Algebraic Curvature Operators and the Ricci Vector Field

Dissertation
der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
Florian Schmidt
aus Sindelfingen

Tübingen
2012

Datum der mündlichen Qualifikation:	17.12.2012
Dekan:	Prof. Dr. Wolfgang Rosenstiel
1. Berichterstatter:	Prof. Dr. Frank Loose
2. Berichterstatter:	Prof. Dr. Christoph Böhm
3. Berichterstatter:	Prof. Dr. Christoph Bohle

Inhaltsverzeichnis

Deutsche Zusammenfassung

Der Ricci Fluss ist ein parabolisches System nichtlinearer partieller Differentialgleichungen zweiter Ordnung für die Riemannsche Metrik g auf einer Mannigfaltigkeit, welches durch die Gleichung

$$\frac{d}{dt}g = -2\mathrm{Ric(g)}$$

gegeben ist. Mit Hilfe des Ricci Flusses wurden in den vergangenen Jahren große Fortschritte erziehlt. Beispielsweise ist es G. Perelman im Jahre 2003 mit Methoden des Ricci Flusses gelungen, die berühmte Poincaré Vermutung zu lösen (siehe [20], [21] and [22]). Sie besagt, dass eine einfach zusammenhängende, geschlossene, dreidimensionale Mannigfaltigkeit homöomorph zur dreidimensionalen Standardsphäre ist. Des Weiteren nutzten S. Brendle und R. Schoen im Jahr 2009 den Ricci Fluss, um den Differenzierbaren Sphärensatz zu beweisen [6]. Der diffenenzierbare Sphärensatz sagt aus, dass jede einfach zusammenhängende, geschlossene n-dimensionale Riemannsche Mannigfaltigkeit, deren Schnittkrümmungen im Intervall $(1/4, 1]$ liegen, diffeomorph zur Standard n-Sphäre ist. Daneben zeigten C. Böhm und B. Wilking 2008, dass jede geschlossene Riemannsche Mannigfaltigkeit mit 2-positivem Krümmungsoperator diffeomrorph zu einer sphärischen Raumform ist [5] und verallgemeinerten damit Hamiltons Resultat aus dem Jahre 1986: Jede geschlossene vierdimensionale Riemannsche Mannigfaltigkeit mit positivem Krümmungsoperator ist sphärisch [11].
Die Beweise der oben genannten Resultate basieren zu großen Teilen auf der geschickten Anwendung der Tensor Maximum Prinzipien Richard S. Hamiltons auf die Evolutionsgleichung

$$\frac{d}{dt}\mathcal{R} = \Delta\mathcal{R} + \Phi(\mathcal{R})$$

des Krümmungsoperators unter dem Ricci Fluss (siehe [11]). Hierbei ist Φ ein vertikales Vektorfeld auf dem Bündel der algebraischen Krümmungsoperatoren. Es ist gegeben durch

$$\Phi(\mathcal{R}) = 2(\mathcal{R}^2 + \mathcal{R}\#\mathcal{R})$$

und wird im Folgenden als das Ricci Vektorfeld bezeichnet. Seine Untersuchung ist eines der Hauptanliegen dieses Textes. Denn: Hamiltons Tensor

Maximum Prinzip erlaubt, aus dem Verhalten der Lösungen der gewöhnlichen Differentialgleichung

$$\frac{d}{dt}\mathcal{R} = \Phi(\mathcal{R})$$

Rückschlüsse auf das Verhalten der Lösungen der partiellen Differentialgleichung

$$\frac{d}{dt}\mathcal{R} = \Delta\mathcal{R} + \Phi(\mathcal{R})$$

zu ziehen.

Neben einer systematischen, breit angelegten Einführung in das Gebiet der algebraischen Krümmungsoperatoren auf Euklidischen Vektorräumen und des #-Produkts (gelesen: "Sharp Produkt"), bietet diese Arbeit zahlreiche neue algebraische Einzelergebnisse, die im Zusammenhang mit der Untersuchung des Ricci Flusses auf Mannigfaltigkeiten stehen. Unter anderem wären hier eine tiefe strukturelle Beziehung zwischen Lie Unteralgebren der $\mathfrak{so}(n)$ zu nennen (siehe 2.3.0.23) sowie eine scharfe Abschätzung der Eigenwerte von $\mathcal{R}\#\mathcal{R}$ durch bestimmte Produkte von Eigenwerten eines Krümmungsoperators $\mathcal{R}$ (vergleiche 2.7.0.28). Außerdem ein Beweis der Tatsache, dass die Böhm-Wilking Identität

$$\mathrm{id}\#\mathcal{R} = \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id} - \mathcal{R}$$

für einen selbstadjungierten Endomorphismus von $\bigwedge^2 V$ nicht nur ein notwendiges Kriterium dafür ist, ein algebraischer Krümmungsoperator zu sein, wie C. Böhm und B. Wilking in [5] zeigten, sondern auch ein hinreichendes (siehe 3.2.3.4, 3.2.3.5, 3.2.3.6 und 3.2.3.7). Ferner, ein rein algebraischer Beweis der bereits bekannten Tatsache, dass das Ricci Vektorfeld tangential zum Raum der algebraischen Krümmungsoperatoren ist (3.3.1.1), rein algebraische Berechnungen und weitergehende Betrachtungen der (irreduziblen) invarianten Komponenten entlang des Ricci Vektorfeldes (siehe 3.3.2.11), welche unter bestimmten Voraussetzungen Rückschlüsse auf geometrische bzw. algebraische Besonderheiten der zugrundeliegenden Krümmungsoperatoren erlauben (3.3.1.3 bzw. Abschnitt 3.3.2).

Abschließend beschäftigen wir uns in Kapitel 4 mit der Dynamik des Ricci Vektorfeldes auf dem Raum der algebraischen Krümmungsoperatoren. Dabei erweist es sich gelegentlich als vorteilhaft, anstelle des Ricci Vektorfeldes dessen sphärischen Tangentialanteil $\overline{\Phi}$ zu betrachten und die radiale Fluchtgeschwindigkeit $\nu(\mathcal{R}) = \tau(\mathcal{R})/\left\|\mathcal{R}\right\|^2 \cdot \mathcal{R}$, $\tau(\mathcal{R}) = \langle \Phi(\mathcal{R}), \mathcal{R}\rangle$, zunächst zu vernachlässigen. Unsere Überlegungen führen unter Anderem zu der folgenden Erkenntnis über Gleichgewichtslagen des normalisierten Ricci-Vektorfeldes: Ist $\mathcal{R}$ ein stationärer Punkt von $\overline{\Phi}$, und spaltet $\mathcal{R}$ geometrisch als Produkt der von Null verschiedenen Krümmungsoperatoren $\mathcal{S}$ und $\mathcal{T}$, so gilt

$$\frac{\tau(\mathcal{S})}{\|\mathcal{S}\|^2} = \frac{\tau(\mathcal{T})}{\|\mathcal{T}\|^2} = \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}.$$

Introduction

If a smooth family of Riemannian metrics $g(t)_{t\in[0,T)}$, $T > 0$, on a manifold M solves the Ricci flow equation

$$\frac{d}{dt}g(t) = -2\mathrm{Ric}(\mathrm{g(t)}),$$

then the whole set of related geometric quantities is changing as well. As the curvature operator carries the full information of the underlying geometry, it is convenient first looking at the evolution of the curvature operators $\mathcal{R}(t) : \bigwedge^2 \mathrm{TM} \to \bigwedge^2 \mathrm{TM}$ before looking at other quantities like Ricci or scalar curvature. As R.S. Hamilton showed in [11], we have that the curvature operator evolves like

$$\frac{d}{dt}\mathcal{R} = \Delta\mathcal{R} + \Phi(\mathcal{R}),$$

where $\Delta = \Delta(g(t))$ is the time dependent spatial Laplacian and $\Phi = \Phi(g(t))$ is a time-dependent vertical vector field on the space of linear bundle endomorphisms $\mathrm{End}(\bigwedge^2 \mathrm{TM})$ of $\bigwedge^2 \mathrm{TM}$. Φ is given by $\Phi(\mathcal{R}) = 2(\mathcal{R}^2 + \mathcal{R}\#\mathcal{R})$, where $\#$ is the sharp product depending on the metric $g(t)$, and will be called the Ricci vector field.
The differential equation $\frac{d}{dt}\mathcal{R} = \Delta\mathcal{R} + \Phi(\mathcal{R})$ is a parabolic partial differential equation for sections of $\mathrm{End}(\bigwedge^2 \mathrm{TM})$ and, hoping to understand the dynamics of the Ricci flow, we are lucky to have Hamilton's tensor maximum principle (proved in [11], and described in the appendix B.8) to our disposal, which states that each closed $\mathcal{C}^0-$subbundle $C \subseteq \mathrm{E}$, which is fiberwise convex and parallel in spatial direction, is preserved by the flow of the PDE, if it is preserved by the flow of the vector field Φ. We are lucky, since this allows us to control the solutions of a parabolic PDE for sections of a Euclidean vector bundle by controlling the fiberwise ODEs, which arise if we drop the Laplace term.
Almost every application of the Ricci flow is also based on an elegant use of Hamilton's tensor maximum principle.
For example, after having invented the Ricci flow in 1982 and proving that every compact three-manifold with positive Ricci curvature is spherical [10], i.e. a manifold which is diffeomorphic to a spherical spaceform (and by the way, one of the main tools in the proof were the scalar parabolic maximum

principles), Hamilton proved in 1986, that every compact 4-manifold with positive curvature operator is also spherical [11] using his tensor maximum principle. Moreover, in 1993 Hamilton found a Harnack inequality for the Ricci flow [12] and proved it using his tensor maximum principle. Later, in 1999, he classified all the non-singular solutions of the Ricci flow in dimension 3 [15]. There, he uses the Hamilton-Ivey Long-time pinching estimate, which is a direct application of the tensor maximum principle. After that, in 2003, Perelman proved the Thurston geometrization conjecture, which includes the proof of the famous Poincaré-conjecture (compare [20], [21] and [22]). Clearly, this is not only a simple application of the maximum principle, but it is involved. Then, in 2006, C. Böhm and B. Wilking generalized Hamilton's theorem from 1982 to higher dimensions. They showed that every n-dimensional Riemannian manifold with 2-positive curvature operator is spherical [5] and again, Hamilton's tensor maximum principle plays a central structural role in the proof. This method of proving didn't stop until 2007, where S. Brendle and R. Schoen proved the differential sphere theorem, which states that every simply connected $\frac{1}{4}-$pinched Riemannian manifold is diffeomorphic to the standard sphere [6]. All in all, we can say that the study of the Ricci vector field is essential in the study of the Ricci flow.

In this work we provide a detailed and systematic introduction to the field of algebraic curvature operators on Euclidean vector spaces, the #-product and the Ricci vector field Φ. The main results are the following:

In chapter 2 we present a new relation between the Killing form κ of a subalgebra $\mathfrak{h}$ of $\bigwedge^2 V \cong \mathfrak{so}(V)$ with the #-product and the orthogonal projection π onto $\mathfrak{h}$. More precisely, our considerations show

$$\kappa(\varepsilon, \delta) = -2 \langle \pi \# \pi(\varepsilon), \delta \rangle$$

for all $\varepsilon, \delta \in \bigwedge^2 V$ (see corollary 2.3.0.23). Hence, we have found a structural relation between the #-product and Lie subalgebras of $\bigwedge^2 V$.

At the end of chapter 2 we present a sharp estimate of the eigenvalues of $\mathcal{R}\#\mathcal{R}$ in terms of the eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$ of a self-adjoint endomorphism $\mathcal{R}$ of $\bigwedge^2 V$. We show that the eigenvalues of $\mathcal{R}\#\mathcal{R}$ lie in the interval

$$\left[(n-2) \min_{i<j} \lambda_i \lambda_j, (n-2) \max_{i<j} \lambda_i \lambda_j\right],$$

where n is the dimension of V and $N = \dim \bigwedge^2 V = \binom{n}{2}$ (compare theorem 2.7.0.28).

Chapter 3 offers some more insights then the other chapters. In [5], C. Böhm and B. Wilking showed that every algebraic curvature operator $\mathcal{R}$

satisfies the Böhm-Wilking identity

$$\mathrm{id}\#\mathcal{R} = \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id} - \mathcal{R}.$$

Encouraged by this result, we found out that the opposite direction is also true: A self-adjoint linear map $\mathcal{R} : \bigwedge^2 V \to \bigwedge^2 V$, V a Euclidean vector space, is an algebraic curvature operator, if it satisfies the Böhm-Wilking identity (see theorem 3.2.3.4, proposition 3.2.3.5, theorem 3.2.3.6 and theorem 3.2.3.7).
Moreover, we show that a self-adjoint endomorphism $\mathcal{R}$ of $\bigwedge^2 V$ is an algebraic curvature operator if and only if it is the sum of wedge-product of certain skew-adjoint linear maps (compare theorem 3.2.3.1). We use this result to show that every algebraic curvature operator $\mathcal{R}$ with range equal to 1, is a multiple of orthogonal projection π onto a subspace of the form $\bigwedge^2 U$, where $U \subseteq V$ is a two dimensional subspace. Thus, there are subalgebras of $\bigwedge^2 V$, which do not come from holonomy groups of Riemannian manifolds.
In section 3.3, when it comes to the Ricci vector field, we give a purely algebraic proof of the well known fact (compare [11]), that the Ricci vector field is tangent to the space of algebraic curvature operators (see theorem 3.3.1.1 for the algebraic proof). Also, only using algebraic data, we show in theorem 3.3.1.3 that the Ricci curvature and the scalar curvature of $\Phi(\mathcal{R})$ of an algebraic curvature operator $\mathcal{R}$ are given by

$$\mathrm{Ric}(\Phi(\mathcal{R}))(\mathrm{x}) = 2\sum_{\mathrm{i}} \mathcal{R}^{\rho}(\mathrm{x}, \mathrm{e_i})\mathrm{Ric}(\mathcal{R})(\mathrm{e_i}),$$

for all $x \in V$, where $\{e_i\}$ is an arbitrary orthonormal basis of V, and

$$\mathrm{scal}(\mathcal{R}) = 2\mathrm{tr}(\Phi(\mathcal{R})) = 2\left\|\mathrm{Ric}(\mathcal{R})\right\|^2.$$

(These formulas are due to R.S. Hamilton [10].)

This means that we obtain the evolution of the Ricci curvature and the scalar curvature, and therefore also the evolution of the Weyl curvature, under the flow of Φ on the space of algebraic curvature operators, by dropping the Laplace terms of the corresponding original evolution equations under the Ricci flow which have been published by R. S. Hamilton in [10]. Recall, that we have

$$\frac{d}{dt}\mathrm{Ric} = \Delta\mathrm{Ric} + \sum_{\mathrm{i}} \mathrm{R}(\mathrm{x}, \mathrm{e_i})\mathrm{Ric}(\mathcal{R})(\mathrm{e_i}),$$

and

$$\frac{d}{dt}\mathrm{scal} = \Delta\mathrm{scal} + 2\left\|\mathrm{Ric}\right\|^2$$

under the Ricci flow. This is due to the fact that the Ricci vector field on the bundle LC of algebraic curvature operators over a Riemannian manifold M is parallel, as we show in the text (see remark 3.3.1.2).

The formula $\mathrm{tr}(\Phi(\mathcal{R})) = \|\mathrm{Ric}(\mathcal{R})\|^2$ can be used to prove very quickly that Ricci flat symmetric spaces are flat, as we do it in corollary 3.4.0.22. Further, this equation allows us to make a first observation concerning the equilibrium positions of the Ricci vector field: they are all Ricci flat. After that we examine the irreducible components of $\Phi(\mathcal{R})$, express them in terms of the irreducible components of $\mathcal{R}$ and discuss some situations, where some components of $\Phi(\mathcal{R})$ vanish. We will see that is sometimes possible to regain geometric or algebraic knowledge about $\mathcal{R}$ from this. For example, assuming that $\mathcal{R}$ is an algebraic curvature operator of traceless Ricci type, we found out that $\mathrm{Ric}_0(\Phi(\mathcal{R})) = 0$ implies $\mathcal{R} = 0$. Also, we have established theorem 3.3.2.4: If $\mathrm{Ric}_0(\Phi(\mathcal{R}))$ vanishes, then $\mathcal{R} = 0$ if the dimension n of the underlying vector space is odd, and, if the dimension is even, $n = 2m$, say, then $\mathcal{R}$ is a multiple of the curvature operator of $\mathbb{S}^m \times \mathbb{H}^m$. Further, we have theorem 3.3.2.10, saying that if the Weyl curvature of $\Phi(\mathcal{R})$ vanishes, then $\mathcal{R}$ is of the form $\mathcal{R} = F \wedge \mathrm{id}$, where F is the tracefree part of a multiple of an orthogonal projection onto a one-dimensional subspace of V.

The Ricci flow preserves products. Does the flow of the Ricci vector field preserve products as well? And what is meant with the term "product of algebraic curvature operators"? As each algebraic curvature operator comes with a canonical geometric realization as the curvature operator of a Riemannian manifold, this question is easy to answer. Geometric products of algebraic curvature operators are direct sums $\mathcal{R}_1 + ... + \mathcal{R}_r$ of algebraic curvature operators $\mathcal{R}_i$ on Euclidean vector spaces V_i. There is a slightly finer version of algebraic product curvature operators, using the holonomy algebra $\mathfrak{h}_\mathcal{R}$ of algebraic curvature operators. $\mathfrak{h}_\mathcal{R}$ is by definition the smallest Lie subalgebra of $\bigwedge^2 V$ containing the image of $\mathcal{R}$. We say that an algebraic curvature operator $\mathcal{R}$ is an algebraic product of the algebraic curvature operators $\mathcal{S}$ and $\mathcal{T}$, if $\mathfrak{h}_\mathcal{S}$ and $\mathfrak{h}_\mathcal{T}$ form $\mathcal{R}$-invariant ideals in $\mathfrak{h}_\mathcal{R}$ and if the holonomy algebra of $\mathcal{R}$ splits orthogonally as the direct sum of the holonomy algebras of $\mathcal{S}$ and $\mathcal{T}$,

$$\mathfrak{h}_\mathcal{R} = \mathfrak{h}_\mathcal{S} \oplus \mathfrak{h}_\mathcal{T}.$$

The flow of Φ preserves both, algebraic and geometric products of algebraic curvature operators, as we show in theorem 4.1.0.3. In order to understand the dynamics of the Ricci vector field, we can now restrict ourselves to the study of the flow on algebraically (or geometrically) irreducible curvature operators. The main result of chapter 4, theorem 4.4.0.17, states that if a fixed point $\mathcal{R}$ of $\overline{\Phi}$ decomposes algebraically (or geometrically) as a product of algebraic curvature operators $\mathcal{R}_1$,..., $\mathcal{R}_r$, then each factor $\mathcal{R}_i$ is also a fixed point and we have the identity

$$\frac{\tau(\mathcal{R}_i)}{\|\mathcal{R}_i\|^2} = \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}$$

for all i.

To deal with the "algebraic aspects of the Ricci flow", in particular with the Ricci vector field, it is most comfortable viewing Riemannian curvature operators as certain self-adjoint endomorphisms of $\bigwedge^2 \mathrm{TM}$. As this approach to the subject is fairly unusual and because we wanted this work to be as self-contained as possible, we have decided to start from zero and develop the theory of algebraic curvature operators as far as we need it, and maybe a little more than this, from our perspective.
A sharp look at the table of contents leads to a good picture of what we are doing here.

For reasons of self-containedness again, we have decided to write an appendix, where we present the material, which is needed to follow the text. This includes one part on multilinear algebra and the very basic theory of linear representations, and second part, which is concerned with Riemannian geometry. A third is concerned with the very basic aspects of Ricci flow, for example, we present the evolution equations of several curvature quantities.

Acknowledgements

I want to give special thanks to my supervisor Prof. Dr. Frank Loose for his help and his constant support and I would to thank Prof. Dr. Christoph Böhm and Prof. Dr. Christoph Bohle for preparing the second and the third opinion, respectively. Moreover, I would like to thank Prof. Dr. Franz Pedit for his support, thanks to Prof. Dr. Gerhard Huisken for helpful discussions and the nice friday morning coffee breaks in his office. Further, I would like to thank my collegues David, Maren, Taki, Meru, Achim, Lynn, Sebastian and Slawa for helpful discussions and a really nice time. Of course, I want to thank my family and friends for always being there for me and supporting me all the time. Finally, I would like to thank the Friedrich-Ebert-Stiftung for donating a very helpful scholarship to me.

Kapitel 1

The Lie Algebra of Bivectors

In this section we will show how the exterior power $\bigwedge^2 V$ of a given n-dimensional Euclidean vector space V can be given the structure of a certain Lie Algebra, the Lie Algebra $\mathfrak{so}(V)$, which is given by the space of skew-adjoint endomorphisms of $(V, \langle\cdot\,,\cdot\rangle)$ together with the canonical Lie Bracket $[A,B] = AB - BA$. We will proceed as follows: first we construct a certain Lie bracket on $\bigwedge^2 V$ using a fixed but arbitrarily chosen scalar product on V and the canonical Lie structure of $\mathfrak{so}(V)$. Then we will see that the resulting Lie algebra is actually isomorphic to $\mathfrak{so}(V)$. After that we list some formulas which will be very important in later computations. Then we discuss some structural properties of $\bigwedge^2 V$: For example, we show that $\bigwedge^2 V$ is semi-simple if the dimension of V is at least 3, prove that it is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ as a Lie algebra if V has dimension 4 and that it is simple in higher dimensions. Next we compare the structures arising from different choices of the underlying scalar products and clarify the relations between them. At the end of this chapter we will translate our construction to the vector bundle setting in a pointwise manner.

1.1 $\bigwedge^2 V$ as a Lie Algebra

Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional Euclidean vector space. Recall that $\langle\cdot\,,\cdot\rangle$ induces a scalar product on $\bigwedge^2 V$, which will also be denoted by $\langle\cdot\,,\cdot\rangle$, in the following way: First, we define a four-linear map $\langle\cdot\,,\cdot\rangle' : V \times V \times V \times V \to \mathbb{R}$,

$$(x,y,u,v) \mapsto \det \begin{pmatrix} \langle x,u\rangle & \langle x,v\rangle \\ \langle y,u\rangle & \langle y,v\rangle \end{pmatrix}.$$

This map obviously satisfies:

- $\langle (x,y,u,v)\rangle' = -\langle (y,x,u,v)\rangle' = -\langle (x,y,v,u)\rangle'$ and
- $\langle (x,y,u,v)\rangle' = \langle (u,v,x,y)\rangle'$.

Then, using the universal property of exterior powers, we get the desired scalar product

$$\langle \cdot\ ,\cdot\rangle : \bigwedge^2 V \times \bigwedge^2 V \mapsto \mathbb{R},$$

satisfying

$$\langle x \wedge y, u \wedge v\rangle = \det \begin{pmatrix} \langle x,u\rangle & \langle x,v\rangle \\ \langle y,u\rangle & \langle y,v\rangle \end{pmatrix}$$

for all $x, y, u, v \in V$. See the appendix A.2 for the details. Now let

$$\tilde{\rho} : V \times V \to \mathrm{End(V)} : \mathrm{(x,y)} \mapsto \mathrm{x^* \otimes y - y^* \otimes x},$$

where x^* is the linear form on V defined by $x^*(y) = \langle x, y\rangle$. Note that the map $x \mapsto x^*$ depends essentially on the choice of the underlying scalar product. Now, $\tilde{\rho}$ is bilinear and skew-symmetric. Thus, it induces a unique linear map $\rho : \bigwedge^2 V \to \mathrm{End(V)}$ by the universal property of exterior powers.

Lemma 1.1.0.1. *The image of ρ is $\mathfrak{so}(V, \langle\cdot,\cdot\rangle)$, the vector space of skew-adjoint endomorphisms of $(V, \langle\cdot\ ,\cdot\rangle)$*

Beweis. First we show that the image of ρ is contained in $\mathfrak{so}(V, \langle\cdot\ ,\cdot\rangle)$:
Let $u, v, x, y \in V$ be arbitrary.Then

$$\begin{aligned} \langle \rho(x \wedge y)u, v\rangle &= \langle \langle x,u\rangle\, y - \langle y,u\rangle\, x, v\rangle \\ &= \langle x,u\rangle \langle y,v\rangle - \langle y,u\rangle \langle x,v\rangle \\ &= \det \begin{pmatrix} \langle x,u\rangle & \langle x,v\rangle \\ \langle y,u\rangle & \langle y,v\rangle \end{pmatrix} \\ &= \langle x \wedge y, u \wedge v\rangle \end{aligned}$$

Since every vector $\varepsilon \in \bigwedge^2 V$ is a sum of vectors of the form $x \wedge y$ with $x, y \in V$, this shows

$$\langle \rho(\varepsilon)u, v\rangle = \langle \varepsilon, u \wedge v\rangle$$

for all $\varepsilon \in \bigwedge^2 V$ and $u, v \in V$.
Hence, using the symmetry of $\langle\cdot\ ,\cdot\rangle$ and skew-symmetry of $\wedge$, we get

$$\langle u, \rho(\varepsilon)v\rangle = \langle v \wedge u, \varepsilon\rangle = -\langle \varepsilon, u \wedge v\rangle = -\langle \rho(\varepsilon)u, v\rangle\,,$$

so we are done with the first part of the proof.
Now let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $(V, \langle\cdot\ ,\cdot\rangle)$. Then

$$\{\rho(e_i \wedge e_j)\}_{1\le i<j\le n} = \{e_i^* \otimes e_j - e_j^* \otimes e_i\}_{1\le i<j\le n}$$

is a basis of $\mathfrak{so}(V, \langle\cdot\ ,\cdot\rangle)$. (If we represent these endomorphisms as matrices w.r.t. the chosen basis $\{e_i\}$, we get that $\rho(e_i \wedge e_j)$ is represented by the matrix $\mathrm{E_{i,j} = (e_{kl})_{kl}}$, with $e_{ij} = 1, e_{ji} = -1$ and $e_{kl} = 0$ otherwise. These matrices are known to form a basis of $\mathfrak{so}(n)$.) Thus, ρ is an isomorphism. □

Now we are ready to define a Lie Algebra structure on $\bigwedge^2 V$.

Definition 1.1.0.2. *For $\varepsilon, \delta \in \bigwedge^2 V$ define*

$$[\varepsilon, \delta] := \rho^{-1}[\rho(\varepsilon), \rho(\delta)],$$

where the bracket on the right hand side is the Lie Bracket of $\mathfrak{so}(V)$.

It is clear that, having constructed the Lie bracket on $\bigwedge^2 V$, we get a Lie bracket on $\left(\bigwedge^2 V\right)^*$ in a natural way: The canonical isomorphism $* : V \to V^* : x \mapsto (y \mapsto \langle x, y\rangle)$ induces an isomorphism $\bigwedge^2 V \to \left(\bigwedge^2 V\right)^*$ via $(x \wedge y)^* = x^* \wedge y^*$ (Note that the star on the left hand side agrees with the star induced by the induced scalar product on $\bigwedge^2 V$.) So we can define the Lie bracket on $\left(\bigwedge^2 V\right)^*$ by pulling back the Lie bracket on $\bigwedge^2 V$ using the $*$-isomorphism. It is also clear that we can use V^* with the induced scalar product as a starting point for our constructions and end up with a Lie algebra structure on $\bigwedge^2 V^*$. As we shall see in later calculations, it will be very useful to be able to switch between the Lie algebra structures of $\bigwedge^2 V$, $\left(\bigwedge^2 V\right)^*$ and $\bigwedge^2 V^*$. Thus, we will now clarify the relations between these spaces. Consider the following diagram:

$$\begin{array}{ccccc}
\left(\bigwedge^2 V\right)^* & \xleftarrow{\;*\;} & \bigwedge^2 V & \xrightarrow{\;*\wedge *\;} & \bigwedge^2 V^* \\
\uparrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\
\mathfrak{so}(V)^* & \xleftarrow{\;*\;} & \mathfrak{so}(V) & \xrightarrow{\;\iota\;} & \mathfrak{so}(V^*)
\end{array}$$

where $\iota(F)(\alpha) = -\alpha \circ F = -F^*\alpha$. One can show easily that this diagram is commutative and that all maps in play are actually isomorphisms of Lie algebras. Moreover, one can show, that the horizontal maps are even isometric, while the vertical maps are only "isometric up to the factor $\frac{1}{2}$ ".

The following lemma summarizes some properties of ρ and $[\cdot, \cdot]$, which are of fundamental importance. In a technical sense, it is the heart of the underlying section.

Lemma 1.1.0.3. *Let $\varepsilon, \delta \in \bigwedge^2 V$, $u, v \in V$ be arbitrary and fix an orthonormal basis $e_1, e_2, ..., e_n$ of V. Then:*

1. $\sum_i \rho(\varepsilon)(e_i) \wedge e_i = -2\varepsilon$
2. $\langle \rho(\varepsilon)(u), v\rangle = \langle \varepsilon, u \wedge v\rangle$
3. $[\varepsilon^*, \delta^*] = \sum_{k=1}^n \left(\iota_{e_k}(\varepsilon^*)\right) \wedge \left(\iota_{e_k}(\delta^*)\right)$,
   where $\iota_u\omega$ is the contraction of the two form ω with the vector u, i.e. $\iota_u\omega(v) = \omega(u, v)$.

4. $(\rho(\varepsilon)(u))^* = \iota_u(\varepsilon^*)$

5. $[\varepsilon, \delta] = \sum_{k=1}^{n} (\rho(\varepsilon)(e_k)) \wedge (\rho(\delta)(e_k))$
   In particular we have

$$[e_i \wedge e_k, e_j \wedge e_k] = e_i \wedge e_j$$

   if i, j *and* k *are mutually distinct and zero otherwise.*
   Moreover, we have

$$[e_i \wedge e_j, e_k \wedge e_l] = 0$$

   if i, j, k *and* l *are mutually distinct.*

6. $\mathrm{ad}_\varepsilon = 2\rho(\varepsilon) \wedge \mathrm{id_V}$, ad *the adjoint representation of* $\mathfrak{so}(V, \langle \cdot\ , \cdot \rangle)$.
   And therefore, ad_ε *is skew-adjoint for all* $\varepsilon \in \bigwedge^2 V$

Before giving the proof, we want to say something concerning the terminology and the notation of lemma 1.1.0.3:
The adjoint representation ad of a Lie algebra $\mathfrak{g}$ is given by the map

$$x \mapsto (y \mapsto [x, y]).$$

(see the appendix B.3, there is a short introduction to Lie groups and Lie algebras.)
The wedge product "$\wedge$" between linear maps is something totally different than the wedge product of vectors! It is defined in the following way:
If F and G are endomorphisms of V, then we define an endomorphism $F \wedge G$ of $\bigwedge^2 V$, letting

$$F \wedge G(x \wedge y) := \frac{1}{2}(Fx \wedge Gy + Gx \wedge Fy).$$

For the details we refer to appendix A.2. Now we proof the lemma.

Beweis. By arguments of linearity an bilinearity it is sufficient to proof the formulas from above on generators of $\bigwedge^2 V$. These have the form $x \wedge y$, $x, y \in V$.

1. We compute

$$\begin{aligned}\sum_i \rho(x \wedge y)(e_i) \wedge e_i &= \sum_i (\langle x, e_i \rangle\, y - \langle y, e_i \rangle\, x) \wedge e_i \\ &= -\sum_i \langle x, e_i \rangle\, e_i \wedge y - x \wedge \sum_i \langle y, e_i \rangle\, e_i \\ &= -2x \wedge y\end{aligned}$$

2. This has already been done in the proof of the previous lemma 1.1.0.1.

3. On the one hand we have
$$
\begin{aligned}
[(u \wedge v)^*, (x \wedge y)^*] &= [u \wedge v, x \wedge y]^* \\
&= \left(\rho^{-1}\left([\rho(u \wedge v), \rho(x \wedge y)]\right)\right)^* .
\end{aligned}
$$
A short computation using the definition of ρ and $[\cdot,\cdot]$ shows
$$
\begin{aligned}
\left(\rho^{-1}\left([\rho(u \wedge v), \rho(x \wedge y)]\right)\right)^* = (&\langle u, x\rangle\, v \wedge y + \langle v, y\rangle\, u \wedge x \\
&- \langle u, y\rangle\, v \wedge x - \langle v, x\rangle\, u \wedge y)^*
\end{aligned}
$$
which equals
$$
\langle u, x\rangle\, v^* \wedge y^* + \langle v, y\rangle\, u^* \wedge x^* - \langle u, y\rangle\, v^* \wedge x^* - \langle v, x\rangle\, u^* \wedge y^*.
$$

And on the other hand we have
$$
\begin{aligned}
&\sum_k (\iota_{e_k}(u^* \wedge v^*)) \wedge (\iota_{e_k}(x^* \wedge y^*)) \\
&\qquad = \sum_k (\langle u, e_k\rangle\, v^* - \langle v, e_k\rangle\, u^*) \wedge (\langle x, e_k\rangle\, y^* - \langle y, e_k\rangle\, x^*) \\
&\qquad = \langle u, x\rangle\, v^* \wedge y^* + \langle v, y\rangle\, u^* \wedge x^* \\
&\qquad\;\; - \langle u, y\rangle\, v^* \wedge x^* - \langle v, x\rangle\, u^* \wedge y^*
\end{aligned}
$$
which gives the result.

4. Using 1. we compute
$$
\begin{aligned}
(\rho(u \wedge v)(x))^*\,(y) &= \langle \rho(u \wedge v)(x), y\rangle \\
&= \langle u \wedge v, x \wedge y\rangle \\
&= u^* \wedge v^*(x, y) \\
&= \iota_x(u^* \wedge v^*)(y) \\
&= \iota_x((u \wedge v)^*)(y)
\end{aligned}
$$

5. follows from 3. and 4. More precisely, we compute
$$
[\varepsilon, \delta]^* = [\varepsilon^*, \delta^*]\,,
$$
By 3. this is equal to
$$
\sum_k (\iota_{e_k}(\varepsilon^*)) \wedge (\iota_{e_k}(\delta^*))\,.
$$
Now, 4. implies that the last expression may be written as
$$
\left(\sum_{k=1}^{n} (\rho(\varepsilon)(e_k)) \wedge (\rho(\delta)(e_k))\right)^*,
$$
and the claim of the first part follows.
The rest follows from a straight forward computation.

6. This is easily established: we have

$$\begin{aligned} 2\rho(u \wedge v) \wedge id_V(x \wedge y) &= \rho(u \wedge v)(x) \wedge y + x \wedge \rho(u \wedge v)(y) \\ &= \langle u, x \rangle\, v \wedge y + \langle v, y \rangle\, u \wedge x \\ &\quad - \langle u, y \rangle\, v \wedge x - \langle v, x \rangle\, u \wedge y \end{aligned}$$

which is equal to $\mathrm{ad_{u \wedge v}(x \wedge y)}$ by the first computation in 3. The proof of the second statement follows from the formula

$$(F \wedge G)^* = F^* \wedge G^*,$$

which is valid for all endomorphisms F, G of V.

□

Remark 1.1.0.4. *Lemma 1.1.0.3.5 implies that if $U \subseteq V$ is a codimension 1 subspace, then $\bigwedge^2 U \subseteq \bigwedge^2 V$ is maximal in the following sense: Whenever $\mathfrak{h}$ is a Lie subalgebra of $\bigwedge^2 V$ with $\bigwedge^2 U \subseteq \mathfrak{h}$, then $\mathfrak{h} = \bigwedge^2 U$ or $\mathfrak{h} = \bigwedge^2 V$. To see this, observe that $U^\perp$ is generated by a single element $v \in V$ with norm equal to 1, say. Therefore, the orthogonal complement $\left(\bigwedge^2 U\right)^\perp$ of $\bigwedge^2 U$ in $\bigwedge^2 V$ is generated by elements of the form $u \wedge v$ with $u \in U$.*
Now let $\mathfrak{h}$ be a Lie subalgebra of $\bigwedge^2 V$ containing $\bigwedge^2 U$ as a proper subspace. Then $\mathfrak{h}$ has nontrivial intersection with $\left(\bigwedge^2 U\right)^\perp$, so it must contain at least one nonzero element of the form $u \wedge v$ with $u \in U$. We assume the norm of u to be 1.
Now let $\{e_1 = u, e_2, ..., e_{n-1}, e_n = v\}$ be an orthonormal basis of V. By now, We know that $e_i \wedge e_j$ lies in $\mathfrak{h}$, provided that $i, j < n$, and that $e_1 \wedge e_n$ lies in $\mathfrak{h}$. Using lemma 1.1.0.3.5 we get

$$e_k \wedge e_n = [e_k \wedge e_1, e_n \wedge e_1] \in \mathfrak{h}$$

for all $1 < k < n$ showing that $\mathfrak{h}$ actually contains the whole Lie algebra $\bigwedge^2 V$ and we are done.

1.2 Structure of $\bigwedge^2 V$

In this section we make some statements about the structure of the Lie algebra $\bigwedge^2 V$ we constructed earlier. These statements are well known. But the proofs are so easy if we chose to represent $\mathfrak{so}(n)$ as $\bigwedge^2 V$, that we will do it nevertheless. The reader who is not familiar with the basic concepts and definitions concerning Lie algebras, such as the Killing form and (semi-) simplicity, may visit the appendix B.3, where we introduce the basic material of the subject.

Proposition 1.2.0.5. *Let* κ *be the Killing form of* $\left(\bigwedge^2 V, [\cdot,\cdot]\right)$ *and* $n = \dim V \geq 2$. *Then*

$$\kappa = 2(2-n)\langle\cdot\ ,\cdot\rangle.$$

So the Killing form of $\bigwedge^2 V$ *is negative definite, if* $n > 2$ *and zero otherwise.*

Once we have established the following lemma, the proof is very easy.

Lemma 1.2.0.6. *For all* $\varepsilon \in \bigwedge^2 V$ *holds* $\|\rho(\varepsilon)\|^2 = 2\|\varepsilon\|^2$ *and* $\|\mathrm{ad}_\varepsilon\|^2 = 2(n-2)\|\varepsilon\|^2$.

Beweis. Let $\varepsilon \in \bigwedge^2 V$ be arbitrary and $\{e_i\}$ an orthonormal basis of V. Then, using lemma 1.1.0.3,

$$\|\rho(\varepsilon)\|^2 = \sum_i \langle\rho(\varepsilon)e_i, \rho(\varepsilon)e_i\rangle = \sum_i \langle\varepsilon, e_i \wedge \rho(\varepsilon)e_i\rangle = 2\|\varepsilon\|^2$$

Further, using lemma 1.1.0.3 again,

$$\begin{aligned}
\|\mathrm{ad}_\varepsilon\|^2 &= \sum_{i<j} \langle \mathrm{ad}_\varepsilon \mathrm{e_i} \wedge \mathrm{e_j}, \mathrm{ad}_\varepsilon \mathrm{e_i} \wedge \mathrm{e_j}\rangle \\
&= 2\sum_{i,j} \langle \mathrm{id} \wedge \rho(\varepsilon)(\mathrm{e_i} \wedge \mathrm{e_j}), \mathrm{id} \wedge \rho(\varepsilon)(\mathrm{e_i} \wedge \mathrm{e_j})\rangle \\
&= \frac{1}{2}\sum_{i,j} \langle e_i \wedge \rho(\varepsilon)e_j + \rho(\varepsilon)e_i \wedge e_j, e_i \wedge \rho(\varepsilon)e_j + \rho(\varepsilon)e_i \wedge e_j\rangle \\
&= \frac{1}{2}\left((n-1)\|\rho(\varepsilon)\|^2 - \|\rho(\varepsilon)\|^2 - \|\rho(\varepsilon)\|^2 + (n-1)\|\rho(\varepsilon)\|^2\right) \\
&= 2(n-2)\|\varepsilon\|^2
\end{aligned}$$

□

Proof of proposition 1.2.0.5.

$$\begin{aligned}
\kappa(\varepsilon,\delta) &= \mathrm{tr}(\mathrm{ad}_\varepsilon \circ \mathrm{ad}_\delta) \\
&= -\langle \mathrm{ad}_\varepsilon, \mathrm{ad}_\delta\rangle \\
&= -\frac{1}{2}\left(\|\mathrm{ad}_{\varepsilon+\delta}\|^2 - \|\mathrm{ad}_\varepsilon\|^2 - \|\mathrm{ad}_\delta\|^2\right) \\
&= (2-n)\left(\|\varepsilon+\delta\|^2 - \|\varepsilon\|^2 - \|\delta\|^2\right) \\
&= 2(2-n)\langle\varepsilon,\delta\rangle
\end{aligned}$$

□

Corollary 1.2.0.7. $\left(\bigwedge^2 V, [\cdot,\cdot]\right)$ *is semisimple if* $\dim V > 2$.

Beweis. Clear, since the Killing form is negative definite by proposition 1.2.0.5 □

Next we examine the question of simplicity of $\bigwedge^2 V$. It is clear that $\bigwedge^2 V$ is simple if the dimension of V is 3: First of all, we know that the Killing form is negative definite in this case, so $\bigwedge^2 V$ is semisimple by definition. Further, $\bigwedge^2 V$ has no 2-dimensional subalgebra ($[\varepsilon, \delta]$ is always perpendicular to the span of ε and δ, as one easily shows) and therefore no 2- or 1- dimensional ideal. Thus, it must be simple.
Things are different in dimension 4. In order to explain the phenomena occurring in dimension 4, we have to introduce the Hodge $*$-operator:
For the moment, let the dimension of V be arbitrary again, $\dim V = n \in \mathbb{N}$, say. Chose an orientation on V and an orienting volume form ω on $(V, \langle \cdot\ , \cdot \rangle)$, i.e. an n-form ω on V satisfying

$$\omega(e_1, ..., e_n) = 1,$$

on every positively oriented orthonormal basis $(e_1, ..., e_n)$ of $V = (V, \langle \cdot\ , \cdot \rangle)$. ω induces a pairing

$$\bigwedge^k V \times \bigwedge^{n-k} V \mapsto \mathbb{R} : (\varepsilon, \delta) \mapsto \omega(\varepsilon \wedge \delta),$$

where $0 \leq k \leq n$. This pairing is easily shown to be non-degenerate. Thus, it induces a uniquely determined linear isomorphism $* : \bigwedge^k V \to \bigwedge^{n-k} V$ with

$$\omega(\varepsilon \wedge *\delta) = \langle \varepsilon, \delta \rangle$$

for all $\varepsilon, \delta \in \bigwedge^k V$. This isomorphism is known as the Hodge $*$-operator. Note that for all $\varepsilon \in \bigwedge^k V$ with $\|\varepsilon\| = 1$. (The induced scalar product on $\bigwedge^k V$ is explained in the appendix A.2.) holds

$$\omega(\varepsilon \wedge *\varepsilon) = 1.$$

Note further that if (e_i) is any positively oriented orthonormal basis of V, then

$$*(e_1 \wedge ... \wedge e_k) = e_{k+1} \wedge ... \wedge e_n.$$

This equation determines the values of $*$ on the whole basis $\{e_{i_1} \wedge ... \wedge e_{i_k}\}_{1 \leq i_1 < ... < i_k}$ of $\bigwedge^k V$. We just have to permute the elements of the basis (e_i) appropriately and relabel the indices to get into the situation above. Now we are ready to state the theorem.

