwhile $r_n \to 0$. Let $z_m = \omega_{r_0^{-2^m}} \cdot o$, then $\gamma^{2^n} z_m \to \theta_{w_\infty}\,\omega_0 \cdot o$ as $n \to \infty$ (for any fixed m). As $m \to \infty$, $z_m \to \omega_\infty \cdot o$, so we get $\gamma^{2^n}\omega_\infty \cdot o \to \theta_{w_\infty}\,\omega_0 \cdot o$ as $n \to \infty$, a contradiction. Thus $r_0 = 1$, i.e., $g \in NM$. □

Remark 5.1.2. *The subgroup Γ_∞ is a uniform lattice in NM.*

Assume that $h_j\omega_\infty \cdot o = p_j^*$ for some $h_j \in G$, $p_j^* \in \widetilde{p_j}$ (the set of fibres of p_j). Let $\Gamma_{j,\infty}$ be the stabilizer of p_j^* in Γ. Clearly this stabilizer is conjugate to Γ_∞ via h_j: $\Gamma_{j,\infty} = h_j\Gamma_\infty h_j^{-1}$.

Recall that, for each cusp p_j, the *Eisenstein series* $E_j(z,\, s)$ over X is defined to be

$$E_j(z,\, s) := \sum_{\gamma\in\Gamma_{j,\infty}\backslash\Gamma} \operatorname{Im}\left(h_j^{-1}\gamma z\right)^s, \quad z\in X, \quad s\in\mathbb{C}.$$

It is known that, for each j, $E_j(z,\, s)$ is absolutely and locally uniformly convergent for $\operatorname{Re}(s) > d-1$. For $h \in L^2(\Gamma\backslash G/K)$, we have the following *spectral decomposition* (see Chap. 7 of [**CS**]):

$$h(z) = \sum_{n\geqslant 0}\langle h,\, \phi_n\rangle\cdot\phi_n(z) + \sum_{j=0}^{k}\int_0^\infty g_j(t)\, E_j\left(z,\, \frac{d-1}{2}+it\right)dt$$

where $\langle\, ,\, \rangle$ is the L^2-inner product on $\Gamma\backslash G/K$ with respect to the measure μ',

$$g_j(t) = \frac{1}{2\pi}\int_{w\in X} h(w)\overline{E_j\left(w,\, \frac{d-1}{2}+it\right)}dw$$

and $\{\phi_n\}_{n=0}^\infty$ is a complete family of orthonormal *cusp forms* on X, i.e., Laplace eigenfunctions with constant terms being zero for their Fourier expansions around the cusps. Be careful that here the Eisenstein series (over $\operatorname{Re}(s) = \frac{d-1}{2}$) is the continuation of the original one, and it is holomorphic in a neighborhood of the line $\operatorname{Re}(s) = \frac{d-1}{2}$ (see Theorem 7.24 of [**Mü**]). Let f be a smooth function over G which is bi-K-invariant and decays rapidly at infinity by which we means the following: the space $K\backslash G/K$ can be identified with the half line $[0,\, \infty)$ thanks to the KAK-decomposition and the fact G is of split rank one, then we require f to be such that $f(t) = \mathcal{O}\left(t^{-\alpha}\right)$ (for any $\alpha > 0$) as $t\to\infty$. Define $k_f(g,\, h) = f\left(h^{-1}g\right)$ for $g,\, h\in G$. There is an integral operator

$$T_f:\, L^2(\Gamma\backslash G/K)\to L^2(\Gamma\backslash G/K), \quad \phi\mapsto T_f(\phi):\, g\mapsto \int\limits_G k_f(g,\, h)\phi(h)dh$$

One can easily show that

$$\left(T_f(\phi)\right)(z) = \int\limits_{\Gamma\backslash G/K} K_f(z,\, w)\phi(w)d\mu(w)$$

where

$$K_f(z,\, w) = \sum_{\gamma\in\Gamma} f\left(w^{-1}\gamma z\right)$$

lies in $L^2(\Gamma\backslash G/K)$ with respect to the variable z (or w) when w (z respectively) is fixed. As f is bi-K-invariant, the term w^{-1} is justified for $w\in\Gamma\backslash G/K$. To expand K_f by use of the spectral decomposition, we first check the following properties:

$$\left\langle K_f(\,\cdot\,,\, w),\, \phi_n(\,\cdot\,)\right\rangle = h_f(\lambda_n)\,\overline{\phi_n(w)},$$

$$\left\langle K_f(\,\cdot\,,\, w),\, E_j\left(\,\cdot\,,\, \frac{d-1}{2}+ir\right)\right\rangle = h_f(\lambda_r)\,\overline{E_j\left(w,\, \frac{d-1}{2}+ir\right)}.$$