Theorem 1.2.0.8. *Let* $\dim V = 4$. *Then* $\bigwedge^2 V \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ *as a Lie algebra. More precisely, we have*

$$\bigwedge^2 V = \bigwedge^+ V \oplus \bigwedge^- V,$$

where $\bigwedge^\pm V$ *is the* ± 1*-eigenspace of the Hodge* $*$*-operator associated to an orienting volume form* $\omega \in \bigwedge^4 V^*$ *of* $(V, \langle \cdot\ , \cdot \rangle)$. *Moreover, if* $\{e_i\}$ *is any oriented orthonormal basis of* V, *then*

$$\bigwedge^+ V = \langle \{e_1 \wedge e_2 + e_3 \wedge e_4,\, e_1 \wedge e_3 + e_4 \wedge e_2,\, e_1 \wedge e_4 + e_2 \wedge e_3\} \rangle$$

and

$$\bigwedge^- V = \langle \{e_1 \wedge e_2 - e_3 \wedge e_4,\, e_1 \wedge e_3 - e_4 \wedge e_2,\, e_1 \wedge e_4 - e_2 \wedge e_3\} \rangle .$$

Beweis. Chose an orientation on V and let $\omega \in \bigwedge^4 V^*$ be an orienting volume form of $(V, \langle \cdot\,, \cdot \rangle)$. We know from above that if (e_1, e_2, e_3, e_4) is any oriented orthonormal basis of V, then $*e_1 \wedge e_2 = e_3 \wedge e_4$ and $*e_3 \wedge e_4 = e_1 \wedge e_2$. This implies $*^2 = \mathrm{id}$, so $\bigwedge^2 V$ decomposes as the direct sum $\bigwedge^+ V \oplus \bigwedge^- V$ of the ± 1-eigenspaces of the Hodge $*$operator. It is clear that

$$*(\varepsilon + *\varepsilon) = (\varepsilon + *\varepsilon,)$$

and

$$*(\varepsilon - *\varepsilon) = -(\varepsilon - *\varepsilon)$$

for all $\varepsilon \in \bigwedge^2 V$, so $\varepsilon \pm *\varepsilon$ belongs to $\bigwedge^{\pm} V$. This tells us

$$e_1 \wedge e_2 + e_3 \wedge e_4,\, e_1 \wedge e_3 + e_4 \wedge e_2,\, e_1 \wedge e_4 + e_2 \wedge e_3 \in \bigwedge^+ V$$

and

$$e_1 \wedge e_2 - e_3 \wedge e_4,\, e_1 \wedge e_3 - e_4 \wedge e_2,\, e_1 \wedge e_4 - e_2 \wedge e_3 \in \bigwedge^- V.$$

These vectors form a linearly independent set, they are even perpendicular to each other. Thus, we get

$$\bigwedge^+ V = \langle \{e_1 \wedge e_2 + e_3 \wedge e_4,\, e_1 \wedge e_3 + e_4 \wedge e_2,\, e_1 \wedge e_4 + e_2 \wedge e_3\} \rangle$$

and

$$\bigwedge^- V = \langle \{e_1 \wedge e_2 - e_3 \wedge e_4,\, e_1 \wedge e_3 - e_4 \wedge e_2,\, e_1 \wedge e_4 - e_2 \wedge e_3\} \rangle$$

for reasons of dimensions. Note that this implies also that each element of $\bigwedge^{\pm} V$ is of the form $\varepsilon \pm *\varepsilon$ with $\varepsilon \in \bigwedge^2 V$.
Straight forward computations using lemma 1.1.0.3.5 show that $\bigwedge^+ V$ and $\bigwedge^- V$ are ideals in $\bigwedge^2 V$. For example, we compute

$$[e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 + e_4 \wedge e_2] = e_1 \wedge e_4 + e_2 \wedge e_3 \in \bigwedge^+ V$$

and

$$[e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_2 - e_3 \wedge e_4] = 0.$$

The other cases are to be treated in the same way. Now we show that the ± 1-eigenspaces of the Hodge $*$-operator are isomorphic to $\mathfrak{so}(3)$. Consider the maps $\iota^{\pm} : \bigwedge^2 U \to \bigwedge^{\pm} V : \varepsilon \mapsto \frac{1}{2}(\varepsilon \pm *\varepsilon)$, where U is any 3-dimensional subspace of V. It is clear that they are isomorphisms of vector spaces. We show that ι^+ is a Lie algebra isomorphism. The proof that ι^- is a Lie algebra isomorphism is almost the same. Let (e_1, e_2, e_3, e_4) be an oriented orthonormal basis of V with $e_1, e_2, e_3 \in U$. We show, as an example, that

$$\iota^+ \left([e_1 \wedge e_2, e_1 \wedge e_3] \right) = \left[\iota^+(e_1 \wedge e_2), \iota^+(e_1 \wedge e_3) \right] .$$

The other non-trivial cases are almost the same. Lemma 1.1.0.3 gives

$$[e_1 \wedge e_2, e_1 \wedge e_3] = e_2 \wedge e_3.$$

Thus, we have

$$\iota^+ ([e_1 \wedge e_2, e_1 \wedge e_3]) = \frac{1}{2}(e_2 \wedge e_3 + e_1 \wedge e_4).$$

On the other hand, we have

$$\begin{aligned} \left[\iota^+(e_1 \wedge e_2), \iota^+(e_1 \wedge e_3)\right] &= \frac{1}{4}\left([e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 + e_4 \wedge e_2]\right) \\ &= \frac{1}{4}\left(e_2 \wedge e_3 + e_1 \wedge e_4 + e_1 \wedge e_4 + e_2 \wedge e_3\right) \\ &= \frac{1}{2}(e_2 \wedge e_3 + e_1 \wedge e_4) \end{aligned}$$

Now the claim follows from the fact that $\bigwedge^2 U$ is isomorphic to $\mathfrak{so}(3)$ as a Lie algebra.

□

It is worthwhile to mention that the splitting $\bigwedge^2 V = \bigwedge^+ V \oplus \bigwedge^- V$ is only SO(V)-invariant. For if we take an orthonormal transformation $G \in \mathrm{O(V)}$ with $\det(G) = -1$, then $G \wedge G$ interchanges $\bigwedge^+ V$ and $\bigwedge^- V$:

Lemma 1.2.0.9. *Let $G \in \mathrm{O(V)}$ with $\det(G) = -1$. Then $G \wedge G$ interchanges $\bigwedge^+ V$ and $\bigwedge^- V$.*

Beweis. Any $G \in \mathrm{O(V)}$ with $\det(G) = -1$ is a product of an odd number of reflections on 3-dimensional subspaces of V. Therefore, it is sufficient to proof the statement for such reflections. Any such reflection is of the form $Ge_1 = -e_1$, $Ge_i = e_i$ for $i = 2, 3, 4$, where $\{e_1, e_2, e_3, e_4\}$ be a positively oriented orthonormal basis of V. A sharp look at the precise description of $\bigwedge^+ V$ and $\bigwedge^- V$ in theorem 1.2.0.8 gives the result. □

Corollary 1.2.0.10. $\mathrm{SO(V)} \cong \mathrm{SO}(\bigwedge^+ \mathrm{V}) \times \mathrm{SO}(\bigwedge^- \mathrm{V})$

Beweis. If $G \in \mathrm{SO(V)}$, then $G \wedge G$ preserves $\bigwedge^+ V$ and $\bigwedge^- V$. Thus, we get a SO(V)-equivariant group homomorphism $\varphi : \mathrm{SO(V)} \to \mathrm{SO}(\bigwedge^+ \mathrm{V}) \times \mathrm{SO}(\bigwedge^- \mathrm{V})$,

$$\varphi(G) := (\pi^+ \circ G \wedge G, \pi^- \circ G \wedge G).$$

The kernel of φ is a normal Lie subgroup of SO(V). Its Lie algebra $\mathrm{Lie}(\ker \varphi)$ is given by the kernel of $\mathrm{D_{id}}\varphi$. But the differential of φ at the identity fulfills

$$\mathrm{D_{id}}\varphi(\rho(\varepsilon)) = 2(\pi^+(\mathrm{id} \wedge \rho(\varepsilon)), \pi^-((\mathrm{id} \wedge \rho(\varepsilon))) = 2(\mathrm{id} \wedge \rho(\varepsilon)),$$

so $D_{\mathrm{id}}\varphi$ is an injection, which implies that $\mathrm{Lie}(\ker\varphi)$ is 0 and hence, we get $\ker\varphi = \{\mathrm{id}\}$. Moreover, this shows that φ is an injective, immersive homomorphism of compact and connected Lie groups. So it is an isomorphism of Lie groups. We note:

$$\mathrm{SO(V)} \cong \mathrm{SO}(\textstyle\bigwedge^+ \mathrm{V}) \times \mathrm{SO}(\textstyle\bigwedge^- \mathrm{V}).$$

□

Theorem 1.2.0.11. *$\bigwedge^2 V$ is simple, if $n \geq 3$ and $n = \dim V \neq 4$.*

Beweis. The case $n = 3$ has been done above.
Strategy for the case $n > 4$:
First, we will show that if $n = \dim V > 4$ and $I \subseteq \bigwedge^2 V$ is an ideal, then for any $(n-1)$-dimensional subspace $U \subseteq V$, $\bigwedge^2 U \subseteq \bigwedge^2 V$ has nontrivial intersection with I or $I^\perp$. And in the second step we will use this fact to prove simplicity by induction over n.

- If J is any ideal in $\bigwedge^2 V$, such that $J \cap \bigwedge^2 U = \{0\}$ for an $(n-1)$-dimensional subspace U of V, then $J \oplus \bigwedge^2 U$ is a subspace of $\bigwedge^2 V$, implying that the dimension of J cannot be greater than $n-1$. So if both of the ideals I and $I^\perp$ have trivial intersection with $\bigwedge^2 U$ for some $(n-1)$-dimensional subspace of V, then $\binom{n}{2} = \dim \bigwedge^2 V = \dim I + \dim I^\perp \leq 2(n-1)$, which implies $n \leq 4$.

- Induction over n:

  The case $n = 5$:

  Let $I \neq 0$ be an ideal in $\bigwedge^2 V$ and let U be a 4-dimensional subspace of V, spanned by an orthonormal basis $\{e_1, e_2, e_3, e_4\}$. We may assume that $I' := I \cap \bigwedge^2 U \neq 0$. Then, by lemma B.3.2.2 and theorem 1.2.0.8, there are the following 3 possibilities:

  1. $I' = \bigwedge^2 U$
  2. $I' = \bigwedge^+ U$
  3. $I' = \bigwedge^- U$

  This is due to the fact that I' is an ideal in $\bigwedge^2 U$.
  Now, 1. gives $I = \bigwedge^2 V$, since $I' = \bigwedge^2 U$ is a subspace of I and $\bigwedge^2 U$ is maximal in $\bigwedge^2 V$ by remark 1.1.0.4, but no ideal at all.
  2. and 3. are done by the same arguments. Let us concentrate on 2.
  Recall that

$$\bigwedge^+ U = \langle\{e_1 \wedge e_2 + e_3 \wedge e_4,\ e_1 \wedge e_3 + e_4 \wedge e_2,\ e_1 \wedge e_4 + e_2 \wedge e_3\}\rangle$$

Let e_5 with norm equal to 1 be perpendicular to U. Then, by lemma 1.1.0.3, $[e_1 \wedge e_5, e_1 \wedge e_2 + e_3 \wedge e_4] = e_5 \wedge e_2 \in I$. In the same way we get $e_5 \wedge e_1, e_5 \wedge e_3, e_5 \wedge e_4 \in I$. Using lemma 1.1.0.3 once more, we get $\bigwedge^2 U \subseteq I$ and the claim follows as in 1.

The case $n > 5$:

If I is an ideal in $\bigwedge^2 V$, and $I \cap \bigwedge^2 U \neq 0$ for some $(n-1)$-dimensional subspace U of V, then $I \cap \bigwedge^2 U \neq 0$ is an ideal in $\bigwedge^2 U$, which implies $\bigwedge^2 U \subseteq I$ by the induction hypotheses. But $\bigwedge^2 U$ is a maximal Lie subalgebra of $\bigwedge^2 V$, so we get $I = \bigwedge^2 V$. If $I \cap \bigwedge^2 U = 0$, then we have $I^\perp \cap \bigwedge^2 U \neq 0$, which implies $I^\perp = \bigwedge^2 V$, so $I = 0$.

□

1.3 Changing the Scalar Product

Any scalar product on V is of the form $(x, y) \mapsto \langle Ax, Ay\rangle$, with $A \in GL(V)$, which is simply the pullback $A^*\langle\cdot\ ,\cdot\rangle$ of $\langle\cdot\ ,\cdot\rangle$ under A. So let us pick some $A \in GL(V)$, define $\langle\cdot\ ,\cdot\rangle_A := A^*\langle\cdot\ ,\cdot\rangle$ and construct the corresponding map ρ_A and the Lie Bracket $[\cdot,\cdot]_A$. We wish to see that $\left(\bigwedge^2 V, [\cdot,\cdot]_A\right)$ is canonically isomorphic to $\left(\bigwedge^2 V, [\cdot,\cdot]\right)$ and express $[\cdot,\cdot]_A$ in terms of $[\cdot,\cdot]$ and A in the following way:

$$[\varepsilon, \delta]_A = (A \wedge A)^{-1} \left[A \wedge A(\varepsilon), A \wedge A(\delta)\right],$$

with $\varepsilon, \delta \in \bigwedge^2 V$. This implies that the canonical isomorphism is given by $A \wedge A$. In order to prove this fact, we define a Lie algebra isomorphism Φ step by step and prove that it agrees with $A \wedge A$ after.
It is easy to see that the map

$$\varphi : \mathfrak{so}(V, \langle\cdot\ ,\cdot\rangle_A) \to \mathfrak{so}(V, \langle\cdot\ ,\cdot\rangle) : H \mapsto AHA^{-1}$$

is an isomorphism of Lie Algebras. Now we define $\Phi := \rho^{-1} \circ \varphi \circ \rho_A$, which is clearly an isomorphism of the Lie algebras $\left(\bigwedge^2 V, [\cdot,\cdot]_A\right)$ and $\left(\bigwedge^2 V, [\cdot,\cdot]\right)$. Next we prove a little lemma, which will help us to establish the desired result.

Lemma 1.3.0.12. *For all $\varepsilon \in \bigwedge^2 V$ holds*

1. $\rho_A(\varepsilon) = \rho(\varepsilon) \circ A^*A$ *and*

2. $A\rho(\varepsilon)A^* = \rho(A \wedge A(\varepsilon))$,

where A^* *the adjoint of* A *w.r.t.* $\langle \cdot\ , \cdot \rangle$

Beweis. By arguments of linearity it is sufficient to consider ε of the form $\varepsilon = x \wedge y$ with $x, y \in V$. For all $z \in V$ holds:

1.

$$\begin{aligned}\rho_A(x \wedge y)(z) &= \langle x, z \rangle_A\, y - \langle y, z \rangle_A\, x \\ &= \langle Ax, Az \rangle\, y - \langle Ay, Az \rangle\, x \\ &= \langle x, A^*Az \rangle\, y - \langle y, A^*Az \rangle\, x \\ &= (\rho(x \wedge y) \circ A^*A)(z),\end{aligned}$$

and

2.

$$\begin{aligned}A\rho(x \wedge y)A^*(z) &= A(\langle x, A^*z \rangle\, y - \langle y, A^*z \rangle\, x \\ &= \langle Ax, z \rangle\, Ay - \langle Ay, z \rangle\, Ax \\ &= \rho(A \wedge A(x \wedge y))(z)\end{aligned}$$

□

Now we show that ϕ equals $A \wedge A$. Let $\varepsilon \in \bigwedge^2 V$ be arbitrary. Using the lemma we compute

$$\begin{aligned}\Phi(\varepsilon) &= \rho^{-1}(\varphi(\rho_A(\varepsilon))) \\ &= \rho^{-1}(\varphi(\rho(\varepsilon)) \circ A^*A) \\ &= \rho^{-1}(A(\rho(\varepsilon) \circ A^*A)A^{-1}) \\ &= \rho^{-1}(A\rho(\varepsilon) \circ A^*) \\ &= \rho^{-1}(\rho(A \wedge A(\varepsilon))) \\ &= A \wedge A(\varepsilon)\end{aligned}$$

This shows that

$$A \wedge A : \left(\bigwedge^2 V, [\cdot, \cdot]_A\right) \to \left(\bigwedge^2 V, [\cdot, \cdot]\right)$$

is an isomorphism of Lie algebras and hence, we arrive at the desired formula

$$[\varepsilon, \delta]_A = (A \wedge A)^{-1} \left[A \wedge A(\varepsilon), A \wedge A(\delta)\right],$$

$\varepsilon, \delta \in \bigwedge^2 V$.

1.4 Translation to the Vector Bundle Setting

Now let M be a smooth manifold and $\pi : (E, \langle \cdot\, , \cdot \rangle) \to M \times [0, T)$, $T > 0$, be a smooth range n Euclidean vector bundle equipped with a metric connection ∇. Using the inner product on E we can turn each fiber of $\bigwedge^2 E$ into a Lie algebra isomorphic to $\mathfrak{so}(n)$. For this purpose we define $\rho_{(p,t)} : \bigwedge^2 E_{(p,t)} \to \mathfrak{so}(E_{(p,t)}, \langle \cdot\, , \cdot \rangle_{(p,t)})$ and then $[\cdot, \cdot]_{(p,t)} : \bigwedge^2 E_{(p,t)} \times \bigwedge^2 E_{(p,t)} \to \bigwedge^2 E_{(p,t)}$ in the same way as above, for any $(p, t) \in M \times [0, T)$. This gives us two bundle maps

$$\rho : \bigwedge^2 E \to \mathfrak{so}(E, \langle \cdot\, , \cdot \rangle),$$

$\mathfrak{so}(E, \langle \cdot\, , \cdot \rangle)$ the bundle of fiberwise skew-adjoint bundle maps of E, and

$$[\cdot, \cdot] : \bigwedge^2 E \times \bigwedge^2 E \to \bigwedge^2 E.$$

Since $\langle \cdot\, , \cdot \rangle$ is smooth, these maps will be smooth as well. But we can say a little more about these maps. They are actually parallel with respect to the corresponding induced connections, as we will show in the proposition below. See the appendix B.1 to get a rough introduction to connections on vector bundles. After that one question is still left open: What about the induced connection on $\mathfrak{so}(V, \langle \cdot\, , \cdot \rangle)$? The answer is the following. The metric connection ∇ on E induces a metric connection on the space of bundle endomorphims End(E), which is also denoted by ∇. Parallel transport w.r.t. metric connections preserves skew-adjointness, as one easily shows. Thus, $\mathfrak{so}(V, \langle \cdot\, , \cdot \rangle)$ is a parallel subbundle of End(V), which implies that it is invariant under covariant differentiation. Thus, we get a metric connection on $\mathfrak{so}(V, \langle \cdot\, , \cdot \rangle)$ by restricting ∇ to $\mathfrak{so}(V, \langle \cdot\, , \cdot \rangle)$.

Proposition 1.4.0.13. *Consider ρ as a section of $\bigwedge^2 E^* \otimes \mathfrak{so}(E, \langle \cdot\, , \cdot \rangle)$ and $[\cdot, \cdot]$ as a section of $(\bigwedge^2 E \oplus \bigwedge^2 E)^* \otimes \bigwedge^2 E$. Then ρ and $[\cdot, \cdot]$ are parallel w.r.t. the induced connections on these bundles.*

Beweis. We show $\nabla \rho = 0$ first. Pick a smooth section s of $\bigwedge^2 E$, a smooth section e of E, $(p, t) \in M \times [0, T)$, a direction $x \in T_{(p,t)} M \times [0, T)$ and compute $(\nabla_x \rho)(s, e)$ in (p, t): Since ∇ is a local operator and each section s of $\bigwedge^2 E$ is (at least locally) a linear combination of wedge-products of sections of E, we can assume $s = s_1 \wedge s_2$, s_1 and s_2 sections of E. Moreover, since ρ is tensorial in s_1, s_2 and e, we can assume $\nabla s_1 = \nabla s_2 = \nabla e = 0$ in (p, t). Let's go:

$$\begin{aligned}(\nabla_x \rho)(s_1 \wedge s_2, e) &= \nabla_x \rho(s_1 \wedge s_2)(e) = \nabla_x (\langle s_1, e \rangle\, s_2 - \langle s_2, e \rangle\, s_1) \\ &= 0\end{aligned}$$

The second statement is now obvious, because $\nabla \rho = 0$ implies $\nabla \rho^{-1} = 0$ and building fiberwise commutators is clearly a tensorial and parallel operation w.r.t. any connection coming from E.

□

Kapitel 2

The #-Product

Besides multiplication of endomorphisms of $\bigwedge^2 V$, there is another $O(V)$-equivariant binary operation on this space, the so called #-product (sharp product), which arises as a summand in the reaction term of the evolution equation of curvature operators under the Ricci flow. In this section we define this operation in several ways and prove equivalence of these definitions. Then we discuss some of its properties. As we shall see, there is a deep structural relation between the Lie sub algebras $\mathfrak{h}$ of $\bigwedge^2 V$ and the #-product. Roughly speaking, it turns out that the Killing form of $\mathfrak{h}$ is given by $\pi\#\pi$, where π is the orthogonal projection onto $\mathfrak{h}$. Next we clarify the relations between #-products arising from different choices of the underlying Euclidean structure of V. As in the previous chapter, we explain shortly, how to define #-products on bundles and say a few words about their geometric properties. At the end of this section we present a sharp estimate on the eigenvalues of $\mathcal{R}\#\mathcal{R}$ in terms of the eigenvalues of a given self adjoint endomorphism $\mathcal{R}$ of $\bigwedge^2 V$.

2.1 Definition and Basic Properties

The bilinear skew-symmetric map $[\cdot,\cdot] : \bigwedge^2 V \times \bigwedge^2 V \to \bigwedge^2 V : (\varepsilon, \delta) \mapsto [\varepsilon, \delta]$ induces a linear map $\alpha : \bigwedge^2(\bigwedge^2 V) \to \bigwedge^2 V$ by the universal property of exterior powers. The scalar product $\langle \cdot\,, \cdot\rangle$ on $\bigwedge^2 V$ induces a scalar product on $\bigwedge^2(\bigwedge^2 V) \to \bigwedge^2 V$, which will also be denoted by $\langle \cdot\,, \cdot\rangle$. Let α^* be the adjoint of α w.r.t. these scalar products. For endomorphisms $\mathcal{R}$ and $\mathcal{S}$ of $\bigwedge^2 V$ we define the #-product $\mathcal{R}\#\mathcal{S}$ of $\mathcal{R}$ and $\mathcal{S}$ by

$$\mathcal{R}\#\mathcal{S} := \alpha \circ \mathcal{R} \wedge \mathcal{S} \circ \alpha^*.$$

This definition of the #-product was first given by C. Böm and B. Wilking in [5]. Before listing some obvious properties of #, we would like to say something about the spaces in play. $\bigwedge^2(\bigwedge^2 V)$ must not be confused with $\bigwedge^4 V$. They are totally different spaces. $\bigwedge^2(\bigwedge^2 V)$ has dimension $\binom{\binom{n}{2}}{2}$ and

$\bigwedge^4 V$ has dimension $\binom{n}{4}$. They can only be isomorphic, if the dimension of V is less than 2. So we have to be careful with notation. For, if ε and δ are elements of $\bigwedge^2 V$, then we may view $\varepsilon \wedge \delta$ as an element of $\bigwedge^2(\bigwedge^2 V)$ or as an element of $\bigwedge^4 V$, just as we like. We want $\varepsilon \wedge \delta$ to belong to $\bigwedge^2(\bigwedge^2 V)$. By times may wish to represent elements $\varepsilon \in \bigwedge^2 V$ as sums of elements of the form $x \wedge y$ with x and $y \in V$. If we want to indicate that the wedge product of $x \wedge y$ and $u \wedge v$, $x, y, u, v \in V$, belongs to $\bigwedge^2(\bigwedge^2 V)$ we put braces around them:

$$(x \wedge y) \wedge (u \wedge v) \in \bigwedge^2(\bigwedge^2 V),$$

while

$$x \wedge y \wedge u \wedge v \in \bigwedge^4 V.$$

Further, we would like to mention that $\bigwedge^2(\bigwedge^2 V)$ is generated by elements of the form $(x \wedge y) \wedge (u \wedge v)$, $x, y, u, v \in V$.

Lemma 2.1.0.14. *1. # is bilinear and symmetric*

2. # is $O(V)$-equivariant

3. If $\mathcal{R}$ and $\mathcal{S}$ are both self-adjoint or skew-adjoint, $\mathcal{R}\#\mathcal{S}$ is self-adjoint.

4. If $\mathcal{R}$ is self-adjoint and $\mathcal{R}$ is (semi-) definite, then $\mathcal{R}\#\mathcal{R}$ is positive (semi-) definite. ((Semi-) definiteness of a self-adjoint endomorphism means that the associated quadratic form is (semi-) definite)

Beweis. 1. The map $(\mathcal{R}, \mathcal{S}) \mapsto \mathcal{R} \wedge \mathcal{S}$ is bilinear and symmetric and therefore # has also these properties.

2. $O(V)$ acts on $\bigwedge^2 V$ via $(G, \varepsilon) \mapsto G \wedge G(\varepsilon)$. The Lie bracket on $\bigwedge^2 V$ is $O(V)$-equivariant, since ρ and ρ^{-1} are $O(V)$-equivariant. Thus, the maps α and α^* are $O(V)$-equivariant either. Since $(\mathcal{R}, \mathcal{S}) \mapsto \mathcal{R} \wedge \mathcal{S}$ is even $O(\bigwedge^2 V)$-equivariant, the claim follows.

3. Functoriality of $\wedge$ and $*$ implies

$$\begin{aligned}(\mathcal{R}\#\mathcal{S})^* &= (\alpha \circ \mathcal{R} \wedge \mathcal{S} \circ \alpha^*)^* \\ &= (\alpha^*)^* \circ \mathcal{R}^* \wedge \mathcal{S}^* \circ \alpha^* \\ &= \alpha \circ \mathcal{R} \wedge \mathcal{S} \circ \alpha^* \\ &= \mathcal{R}\#\mathcal{S}\end{aligned}$$

4. If ε and δ are linearly independent eigenvectors of $\mathcal{R}$ corresponding to eigenvalues λ and μ, then $\varepsilon \wedge \delta$ is an eigenvector of $\mathcal{R} \wedge \mathcal{R}$ with eigenvalue $\lambda\mu$. This implies that $\mathcal{R} \wedge \mathcal{R}$ is positive (semi-) definite,

whenever $\mathcal{R}$ is self-adjoint and (semi-) definite, since every eigenvalue of $\mathcal{R} \wedge \mathcal{R}$ has this form in this case. Now the claim follows from

$$\langle \mathcal{R}\#\mathcal{R}\varepsilon, \varepsilon\rangle = \langle \mathcal{R} \wedge \mathcal{R}\alpha^*\varepsilon, \alpha^*\varepsilon\rangle$$

□

Proposition 2.1.0.15. *Let $\mathcal{R}$ and $\mathcal{S}$ be endomorphisms of $\bigwedge^2 V$ and $x, y \in V$ be arbitrary. Fix an orthonormal basis $\{e_i\}$ of V. Then*

1. $\alpha^*(x \wedge y) = \sum_i (x \wedge e_i) \wedge (y \wedge e_i)$
2. $\mathcal{R}\#\mathcal{S}(x \wedge y) = \frac{1}{2}\sum_i [\mathcal{R}(x \wedge e_i), \mathcal{S}(y \wedge e_i)] + [\mathcal{S}(x \wedge e_i), \mathcal{R}(y \wedge e_i)]$

Beweis. 1. Let $a, b, c, d \in V$ be arbitrary.Then, using 1.1.0.3, we compute

$$\begin{aligned}
&\langle \alpha^*(x \wedge y), (a \wedge b) \wedge (c \wedge d)\rangle = \langle x \wedge y, [a \wedge b, c \wedge d]\rangle \\
&= \sum_k \langle x \wedge y, (\rho(a \wedge b)(e_k)) \wedge (\rho(c \wedge d)(e_k))\rangle \\
&= \sum_k \det \begin{pmatrix} \langle x, \rho(a \wedge b)(e_k)\rangle & \langle x, \rho(c \wedge d)(e_k)\rangle \\ \langle y, \rho(a \wedge b)(e_k)\rangle & \langle y, \rho(c \wedge d)(e_k)\rangle \end{pmatrix} \\
&= \sum_k \det \begin{pmatrix} \langle x \wedge e_k, a \wedge b\rangle & \langle x \wedge e_k, c \wedge d\rangle \\ \langle y \wedge e_k, a \wedge b\rangle & \langle y \wedge e_k, c \wedge d\rangle \end{pmatrix} \\
&= \sum_k \langle (x \wedge e_k) \wedge (y \wedge e_k), (a \wedge b) \wedge (c \wedge d)\rangle \\
&= \left\langle \sum_k (x \wedge e_k) \wedge (y \wedge e_k), (a \wedge b) \wedge (c \wedge d) \right\rangle
\end{aligned}$$

2. Let x and y be elements of V. We have

$$\mathcal{R}\#\mathcal{S}(x \wedge y) = \alpha \circ \mathcal{R} \wedge \mathcal{S} \circ \alpha^*(x \wedge y)$$

by definition of the #-product. Using 1., we compute

$$\begin{aligned}
\alpha \circ \mathcal{R} \wedge \mathcal{S} \circ \alpha^*(x \wedge y) &= \sum_i \alpha \circ \mathcal{R} \wedge \mathcal{S}(x \wedge e_i) \wedge (y \wedge e_i) \\
&= \frac{1}{2}\alpha \left(\mathcal{R}(x \wedge e_i) \wedge \mathcal{S}(y \wedge e_i) + \mathcal{S}(x \wedge e_i) \wedge \mathcal{R}(y \wedge e_i)\right) \\
&= \frac{1}{2}\sum_i \left([\mathcal{R}(x \wedge e_i), \mathcal{S}(y \wedge e_i)] + [\mathcal{S}(x \wedge e_i), \mathcal{R}(y \wedge e_i)]\right)
\end{aligned}$$

□

Next we summarize some properties of the maps α and α^*:

Proposition 2.1.0.16. *1.* $\alpha \circ \alpha^* = (n-2)id$,

2. $\langle \alpha^*\varepsilon, \alpha^*\delta \rangle = (n-2) \langle \varepsilon, \delta \rangle$ *for all* $\varepsilon, \delta \in \bigwedge^2 V$ *and*

3. $\alpha^* \circ \alpha = (n-2)\pi$, *where* π *is the orthogonal projection onto the image of* α^*.

4. In any orthonormal basis $\{\varepsilon_i\}$ *of* $\bigwedge^2 V$ *and for every* $\varepsilon \in \bigwedge^2 V$ *holds*

$$\alpha^*\varepsilon = \frac{1}{2} \sum_i [\varepsilon_i, \varepsilon] \wedge \varepsilon_i$$

5.

$$\alpha^*\varepsilon = -\frac{1}{2} \sum_i \alpha(\varepsilon \wedge \varepsilon_i) \wedge \varepsilon_i$$

Beweis. 1. Let $\{e_i\}$ be an orthonormal basis of V. The $\{e_i \wedge e_j\}_{i<j}$ is an orthonormal basis of $\bigwedge^2 V$. Lemma 1.1.0.3 tells us that $[e_i \wedge e_k, e_j \wedge e_k] = e_i \wedge e_j$ if i, j and k are mutually distinct and zero otherwise. Using this, we compute

$$\alpha \circ \alpha^*(e_i \wedge e_j) = \sum_k [e_i \wedge e_k, e_j \wedge e_k] = \sum_{k \neq i,j} e_i \wedge e_j = (n-2) e_i \wedge e_j.$$

2. This is an immediate consequence of 1.

3. This follows using 1. and 2.

4. We compute

$$\begin{aligned} 2\alpha^*\varepsilon &= \sum_{i,j} \langle \alpha^*\varepsilon, \varepsilon_i \wedge \varepsilon_j \rangle \varepsilon_i \wedge \varepsilon_j = \sum_{i,j} \langle \varepsilon, [\varepsilon_i, \varepsilon_j] \rangle \varepsilon_i \wedge \varepsilon_j \\ &= \sum_{i,j} \langle \varepsilon_j, [\varepsilon, \varepsilon_i] \rangle \varepsilon_i \wedge \varepsilon_j = \sum_{i,j} \varepsilon_i \wedge \langle \varepsilon_j, [\varepsilon, \varepsilon_i] \rangle \varepsilon_j \\ &= \sum_i \varepsilon_i \wedge [\varepsilon, \varepsilon_i] = \sum_i [\varepsilon_i, \varepsilon] \wedge \varepsilon_i \end{aligned}$$

5. 5. is just a reformulation of 4.

□

Corollary 2.1.0.17. *We have*

$$\mathrm{id} \# \mathrm{id} = (\mathrm{n} - 2)\mathrm{id}$$

Beweis. This is just a reformulation of proposition 2.1.0.16.1. □

Proposition 2.1.0.18. α^* *is a homomorphism of Lie algebras.*

Beweis. The map $\iota : O(V) \to O(\bigwedge^2 V) : G \mapsto G \wedge G$ is a homomorphism of Lie groups. Its differential at the identity of $O(V)$, $D\iota_{id}$, is a Lie algebra homomorphism. Consider the following diagram:

$$\begin{array}{ccc} \bigwedge^2 V & \xrightarrow{\alpha^*} & \bigwedge^2 \left(\bigwedge^2 V\right) \\ \big\downarrow \rho & & \big\downarrow \rho \\ \mathfrak{so}(V) & \xrightarrow{D\iota_{id}} & \mathfrak{so}(\bigwedge^2 V) \end{array}$$

We are done if we can show that this diagram commutes, since the vertical maps are Lie algebra isomorphisms.
It is easy to see that

$$D\iota_{id} H = 2id \wedge H, H \in T_{id}O(V).$$

Using lemma 1.1.0.3 we get

$$D\iota_{id} \circ \rho = \mathrm{ad}.$$

On the other hand

$$\begin{aligned} \rho \circ \alpha^*(x \wedge y) &= \sum_k \rho((x \wedge e_k) \wedge (y \wedge e_k)) \\ &= \sum_k (x \wedge e_k)^* \otimes (y \wedge e_k) - (y \wedge e_k)^* \otimes (x \wedge e_k) \\ &= 2\sum_k x^* \otimes y \wedge e_k^* \otimes e_k - y^* \otimes x \wedge e_k^* \otimes e_k \\ &= 2\rho(x \wedge y) \wedge id \end{aligned}$$

In the third step we used the identity

$$(x \wedge y)^* \otimes (a \wedge b) = 2(x^* \otimes a) \wedge (y^* \otimes b),$$

which follows from a straight forward computation.

□

2.2 Alternative Definitions of the #-Product

There is an alternative definition of the #-product, which was first given by R. S. Hamilton in [11]. Let $\alpha, \beta \in \bigwedge^2 V^*$ and $a, b \in \bigwedge^2 V$. Then $\alpha \otimes a$ and $\beta \otimes b$ are endomorphisms of $\bigwedge^2 V$. Now define

$$\alpha \otimes a \#_2 \beta \otimes b := \frac{1}{2} [\alpha, \beta] \otimes [a, b].$$

Now extend $\#_2$ bilinearly to the whole vector space of endomorphisms of $\bigwedge^2 V$. This is possible by the universal property since the map

$$\bigwedge^2 V^* \times \bigwedge^2 V \times \bigwedge^2 V^* \times \bigwedge^2 V \to \bigwedge^2 V^* \times \bigwedge^2 V$$

$$(\alpha, a, \beta, b) \mapsto \frac{1}{2}\,[\alpha, \beta] \otimes [a, b]$$

is 4-linear and induces $\#_2$. Note that up to multiplication with $\frac{1}{2}$ $\#_2$ is just the algebra tensor product of the Lie Brackets on $\bigwedge^2 V^*$ and $\bigwedge^2 V$,

$$\#_2 = \frac{1}{2} \otimes_{Alg} .$$

See the appendix A.3 for the definition of algebra tensor products. The following observation is due to C. Böm and B. Wilking [5]:

Lemma 2.2.0.19. *Let $\mathcal{R}$ and $\mathcal{S}$ be endomorphisms of $\bigwedge^2 V$. Then*

$$\mathcal{R}\#\mathcal{S} = \frac{1}{2}\mathcal{R} \otimes_{Alg} \mathcal{S} = \mathcal{R}\#_2\mathcal{S}.$$

Beweis. It is sufficient to consider range 1 endomorphisms of $\bigwedge^2 V$ as above. We will make use of lemma 1.1.0.3 and proposition 2.1.0.15. On the one hand we have

$$\begin{aligned}(\alpha \otimes a) \otimes_{Alg} (\beta \otimes b)(x \wedge y) &= \frac{1}{2}\,[\alpha, \beta] \otimes [a, b]\,(x \wedge y)\\ &= \frac{1}{2} \sum_k \left(\iota_{e_k}\alpha\right) \wedge \left(\iota_{e_k}\beta\right)(x, y) \cdot [a, b]\end{aligned}$$

On the other hand

$$\begin{aligned}&(\alpha \otimes a)\#(\beta \otimes b)(x \wedge y)\\ &= \frac{1}{2} \sum_k \left[\alpha \otimes a(x \wedge e_k), \beta \otimes b(y \wedge e_k)\right] + \left[\beta \otimes b(x \wedge e_k), \alpha \otimes a(y \wedge e_k)\right]\\ &= \frac{1}{2} \sum_k (\alpha(x \wedge e_k)\beta(y \wedge e_k) - \beta(x \wedge e_k)\alpha(y \wedge e_k)) \cdot [a, b]\\ &= \frac{1}{2} \sum_k \left(\iota_{e_k}\alpha(x)\iota_{e_k}\beta(y) - \iota_{e_k}\beta(x)\iota_{e_k}\alpha(y)\right) \cdot [a, b]\\ &= \frac{1}{2} \sum_k \left(\iota_{e_k}\alpha\right) \wedge \left(\iota_{e_k}\beta\right)(x, y) \cdot [a, b]\end{aligned}$$

□

Corollary 2.2.0.20. *We have $\mathcal{R}\#\mathcal{S} = \frac{1}{2}\sum_{i,j}\,[\varepsilon_i, \varepsilon_j]^* \otimes [\mathcal{R}\varepsilon_i, \mathcal{S}\varepsilon_j]$ with respect to any given orthonormal basis $\{\varepsilon_i\}$ of $\bigwedge^2 V$.*

Beweis. Evolve $\mathcal{R}$ and $\mathcal{S}$ in a basis $\{\varepsilon_i\}$ of $\bigwedge^2 V$,

$$\mathcal{R} = \sum_{i,j} r_i^j \varepsilon_i^* \otimes \varepsilon_j,$$

and

$$\mathcal{S} = \sum_{i,j} s_i^j \varepsilon_i^* \otimes \varepsilon_j.$$

Then, by lemma 2.2.0.19,

$$\begin{aligned}
\mathcal{R}\#\mathcal{S} &= \frac{1}{2}\mathcal{R} \otimes_{Alg} \mathcal{S} \\
&= \frac{1}{2} \sum_{i,j,k,l} r_i^j s_k^l \, [\varepsilon_i^*, \varepsilon_k^*] \otimes [\varepsilon_j, \varepsilon_l] \\
&= \frac{1}{2} \sum_{i,j,k,l} [\varepsilon_i, \varepsilon_k]^* \otimes \left[r_i^j \varepsilon_j, s_k^l \varepsilon_l\right] \\
&= \frac{1}{2} \sum_{i,k} [\varepsilon_i, \varepsilon_k]^* \otimes [\mathcal{R}\varepsilon_i, \mathcal{S}\varepsilon_k]
\end{aligned}$$

and the claim follows replacing k by j. □

Remark 2.2.0.21. Corollary 2.2.0.20 may be viewed as a third possible definition of the #-product. Each of these definitions has it's own advantages in different contexts and we will use the possibility to switch between these definitions extensively in coming calculations. Further, if we replace the $*$ coming from the scalar product by the usual one, the we get

$$\mathcal{R}\#\mathcal{S} = \frac{1}{2} \sum_{i,j} \left[\varepsilon_i^*, \varepsilon_j^*\right] \otimes [\mathcal{R}\varepsilon_i, \mathcal{S}\varepsilon_j]$$

for any basis $\{\varepsilon_i\}$ of $\bigwedge^2 V$. Here, $\{\varepsilon_i^*\}$ is the dual basis of $\{\varepsilon_i\}$.

2.3 Relations to Lie Subalgebras of $\bigwedge^2 V$

The reader, who is not familiar with the basic concepts of the theory of compact (or reductive) and semisimple Lie-Algebras, is invited to visit the appendix B.3. We start with a corollary of lemma 2.2.0.19. The formula presented was first published by C. Böhm and B. Wilking in [5].

Corollary 2.3.0.22. *For any self-adjoint endomorphism $\mathcal{R}$ of $\bigwedge^2 V$ and any $\varepsilon \in \bigwedge^2 V$ holds*

$$\langle \mathcal{R}\#\mathcal{R}\varepsilon, \varepsilon \rangle = -\frac{1}{2} \mathrm{tr}\left((\mathrm{ad}_\varepsilon \circ \mathcal{R})^2\right)$$

Beweis. Let $\{\varepsilon_i\}$ be an orthonormal eigenbasis of $\mathcal{R}$, $\mathcal{R}\varepsilon_i = \lambda_i\varepsilon_i$, $\lambda_i \in \mathbb{R}$, and $\varepsilon \in \bigwedge^2 V$. Then

$$\begin{aligned}
2\langle \mathcal{R}\#\mathcal{R}\varepsilon, \varepsilon\rangle &= \sum_{i,j} \lambda_i\lambda_j \langle [\varepsilon_i, \varepsilon_j], \varepsilon\rangle \langle [\varepsilon_i, \varepsilon_j], \varepsilon\rangle \\
&= \sum_{i,j} \lambda_i\lambda_j \langle [\varepsilon_j, \varepsilon], \varepsilon_i\rangle \langle [\varepsilon_i, \varepsilon_j], \varepsilon\rangle \\
&= \sum_{i,j} \lambda_j \langle [\varepsilon_j, \varepsilon], \lambda_i\varepsilon_i\rangle \langle [\varepsilon_i, \varepsilon_j], \varepsilon\rangle \\
&= \sum_{i,j} \lambda_j \langle \mathcal{R}[\varepsilon_j, \varepsilon], \varepsilon_i\rangle \langle [\varepsilon_i, \varepsilon_j], \varepsilon\rangle \\
&= \sum_{i} \lambda_j \langle [\mathcal{R}[\varepsilon_j, \varepsilon], \varepsilon_j], \varepsilon\rangle \\
&= -\sum_{i} \lambda_j \langle [\mathcal{R}[\varepsilon_j, \varepsilon], \varepsilon], \varepsilon_j\rangle \\
&= -\sum_{i} \langle \mathcal{R}[\mathcal{R}[\varepsilon_j, \varepsilon], \varepsilon], \varepsilon_j\rangle \\
&= -\mathrm{tr}\left((\mathrm{ad}_\varepsilon \circ \mathcal{R})^2\right)
\end{aligned}$$

□

Corollary 2.3.0.23. *If $\mathfrak{h}$ is a Lie subalgebra of $\bigwedge^2 V$ with Killing form $\kappa_{\mathfrak{h}}$ and $\pi : \bigwedge^2 V \to \bigwedge^2 V$ is the orthogonal projection onto $\mathfrak{h}$, then, for all $\varepsilon, \delta \in \mathfrak{h}$,*

$$\kappa_{\mathfrak{h}}(\varepsilon, \delta) = -2\langle \pi\#\pi\varepsilon, \delta\rangle .$$

Beweis. We have that π commutes with ad_ε for any $\varepsilon \in \mathfrak{h}$ and that π maps $\mathfrak{h}^\perp$ to 0, so

$$\kappa_{\mathfrak{h}}(\varepsilon, \varepsilon) = \mathrm{tr}_{\mathfrak{h}}(\mathrm{ad}_\varepsilon^2) = \mathrm{tr}_{\mathfrak{h}}((\mathrm{ad}_\varepsilon \circ \pi)^2) = \mathrm{tr}_{\bigwedge^2 \mathrm{V}}((\mathrm{ad}_\varepsilon \circ \pi)^2) = -2\langle \pi\#\pi\varepsilon, \varepsilon\rangle$$

This shows that $\kappa_{\mathfrak{h}}$ is negative semidefinite. So it is compact by definition.

□

2.4 Useful Properties of

The following proposition is due to G. Huisken [17]:

Proposition 2.4.0.24. *The trilinear map $\tau_0 : End(\bigwedge^2 V)^3 \to \mathbb{R} : (\mathcal{R}, \mathcal{S}, \mathcal{T}) \mapsto \langle \mathcal{R}\#\mathcal{S}, \mathcal{T}\rangle$ is fully symmetric.*

Beweis. τ_0 is symmetric in $\mathcal{R}$ and $\mathcal{S}$. So we are done if we show that it is symmetric in $\mathcal{S}$ and $\mathcal{T}$. Let $\{\varepsilon_i\}$ be an orthonormal basis of $\bigwedge^2 V$

$$\begin{aligned}
\tau_0(\mathcal{R},\mathcal{S},\mathcal{T}) = \langle \mathcal{R}\#\mathcal{S},\mathcal{T}\rangle &= \sum_k \langle \mathcal{R}\#\mathcal{S}\varepsilon_k, \mathcal{T}\varepsilon_k\rangle \\
&= \sum_k \left\langle \frac{1}{2}\sum_{i,j} [\varepsilon_i,\varepsilon_j]^* \otimes [\mathcal{R}\varepsilon_i,\mathcal{S}\varepsilon_j]\,\varepsilon_k, \mathcal{T}\varepsilon_k \right\rangle \\
&= \frac{1}{2}\sum_{i,j,k} \langle\langle [\varepsilon_i,\varepsilon_j], \varepsilon_k\rangle\, [\mathcal{R}\varepsilon_i,\mathcal{S}\varepsilon_j]\,\varepsilon_k, \mathcal{T}\varepsilon_k\rangle \\
&= \frac{1}{2}\sum_{i,j,k} \langle [\varepsilon_i,\varepsilon_j], \varepsilon_k\rangle \langle [\mathcal{R}\varepsilon_i,\mathcal{S}\varepsilon_j]\,\varepsilon_k, \mathcal{T}\varepsilon_k\rangle \\
&= \frac{1}{2}\sum_{i,j,k} \langle [\varepsilon_i,\varepsilon_k], \varepsilon_j\rangle \langle [\mathcal{R}\varepsilon_i,\mathcal{T}\varepsilon_k]\,\varepsilon_k, \mathcal{S}\varepsilon_j\rangle
\end{aligned}$$

In the last step we used that ad_ε is skew-adjoint for every $\varepsilon \in \bigwedge^2 V$. It is clear that the last expression in the above calculation equals $\tau_0(\mathcal{R},\mathcal{T},\mathcal{S})$.

□

Corollary 2.4.0.25. *The linear map $\mathcal{R} \mapsto \mathcal{R}\#\mathcal{S}$ is self-adjoint for all endomorphisms $\mathcal{S}$ of $\bigwedge^2 V$ and $\mathrm{tr}(\mathcal{R}\#\mathcal{S}) = \langle \mathrm{id}\#\mathcal{R},\mathcal{S}\rangle$.*

Beweis. The first statement is clear. To prove the second statement, simply recall that $\mathrm{tr}(\mathcal{R}) = \langle \mathcal{R}, \mathrm{id}\rangle$.

□

2.5 Changing the Scalar Product

According to section 1.3 we have that the Lie bracket $[\cdot,\cdot]_A$ on $\bigwedge^2 V$, which is associated to the scalar product $A^*\langle\cdot\,,\cdot\rangle$ with $A \in \mathrm{GL(V)}$, is given by

$$[\varepsilon,\delta]_A = (A\wedge A)^{-1}\,[A\wedge A\varepsilon, A\wedge A\delta]\,.$$

This leads directly to the formula

$$\mathcal{R}\#_A\mathcal{S} = (A\wedge A)^{-1}\circ(A\wedge A\circ\mathcal{R}\circ(A\wedge A)^*)\#(A\wedge A\circ\mathcal{S}\circ(A\wedge A)^*)\circ A\wedge A,$$

for endomorphisms $\mathcal{R}$ and $\mathcal{S}$ of $\bigwedge^2 V$, where $\#_A$ is the sharp product w.r.t. $\langle\cdot\,,\cdot\rangle_A$ and $(A\wedge A)^*$ is the adjoint of $A\wedge A$ w.r.t. $\langle\cdot\,,\cdot\rangle$.

2.6 Translation to the Bundle Setting

As in subsection 1.4, let M be a smooth manifold (possibly with boundary) and $\pi : \mathrm{E} \to \mathrm{M}\times[0,\mathrm{T})$, $T > 0$, be a n-dimensional smooth Euclidean vector

bundle, equipped with a metric connection ∇. The inner product on E will be denoted by $\langle \cdot\ , \cdot \rangle$.
For any two bundle endomorphisms $\mathcal{R}$ and $\mathcal{S}$ of $\bigwedge^2 \mathrm{E}$ we define

$$\mathcal{R}\#\mathcal{S}(p,t) := \mathcal{R}_{(p,t)}\#_{(p,t)}\mathcal{S}_{(p,t)},$$

where (p,t) runs through $M \times [0,T)$ and $\#_{(p,t)}$ is the sharp product on $\mathrm{End}\left(\bigwedge^2 \mathrm{E}_{(\mathrm{p,t})}\right)$ w.r.t. $\langle \cdot\ , \cdot \rangle_{(p,t)}$. It is clear that $\mathcal{R}\#\mathcal{S}$ is smooth, provided that $\mathcal{R}$ and $\mathcal{S}$ are smooth. We may view $\#$ as a smooth section of the bundle $\mathrm{End}\left(\bigwedge^2 \mathrm{E}\right)^* \otimes \mathrm{End}\left(\bigwedge^2 \mathrm{E}\right)^* \otimes \mathrm{End}\left(\bigwedge^2 \mathrm{E}\right)$. From this perspective $\#$ turns out to be parallel w.r.t. the induced metric connection ∇ on the bundle from above. We know from section 1.3 that the Lie bracket on $\bigwedge^2 \mathrm{E}$, and therefore also the induced Lie bracket on $\bigwedge^2 \mathrm{E}^*$, is parallel (w.r.t the induced connection on $\bigwedge^2 \mathrm{E}$). Since the $\#$-product is simply the algebra tensor product of the Lie brackets in play, it is clearly parallel.

2.7 A Sharp Estimate on the Eigenvalues of $\mathcal{R}\#\mathcal{R}$ in terms of the Eigenvalues of $\mathcal{R}$

Expressing the eigenvalues of $\mathcal{R}\#\mathcal{R}$ in terms of the eigenvalues of $\mathcal{R}$ is a very hard task. According to corollary 2.2.0.20 we have

$$\mathcal{R}\#\mathcal{R} = \frac{1}{2}\sum_{i,j} \lambda_i\lambda_j \,[\varepsilon_i,\varepsilon_j]^* \otimes [\varepsilon_i,\varepsilon_j] = \frac{1}{2}\sum_{i,j,k,l} \lambda_i\lambda_j c_{ijk}c_{ijl}\varepsilon_k^* \otimes \varepsilon_l,$$

if $\{\varepsilon_i\}$ is an orthonormal eigenbasis of $\mathcal{R}$, $\mathcal{R}\varepsilon_i = \lambda_i\varepsilon_i$, and the c_{ijk} are the structure constants of $\bigwedge^2 V$ w.r.t. the basis $\{\varepsilon_i\}$. So this task is at least as hard as determining the structure constants of a generic orthonormal basis of $\bigwedge^2 V$. Even if we assume $\mathcal{R}\#\mathcal{R}$ to be diagonal w.r.t. this basis, we have no chance to compute the eigenvalues of $\mathcal{R}\#\mathcal{R}$ from the knowledge of the eigenvalues of $\mathcal{R}$, for we have

$$\mathcal{R}\#\mathcal{R}\varepsilon_m = \left(\frac{1}{2}\sum_{i,j,m} \lambda_i\lambda_j c_{ijm}c_{ijm}\right)\varepsilon_m$$

in this case and we still have the problem of computing the structure constants. And things become even worse, if we ask for the eigenvectors of $\mathcal{R}\#\mathcal{R}$. However, if we restrict ourselves to self-adjoint linear maps $\mathcal{R}$, which are diagonal w.r.t. orthonormal bases of the form $\{e_i \wedge e_j\}_{i<j}$, where $\{e_i\}$ an orthonormal basis of V, then things become very easy:

Proposition 2.7.0.26. *Let $\mathcal{R}$ be a self-adjoint endomorphism of $\bigwedge^2 V$ and suppose that $\mathcal{R}$ is diagonal w.r.t. an orthonormal basis $\{e_i \wedge e_j\}_{i<j}$, where*

$\{e_i\}$ an orthonormal basis of V, i.e. $\mathcal{R}e_i \wedge e_i = \lambda_{ij} e_i \wedge e_j$ for $i < j$. Then $\mathcal{R}\#\mathcal{R}$ is also diagonal w.r.t. this basis and the eigenvalues $\{\mu_{ij}\}_{i<j}$ of $\mathcal{R}\#\mathcal{R}$ are given by

$$\mu_{ij} = \sum_{k \neq i,j} \lambda_{ik}\lambda_{jk} = \sum_{k} \lambda_{ik}\lambda_{jk}.$$

Here, we have made the following conventions: We put

1. *$\lambda_{ji} := \lambda_{ij}$ for all $i < j$ and*
2. *$\lambda_{ii} := 0$ for all i.*

Beweis. The conventions are to be justified easily: We have

$$\lambda_{ij} = \langle \mathcal{R}e_i \wedge e_j, e_i \wedge e_j \rangle = \langle \mathcal{R}e_j \wedge e_i, e_j \wedge e_i \rangle = \lambda_{ji}$$

and

$$\lambda_{ii} = \langle \mathcal{R}e_i \wedge e_i, e_i \wedge e_i \rangle = 0.$$

Using corollary 2.2.0.20 and lemma 1.1.0.3 we see

$$\begin{aligned}
\mathcal{R}\#\mathcal{R}e_i \wedge e_j &= \sum_k [Re_i \wedge e_k, \mathcal{R}e_j \wedge e_k] \\
&= \sum_k \lambda_{ik}\lambda_{jk} [e_i \wedge e_k, e_j \wedge e_k] \\
&= \left(\sum_{k \neq i,j} \lambda_{ik}\lambda_{jk} \right) e_i \wedge e_j
\end{aligned}$$

□

Remark 2.7.0.27. The assumptions of proposition 2.7.0.26 are always fulfilled if $\mathcal{R}$ takes the form $\mathcal{R} = F \wedge \mathrm{id}$ or $\mathcal{R} = F \wedge F$, where F is a self-adjoint endomorphism of V.

One sees immediately that each eigenvalue $\mu_{ij} := \lambda_{ij}(\mathcal{R}\#\mathcal{R})$ of $\mathcal{R}\#\mathcal{R}$ will lie in the interval

$$\left[(n-2) \min_{k \neq i,j} \lambda_{ik}\lambda_{jk}, (n-2) \max_{k \neq i,j} \lambda_{ik}\lambda_{jk} \right]$$

if $\mathcal{R}$ is as in proposition 2.7.0.26. Note that we get an estimate for each eigenvalue of $\mathcal{R}\#\mathcal{R}$ in terms of the eigenvalues of $\mathcal{R}$ in this case.
In the general case, it is at least possible to describe the range of the eigenvalues of $\mathcal{R}\#\mathcal{R}$ in terms of the eigenvalues of $\mathcal{R}$:

Theorem 2.7.0.28. *If $\mathcal{R}$ is a self-adjoint endomorphism of $\bigwedge^2 V$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$, then for each eigenvalue μ of $\mathcal{R}\#\mathcal{R}$ holds*

$$(n-2) \min_{k \neq l} \lambda_k \lambda_l \leq \mu \leq (n-2) \max_{k \neq l} \lambda_k \lambda_l.$$

This estimate is sharp.

We need a little lemma to prove 2.7.0.28:

Lemma 2.7.0.29. *Let $F : V \to V$ be a self-adjoint endomorphism and U a subspace of V. Let $\pi : V \to U$ be the orthogonal projection of V onto U and $\iota : U \to V$ the inclusion. Set $\tilde{F} := \pi \circ F \circ \iota$. Then*

$$\lambda_1(\tilde{F}) \geq \lambda_1(F) \text{ and } \lambda_n(\tilde{F}) \leq \lambda_n(F).$$

Beweis. $\tilde{F} : U \to U$ is self-adjoint. So

$$\lambda_1(\tilde{F}) = \min_{v \in \mathbb{S}^{n-1} \cap U} \left\langle \tilde{F}v, v \right\rangle = \min_{v \in \mathbb{S}^{n-1} \cap U} \langle F(v), v \rangle \geq \min_{v \in \mathbb{S}^{n-1}} \langle F(v), v \rangle = \lambda_1(F)$$

and

$$\lambda_n(\tilde{F}) = \max_{v \in \mathbb{S}^{n-1} \cap U} \left\langle \tilde{F}v, v \right\rangle = \max_{v \in \mathbb{S}^{n-1} \cap U} \langle F(v), v \rangle \leq \max_{v \in \mathbb{S}^{n-1}} \langle F(v), v \rangle = \lambda_n(F)$$

□

Proof of Theorem 2.7.0.28. Let $\{\delta_i\}$ an orthonormal eigenbasis of $\mathcal{R}\#\mathcal{R}$, i.e. $\mathcal{R}\#\mathcal{R}\delta_i = \mu_i\delta_i$, $\mu_i \in \mathbb{R}$, such that $\mu_1 \leq ... \leq \mu_N$. Then Proposition 2.1.0.16 implies that $\{\tilde{\delta}_i = \frac{1}{\sqrt{n-2}}\alpha^*\delta_i\}$ is an orthonormal basis of the image of α^*. Let $\pi : \bigwedge^2 \left(\bigwedge^2 V\right) \to \bigwedge^2 \left(\bigwedge^2 V\right)$ be the orthogonal projection onto the image of α^* and $\iota : \mathrm{im}(\alpha^*) \to \bigwedge^2 \left(\bigwedge^2 \mathrm{V}\right)$ the inclusion.
A simple computation using Proposition 2.1.0.16 shows that

$$(\pi \circ \mathcal{R} \wedge \mathcal{R} \circ \iota)(\tilde{\delta}_i) = \frac{1}{n-2}\mu_i\tilde{\delta}_i.$$

So the previous lemma implies

$$(n-2)\lambda_{min}(\mathcal{R} \wedge \mathcal{R}) \leq \mu_i \leq (n-2)\lambda_{max}(\mathcal{R} \wedge \mathcal{R})$$

Now we are done proving the estimate, since every eigenvalue of $\mathcal{R} \wedge \mathcal{R}$ is a product $\lambda_i\lambda_j$ of the eigenvalues $\lambda_1, ..., \lambda_N$ of $\mathcal{R}$. The estimate is sharp, since we have equality, if we take $\mathcal{R} = \mathrm{id}$.

□

Kapitel 3

Algebraic Curvature Operators

This chapter is devoted to the study of algebraic curvature operators on Euclidean vector spaces. We begin with a geometric motivation of the subject and show for example, that the 2-jets of Riemannian metrics at $0 \in \mathbb{R}^n$ are in one-to-one correspondence with the algebraic curvature operators on $\mathbb{R}^n$. After that we focus on topics concerning the structure of the space of algebraic curvature operators, such as the irreducible decomposition w.r.t. the action of SO(V) or special curvature features depending on the dimension. Further we give alternative characterizations of algebraic curvature operators in subsection 3.2.3, where we show that a self-adjoint endomorphism $\mathcal{R}$ of $\bigwedge^2 V$ is an algebraic curvature operator if it satisfies the Böhm-Wilking identity

$$\mathrm{id}\#\mathcal{R} = \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id} - \mathcal{R}.$$

The other direction is due to Böhm and Wilking and has been proved in [5]. Later in the text we introduce the Ricci vector field Φ, which arises as the reaction term in the evolution equation of the curvature operator under the Ricci flow, and show by purely algebraic means that Φ is tangent to the space of algebraic curvature operators. Next we discuss the interesting class of algebraic curvature operators which arise as curvature operators of symmetric spaces using the results of the previous chapters. For example, we will prove very quickly, that Ricci-flat symmetric spaces are flat, using the Ricci vector field. At the end of this chapter we introduce algebraic product curvature operators using holonomy algebras and we show that the Ricci vector field respects these products. We will use several objects and facts of Riemannian geometry, such as Riemannian manifolds, connections on vectorbundels, Holonomy, Lie groups and Lie algebras, Symmetric spaces and related stuff. The reader who is not familiar with these topics is invited to visit the appendix B for a short introduction.

3.1 Definition and Geometric Realization of Algebraic Curvature Operators

3.1.1 Geometric Motivation

Let (M,g) be a Riemannian manifold, $p \in M$ and R^g the curvature tensor of g. Consider the restriction R^g_p of R^g to the fiber $V := \mathrm{T_pM}$ of the tangent bundle,

$$R^g_p : V \times V \times V \to V : (x,y,z) \mapsto R^g(x,y)z$$

and the restriction g_p of g to $V \times V$,

$$g_p : V \times V \to \mathbb{R} : (x,y) \mapsto g(x,y).$$

Note that $(V, \langle \cdot\ , \cdot \rangle) := (\mathrm{T_pM}, \mathrm{g_p})$ is a Euclidean vector space. We get that R^g_p fulfills the following identities for all $x,y,z,w \in V$:

1. $g_p(R^g_p(x,y)z,w) = -g_p(R^g_p(y,x)z,w) = -g_p(R^g_p(x,y)w,z)$
2. $g_p(R^g_p(x,y)z,w) = g_p(R^g_p(z,w)x,y)$
3. $R^g_p(x,y)z + R^g_p(y,z)x + R^g_p(z,x)y = 0$

The latter identity is known as the 1. Bianchi identity. Varying g we get a whole set of trilinear maps

$$\left\{R^g_p : g \text{ Riemannian metric on M, } R^g \text{ curvature tensor of } g\right\},$$

each R^g_p satisfying 1.2. and 3. w.r.t. its own g_p. Now we focus on the set $\mathrm{LC}\,(\mathrm{M},\mathrm{g})_\mathrm{p}$, whose elements are by definition the trilinear maps $R : V \times V \times V \to V$ satisfying 1.2 and 3. w.r.t. $\langle \cdot\ , \cdot \rangle = g_p$. $R \in \mathrm{LC}\,((\mathrm{M},\mathrm{g}))_\mathrm{p}$ is called an algebraic curvature operator on (M,g) at p and $\mathrm{LC}\,(\mathrm{M},\mathrm{g})_\mathrm{p}$ is called the space of algebraic curvature operators on (M,g) at p. As the conditions 1.2. and 3. are linear in R, $\mathrm{LC}\,(\mathrm{M},\mathrm{g})_\mathrm{p}$ is a finite dimensional vector space. We ask if the space of algebraic curvature operators on (M,g) at p agrees with the set

$$\mathrm{LC}\,(\mathrm{M},\mathrm{g})_\mathrm{p} = \left\{\mathrm{R^h_p} : \mathrm{h} \text{ Riemannian metric on M, } \mathrm{h_p} = \mathrm{g_p}\right\}?$$

The answer is yes, as we will prove in the following subsection. Thus, on this infinitesimal level, it makes no difference whether we examine Riemannian metrics or algebraic curvature operators.
We want to use the results of the proceeding sections in our studies of the space of algebraic curvature operators. Thus, we have to transform the notion of algebraic curvature operators to the right setup. Recall that if R^g is the curvature tensor belonging to a Riemannian metric g on a manifold M,

then, using the universal property of exterior powers, it is always possible to define an endomorphism field $\mathcal{R}^g : \bigwedge^2 \mathrm{TM} \to \bigwedge^2 \mathrm{TM}$ requiring

$$g(\mathcal{R}^g(X \wedge Y), Z \wedge W) = g(R^g(X,Y)W, Z)$$

for all vector fields X, Y, Z and W on M, where the g on the left hand side is the induced Euclidean structure on $\bigwedge^2 \mathrm{TM}$. $\mathcal{R}^g$ is called the curvature operator of g. The symmetries of R^g imply that $\mathcal{R}^g$ is fiberwise self-adjoint w.r.t. the induced bundle metric on $\bigwedge^2 \mathrm{TM}$. On the other hand, note that if we are given an endomorphism field $\mathcal{R} : \bigwedge^2 \mathrm{TM} \to \bigwedge^2 \mathrm{TM}$, we may turn it into a tensor field $R : \mathrm{TM} \times \mathrm{TM} \times \mathrm{TM} \to \mathrm{TM}$, requiring

$$g(R(X,Y)W, Z) = g(\mathcal{R}(X \wedge Y), Z \wedge W)$$

for all vector fields X, Y, Z and W on M. Using this identification of $\mathcal{R}$ and R we define the space of algebraic curvature operators on $(\bigwedge^2 V, \langle \cdot \ , \cdot \rangle)$

$$\mathrm{LC}\left(\mathrm{M}, \mathrm{g}_{\bigwedge^2 \mathrm{TM}}\right)_{\mathrm{p}} := \left\{\mathcal{R} \in \mathrm{End}(\textstyle\bigwedge^2 \mathrm{V}) : \mathrm{R \ algebraic \ curvature \ operator \ on}(\mathrm{V}, \langle \cdot \ , \cdot \rangle)\right\}.$$

3.1.2 Definition of Algebraic Curvature Operators

We start from zero. As in the proceeding sections, $(V, \langle \cdot \ , \cdot \rangle)$ denotes an n-dimensional Euclidean vector space and $N = \binom{n}{2}$ denotes the dimension of $\bigwedge^2 V$. Given an endomorphism $\mathcal{R}$ of $\bigwedge^2 V$, define a trilinear map $\mathcal{R}^\rho : V \times V \times V \to V$ by

$$\mathcal{R}^\rho(x,y)z := -\rho(\mathcal{R}(x \wedge y))(z),$$

where $\rho : \bigwedge^2 V \to \mathfrak{so}(V, \langle \cdot \ , \cdot \rangle)$ is the canonical map defined in section 1. Recall that ρ satisfies $\rho(x \wedge y) = x^* \otimes y - y^* \otimes x$ for all $x, y \in V$. Further, we have the formula

$$\langle \rho(\varepsilon)(x), y \rangle = \langle \varepsilon, x \wedge y \rangle,$$

provided by lemma 1.1.0.3. This formula guaranties, that for all elements x, z, y and w of V holds

$$\langle \mathcal{R}^\rho(x,y)w, z \rangle = \langle \mathcal{R}(x \wedge y), z \wedge w \rangle.$$

This shows that we have chosen the right identification of endomorphisms of $\bigwedge^2 V$ with $(1,3)$-tensors on V. In order to define the space of algebraic curvature operators on $\bigwedge^2 V$ we introduce the Bianchi map $\mathrm{B} : \mathrm{End}(\bigwedge^2 \mathrm{V}) \to \mathrm{T}^3_1\mathrm{V}$, where $\mathrm{T}^3_1\mathrm{V}$ is the space of trilinear maps $V \times V \times V \to V$. The Bianchi map is defined by

$$\mathrm{B}(\mathcal{R})(\mathrm{x},\mathrm{y})\mathrm{z} := \mathcal{R}^\rho(\mathrm{x},\mathrm{y})\mathrm{z} + \mathcal{R}^\rho(\mathrm{y},\mathrm{z})\mathrm{x} + \mathcal{R}^\rho(\mathrm{z},\mathrm{x})\mathrm{y}.$$

We say that $\mathcal{R}$ satisfies the 1. Bianchi identity, if $\mathcal{R}$ lies in the kernel of B.

Definition 3.1.2.1. *An endomorphism $\mathcal{R}$ of $\bigwedge^2 V$ is called an algebraic curvature operator on $\left(\bigwedge^2 V, \langle \cdot\ ,\cdot\rangle\right)$ if*

1. *$\mathcal{R}$ is self-adjoint*
2. *$\mathcal{R}$ satisfies the first Bianchi Identity.*

The space of algebraic curvature operators on $\left(\bigwedge^2 V, \langle \cdot\ ,\cdot\rangle\right)$ will be denoted by $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}, \langle \cdot\ ,\cdot\rangle\right)$ *or simply* $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$*, if it is clear, which scalar product is used.*

We shall give an algebraic example.

Example 3.1.2.2. *If F and G are self-adjoint endomorphisms of V, then $F \wedge G$ is an algebraic curvature operator on $\bigwedge^2 V$. In particular, if $U \subseteq V$ is a subspace and $\pi_U : V \to V$ is the orthogonal projection onto U, then $\pi_{\bigwedge^2 U} := \pi_U \wedge \pi_U \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$. Positive multiples of the curvature operators $\mathcal{S}_U = \pi_{\bigwedge^2 U}$ and $\mathcal{H}_U = -\pi_{\bigwedge^2 U}$ will be called weakly spherical and weakly hyperbolic, respectively. The positive multiples of $\mathcal{S}_V$ and $\mathcal{H}_V$ are called spherical and hyperbolic. Notice that $\mathcal{S}_V$ is the curvature operator of the round sphere $\mathbb{S}^n$ and $\mathcal{H}_V$ is the curvature operator of the n-dimensional hyperbolic space $\mathbb{H}^n$.*

Beweis. It is sufficient to proof the statement only for $F \wedge F$, since we have

$$2F \wedge G = (F + G) \wedge (F + G) - F \wedge F - G \wedge G$$

for all endomorphisms F and G of V and $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ is a vector space: It is clear that $F \wedge F$ is self-adjoint, since F is self-adjoint by assumption. So it remains to check the first Bianchi identity: Let $x, y, z \in V$ be arbitrary. Then

$$\begin{aligned}(F \wedge F)^{\rho}(x, y)z &= -\rho(Fx \wedge Fy)(z) \\ &= -(\langle Fx, z\rangle\, Fy - \langle Fy, z\rangle\, Fx) \\ &= -F(\langle x, Fz\rangle\, y - \langle y, Fz\rangle\, x)\end{aligned}$$

In the last step we used that F is self-adjoint. From the above computation we get

$$\begin{aligned}B(F \wedge F)(x, y)z = -F(&\langle x, Fz\rangle\, y - \langle y, Fz\rangle\, x + \langle y, Fx\rangle\, z \\ &- \langle z, Fx\rangle\, y + \langle z, Fy\rangle\, x - \langle x, Fy\rangle\, z) \\ = 0\ &\end{aligned}$$

since F is self-adjoint.

□

Now we may ask the question whether a given algebraic curvature operator is of the form $F \wedge G$ with F and G as in the previous lemma or at least a linear combination of curvature operators of this form. As we shall see later, the answer to the first question is yes in many cases while the answer to the second is yes in any case. One can even show that the space of algebraic curvature operators on $\bigwedge^2 V$ is generated by weakly spherical algebraic curvature operators.

Remark 3.1.2.3. *1.* Usually, algebraic curvature operators are defined as 4-linear maps R on V, satisfying

(a) $R(x,y,u,v) = -R(y,x,u,v)$

(b) $R(x,y,u,v = R(u,v,x,y))$

(c) $R(x,y,u,v) + R(u,x,y,v) + R(y,u,x,v) = 0$

for all $x,y,u,v \in V$. And the space of algebraic curvature operators on V w.r.t. $\langle \cdot\,,\cdot\rangle$ is defined as

$$\mathrm{LC}\,(\mathrm{V}, \langle \cdot\,,\cdot\rangle) := \{\mathrm{R} : \mathrm{R} \text{ a 4-linear map on V satisfying a),b), and c) }\}.$$

But, examining the relations between the #-product and algebraic curvature operators, for example, it will prove to be useful to regard algebraic curvature operators as endomorphisms of $\bigwedge^2 V$.

2. If we are given two symmetric bilinear maps β and β' on V, we can use these maps to define a 4-linear map $\beta \circ \beta'$ on V, letting

$$\begin{aligned}\beta \circ \beta'(x,y,u,v) := {}& \beta(x,u)\beta'(y,v) + \beta(y,v)\beta'(x,u) \\ & - \beta(y,u)\beta'(x,v) - \beta(x,v)\beta'(y,u)\end{aligned}$$

The operation "$\circ$" is known as the Kulkarni-Nomizu product. One can show that the Kulkarni-Nomizu product of β and β' behaves like an algebraic curvature operator on V: We have

(a) $\beta \circ \beta'(x,y,u,v) = -\beta \circ \beta'(y,x,u,v)$

(b) $\beta \circ \beta'(x,y,u,v) = \beta \circ \beta'(u,v,x,y)$

(c) $\beta \circ \beta'(x,y,u,v) + \beta \circ \beta'(u,x,y,v) + \beta \circ \beta'(y,u,x,v) = 0$

for all $x,y,u,v \in V$.
Writing β and β' as

$$\beta(x,y) = \langle F(x), y\rangle \text{ and } \beta'(x,y) = \left\langle F'(x), y\right\rangle,$$

$F, F' : V \to V$ self-adjoint, we may convince ourselves that

$$\left\langle F \wedge F'(x \wedge y), u \wedge v\right\rangle = \frac{1}{2}\beta \circ \beta'(x,y,u,v).$$

3.1.3 Geometric Realization of Algebraic Curvature Operators

The most famous examples of algebraic curvature operators are given by curvature operators $\mathcal{R}$ of Riemannian manifolds (M,g): We have that $\mathcal{R}_p$ is an algebraic curvature operator on $\left(\bigwedge^2 \mathrm{T_pM}, \mathrm{g_p}\right)$ for all $p \in M$. On the other hand every algebraic curvature operator $\mathcal{R}$ has a geometric realization: Given an algebraic curvature operator $\mathcal{R}$, there exists a Riemannian metric g on an neighborhood U of $0 \in V$, such that $\mathcal{R}$ is the curvature operator of g in 0. This tells us that the curvature operators of Riemannian manifolds are the only possible examples of algebraic curvature operators.
To prove the existence of such a Riemannian metric, let

$$g_p(x,y) := \langle x,y\rangle - \frac{1}{3}\langle \mathcal{R}x\wedge p, y\wedge p\rangle .$$

$\mathcal{R}$ is self-adjoint w.r.t. $\langle \cdot\, , \cdot\rangle$, so g_p is a symmetric bilinear map for all $p \in V$. Positive definiteness is an open condition, which implies that g is indeed a Riemannian metric on a sufficiently small neighborhood U of 0. In the following we will always take $U = \{p \in V : g_p > 0\}$ to be the maximal domain of definition of g. Later we will see that U is starshaped w.r.t. $0 \in U$. Note that (U,g) is an analytic Riemannian manifold.
Now we compute the curvature operator of g in 0:
Let $\mathcal{R}^g$ be the curvature operator of g, R^g the $(3,1)$-curvature tensor of g and $R = \mathcal{R}^\rho$. $\mathcal{R}^g$ and R^g are algebraically equivalent and related by the formula

$$g(\mathcal{R}^g x\wedge y, u\wedge v) = g(R^g(x,y)v,u).$$

Now we start computing R^g: First of all, we need to compute the Levi-Civita connection ∇ of g. We will proceed as follows: First we compute $\nabla_X Y$ on constant vector fields X and Y and then we use that ∇ is derivative in Y to obtain the full information about ∇
Let $X \equiv x, Y \equiv y$ and $Z \equiv z$, $x,y,z \in V$, be constant vector fields on U and $p \in U$. Then, using the Koszul formula, we get

$$\begin{aligned} 2g_p(\nabla_X Y, Z) &= Xg(Y,Z)|_p + Yg(Z,X)|_p - Zg(X,Y)|_p \\ &\quad + g_p([X,Y],Z) - g_p([Y,Z],X) + g_p([Z,X],Y) \\ &= Xg(Y,Z)|_p + Yg(Z,X)|_p - Zg(X,Y)|_p \end{aligned}$$

since the terms with the Lie brackets vanish. Using the symmetries of R, we get

$$\begin{aligned} Xg(Y,Z)|_p &= \frac{d}{dt}\bigg|_{t=0}\left(\langle x,y\rangle - \frac{1}{3}\langle \mathcal{R}((p+tx)\wedge y),(p+tx)\wedge z\rangle\right) \\ &= -\frac{1}{3}(\langle \mathcal{R}y\wedge x, z\wedge p\rangle + \langle \mathcal{R}y\wedge p, z\wedge x\rangle \\ &= \frac{1}{3}(\langle R(x,y)p,z\rangle + \langle R(p,y)x,z\rangle). \end{aligned}$$

Interchanging the roles of X, Y and Z and using the symmetries of $\mathcal{R}$ leads to

$$Yg(Z,X)|_p = -\frac{1}{3}(\langle R(x,y)p,z\rangle + \langle R(x,p)y,z\rangle)$$

and

$$-Zg(X,Y)|_p = \frac{1}{3}(\langle R(p,x)y,z\rangle + \langle R(p,y)x,z\rangle).$$

This gives

$$g_p(\nabla_X Y, Z) = \frac{1}{3}\langle R(p,x)y + R(p,y)x, z\rangle\,.$$

Now let $\{e_i\}$ be a g_p-orthonormal basis of V and $E_i \equiv e_i$ the corresponding constant vector fields on U. We find

$$\nabla_X Y|_p = \frac{1}{3}\sum_i g_p(\nabla_X Y, E_i)E_i(p) = \frac{1}{3}\sum_i \langle R(p,x)y + R(p,y)x, e_i\rangle\, e_i.$$

Since every vector field on U is a sum of vector fields fY, where $f : U \to \mathbb{R}$ is a smooth function and Y is a constant vector field on U, we get the full information about ∇ from our knowledge about ∇ on constant vector fields and the Leibniz-rule. More precisely, we have

$$\begin{aligned} \nabla_X fY|_p &== (Xf)Y|_p + f(p)\nabla_X Y|_p \\ &= \mathrm{D_X fY|_p} + \frac{1}{3}\sum_{\mathrm{i}} \langle \mathrm{R(p,x)f(p)y + R(p,f(p)y)x, e_i}\rangle\, \mathrm{e_i}. \end{aligned}$$

We summarize our results in the following proposition.

Proposition 3.1.3.1. *Let (U,g) be the geometric realization of* $\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$, *$p \in U$ and X, Y, Z vector fields on U with $X(p) = x, Y(p) = y, Z(p) = z$. Further, let $\{e_i\}$ be a g_p-orthonormal basis of V and $S \in \mathrm{Sym}\,(\mathrm{TU,g})$ be the fiberwise inverse of $G \in \mathrm{Sym}\,(\mathrm{TU,g})$, $G_p(X_p) := X_p - \frac{1}{3}R(X_p,p)p$. Then,*

1. $\nabla_X Y|_p = \mathrm{D_X Y|_p} + \frac{1}{3}\sum_{\mathrm{i}} \langle \mathrm{R(p,x)y + R(p,y)x, e_i}\rangle\, \mathrm{e_i}$,
2. $\nabla_X Y|_p = \mathrm{D_X Y|_p} + \frac{1}{3}\mathrm{S_p(R(p,x)y + R(p,y)x)}$
3. $g_p(\nabla_X Y, Z) = \left\langle \mathrm{D_X Y|_p} + \frac{1}{3}(-\mathrm{R(D_X Y|_p, p)p + R(p,x)y + R(p,y)x), z}\right\rangle$.

Beweis. 1. There is nothing left to show.

2. Using 1., it follows that

$$\begin{aligned}
g_p(\nabla_X Y, Z) &= g_p(\mathrm{D_X Y}|_\mathrm{p}, \mathrm{z}) + \frac{1}{3}\sum_\mathrm{i} \mathrm{g_p}(\langle \mathrm{R(p,x)y + R(p,y)x, e_i}\rangle\, \mathrm{e_i, z}) \\
&= g_p(\mathrm{D_X Y}|_\mathrm{p}, \mathrm{z}) + \frac{1}{3}\sum_\mathrm{i} \langle \mathrm{R(p,x)y + R(p,y)x, e_i}\rangle\, \mathrm{g_p(e_i, z)} \\
&= g_p(\mathrm{D_X Y}|_\mathrm{p}, \mathrm{z}) + \frac{1}{3}\left\langle \mathrm{R(p,x)y + R(p,y)x}, \sum_\mathrm{i} \mathrm{g_p(z, e_i) e_i}\right\rangle \\
&= g_p(\mathrm{D_X Y}|_\mathrm{p}, \mathrm{z}) + \frac{1}{3}\langle \mathrm{R(p,x)y + R(p,y)x, z}\rangle \\
&= g_p(\mathrm{D_X Y}|_\mathrm{p}, \mathrm{z}) + \frac{1}{3}\mathrm{g_p(S_p(R(p,x)y + R(P,y)x), z)} \\
&= g_p(\mathrm{D_X Y}|_\mathrm{p} + \frac{1}{3}\mathrm{S_p(R(p,x)y + R(p,y)x), z)}.
\end{aligned}$$

3. This follows from 2. using $g_p(x, y) = \langle G_p(x), y\rangle$.

□

We are ready to compute R^g:

Theorem 3.1.3.2. *We have*

$$\begin{aligned}
g_p(R^g_p(X,Y)Z, W) = &\langle R(X,Y)Z, W\rangle \\
&+ \frac{1}{9}\sum_i \langle (\mathrm{D_{e_i} R(-,p)p \wedge D_{e_i} R(-,p)p})^\rho (X,Y)Z, W\rangle
\end{aligned}$$

for all $p \in U$ and all vector fields X, Y, Z and W on U, where $\{e_i\}$ is a g_p-orthonormal basis of V. In particular, we have

$$R^g_0 = R.$$

and

$$\nabla R^g_p|_{p=0} = 0.$$

Beweis. Let X, Y, Z and W be vector fields on U. We have

$$\begin{aligned}
g(R^g(X,Y)Z, W) &= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W) \\
&= Xg(\nabla_Y Z, W) - g(\nabla_Y Z, \nabla_X W) \\
&\quad - Yg(\nabla_X Z, W) + g(\nabla_X Z, \nabla_Y W) - g(\nabla_{[X,Y]} Z, W).
\end{aligned}$$

everywhere on U. From now on we assume X, Y, Z and W to be constant. In p holds

$$\begin{aligned} Xg(\nabla_Y Z, W) &= X \left\langle \mathrm{D_Y Z} + \frac{1}{3}(-\mathrm{R(D_Y Z, p)p} + \mathrm{R(p, Y)Z} + \mathrm{R(p, Z)Y), W} \right\rangle \\ &= \frac{1}{3} X \langle R(p, Y)Z + R(p, Z)Y, W \rangle \end{aligned}$$

since the derivative of Z vanishes identically everywhere. Thus,

$$\begin{aligned} Xg(\nabla_Y Z, W) &= X \left\langle \mathrm{D_Y Z} + \frac{1}{3}(-\mathrm{R(D_Y Z, p)p} + \mathrm{R(p, Y)Z} + \mathrm{R(p, Z)Y), W} \right\rangle \\ &= \frac{1}{3} \langle R(X, Y)Z + R(X, Z)Y, W \rangle \end{aligned}$$

This gives

$$\begin{aligned} g_p(R^g(X, Y)Z, W) =& \frac{1}{3} \langle (R(X, Y)Z + R(X, Z)Y - R(Y, X)Z - R(Y, Z)X), W \rangle \\ & - g_p(\nabla_Y Z, \nabla_X W) + g_p(\nabla_X Z, \nabla_Y W) \end{aligned}$$

Using the symmetries of R and the 1. Bianchi identity, it follows that

$$\frac{1}{3} \langle (R(X, Y)Z + R(X, Z)Y - R(Y, X)Z - R(Y, Z)X), W \rangle = \langle R(X, Y)Z, W \rangle,$$

which leads to

$$g_p(R^g(X, Y)Z, W) = \langle R(X, Y)Z, W \rangle - g_p(\nabla_Y Z, \nabla_X W) + g_p(\nabla_X Z, \nabla_Y W).$$

Now we treat the terms involving the covariant derivatives:
We have

$$\nabla_X Y|_p = \mathrm{D_X Y}|_\mathrm{p} + \frac{1}{3} \sum_\mathrm{i} \langle \mathrm{R(p, X)Y + R(p, Y)X, e_i} \rangle \, \mathrm{e_i}$$

for all vector fields X and Y. Since X, Y, Z and W are constant, this leads to

$$\begin{aligned} & g_p(\nabla_X Z, \nabla_Y W) - g_p(\nabla_Y Z, \nabla_X W) \\ &= \frac{1}{9} \sum_i \begin{pmatrix} \langle R(p, X)Z + R(p, Z)X, e_i \rangle \langle R(p, Y)W + R(p, W)Y, e_i \rangle \\ - \langle R(p, Y)Z + R(p, Z)Y, e_i \rangle \langle R(p, X)W + R(p, W)X, e_i \rangle \end{pmatrix} \\ &= \frac{1}{9} \sum_i \begin{pmatrix} \langle R(X, p)e_i + R(X, e_i)p, Z \rangle \langle R(Y, p)e_i + R(Y, e_i)p, W \rangle \\ - \langle R(Y, p)e_i + R(Y, e_i)p, Z \rangle \langle R(X, p)e_i + R(X, e_i)p, W \rangle \end{pmatrix} \\ &= -\frac{1}{9} \sum_i \begin{pmatrix} \langle (\mathrm{D_{e_i} R(-, p)p}) (X), Z \rangle \langle (\mathrm{D_{e_i} R(-, p)p}) (Y), W \rangle \\ - \langle (\mathrm{D_{e_i} R(-, p)p}) (Y), Z \rangle \langle (\mathrm{D_{e_i} R(-, p)p}) (X), W \rangle \end{pmatrix} \\ &= -\frac{1}{9} \sum_i \langle \mathrm{D_{e_i} R(-, p)p \wedge D_{e_i} R(-, p)p(X \wedge Y)), Z \wedge W} \rangle \\ &= \frac{1}{9} \sum_i \langle (\mathrm{D_{e_i} R(-, p)p \wedge D_{e_i} R(-, p)p})^\rho (X, Y)Z, W \rangle \end{aligned}$$

This shows the first and the second claim. We are left showing $\nabla R^g|_{p=0} = 0$: The map $p \mapsto \iota(p) = -p$ is an isometry of (U, g). This shows $\nabla R^g|_0 = 0$, since the curvature R^g and therefore also the covariant derivative ∇R^g is preserved by isometric transformations: In $p = 0$, we have

$$\begin{aligned} -(\nabla_x R^g)|_0(u,v)w &= \mathrm{D}\iota_0((\nabla_\mathrm{x} \mathrm{R^g})|_0(\mathrm{u},\mathrm{v})\mathrm{w}) \\ &= (\nabla_{\mathrm{D}\iota_0(\mathrm{x})} R^g)|_0(\mathrm{D}\iota_0(\mathrm{u}), \mathrm{D}\iota_0(\mathrm{v}))\mathrm{D}\iota_0(\mathrm{w}) \\ &= (\nabla_x R^g)|_0(u,v)w. \end{aligned}$$

□

Now we are able to prove the statement mentioned in the motivating part at the beginning of this chapter.

Corollary 3.1.3.3. *If (M, g) is a Riemannian manifold and $p \in M$ then*

$$\mathrm{LC}\,(\mathrm{M},\mathrm{g})_\mathrm{p} = \left\{ \mathrm{R}^\mathrm{h}_\mathrm{p} : \mathrm{h} \textit{ Riemannian metric on M, } \mathrm{h_p} = \mathrm{g_p} \right\}.$$

Beweis. It is clear that every curvature tensor R^h of a Riemannian metric h lies in $\mathrm{LC}\,(\mathrm{M},\mathrm{g})_\mathrm{p}$, if $h_p = g_p$. We have to show that every element R of $\mathrm{LC}\,(\mathrm{M},\mathrm{g})_\mathrm{p}$ determines a Riemannian metric h on M with $R^h_p = R$. Let (φ, U') be a chart around p with $\varphi(p) = 0$. Put $\langle \cdot\,, \cdot \rangle := (\varphi_* g)|_0$, i.e.

$$\langle x, y \rangle = g_p(\mathrm{D}\varphi_\mathrm{p}^{-1}\mathrm{x}, \mathrm{D}\varphi_\mathrm{p}^{-1}\mathrm{y})$$

and define $R' := \varphi_* R$, i.e.

$$R'(x,y)z = \mathrm{D}\varphi_\mathrm{p}\mathrm{R}(\mathrm{D}\varphi_\mathrm{p}^{-1}\mathrm{x}, \mathrm{D}\varphi_\mathrm{p}^{-1}\mathrm{y})\mathrm{D}\varphi_\mathrm{p}^{-1}\mathrm{z}.$$

Now let h' be the geometric realization of R' in a small neighborhood $U \subseteq \varphi(U')$,

$$h'_q(x,y) = \langle x, y \rangle - \frac{1}{3} \left\langle R'(x,q)q, y \right\rangle.$$

For vector fields X, Y on $U'' := \varphi^{-1}(U)$ define

$$h''(X,Y) := h'(\mathrm{D}\varphi\mathrm{X}, \mathrm{D}\varphi\mathrm{Y}).$$

This gives a Riemannian metric on U'' with $R^{h''}_p = R$. Now let $C \subseteq U''$ a compact neighborhood of p and $\psi : M \to [0,1]$ be a bump function with $\psi|_C \equiv 1$ and $\psi|_{M \setminus U''} \equiv 0$. To finish the proof put $h := (1-\psi)g + \psi h''$.

□

Note that if the algebraic curvature operators $\mathcal{R}$ and $\mathcal{S}$ differ by conjugation with an element G of $O(V)$ ($\mathcal{S} = G \wedge G\mathcal{R}G^{-1} \wedge G^{-1}$), then their geometric realizations are isometric via the map $p \mapsto Gp$.
On the other hand, if the geometric realizations $g_\mathcal{R}$ and $g_\mathcal{S}$ of $\mathcal{R}$ and $\mathcal{S}$ are

isometric via a map φ, i.e. $\varphi^* g_{\mathcal{S}} = g_{\mathcal{R}}$, with $\varphi(0) = 0$, then $\mathcal{R}$ and $\mathcal{S}$ belong to the same O(V)-orbit. For if $\varphi(0) = 0$, then $\mathrm{D}\varphi_0 = \mathrm{G} \in \mathrm{O(V)}$, which implies $\mathcal{S} = G \wedge G\mathcal{R}G^{-1} \wedge G^{-1}$, as claimed. We use this result to prove the following

Proposition 3.1.3.4. *Let $\mathcal{R}$ be an algebraic curvature operator on $\bigwedge^2 V$ and (U, g) its geometric realization. Then*

$$\mathrm{Isom}_0(\mathrm{U}, \mathrm{g}) = \mathrm{Stab}_{\mathrm{O(V},\langle\cdot\ ,\cdot\rangle)}(\mathcal{R}),$$

where $\mathrm{Isom}_0(\mathrm{U}, \mathrm{g})$ *is the isotropy group of* (U, g) *in* $0 \in U$ *and* $\mathrm{Stab}_{\mathrm{O(V},\langle\cdot\ ,\cdot\rangle)}(\mathcal{R})$ *is the stabilizer of* $\mathcal{R}$ *w.r.t. the induced action of* O(V) *on the space of algebraic curvature operators.*

Beweis. It is clear from above that the stabilizer of $\mathcal{R}$ lies completely in the isotropy group of the geometric realization. Let $\varphi \in \mathrm{Isom}_0(\mathrm{u}, \mathrm{g})$ be arbitrary. Then, arguing as above, we get $\varphi = G$ for some $G \in \mathrm{O(V)}$. We have to show that G lies in the stabilizer of $\mathcal{R}$. Let $p \in U$ and $v, w \in V$ be arbitrary. Then we get

$$g_{G(p)}(G(v), G(w)) = g_p(v, w),$$

where

$$g_{G(p)}(G(v), G(w)) = \langle v, w\rangle - \frac{1}{3}\langle \mathcal{R}G(p) \wedge G(v), G(p) \wedge G(w)\rangle$$

and

$$g_p(v, w) = \langle v, w\rangle - \frac{1}{3}\langle \mathcal{R}p \wedge v, p \wedge w\rangle .$$

This leads to

$$\left\langle \left((G \wedge G)^{-1} \circ \mathcal{R} \circ G \wedge G - \mathcal{R}\right) p \wedge v, p \wedge w \right\rangle = 0$$

for all $p \in U$ and all $v, w \in V$. The map $\mathcal{S} := (G \wedge G)^{-1} \circ \mathcal{R} \circ G \wedge G - \mathcal{R}$ is an algebraic curvature operator with geometric realization $g_{\mathcal{S}} \equiv \langle\cdot\ ,\cdot\rangle$ which implies $\mathcal{S} = 0$. Thus, we have that G lies in the stabilizer of $\mathcal{R}$ as claimed. □

Proposition 3.1.3.5. *A curve γ in U is a geodesic if and only if*

$$\frac{d^2}{dt^2}\gamma(t) = -\frac{2}{3} S_{\gamma(t)}(\mathcal{R}^\rho(\gamma, \dot\gamma)\dot\gamma).$$

Consequently, the radial line segments $\gamma_v : \mathbb{R} \to V.t \mapsto tv$, $\|v\| = 1$, through $0 \in U$ are geodesics as long as they stay in U.