Here $\lambda_r = \left(\frac{d-1}{2}\right)^2 + r^2$ and $h_f(\lambda)$ is as before, see (1.9). From these two formulas we get the expansion of $K_f(z,\, w)$ (see Chap. 7 of [**CS**]):

$$\sum_{n=0}^{\infty} h_f(\lambda_n)\phi_n(z)\overline{\phi_n(w)} + \frac{1}{4\pi}\sum_{j=0}^{k}\int_{-\infty}^{\infty} h_f(\lambda_r)E_j\left(z,\, \frac{d-1}{2}+ir\right)E_j\left(w,\, \frac{d-1}{2}-ir\right)dr \tag{5.1}$$

for a test function $f \in C_c^\infty(\mathbb{R}_{>0})$. In what follows, we choose the test function $f = \Phi_\mu$ as before, see Sect. 2.1. Under this function, the expansion (5.1) is locally absolutely and uniformly convergent. Such convergence is needed for the exchange the order of summation and integration when we do the integral along the geodesic C. In surface case, there is a widely-used weaker condition (see e.g. the conditions 1.63 of [**Iw**]) for which the test functions need not be compactly supported while still the expansion is locally absolutely and uniformly convergent. We remark that for any $f \in \mathcal{S}(\mathbb{R}_{>0})$, the space of Schwarz functions, the series on the right hand side of (5.1) converges absolutely and uniformly on compact subsets of $X \times X$. This follows from the property of Eisenstein series, namely the local uniform convergence of Eisenstein series. For 3-dimensional case, see Theorem 4.1, p.278 of [**El**]. The necessary properties of Eisenstein series (Proposition 1.3 on p.84 of [**El**]) for higher dimensional situation still hold (see [**CS**] or [**Mü**]).

5.2 Periods along the geodesic

Integrating the kernel function $K_f(z,\, w)$ over $(z,\, w) \in C \times C$, the spectral expansion (5.1) gives rise to an equality between two terms, the "geometric side" :

$$\int_C \int_C \sum_{\gamma\in\Gamma} f\left(w^{-1}\gamma z\right) dwdz$$

and the "spectral side":

$$\sum_{n=0}^{\infty} h_f(\lambda_n)\left|P_C(\phi_n)\right|^2 +$$
$$\frac{1}{4\pi}\sum_{j=0}^{k}\int_{-\infty}^{\infty}\int_{z\in C}\int_{w\in C} h_f(\lambda_r)E_j\left(z,\, \frac{d-1}{2}+ir\right)E_j\left(w,\, \frac{d-1}{2}-ir\right)dwdzdr. \tag{5.2}$$

Note that we have changed the order of integrations and the summation on the spectral side. The geodesic C is compact, so this is available for the first term in which cusp forms get involved, thanks to the locally uniform convergence of (5.1). As for the second term of (5.2), we apply Fubini's Theorem after Proposition **??**. There is nothing to say about the geometric side because our work on this term for compact manifolds (see Sect. 2.3) essentially depends only on the property that Γ is torsion-free, i.e., we did not use the assumption that Γ is uniform. Now we focus on the second term of (5.2).

Proposition 5.2.1. $\int_0^T \left|E_j\left(z, \frac{d-1}{2} + ir\right)\right|^2 dr = \mathcal{O}\left(T^d\right)$.