Beweis. γ is a geodesic in U if and only if

$$0 = \nabla_{\dot\gamma}\dot\gamma = \mathrm{D}_{\dot\gamma}\dot\gamma + \frac{2}{3}\mathrm{S}_\gamma(\mathcal{R}^\rho(\gamma, \dot\gamma)\dot\gamma).$$

We have

$$\mathrm{D}_{\dot\gamma}\dot\gamma|_{\gamma(\mathrm{t_0})} = \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{\mathrm{t=t_o}} \dot\gamma(\gamma(\mathrm{t_0}+\mathrm{t})) = \frac{\mathrm{d}^2}{\mathrm{dt}^2}\Big|_{\mathrm{t=t_0}} \gamma(\mathrm{t})$$

for all t. Thus, the first claim follows. Pick $v \in V$ with $\|v\| = 1$. Then $\gamma_v(t) = tv$ and $\dot{\gamma_v} = v$ so $\mathcal{R}^\rho(\gamma_v, \dot\gamma_v)\dot\gamma_v = 0$. On the other hand, we have $\frac{d^2}{dt^2}\gamma_v = 0$. Thus, γ_v is a geodesic in U.

□

Next we state some further geometric properties of geometric realizations of algebraic curvature operators. For $v \in V$ with $\|v\| = 1$ define

$$\lambda(v) := \max_{w\perp v, \|w\|=1} \langle \mathcal{R}v \wedge w, v \wedge w\rangle .$$

Note that $\lambda(v)$ is the maximal eigenvalue of the Jacobi operator $J_v : V \to V : w \mapsto \mathcal{R}^\rho(w, v)v$. Moreover define

$$f : S^{n-1} \subseteq V \to [0,\infty) : v \mapsto \max(\lambda(v), 0).$$

Proposition 3.1.3.6. *Let $\mathcal{R}$ be an algebraic curvature operator with $\|\mathcal{R}\| = 1$ and geometric realization (U, g). Then,*

1. *$U = \left\{p \in V \setminus \{0\} : \|p\|^2 f(\frac{p}{\|p\|}) < 3\right\} \cup \{0\}$. Thus, U is starshaped w.r.t. the point 0.*
2. *$U = V$ if and only if the Jacobi operators J_v of $\mathcal{R}$ are non positive for each $v \in V$.*
3. *λ constant implies $U = B_r$ for some $r > 0$ or $U = V$.*

Beweis. 1. By construction of g U is the set of points $p \in V$, where g_p is positive definite. Now let $p \in V$ with $\|p\| = 1$ be given. Then for all $t \geq 0$ and $v \perp p$ with $\|v\| = 1$ we have

$$g_{tp}(v, v) = \|v\|^2 - \frac{1}{3}t^2 \langle J_p(v), v\rangle \geq 1 - \frac{t^2}{3}\lambda(p) \geq 1 - \frac{t^2}{3} f(p).$$

and

$$g_{tp}(p, p) = 1.$$

This tells us that g_{tp} is positive definite as long as $t^2 f(p) < 3$. If If $f(p) > 0$ and $t(p)$ denotes the first time, where we have equality $t(p)^2 f(p) = 3$, then, by definition of f, there exists an eigenvector v of J_p with $\langle J_p(v), v\rangle = f(p)$. Hence we have $g_{t(p)p}(v, v) = 0$, telling us that $g_{t(p)p}$ is not positive definite. Moreover, if we take $t > t(p)$ then we even get $g_{tp}(v, v) < 0$. Putting things together it follows that g_{tp} is positive definite if and only if $t^2 f(p) < 3$ and the claim follows.

2. If $U = V$, then $\|p\|^2 f(\frac{p}{\|p\|}) < 3$ for all $p \in V$. This implies $f \equiv 0$, so the Jacobi operators J_v of $\mathcal{R}$ are clearly nonpostive for each $v \in V$. The other direction is clear.

3. This follows immediately from 1.

□

3.2 The Structure of the Space of Algebraic Curvature Operators

Now we take a closer look at the structure of algebraic curvature operators $\mathcal{R}$ in $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$. It is clear that the space of algebraic curvature operators is not interesting if the dimension of the underlying vector space V is less than 2. In this cases we have $\bigwedge^2 V = 0$, so we get $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) = 0$ either. In the following we will restrict our considerations to the case $n = \dim V \geq 2$.

3.2.1 Fundamental Properties of the Space of Algebraic Curvature Operators

Here, we deal with topics like the dependence of $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ on the underlying Euclidean structure, the invariant components of $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$, we answer the question, how $\mathrm{LC}\left(\bigwedge^2 \mathrm{U}\right)$ lies in $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ if $U \subseteq V$ is a subspace and then we say a few words about bundles of algebraic curvature operators over Riemannian manifolds.

3.2.1.1 Changing the Scalar Product

As in section 1.3, let $A \in \mathrm{GL(V)}$, define a new scalar product $\langle \cdot\ , \cdot \rangle_A := A^* \langle \cdot\ , \cdot \rangle$ on V and let $\rho_A : \left(\bigwedge^2 V, \langle \cdot\ , \cdot \rangle_A\right) \to \mathfrak{so}(V, \langle \cdot\ , \cdot \rangle_A)$ the corresponding representation. The induced scalar scalar product $\langle \cdot\ , \cdot \rangle_A$ on $\bigwedge^2 V$ takes the form $\langle \cdot\ , \cdot \rangle_A = (A \wedge A)^* \langle \cdot\ , \cdot \rangle$. It is clear how to transform self-adjoint endomorphism $\mathcal{R}$ of $\left(\bigwedge^2 V, \langle \cdot\ , \cdot \rangle\right)$ into self-adjoint endomorphisms $\mathcal{R}_A$ of $\left(\bigwedge^2 V, \langle \cdot\ , \cdot \rangle_A\right)$: simply define

$$\mathcal{R}_A := A^{-1} \wedge A^{-1} \circ \mathcal{R} \circ A \wedge A.$$

It is not very surprising that the $\mathcal{R} \mapsto \mathcal{R}_A$ also preserves the property of being an algebraic curvature operator.

Proposition 3.2.1.1. *An endomorphism $\mathcal{R}$ of $\bigwedge^2 V$ is an algebraic curvature operator on $\left(\bigwedge^2 V, \langle \cdot \ , \cdot \rangle\right)$ if and only if $\mathcal{R}_A := A^{-1} \wedge A^{-1} \circ \mathcal{R} \circ A \wedge A$ is an algebraic curvature operator on $\left(\bigwedge^2 V, \langle \cdot \ , \cdot \rangle_A\right)$*

Beweis. We have to check the 1. Bianchi identity. Using lemma 1.3.0.12.2, we compute

$$
\begin{aligned}
\left\langle \mathcal{R}_A^{\rho_A}(x,y)z, w \right\rangle_A &= \left\langle A(\mathcal{R}_A^{\rho_A}(x,y)z), A(w) \right\rangle \\
&= \left\langle A(\mathcal{R}_A^{\rho_A}(x,y)z), A^*A(w) \right\rangle \\
&= -\left\langle \rho_A(\mathcal{R}_A(x,y)z)), A^*A(w) \right\rangle \\
&= -\left\langle \rho(\mathcal{R}_A(x \wedge y)(A^*A(z)), A^*A(w) \right\rangle \\
&= \left\langle A^{-1} \wedge A^{-1} \circ \mathcal{R} \circ A \wedge A(x \wedge y), A^*A \wedge A^*A(z \wedge w) \right\rangle \\
&= \left\langle \mathcal{R}(A \wedge A(x \wedge y)), A \wedge A(z \wedge w) \right\rangle \\
&= \left\langle \mathcal{R}^\rho(A(x), A(y))A(z), A(w) \right\rangle \\
&= \left\langle A^{-1}(\mathcal{R}^\rho(A(x), A(y))A(z)), w \right\rangle_A,
\end{aligned}
$$

and the claim follows.

□

3.2.1.2 The Invariant Components of the Space of Algebraic Curvature Operators

The Bianchi map B is O(V)-equivariant and hence, the space of algebraic curvature operators is an O(V)-invariant subspace of Sym $\left(\bigwedge^2 \mathrm{V}\right)$. This leads immediately to the question, if the space of algebraic curvature operators decomposes into further invariant subspaces. In this subsection we will idendify and investigate the invariant subspaces SCAL, RIC_0 and WEYL and introduce them as eigenspaces of the (self-adjoint) O(V)-equivariant Ricci operator. After that, we compute the dimensions of these spaces and construct examples of Weyl curvature operators. Finally, we prove irreducibility of SCAL and RIC_0.

Lemma 3.2.1.2. *If $\mathcal{S}$ is an algebraic curvature operator on $\bigwedge^2 V$, then we have*

$$\mathcal{S} = 0$$

if and only if for all $x, y \in V$ holds

$$\langle \mathcal{S}(x \wedge y), x \wedge y \rangle = 0$$

Beweis. This follows easily, since if we have

$$\langle \mathcal{S}x \wedge y, x \wedge y \rangle = 0$$

for all $x, y \in V$, then $\mathcal{S}$ must be skew-adjoint. But on the other hand, $\mathcal{S}$ is self-adjoint by definition. Thus it is zero. The other direction is clear.

□

Lemma 3.2.1.3. *Consider the linear map* $\sigma : \mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right) \to \mathrm{Tri(V;V)}$,

$$\sigma(\mathcal{R})(x, y)z := \mathcal{R}^\rho(x, y)z + \mathcal{R}^\rho(x, z)y$$

Then, $\mathcal{R}$ is perpendicular to the space of algebraic curvature operators if and only if $\sigma(\mathcal{R}) = 0$

Beweis. Let $\mathcal{R}$ be perpendicular to the space of algebraic curvature operators. Then we have $\langle \mathcal{R}, \pi \rangle = 0$ for any orthogonal projection π of the form $\langle x \wedge y, \cdot \rangle \otimes x \wedge y$, where $x, y \in V$ with $x \perp y$ and $\|x\| = \|y\| = 1$ - notice that π equals $2\pi_x \wedge \pi_y$, $\pi_x := x^* \otimes x$, $\pi_y := y^* \otimes y$, so π is an algebraic curvature operator by example 3.1.2.2. Now define $e_1 = x$, $e_2 = y$ and extend $\{e_1, e_2\}$ to an orthonormal basis $\{e_1, e_2, e_3, ..., e_n\}$ of V.

$$\begin{aligned} 0 = 2\langle \mathcal{R}, \pi \rangle &= \sum_{i,j} \langle \mathcal{R}(e_i \wedge e_j), \pi(e_i \wedge e_j) \rangle \\ &= \langle \mathcal{R}(x \wedge y), x \wedge y \rangle = \langle \mathcal{R}^\rho(x, y)y, x \rangle \end{aligned}$$

This implies $\sigma(\mathcal{R})(x, y)z = 0$, whenever x, y and z are mutually perpendicular, since $\sigma(\mathcal{R})$ is symmetric in y and z. Hence, $\sigma(\mathcal{R})$ must be 0.
Now assume that $\sigma(\mathcal{R}) = 0$. Decompose $\mathcal{R} = \mathcal{S} + \mathcal{T}$ with $\mathcal{S} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ and $\mathcal{T} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)^\perp$. Then, $\sigma(\mathcal{R}) = \sigma(\mathcal{S}) + \sigma(\mathcal{T}) = \sigma(\mathcal{S})$, since $\mathcal{T}$ is perpendicular to $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$. This gives

$$\sigma(\mathcal{S}) = 0.$$

By lemma 3.2.1.2 it follows that $\mathcal{S} = 0$, so that $\mathcal{R} = \mathcal{T}$ is actually perpendicular to $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$. □

Corollary 3.2.1.4. *The orthogonal complement of* $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ *in* $\mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$ *is canonically isomorphic to* $\bigwedge^4 V$.

Beweis. For any $\mathcal{R} \in \mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$ we may define a 4-linear map $\omega_\mathcal{R}$ on V, letting

$$\omega_\mathcal{R}(x, y, z, w) := -\langle \mathcal{R}^\rho(x, y)z, w \rangle.$$

If $\mathcal{R}$ is perpendicular to the space of algebraic curvature operators, then $\omega_\mathcal{R}$ is a 4-form on V by lemma 3.2.1.3.
Conversely, if ω is a 4-form on V, we may interpret ω as a symmetric bilinear map $\widetilde{\omega}$ on $\bigwedge^2 V$,

$$\widetilde{\omega}(\varepsilon, \delta) := \omega(\varepsilon \wedge \delta).$$

Since $\widetilde{\omega}$ is symmetric, there exists a uniquely defined self-adjoint endomorphism $\mathcal{R}_\omega$ of $\bigwedge^2 V$, such that

$$\widetilde{\omega}(\varepsilon.\delta) = \langle \mathcal{R}_\omega \varepsilon, \delta \rangle .$$

This gives

$$\omega(x,y,z,w) = -\left\langle \mathcal{R}^\rho_\omega(x,y)z, w \right\rangle ,$$

which implies $\sigma(\mathcal{R}_\omega) = 0$. Using lemma 3.2.1.3, it follows that $\mathcal{R}_\omega$ is perpendicular to $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$.
These to linear constructions are obviously inverse to each other:

$$\mathcal{R}_{\omega_\mathcal{R}} = \mathcal{R}, \omega_{\mathcal{R}_\omega} = \omega.$$

This shows that the map $\omega : \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)^\perp \to \bigwedge^4 \mathrm{V} : \mathcal{R} \mapsto \omega_\mathrm{R}$ is an isomorphism. □

Next we consider the O(V)-equivariant linear map $\mathrm{Ric} : \mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right) \to \mathrm{Sym}\,(\mathrm{V})$, defined by

$$\mathrm{Ric}(\mathcal{R})(\mathrm{x}) = \sum_\mathrm{i} \mathcal{R}^\rho(\mathrm{x}, \mathrm{e_i})\mathrm{e_i},$$

where $\{e_i\}$ is an orthonormal basis of V. Note that for all $x, y \in V$ holds:

$$\langle \mathrm{Ric(x), y} \rangle = \sum_i \langle \mathcal{R}^\rho(x, e_e)e_i), y\rangle = \sum_i \langle \mathcal{R}^\rho(e_i, x)y, e_i\rangle = \mathrm{tr(z} \mapsto \mathcal{R}^\rho\mathrm{(z,x)y)},$$

so our definition of Ric is clearly independent of the choice of the orthonormal basis $\{e_i\}$. Ric is called the Ricci operator.
The Ricci operator gives rise to a "new" O(V)-equivariant operator $\mathrm{Ric} : \mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right) \to \mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$,

$$\mathcal{R} \mapsto \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id_V}.$$

The O(V)-equivariance of Ric guaranties that the space of algebraic curvature operators is mapped onto itself. But Ric has another property, which allows us to compute the decomposition of $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ into O(V)-invariant subspaces. We have

Lemma 3.2.1.5. *The map* $\mathcal{R} \mapsto \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id_V}$ *is self-adjoint w.r.t. the induced scalar product on* $\mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$

Beweis. Let $\mathcal{R}$ and $\mathcal{S}$ be self-adjoint endomorphisms of $\bigwedge^2 V$ and $\{e_i\}$ an orthonormal basis of V. Then, using lemma 1.1.0.3.2 in the third step of the following computation, we get

$$\begin{aligned}
\langle \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id_V}, \mathcal{S} \rangle &= \langle \mathcal{S}, \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id_V} \rangle \\
&= \frac{1}{4} \sum_{i,j} \langle \mathcal{S}(e_i \wedge e_j), \mathrm{Ric}(\mathcal{R})(\mathrm{e_i}) \wedge \mathrm{e_j} + \mathrm{e_i} \wedge \mathrm{Ric}(\mathcal{R})(\mathrm{e_j}) \rangle \\
&= \frac{1}{4} \sum_{i,j} \langle \mathcal{S}^\rho(e_i, e_j) e_j), \mathrm{Ric}(\mathcal{R})(\mathrm{e_i}) \rangle + \langle \mathcal{S}^\rho(e_j, e_i) e_i), \mathrm{Ric}(\mathcal{R})(\mathrm{e_j}) \rangle \\
&= \frac{1}{2} \langle \mathrm{Ric}(\mathcal{S}), \mathrm{Ric}(\mathcal{R}) \rangle
\end{aligned}$$

This shows that the map $(\mathcal{R}, \mathcal{S}) \mapsto \langle \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id_V}, \mathcal{S} \rangle$ is symmetric and we are done. □

The eigenspaces $\mathrm{E}_\lambda(\mathrm{Ric})$, $\lambda \in \mathbb{R}$, of Ric are O(V)-invariant subspaces of $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) \oplus \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)^\perp$. Lemma 3.2.1.3 assures $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)^\perp$ lies in the kernel of Ric. Thus, $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ splits orthogonally and O(V)-invariantly into $\mathrm{im(Ric)} \oplus \left(\ker(\mathrm{Ric}) \cap \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)\right)$. We define

$$\mathrm{WEYL} := \ker(\mathrm{Ric}) \cap \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$$

and call it the space of algebraic Weyl curvature operators. On the other hand, it is clear, that every non-zero eigenvector of Ric has the form $F \wedge \mathrm{id_v}$ with $F \in \mathrm{Sym}\,(\mathrm{V})$. The following technical lemma will help us to find all the nonzero eigenvalues.

Lemma 3.2.1.6. *If F is a self-adjoint endomorphism of V,then*

1. $\mathrm{tr}(\mathrm{F} \wedge \mathrm{id}) = \frac{\mathrm{n}-1}{2} \mathrm{tr}(\mathrm{F})$
2. $\mathrm{tr}(\mathrm{F} \wedge \mathrm{F}) = \frac{1}{2}(\mathrm{tr}(\mathrm{F})^2 - \|\mathrm{F}\|^2)$
3. $\mathrm{Ric}(\mathrm{F} \wedge \mathrm{id}) = \frac{\mathrm{n}-2}{2}\mathrm{F} + \frac{1}{2}\mathrm{tr}(\mathrm{F})\mathrm{id}$
4. $\mathrm{Ric}(\mathrm{F} \wedge \mathrm{F}) = -\mathrm{F}^2 + \mathrm{tr}(\mathrm{F})\mathrm{F}$
5. *If, in addition,* $\mathrm{tr}(\mathrm{F}) = 0$*, then*
   $\mathrm{W}(\mathrm{F} \wedge \mathrm{F}) = \mathrm{F} \wedge \mathrm{F} - \frac{1}{(\mathrm{n}-1)(\mathrm{n}-2)} \|\mathrm{F}\|^2 \,\mathrm{id} + \frac{2}{\mathrm{n}-2}\mathrm{F}^2 \wedge \mathrm{id}$

Beweis. Choose an orthonormal eigenbasis $\{e_i\}$ of F and write

$$F = \sum_i f_i e_i^* \otimes e_i,$$

$f_i \in \mathbb{R}$ the eigenvalues of F. Then

$$F \wedge \mathrm{id}(\mathrm{e_i} \wedge \mathrm{e_j}) = \frac{1}{2}(\mathrm{F(e_i)} \wedge \mathrm{e_j} + \mathrm{F(e_i)} \wedge \mathrm{e_j}) = \frac{\mathrm{f_i} + \mathrm{f_j}}{2}\mathrm{e_i} \wedge \mathrm{e_j}$$

and

$$F \wedge F(e_i \wedge e_j) = F(e_i) \wedge F(e_j) = f_i f_j e_i \wedge e_j.$$

We compute

1.

$$\begin{aligned}
\mathrm{tr(F \wedge id)} &= \sum_{i<j} \langle F \wedge \mathrm{id}(\mathrm{e_i} \wedge \mathrm{e_j}), \mathrm{e_i} \wedge \mathrm{e_j} \rangle \\
&= \sum_{i \neq j} \frac{f_i + f_j}{4} = \frac{1}{4} \sum_i \sum_{j \neq i} f_i + f_j \\
&= \frac{1}{4} \sum_i \left((n-1) f_i + \sum_{j \neq i} f_j \right) \\
&= \frac{1}{4} \sum_i ((n-1) f_i + tr(F) - f_i) \\
&= \frac{n-1}{2} \mathrm{tr(F)}
\end{aligned}$$

2.

$$\begin{aligned}
2\mathrm{tr(F \wedge F)} &= \sum_{i \neq j} f_j f_j = \sum_i f_i \sum_{j \neq i} f_j = \sum_i f_i (\mathrm{tr(F)} - \mathrm{f_i}) \\
&= \mathrm{tr(F)}^2 - \|\mathrm{F}\|^2
\end{aligned}$$

3. For all $i = 1, ..., n$ holds

$$\begin{aligned}
\mathrm{Ric(F \wedge id)(e_i)} &= \sum_j (F \wedge id)^\rho (e_i, e_j) e_j = -\sum_j \rho(F \wedge id(e_i \wedge e_j)) e_j \\
&= -\frac{1}{2} \sum_{j \neq i} \rho(Fe_i \wedge e_j) e_j + \rho(e_i \wedge Fe_j) e_j \\
&= \frac{1}{2} \sum_{j \neq i} (f_i + f_j) e_i = \frac{n-1}{2} F e_i + \frac{1}{2} \sum_{j \neq i} f_j e_i \\
&= \left(\frac{n-2}{2} F + \frac{1}{2} \mathrm{tr(F)id} \right) e_i
\end{aligned}$$

4. For all $i = 1, ..., n$ holds

$$\begin{aligned}
\mathrm{Ric(F \wedge F)(e_i)} &= \sum_j (F \wedge F)^\rho(e_i, e_j)e_j = -\sum_j \rho(F \wedge F(e_i \wedge e_j))e_j \\
&= -\sum_{j \neq i} \rho(Fe_i \wedge Fe_j)e_j = -\sum_{j \neq i} f_i f_j \rho(e_i \wedge e_j)e_j \\
&= \sum_{j \neq i} (f_i f_j)e_i = f_i(\mathrm{tr(F) - f_i)e_i} \\
&= \left(-F^2 + \mathrm{tr(F)F}\right) e_i
\end{aligned}$$

5. We have

$$\mathrm{W(F \wedge F) = F \wedge F - \frac{1}{N}tr(F \wedge F)id_{\bigwedge^2 V} - \frac{2}{n-2}Ric_0(F \wedge F) \wedge id_V}$$

Using 2. we get

$$\mathrm{tr(F \wedge F)} = -\frac{1}{2} \|\mathrm{F}\|^2 .$$

and 4. implies

$$\mathrm{Ric(F \wedge F) = -F^2}$$

so

$$\mathrm{Ric_0(F \wedge F)} = -F^2 + \frac{1}{n} \|F\|^2 id_V$$

Putting all together the claim follows.

□

Corollary 3.2.1.7. *Let* $\dim V = n$, $n \geq 2$. *The spectrum of the Ricci operator equals* $\{0, n-1, \frac{n-2}{2}\}$. *Moreover, if* $\mathcal{R}$ *is an algebraic curvature operator, we have*

1. $\mathcal{R} \in \mathrm{WEYL}$ *if and only if* $\mathrm{Ric}(\mathcal{R}) = 0$
2. $\mathcal{R} = \lambda \mathrm{id}_{\bigwedge^2 \mathrm{V}}$, $\lambda \in \mathbb{R}$, *if and only if* $\mathrm{Ric}(\mathcal{R}) = (\mathrm{n}-1)\mathcal{R}$
3. $\mathcal{R} = F \wedge \mathrm{id_V}$, $F \in \mathrm{Sym_0\,(V)}$, *if and only if* $\mathrm{Ric}(\mathcal{R}) = \frac{\mathrm{n}-2}{2}\mathcal{R}$

Beweis. □

Now we define

- $\mathrm{SCAL} := \mathrm{E_{n-1}(Ric)} = \left\{\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) : \mathcal{R} = \lambda \mathrm{id}, \lambda \in \mathbb{R}\right\}$, the space of algebraic scalar curvature operators ,
- $\mathrm{RIC_0} := \mathrm{E_{\frac{n-2}{2}}(Ric)} = \left\{\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) : \mathcal{R} = \mathrm{F \wedge id, F \in Sym_0\,(V)}\right\}$, the space of algebraic traceless Ricci curvature operators and

- $\mathrm{WEYL} := \ker(\mathrm{Ric}) \cap \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$, the space of algebraic Weyl curvature operators.

From the above we conclude that $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ decomposes orthogonally and O(V)-invariantly as the direct sum

$$\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) = \mathrm{SCAL} \oplus \mathrm{RIC}_0 \oplus \mathrm{WEYL}.$$

Consequently, we can write every given algebraic curvature operator $\mathcal{R}$ in the form

$$\mathcal{R} = \underbrace{\frac{1}{N}\mathrm{tr}(\mathcal{R})\mathrm{id}_{\bigwedge^2 \mathrm{V}}}_{\in \mathrm{SCAL}} + \underbrace{\frac{2}{n-2}\mathrm{Ric}_0(\mathcal{R}) \wedge \mathrm{id}_{\mathrm{V}}}_{\in \mathrm{RIC}_0} + \underbrace{\mathrm{W}(\mathcal{R})}_{\in \mathrm{WEYL}}.$$

The whole space of self-adjoint endomorphisms decomposes orthogonally and $O(V)$-invariantly as

$$\mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right) = \mathrm{SCAL} \oplus \mathrm{RIC}_0 \oplus \mathrm{WEYL} \oplus \ker(\sigma).$$

Having established these results, two questions arise: What about irreducibility of the summands what about their dimensions? The second question is much easier to answer. So we will treat this question first.

Proposition 3.2.1.8. *Let V be an n-dimensional vector space and $N := \dim(\bigwedge^2 V) = \binom{n}{2}$, then*

- $\dim \mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right) = \binom{\mathrm{N}+1}{2}$
- $\dim \ker(\sigma) = \binom{n}{4}$
- $\dim \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) = \frac{1}{12}\mathrm{n}^2(\mathrm{n}^2 - 1)$
- $\dim \mathrm{SCAL} = 1$, *if* $n \geq 2$
- $\dim \mathrm{RIC}_0 = \binom{\mathrm{n}+1}{2} - 1$, *if* $n \geq 2$
- $\dim \mathrm{WEYL} = \frac{\mathrm{n}-3}{2}\binom{\mathrm{n}+2}{3}$, *if* $n \geq 3$ *and* $\dim \mathrm{WEYL} = 0$ *if* $n = 2$.

Beweis. Just compute. There is no trick. □

Proposition 3.2.1.8 asserts that there is always Weyl curvature in dimension $n \geq 4$. How do Weyl curvature operators look like?

Proposition 3.2.1.9. *If F and G are self-adjoint endomorphisms of V with vanishing trace and $FG + GF = 0$ then $F \wedge G$ is an algebraic Weyl curvature operator.*

Beweis. Using the formula

$$2F \wedge G = (F+G) \wedge (F+G) - F \wedge F - G \wedge G.$$

and lemma 3.2.1.6, it follows that

$$\begin{aligned}\mathrm{W(F \wedge G)} = F \wedge G &- \frac{1}{(n-1)(n-2)}(\langle F, G\rangle - \mathrm{tr(F)tr(G)})\mathrm{id}_{\bigwedge^2 \mathrm{V}} \\ &+ \frac{1}{n-2}(FG + GF - \mathrm{tr(F)G} - \mathrm{tr(G)F}) \wedge \mathrm{id_V}\end{aligned}$$

for all self-adjoint endomorphisms F and G of V. But

$$\langle F, G\rangle = \mathrm{tr(F^*G)} = \mathrm{tr(FG)} = \frac{1}{2}\mathrm{tr(FG + GF)} = 0,$$

so

$$\mathrm{W(F \wedge G) = F \wedge G},$$

since we also have $\mathrm{tr(F)} = \mathrm{tr(G)} = 0$ by our assumptions. □

Remark 3.2.1.10. If the dimension of V is greater then three it is always possible to find self-adjoint endomorphisms F and G of V with vanishing trace and $FG + GF = 0$, such that $F \wedge G \neq 0$. Let $\{e_i\}$ be an orthonormal basis of V and define $F := e_1^* \otimes e_2 + e_2^* \otimes e_1$, $G := e_3^* \otimes e_4 + e_4^* \otimes e_3$.

Now we turn our attention towards the question of irreducibility of the summands appearing in the O(V)-invariant decomposition of $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$. As SCAL is 1-dimensional, it is clearly irreducible. What about $\mathrm{RIC_0}$?

Proposition 3.2.1.11. $\mathrm{RIC_0}$ *is irreducible in any dimension.*

Beweis. SO(V) acts on $\mathrm{RIC_0}$ via conjugation on the first wedge-factor. (every element of $\mathrm{RIC_0}$ is of the form $F \wedge \mathrm{id}$, with $\mathrm{tr(F)} = 0$). Therefore, it is sufficient to prove that $\mathrm{Sym_0(V)}$ is irreducible. We will prove this in an extra lemma.

□

Lemma 3.2.1.12. *If V is a Euclidean vector space of finite dimension n, then* $\mathrm{Sym_0(V)}$ *is irreducible w.r.t. the canonical action of* SO(V).

Beweis. If F is a self-adjoint endomorphism of V then we find an orthonormal Basis $\{e_i\}$ of V and real numbers $f_1, f_2, ..., f_n$ such that $F(e_i) = f_i e_i$ for all $i = 1, ..., n$. Thus, we may write F in the form

$$F = \sum_{k=1}^{n-1}\left(\sum_{i=1}^{k} f_i\right)(e^k \otimes e_k - e^{k+1} \otimes e_{k+1}) + \mathrm{tr(F)e^n \otimes e_n}.$$

If F has vanishing trace, this expression reduces to

$$F = \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k} f_i \right) (e^k \otimes e_k - e^{k+1} \otimes e_{k+1}),$$

showing that F lies in the linear span of the set

$$M := \{x^* \otimes x - y^* \otimes y : x, y \in V, \|x\| = \|y\| = 1, y \perp x\}.$$

M is obviously SO(V)-invariant. And, since SO(V) is acting transitively on the set of positively oriented orthonormal n-frames of V, it is also acting transitively on M. This shows, that every SO(V)-invariant subspace U of $\mathrm{Sym}_0\,(\mathrm{V})$ equals $\mathrm{Sym}_0\,(\mathrm{V})$, if it contains at least one element of M.
Let U be a proper SO(V)-invariant subspace of $\mathrm{Sym}_0\,(\mathrm{V})$ and pick some $F \in U \setminus \{0\}$. Then the range of F is greater than or equal to 2, otherwise, F would be the zero map, since F is skew-adjoint. If the range of F equals 2, then we are done. Since in this case, we get $F = f_1(e_1 \otimes e_1 - e_2 \otimes e_2)$ from the equation from above. Thus, we have proved our claim, if we can show:
If U contains an element F of range $m \geq 3$, then U contains an element F' of Rank m' with $2 \leq m' < m$.
Let $F \in U$ be an element of range m. Then we have $F = \sum_{i=1}^{m} f_i e_i^* \otimes e_i$, with $f_1, ..., f_m \neq 0$. If M is even, then the map

$$F' := f_1 e_1 \otimes e_1 + \sum_{i=2}^{m-1} f_{i+1} e_i \otimes e_i + f_2 e_m \otimes e_m$$

is also an element of U and so is the difference

$$F - F' = \sum_{i=2}^{m-1} (f_{i+1} - f_i) e_i \otimes e_i + (f_2 - f_m) e_m \otimes e_m.$$

If the range of $F - F'$ is different from zero, then it lies between 2 and m-1. If the range is zero, then $f_2 = f_3 = ... = f_m$ and we didn't gain anything. But, since F was assumed to be nonzero with vanishing trace, we get

$$f_i = (m-1) f_1$$

for all $i = 2, ..., n$ in this case. So we simply conjugate F with $G \in \mathrm{SO(V)}$ which interchanges e_1 with e_2. Writing $H := GFG^{-1}$, we get $H - H' \neq 0$ and hence, $2 \leq \mathrm{range(H - H')} < \mathrm{m}$. Now we assume m to be odd:
In this case define

$$F' := f_1 e_1 \otimes e_1 + \sum_{i=2}^{m-2} f_{i+1} e_i \otimes e_i + f_2 e_{m-1} \otimes e_{m-1} - f_m e_m \otimes e_m.$$

Then, if $F - F' = 0$, then $f_m = 0$, which is impossible, since the range of F was assumed to be m. This shows that the range of $F - F'$ lies between 2 and $m - 1$. □

What about the question of irreducibility of WEYL? If we consider the action of O(V) on WEYL, the answer is "yes" in every dimension. If we consider the action of SO(V) on WEYL instead, the answer depends on the dimension of V. It turns out that WEYL is not SO(V)-irreducible in dimension 4, but SO(V)-irreducible in any other dimension. The case $n \geq 5$ requires some more work than we can do here. We refer to [4] Exposé IX and [2] pp. 82-83 for the proof. The case $n = 4$ will be treated in section 3.2.2.

3.2.1.3 How does $\mathrm{LC}\left(\bigwedge^2 \mathrm{U}\right)$ embed into $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ if U is a Subspace of V?

Next we want to see how $\mathrm{LC}\left(\bigwedge^2 \mathrm{U}\right)$ embeds into $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$, if U is a subspace of V. By now, we know that every algebraic curvature operator $\mathcal{R}$ on $\bigwedge^2 U$ is a linear combination of curvature operators of the form $F \wedge G$, where F and G are self-adjoint endomorphisms of U. We simply define $\iota(F \wedge G) := \widetilde{F} \wedge \widetilde{G}$, where $\widetilde{F}$ and $\widetilde{G}$ are the standard embeddings of F and G into the space of (self-adjoint) endomorphisms of V ($\widetilde{F} = F \circ \pi$, where π is the orthogonal projection $V \to U$). As one sees easily, ι is O(U)-equivariant and isometric. But it does not preserve the irreducible splittings of $\mathrm{LC}\left(\bigwedge^2 \mathrm{U}\right)$ and $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$: $\mathrm{id}_{\bigwedge^2 \mathrm{U}}$ is mapped to $\pi \wedge \pi$, and $\mathrm{Ric_o}(\pi \wedge \pi) \neq 0 \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$. At least one can say that ι maps the span $\left\langle \mathrm{id}_{\bigwedge^2 \mathrm{U}} \right\rangle \subseteq \mathrm{LC}\left(\bigwedge^2 \mathrm{U}\right)$ to $\mathrm{SCAL} \oplus \mathrm{RIC}_0 \subseteq \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$: The O(U)-equivariance of ι implies that the image of $\left\langle \mathrm{id}_{\bigwedge^2 \mathrm{U}} \right\rangle$ under ι is a one dimensional O(U)-invariant subspace in $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$, so its projections to SCAL,RIC_0 and WEYL will be O(U)-invariant as well and at most one dimensional. This means that the Weyl part $\mathcal{W}$ of $\pi \wedge \pi$ is invariant under conjugation with elements of O(U), so $\mathcal{W}$ must be zero by corollary 3.4.0.22.
A simple computation using lemma 3.2.1.6 shows that $\mathrm{Ric}(\widetilde{\mathrm{F}} \wedge \widetilde{\mathrm{G}})$ equals $\widetilde{\mathrm{Ric}(\mathrm{F} \wedge \mathrm{G})}$, which implies that ι preserves WEYL. Further, we get that ι maps $\mathrm{RIC}_0(\mathrm{U}) \subseteq \mathrm{LC}\left(\bigwedge^2 \mathrm{U}\right)$ to $\mathrm{SCAL} \oplus \mathrm{RIC}_0 \subseteq \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$.

3.2.1.4 The Bundle of Algebraic curvature Operators over a Riemannian Manifold

Let (M, g) be a Riemannian manifold. Within each fiber of the vector bundle of self-adjoint bundle maps $\left(\bigwedge^2 \mathrm{TM}, \mathrm{g}_{\bigwedge^2 \mathrm{TM}}\right) \to \left(\bigwedge^2 \mathrm{TM}, \mathrm{g}_{\bigwedge^2 \mathrm{TM}}\right)$ we have a subspace $\mathrm{LC}\left(\bigwedge^2 \mathrm{T_pM}, \mathrm{g}_{\bigwedge^2 \mathrm{T_pM}}\right)$ of algebraic curvature operators on $\left(\bigwedge^2 \mathrm{T_pM}, \mathrm{g}_{\bigwedge^2 \mathrm{T_pM}}\right)$. The collection of all these spaces forms the vector bundle $\mathrm{LC}(\mathrm{M},\mathrm{g})$ of algebraic curvature operators on (M,g).

Proposition 3.2.1.13. $\mathrm{LC}(\mathrm{M},\mathrm{g})$ *is a parallel subbundle of* $\mathrm{Sym}\left(\bigwedge^2 \mathrm{TM}, \mathrm{g}_{\bigwedge^2 \mathrm{TM}}\right)$

Beweis. We have seen in section 1.4 that the map $\rho : \bigwedge^2 \mathrm{TM} \to \mathfrak{so}(\mathrm{TM})$,

$$\rho(x_p \wedge y_p) = g_p(x_p, \cdot) \otimes y_p - g_p(y_p, \cdot) \otimes x_p$$

is parallel w.r.t. the connection induced by the Levi-Civita connection. Thus, the first Bianchi identity is a parallel condition. □

3.2.2 Curvature and Dimension

The structure of the space of algebraic curvature operators $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ depends on the dimension of the underlying vector space V. For example, we have $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) = 0$ if the dimension of V is less than 2, and $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) = \mathrm{SCAL}$, if $\dim V = 2$, as $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ must be one-dimensional in this case (see proposition 3.2.1.8).

3.2.2.1 Curvature in Dimension n=3

In dimension three, each algebraic curvature operator is completely determined by its Ricci curvature:
An easy computation shows that if $\mathrm{Ric}(\mathcal{R})$ is diagonal w.r.t. the orthonormal basis (e_1, e_2, e_3) with eigenvalues λ_1, λ_2 and λ_3, then $\mathcal{R}$ is diagonal w.r.t. the orthonormal basis $(e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1)$ with eigenvalues $\mathrm{tr}(\mathcal{R}) - \lambda_3, \mathrm{tr}(\mathcal{R}) - \lambda_1$, and $\mathrm{tr}(\mathcal{R}) - \lambda_2$, respectively. This implies that $\mathcal{R} = 0$ if $\mathrm{Ric}(\mathcal{R}) = 0$ ($\mathrm{tr}(\mathcal{R}) = \frac{1}{2}\mathrm{tr}(\mathrm{Ric}(\mathcal{R}))$), so there can be no Weyl curvature in dimension 3 and we get $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) = \mathrm{SCAL} \oplus \mathrm{RIC}_0$.
Recall that remark 3.2.1.10 tells us, that there is always Weyl curvature in dimension greater than 3. The case $\dim V = 4$ is special, since WEYL is not irreducible in this case.

3.2.2.2 Curvature in Dimension n=4

Let V be an oriented Euclidean vector space of dimension 4. As we have seen in theorem 1.2.0.8, $\bigwedge^2 V$ splits as a Lie algebra as the direct sum of the $\pm$-eigenspaces $\bigwedge^{\pm} V$ of the Hodge $*$-operator. This has certain consequences concerning the structure of the space of algebraic curvature operators: The curvature operators of traceless Ricci type will interchange the eigenspace of the Hodge $*$-operator and the space of algebraic Weyl curvature operators will split orthogonally into two irreducible subspaces WEYL^+ and WEYL^-.

Proposition 3.2.2.1. *The traceless Ricci part of any algebraic curvature operator $\mathcal{R}$ on $\bigwedge^2 V$ interchanges $\bigwedge^+ V$ and $\bigwedge^- V$.*

Beweis. Pick an algebraic curvature operator $\mathcal{R}$ of traceless Ricci type. Then $\mathcal{R}$ has the form $\mathcal{R} = F \wedge \mathrm{id}$, where $F \in \mathrm{Sym}_0\,(\mathrm{V})$ is a self-adjoint endomorphism of V with vanishing trace. Let (e_1, e_2, e_3, e_4) be a positively oriented eigenbasis of F and write $F = \sum_i f_i e_i^* \otimes e_i$, $f_i \in \mathbb{R}$ the eigenvalues of F. Then $e_1 \wedge e_2 + e_3 \wedge e_4$ lies in $\bigwedge^+ V$ and

$$\begin{aligned} 2F \wedge \mathrm{id}(\mathrm{e}_1 \wedge \mathrm{e}_2 + \mathrm{e}_3 \wedge \mathrm{e}_4) &= (f_1 + f_2) e_1 \wedge e_2 + (f_3 + f_4) e_3 \wedge e_4 \\ &= (f_1 + f_2) e_1 \wedge e_2 - (f_1 + f_2) e_3 \wedge e_4 \\ &= (f_1 + f_2)(e_1 \wedge e_2 - e_3 \wedge e_4), \end{aligned}$$

so the image of $e_1 \wedge e_2 + e_3 \wedge e_4$ under $F \wedge \mathrm{id}$ lies in $\bigwedge^- V$. The other eigenvectors of the Hodge $*$-operator may be treated in exactly the same way. □

Let $U \subseteq V$ be a three-dimensional subspace. Using the Lie algebra isomorphisms $\iota^{\pm} : \bigwedge^2 U \to \bigwedge^{\pm} V : \varepsilon \mapsto \frac{1}{2}(\varepsilon \pm *\varepsilon)$ (compare theorem 1.2.0.8) and the orthogonal projections $\pi^{\pm} : \bigwedge^2 V \to \bigwedge^{\pm} V$ we can turn any self-adjoint linear operator F on $\bigwedge^2 U$ into a self-adjoint endomorphism F^+ of $\bigwedge^2 V$, which preserves $\bigwedge^+ V$ and satisfies $\bigwedge^- V \subseteq \ker F^+$, letting

$$F^+ := \iota^+ \circ F \wedge \mathrm{id} \circ (\iota^+)^{-1} \circ \pi^+.$$

Analogously, we may define

$$F^- := \iota^- \circ F \wedge \mathrm{id} \circ (\iota^-)^{-1} \circ \pi^-.$$

Straight forward computations, which will be done in the following lemma, show that F^+ and F^- are algebraic curvature operators on $\bigwedge^2 V$ if and only if $F \wedge \mathrm{id_U}$ is an algebraic curvature operator of traceless Ricci type on $\bigwedge^2 U$ and that F^+ and F^- are actually Weyl curvature operators in this case. (They have vanishing trace and preserve $\bigwedge^+ V$ and $\bigwedge^- V$, so their traceless Ricci parts must vanish by proposition 3.2.2.1.)
So let us define

$$\mathrm{WEYL}^{\pm} := \left\{ \mathrm{F}^{\pm} : \mathrm{F} \wedge \mathrm{id_U} \in \mathrm{RIC}_0(\textstyle\bigwedge^2 \mathrm{U}) \right\}.$$

Lemma 3.2.2.2. *F^+ and F^- are algebraic curvature operators if and only if $F : U \to U$ is self-adjoint with vanishing trace, i.e. if $F \wedge \mathrm{id_U} \in \mathrm{RIC}_0(\bigwedge^2 \mathrm{U})$.*

Beweis. We only prove the statement for F^+. The proof of the other statement is almost the same.
First, we show that F^+ is self-adjoint if and only if F is self-adjoint:

Let (e_1, e_2, e_3, e_4) be a positively oriented orthonormal basis of V, such that $U = \langle e_1, e_2, e_3 \rangle$ and $Fe_i == f_i e_i$ for $i = 1, 2, 3$. Then

$$\pi^+(e_i \wedge e_j) = \frac{1}{2}(e_i \wedge e_j + *(e_i \wedge e_j))$$

for all i, j. Thus,

$$F^+(e_1 \wedge e_2 + *(e_1 \wedge e_2)) = 2(f_1 + f_2)(e_1 \wedge e_2 + *(e_1 \wedge e_2)),$$

$$F^+(e_1 \wedge e_3 + *(e_1 \wedge e_3)) = 2(f_1 + f_3)(e_1 \wedge e_3 + *(e_1 \wedge e_3))$$

and

$$F^+(e_2 \wedge e_3 + *(e_2 \wedge e_3)) = 2(f_2 + f_3)(e_2 \wedge e_3 + *(e_2 \wedge e_3)),$$

showing that F^+ is self-adjoint, if F is self-adjoint.
Now, we prove the other direction. If F^+ is self-adjoint, then the restriction of F^+ to $\bigwedge^+ V$ is diagonal in an basis of the form $\{e_1 \wedge e_2 + *(e_1 \wedge e_2), e_1 \wedge e_3 + *(e_1 \wedge e_3), e_2 \wedge e_3 + *(e_2 \wedge e_3)\}$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of U and (e_1, e_2, e_3, e_4) is a positively oriented orthonormal basis of V. Now we have

$$F^+(e_1 \wedge e_2 + *(e_1 \wedge e_2)) = f^+_{1,2}(e_1 \wedge e_2 + *(e_1 \wedge e_2)),$$

and, as an easy computation shows,

$$F^+(e_1 \wedge e_2 + *(e_1 \wedge e_2)) = F \wedge \mathrm{id}(e_1 \wedge e_2) + *(\mathrm{F} \wedge \mathrm{id}(e_1 \wedge e_2)).$$

This gives

$$f^+_{1,2} e_1 \wedge e_2 - \frac{1}{2}(Fe_1 \wedge e_2 + e_1 \wedge Fe_2) + *(f^+_{1,2} e_1 \wedge e_2 - \frac{1}{2}(Fe_1 \wedge e_2 + e_1 \wedge Fe_2)) = 0.$$

This implies that $f^+_{1,2} e_1 \wedge e_2 - \frac{1}{2}(Fe_1 \wedge e_2 + e_1 \wedge Fe_2)$ lies in the -1-eigenspace of the Hodge $*$-operator. Hence, this sum must be 0, for F preserves the linear span of e_1, e_2 and e_3. We get

$$\frac{1}{2}((f^+_{1,2} e_1 - Fe_1) \wedge e_2) + e_1 \wedge (f^+_{1,2} e_2 - Fe_2) = 0$$

which gives

$$Fe_1 = f_1 e_1, Fe_2 = f_2 e_2 \text{ and } f^+_{1,2} = \frac{f_1 + f_2}{2}.$$

Analogously, we get $Fe_3 = f_3 e_3$. Hence, F is self-adjoint.

We are left showing that F^+ is an algebraic curvature operator if and only the self-adjoint map F has vanishing trace:
It is clear that $\mathrm{B}(\mathrm{F}^+)(\mathrm{x}, \mathrm{y})\mathrm{y} = 0$ for all $x, y \in V$, no matter whether F has

vanishing trace or not. We treat the other cases:
Keeping the notation from above, we compute

$$\begin{aligned}(F^+)^\rho(e_1,e_2)e_3 &= -\frac{1}{2}\rho(\iota^+(F\wedge \mathrm{id}((\iota^+)^{-1}(\mathrm{e}_1\wedge \mathrm{e}_2+\mathrm{e}_3\wedge \mathrm{e}_4))))(\mathrm{e}_3)\\ &= -\rho(\iota^+(F\wedge \mathrm{id}(\mathrm{e}_1\wedge \mathrm{e}_2)))(\mathrm{e}_3)\\ &= -\frac{1}{2}(f_1+f_2)\rho(\iota^+(e_1\wedge e_2))(e_3)\\ &= -\frac{1}{4}(f_1+f_2)\rho(e_1\wedge e_2+e_3\wedge e_4)(e_3)\\ &= -\frac{1}{4}(f_1+f_2)e_4\end{aligned}$$

In the same way we get

$$(F^+)^\rho(e_2,e_3)e_1 = -\frac{1}{4}(f_2+f_3)e_4$$

and

$$(F^+)^\rho(e_3,e_1)e_2 = -\frac{1}{4}(f_3+f_1)e_4,$$

showing that the sum of these terms equals

$$-\frac{1}{4}(f_1+f_2+f_3)e_4 = -\frac{1}{4}\mathrm{tr}(\mathrm{F})\mathrm{e}_4$$

Moreover, we have

$$\begin{aligned}(F^+)^\rho(e_1,e_2)e_4 &= -\frac{1}{2}\rho(\iota^+(F\wedge \mathrm{id}((\iota^+)^{-1}(\mathrm{e}_1\wedge \mathrm{e}_2+\mathrm{e}_3\wedge \mathrm{e}_4))))(\mathrm{e}_4)\\ &= -\rho(\iota^+(F\wedge \mathrm{id}(\mathrm{e}_1\wedge \mathrm{e}_2)))(\mathrm{e}_4)\\ &= -\frac{1}{2}(f_1+f_2)\rho(\iota^+(e_1\wedge e_2))(e_4)\\ &= -\frac{1}{4}(f_1+f_2)\rho(e_1\wedge e_2+e_3\wedge e_4)(e_4)\\ &= \frac{1}{4}(f_1+f_2)e_3,\end{aligned}$$

$$\begin{aligned}(F^+)^\rho(e_2,e_4)e_1 &= \frac{1}{2}\rho(\iota^+(F\wedge \mathrm{id}((\iota^+)^{-1}(\mathrm{e}_1\wedge \mathrm{e}_3-\mathrm{e}_2\wedge \mathrm{e}_4))))(\mathrm{e}_1)\\ &= \rho(\iota^+(F\wedge \mathrm{id}(\mathrm{e}_1\wedge \mathrm{e}_3)))(\mathrm{e}_1)\\ &= \frac{1}{2}(f_1+f_3)\rho(\iota^+(e_1\wedge e_3))(e_1)\\ &= \frac{1}{4}(f_1+f_3)\rho(e_1\wedge e_3-e_2\wedge e_4)(e_1)\\ &= \frac{1}{4}(f_1+f_3)e_3\end{aligned}$$

and

$$\begin{aligned}(F^+)^\rho(e_4, e_1)e_2 &= \frac{1}{2}\rho(\iota^+(F \wedge \mathrm{id}((\iota^+)^{-1}(\mathrm{e}_1 \wedge \mathrm{e}_4 + \mathrm{e}_2 \wedge \mathrm{e}_3))))(\mathrm{e}_2)\\ &= \rho(\iota^+(F \wedge \mathrm{id}(\mathrm{e}_2 \wedge \mathrm{e}_3)))(\mathrm{e}_2)\\ &= \frac{1}{2}(f_2 + f_3)\rho(\iota^+(e_2 \wedge e_3))(e_2)\\ &= \frac{1}{4}(f_2 + f_3)\rho(e_1 \wedge e_4 + e_2 \wedge e_3)(e_2)\\ &= \frac{1}{4}(f_2 + f_3)e_3,\end{aligned}$$

so their sum equals

$$\frac{1}{4}(f_1 + f_2 + f_3)e_3 = \frac{1}{4}\mathrm{tr}(\mathrm{F})\mathrm{e}_3$$

This shows

$$\mathrm{B}(\mathrm{F}^+)(\mathrm{e}_1, \mathrm{e}_2)\mathrm{e}_3 = -\frac{1}{4}\mathrm{tr}(\mathrm{F})\mathrm{e}_4$$

and

$$\mathrm{B}(\mathrm{F}^+)(\mathrm{e}_1, \mathrm{e}_2)\mathrm{e}_4 = \frac{1}{4}\mathrm{tr}(\mathrm{F})\mathrm{e}_3$$

Analogously, we may compute

$$\mathrm{B}(\mathrm{F}^+)(\mathrm{e}_1, \mathrm{e}_3)\mathrm{e}_4 = -\frac{1}{4}\mathrm{tr}(\mathrm{F})\mathrm{e}_2$$

and

$$\mathrm{B}(\mathrm{F}^+)(\mathrm{e}_2, \mathrm{e}_3)\mathrm{e}_4 = \frac{1}{4}\mathrm{tr}(\mathrm{F})\mathrm{e}_1.$$

Hence, F^+ satisfies the first Bianchi identity if and only if $\mathrm{tr}(\mathrm{F}) = 0$ and the claim follows.

□

Corollary 3.2.2.3. $\mathrm{WEYL}^\pm \cong \mathrm{Sym}_0\left(\bigwedge^\pm \mathrm{V}\right)$

Beweis. Restricting $\mathcal{W}^\pm \in \mathrm{WEYL}^\pm$ to $\bigwedge^\pm V$, we get an element of $\mathrm{Sym}_0\left(\bigwedge^\pm \mathrm{V}\right)$. The claim follows since both spaces are 5-dimensional. □

The following proposition shows that WEYL decomposes orthogonally as the direct sum of the SO(V)-irreducible subspaces WEYL^+ and WEYL^-.

Proposition 3.2.2.4. *In dimension 4 any algebraic Weyl curvature operator $\mathcal{W}$ on $\bigwedge^2 V$ is of the form*

$$\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^-,$$

where $\mathcal{W}^\pm \in \mathrm{WEYL}^\pm$ and therefore preserves the splitting $\bigwedge^+ V \oplus \bigwedge^- V$. Moreover, WEYL^+ and WEYL^- are $\mathrm{SO(V)}$ irreducible subspaces of WEYL.

Beweis. In dimension 4, the space of algebraic Weyl curvature operators has dimension 10. WEYL^+ and WEYL^- are both 5-dimensional and perpendicular to each other, so they sum up to WEYL. It is easy to see that WEYL^+ and WEYL^- are SO(V)-invariant subspaces of WEYL. Thus, we are left to show that these spaces are SO(V)-irreducible. Actually, we have that $\mathrm{WEYL}^\pm = \mathrm{Sym}_0\left(\bigwedge^\pm \mathrm{V}\right)$, so it is irreducible w.r.t. the action of $\mathrm{SO}(\bigwedge^\pm \mathrm{V})$ on $\mathrm{Sym}\left(\bigwedge^\pm \mathrm{V}\right)$. The claim follows, since we may view $\mathrm{SO}(\bigwedge^\pm \mathrm{V}) \subseteq \mathrm{SO(V)}$ as a subgroup. More precisely, using the notation of corollary 1.2.0.10, we have that the pre-image of $\mathrm{SO}(\bigwedge^\pm \mathrm{V})$ under φ acts irreducibly on $\mathrm{WEYL}^\pm$. □

Combining the results from above, we can say that every algebraic curvature operator $\mathcal{R}$ on $\bigwedge^2 V$ decomposes SO(V)-irreducibly into its SCAL-, RIC_0-, WEYL^+- and WEYL^--part,

$$\mathcal{R} = \frac{1}{6}\mathrm{trRid} + \mathrm{Ric}_0(\mathrm{R}) \wedge \mathrm{id} + \mathrm{W}^+(\mathcal{R}) + \mathrm{W}^-(\mathcal{R}),$$

where $\mathrm{W}^\pm(\mathcal{R}) := \mathrm{W}(\mathcal{R}) \circ \pi^\pm$, $\mathrm{W}(\mathcal{R})$ the Weyl curvature of $\mathcal{R}$ and $\pi^\pm : \bigwedge^2 V \to \bigwedge^\pm V$ the orthogonal projection. Note that the projections $\pi^\pm$ are definitely not algebraic curvature operators. We formulate this fact in an extra lemma:

Lemma 3.2.2.5. *The orthogonal projections* $\pi^\pm \bigwedge^2 V \to \bigwedge^\pm v \subseteq \bigwedge^2 V$ *onto the eigenspaces of the Hodge $*$-operator do not belong to the space of algebraic curvature operators.*

Beweis. If $\pi^\pm$ was actually an algebraic curvature operator, we could decompose it according to the equation from above. Now, $\pi^\pm$ preserves $\bigwedge^\pm V$, which forces $\mathrm{Ric}_0(\pi^\pm)$ to be zero, so we get

$$\pi^\pm = \frac{1}{2}\mathrm{id} + \mathrm{W}(\pi).$$

This implies $\mathrm{W}^\mp(\pi^\pm) = -\frac{1}{2}\pi^\mp$, a contradiction. □

Proposition 3.2.2.6. WEYL *is* O(V)*-irreducible in dimension 4.*

Beweis. Let $U \subseteq \mathrm{WEYL}$ be an O(V)-invariant subspace, $U \neq \{0\}$. Then U is also SO(V)-invariant. From $U \neq 0$ we conclude that U contains WEYL^+ or WEYL^-. W.l.o.g, we assume $\mathrm{WEYL}^+ \subseteq \mathrm{U}$. Let $\mathcal{R} \in \mathrm{WEYL}^+$. Pick $G \in \mathrm{O(V)}$ with $\det(G) = -1$. Then, by lemma 1.2.0.9, G interchanges the eigenspaces of the Hodge -operator $\bigwedge^+ V$ and $\bigwedge^- V$. Thus, $G \wedge G \circ \mathcal{R} \circ (G \wedge G)^{-1}$ is an element of WEYL^-. This implies $U = \mathrm{WEYL}$. □

3.2.2.3 Curvature in Dimension n≥5

In dimension $n \geq 5$, the space of algebraic Weyl curvature operators is irreducible w.r.t. the action of SO(V). We won't prove this fact in our text. We refer to [4] Exposé IX and [2] pp. 82-83 for the proof instead.

Theorem 3.2.2.7. *If the dimension of V is greater than four, then every algebraic Weyl curvature operator is of the form*

$$\sum_{i=1}^{m} F_i \wedge G_i,$$

where F_i and G_i are self-adjoint endomorphisms of V with vanishing trace and $F_iG_i + G_iF_i = 0$ for all $i = 1, ..., m$, $m \in \mathbb{N}$.

Beweis. The bilinear map $\beta : \mathrm{Sym}\,(\mathrm{V}) \times \mathrm{Sym}\,(\mathrm{V}) \to \mathrm{WEYL}$,

$$\beta(F, G) := W(F \wedge G),$$

is SO(V)-equivariant. So it maps SO(V)-invariant subsets of $\mathrm{Sym}\,(\mathrm{V}) \times \mathrm{Sym}\,(\mathrm{V})$ to SO(V)-invariant subsets of WEYL. Since WEYL is irreducible if $\dim V > 4$ each of its nonempty SO(V)-invariant subsets $N \neq \{0\}$ spans WEYL. Thus, it is sufficient to find an SO(V)-invariant subset $M \neq \{0\}$ of $\mathrm{Sym}\,(\mathrm{V}) \times \mathrm{Sym}\,(\mathrm{V})$ with $\beta(M) \neq 0$.
Let $M := \{(F, G) \in \mathrm{Sym}\,(\mathrm{V}) \times \mathrm{Sym}\,(\mathrm{V}) : \mathrm{FG} + \mathrm{GF} = 0\}$. M is obviously invariant under the action of SO(V). Further, by remark 3.2.1.10 and $\dim V > 4$, we have that M and $\beta(M)$ are nonempty.

□

Remark 3.2.2.8. Actually we have shown more: Take F and G of V as in 3.2.1.10. Then the SO(V)-orbit of $F \wedge G$ spans WEYL. It follows that every algebraic Weyl curvature operator W is of the form

$$\mathrm{W} = \sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{w_i g_i . F \wedge G},$$

where $m = \dim \mathrm{WEYL}$, $g_1, ..., g_m \in \mathrm{SO(V)}$ and $w_i \in \mathbb{R}$.

3.2.3 The 1. Bianchi Identity and Alternative Characterizations of Curvature Operators

In this subsection we give two alternative characterizations of algebraic curvature operators, one using the #-product, the other using the representation $\rho : \bigwedge^2 V \to \mathfrak{so}(V)$, eigenvalues and orthonormal eigenbasis. The main difference to the the definition from above is, that these characterizations do not use $(3, 1)$-tensors on V. They only use data of the endomorphisms under consideration and $\bigwedge^2 V$. At the end, this gives a more precise picture of curvature operators in general and much more flexibility in further computations concerning algebraic curvature operators.

Theorem 3.2.3.1. *If $\mathcal{R}$ is a self-adjoint endomorphism of $\bigwedge^2 V$ which is diagonal in an orthonormal basis $\{\varepsilon_i\}$ of $\bigwedge^2 V$, i.e. $\mathcal{R} = \sum_i \lambda_i \varepsilon_i^* \otimes \varepsilon_i$, $\lambda_i \in \mathbb{R}$, then $\mathcal{R}$ is an algebraic curvature operator if and only if*

$$\mathcal{R} = \sum_i \lambda_i \rho(\varepsilon_i) \wedge \rho(\varepsilon_i)$$

Beweis. Let $\mathcal{R} = \sum_i \lambda_i \varepsilon_i^* \otimes \varepsilon_i$ be an algebraic curvature operator and $u, v, x, y \in V$. Then, using Lemma 1.1.0.3 and the first Bianchi identity, we compute

$$\begin{aligned}
\langle \mathcal{R} x \wedge y, u \wedge v \rangle &= \langle \mathcal{R}^\rho(y,x)u, v \rangle = -\langle \mathcal{R}^\rho(x,u)y, v\rangle - \langle \mathcal{R}^\rho(u,y)x, v\rangle \\
&= \langle \rho(\mathcal{R}x \wedge u)y, v\rangle - \langle \rho(\mathcal{R}y \wedge u)x, v\rangle \\
&= \sum_i \langle \rho(\langle \mathcal{R}x \wedge u, \varepsilon_i\rangle \varepsilon_i)y, v\rangle - \langle \rho(\langle \mathcal{R}y \wedge u, \varepsilon_i\rangle \varepsilon_i)x, v\rangle \\
&= \sum_i \langle \rho(\langle x \wedge u, \mathcal{R}\varepsilon_i\rangle \varepsilon_i)y, v\rangle - \langle \rho(\langle y \wedge u, \mathcal{R}\varepsilon_i\rangle \varepsilon_i)x, v\rangle \\
&= \sum_i \lambda_i \left(\langle\langle x \wedge u, \varepsilon_i\rangle \rho(\varepsilon_i)y, v\rangle - \langle\langle y \wedge u, \varepsilon_i\rangle \rho(\varepsilon_i)x, v\rangle\right) \\
&= \sum_i \lambda_i \left(\langle\langle u, \rho(\varepsilon_i)x\rangle \rho(\varepsilon_i)y, v\rangle - \langle\langle u, \rho(\varepsilon_i)y\rangle \rho(\varepsilon_i)x, v\rangle\right) \\
&= \sum_i \lambda_i \left(\langle\langle \rho(\varepsilon_i)x, u\rangle \rho(\varepsilon_i)y - \langle \rho(\varepsilon_i)y, u\rangle \rho(\varepsilon_i)x, v\rangle\right) \\
&= \sum_i \lambda_i \langle \rho\left(\rho(\varepsilon_i)x \wedge \rho(\varepsilon_i)y\right) u, v\rangle \\
&= \left\langle \left(\sum_i \lambda_i \rho(\varepsilon_i) \wedge \rho(\varepsilon_i)\right) x \wedge y, u \wedge v \right\rangle
\end{aligned}$$

Now let $\mathcal{R} = \sum_i \lambda_i \rho(\varepsilon_i) \wedge \rho(\varepsilon_i)$ be given and $x, y, z \in V$. We have to check the 1. Bianchi identity.

$$\begin{aligned}
\mathcal{R}^\rho(x,y)z &= -\sum_i \lambda_i \rho(\rho(\varepsilon_i)x \wedge \rho(\varepsilon_i)y)z \\
&= -\sum_i \lambda_i (\langle \rho(\varepsilon_i)x, z\rangle \rho(\varepsilon_i)y - \langle \rho(\varepsilon_i)y, z\rangle \rho(\varepsilon_i)x) \\
&= -\sum_i \lambda_i \langle \varepsilon_i, x \wedge z\rangle \rho(\varepsilon_i)y - \lambda_i \langle \varepsilon_i, y \wedge z\rangle \rho(\varepsilon_i)x \\
&= -\sum_i \rho(\langle x \wedge z, \lambda_i \varepsilon_i\rangle \varepsilon_i)y + \sum_i \rho(\langle y \wedge z, \lambda_i \varepsilon_i\rangle \varepsilon_i)x \\
&= -\rho(\mathcal{R}x \wedge z)y + \rho(\mathcal{R}y \wedge z)x \\
&= -\mathcal{R}^\rho(z,x)y - \mathcal{R}^\rho(y,z)x
\end{aligned}$$

□

Remark 3.2.3.2. Theorem 3.2.3.1 is somehow remarkable, since every $\rho(\varepsilon_i)$ is skew-adjoint. And if $F : V \to V$ is skew-adjoint, then it is generally not true, that $F \wedge F$ is an algebraic curvature operator.

Corollary 3.2.3.3. *Every algebraic curvature operator $\mathcal{R}$ of range 1 has the form*

$$\mathcal{R} = x^* \wedge y^* \otimes x \wedge y$$

with $x, y \in V$.

Beweis. Pick an algebraic curvature operator $\mathcal{R}$ with range equal to one. $\mathcal{R}$ has the form

$$\mathcal{R} = \varepsilon^* \wedge \varepsilon$$

for some suitable $\varepsilon \in \bigwedge^2 V$. Now, theorem 3.2.3.1 asserts that

$$\varepsilon^* \wedge \varepsilon = \rho(\varepsilon) \wedge \rho(\varepsilon).$$

This implies that the range of $\rho(\varepsilon)$ must be equal to 2, since

1. for any endomorphism F of V holds $\bigwedge^2 \mathrm{im(F)} \subseteq \mathrm{im(F \wedge F)}$, telling us that the dimension of $\mathrm{im}(\rho(\varepsilon))$ cannot be greater that 2, and

2. skew-adjoint endomorphims of V with range lower than two are necessarily zero

From this we conclude that $\rho(\varepsilon)$ takes the form

$$\rho(\varepsilon) = x^* \otimes y - y^* \otimes x$$

for some $x, y \in V$ which tells us

$$\varepsilon = x \wedge y.$$

Now the the claim follows using the last proposition once more. ☐

The following theorem is due to C. Böhm and B. Wilking [5]. We will see later, that the identity below is not only a consequence of the first Bianchi identity, but it is actually equivalent to the 1. Bianchi identity.

Theorem 3.2.3.4. *Any algebraic curvature operator $\mathcal{R}$ satisfies the Böhm-Wilking identity:*

$$\mathrm{id}\#\mathcal{R} = \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id} - \mathcal{R}.$$

Moreover, we have

1. $\mathcal{R} \in \mathrm{SCAL} \Rightarrow \mathrm{id}\#\mathcal{R} = (\mathrm{n}-2)\mathcal{R}$

2. $\mathcal{R} \in \mathrm{RIC}_0 \Rightarrow \mathrm{id}\#\mathcal{R} = \frac{\mathrm{n}-4}{2}\mathcal{R}$

3. $\mathcal{R} \in \mathrm{WEYL} \Rightarrow \mathrm{id}\#\mathcal{R} = -\mathcal{R}$

and the map $\mathcal{R} \mapsto \mathrm{id}\#\mathcal{R}$ *restricts to an isomorphism of* $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$*, if* $\dim V \neq 2, 4$*.*

Beweis. We will use Lemma 1.1.0.3, Proposition 2.1.0.15 and the 1. Bianchi identity. Let $\mathcal{R}$ be an algebraic curvature operator and $\{e_i\}$ be an orthonormal basis of V. Then

$$\begin{aligned}
\mathrm{id}\#\mathcal{R}(\mathrm{x}\wedge \mathrm{y}) &= \frac{1}{2}\sum_i [x\wedge e_i, \mathcal{R}y\wedge e_i] - [\mathcal{R}x\wedge e_i, y\wedge e_i] \\
&= \sum_i -\left(\rho(\mathcal{R}y\wedge e_i)\wedge \mathrm{id}\right)(x\wedge e_i) + (\rho(\mathcal{R}x\wedge e_i)\wedge \mathrm{id})(\mathrm{y}\wedge \mathrm{e_i}) \\
&= \frac{1}{2}(x\wedge \mathrm{Ric}(\mathcal{R})\mathrm{y}) + \mathrm{Ric}(\mathcal{R})\mathrm{x}\wedge \mathrm{y}) \\
&\quad + \frac{1}{2}\sum_i (\mathcal{R}^\rho(y, e_i)x - \mathcal{R}^\rho(x, e_i)y)\wedge e_i \\
&= \mathrm{Ric}(\mathcal{R})\wedge \mathrm{id}(\mathrm{x}\wedge \mathrm{y}) - \frac{1}{2}\sum_{\mathrm{i}} (\mathcal{R}^\rho(\mathrm{x}, \mathrm{y})\mathrm{e_i})\wedge \mathrm{e_i} \\
&= (\mathrm{Ric}(\mathcal{R})\wedge \mathrm{id} - \mathcal{R})\mathrm{x}\wedge \mathrm{y}
\end{aligned}$$

Now we compute the eigenvalues of the $O(V)$-equivariant linear map

$$\mathcal{R} \mapsto \mathrm{id}\#\mathcal{R}$$

using the Böhm-Wilking identity.

1. Pick $\mathcal{R} \in \mathrm{SCAL}$. Then $\mathcal{R} = \lambda\mathrm{id}$ for some $\lambda \in \mathbb{R}$. Now, 1. follows from Proposition 2.1.0.16

2. If $\mathcal{R} \in \mathrm{RIC}_0$ then $\mathcal{R}$ is of the form $F\wedge \mathrm{id}$, where F is self-adjoint and $\mathrm{tr(F)} = 0$. So 2. follows using Lemma 3.2.1.6.

3. $\mathcal{R} \in \mathrm{WEYL}$ implies $\mathrm{Ric}(\mathcal{R}) = 0$, so we get $\mathrm{id}\#\mathcal{R} = -\mathcal{R}$ in this case, as claimed.

Now it is also clear that our map restricts to an isomorphism of the space of algebraic curvature operators, provided that the dimension of V is neither 2 nor 4. □

One can proof that, except in dimension 2 an 4, the map $\mathcal{R} \mapsto id\#\mathcal{R}$ is indeed an isomorphism on the whole space of self-adjoint endomorphisms of $\bigwedge^2 V$:

Proposition 3.2.3.5. *If a self-adjoint endomorphism* $\mathcal{R}$ *of* $\bigwedge^2 V$ *is perpendicular to the space of algebraic curvature operators, then*

$$\mathrm{id}\#\mathcal{R} = 2\mathcal{R}$$

Beweis. For any self-adjoint endomorphism $\mathcal{R}$ of $\bigwedge^2 V$ define a (3,1)-tensor $\sigma(\mathcal{R})$ of V as follows:

$$\sigma(\mathcal{R})(x,y)z := \mathcal{R}^\rho(x,y)z + \mathcal{R}^\rho(x,z)y$$

1. If $\mathcal{R}$ is self-adjoint and perpendicular to the space of algebraic curvature operators, then $\sigma(\mathcal{R}) = 0$:
   Let $\mathcal{S} \in LC(\bigwedge^2 V)$ be arbitrary and $\{e_i\}$ an orthonormal basis of V. Then, applying the 1. Bianchi identity, we get

   $$\begin{aligned} 0 = 4\langle \mathcal{R}, \mathcal{S}\rangle &= \sum_{i,j,k} \langle \mathcal{R}^\rho(e_i,e_j)e_k, \mathcal{S}^\rho(e_i,e_j)e_k\rangle \\ &= -\sum_{i,j,k} \langle \mathcal{R}^\rho(e_i,e_j)e_k, \mathcal{S}^\rho(e_j,e_k)e_i + \mathcal{S}^\rho(e_k,e_i)e_j\rangle \end{aligned}$$

   Shifting the indices, we see that the right hand side equals

   $$-\sum_{i,j,k} \langle \mathcal{R}^\rho(e_k,e_i)e_j + \mathcal{R}^\rho(e_j,e_k)e_i, \mathcal{S}^\rho(e_i,e_j)e_k\rangle$$

   By lemma 1.1.0.3 this is the same as

   $$-\sum_{i,j,k} \langle (\mathcal{R}^\rho(e_k,e_i)e_j - \mathcal{R}^\rho(e_k,e_j)e_i) \wedge e_k, \mathcal{S}e_i \wedge e_j\rangle$$

   Now let $\mathcal{S}$ be the orthogonal projection onto the linear span of $e_i \wedge e_j$. It is clear that $\mathcal{S}$ is an algebraic curvature operator. Then the equation from above reads

   $$-\sum_{k} \langle (\mathcal{R}^\rho(e_k,e_i)e_j - \mathcal{R}^\rho(e_k,e_j)e_i) \wedge e_k, e_i \wedge e_j\rangle$$

   So we get

   $$\begin{aligned} 0 &= \sum_k (\langle (-\mathcal{R}^\rho(e_k,e_j)e_i) \wedge e_k, e_i \wedge e_j\rangle + \langle (\mathcal{R}^\rho(e_k,e_i)e_j) \wedge e_k, e_i \wedge e_j\rangle) \\ &= -2\langle \mathcal{R}^\rho(e_i,e_j)e_j, e_i\rangle \end{aligned}$$

   using the definition of $\langle \cdot\,,\cdot\rangle$ on $\bigwedge^2 V$ in the last step.
   The orthonormal basis $\{e_i\}$ and projection $\mathcal{S}$ were chosen arbitrarily. Thus, we get

   $$\langle \mathcal{R}^\rho(x,y)y, x\rangle = 0,$$

   whenever x and y are perpendicular to each other. Now, since the linear map $x \mapsto \mathcal{R}^\rho(x,y)y$ is self-adjoint for any $y \in V$, we get that the maps $x \mapsto \mathcal{R}^\rho(x,y)y$ must vanish identically, no matter which $y \in V$ we choose. Now 1. follows from

   $$\sigma(\mathcal{R})(x,y)y = 2\mathcal{R}^\rho(x,y)y = 0$$

for all x and y in V and the formula

$$\sigma(\mathcal{R})(x,y)z = \frac{1}{2}(\sigma(\mathcal{R})(x,y+z)(y+z) - \sigma(\mathcal{R})(x,y)y - \sigma(\mathcal{R})(x,z)z),$$

which is true for all x, y and z in V.

2. If $\mathcal{R}$ lies in the kernel of σ, then $id\#\mathcal{R} = 2\mathcal{R}$:
Let $x, y \in V$ be arbitrary. Using lemma 1.1.0.3 we compute:

$$\begin{aligned}
\mathrm{id}\#\mathcal{R}\mathrm{x}\wedge\mathrm{y} &= \frac{1}{2}\sum_k [x\wedge e_k, \mathcal{R}y\wedge e_k] + [\mathcal{R}x\wedge e_k, y\wedge e_k] \\
&= \sum_k -(\rho(\mathcal{R}y\wedge e_k)\wedge id)(x\wedge e_k) + (\rho(\mathcal{R}x\wedge e_k)(y\wedge e_k)) \\
&= \frac{1}{2}\sum_k (\mathcal{R}^\rho(y,e_k)x)\wedge e_k - (\mathcal{R}^\rho(x,e_k)y)\wedge e_k + \mathrm{Ric}(\mathcal{R})\wedge\mathrm{id}(\mathrm{x}\wedge\mathrm{y}) \\
&= \frac{1}{2}\sum_k (\mathcal{R}^\rho(e_k,x)y - \mathcal{R}^\rho(e_k,y)x)\wedge e_k \\
&= \sum_k \mathcal{R}^\rho(e_k,x)y\wedge e_k
\end{aligned}$$

Now pick some $u, v \in V$. Then:

$$\begin{aligned}
\langle \mathrm{id}\#\mathcal{R}\mathrm{x}\wedge\mathrm{y}, \mathrm{u}\wedge\mathrm{v}\rangle &= \left\langle \sum_k \mathcal{R}^\rho(e_k,x)y\wedge e_k, u\wedge v\right\rangle \\
&= \sum_k (\langle \mathcal{R}^\rho(e_k,x)y, u\rangle \langle e_k, v\rangle - \langle \mathcal{R}^\rho(e_k,x)y, v\rangle \langle e_k, u\rangle) \\
&= \langle \mathcal{R}^\rho(u,y)x, v\rangle - \langle \mathcal{R}^\rho(v,y)x, u\rangle \\
&= 2\langle \mathcal{R}^\rho(u,y)x, v\rangle
\end{aligned}$$

since $u \mapsto \mathcal{R}^\rho(u,y)x$ is skew-adjoint. Using $\sigma(\mathcal{R}) = 0$ we get:

$$2\langle \mathcal{R}^\rho(u,y)x, v\rangle = -2\langle \mathcal{R}^\rho(y,u)x, v\rangle = 2\langle \mathcal{R}^\rho(y,x)u, v\rangle = 2\langle \mathcal{R}x\wedge y, u\wedge v\rangle$$

and so we are done.

□

Now we are ready to express the orthogonal projection onto the the space of algebraic curvature operators using the #-product and the Ricci curvature operator. From this we will get that the Böhm-Wilking identity is equivalent to the 1- Bianchi identity as a by-product.

Theorem 3.2.3.6. *Let* $\pi : Sym(\bigwedge^2 V) \to LC\left(\bigwedge^2 V\right)$ *and* $\pi^\perp : Sym(\bigwedge^2 V) \to LC\left(\bigwedge^2 V\right)^\perp$ *be the orthogonal projections. Then*

1. $\pi(\mathcal{R}) = \frac{1}{3}(2\mathcal{R} - id\#\mathcal{R} + Ric(\mathcal{R}) \wedge id)$ *and*

2. $\pi^{\perp}(\mathcal{R}) = \frac{1}{3}(\mathcal{R} + id\#\mathcal{R} - Ric(\mathcal{R}) \wedge id)$.

Beweis. By the previous proposition, $2\mathcal{R} - id\#\mathcal{R}$ is an algebraic curvature operator. As $\mathrm{Ric}\mathcal{R} \wedge \mathrm{id}$ is an algebraic curvature operator as well, it follows that π maps $\mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$ to $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$.
Now, if $\mathcal{R}$ is an algebraic curvature operator, then we get $\pi(\mathcal{R}) = \mathcal{R}$, using the Böhm-Wilking identity. Thus, π is a projection. Further, if $\mathcal{R}$ is perpendicular to the space of algebraic curvature operators, we get that $2\mathcal{R} - id\#\mathcal{R} = 0$ and $Ric(\mathcal{R}) = 0$, which implies $\pi(\mathcal{R}) = 0$ and the first claim follows.
The second formula follows directly from $\pi^{\perp} = id - \pi$.

□

We conclude

Theorem 3.2.3.7. *The Böhm-Wilking identity is equivalent to the 1. Bianchi identity. More precisely, a self-adjoint endomorphism $\mathcal{R}$ of $\bigwedge^2 V$ is an algebraic curvature operator if and only if* $\mathrm{id}\#\mathcal{R} = \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id} - \mathcal{R}$.

□

We close this section with an application of the stuff of this subsection to the 4-dimensional case. As we have seen in subsection 3.2.2.2, any algebraic curvature operator of traceless Ricci type interchanges the eigenspaces $\bigwedge^+ V$ and $\bigwedge^- V$ of the Hodge $*$-operator, while they are preserved by Weyl curvature operators and, which is a rather trivial observation, by the elements of SCAL.

Proposition 3.2.3.8. *Let* $\dim V = 4$ *and* $\mathcal{R}$ *be a self-adjoint endomorphism of* $\bigwedge^2 V$. *Then*

1. *If* $\mathcal{R}$ *interchanges the eigenspaces of the Hodge* $*$*-operator, then* $\mathcal{R}$ *is an algebraic curvature operator of traceless Ricci type.*

2. *If* $\mathcal{R}$ *preserves the eigenspaces of the Hodge* $*$*-operator and has vanishing trace, then* $\mathcal{R}$ *is an algebraic Weyl curvature operator.*

Beweis. Let $\{\varepsilon_i\}$ be an orthonormal basis of $\bigwedge^2 V$, such that $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \bigwedge^+ V$ and $\varepsilon_4, \varepsilon_4, \varepsilon_6 \in \bigwedge^- V$. Then we get

$$[\varepsilon_i, \varepsilon_j] = 0,$$

whenever $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$, since $\bigwedge^+ V$ and $\bigwedge^- V$ are ideals in $\bigwedge^2 V$.

1. Since $\mathcal{R}$ interchanges the ideals $\bigwedge^+ V$ and $\bigwedge^- V$, we get
$$[\mathcal{R}\varepsilon_i, \varepsilon_j] = 0,$$
whenever $i, j \in \{1, 2, 3\}$ or $i, j \in \{4, 5, 6\}$. This implies
$$\mathrm{id}\#\mathcal{R} = 0$$
and the claim follows.

2. $\mathcal{R}$ preserves the ideals $\bigwedge^+ V$ and $\bigwedge^- V$, so we may write it as a direct sum $\mathcal{R}^+ + \mathcal{R}^-$ with $\mathcal{R}^\pm : \bigwedge^\pm V \to \bigwedge^\pm V$. Thus, the traceless Ricci curvature part of $\mathcal{R}^\pm$ vanishes by proposition 3.2.2.1. Hence, the traceless Ricci curvature part of $\mathcal{R}$ must vanish either. Now, $\mathrm{tr}(\mathcal{R}^\pm)$ is O(V)-invariant, while the splitting $\bigwedge^2 V = \bigwedge^+ V \oplus \bigwedge^- V$ is only SO(V)-invariant. We use this to show $\mathrm{tr}(\mathcal{R}^\pm) = 0$. Then the claim will follow from $\mathrm{Sym}\left(\bigwedge^\pm \mathrm{V}\right) = \mathrm{WEYL}^\pm$.
Let $G \in \mathrm{O(V)}$ be an orthogonal transformation with $\det(G) = -1$. Using lemma 1.2.0.9, it follows that $G \wedge G$ interchanges $\bigwedge^+ V$ and $\bigwedge^- V$. This implies $\mathrm{tr}(\mathcal{R}^+) = 0$ and $\mathrm{tr}(\mathcal{R}^-) = 0$, since $\mathcal{R}^+|_{\bigwedge^- V} = 0$ and $\mathcal{R}^-|_{\bigwedge^+ V} = 0$.

□

3.2.4 Relating Algebraic Curvature Operators to Subalgebras of $\bigwedge^2 V$

Every endomorphism $\mathcal{R}$ on $\bigwedge^2 V$ - and in particular every algebraic curvature operator - gives rise to a Lie subalgebra $\mathfrak{h}_\mathcal{R}$ of $\bigwedge^2 V$, which is by definition the smallest Lie subalgebra of $\bigwedge^2 V$ containing the image of $\mathcal{R}$,
$$\mathfrak{h}_\mathcal{R} := \bigcap_{\mathrm{im(R)} \subseteq \mathfrak{h}} \mathfrak{h}.$$
We call $\mathfrak{h}_\mathcal{R}$ the holonomy algebra of $\mathcal{R}$. The following proposition relates the holonomy algebra of an algebraic curvature operator to the Lie algebra of the holonomy group of its geometric realization.

Proposition 3.2.4.1. *Let $\mathcal{R}$ be an algebraic curvature operator with geometric realization (U, g). Let $\mathfrak{hol} \subseteq \mathfrak{so}(V)$ denote the Lie algebra of the holonomy group* $\mathrm{Hol_0(U, g)}$ *based at the point $0 \in U$. Then*
$$\rho(\mathfrak{h}_\mathcal{R}) \subseteq \mathfrak{hol}.$$

Beweis. Recall that the isomorphism $\rho : \bigwedge^2 V \mapsto \mathfrak{so}(V)$ was defined on generators $x \wedge y$ of $\bigwedge^2 V$ by
$$\rho(x \wedge y) := \langle x, \cdot \rangle \otimes y - \langle y, \cdot \rangle \otimes x.$$

ρ maps $\mathfrak{h}_{\mathcal{R}}$ to the subalgebra $\mathfrak{h}$ which is generated by the subset

$$\{\mathcal{R}^\rho(x,y) : x, y \in V\} \subseteq \mathfrak{so}(V).$$

By theorem 3.1.3.2, we have that $\mathcal{R}^\rho$ equals the curvature tensor R^g of g in 0. Thus, the claim follows from the theorem of Ambrose and Singer [1], which also is stated in the appendix B.4. □

Now, we may ask, whether a given Lie subalgebra $\mathfrak{h}$ of $\bigwedge^2 V$ is the holonomy algebra of a suitable algebraic curvature operator or not. In view of corollary 3.2.3.3 this can't be true for every subalgebra, if the dimension of V is at least 4. To see this, simply take a one dimensional subspace $\mathfrak{h}$ of $\bigwedge^2 V$, which is always a subalgebra. If $\mathfrak{h}$ was the holonomy algebra of an algebraic curvature operator $\mathcal{R}$, $\mathfrak{h} = \mathfrak{h}_{\mathcal{R}}$, then $\mathcal{R}$ must have range equal to 1. Now, corollary 3.2.3.3 implies that $\mathfrak{h}$ is generated of an element of the form $x \wedge y$, with $x, y \in V$. Thus, if the dimension of V is at least 4, we may construct counter examples in the following way: Take a linearly independent set $\{u, v, x, y\} \subseteq V$, define $\varepsilon := x \wedge y + u \wedge v$ and $\mathfrak{h} := \langle \varepsilon \rangle$. Then $\mathfrak{h}$ is definitely not the holonomy algebra of an algebraic curvature operator, since it doesn't contain elements of the form $x \wedge y$, $x, y \in V$.
We note

Theorem 3.2.4.2. *Let $\mathfrak{h}$ be a one dimensional Lie subalgebra of $\bigwedge^2 V$ and $\pi : \bigwedge^2 V \to \bigwedge^2 V$ the orthogonal projection onto $\mathfrak{h}$. Then we have $\mathfrak{h} = \mathfrak{h}_{\mathcal{R}}$ for an algebraic curvature operator $\mathcal{R}$ if and only if π is an algebraic curvature operator.*

The theorem from above is false, if we consider Lie subalgebras of higher dimensions. For example, in dimension 4, the orthogonal projections onto the eigenspaces $\bigwedge^{\pm} V$ of the Hodge $*$-operator (see theorem 1.2.0.8, where we give the definition) do not belong to the space of algebraic curvature operators (see lemma 3.2.2.5 for the proof), but we have $\bigwedge^+ V = \mathfrak{h}_{\mathcal{R}}$ for every nonzero algebraic curvature operator $\mathcal{R} \in \mathrm{WEYL}^{\pm}$. More precisely, if $\mathcal{R} \in \mathrm{WEYL}^+ \setminus \{0\}$, say, then $0 \neq \mathfrak{h}_{\mathcal{R}} \subseteq \bigwedge^+ V$. As $\bigwedge^+ V$ does not contain nonzero elements of the form $x \wedge y$, $x, y \in V$, $\mathfrak{h}_{\mathcal{R}}$ is at least 2-dimensional by the previous theorem. But $\bigwedge^{\pm} V$ is isomorphic to $\mathfrak{so}(3)$ as a Lie algebra by theorem 1.2.0.8, so it does not contain 2-dimensional Lie subalgebras. Therefore, we get $\mathfrak{h}_{\mathcal{R}} = \bigwedge^+ V$ for reasons of dimension.
This discussion shows that the question, whether a given Lie subalgebra of $\bigwedge^2 V$ is the holonomy algebra of an algebraic curvature operator or not, is not trivial.

3.2.5 Ricci Curvature

The Ricci curvature of a self-adjoint endomorphism $\mathcal{R}$ of $\bigwedge^2 V$ was defined using an orthonormal basis $\{e_i\}$ of V by the formula

$$\mathrm{Ric}(\mathcal{R})\mathrm{x} = \sum_{\mathrm{i}} \mathcal{R}^\rho(\mathrm{x}, \mathrm{e_i})\mathrm{e_i}.$$

Equivalently, we could have defined it by

$$\langle \mathrm{Ric}(\mathcal{R})\mathrm{x}, \mathrm{y}\rangle = \mathrm{tr}(\mathrm{z} \mapsto \mathcal{R}^\rho(\mathrm{z}, \mathrm{x})\mathrm{y}) = \sum_{\mathrm{i}} \langle \mathcal{R}\mathrm{x} \wedge \mathrm{e_i}, \mathrm{y} \wedge \mathrm{e_i}\rangle .$$

We will see now, that it is possible to describe the Ricci curvature $\mathrm{Ric}(\mathcal{R})$ of any algebraic curvature operator $\mathcal{R}$ in terms of an orthonormal eigenbasis and the corresponding eigenvalues of $\mathcal{R}$ itself, without using orthonormal bases of V. We will use this representation of the Ricci curvature in 3.5.

Proposition 3.2.5.1. *Let $\mathcal{R}$ be a self-adjoint endomorphism of $\bigwedge^2 V$, $\mathcal{R} = \sum_i \lambda_i \varepsilon_i^* \otimes \varepsilon_i$. Then*

1. $\mathrm{Ric}(\mathcal{R}) = -\sum_{\mathrm{i}} \lambda_{\mathrm{i}} \rho(\varepsilon_{\mathrm{i}})^2$
2. $\mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id} = -\frac{1}{2}\sum_{\mathrm{i}} \lambda_{\mathrm{i}} \mathrm{ad}^2_{\varepsilon_{\mathrm{i}}} + \sum_{\mathrm{i}} \lambda_{\mathrm{i}} \rho(\varepsilon_{\mathrm{i}}) \wedge \rho(\varepsilon_{\mathrm{i}})$

Beweis. Let $\mathcal{R} = \sum_i \lambda_i \varepsilon_i^* \otimes \varepsilon_i$ be any self-adjoint endomorphism of $\bigwedge^2 V$, $\{e_i\}$ an orthonormal basis of V and $x \in V$. Then

$$\begin{aligned} \mathrm{Ric}(\mathcal{R})\mathrm{x} &= -\sum_i \rho(\mathcal{R}x \wedge e_i)(e_i) = -\sum_{i,j} \lambda_j \rho(\varepsilon_j^*(x \wedge e_i)\varepsilon_j)(e_i) \\ &= -\sum_{i,j} \lambda_j \langle \varepsilon_j, x \wedge e_i\rangle \rho(\varepsilon_j)(e_i) = -\sum_{i,j} \lambda_j \langle \rho(\varepsilon_j)(x), e_i\rangle \rho(\varepsilon_j)(e_i) \\ &= -\sum_j \lambda_j \rho(\varepsilon_j)^2 x \end{aligned}$$

To prove the second formula, just recall that $2(\mathrm{id} \wedge \mathrm{F})^2 = \mathrm{id} \wedge \mathrm{F}^2 + \mathrm{F} \wedge \mathrm{F}$ and make use of the formula $\mathrm{ad}_\varepsilon = 2\rho(\varepsilon) \wedge \mathrm{id}$, proved in lemma 1.1.0.3.