Proof. See Corollary 7.7 of [**CS**]. □

Proposition 5.2.2. *For $x > 0$, there is a uniform bound*

$$|K_{ir}(x)| \leqslant e^{-\delta|r|}K_0(x\cos\delta), \quad 0 \leqslant \delta \leqslant \delta_0 < \frac{\pi}{2}.$$

Proof. See formula 1.100 of [**Ya**]. □

Denote the second term on the right hand side of (5.2) by H_μ, and $\left|\int_C E_j\left(z, \frac{d-1}{2} + ir\right) dz\right|^2$ by $F(r)$. For r large, by Proposition 5.2.1 and Hölder's inequality, we have:

$$\int_0^r F(x)dx \leqslant \operatorname{len}(C) \int_0^r \left|E_j\left(z, \frac{d-1}{2} + ir\right)\right|^2 dr \ll r^d. \tag{5.3}$$

The integration by parts shows that

$$\int_S^\infty F(r)h_f(\lambda_r)dr = \left(\int_0^r F(x)dx + A\right) h_f(\lambda_r)\Big|_S^\infty - \int_S^\infty \left(\int_0^r F(x)dx + A\right) \frac{\partial h_f(\lambda_r)}{\partial r}dr \tag{5.4}$$

for some number A. In view of (4.12) and (5.3), for S large and $S > \mu$, one has

$$\int_0^r F(x)dx \cdot h_f(\lambda_r)\Big|_S^\infty = \int_0^S F(x)dx \cdot h_f(\lambda_S) \ll S^d \cdot e^{-\frac{\pi}{2}S}\mu^{-\frac{d-1}{2}} \ll_{\epsilon,\mu} e^{-\left(\frac{\pi}{2}-\epsilon\right)S} < e^{-\left(\frac{\pi}{2}-\epsilon\right)\mu}$$

where the second $\mathcal{O}$-constant, depending on ϵ and μ, tends to 0 as $\mu \to \infty$. Besides, $Ah_f(\lambda_r)\Big|_S^\infty = Ah_f(\lambda_S) \ll e^{-\frac{\pi}{2}S} < e^{-\frac{\pi}{2}\mu}$, as $\mu \to \infty$. In short, we get: for $S > \mu$,

$$\left(\int_0^r F(x)dx + A\right) h_f(\lambda_r)\Big|_S^\infty \ll_\epsilon e^{-\left(\frac{\pi}{2}-\epsilon\right)\mu}, \quad \text{as } \mu \to \infty. \tag{5.5}$$

As for the second part on the right hand side of (5.4), by (4.12) and (5.3), we have:

$$\int_S^\infty \left(\int_0^r F(x)dx\right) \frac{\partial h_f(\lambda_r)}{\partial r}dr \ll \int_S^\infty r^d e^{-\frac{\pi}{2}r}dr \ll S^d e^{-\frac{\pi}{2}S} \ll_\epsilon e^{-\left(\frac{\pi}{2}-\epsilon\right)S} < e^{-\left(\frac{\pi}{2}-\epsilon\right)\mu}.$$

Besides, $\int_S^\infty A\frac{\partial h_f(\lambda_r)}{\partial r}dr \ll \int_S^\infty e^{-\frac{\pi}{2}r}dr = e^{-\frac{\pi}{2}S} < e^{-\frac{\pi}{2}\mu}$, as $\mu \to \infty$. In short, we get: for $S > \mu$,

$$\int_S^\infty \left(\int_0^r F(x)dx + A\right) \frac{\partial h_f(\lambda_r)}{\partial r}dr \ll_\epsilon e^{-\left(\frac{\pi}{2}-\epsilon\right)\mu}, \quad \text{as } \mu \to \infty. \tag{5.6}$$

For $S > \mu$, by (5.5) and (5.6),

$$\int_S^\infty F(r)h_f(\lambda_r)dr \ll_\epsilon e^{-\left(\frac{\pi}{2}-\epsilon\right)\mu}, \quad \text{as } \mu \to \infty.$$

Now let's consider the remaining term:

$$\int_0^S F(r)h_f(\lambda_r)dr.$$

Choose $S = \mu + \epsilon_\mu$ where $\epsilon_\mu > 0$ decays rapidly enough (with respect to μ) such that the intergal $\sqrt{\frac{2\mu}{\pi}}e^\mu \int_\mu^{\mu+\epsilon_\mu} F(r)h_f(\lambda_r)dr$ also decays rapidly as $\mu \to \infty$ (we multiply a factor ahead because later we shall do this on the both spectral and geometric sides). This is available since

$$\int_\mu^{\mu+\epsilon_\mu} F(r)h_f(\lambda_r)dr = \int_\mu^{\mu+\epsilon_\mu} F(r)dr \cdot h_f(\lambda_{r_\mu})$$

for some $r_\mu \in (\mu,\, \mu+\epsilon_\mu)$. By (4.12),

$$h_f(\lambda_{r_\mu}) = 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_{ir_\mu}(\mu) \ll \mu^{-\frac{d-1}{2}} e^{-\frac{\pi}{2}r_\mu} < \mu^{-\frac{d-1}{2}} e^{-\frac{\pi}{2}\mu}.$$