□

Remark 3.2.5.2.

1. Proposition 3.2.5.1, theorem 3.2.3.4 and theorem 3.2.3.1 imply that for algebraic curvature operators $\mathcal{R} = \sum_i \lambda_i \varepsilon_i^* \otimes \varepsilon_i$ holds

$$\mathrm{id}\#\mathcal{R} = -\frac{1}{2}\sum_{\mathrm{i}} \lambda_{\mathrm{i}} \mathrm{ad}^2_{\varepsilon_{\mathrm{i}}}$$

2. Proposition 3.2.5.1 implies

$$\sum_i \sigma_i \mathrm{ad}^2_{\varepsilon_{\mathrm{i}}} = 2\sum_{\mathrm{i}} \sigma_{\mathrm{i}} \rho(\varepsilon_{\mathrm{i}}) \wedge \rho(\varepsilon_{\mathrm{i}})$$

for all self-adjoint endomorphisms $\mathcal{S} = \sum_i \sigma_i \varepsilon_i^* \otimes \varepsilon_i$, which are perpendicular to the space of algebraic curvature operators.

3.3 The Ricci Vector Field

The Ricci vector field Φ arises naturally as the reaction term in the evolution equation of the curvature operator under the Ricci flow, see [11]. More precisely, the evolution equation reads

$$\nabla_{\frac{\partial}{\partial t}} \mathcal{R} = \Delta \mathcal{R} + \Phi(\mathcal{R}),$$

where $\Phi(\mathcal{R}) = 2(\mathcal{R}^2 + \mathcal{R}\#\mathcal{R})$. (See theorem C.2.0.9 for a more detailed description of the quantities in play and [24], where the evolution equation is proved in coordinate-free way.) First, we show that Φ preserves the space of algebraic curvature operators by purely algebraic means and compute trace and Ricci curvature along the Ricci vector field. After that we examine the irreducible components of $\Phi(\mathcal{R})$ and express them in terms of the irreducible components of $\mathcal{R}$. Further, we discuss some situations, where some components of $\Phi(\mathcal{R})$ vanish. We will see that it is sometimes possible to regain knowledge about $\mathcal{R}$ from this.

3.3.1 The Ricci Vector Field on the Space of Algebraic Curvature Operators

From the analytic viewpoint it is clear that Φ maps curvature operators to curvature operators, since the algebraic properties of curvature operators $\mathcal{R}$ carry over to their covariant derivatives of arbitrary order. But, also for reasons of consistency, we want to establish this result in a purely algebraic way.

Theorem 3.3.1.1. *If $\mathcal{R}$ is an algebraic curvature operator on $\bigwedge^2 V$, so is $\Phi(\mathcal{R})$.*

Beweis. We have to check the 1. Bianchi identity, the rest is clear. Let $\{e_i\}$ be an orthonormal basis of V. Using Lemma 1.1.0.3.1 we get

$$(\mathcal{R}^2)^\rho(x,y)z = \frac{1}{2}\sum_i \mathcal{R}^\rho(\mathcal{R}^\rho(x,y)e_i, e_i)z$$

and proposition 2.1.0.15.2 gives

$$\mathcal{R}\#\mathcal{R}^\rho(x,y)z = -\sum_i \left[\mathcal{R}^\rho(x,e_i), \mathcal{R}^\rho(y,e_i)\right](z).$$

Then, applying the Bianchi map B to the sum of these terms, we get

$\mathrm{B}(\mathcal{R}^2+\mathcal{R}\#\mathcal{R})(x,y)z$

$$= \sum_i \begin{pmatrix} \frac{1}{2}\mathcal{R}^\rho(\mathcal{R}^\rho(x,y)e_i,e_i)z - \mathcal{R}^\rho(x,e_i)\mathcal{R}^\rho(y,e_i)z + \mathcal{R}^\rho(y,e_i)\mathcal{R}^\rho(x,e_i)z \\ \frac{1}{2}\mathcal{R}^\rho(\mathcal{R}^\rho(z,x)e_i,e_i)y - \mathcal{R}^\rho(z,e_i)\mathcal{R}^\rho(x,e_i)y + \mathcal{R}^\rho(x,e_i)\mathcal{R}^\rho(z,e_i)y \\ \frac{1}{2}\mathcal{R}^\rho(\mathcal{R}^\rho(y,z)e_i,e_i)x - \mathcal{R}^\rho(y,e_i)\mathcal{R}^\rho(z,e_i)x + \mathcal{R}^\rho(z,e_i)\mathcal{R}^\rho(y,e_i)x \end{pmatrix}.$$

Rearranging the terms we get

$$\sum_i \begin{pmatrix} \frac{1}{2}\mathcal{R}^\rho(\mathcal{R}^\rho(x,y)e_i,e_i)z + \mathcal{R}^\rho(x,e_i)(\mathcal{R}^\rho(e_i,y)z + \mathcal{R}^\rho(z,e_i)y) \\ \frac{1}{2}\mathcal{R}^\rho(\mathcal{R}^\rho(z,x)e_i,e_i)y + \mathcal{R}^\rho(y,e_i)(\mathcal{R}^\rho(e_i,z)x + \mathcal{R}^\rho(x,e_i)z) \\ \frac{1}{2}\mathcal{R}^\rho(\mathcal{R}^\rho(y,z)e_i,e_i)x + \mathcal{R}^\rho(z,e_i)(\mathcal{R}^\rho(e_i,x)y + \mathcal{R}^\rho(y,e_i)x) \end{pmatrix}.$$

Applying the 1. Bianchi identity to the terms in the second column gives

$$\sum_i \begin{pmatrix} \frac{1}{2}\mathcal{R}^\rho(\mathcal{R}^\rho(x,y)e_i,e_i)z - \mathcal{R}^\rho(x,e_i)\mathcal{R}^\rho(y,z)e_i \\ \frac{1}{2}\mathcal{R}^\rho(\mathcal{R}^\rho(z,x)e_i,e_i)y - \mathcal{R}^\rho(y,e_i)\mathcal{R}^\rho(z,x)e_i \\ \frac{1}{2}\mathcal{R}^\rho(\mathcal{R}^\rho(y,z)e_i,e_i)x - \mathcal{R}^\rho(z,e_i)\mathcal{R}^\rho(x,y)e_i \end{pmatrix}.$$

Now apply the 1. Bianchi identity to the terms in the first column and rearrange the terms in the third to get

$$\sum_i \begin{pmatrix} -\frac{1}{2}\mathcal{R}^\rho(e_i,z)\mathcal{R}^\rho(x,y)e_i - \frac{1}{2}\mathcal{R}^\rho(z,\mathcal{R}^\rho(x,y)e_i)e_i - \mathcal{R}^\rho(z,e_i)\mathcal{R}^\rho(x,y)e_i \\ -\frac{1}{2}\mathcal{R}^\rho(e_i,y)\mathcal{R}^\rho(z,x)e_i - \frac{1}{2}\mathcal{R}^\rho(y,\mathcal{R}^\rho(z,x)e_i)e_i - \mathcal{R}^\rho(y,e_i)\mathcal{R}^\rho(z,x)e_i \\ -\frac{1}{2}\mathcal{R}^\rho(e_i,x)\mathcal{R}^\rho(y,z)e_i - \frac{1}{2}\mathcal{R}^\rho(x,\mathcal{R}^\rho(y,z)e_i)e_i - \mathcal{R}^\rho(x,e_i)\mathcal{R}^\rho(y,z)e_i \end{pmatrix}$$

which equals

$$-\frac{1}{2}\sum_i \begin{pmatrix} \mathcal{R}^\rho(z,e_i)\mathcal{R}^\rho(x,y)e_i + \mathcal{R}^\rho(z,\mathcal{R}^\rho(x,y)e_i)e_i \\ \mathcal{R}^\rho(y,e_i)\mathcal{R}^\rho(z,x)e_i + \mathcal{R}^\rho(y,\mathcal{R}^\rho(z,x)e_i)e_i \\ \mathcal{R}^\rho(y,e_i)\mathcal{R}^\rho(z,x)e_i + \mathcal{R}^\rho(x,\mathcal{R}^\rho(y,z)e_i)e_i \end{pmatrix}$$

Now let $v \in V$ be arbitrary. Then, using the symmetries of $\mathcal{R}^\rho$ and the results from above, we compute

$$
\begin{aligned}
2\left\langle B(\mathcal{R}^2+\mathcal{R}\#\mathcal{R})(x,y)z,v\right\rangle &= \begin{pmatrix} \mathrm{tr}\left(\mathcal{R}^\rho(\mathrm{x},\mathrm{y})\circ(\mathcal{R}^\rho(\cdot,\mathrm{v})\mathrm{z}+\mathcal{R}^\rho(\cdot,\mathrm{z})\mathrm{v})\right)\\ \mathrm{tr}\left(\mathcal{R}^\rho(\mathrm{z},\mathrm{x})\circ(\mathcal{R}^\rho(\cdot,\mathrm{v})\mathrm{y}+\mathcal{R}^\rho(\cdot,\mathrm{y})\mathrm{v})\right)\\ \mathrm{tr}\left(\mathcal{R}^\rho(\mathrm{y},\mathrm{z})\circ(\mathcal{R}^\rho(\cdot,\mathrm{v})\mathrm{x}+\mathcal{R}^\rho(\cdot,\mathrm{x})\mathrm{v})\right)\end{pmatrix}\\
&= -\begin{pmatrix}\langle\mathcal{R}^\rho(x,y),\mathcal{R}^\rho(\cdot,v)z+\mathcal{R}^\rho(\cdot,z)v\rangle\\ \langle\mathcal{R}^\rho(z,x),\mathcal{R}^\rho(\cdot,v)y+\mathcal{R}^\rho(\cdot,y)v\rangle\\ \langle\mathcal{R}^\rho(y,z),\mathcal{R}^\rho(\cdot,v)x+\mathcal{R}^\rho(\cdot,x)v\rangle\end{pmatrix}\\
&= 0
\end{aligned}
$$

Let us explain this computation step by step:
To see that the first step is correct, we compute the first row as an example:

$$
\begin{aligned}
&\sum_i \langle\mathcal{R}^\rho(z,e_i)\mathcal{R}^\rho(x,y)e_i+\mathcal{R}^\rho(z,\mathcal{R}^\rho(x,y)e_i)e_i,v\rangle\\
&\quad=\sum_i(\langle\mathcal{R}^\rho(z,e_i)\mathcal{R}^\rho(x,y)e_i,v\rangle+\langle\mathcal{R}^\rho(z,\mathcal{R}^\rho(x,y)e_i)e_i,v\rangle)\\
&\quad=\sum_i(-\langle\mathcal{R}^\rho(z,e_i)v,\mathcal{R}^\rho(x,y)e_i\rangle+\langle\mathcal{R}^\rho(e_i,v)z,\mathcal{R}^\rho(x,y)e_i\rangle)\\
&\quad=-\sum_i\langle e_i,\mathcal{R}^\rho(x,y)(\mathcal{R}^\rho(e_i,z)v+\mathcal{R}^\rho(e_iv)z)\rangle\\
&\quad=-\mathrm{tr}\left(\mathcal{R}^\rho(\mathrm{x},\mathrm{y})\circ(\mathcal{R}^\rho(\cdot,\mathrm{v})\mathrm{z}+\mathcal{R}^\rho(\cdot,\mathrm{z})\mathrm{v})\right).
\end{aligned}
$$

In the second step we simply used the definition of the induced scalar product on the space of endomorphisms of V and the fact that the map $z\mapsto\mathcal{R}^\rho(x,y)z$ is skew-adjoint for all $x,y\in V$.
The last step now follows since the map $z\mapsto\mathcal{R}^\rho(z,x)y+\mathcal{R}^\rho(z,y)x$ is self-adjoint for all $x,y\in V$ and the spaces of skew-adjoint and self-adjoint linear maps are perpendicular to each other. □

Remark 3.3.1.2 (Translation to the bundle setting)**.** As in subsection 1.4 and subsection 2.6, let M be a smooth manifold (possibly with boundary) and $(\mathrm{E},\langle\cdot\,,\cdot\rangle)\to\mathrm{M}\times[0,\mathrm{T})$, $T>0$, a Euclidean vector bundle with a metric connection ∇. Following these subsections, we may consider Φ as a vertical vector field on the bundle of endomorphism fields $\mathrm{End}(\bigwedge^2\mathrm{E})$. Φ preserves the parallel subbundle $\mathrm{Sym}\left(\bigwedge^2\mathrm{E},\langle\cdot\,,\cdot\rangle\right)$ of self-adjoint endomorphism fields of $\left(\bigwedge^2\mathrm{E},\langle\cdot\,,\cdot\rangle\right)$ and the last proposition tells us, that Φ even preserves the parallel subbundle $\mathrm{LC}\left(\bigwedge^2\mathrm{TM},\langle\cdot\,,\cdot\rangle\right)$ of algebraic curvature operators. As the usual multiplication of endomorphism fields is clearly parallel and the #- product is parallel either, it follows that Φ is actually a parallel vertical vector field on $\mathrm{LC}\left(\bigwedge^2\mathrm{TM},\mathrm{g}\right)$.

Next we consider the curvature quantities scal and Ric of $\Phi(\mathcal{R})$. The derived formulas agree with the reaction terms in the evolution equations of scal and Ric under the Ricci flow, which have been computed by R.S. Hamilton in [10], [11]. This is actually not very surprising, since Φ is parallel. But it is important. We will use these formulas in chapter 4, where we examine the dynamics of the flow of Φ on the space of algebraic curvature operators.

Theorem 3.3.1.3. *For any algebraic curvature operator holds*

1. $\mathrm{tr}(\Phi(\mathcal{R})) = \|\mathrm{Ric}(\mathcal{R})\|^2$ *and*
2. $\mathrm{Ric}(\Phi(\mathcal{R})) = 2\sum_{\mathrm{i}} \mathcal{R}^\rho(\cdot, \mathrm{e_i})\mathrm{Ric}(\mathcal{R})(\mathrm{e_i})$
   for any given orthonormal basis $\{e_i\}$ of V.

Beweis. 1. The following computation shows that if F is a self-adjoint endomorphism of V, then

$$\|F \wedge id\|^2 = \frac{n-2}{4}\|F\|^2 + \frac{1}{4}(\mathrm{tr(F)})^2.$$

Let $\{e_i\}$ be an orthonormal eigenbasis of F, i.e. $Fe_i = f_i e_i, f_i \in \mathbb{R}$. Then

$$\begin{aligned}
\|F \wedge id\|^2 &= \sum_{i<j} \|F \wedge id(e_i \wedge e_j)\|^2 = \frac{1}{8}\sum_{i \neq j}(f_i + f_j)^2 \\
&= \frac{1}{8}\sum_i \sum_{j \neq i} f_i^2 + f_j^2 + 2f_i f_j \\
&= \frac{1}{8}\sum_i \left((n-1)f_i^2 + \sum_{j \neq i} f_j^2\right) + \frac{1}{4}\sum_i f_i \sum_{j \neq i} f_j \\
&= \frac{1}{8}\sum_i \left((n-2)f_i^2 + \|F\|^2\right) + \frac{1}{4}\sum_i f_i(\mathrm{tr(F)} - \mathrm{f_i}) \\
&= \frac{1}{8}\left((n-2)\|F\|^2 + n\|F\|^2\right) + \frac{1}{4}\left((\mathrm{tr(F)}^2 - \|\mathrm{F}\|^2)\right) \\
&= \frac{n-2}{4}\|F\|^2 + \frac{1}{4}(\mathrm{tr(F)})^2.
\end{aligned}$$

Now let's compute the trace of $\Phi(\mathcal{R})$. Using that the map $\mathcal{R} \mapsto \mathrm{id}\#\mathcal{R}$ is self-adjoint, we get

$$\mathrm{tr}(\Phi(\mathcal{R})) = \langle \Phi(\mathcal{R}), \mathrm{id}\rangle = 2\left\langle \mathcal{R}^2 + \mathcal{R}\#\mathcal{R}, \mathrm{id}\right\rangle = 2\left\langle \mathcal{R} + \mathrm{id}\#\mathcal{R}, \mathcal{R}\right\rangle,$$

which is equal to

$$2\left\langle \mathrm{Ric}(\mathcal{R}) \wedge \mathrm{id}, \mathcal{R}\right\rangle$$

by the Böhm-Wilking identity. Thus,

$$
\begin{aligned}
2\left\langle \mathrm{Ric}(\mathcal{R})\wedge \mathrm{id},\mathcal{R}\right\rangle &= 2\left\langle \mathrm{Ric}_0(\mathcal{R})\wedge \mathrm{id},\mathcal{R}\right\rangle + \frac{2}{n}\mathrm{tr}(\mathrm{Ric}(\mathcal{R}))^2\\
&= \frac{4}{n-2}\left\|\mathrm{Ric}_0(\mathcal{R})\wedge \mathrm{id}\right\|^2 + \frac{2}{n}\mathrm{tr}(\mathrm{Ric}(\mathcal{R}))^2\\
&= \left\|\mathrm{Ric}(\mathcal{R})\right\|^2.
\end{aligned}
$$

2. We will proof the following identities:

   (a) $\mathrm{Ric}(\mathcal{R}\#\mathcal{R})\mathrm{x} = \sum_{\mathrm{i}} \mathcal{R}^\rho(\mathrm{x},\mathrm{e_i})\mathrm{Ric}(\mathcal{R})(\mathrm{e_i}) + \sum_{\mathrm{i,j}} \mathcal{R}^\rho(\mathrm{e_i},\mathrm{e_j})\mathcal{R}^\rho(\mathrm{x},\mathrm{e_j})\mathrm{e_i}$

   (b) $\mathrm{Ric}(\mathcal{R}^2)\mathrm{x} = -\frac{1}{2}\sum_{\mathrm{i,j}}\big(\mathcal{R}^\rho(\mathrm{e_j},\mathrm{e_i})^2\mathrm{x} + \mathcal{R}^\rho(\mathrm{e_i},\mathcal{R}^\rho(\mathrm{e_i},\mathrm{e_j})\mathrm{x})\mathrm{e_j}\big)$

   (c) $2\sum_{i,j}\mathcal{R}^\rho(e_i,e_j)\mathcal{R}^\rho(x,e_j)e_i = \sum_{i,j}\mathcal{R}^\rho(e_i,e_j)^2x$

   (d) $\sum_{i,j}\mathcal{R}^\rho(e_i,\mathcal{R}^\rho(e_i,e_j)x)e_j = 0$

   Then the claim follows immediately combing (a), (b), (c) and (d). In order to proof (b) and (d) we need the following lemma:

   Lemma 3.3.1.4. $\sum_{i,j}\left[\mathcal{R}^\rho(e_i,e_j),\mathcal{R}^\rho(\cdot,e_i)e_j\right] = 0$

   Beweis. Using the 1. Bianchi identity in the second step of the following computation, we get

$$
\begin{aligned}
\sum_{i,j}\left[\mathcal{R}^\rho(e_i,e_j),\mathcal{R}^\rho(\cdot,e_i)e_j\right] &= -\sum_{i,j}\left[\mathcal{R}^\rho(e_j,e_i),\mathcal{R}^\rho(\cdot,e_i)e_j\right]\\
&= \sum_{i,j}\left[\mathcal{R}^\rho(e_j,e_i),\mathcal{R}^\rho(e_i,e_j)\right]\\
&\quad+\sum_{i,j}\left[\mathcal{R}^\rho(e_j,e_i),\mathcal{R}^\rho(e_j,\cdot)e_i\right]\\
&= \sum_{i,j}\left[\mathcal{R}^\rho(e_j,e_i),\mathcal{R}^\rho(e_j,\cdot)e_i\right]\\
&= -\sum_{i,j}\left[\mathcal{R}^\rho(e_j,e_i),\mathcal{R}^\rho(\cdot,e_j)e_i\right]\\
&= -\sum_{i,j}\left[\mathcal{R}^\rho(e_i,e_j),\mathcal{R}^\rho(\cdot,e_i)e_j\right].
\end{aligned}
$$

□

(a) We have

$$\begin{aligned}
\mathrm{Ric}(\mathcal{R}\#\mathcal{R})\mathrm{x} &= -\sum_i \rho(\mathcal{R}\#\mathcal{R}x \wedge e_i)(e_i) \\
&= -\sum_{i,j} \rho([\mathcal{R}x \wedge e_j, \mathcal{R}e_i \wedge e_j])(e_i) \\
&= -\sum_{i,j} [\mathcal{R}^\rho(x, e_j), \mathcal{R}^\rho(e_i, e_j)]\,(e_i) \\
&= -\sum_{i,j} \mathcal{R}^\rho(x, e_j)\mathcal{R}^\rho(e_i, e_j)(e_i) \\
&\quad + \sum_{i,j} \mathcal{R}^\rho(e_i, e_j)\mathcal{R}^\rho(x, e_j)(e_i) \\
&= \sum_i \mathcal{R}^\rho(x, e_i) Ric(\mathcal{R})(e_i) \\
&\quad + \sum_{i,j} \mathcal{R}^\rho(e_i, e_j)\mathcal{R}^\rho(x, e_j)e_i.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathrm{Ric}(\mathcal{R}^2)\mathrm{x} &= -\sum_i \rho(\mathcal{R}^2 x \wedge e_i)(e_i) = -\sum_i \rho(\mathcal{R}(\mathcal{R}x \wedge e_i))(e_i) \\
&= -\sum_i \rho(\mathcal{R}(-\frac{1}{2}\sum_j \rho(\mathcal{R}x \wedge e_i)(e_j) \wedge e_j))(e_i) \qquad (3.1)
\end{aligned}$$

by Lemma 1.1.0.3. So

$$\mathrm{Ric}(\mathcal{R}^2)\mathrm{x} = \frac{1}{2}\sum_{\mathrm{i,j}} \mathcal{R}^\rho(\mathcal{R}^\rho(\mathrm{x}, \mathrm{e_i})\mathrm{e_j}, \mathrm{e_j})\mathrm{e_i}$$

After applying the 1. Bianchi identity for two times, the right hand side becomes

$$\frac{1}{2}\left(\sum_{i,j} \mathcal{R}^\rho(e_j, e_i)\mathcal{R}^\rho(e_i, e_j)x - \sum_{i,j} \mathcal{R}^\rho(e_j, e_i)\mathcal{R}^\rho(e_j, x)e_i \right.$$
$$\left. - \sum_{i,j} \mathcal{R}^\rho(e_i, \mathcal{R}^\rho(e_i, e_j)x)e_j + \sum_{i,j} \mathcal{R}^\rho(e_i, \mathcal{R}^\rho(e_j, x)e_i)e_j \right).$$

Since the third term equals

$$\sum_{i,j} \mathcal{R}^\rho(e_j, \mathcal{R}^\rho(e_j, e_i)x)e_i$$

we get

$$\mathrm{Ric}(\mathcal{R}^2)\mathrm{x} = \frac{1}{2}\left(-\sum_{i,j} \mathcal{R}^\rho(e_i, e_j)^2 x + \sum_{i,j} \left[\mathcal{R}^\rho(e_j, e_i), \mathcal{R}^\rho(e_j, \cdot)e_i)\right] x \right.$$
$$\left. - \sum_{i,j} \mathcal{R}^\rho(e_i, \mathcal{R}^\rho(e_j, e_i)x)e_j \right).$$

By the previous lemma we have

$$\sum_{i,j} \left[\mathcal{R}^\rho(e_j, e_i), \mathcal{R}^\rho(e_j, \cdot)e_i)\right] x = 0,$$

so

$$\mathrm{Ric}(\mathcal{R}^2)\mathrm{x} = -\frac{1}{2}\sum_{\mathrm{i,j}} \mathcal{R}^\rho(\mathrm{e_i}, \mathrm{e_j})^2\mathrm{x} - \mathcal{R}^\rho(\mathrm{e_i}, \mathcal{R}^\rho(\mathrm{e_j}, \mathrm{x})\mathrm{e_i})\mathrm{e_j}.$$

(c) Applying the 1. Bianchi identity and the symmetries of $\mathcal{R}^\rho$, it follows that

$$\begin{aligned}
\sum_{i,j} \mathcal{R}^\rho(e_i, e_j)\mathcal{R}^\rho(x, e_j)e_i &= \sum_{i,j} \mathcal{R}^\rho(e_i, e_j)\mathcal{R}^\rho(e_i, e_j)x \\
&\quad + \sum_{i,j} \mathcal{R}^\rho(e_i, e_j)\mathcal{R}^\rho(x, e_i)e_j \\
&= \sum_{i,j} \mathcal{R}^\rho(e_i, e_j)^2(e_i, e_j)x \\
&\quad - \sum_{i,j} \mathcal{R}^\rho(e_j, e_i)\mathcal{R}^\rho(x, e_i)e_j \\
&= \sum_{i,j} \mathcal{R}^\rho(e_i, e_j)^2(e_i, e_j)x \\
&\quad - \sum_{i,j} \mathcal{R}^\rho(e_i, e_j)\mathcal{R}^\rho(x, e_j)e_i,
\end{aligned}$$

so

$$2\sum_{i,j} \mathcal{R}^\rho(e_i, e_j)\mathcal{R}^\rho(x, e_j)e_i = \sum_{i,j} \mathcal{R}^\rho(e_i, e_j)^2 x.$$

(d) The previous lemma implies

$$\sum_{i,j} \mathcal{R}^\rho(e_i, \mathcal{R}^\rho(e_i, e_j)x)e_j = \sum_{i,j} \mathcal{R}^\rho(e_i, e_j)\mathcal{R}^\rho(x, e_i)e_j$$

Hence,

$$\begin{aligned}2\sum_{i,j}\mathcal{R}^{\rho}(e_i,\mathcal{R}^{\rho}(e_i,e_j)x)e_j &= \sum_{i,j}\mathcal{R}^{\rho}(e_i,\mathcal{R}^{\rho}(e_i,e_j)x)e_j\\ &\quad+\sum_{i,j}\mathcal{R}^{\rho}(e_i,e_j)\mathcal{R}^{\rho}(x,e_i)e_j\\ &=-\sum_{i,j}\mathcal{R}^{\rho}(\mathcal{R}^{\rho}(e_i,e_j)x,e_i)e_j\\ &\quad+\sum_{i,j}\mathcal{R}^{\rho}(e_i,e_j)\mathcal{R}^{\rho}(x,e_i)e_j\\ &=\sum_{i,j}\left[\mathcal{R}^{\rho}(e_i,e_j),\mathcal{R}^{\rho}(\cdot,e_i)e_j\right](x)\\ &=0,\end{aligned}$$

where we used the lemma again.

□

Corollary 3.3.1.5. *$\mathcal{R}\in WEYL$ implies $\Phi(\mathcal{R})\in WEYL$, so the flow of Φ preserves the space of Weyl curvature operators.*

Beweis. Clear by theorem 3.3.1.3. □

3.3.2 On the Irreducible Decomposition of $\Phi(\mathcal{R})$

Now we are ready to describe the irreducible components of $\Phi(\mathcal{R})$ in terms of the irreducible components of a given algebraic curvature operator $\mathcal{R}$. Decomposing the algebraic curvature operator $\mathcal{R}$ into its irreducible components $I\in\mathrm{SCAL}$, $\mathcal{T}\in\mathrm{RIC}_0$ and $\mathrm{W}\in\mathrm{WEYL}$, we see

$$\Phi(\mathcal{R})=\Phi(I)+\Phi(\mathcal{T})+\Phi(\mathrm{W})+2\varphi(\mathrm{I},\mathcal{T})+2\varphi(\mathrm{I},\mathrm{W})+2\varphi(\mathcal{T},\mathrm{W}),$$

where

$$\varphi(\mathcal{R}_1,\mathcal{R}_2):=\mathcal{R}_1\mathcal{R}_2+\mathcal{R}_2\mathcal{R}_1+2\mathcal{R}_1\#\mathcal{R}_2$$

for any two endomorphisms $\mathcal{R}_1,\mathcal{R}_2$ of $\bigwedge^2 V$. φ is symmetric and satisfies

$$\varphi(\mathcal{R}_1,\mathcal{R}_2)=\frac{1}{2}(\Phi(\mathcal{R}_1+\mathcal{R}_2)-\Phi(\mathcal{R}_1)-\Phi(\mathcal{R}_2)).$$

This ensures that $\varphi(\mathcal{R}_1,\mathcal{R}_2)$ is an algebraic curvature operator provided that $\mathcal{R}_1$ and $\mathcal{R}_2$ are algebraic curvature operators. Further, using proposition 2.4.0.24, we see

Proposition 3.3.2.1. *The trilinear map $(\mathcal{R},\mathcal{S},\mathcal{T})\mapsto\langle\varphi(\mathcal{R},\mathcal{S}),\mathcal{T}\rangle$ is fully symmetric.*

Now we want to identify the irreducible components of $\Phi(\mathcal{R})$:

3.3.2.1 The Trace Part of $\Phi(\mathcal{R})$

It is clear that $\Phi(I)$ lies completely in SCAL. Theorem 3.3.1.3 tells us that $\mathrm{tr}(\Phi(\mathcal{T})) = \|\mathrm{Ric}(\mathcal{T})\|^2 = \|\mathrm{Ric}_0(\mathcal{T})\|^2$, so $\mathrm{tr}(\Phi(\mathcal{R}))$ will depend on this term either. Corollary 3.3.1.5 implies $\mathrm{tr}(\Phi(\mathrm{W})) = 0$. Using proposition 3.3.2.1 and theorem 3.2.3.4, we see $\varphi(I, \mathcal{T}) \in \mathrm{RIC}_0$, $\varphi(I, \mathrm{W}) = 0$ and $\mathrm{tr}(\varphi(\mathcal{T}, \mathrm{W})) = \langle \mathrm{id}, \varphi(\mathcal{T}, \mathrm{W})\rangle = \langle \mathcal{T}, \varphi(\mathrm{id}, \mathrm{W})\rangle = 0$. Thus,

$$\mathrm{tr}(\Phi(\mathcal{R})) = \mathrm{tr}(\Phi(\mathrm{I})) + \mathrm{tr}(\Phi(\mathcal{T})) = \|\mathrm{Ric}(\mathrm{I} + \mathcal{T})\|^2 .$$

This tells us that the trace part of $\Phi(\mathcal{R})$ depends quadratically on the Ricci curvature of $\mathcal{R}$ and that it is independent of the Weyl part of $\mathcal{R}$.

3.3.2.2 The Traceless Ricci Part of $\Phi(\mathcal{R})$

By what we have done above, we are left with the examination of the terms $\varphi(I, \mathcal{T})$, $\varphi(\mathcal{T}, \mathrm{W})$ and $\Phi(\mathcal{T})$:
We observe immediately that the tracefree Ricci part of $\Phi(\mathcal{R})$ depends linearly on I and W and quadratically on $\mathcal{T}$.
Let us treat the term $\varphi(I, \mathcal{T})$ first. Theorem 3.2.3.4 gives $\varphi(I, \mathcal{T}) \in \mathrm{RIC}_0$. If the dimension of V equals 2, then $\varphi(I, \mathcal{T}) = 0$. But if it is different from 2, then $\varphi(I, \mathcal{T}) = 0$ if and only if $\mathcal{T} = 0$.
We note

Lemma 3.3.2.2. *For all $I \in \mathrm{SCAL}$ and $\mathcal{T} \in \mathrm{RIC}_0$ holds*

$$\varphi(I, \mathcal{T}) = (n-2)I\mathcal{T}.$$

Now we treat the term $\varphi(\mathcal{T}, \mathrm{W})$. By proposition 2.4.0.24 we get

$$\left\langle \varphi(\mathcal{T}, \mathrm{W}), \mathrm{W}' \right\rangle = \left\langle \varphi(\mathrm{W}, \mathrm{W}'), \mathcal{T} \right\rangle = 0$$

for every algebraic Weyl curvature operator W', since $2\varphi(\mathrm{W}, \mathrm{W}') = \Phi(\mathrm{W} + \mathrm{W}') - \Phi(\mathrm{W}) - \Phi(\mathrm{W}')$ and each of the terms on the right hand side lies in WEYL. Hence, $\varphi(\mathcal{T}, \mathrm{W}) \in \mathrm{RIC}_0$. We claim that $\varphi(\mathcal{T}, \mathrm{W})$ does not vanish in general. Using theorem 3.3.1.3 we compute

$$\mathrm{Ric}(\varphi(\mathcal{T}, \mathrm{W})) = \sum_{\mathrm{i}} \mathrm{W}^{\rho}(\cdot, \mathrm{e_i})\mathrm{Ric}(\mathcal{T})\mathrm{e_i},$$

where $\{e_i\}$ is an orthonormal basis of V. Now let $x, y \in V$ and compute

$$\begin{aligned}
\langle \mathrm{Ric}(\varphi(\mathcal{T},\mathrm{W}))\mathrm{x},\mathrm{y}\rangle &= \sum_i \langle \mathrm{W}^\rho(\mathrm{x},\mathrm{e_i})\mathrm{Ric}(\mathcal{T})\mathrm{e_i},\mathrm{y}\rangle \\
&= \sum_i \langle \mathrm{W}(\mathrm{x}\wedge \mathrm{e_i}), \mathrm{y}\wedge \mathrm{Ric}(\mathcal{T})\mathrm{e_i}\rangle \\
&= 2\sum_i \langle \mathrm{W}(\mathrm{x}\wedge \mathrm{e_i}), \mathrm{Ric}(\mathcal{T})\wedge \mathrm{id}(\mathrm{y}\wedge \mathrm{e_i})\rangle \\
&\quad - \sum_i \langle \mathrm{W}(\mathrm{x}\wedge \mathrm{e_i}), \mathrm{Ric}(\mathcal{T})\mathrm{y}\wedge \mathrm{e_i}\rangle \\
&= (n-2)\sum_i \langle \mathrm{W}(\mathrm{x}\wedge \mathrm{e_i}), \mathcal{T}(\mathrm{y}\wedge \mathrm{e_i})\rangle \\
&= (n-2)\sum_i \langle \mathcal{T}\mathrm{W}(\mathrm{x}\wedge \mathrm{e_i}), \mathrm{y}\wedge \mathrm{e_i}\rangle \\
&= (n-2)\,\langle \mathrm{Ric}(\mathcal{T}\mathrm{W})\mathrm{x},\mathrm{y}\rangle
\end{aligned}$$

Hence,

$$\mathrm{Ric}(\varphi(\mathcal{T},\mathrm{W})) = (\mathrm{n}-2)\mathrm{Ric}(\mathcal{T}\mathrm{W}).$$

This expression is symmetric in $\mathcal{T}$ and W, which gives

$$\varphi(\mathcal{T},\mathrm{W}) = \mathrm{Ric}_0(\mathcal{T}\mathrm{W}+\mathrm{W}\mathcal{T})\wedge \mathrm{id}.$$

Here, we used that $\varphi(\mathcal{T},\mathrm{W})\in \mathrm{RIC}_0$ implies $\varphi(\mathcal{T},\mathrm{W}) = \frac{2}{\mathrm{n}-2}\mathrm{Ric}_0(\varphi(\mathcal{T},\mathrm{W}))\wedge$ id.
We note this result in a proposition

Proposition 3.3.2.3. *For all* $\mathcal{T}\in \mathrm{RIC}_0$ *and* $\mathrm{W}\in \mathrm{WEYL}$ *holds*

$$\varphi(\mathcal{T},\mathrm{W}) = \mathrm{Ric}_0(\mathcal{T}\mathrm{W}+\mathrm{W}\mathcal{T})\wedge \mathrm{id}.$$

Assume for a moment that the dimension of V equals 4. Using the results from subsection 3.2.2.2 it follows that W preserves the eigenspaces of the Hodge $*$-operator and $\mathcal{T}$ interchanges them. Proposition 3.2.3.8 then implies that $\mathcal{T}\mathrm{W}+\mathrm{W}\mathcal{T}$ is an algebraic curvature operator of traceless Ricci type. Thus, we get

$$\varphi(\mathcal{T},\mathrm{W}) = \mathcal{T}\mathrm{W}+\mathrm{W}\mathcal{T}$$

in dimension 4. Now we give an example for which $\varphi(\mathcal{T},\mathrm{W})\neq 0$:
Define

$$\mathcal{T} := \iota^+\circ(\iota^-)^{-1}\circ\pi^- + \iota^-\circ(\iota^+)^{-1}\circ\pi^+,$$

where $\iota^\pm$ was defined in theorem 1.2.0.8 and $\pi^\pm$ is the orthogonal projection onto the ±1-eigenspace of the Hodge $*$-operator. $\mathcal{T}$ simply interchanges the eigenspaces of the Hodge $*$-operator and is therefore an algebraic curvature of traceless Ricci type (show $\mathrm{id}\#\mathcal{T} = 0$). Now let $\mathrm{W} \in \mathrm{WEYL}^+$, $\mathrm{W} \neq 0$. Then we clearly have

$$\mathcal{T}\mathrm{W} + \mathrm{W}\mathcal{T} \neq 0.$$

We may use this result to construct algebraic curvature operators $\mathcal{T}$ and W in higher dimensions, such that $\varphi(\mathcal{T}, \mathrm{W}) \neq 0$.

Now we come to treat the last term $\Phi(\mathcal{T})$. Theorem 3.3.1.3 says

$$\begin{aligned}\mathrm{Ric}_0(\Phi(\mathcal{T})) &= 2\sum_i \mathcal{T}^\rho(\cdot, e_i)\mathrm{Ric}(\mathcal{T})\mathrm{e_i} - \frac{2}{\mathrm{n}}\left\|\mathrm{Ric}(\mathcal{T})\right\|^2 \mathrm{id} \\ &= 2\sum_i \mathcal{T}^\rho(\cdot, e_i)\mathrm{Ric}_0(\mathcal{T})\mathrm{e_i} - \frac{2}{\mathrm{n}}\left\|\mathrm{Ric}_0(\mathcal{T})\right\|^2 \mathrm{id},\end{aligned}$$

which is usually different from 0. For, if $\mathrm{Ric}_0(\Phi(\mathcal{T})) = 0$, then

$$\sum_i \mathcal{T}^\rho(\cdot, e_i)\mathrm{Ric}_0(\mathcal{T})\mathrm{e_i} = \frac{1}{\mathrm{n}}\left\|\mathrm{Ric}_0(\mathcal{T})\right\|^2 \mathrm{id}.$$

$\mathcal{T} \in \mathrm{RIC}_0$ implies $\frac{2}{n-2}\mathrm{Ric}_0(\mathcal{T}) \wedge \mathrm{id} = \mathcal{T}$. Hence, if $\{e_i\}$ is an orthonormal eigenbasis of $\mathrm{Ric}_0(\mathcal{T})$ and $\tau_1, ..., \tau_n$ are the corresponding eigenvalues, then $\{e_i \wedge e_j\}$ is an orthonormal eigenbasis of $\mathcal{T}$ with corresponding eigenvalues $\frac{\tau_i+\tau_j}{n-2}$. This gives

$$\begin{aligned}\sum_i \mathcal{T}^\rho(e_j, e_i)\mathrm{Ric}_0(\mathcal{T})\mathrm{e_i} &= \sum_{i\neq j} \tau_i \mathcal{T}^\rho(e_j, e_i)e_i \\ &= -\sum_{i\neq j} \tau_i \rho(\mathcal{T}(e_j \wedge e_i))e_i \\ &= \sum_{i\neq j} \frac{\tau_i^2 + \tau_i\tau_j}{n-2} e_j \\ &= \frac{1}{n-2}\left(\left\|\mathrm{Ric}_0(\mathcal{T})\right\|^2 - 2\tau_j^2\right)e_j,\end{aligned}$$

leading to

$$\tau_j^2 = \frac{n-2}{2}\left(\frac{1}{n-2} - \frac{1}{n}\right)\left\|\mathrm{Ric}_0(\mathcal{T})\right\|^2 = \frac{1}{n}\left\|\mathrm{Ric}_0(\mathcal{T})\right\|^2$$

for all $j = 1, ..., n$. This leads to the following theorem

Theorem 3.3.2.4. *Assume that* $\mathrm{Ric}_0(\Phi(\mathcal{T})) = 0$.

1. *If the dimension of V is odd, then* $\mathrm{Ric}_0(\mathcal{T}) = 0$ *and hence even* $\mathcal{T} = 0$.

2. If the dimension of V is even, then

$$\mathcal{T} = \frac{1}{\sqrt{n}} \|\mathrm{Ric}_0(\mathcal{T})\| (\pi - \pi^{\perp}),$$

where $\pi : \bigwedge^2 V \to \bigwedge^2 E$ is the orthogonal projection and E denotes the eigenspace associated with the eigenvalue $\frac{1}{\sqrt{n}} \|\mathrm{Ric}_0(\mathcal{T})\|$ of $\mathcal{T}$.

Beweis. Clear □

Again, there is something special happening in dimension 4:
Both $\mathcal{T}^2$ and $\mathcal{T}\#\mathcal{T}$ preserve the eigenspaces of the Hodge $*$-operator. Thus, $\Phi(\mathcal{T})$ is an algebraic curvature operator on $\bigwedge^2 V$ preserving these spaces as well. It follows that its traceless Ricci part must vanish. As a consequence, the traceless Ricci part of $\Phi(\mathcal{R})$ reads

$$\mathrm{Ric}_0(\Phi(\mathcal{R})) \wedge \mathrm{id} = 2\mathrm{I}\mathcal{T} + \mathcal{T}\mathrm{W} + \mathrm{W}\mathcal{T}.$$

We note

Proposition 3.3.2.5. *In dimension 4 holds*

$$\mathrm{Ric}_0(\Phi(\mathcal{R})) \wedge \mathrm{id} = 2\mathrm{I}\mathcal{T} + \mathcal{T}\mathrm{W} + \mathrm{W}\mathcal{T}$$

for any algebraic curvature operator $\mathcal{R}$.

In order to get a better understanding of the traceless Ricci part of $\Phi(R)$ in general, we have to take a closer look at the term $\Phi(\mathcal{T}$:)
$\mathcal{T}$ is of the form $\mathcal{T} = F \wedge \mathrm{id}$ with $F \in \mathrm{Sym}_0\,(\mathrm{V})$. Let $\{e_i\}$ be an orthonormal eigenbasis of F, i.e. $Fe_i = f_i e_i$, $f_i \in \mathbb{R}$. Now we start computing $\Phi(F \wedge \mathrm{id})$:
It is easy to see that

$$2(F \wedge \mathrm{id})^2 = \mathrm{F}^2 \wedge \mathrm{id} + \mathrm{F} \wedge \mathrm{F}.$$

Treating the term $2(F \wedge \mathrm{id})\#(\mathrm{F} \wedge \mathrm{id})$ requires some more work. We compute:

$$\begin{aligned}
2(F \wedge id)\#(F \wedge id)e_i \wedge e_j &= \sum_k [F \wedge id(e_i \wedge e_k), F \wedge id(e_j \wedge e_k)] \\
&= \frac{1}{2} \sum_{k \neq i,j} (f_i + f_k)(f_j + f_k) e_i \wedge e_j \\
&= \frac{1}{2} \sum_{k \neq i,j} (f_i f_j + f_k(f_i + f_j) + f_k^2) e_i \wedge e_j \\
&= \frac{1}{2} \Big((n-2) f_i f_j + (tr(F) - (f_i + f_j))(f_i + f_j) + (\|F\|^2 - (f_i^2 + f_j^2)) \Big) e_i \wedge e_j \\
&= \left(\frac{n-2}{2} F \wedge F + tr(F) F \wedge \mathrm{id} - 2(\mathrm{F} \wedge \mathrm{id})^2 + \frac{1}{2} \|\mathrm{F}\|^2 \mathrm{id} - \mathrm{F}^2 \wedge \mathrm{id} \right) e_i \wedge e_j \\
&= \left(\frac{n-2}{2} F \wedge F - 2(F \wedge \mathrm{id})^2 + \frac{1}{2} \|\mathrm{F}\|^2 \mathrm{id} - \mathrm{F}^2 \wedge \mathrm{id} \right) e_i \wedge e_j
\end{aligned}$$

This yields the proposition

Proposition 3.3.2.6. *Let F be a self-adjoint endomorphism of V with vanishing trace. Then*

$$\Phi(F \wedge \mathrm{id}) = \frac{\mathrm{n}-2}{2} \mathrm{F} \wedge \mathrm{F} + \frac{1}{2} \|\mathrm{F}\|^2 \mathrm{id} - \mathrm{F}^2 \wedge \mathrm{id},$$

or

$$\Phi(F \wedge \mathrm{id}) = \frac{\mathrm{n}}{2} \mathrm{F} \wedge \mathrm{F} - 2(\mathrm{F} \wedge \mathrm{id})^2 + \frac{1}{2} \|\mathrm{F}\|^2 \mathrm{id}.$$

Now we compute the traceless Ricci part of $\Phi(\mathcal{T})$, $\mathcal{T} = F \wedge \mathrm{id}$.