It suffices to control $\int_\mu^{\mu+\epsilon_\mu} F(r)dr$ so that it decays rapidly (as fast as we want), which is clearly possible. So we may focus on $\int_0^\mu F(r)h_f(\lambda_r)dr$. By Proposition 5.2.2 (letting $\delta = 0$) and 5.2.1,

$$\left|\int_0^\mu F(r)h_f(\lambda_r)dr\right| \leqslant 2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} K_0(\mu) \int_0^\mu F(r)dr \ll \mu^{-\frac{d}{2}+d} e^{-\mu} \tag{5.7}$$

We summarize what we have done as follows:

$$H_\mu = \mathcal{O}\left(\mu^{-\frac{d}{2}+d}e^{-\mu}\right) + \mathcal{O}\left(e^{-\left(\frac{\pi}{2}-\epsilon\right)\mu}\right)$$

Multiplying $\sqrt{\frac{2\mu}{\pi}}e^\mu$ on spectral and geometric sides and taking the limitation $\mu \to \infty$, one has

Proposition 5.2.3. *For any d-dimensional hyperbolic manifold of hyperbolic nolume and a closed geodesic C over it, the following holds*

$$2^d \cdot e^\mu \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-2} \sum_{i=0}^{\infty} K_{\nu_i}(\mu)\, |P_C(\phi_i)|^2 = 2\|E\| \operatorname{len}(C) + o\left(\mu^{\frac{d+1}{2}}\right), \quad \mu \to \infty.$$

The term $o\left(\mu^{\frac{d+1}{2}}\right)$ makes the formula far from (2.5.1). To save the power, one is led to (5.7). If we could save the average growth order of the Eisenstein series on the critical line such that

$$\int_0^T \left|E_j\left(z,\, \frac{d-1}{2}+ir\right)\right|^2 dr = \mathcal{O}\left(T^{\frac{d-1}{2}-\epsilon}\right), \quad \epsilon > 0,$$

then only $2\|E\| \operatorname{len}(C)$ remains in the above formula. By this, the important Selberg-Roelcke conjecture, which asserts that there are infinitely many cusp forms on any hyperbolic manifold of finite volume, follows since there are infinitely many nonvanishing periods along the closed geodesic. However, there are evidences ([**DIPS**], [**PS**], [**Sa**]) indicating that the Selberg-Roelcke conjecture might not hold when the lattice lacks certain arthmetic and symmetric properties.

The conclusions corresponding to weighted periods (Sect. 2.6), twisted periods (Sect. 2.7) and special totally geodesic submanifold (Chapter 3) can also be extended to noncompact situation, with the extra term $o\left(\mu^{\frac{d+1}{2}}\right)$ on the spectral side. We omit the details.

As we have noticed (see Remark 4.1.6), the bound in Proposition 4.1.4 on periods does not rely on the compactness of X. So we can refine the formula in Proposition 5.2.3 as:

$$2^d \left(\sqrt{\frac{\pi}{2\mu}}\right)^{d-1} \sum_{i=0}^{\infty} e^{-\frac{r_i^2}{2\mu}} \left|P_C(\phi_i)\right|^2 = 2\|E\| \operatorname{len}(C) + o\left(\mu^{\frac{d+1}{2}}\right), \quad \mu \to \infty.$$

The main part of the L.H.S. of this formula is for those i such that $r_i \leqslant \mu^{\frac{1}{2}+\epsilon}$ (the sum of those rest terms tend to 0 as $\mu \to \infty$, see Sect. 4.2). In this interval, $h_f(r_i) > 0$, so the L.H.S. is smaller than $2\|E\| \operatorname{len}(C)$, but we do not know how small it is as $\mu \to \infty$.

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