Proposition 3.3.2.7. *Consider $\mathcal{T} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ with $\mathcal{T} = F \wedge \mathrm{id}$ The traceless Ricci part of $\Phi(\mathcal{T})$ is given by*

$$\frac{2}{n-2} \mathrm{Ric}_0(\Phi(\mathrm{F} \wedge \mathrm{id})) = -2 \left(\mathrm{F}^2 \wedge \mathrm{id} - \frac{1}{\mathrm{n}} \|\mathrm{F}\|^2 \mathrm{id} \right).$$

Beweis. It is clear that the traceless Ricci part of $\frac{1}{2} \|F\|^2 \mathrm{id}$ is zero. Using lemma 3.2.1.6, we compute

$$\mathrm{Ric}_0(\mathrm{F} \wedge \mathrm{F}) = -\mathrm{F}^2 + \frac{1}{\mathrm{n}} \|\mathrm{F}\|^2 \mathrm{id}$$

and

$$\mathrm{Ric}_0(\mathrm{F}^2 \wedge \mathrm{id}) = \frac{\mathrm{n}-2}{2} \left(\mathrm{F}^2 - \frac{1}{\mathrm{n}} \|\mathrm{F}\|^2 \mathrm{id} \right).$$

This gives

$$\mathrm{Ric}_0(\Phi(\mathrm{F} \wedge \mathrm{id})) = (2-\mathrm{n}) \left(\mathrm{F}^2 - \frac{1}{\mathrm{n}} \|\mathrm{F}\|^2 \mathrm{id} \right)$$

and the claim follows. □

Corollary 3.3.2.8. *Let $\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ be an algebraic curvature operator with invariant decomposition $\mathcal{R} = I + \mathcal{T} + \mathrm{W}$, $I \in \mathrm{SCAL}$, $\mathcal{T} = F \wedge \mathrm{id} \in \mathrm{RIC}_0$ and $\mathrm{W} \in \mathrm{WEYL}$. Then the traceless Ricci part of $\Phi(\mathcal{R})$ is given by*

$$\frac{2}{n} \|F\|^2 \mathrm{id} + 2(\mathrm{n}-2)\mathrm{I}\mathcal{T} - 2\mathrm{F}^2 \wedge \mathrm{id} + 2\mathrm{Ric}_0(\mathcal{T}\mathrm{W} + \mathrm{W}\mathcal{T}) \wedge \mathrm{id}$$

3.3.2.3 The Weyl Part of $\Phi(\mathcal{R})$

We know already that $\Phi(\mathrm{W})$ lies in WEYL. The only other term producing Weyl curvature is $\Phi(\mathcal{T})$. We get

$$\mathrm{W}(\Phi(\mathcal{R})) = \Phi(\mathrm{W}) + \mathrm{W}(\Phi(\mathcal{T})).$$

We will now take a closer look at the term $\Phi(\mathcal{T})$ and its Weyl curvature: Proposition 3.3.2.6 gives

$$\Phi(F \wedge \mathrm{id}) = \frac{\mathrm{n}-2}{2} \mathrm{F} \wedge \mathrm{F} + \frac{1}{2} \|\mathrm{F}\|^2 \mathrm{id} - \mathrm{F}^2 \wedge \mathrm{id}.$$

It is clear that the Weyl part of $\Phi(F \wedge \mathrm{id})$ comes from $F \wedge F$, since the other two terms obviously lie in SCAL and SCAL $\oplus$ RIC_0. Hence, the Weyl curvature of $\Phi(F \wedge \mathrm{id})$ takes the form

$$\mathrm{W}(\Phi(\mathrm{F} \wedge \mathrm{id})) = \frac{\mathrm{n}-2}{2}\mathrm{F} \wedge \mathrm{F} - \frac{1}{2(\mathrm{n}-1)} \|\mathrm{F}\|^2 \,\mathrm{id} + \mathrm{F}^2 \wedge \mathrm{id}.$$

An easy calculation shows that this expression is always zero in dimension $n \leq 3$. What happens if the Weyl curvature of $\Phi(F \wedge \mathrm{id})$ vanishes in dimension $n \geq 4$? If this is the case, the eigenvalues of F will fulfill the relations

$$(f_i - f_j)^2 + n f_i f_j = \frac{1}{n-1} \|F\|^2, \text{ if } i \neq j.$$

So, if $F \neq 0$, this situation can only occur if the kernel of F has dimension less than 2. Moreover, assuming $\dim(\ker(F)) = 1$, it follows that the nonzero eigenvalues of F are equal up to sign. Now, F has vanishing trace, which implies that the number of positive eigenvalues equals the number of negative eigenvalues. Therefore, V splits as a direct sum

$$V = \mathrm{E}^+ \oplus \mathrm{E}^- \oplus \ker(\mathrm{F}), \ \dim(\mathrm{E}^+) = \dim(\mathrm{E}^-)$$

where E^+ and E^- are the eigenspaces of F corresponding to the positive and the negative eigenvalue f and $-f$ of F, respectively. By the way, note that the dimension of V has to be odd in this case. Now the formula from above gives

$$nf^2 = (4-n)f^2,$$

which implies $f = 0$. Otherwise we would have $n = 2$, which was excluded. We have shown

Proposition 3.3.2.9. *Let $\mathcal{T} = F \wedge \mathrm{id} \neq 0$ be an algebraic curvature operator of traceless Ricci type in dimension $n \geq 4$. If the Weyl curvature of $\Phi(\mathcal{T})$ vanishes, then F is an isomorphism of V.*

From now on we assume the dimension of V to be greater than 3. We are left with the case, where F is an isomorphism of V. The formula from above implies

$$(f_i - f_j)((n-2)f_k + f_i + f_j) = 0.$$

provided that i, j and k are mutually distinct. $\mathrm{tr}(\mathrm{F}) = 0$ implies that there are at least two distinct eigenvalues of F. Assume $f_1 \neq f_2$. Then we get

$$f_k = -\frac{f_1 + f_2}{n-2}$$

for all $k \geq 3$. We conclude that F has at most 3 distinct eigenvalues f_1, f_2 and f_3. At least two of them are different from each other. We keep the assumption $f_1 \neq f_2$. Now we show that there are precisely two distinct eigenvalues. Assuming f_1, f_2 and f_3 to be mutually distinct we get

$$(n-2)f_2 + f_1 + f_3 = 0 \text{ and } (n-2)f_1 + f_2 + f_3 = 0,$$

which leads to $f_1 = f_2$, a contradiction. We note

Proposition 3.3.2.10. *Let $\mathcal{T} = F \wedge \mathrm{id}$ be an algebraic curvature operator of traceless Ricci type in dimension $n \geq 4$. Suppose that the Weyl curvature of $\Phi(\mathcal{T})$ vanishes. Then F takes the form*

$$F = (v^* \otimes v)_0$$

for some $v \in V$, i.e. F is the tracefree part of the map $x \mapsto \langle v, x \rangle v$.

At the end of this paragraph we would like to state our results about the irreducible decomposition of $\Phi(\mathcal{R})$ in one theorem.

Theorem 3.3.2.11. *Let $\mathcal{R}$ be an algebraic curvature operator decomposing as $\mathcal{R} = \mathcal{I} + \mathcal{T} + \mathcal{W}$ with $\mathcal{I} \in \mathrm{SCAL}$, $\mathcal{T} = F \wedge \mathrm{id} \in \mathrm{RIC}_0$ and $\mathcal{W} \in \mathrm{WEYL}$. Then,*

1. *the trace-part of $\Phi(\mathcal{R})$ is given by*

$$\frac{1}{N} \|\mathrm{Ric}(\mathcal{R})\|^2 \,\mathrm{id},$$

2. *the traceless Ricci part of $\Phi(\mathcal{R})$ is given by*

$$\frac{2}{n} \|F\|^2 \,\mathrm{id} + 2(\mathrm{n}-2)\mathrm{I}\mathcal{T} - 2\mathrm{F}^2 \wedge \mathrm{id} + 2\mathrm{Ric}_0(\mathcal{T}\mathcal{W} + \mathcal{W}\mathcal{T}) \wedge \mathrm{id}$$

3. *the Weyl part of $\Phi(\mathcal{R})$ is given by*

$$\Phi(\mathcal{W}) + \frac{n-2}{2} F \wedge F - \frac{1}{2(n-1)} \|F\|^2 \, id + F^2 \wedge id.$$

3.4 Algebraic Symmetric Curvature Operators

Suppose that we are given a simply connected and irreducible symmetric space (M, g) with curvature operator $\mathcal{R}^g$ and $p \in M$. Since $\mathcal{R}^g$ is parallel, the theorem of Ambrose and Singer [1], see appendix B.4 for the formulation of this theorem, gives us $\mathfrak{h}_{\mathcal{R}_p^g} = \rho_p^{-1}(\mathfrak{hol}_p(M, g))$. Theorem B.7.3.2 of the appendix tells us that the Lie algebra $\mathfrak{hol}_p(M, g)$ of the holonomy group of (M, g) coincides with the Lie algebra $\mathfrak{iso}_p(M, g)$ of the isotropy group $\mathrm{Isom_p(M, g)}$. This implies that $\mathfrak{h}_{\mathcal{R}_p^g} = \rho_p^{-1}(\mathfrak{iso}_p(M, g))$. Now, $\mathrm{Isom_p(M, g)}$ acts isometrically on the space of algebraic curvature operators $\mathrm{LC}\left(\bigwedge^2(\mathrm{T_pM, g_p})\right)$ via conjugation,

$$(\varphi, \mathcal{R}) \mapsto \mathrm{D}\varphi_\mathrm{p} \wedge \mathrm{D}\varphi_\mathrm{p} \circ \mathcal{R} \circ \mathrm{D}\varphi_\mathrm{p}^{-1} \wedge \mathrm{D}\varphi_\mathrm{p}^{-1}.$$

It is clear that $\mathrm{Isom_p(M, g)}$ lies completely in the stabilizer of $\mathcal{R}_p^g$ w.r.t. this action. This shows that $\mathfrak{h}_{\mathcal{R}_p^g}$ lies completely in the Lie algebra of the stabilizer of $\mathcal{R}_p^g$ w.r.t. the action of $\mathrm{SO(T_pM, g_p)}$ on $\mathrm{LC}\left(\bigwedge^2(\mathrm{T_pM, g_p})\right)$:

$$\mathfrak{h}_{\mathcal{R}_p^g} \subseteq \mathfrak{stab}_{\mathrm{SO(T_pM, g_p)}}(\mathcal{R}_p^g) = \left\{\varepsilon \in \bigwedge^2(\mathrm{T_pM, g_p}) : [(\mathrm{ad_p})_\varepsilon, \mathcal{R}] = 0\right\}.$$

We will use this fact to define Algebraic symmetric curvature operators on a given n-dimensional Euclidean vector space $(V, \langle \cdot\, , \cdot \rangle)$ and see what we get.

Definition 3.4.0.12. *An algebraic curvature operator $\mathcal{R}$ is called an algebraic symmetric curvature operator (or simply a symmetric curvature operator) on $\bigwedge^2 V$, if it commutes with* ad_ε *for every $\varepsilon \in \mathfrak{h}_\mathcal{R}$.*

Example 3.4.0.13. Every range 1 algebraic curvature operator is symmetric. Moreover, any algebraic curvature operator of the form $\mathcal{R} = \pm\lambda\pi$, where $\pi : \bigwedge^2 V \to \bigwedge^2 V$ is an orthogonal projection onto a subspace $\bigwedge^2 U$, $U \leq V$ a subspace, and $\lambda \in \mathbb{R}$, is symmetric.

Each algebraic curvature operator $\mathcal{R}$ has a geometric realization on an open subset U of V. We will see in a moment that the symmetric curvature operators even have geometric realizations as curvature operators of simply connected symmetric spaces. This theorem is due to É. Cartan [7].

Theorem 3.4.0.14. *If $\mathcal{R} \in LC(\bigwedge^2 V)$ is symmetric, then $\mathcal{R}$ is the curvature operator of a simply connected symmetric space (M, g). (M, g) is unique up to isometry.*

Beweis. We define a Lie algebra structure on $V \oplus \mathfrak{h}_R$ as follows:

1. For $v, w \in V$ define $[v, w] := \mathcal{R}v \wedge w$
2. For $v \in V$ and $\varepsilon \in \mathfrak{h}_R$ define $[\varepsilon, v] := -[v, \varepsilon] := \rho(\varepsilon)v$

3. For $\varepsilon, \delta \in \mathfrak{h}_R$ define $[\varepsilon, \delta] := [\varepsilon, \delta]_{\bigwedge^2 V}$

It is clear that $[\cdot, \cdot]$ is $\mathbb{R}$-bilinear and skew-symmetric. It remains to check the Jacobi identity:

- For $u, v, w \in V$ holds

$$[u, [v, w]] = [u, \mathcal{R}v \wedge w] = -\rho(\mathcal{R}v \wedge w)u = \mathcal{R}^\rho(v, w)u$$

  In this case the Jacobi identity follows from the fact that $\mathcal{R}^\rho$ satisfies the 1. Bianchi identity.

- It is clear that the Jacobi identity is valid for all $\varepsilon, \delta, \gamma \in \mathfrak{h}_R$.

- For $\varepsilon, \delta \in \mathfrak{h}_R$ and $v \in V$ we get

$$\begin{aligned}
[\varepsilon, [\delta, v]] + [\delta, [v, \varepsilon]] + [v, [\varepsilon, \delta]] &= \rho(\varepsilon)\rho(\delta)v - \rho(\delta)\rho(\varepsilon)v - \rho([\varepsilon, \delta])v \\
&= [\rho(\varepsilon), \rho(\delta)]\, v - \rho([\varepsilon, \delta])v \\
&= 0
\end{aligned}$$

  since ρ is a homomorphism of Lie algebras.

- And for $\varepsilon \in \mathfrak{h}_R$ and $v, w \in V$ we have

$$\begin{aligned}
[v, [w, \varepsilon]] + [w, [\varepsilon, v]] + [\varepsilon, [v, w]] &= -\,[v, \rho(\varepsilon)w] + [w, \rho(\varepsilon)v] + [\varepsilon, \mathcal{R}v \wedge w] \\
&= -\mathcal{R}v \wedge \rho(\varepsilon)w + \mathcal{R}w \wedge \rho(\varepsilon)v + \mathrm{ad}_\varepsilon \circ \mathcal{R}\mathrm{v} \wedge \mathrm{w} \\
&= (-\mathcal{R} \circ 2\rho(\varepsilon) \wedge id + \mathrm{ad}_\varepsilon \circ \mathcal{R})\mathrm{v} \wedge \mathrm{w} \\
&= [\mathrm{ad}_\varepsilon, \mathcal{R}]\, v \wedge w \\
&= 0
\end{aligned}$$

  since $\mathcal{R}$ is symmetric.

Now let $L : V \oplus \mathfrak{h}_R \to V \oplus \mathfrak{h}_R$ be the linear map defined by

$$L|_V = -id_V,\ L_{\mathfrak{h}_R} = id_{\mathfrak{h}_R}.$$

Then L is an involutive homomorphism of Lie algebras:

- $L^2 = id$ is clear.

- For $v, w \in V$ we get

$$L\,[v, w] = L\mathcal{R}v \wedge w = \mathcal{R}v \wedge w = [v, w]$$

  and

$$[Lv, Lw] = [v, w]$$

- For $\varepsilon \in V\mathfrak{h}_R$ and $v \in V$

$$L[\varepsilon, v] = L\rho(\varepsilon)v = -\rho(\varepsilon)v = [-\varepsilon, v] = [L\varepsilon, Lv]$$

- And for $\varepsilon, \delta \in \mathfrak{h}_R$

$$L[\varepsilon, \delta][\varepsilon, \delta] = [L\varepsilon, L\delta]$$

Hence, the pair $(V \oplus \mathfrak{h}_R, L)$ is an effective orthogonal symmetric Lie algebra and therefore there exists a symmetric space $M = G/H$ with $Lie(G) = V \oplus \mathfrak{h}_R$, $Lie(H) = \mathfrak{h}_R$ and with curvature tensor S, such that

$$S(x, y)z = -[x, [y, z]] = \mathcal{R}^\rho(x, y)z$$

for all $x, y, z \in V$.

□

Remark 3.4.0.15. *It is clear that every symmetric algebraic curvature operator $\mathcal{R}$ has a geometric realization M as a symmetric space, such that M is simply connected. Recall that the universal cover $\widetilde{M}$ of M with the pullback metric is a symmetric space, whenever M is a symmetric space.*

Lemma 3.4.0.16. *Let $\mathcal{R} \in LC(\bigwedge^2 V)$ be symmetric. Choose an orthonormal basis ε_i, such that $\mathcal{R}\varepsilon_i = \lambda_i \varepsilon_i$.Then*

1. ad_ε *preserves the eigenspaces of $\mathcal{R}$ for every $\varepsilon \in \mathfrak{h}_\mathcal{R}$*
2. *$\lambda_i \neq \lambda_j$ and $\lambda_i, \lambda_j \neq 0$, then $[\varepsilon_i, \varepsilon_j] = 0$*

Beweis.

1. $\mathcal{R}\mathrm{ad}_\varepsilon \varepsilon_\mathrm{i} = \mathrm{ad}_\varepsilon \mathcal{R}\varepsilon_\mathrm{i} = \lambda_\mathrm{i}\mathrm{ad}_\varepsilon \varepsilon_\mathrm{i}$
2. From 1. we conclude that $\lambda_i \neq \lambda_j$ and $\lambda_i, \lambda_j \neq 0$ imply $[\varepsilon_i, \varepsilon_j]$ lies in the λ_i-eigenspace and in the λ_j-eigenspace of $\mathcal{R}$ at the same time, which is only possible if $[\varepsilon_i, \varepsilon_j] = 0$.

□

Corollary 3.4.0.17. *If $\mathcal{R} \in LC(\bigwedge^2 V)$ is symmetric, $\mathfrak{h}_\mathcal{R}$ is the direct sum of the eigenspaces of $\mathcal{R}$ corresponding to the nonzero eigenvalues of $\mathcal{R}$. Each of the eigenspaces forms an ideal in $\mathfrak{h}_R$ and $\mathfrak{h}_\mathcal{R} = \mathrm{im}(\mathcal{R})$.*

Beweis. This is an immediate consequence of the previous lemma. □

Theorem 3.4.0.18. *If $\mathcal{R} \in LC(\bigwedge^2 V)$ is symmetric, then $\mathcal{R}\#\mathcal{R}$ preserves the simple $\mathcal{R}$-invariant ideals of $\mathfrak{h}_R$ In particular, we have*

$$\mathcal{R}\#\mathcal{R} = \pi_\mathcal{R}\#\pi_\mathcal{R} \circ \mathcal{R}^2$$

in this case, where $\pi_\mathcal{R}$ is the orthogonal projection of $\bigwedge^2 V$ onto the image of $\mathcal{R}$.

Beweis. Write $\mathcal{R} = \sum_i \lambda_i \varepsilon_i^* \otimes \varepsilon_i$, $\lambda_i \in \mathbb{R}$ and $\{\varepsilon_i\}$ an orthonormal eigenbasis of $\mathcal{R}$. Then, by corollary 2.2.0.20, we have

$$\mathcal{R}\#\mathcal{R}\varepsilon_i = \frac{1}{2}\sum_{k,l} \lambda_k \lambda_l \left\langle [\varepsilon_k, \varepsilon_l], \varepsilon_i \right\rangle [\varepsilon_k, \varepsilon_l]$$

for each i. Now we fix some i between 1 and N. If the term

$$\lambda_k \lambda_l \left\langle [\varepsilon_k, \varepsilon_l], \varepsilon_i \right\rangle [\varepsilon_k, \varepsilon_l]$$

is nonzero for some k and l , then

$$\lambda_k, \lambda_l \neq 0, \left\langle [\varepsilon_k, \varepsilon_l], \varepsilon_i \right\rangle \neq 0 \text{ and } [\varepsilon_k, \varepsilon_l], [\varepsilon_k, \varepsilon_i], [\varepsilon_l, \varepsilon_i] \neq 0.$$

Now, lemma 3.4.0.16 implies

$$\lambda_k = \lambda_l = \lambda_i$$

, if $\lambda_i \neq 0$ in this case. Moreover, since the simple $\mathcal{R}$-invariant ideals of $\mathfrak{h}_{\mathcal{R}}$ are perpendicular to each other, $\left\langle [\varepsilon_k, \varepsilon_l], \varepsilon_i \right\rangle \neq 0$ forces ε_k and ε_l to lie in the simple $\mathcal{R}$-invariant ideal of $\mathfrak{h}_{\mathcal{R}}$ containing ε_i. This shows that $\mathcal{R}\#\mathcal{R}$ preserves the simple $\mathcal{R}$-invariant ideals of $\mathfrak{h}_{\mathcal{R}}$.
Further, it turns out that

$$\begin{aligned}
\mathcal{R}\#\mathcal{R}\varepsilon_i &= \frac{1}{2}\sum_{k,l} \lambda_k \lambda_l \left\langle [\varepsilon_k, \varepsilon_l], \varepsilon_i \right\rangle [\varepsilon_k, \varepsilon_l] \\
&= \frac{\lambda_i^2}{2} \sum_{k,l;\lambda_k,\lambda_l \neq 0} \left\langle [\varepsilon_k, \varepsilon_l], \varepsilon_i \right\rangle [\varepsilon_k, \varepsilon_l] \\
&= \lambda_i^2 \pi_{\mathcal{R}}\#\pi_{\mathcal{R}}(\varepsilon_i) \\
&= (\pi_{\mathcal{R}}\#\pi_{\mathcal{R}} \circ \mathcal{R}^2)(\varepsilon_i)
\end{aligned}$$

as claimed.

□

Remark 3.4.0.19. *Note that $\pi_{\mathcal{R}}\#\pi_{\mathcal{R}}$ commutes with $\mathcal{R}$, so $\pi_{\mathcal{R}}\#\pi_{\mathcal{R}}$ and $\mathcal{R}$ are simultaneously diagonalizable. Thus, $\pi_{\mathcal{R}}\#\pi_{\mathcal{R}}$ takes the form*

$$\pi_{\mathcal{R}}\#\pi_{\mathcal{R}} = \sum_{\{i|\lambda_i \neq 0\}} \|\mathrm{ad}_{\varepsilon_i}\|^2 \varepsilon^i \otimes \varepsilon_i$$

in a common orthonormal eigenbasis.
This leads to the following formula for $\mathcal{R}\#\mathcal{R}$:

$$\mathcal{R}\#\mathcal{R} = \sum_i \lambda_i^2 \|\mathrm{ad}_{\varepsilon_i}\|^2 \varepsilon^i \otimes \varepsilon_i.$$

Beweis. If ε_k lies in the kernel of $\mathcal{R}$, so does $[\varepsilon_k, \varepsilon_i]$ if $\lambda_i \neq 0$, since

$$\mathcal{R}\,[\varepsilon_i, \varepsilon_k] = (\mathcal{R} \circ \mathrm{ad}_{\varepsilon_\mathrm{i}})(\varepsilon_\mathrm{k}) = (\mathrm{ad}_{\varepsilon_\mathrm{i}} \circ \mathcal{R})(\varepsilon_\mathrm{k}) = 0.$$

So the eigenvalues of $\pi_{\mathcal{R}} \# \pi_{\mathcal{R}}$ are given by

$$\langle \pi_{\mathcal{R}} \# \pi_{\mathcal{R}} \varepsilon_i, \varepsilon_i \rangle = \sum_{k,l} \langle \mathrm{ad}_{\varepsilon_\mathrm{i}} \varepsilon_\mathrm{k}, \varepsilon_\mathrm{l} \rangle^2 = \sum_k \|\mathrm{ad}_{\varepsilon_\mathrm{i}} \varepsilon_\mathrm{k}\|^2 = \|\mathrm{ad}_{\varepsilon_\mathrm{i}}\|^2$$

□

Corollary 3.4.0.20. *If $\mathcal{R}$ is symmetric, so is $\Phi(\mathcal{R})$.*

Beweis. Clear. □

Corollary 3.4.0.21. *If $\mathcal{R} \in LC(\bigwedge^2 V)$ is symmetric, then $tr(\mathcal{R} \# \mathcal{R}) \geq 0$*

Beweis. Clear. □

Corollary 3.4.0.22. *If $\mathcal{R} \in LC(\bigwedge^2 V)$ is symmetric and $\mathcal{R} \in WEYL$, then $\mathcal{R} = 0$.*

Beweis. $\mathcal{R} \in WEYL$ implies

$$0 = tr\Phi(\mathcal{R}) = 2\,\|\mathcal{R}\|^2 + 2tr\mathcal{R}\#\mathcal{R},$$

so

$$tr\mathcal{R}\#\mathcal{R} \leq 0$$

which implies $tr\mathcal{R}\#\mathcal{R} = 0$, since $\mathcal{R}$ is symmetric. But then, each eigenvalue of $\mathcal{R}$ must be zero, so $\mathcal{R}$ is zero. □

Corollary 3.4.0.23. *If $\mathcal{R} \in LC(\bigwedge^2 V)$ is symmetric with $\Phi(\mathcal{R}) = \tau(\mathcal{R})\mathcal{R}$, then all nonzero eigenvalues of $\mathcal{R}$ have the same sign. If it has vanishing trace, then it is trivial*

Beweis. This is clear, because $\Phi(\mathcal{R})$ is nonnegative, whenever $\mathcal{R}$ is symmetric. □

3.5 Algebraic Product Curvature Operators

Suppose that $\mathcal{R}$ is the curvature operator of a product of Riemannian manifolds M and N. Then $\mathcal{R}$ is the orthogonal direct sum of the curvature operators $\mathcal{R}_M$ and $\mathcal{R}_N$ of M and N and the images of the summands form ideals in the smallest Lie subalgebra of $\bigwedge^2 T(M \times N)$ containing the image of $\mathcal{R}$. We use this as a starting point for our definition of algebraic product curvature operators.

Definition 3.5.0.24. *An algebraic curvature operator $\mathcal{R}$ on $\bigwedge^2 V$ is called a geometric product of the algebraic curvature operators $\mathcal{R}_1, ..., \mathcal{R}_r$, $r \in \mathbb{N}$, if*

$$\mathcal{R} = \sum_i \mathcal{R}_i$$

and if there exists an orthonormal decomposition $V = V_1 \oplus ... \oplus V_r$ into subspaces U_i of V with

$$\mathrm{im}(\mathcal{R}_\mathrm{i}) \subseteq \bigwedge^2 \mathrm{V}_\mathrm{i}$$

for each $i = 1, ..., r$.
$\mathcal{R}$ is called geometrically irreducible, if $\mathcal{R} = 0$ or if $\bigwedge^2 V$ does not contain any proper $\mathcal{R}$-invariant subspaces of the form $\bigwedge^2 W$, where $W \leq V$ is a subspace.

The following observation is obviously true: $\mathcal{R}$ is the geometric product of the algebraic curvature operators $\mathcal{R}_1, ..., \mathcal{R}_r$ if and only if the geometric realization $g_{\mathcal{R}}$ of $\mathcal{R}$ splits isometrically as the product of the geometric realizations g_i of the restrictions of the curvature operators $\mathcal{R}_i$ to the $\bigwedge^2 V_i$, V_i taken from the definition of geometric products. Thus, from a geometric viewpoint, this is precisely the definition of products of algebraic curvature operators we need. But there is another definition available, which is slightly more algebraic, a little finer and easier to handle within our context.
Recall that, for any given algebraic curvature operator $\mathcal{R}$, its holonomy algebra $\mathfrak{h}_{\mathcal{R}}$ was defined to be the smallest Lie subalgebra of $\bigwedge^2 V$ containing the image of $\mathcal{R}$ (compare subsection 3.2.4).

Definition 3.5.0.25. *An algebraic curvature operator $\mathcal{R}$ is called an algebraic product of algebraic curvature operators $\mathcal{R}_1, ..., \mathcal{R}_r$, if each $\mathfrak{h}_{\mathcal{R}_i}$ is an $\mathcal{R}$-invariant ideal in $\mathfrak{h}_{\mathcal{R}}$, $\mathcal{R}|_{\mathfrak{h}_{\mathcal{R}_i}} = \mathcal{R}_i$ and $\mathfrak{h}_{\mathcal{R}} = \mathfrak{h}_{\mathcal{R}_1} \oplus ... \oplus \mathfrak{h}_{\mathcal{R}_r}$.*
$\mathcal{R}$ is called algebraically irreducible, if $\mathfrak{h}_{\mathcal{R}}$ does not contain any proper $\mathcal{R}$-invariant ideals. Otherwise, $\mathcal{R}$ is called algebraically reducible. Those R-invariant ideals of $\mathfrak{h}_{\mathcal{R}}$, which do not contain any further proper $\mathcal{R}$-invariant ideals will be called $\mathcal{R}$-irreducible.

First of all, why do we have to distinguish the definitions from above? It is clear from the definitions that every geometric product of algebraic curvature operators is also an algebraic product. But it is in general not true that algebraic products are geometric. For example, if V is 4-dimensional, we can take $\mathcal{R} = \mathcal{W}^+ + \mathcal{W}^-$ with $\mathcal{W}^+ \in \mathrm{WEYL}^+$ and $\mathcal{W}^- \in \mathrm{WEYL}^-$ with $\mathcal{W}^+ \neq 0$ and $\mathcal{W}^- \neq 0$. Then we have that $\mathcal{R}$ is geometrically irreducible, but not irreducible in the sense of the previous definition, since we have $\mathfrak{h}_{\mathcal{W}^+} = \bigwedge^+ V$, $\mathfrak{h}_{\mathcal{W}^-} = \bigwedge^- V$, where $\bigwedge^{\pm} V$ is the ± 1-eigenspace of the Hodge $*$-operator (see theorem 1.2.0.8 for the definition of the spaces in play). As the eigenspaces of the Hodge $*$-operator form proper $\mathcal{R}$-invariant (and even irreducible) ideals in $\bigwedge^2 V = \mathfrak{h}_{\mathcal{R}}$, $\mathcal{R}$ is the algebraic product of $\mathcal{W}^+$ and $\mathcal{W}^-$ and hence, it is reducible in the algebraic sense. On the other hand this shows that $\mathcal{R}$ is geometrically irreducible, since every nontrivial $\mathcal{R}$-invariant subspace of $\bigwedge^2 V$ of the form $\bigwedge^2 W$, $W \leq V$ a subspace, must contain at least one of the spaces $\mathfrak{h}_{\mathcal{W}^+}$ or $\mathfrak{h}_{\mathcal{W}^+}$. This follows from the fact that the decomposition of $\bigwedge^2 V = \mathfrak{h}_{\mathcal{W}^+} \oplus \mathfrak{h}_{\mathcal{W}^-}$ is $\mathcal{R}$-irreducible together with the fact that geometric products of algebraic curvature operators are also products in the algebraic sense. At least, we can say that every algebraic curvature operator, which is irreducible in the algebraic sense, is also geometrically irreducible.
We note

Proposition 3.5.0.26. *Every geometric product of algebraic curvature operators is an algebraic product as well. Further, we have that every algebraic curvature operator, which is algebraically irreducible, is also geometrically irreducible.*

□

Further, we find that it is important to point out that $\mathcal{R}$-irreducibility does not imply irreducibility at all. For example, if we assume V to be four-dimensional again, we can take any algebraic curvature operator of the form $\mathcal{R} = F \wedge \mathrm{id}$ with vanishing trace and trivial kernel. Such an algebraic curvature operator interchanges the eigenspaces of the Hodge $*$-operator, so the only possible ideals of $\mathfrak{h}_{\mathcal{R}} = \bigwedge^2 V$ are definitely not $\mathcal{R}$-invariant. Another example is given by $\mathcal{R} = \mathrm{id}$ in the four-dimensional case. Here, we have two $\mathcal{R}$-invariant ideals of $\bigwedge^2 V$: the eigenspaces of the Hodge $*$-operator. But the restrictions of $\mathcal{R}$ to each of these spaces do not give algebraic curvature operators (compare lemma 3.2.2.5).

Examples 3.5.0.27. *Now we give some examples of irreducible algebraic curvature operators $\mathcal{R}$ with the property that $\mathfrak{h}_{\mathcal{R}}$ does not contain any proper ideals:*

1. *Irreducible symmetric algebraic curvature operators.*

2. Irreducible algebraic Weyl curvature operators.

Beweis. 1. This follows easily using corollary 3.4.0.17.

2. Recall that if $\mathcal{R}$ is any algebraic curvature operator, which is diagonal in the orthonormal basis $\{\varepsilon_i\}$, $\mathcal{R}\varepsilon_i = \lambda_i\varepsilon_i$, $\lambda_i \in \mathbb{R}$, then

$$\mathrm{id}\#\mathcal{R} = -\frac{1}{2}\sum_{\mathrm{i}} \lambda_{\mathrm{i}} \mathrm{ad}^2_{\varepsilon_{\mathrm{i}}},$$

(see remark 3.2.5.2) so $\mathrm{id}\#\mathcal{R}$ preserves the ideals of $\mathfrak{h}_{\mathcal{R}}$.
On the other hand we know that the Weyl curvature operators are eigenvectors of $\mathcal{S} \mapsto \mathrm{id}\#\mathcal{S}$ with eigenvalue -1.
This implies, that for Weyl curvature operators $\mathcal{W}$, any ideal of $\mathfrak{h}_{\mathcal{W}}$ is $\mathcal{W}$-invariant, so $\mathfrak{h}_{\mathcal{W}}$ cannot contain any proper ideal, if $\mathcal{W}$ is assumed to be irreducible.

□

3.5.1 Geometric Realization of Algebraic Product Curvature Operators

It is clear that every algebraic curvature operator splits as an algebraic product of irreducible algebraic curvature operators: Let $\mathcal{R}$ be an algebraic curvature operator. Then, first of all, $\mathfrak{h}_{\mathcal{R}}$ is an ideal of itself. So if there are no proper $\mathcal{R}$-invariant ideals, we are done. But if $I \subseteq \mathfrak{h}_{\mathcal{R}}$ is a proper $\mathcal{R}$-invariant ideal in $\mathfrak{h}_{\mathcal{R}}$, so is $I^{\perp}$, since $\mathcal{R}$ is self-adjoint and $\mathrm{ad}_{\varepsilon}$ is skew-adjoint for any $\varepsilon \in \bigwedge^2 V$. Repeating this procedure with $I = \mathfrak{h}_{\mathcal{R}|_I}$ and $I^{\perp} = \mathfrak{h}_{\mathcal{R}|_{I^{\perp}}}$ instead of $\mathfrak{h}_{\mathcal{R}}$ again and again will finally give us the desired irreducible decomposition after finitely many steps. This decomposition is unique. We note

Theorem 3.5.1.1. *Every algebraic curvature operator owns a unique decomposition as an algebraic product of irreducible algebraic curvature operators.*

□

If we consider geometric decompositions of algebraic curvature operators, we get a stronger result.

Theorem 3.5.1.2. *If $\mathcal{R}$ is an algebraic curvature operator, then $\mathcal{R}$ owns a unique geometrically irreducible decomposition $\mathcal{R} = \mathcal{R}_1 + ... + \mathcal{R}_r$ and the geometric realization $(U, g_{\mathcal{R}})$ of $\mathcal{R}$ splits isometrically as*

$$(U, g_{\mathcal{R}}) = (\mathbb{R}^m, g_{\mathbb{R}^m}) \times (U_1, g_{\mathcal{R}_1}) \times ... \times (U_r, g_{\mathcal{R}_r})$$

($g_{\mathbb{R}^m}$ the standard metric on $\mathbb{R}^m$, $m = n - \sum_{i=1}^r d_i$, $d_i = \dim \mathrm{T_0U_i}$), such that each $\mathcal{R}_i$ is the curvature operator of $g_{\mathcal{R}_i}$ at 0 in V and $\mathfrak{h}_{\mathcal{R}_i} \subseteq \bigwedge^2 V_{\mathcal{R}_i}$, $V_{\mathcal{R}_i} = \mathrm{T_oU_i}$, for all $i = 1, ..., r$.

Beweis. Given an algebraic curvature operator $\mathcal{R}$, we consider its geometrical realization $g = g_{\mathcal{R}}$ on a (possibly small) simply connected neighborhood U of $0 \in V$. The tangent bundle TU splits into holonomy-irreducible subbundles

$$\mathrm{TU} = \mathrm{E_0} \oplus ... \oplus \mathrm{E_r},$$

where $r \in \mathbb{N}$ and $\sum_{i=0}^{r} \mathrm{rank(E_i)} = \mathrm{n}$ and $\mathrm{E_0|_0} \subseteq \mathrm{V}$ is the subspace, on which the holonomy group $\mathrm{Hol(U, g)}$ acts trivially. The De Rham decomposition theorem (compare appendix B.5) implies that U splits isometrically as a product

$$(U, g) = (\mathbb{R}^m, g_{\mathbb{R}^m}) \times (U_1, g_1) \times ... \times (U_r, g_r),$$

($m = \mathrm{rank(E_0)}$ and g_{R^m} the standard metric on $\mathbb{R}^m$,) which implies that the curvature operator $\mathcal{R}^g$ of g decomposes orthogonally as the direct sum of the curvature operators of the involved factors. Let $V_i := \mathrm{T_0U_i} \subseteq \mathrm{V}$ and $\mathcal{R}_i : \bigwedge^2 V_i \to \bigwedge^2 V_i$ be the curvature operator of g_i at 0. It is clear that $\mathcal{R}|_{\bigwedge^2 V_i} = \mathcal{R}_i$ for all $i = 1, ..., r$ and $\mathcal{R}|_{\bigwedge^2 \mathbb{R}^m} = 0$. Thus, we have that $\mathcal{R}$ is the direct orthogonal sum of the $\mathcal{R}_i$ and $\mathfrak{h}_{\mathcal{R}_i} = \mathfrak{h}_{\mathcal{R}} \cap \bigwedge^2 V_i$ for all $i = 1, ..., r$. Obviously, each $\mathfrak{h}_{\mathcal{R}_i}$ is $\mathcal{R}$-invariant and even an ideal in $\mathfrak{h}_{\mathcal{R}}$, finally showing that $\mathcal{R}$ is a product of the $\mathcal{R}_i$ in the sense of definition 3.5.0.24. Now we are left to show that this splitting is geometrically irreducible. But this follows from the fact, that $\mathrm{Hol(U, g)}$, the holonomy group of the geometric realization of $\mathcal{R}$, is acting irreducibly on each V_i by construction. □

Theorem 3.5.1.2 implies that for any algebraic curvature operator $\mathcal{R}$ on $\bigwedge^2 V$, there exists a uniquely defined minimal subspace $V_{\mathcal{R}}$ of V, such that $\mathfrak{h}_{\mathcal{R}} \subseteq \bigwedge^2 V_{\mathcal{R}}$. As we may view any algebraic curvature operator $\mathcal{R}$ on $\bigwedge^2 V$ as an algebraic curvature operator on $\bigwedge^2 V_{\mathcal{R}}$, we can restrict our further considerations to algebraic curvature operators with $V_{\mathcal{R}} = V$. Algebraic curvature operators, sharing this property, will be called reduced.
Moreover, since any algebraic curvature operator turned out to be a product of irreducible algebraic curvature operators, we may restrict our considerations to irreducible algebraic curvature operators. Note that if $\mathcal{R}$ is reduced and irreducible it follows that the geometric realization of $\mathcal{R}$ is holonomy-irreducible.

3.5.2 The Holonomy Algebra and the Ricci Vector Field

There is another property of the Lie algebra $\mathfrak{h}_{\mathcal{R}}$ of an algebraic curvature operator, which is fundamental in our further considerations: It is $\mathcal{R}\#\mathcal{R}$-invariant and therefore invariant under $\Phi(\mathcal{R})$ as well. But one can say more: $\mathcal{R}\#\mathcal{R}$ and $\Phi(\mathcal{R})$ even preserve the $\mathcal{R}$-invariant ideals of $\mathfrak{h}_{\mathcal{R}}$, so its irreducible $\mathcal{R}$-invariant splitting will be preserved by these maps either.
To see this, evolve $\mathcal{R}$ in an orthonormal eigenbasis $\{\varepsilon_i\}$, i.e. $\mathcal{R} = \sum_i \lambda_i \varepsilon_i^* \otimes \varepsilon_i$

with suitable $\lambda_i \in \mathbb{R}$, where we choose the eigenbasis in a way such that each ε_i is contained in some $\mathcal{R}$-irreducible ideal. Then, by corollary 2.2.0.20, $\mathcal{R}\#\mathcal{R}$ reads

$$\mathcal{R}\#\mathcal{R} = \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j \left[\varepsilon_i^*, \varepsilon_j^*\right] \otimes [\varepsilon_i, \varepsilon_j] \,.$$

so the image of $\mathcal{R}\#\mathcal{R}$ clearly lies in $\mathfrak{h}_{\mathcal{R}}$ and the $\mathcal{R}$-invariant ideals are obviously preserved. Note that this gives us $\mathfrak{h}_{\Phi(\mathcal{R})} \subseteq \mathfrak{h}_{\mathcal{R}}$ as a by-product. Furthermore, the geometric splitting of $\mathcal{R}$ is also preserved by $\Phi(\mathcal{R})$. We note

Theorem 3.5.2.1. *Let $\mathcal{R}$ be an algebraic curvature operator. Then*

$$\mathfrak{h}_{\Phi(\mathcal{R})} \subseteq \mathfrak{h}_{\mathcal{R}}.$$

Further, any $\mathcal{R}$-invariant ideal $\mathfrak{I}$ in $\mathfrak{h}_{\mathcal{R}}$ is also $\Phi(\mathcal{R})$-invariant. Consequently, $\Phi(\mathcal{R})$ respects both, the $\mathcal{R}$-irreducible algebraic splitting of $\mathfrak{h}_{\mathcal{R}}$ and the $\mathcal{R}$-irreducible geometric splitting of V.

Beweis. There is nothing left to be done. □

.

Kapitel 4

On the Dynamic of the Ricci Vector Field

In this final chapter we study the flow of the Ricci vector field Φ on the space of algebraic curvature operators. Recall that Φ was defined by

$$\Phi(\mathcal{R}) = 2(\mathcal{R}^2 + \mathcal{R}\#\mathcal{R})$$

for $\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$.

Φ decomposes orthogonally as a sum of a spherical part $\overline{\Phi}$, which is tangent to a central sphere in $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ and a radial part ν, which is given by

$$\nu(\mathcal{R}) = \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}\mathcal{R},$$

where $\tau(\mathcal{R}) = \langle \Phi(\mathcal{R}), \mathcal{R}\rangle$. One of the highlights of this chapter is theorem 4.4.0.17, which states that the equilibrium positions of $\overline{\Phi}$ split algebraically (and also geometrically) as sums of irreducible equilibrium positions of $\overline{\Phi}$ and relates the radial parts of the summands to the radial part of their sum. Another highlight is given by theorem **??**, which shows that if the Φ-trajectory $t \mapsto \mathcal{R}(t)$ through an algebraic curvature operator $\mathcal{R}$ does not converge to zero, then $\mathcal{R}$ must either be an equilibrium of Φ or $\lim_{t\to T_+} \tau\left(\frac{\mathcal{R}(t)}{\|\mathcal{R}(t)\|}\right) > 0$, T_+ the maximal lifetime of $\mathcal{R}(t)$.

4.1 Invariant Subspaces

We start with a simple but fundamental observation.

Lemma 4.1.0.2. *$\mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$ and $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ are preserved by the flow of Φ.*

Beweis. If $\mathcal{R}$ is self-adjoint, so is $\mathcal{R}^2$ and $\mathcal{R}\#\mathcal{R}$ (compare lemma 2.1.0.14). Thus, Φ is tangent to $\mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$. And theorem 3.3.1.1 implies that Φ is tangent to $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$. The first claim follows.

□

Recall that the holonomy algebra $\mathfrak{h}_\mathcal{R}$ of an algebraic curvature operator $\mathcal{R}$ on $\bigwedge^2 V$ was defined to be the smallest Lie subalgebra of $\bigwedge^2 V$ containing the image of $\mathcal{R}$. This has been done in subsection 3.2.4. It is clear that one can associate a holonomy algebra $\mathfrak{h}_\mathcal{S}$ to any endomorphism $\mathcal{S}$ of $\bigwedge^2 V$. The theorem below states that the flow of the Ricci vector field Φ respects holonomy algebras and their irreducible decompositions.

Theorem 4.1.0.3. *Let $\mathcal{R} \in \mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$, then $\mathfrak{h}_\mathcal{R}$ itself and any of its $\mathcal{R}$-invariant ideals are preserved by the flow of Φ. More precisely, if $t \mapsto \mathcal{R}(t)$, $t \in I := (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, is a solution curve of Φ with $\mathcal{R}(0) = \mathcal{R}$, then $\mathfrak{h}_{\mathcal{R}(t)} = \mathfrak{h}_\mathcal{R}$ for all $t \in I$ and any $\mathcal{R}$-invariant ideal of $\mathfrak{h}_\mathcal{R}$ is also $\mathcal{R}(t)$-invariant and vice versa.*

Beweis. Let $s \in I$ and define

$$M_\mathcal{R}^s := \left\{\mathcal{S} \in \mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right) : \mathfrak{h}_\mathcal{S} \subseteq \mathfrak{h}_{\mathcal{R}(\mathrm{s})}\right\}.$$

Since $\mathrm{im}(\mathcal{S}+\mathcal{T}) \subseteq \mathrm{im}\mathcal{S}+\mathrm{im}\mathcal{T} \subseteq \mathfrak{h}_\mathcal{S}+\mathfrak{h}_\mathcal{T} \subseteq \mathfrak{h}_{\mathcal{R}(\mathrm{s})}$ holds for all $\mathcal{S}, \mathcal{T} \in M_\mathcal{R}^s$, we have that $M_\mathcal{R}^s$ is a subspace of $\mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$. Section 3.5.2 gives $\mathfrak{h}_{\Phi(\mathcal{S})} \subseteq \mathfrak{h}_\mathcal{S}$ for every $\mathcal{S} \in \mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$, which implies $\Phi(\mathcal{S}) \in M_\mathcal{R}^s$, whenever $\mathcal{S} \in M_\mathcal{R}^s$, so Φ is tangent to $M_\mathcal{R}^s$. This gives $\mathfrak{h}_{\mathcal{R}(t)} \subseteq \mathfrak{h}_{\mathcal{R}(s)}$, whenever $t \geq s$. The same arguments show that $-\Phi$ is tangent to $M_\mathcal{R}^s$, which gives $\mathfrak{h}_{\mathcal{R}(s)} \subseteq \mathfrak{h}_{\mathcal{R}(t)}$, whenever $t \leq s$.
Now let $\mathfrak{I}$ be any $\mathcal{R}$-invariant ideal of $\mathfrak{h}_\mathcal{R}$ and define

$$N_\mathfrak{I} := \{\mathcal{S} \in M_\mathcal{R} : \mathcal{S}(\mathfrak{I}) \subseteq \mathfrak{I}\},$$

$M_\mathcal{R} = M_\mathcal{R}^0$. Now, $N_\mathfrak{I}$ is a subspace of $M_\mathcal{R}$. Following section 3.5.2, we see that $\Phi(\mathcal{S})$ preserves $\mathfrak{I}$ if $\mathcal{S}$ is an element of $N_\mathfrak{I}$, showing that Φ and $-\Phi$ are tangent to $N_\mathfrak{I}$. The claim follows. □

Corollary 4.1.0.4. *Let $U \leq V$ be a subspace. Then $\mathrm{LC}\left(\bigwedge^2 \mathrm{U}\right) \subseteq \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ is preserved by the flow of Φ.*

Beweis. Let $\mathcal{R}$ be an algebraic curvature operator on $\bigwedge^2 V$ which reduces to a curvature operator on $\bigwedge^2 U$, i.e. $\mathcal{R}$ satisfies

$$\mathcal{R} = \iota \circ \mathcal{R} \circ \pi,$$

where $\iota : \bigwedge^2 U \to \bigwedge^2 V$ is the canonical inclusion and $\pi : \bigwedge^2 V \to \bigwedge^2 V$ is the orthogonal projection onto $\bigwedge^2 U$. Theorem 4.1.0.3 implies that the flow of Φ preserves the holonomy algebra $\mathfrak{h}_{\mathcal{R}}$. But $\mathfrak{h}_{\mathcal{R}} \subseteq \bigwedge^2 U$. The claim follows. □

We have already seen that the flow of Φ preserves the space of algebraic curvature operators (compare theorem 4.1.0.3). The following theorem provides some examples of Φ-invariant subspaces of $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$.

Theorem 4.1.0.5. 1. *The flow of* Φ *preserves* SCAL *and* WEYL. *If the underlying dimension is 4, then* Φ *even preserves* WEYL^+ *and* WEYL^-.

2. *The flow of* Φ *preserves the set of algebraic symmetric curvature operators.*

3. *Let* $G \in \mathrm{O(V)}$ *and define*

$$\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)_{\mathrm{G}} := \left\{\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) : \mathrm{G} \wedge \mathrm{G} \circ \mathcal{R} \circ (\mathrm{G} \wedge \mathrm{G})^{-1} = \mathcal{R}\right\}.$$

The flow of Φ *preserves* $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)_{\mathrm{G}}$.

Beweis. 1. Corollary 2.1.0.17 states that $\mathrm{id}\#\mathrm{id} = (n-2)\mathrm{id}$, so

$$\Phi(\lambda \mathrm{id}) = 2(\mathrm{n}-1)\lambda^2 \mathrm{id}$$

for every $\lambda \in \mathbb{R}$, which implies that the flow of Φ preserves SCAL.
By corollary 3.3.1.5, Φ is tangent to WEYL, so WEYL is preserved by the flow of Φ.
Now let $\dim V = 4$, $\mathcal{R} \in \mathrm{WEYL}^+$ and $t \mapsto \mathcal{R}(t), t \in (T^-, T^+)$ the maximal solution of Φ with $\mathcal{R}(0) = \mathcal{R}$. Then $\mathrm{im}(\mathcal{R}) \subseteq \bigwedge^+ \mathrm{V}$, where $\bigwedge^+ V$ is the +1-eigenspace of the Hodge $*$-operator on V. $\bigwedge^+ V$ is an ideal in $\bigwedge^2 V$ (compare theorem 1.2.0.8 and corollary 3.2.2.3). This gives $\mathfrak{h}_{\mathcal{R}} \subseteq \bigwedge^+ V$. Theorem 4.1.0.3 says that the holonomy algebra does not change along the flow of Φ. Hence, we have $\mathrm{im}(\mathcal{R}(\mathrm{t})) \subseteq \bigwedge^+ \mathrm{V}$ for all $t \in (T^-, T^+)$, which implies $\mathcal{R}(t) \in \mathrm{WEYL}^+$ by corollary 3.2.2.3. This shows that WEYL^+ is Φ-invariant. The proof that WEYL^- is Φ-invariant is almost the same.

2. Let $\mathcal{R}$ be an algebraic symmetric curvature operator. Write

$$\mathcal{R} = \sum_i \lambda_i \varepsilon_i^* \otimes \varepsilon_i,$$

where $\lambda_1, ..., \lambda_N$, $N = \binom{n}{2}$, denote the eigenvalues of $\mathcal{R}$ and $\{\varepsilon_i\}$ is an eigenbasis. Then, by remark 3.4.0.19, we have

$$\Phi(\mathcal{R}) = 2\sum_i \left(1 + \|\mathrm{ad}_{\varepsilon_\mathrm{i}}\|^2\right) \lambda_i^2 \varepsilon_i^* \otimes \varepsilon_i.$$

Now we observe that every algebraic curvature operator $\mathcal{S}$ which is diagonal w.r.t. the chosen orthonormal basis and satisfies $\mathrm{im}(\mathcal{S}) \subseteq \mathrm{im}(\mathcal{R})$, is automatically symmetric. So the idea is now to evolve the eigenvalues of $\mathcal{R}$ appropriately and keep the eigenbasis fixed: For each i solve

$$\frac{d}{dt}\lambda_i(t) = 2\left(1 + \|\mathrm{ad}_{\varepsilon_\mathrm{i}}\|^2\right)\lambda_i^2(t),$$

with $\lambda_i(0) = \lambda_i$ and let

$$\mathcal{R}(t) := \sum_i \lambda_i(t)\varepsilon_i^* \otimes \varepsilon_i$$

on the maximal interval $[0, T)$, where each $\lambda_i(t)$ is defined.

Using remark 3.4.0.19 and arguing as in the computation in theorem 3.4.0.18, we see that

$$\frac{d}{dt}\mathcal{R}(t) = \Phi(\mathcal{R}(t))$$

for each t. As we have $\mathcal{R}(0) = \mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$, we get that the whole solution curve lies in the space of algebraic curvature operators.

3. According to lemma 2.1.0.14 # is an O(V)-equivariant operation. Hence, Φ is equivariant either. This implies that Φ is tangent to $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)_\mathrm{G}$. □

Remark 4.1.0.6. *The flow of* Φ *does not preserve* RIC_0. *Every trajectory starting in* RIC_0 *must leave* RIC_0 *instantaneously unless it is constantly zero.*

Beweis. If the maximal solution curve $t \mapsto \mathcal{R}(t)$, $t \in (T^-, T^+)$, through $\mathcal{R} \in \mathrm{RIC}_0$ stays in RIC_0 for $t \in [0, \varepsilon]$, $\varepsilon > 0$, then, using tneorem 3.3.1.3, we get

$$0 = \frac{d}{dt}\mathrm{tr}(\mathcal{R}(\mathrm{t})) = \mathrm{tr}(\Phi(\mathcal{R}(\mathrm{t}))) = \|\mathrm{Ric}(\mathcal{R}(\mathrm{t}))\|^2$$

for $t \in [0, \varepsilon)$, which implies $\mathcal{R} = 0$ and we are done. □

Theorem 4.1.0.7. *The flow of* Φ *preserves stabilizers. More precisely we have*

$$\mathrm{Stab}_{\mathrm{O(V)}}(\mathcal{R}) \subset \mathrm{Stab}_{\mathrm{O(V)}}(\Phi(\mathcal{R}))$$

w.r.t. the canonical action of O(V) *on* $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$.

Beweis. This is an immediate consequence of the fourth item in theorem 4.1.0.5. □

4.2 The Curvature Normalized Flow

The Ricci vector field Φ is homogeneous of order 2, which means that for all $\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ and every $\lambda > 0$ holds

$$\Phi(\lambda\mathcal{R}) = \lambda^2\Phi(\mathcal{R}).$$

This implies that the flow of Φ behaves nicely under rescaling of space and time.

Lemma 4.2.0.8. *Let $F : \mathcal{D}^\Phi \to \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ be the flow of Φ, where $\mathcal{D}^\Phi = \{(\mathcal{R}, t) : t \in (T_-(\mathcal{R}), T_+(\mathcal{R}))\} \subseteq \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) \times \mathbb{R}$ is the maximal domain of definition of the flow of Φ, i.e. for each $\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ the curve*
$F_\mathcal{R} : (T_-(\mathcal{R}), T_+(\mathcal{R})) \to \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$,

$$F_\mathcal{R}(t) := F(\mathcal{R}, t)$$

is the maximal trajectory of Φ through $\mathcal{R}$. Then we have

$$F(\lambda\mathcal{R}, t) = \lambda F(\mathcal{R}, \lambda t),$$

whenever $(\lambda\mathcal{R}, t) \in \mathcal{D}^\Phi$, $\lambda > 0$. Moreover, we have $T_\pm(\lambda\mathcal{R}) = \frac{1}{\lambda}T_\pm(\mathcal{R})$.

Beweis. F is the flow of Φ, so

$$\frac{d}{dt}F(\lambda\mathcal{R}, t) = \Phi(F(\lambda\mathcal{R}, t)).$$

A simply computation shows

$$\frac{d}{dt}\lambda F(\mathcal{R}, \lambda t) = \lambda^2\Phi(F(\mathcal{R}, \lambda t) = \Phi(\lambda F(\mathcal{R}, \lambda t),$$

so both sides of the equation fulfill the same differential equation, also with the same initial condition $\lambda\mathcal{R}$. Now the desired formula

$$F(\lambda\mathcal{R}, t) = \lambda F(\mathcal{R}, \lambda t)$$

follows from the uniqueness of solutions of ordinary differential equations. The rest is clear. □

Lemma 4.2.0.8 implies that for each $\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$, $\mathcal{R} \neq 0$, and each $\lambda > 0$ the sets

$$\left\{\frac{F(\mathcal{R}, t)}{\|F(\mathcal{R}, t)\|} : t \in (T_-(\mathcal{R}), T_+(\mathcal{R}))\right\}$$

and

$$\left\{\frac{F(\lambda\mathcal{R}, t)}{\|F(\lambda\mathcal{R}, t)\|} : t \in (T_-(\lambda\mathcal{R}), T_+(\lambda\mathcal{R}))\right\}$$

agree. Φ induces a vector field $\overline{\Phi}$ on $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ which is tangent to the central spheres $\mathbb{S}_r = \left\{\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right) : \|\mathrm{R}\| = \mathrm{r}\right\}$, $r > 0$,

$$\overline{\Phi}(\mathcal{R}) = \Phi(\mathcal{R}) - \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}\mathcal{R}, \text{ with } \tau(\mathcal{R}) = \langle \Phi(\mathcal{R}), \mathcal{R} \rangle .$$

We call $\overline{\Phi}$ the spherical Ricci vector field on the space of algebraic curvature operators. Let $\overline{F}$ be the flow of $\overline{\Phi}$. $\overline{F}$ preserves the norm of the curvature operators. In the following we will examine the flow of $\overline{\Phi}$ to $\mathbb{S} = \mathbb{S}_1$ as well as the flow of Φ. This will help us to deal with the solution curves tending to infinity. We will now use the formula in lemma 4.2.0.8 to construct solution curves of $\overline{\Phi}$ from solution curves of Φ and vice versa.

Proposition 4.2.0.9. *Let $\mathcal{R}$ be a curvature operator with $\|\mathcal{R}\| = 1$ and T the lifetime of $\mathcal{R}$. If φ solves the ODE $\dot{\varphi}(t) = \frac{1}{\|F(\mathcal{R},\varphi(t))\|}$ with initial data $\varphi(0) = 0$, then $\frac{F(\mathcal{R},\varphi(t))}{\|F(\mathcal{R},\varphi(t))\|}$ is a solution curve of $\overline{\Phi}$ starting at $\mathcal{R}$.*

Beweis. We have

$$\frac{d}{dt}F(\mathcal{R}, \varphi(t)) = \frac{1}{\|F(\mathcal{R}, \varphi(t))\|}\Phi(F(\mathcal{R}, \varphi(t)))$$

and

$$\frac{d}{dt}\|F(\mathcal{R}, \varphi(t))\|^2 = \frac{2\langle \Phi(F(\mathcal{R}, \varphi(t))), F(\mathcal{R}, \varphi(t))\rangle}{\|F(\mathcal{R}, \varphi(t))\|} = \frac{2\tau(F(\mathcal{R}, \varphi(t)))}{\|F(\mathcal{R}, \varphi(t))\|}$$

which gives

$$\begin{aligned}\frac{d}{dt}\frac{F(\mathcal{R}, \varphi(t))}{\|F(\mathcal{R}, \varphi(t))\|} &= \Phi\left(\frac{F(\mathcal{R}, \varphi(t))}{\|F(\mathcal{R}, \varphi(t))\|}\right) - \frac{\tau(F(\mathcal{R}, \varphi(t)))}{\|F(\mathcal{R}, \varphi(t))\|^4}F(\mathcal{R}, \varphi(t)) \\ &= \overline{\Phi}\left(\frac{F(\mathcal{R}, \varphi(t))}{\|F(\mathcal{R}, \varphi(t))\|}\right)\end{aligned}$$

Finally, $\varphi(0) = 0$ gives the result. □

Proposition 4.2.0.10. *Let $\mathcal{R}$ be a curvature operator with $\|\mathcal{R}\| = 1$. If φ and ψ solve the ODEs $\dot{\varphi} = -\varphi^2\tau(\overline{F}(\mathcal{R}, \psi))$ and $\dot{\psi} = \varphi$ with $\varphi(0) = 1$ and $\psi(0) = 0$, then $\varphi\overline{F}(\mathcal{R}, \psi)$ is a solution of Φ starting at $\mathcal{R}$.*

Beweis. Just compute

$$\begin{aligned}\frac{d}{dt}\varphi\overline{F}(\mathcal{R}, \psi) &= -\varphi^2\tau(\overline{F}(\mathcal{R}, \psi))\overline{F}(\mathcal{R}, \psi) + \varphi^2\overline{\Phi}(\overline{F}(\mathcal{R}, \psi)) \\ &= \Phi(\varphi\overline{F}(\mathcal{R}, \psi))\end{aligned}$$

Now, the claim follows from $\varphi(0) = 1$ and $\psi(0) = 0$. □

4.3 Evolution equations

We start with a fundamental observation, which has been made by G. Huisken in [17] and states that the flows of the vector fields Φ and $\overline{\Phi}$ are actually gradient flows associated to the cubic potential τ, which is given by

$$\tau(\mathcal{R}) = \langle \Phi(\mathcal{R}), \mathcal{R} \rangle .$$

Proposition 4.3.0.11. *We have that*

1. $\frac{d}{dt}\mathcal{R}(t) = \frac{1}{3}\nabla\tau(\mathcal{R}(t))$, *if* $t \mapsto \mathcal{R}(t)$ *is a solution curve of* Φ *and*

2. $\frac{d}{dt}\mathcal{R}(t) = \frac{1}{3}\overline{\nabla}\tau(\mathcal{R}(t))$, *if* $t \mapsto \mathcal{R}(t)$ *is a solution curve of* $\overline{\Phi}$ *on the unit sphere in* $\mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$.

Here, $\nabla\tau$ *is the gradient of* τ *and* $\overline{\nabla}\tau$ *is the gradient of* τ *on the unit sphere* $\mathbb{S} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$.

Beweis. The map $(\mathcal{R}, \mathcal{S}, \mathcal{T}) \mapsto \langle \varphi(\mathcal{R}, \mathcal{S}), \mathcal{T} \rangle$ is trilinear and fully symmetric by proposition 3.3.2.1. This gives

$$\left.\frac{d}{dt}\right|_{t=0} \tau(\mathcal{R} + t\mathcal{H}) = 3 \langle \Phi(\mathcal{R}), \mathcal{H} \rangle .$$

Now, 1. is immediate.
The gradient $\overline{\nabla}\tau$ on the unit sphere is simply the projection of $\nabla\tau$ to the tangent bundle of the sphere, which gives

$$\overline{\nabla}\tau(\mathcal{R}) = \nabla\tau - \langle \nabla\tau, \mathcal{R} \rangle$$

and the claim follows. □

Corollary 4.3.0.12. *Let* $t \mapsto \mathcal{R}(t)$, $t \in (T_-, T_+)$, *be the maximal solution curve of* Φ *starting from* $\mathcal{R}(0) = \mathcal{R}_0$. *Then*

1. $t \mapsto \tau(\mathcal{R}(t))$ *is strictly monotonically increasing, unless* $\mathcal{R}_0$ *is an equilibrium position of* Φ

2. $t \mapsto \|\mathcal{R}(t)\|^2$ *is strictly monotonically decreasing on* $\{\tau < 0\}$ *and strictly monotonically increasing on* $\{\tau > 0\}$. *Moreover, it is strictly convex, unless* $\mathcal{R}_0$ *is an equilibrium position of* Φ, ,

3. $t \mapsto \mathcal{R}(t)$ *has infinite lifetime, if* $t \mapsto \tau(\mathcal{R}(t))$ *stays nonpositive. Further,* $\tau(\mathcal{R}(t))$ *converges to 0 as* t *reaches infinity, in this case*

Beweis. 1. We have

$$\frac{d}{dt}\tau(\mathcal{R}(t)) = 3\,\|\Phi(\mathcal{R}(t))\|^2 \geq 0,$$

so $t \mapsto \mathcal{R}(t)$ is monotonically increasing. Now suppose that $t \mapsto \mathcal{R}(t)$ is constant on $I = (t_0, t_1)$. Then

$$0 = \frac{d}{dt}\tau(\mathcal{R}(t)) = 3\,\|\Phi(\mathcal{R}(t))\|^2$$

on I, which implies that $\mathcal{R}(t) = \mathcal{R}_0$ for all t.

2. We have
$$\frac{d}{dt}\,\|\mathcal{R}(t)\|^2 = 2\tau(\mathcal{R}(t)),$$
proving the first two statements, and
$$\frac{d^2}{dt^2}\,\|\mathcal{R}(t)\|^2 = 6\,\|\Phi(\mathcal{R})\|^2 \geq 0,$$
showing that $t \mapsto \|\mathcal{R}(t)\|^2$ is convex.
If it is not strictly convex, then there exists an interval $I = [t_0, t_1]$, on which it is affine linear, forcing its second derivative
$$\frac{d^2}{dt^2}\,\|\mathcal{R}(t)\|^2 = 6\,\|\Phi(\mathcal{R})\|^2$$
to be zero on I. This gives the result.

3. If $\tau(\mathcal{R}(t))$ is nonpositive during the whole flow, then the norm $\|\mathcal{R}(t)\|^2$ is nonincreasing in t. Thus, our solution doesn't leave the compact ball of radius $\|\mathcal{R}_0\|$, say, so it has infinite lifetime.
Assume that $\tau(\mathcal{R}(t))$ is bounded from above by $c < 0$. Then we get that $\|\mathcal{R}(t)\|$ becomes zero before time $t_0 = -\frac{\|\mathcal{R}_0\|^2}{c}$, which implies $\tau(\mathcal{R}(t_0) = 0)$, which is impossible.

□

Corollary 4.3.0.13. *Let $t \mapsto \mathcal{R}(t)$, $t \in \mathbb{R}$, be a trajectory of $\overline{\Phi}$. Then $t \mapsto \tau(\mathcal{R}(t))$ is strictly monotonically increasing unless $\mathcal{R}$ is an equilibrium position of $\overline{\Phi}$.*

Beweis. Clear, since we have $\overline{\Phi} = \frac{1}{3}\overline{\nabla}\tau$ by proposition 4.3.0.11. □

Understanding the asymptotic behavior of the solution curves of Φ requires understanding the evolution of the irreducible components of curvature. The first step towards this direction is to determine the evolution equation of the geometrical quantities in play. Following theorem 3.3.1.3, the evolution of scalar and Ricci curvature under Φ are obvious:

Proposition 4.3.0.14. *Let $t \mapsto \mathcal{R}(t)$, $t \in (T_-, T_+)$, be the maximal solution curve of Φ starting from $\mathcal{R}(0) = \mathcal{R}_0$.*

1. *Evolution of the irreducible components:*

   (a) $\frac{d}{dt}\mathrm{tr}(\mathcal{R}(\mathrm{t})) = \|\mathrm{Ric}(\mathcal{R}(\mathrm{t}))\|^2$,

   (b) $\frac{d}{dt}\mathrm{Ric}(\mathcal{R}(\mathrm{t})) = 2\sum_\mathrm{i} \mathcal{R}^\rho(\mathrm{t})(\cdot, \mathrm{e_i})\mathrm{Ric}(\mathcal{R}(\mathrm{t}))\mathrm{e_i}$, $\{e_i\}$ *an arbitrary orthonormal basis of* $(V, \langle \cdot\ , \cdot\rangle)$.

   (c) $\frac{d}{dt}\mathrm{Ric_0}(\mathcal{R}(\mathrm{t})) = 2\sum_\mathrm{i} \mathcal{R}^\rho(\mathrm{t})(\cdot, \mathrm{e_i})\mathrm{Ric}(\mathcal{R}(\mathrm{t}))\mathrm{e_i} - \frac{2}{\mathrm{n}}\|\mathrm{Ric}(\mathcal{R}(\mathrm{t}))\|^2 \,\mathrm{id}$

   (d) $\frac{d}{dt}\mathrm{W}(\mathcal{R}(\mathrm{t})) = \Phi(\mathrm{W}(\mathcal{R}(\mathrm{t}))) + \frac{2}{\mathrm{n}-2}\mathrm{Ric_0}(\mathcal{R}(\mathrm{t})) \wedge \mathrm{Ric_0}(\mathcal{R}(\mathrm{t})) - \frac{2}{(\mathrm{n}-1)(\mathrm{n}-2)^2}\|\mathrm{Ric_0}(\mathcal{R}(\mathrm{t}))\|^2 \,\mathrm{id} + \frac{4}{(\mathrm{n}-2)^2}\mathrm{Ric_0}(\mathcal{R}(\mathrm{t}))^2 \wedge \mathrm{id}$

2. *Evolution of the norms of the irreducible components:*

   (a) $\frac{d}{dt}\frac{1}{N}(\mathrm{tr}(\mathcal{R}(\mathrm{t})))^2 = \frac{2}{\mathrm{N}}\mathrm{tr}(\mathcal{R}(\mathrm{t}))\,\|\mathrm{Ric}(\mathcal{R}(\mathrm{t}))\|^2$

   (b) $\frac{d}{dt}\|\mathrm{W}(\mathcal{R}(\mathrm{t}))\|^2 = 2\tau(\mathrm{W}(\mathcal{R}(\mathrm{t})))$

   (c) $\frac{d}{dt}\frac{2}{n-2}\|\mathrm{Ric_0}(\mathcal{R}(\mathrm{t})) \wedge \mathrm{id}\|^2 = -\frac{4}{N^2}\mathrm{tr}(\mathcal{R}(\mathrm{t}))\frac{2}{\mathrm{n}-2}\|\mathrm{Ric_0}(\mathcal{R}(\mathrm{t})) \wedge \mathrm{id}\|^2 - \frac{8}{\mathrm{n^2N}}(\mathrm{tr}(\mathcal{R}(\mathrm{t})))^3 + 2(\tau(\mathcal{R}(\mathrm{t})) - 2\tau(\mathrm{W}(\mathcal{R}(\mathrm{t})))$.

Beweis. 1.(a) to 1.(c) and 2.(a) follow directly from theorem 3.3.1.3. 1.(d) follows using the formulas for the Weyl curvatures of $\Phi(\mathcal{R})$ and $\Phi(\frac{2}{n-2}\mathrm{Ric_0}(\mathcal{R})\wedge \mathrm{id})$ from subsection 3.3.2.3. Now we proof 2.(b):
We have

$$\begin{aligned}
\frac{d}{dt}\|\mathrm{W}(\mathcal{R}(\mathrm{t}))\|^2 &= 2\,\langle \mathrm{W}(\Phi(\mathcal{R}(\mathrm{t}))), \mathrm{W}(\mathcal{R}(\mathrm{t}))\rangle \\
&= 2\,\langle \Phi(\mathcal{R}(t)), \mathrm{W}(\mathcal{R}(\mathrm{t}))\rangle \\
&= 2\,\langle \varphi(\mathcal{R}(t), \mathrm{W}(\mathcal{R}(\mathrm{t}))), \mathrm{W}(\mathcal{R}(\mathrm{t}))\rangle \\
&= 2\,\langle \Phi(\mathrm{W}(\mathcal{R}(\mathrm{t}))), \mathrm{W}(\mathcal{R}(\mathrm{t}))\rangle \\
&= 2\tau(\mathrm{W}(\mathcal{R}(\mathrm{t})))
\end{aligned}$$

Here we used that the trilinear map $(\mathcal{R}, \mathcal{S}, \mathcal{T} \mapsto \langle\varphi(\mathcal{R}, \mathcal{S}), \mathcal{T}\rangle)$ is fully symmetric, that $\varphi(\mathrm{id}, \mathrm{W}(\mathcal{R})) = 0$ and that $\varphi(\mathrm{Ric_0}(\mathcal{R}) \wedge \mathrm{id}, \mathrm{W}(\mathcal{R}))$ lies in $\mathrm{RIC_0}$ (compare proposition 3.3.2.1 and the subsections 3.3.2.1 and 3.3.2.2).
2.(c) now follows using 2.(a) and 2.(b).

□

First of all we observe that the evolution of scalar curvature does not depend on the Weyl curvature. From the evolution of scalar curvature we get immediately that the lifetime of any solution curve of Φ is finite, provided that the scalar curvature becomes positive in finite time. To see this, first observe that $\|\mathrm{Ric}(\mathcal{R})\|^2 \geq \frac{2}{n}\mathrm{tr}\mathcal{R}^2$, which allows us to compare the scalar curvature with solutions of the explosion equation $\dot{\varphi} = \frac{2}{n}\varphi^2$ with initial data $\varphi(0) = \mathrm{tr}(\mathcal{R}(\mathrm{t_0}))$. This comparison gives $\mathrm{tr}(\mathcal{R}(\mathrm{t_0} + \mathrm{t})) \geq \varphi(\mathrm{t})$ for all

$t \geq 0$, whenever it makes sense. The solution of $\dot{\varphi} = \varphi^2$ with initial data $\varphi(0) = c$ is given by

$$\varphi(t) = \frac{nc}{n - 2ct}.$$

This tells us that the lifetime T_+ of the solution curve through $\mathcal{R}_0$ is bounded from above by $T \leq \frac{n}{c\mathrm{tr}(\mathcal{R}(0))}$, if $\mathrm{tr}(\mathcal{R}_0) > 0$ and that $\mathrm{tr}(\mathcal{R})$ becomes infinitely large before this time. Remembering corollary 4.3.0.12, we see that this forces τ to become positive during the flow.
If the lifetime is finite, then the solution leaves every compact subset of the space of algebraic curvature operators. So the norm $\|\mathcal{R}(t)\|$ tends to infinity, as t tends to T_+. We know from corollary 4.3.0.12 that $\tau(\mathcal{R}(t))$ will become positive. Actually, $\tau(\mathcal{R}(t))$ will become infinitely large, since

$$\|\mathcal{R}(t)\|^2 = \|\mathcal{R}(0)\|^2 + 2\int_0^t \tau(\mathcal{R}(s))\mathrm{ds}$$

tends to infinity as t tends to $T_+ < \infty$.

Now we treat the case, where we have infinite lifetime of the solution curve and therefore necessarily nonpositive scalar curvature.
First, if the scalar curvature becomes zero at time t_0, then we must have that the Ricci curvature is zero at this time either. Otherwise, the scalar curvature would become positive, which is impossible, since we have infinite lifetime. Hence, we have that $\mathcal{R}(t_0)$ is a Weyl curvature operator, which tells us that the whole solution curve lies in WEYL, since WEYL is invariant under the flow of Φ.
We are left considering the case where the scalar curvature remains strictly negative for all times. As we have

$$\mathrm{tr}(\mathcal{R}(\mathrm{t})) = \mathrm{tr}(\mathcal{R}(0)) + \int_0^{\mathrm{t}} \|\mathrm{Ric}(\mathcal{R}(\mathrm{s}))\|^2\,\mathrm{ds},$$

this tells us that there is a sequence (t_n) with $t_n \to \infty$ and $\mathrm{Ric}(\mathcal{R}(\mathrm{t_n})) \to 0$. But it may happen, that $\limsup_{t\to\infty} \|\mathrm{Ric}(\mathcal{R}(\mathrm{t}))\| > 0$. We note our results in the following theorem.

Theorem 4.3.0.15. *Let $t \mapsto \mathcal{R}(t)$, $t \in (T_-, T_+)$, be the maximal solution curve of Φ starting from $\mathcal{R}(0) = \mathcal{R}_0$. Then:*

1. *We have $T_+ < \infty$, if $\mathrm{tr}(\mathcal{R}(\mathrm{t}))$ becomes positive in finite time.*
2. *We have $T_+ < \infty$, if $\mathrm{tr}(\mathcal{R}(\mathrm{t_0})) = 0$ and $\mathrm{Ric_0}(\mathcal{R}(\mathrm{t_0})) \neq 0$ for some $t_0 \in (T_-, T_+)$.*
3. *If we have $T^+ = \infty$, then $\mathrm{tr}(\mathcal{R}(\mathrm{t})) \leq 0$ for all t and*

$$\liminf_{t\mapsto\infty} \|\mathrm{Ric}(\mathcal{R}(\mathrm{t}))\| = 0.$$

Moreover, if we have $\mathrm{tr}(\mathcal{R}(\mathrm{t_0})) = 0$ for some $t_0 \in (T_-, \infty)$, then we have $\mathrm{Ric}(\mathcal{R}(\mathrm{t})) = 0$ for all $t \in (T_-, \infty)$.

4.4 Equilibrium Positions

For a given $\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$, let $\Omega_\Phi(\mathcal{R})$ be the Ω-limitset of $\mathcal{R}$ w.r.t. the flow of Φ and, if $\|\mathcal{R}\| = 1$, let $\Omega_{\overline{\Phi}}(\mathcal{R})$ be the Ω-limitset of $\mathcal{R}$ w.r.t. the flow of $\overline{\Phi}$. We are interested in the asymptotic behavior of the solution curves of Φ, which reflects in the behavior of $\overline{\Phi}$-trajectories near ω-limitsets of $\overline{\Phi}$. The elements of these sets are equilibrium positions of $\overline{\Phi}$. As we have

$$\overline{\Phi}(\mathcal{R}) = \Phi(\mathcal{R}) - \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}\mathcal{R}$$

the equilibrium positions of $\overline{\Phi}$ on $\mathbb{S}$ are in 1-1 correspondence to the one-dimensional Φ-invariant subspaces of $LC(\bigwedge^2 V)$. We give shall some examples:

Examples 4.4.0.16. *1. Every range 1 algebraic curvature operator defines a Φ-invariant 1-dim subspace*

2. Spherical curvature operators:

(i) Let $\{e_i\}$ be an orthonormal basis of V and $\mathcal{R}_{ij} := e_i^* \wedge e_j^* \otimes e_i \wedge e_j$ for $i, j \in \{1, ..., n\}$, then

$$\Phi(\mathcal{R}_{ij} + \mathcal{R}_{jk} + \mathcal{R}_{ki}) = 4(\mathcal{R}_{ij} + \mathcal{R}_{jk} + \mathcal{R}_{ki}),$$

if i, j and k are mutually distinct. $\mathcal{R} = \mathcal{R}_{ij} + \mathcal{R}_{jk} + \mathcal{R}_{ki}$ is the curvature operator of $\mathbb{S}^2 \times \mathbb{R}^{n-2}$.

(ii) Let $l \leq n$ and σ a permutation of $\{1, ..., n\}$, then

$$\Phi(\sum_{1 \leq i < j \leq l} \mathcal{R}_{\sigma(i)\sigma(j)}) = 2(l-1) \sum_{1 \leq i < j \leq l} \mathcal{R}_{\sigma(i)\sigma(j)}.$$

$\mathcal{R} = \sum_{1 \leq i < j \leq l} \mathcal{R}_{\sigma(i)\sigma(j)}$ is the curvature operator of $\mathbb{S}^l \times \mathbb{R}^{n-l}$.

(iii) Let $1 \leq l_1 < k_1 < l_2 < k_2 < ... < l_m < k_m \leq n$, $\mathcal{R}^s := \sum_{l_s \leq i < j \leq k_s} \mathcal{R}_{ij}$ for $s \in \{1, ..., m\}$ and

$$\mathcal{R} := \sum_{s=1}^{m} \frac{1}{2(k_s - l_s - 1)} \mathcal{R}^s$$

then

$$\Phi(\mathcal{R}) = \mathcal{R}.$$

$\mathcal{R}$ is the curvature operator of $\mathbb{S}_{r_1}^{k_1-l_1} \times \mathbb{S}_{r_2}^{k_2-l_2} \times \mathbb{S}_{r_m}^{k_m-l_m} \times \mathbb{R}^M$, where $r_s = \frac{1}{\sqrt{2(k_s-l_s)}}$ and $M = n - \sum_{s=1}^{m} k_s - l_s$

3. *Hyperbolic curvature operators:*
   Replacing $\mathcal{R}$ by $-\mathcal{R}$ in $(i), (ii)$ and (iii) from above we get the hyperbolic analogues and

   (i) $\Phi(-\mathcal{R}) = 4\mathcal{R}$ *in case* (i)

   (ii) $\Phi(-\mathcal{R}) = \mathcal{R}$ *in the cases* (ii) *and* (iii).

Beweis. 1. If $\mathcal{R}$ has range 1, then $\mathcal{R} = \pm\varepsilon^* \wedge \varepsilon$ for some $\varepsilon \in \bigwedge^2 V$. Then

$$\mathcal{R}^2 = \pm \|\mathcal{R}\|^2 \mathcal{R}$$

and

$$\mathcal{R}\#\mathcal{R} = 0.$$

So

$$\Phi(\mathcal{R}) = \pm 2 \left\|\mathcal{R}^2\right\| \mathcal{R}.$$

2. (a) $\mathcal{R}_{ij}$ is a range 1 algebraic curvature operator, so $\mathcal{R}_{ij}\#\mathcal{R}_{ij} = 0$. This implies

$$(\mathcal{R}_{ij} + \mathcal{R}_{jk})\#(\mathcal{R}_{ij} + \mathcal{R}_{jk}) = 2\mathcal{R}_{ij}\#\mathcal{R}_{jk} = \mathcal{R}_{ik},$$

so

$$\begin{aligned} \mathcal{R}\#\mathcal{R} &= (\mathcal{R}_{ij} + \mathcal{R}_{jk} + \mathcal{R}_{ki})\#(\mathcal{R}_{ij} + \mathcal{R}_{jk} + \mathcal{R}_{ki}) \\ &= 2(\mathcal{R}_{ij}\#\mathcal{R}_{jk} + \mathcal{R}_{jk}\#\mathcal{R}_{ki} + \mathcal{R}_{ki}\#\mathcal{R}_{ij}) \\ &= \mathcal{R}_{ij} + \mathcal{R}_{jk}\mathcal{R}_{ki} \\ &= \mathcal{R} \end{aligned}$$

Since $\mathcal{R}_{ij}\mathcal{R}_{kl} = 0$, whenever $\{i,j\} \neq \{k,l\}$ and $\mathcal{R}_{ij}^2 = \mathcal{R}_{ij}$, we get

$$\mathcal{R}^2 = (\mathcal{R}_{ij}+\mathcal{R}_{jk}+\mathcal{R}_{ki})^2 = \mathcal{R}_{ij}^2+\mathcal{R}_{jk}^2+\mathcal{R}_{ki}^2 = \mathcal{R}_{ij}+\mathcal{R}_{jk}+\mathcal{R}_{ki} = \mathcal{R}.$$

This shows

$$\Phi(\mathcal{R}) = 4\mathcal{R}.$$

(b) W.l.o.g. $\sigma = id$. Then by the same arguments as above:

$$\mathcal{R}^2 = \mathcal{R}.$$

Now we treat the term $\mathcal{R}\#\mathcal{R}$:

$$\begin{aligned}\mathcal{R}\#\mathcal{R} &= \frac{1}{4}\sum_{1\le i,j,r,s\le l}\mathcal{R}_{ij}\#\mathcal{R}_{rs}\\ &= \frac{1}{2}\sum_{1\le i,j,r\le l}\mathcal{R}_{ij}\#\mathcal{R}_{jr}\\ &= \frac{1}{2}\sum_{1\le i,r\le l}\sum_{1\le j\le l, j\neq i,r}\mathcal{R}_{ir}\\ &= \frac{l-2}{2}\sum_{1\le i,r\le l}\mathcal{R}_{ir}\\ &= (l-2)\mathcal{R}\end{aligned}$$

This gives the result.

(c) For all $s = 1, ..., m$ holds

$$\Phi\left(\frac{1}{2(k_s-l_s)}\mathcal{R}^s\right) = \frac{1}{2(k_s-l_s)}\mathcal{R}^s$$

And if $r \neq s$, then

$$\mathcal{R}^r\mathcal{R}^s = \mathcal{R}^s\mathcal{R}^r = 0 \text{ and } \mathcal{R}^r\#\mathcal{R}^s = 0.$$

Thus,

$$\begin{aligned}\Phi(\mathcal{R}) &= \Phi\left(\sum_{s=1}^{m}\frac{1}{2(k_s-l_s-1)}\mathcal{R}^s\right)\\ &= \sum_{s=1}^{m}\Phi\left(\frac{1}{2(k_s-l_s-1)}\mathcal{R}^s\right)\\ &= \sum_{s=1}^{m}\frac{1}{2(k_s-l_s-1)}\mathcal{R}^s\\ &= \mathcal{R}\end{aligned}$$

□

The examples from above show that there exist many equilibrium positions and that we can construct new equilibriums building geometric products of certain equilibrium positions. But it also shows that we are not allowed to mix hyperbolic and spherical curvature operators. The following theorem generalizes our thoughts to arbitrary equilibrium positions and builds the core of this section.

Theorem 4.4.0.17. *Let $\mathcal{R} \neq 0$ be an algebraic curvature operator on V with*

$$\Phi(\mathcal{R}) = \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}\mathcal{R},$$

in other words, $\frac{\mathcal{R}}{\|\mathcal{R}\|}$ is an equilibrium of $\overline{\Phi}$. Suppose that $\mathcal{R}$ splits as an algebraic product of the algebraic curvature operators $\mathcal{R}_i \neq 0$, $i = 1, ..., r$, $r \in \mathbb{N}$. Then each of the $\mathcal{R}_i$ satisfies

$$\Phi(\mathcal{R}_i) = \frac{\tau(\mathcal{R}_i)}{\|\mathcal{R}_i\|^2}\mathcal{R}_i$$

and we have the relations

$$\frac{\tau(\mathcal{R}_i)}{\|\mathcal{R}_i\|^2} = \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}$$

for each i.

Beweis. On the one hand we have

$$\Phi(\mathcal{R}) = \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}\mathcal{R} = \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}\sum_i \mathcal{R}_i$$

and on the other hand Φ preserves products, thus we get

$$\Phi(\mathcal{R}) = \sum_i \Phi(\mathcal{R}_i)$$

as an algebraic product of algebraic curvature operators and

$$\mathfrak{h}_{\Phi(\mathcal{R}_i)} \subseteq \mathfrak{h}_{\mathcal{R}_i}.$$

Hence, we get

$$\Phi(\mathcal{R}_i) = \frac{\tau(\mathcal{R}_i)}{\|\mathcal{R}_i\|^2}\mathcal{R}_i$$

for all i and also

$$\frac{\tau(\mathcal{R}_i)}{\|\mathcal{R}_i\|^2} = \frac{\tau(\mathcal{R})}{\|\mathcal{R}\|^2}$$

as claimed.

□

Remark 4.4.0.18. Obviously, theorem 4.4.0.17 is also true for geometric splittings.

Example 4.4.0.19. Let $U \subseteq V$ be a proper subspace and consider the algebraic (and geometric) product curvature operator

$$\mathcal{R} = \frac{1}{\|\mathcal{S}+\mathcal{H}\|}(\mathcal{S}+\mathcal{H}),$$

where $\mathcal{S} := \mathcal{S}_U$ and $\mathcal{H} := \mathcal{H}_{U^\perp}$ are weakly spherical and weakly hyperbolic curvature operators as in example 3.1.2.2. Following theorem 4.4.0.17 we see that the normalized curvature operator $\frac{1}{\sqrt{2N}}(\mathcal{S}+\mathcal{H})$ of $\mathbb{S}^n \times \mathbb{H}^n$ is not a fixed point of $\overline{\Phi}$, since $\tau(\mathcal{S}) > 0$ while $\tau(\mathcal{H}) < 0$.

Corollary 4.4.0.20. *Let* $\mathcal{R} \in \mathrm{LC}\left(\bigwedge^2 \mathrm{V}\right)$ *with* $\|R\| = 1$ *and* $\tau(\mathcal{R}) \geq 0$*. Suppose that* $\mathcal{S} \in \Omega_{\overline{\Phi}}(\mathcal{R})$ *is an algebraic (or geometric) product of the curvature operators* $\mathcal{S}_1, ..., \mathcal{S}_r$*,* $\mathcal{S}_i \neq 0$*. Then, for all i holds*

$$\tau(\mathcal{S}_i) \geq 0.$$

If we have $\tau(\mathcal{S}_i) = 0$ *for some* $i \in \{1, ..., r\}$*, then* $\mathcal{S} = \mathcal{R}$ *is an equilibrium of* Φ *and therefore* $\mathcal{R} \in \mathrm{WEYL}$*.*

Beweis. $\mathcal{S}$ is an equilibrium position of $\overline{\Phi}$ with $\|\mathcal{S}\| = 1$. Hence,

$$\Phi(\mathcal{S}) = \tau(\mathcal{S})\mathcal{S}.$$

As τ is monotonically increasing along the flow of $\overline{\Phi}$ $\tau(\mathcal{S})$ is nonnegative. The first claim follows from corollary 4.4.0.17.
Now let $i \in \{1, ..., r\}$ such that $\tau(\mathcal{S}_i) = 0$. Then, by corollary 4.4.0.17 again, we get $\tau(\mathcal{S}) = 0$, which gives $\mathcal{S} = \mathcal{R}$, since τ is strictly increasing along the non stationary flow lines of $\overline{\Phi}$ by corollary 4.3.0.13. It follows that

$$\Phi(\mathcal{R}) = \tau(\mathcal{R})\mathcal{R} = 0,$$

from which we conclude that $\mathcal{R}$ is an algebraic Weyl curvature operator using theorem 3.3.1.3. □

Example 4.4.0.21 (Example 4.4.0.19 continued)**.** Let $\mathcal{R}$ as in example 4.4.0.19. Corollary 4.4.0.20 implies $\Omega(\mathcal{R}) = \left\{\frac{1}{\|\mathcal{S}\|}\mathcal{S}\right\}$.

The equilibriums of the Ricci vector field Φ lie in the space of Weyl curvature operators. For, if $\Phi(\mathcal{R}) = 0$ for an algebraic curvature operator $\mathcal{R}$, then $\mathrm{tr}(\Phi(\mathcal{R})) = 0$ as well and therefore $\mathrm{Ric}(\mathcal{R}) = 0$, since we have $\mathrm{tr}(\Phi(\mathcal{R})) = \|\mathrm{Ric}(\mathcal{R})\|^2$ by theorem 3.3.1.3.
What about the equilibriums of $\overline{\Phi}$? We know that the (weakly) spherical and hyperbolic algebraic curvature operators are fixed by the flow of $\overline{\Phi}$. They lie in $\mathrm{SCAL} \oplus \mathrm{RIC}_0$. Since WEYL is $\overline{\Phi}$-invariant, there must exist equilibrium positions in WEYL as well. SCAL is invariant either. This implies that there

are also equilibrium positions $\mathcal{R}$ in $\mathrm{SCAL} \oplus \mathrm{WEYL}$ with $\mathrm{tr}(\mathcal{R}) \neq 0$ and $\mathrm{W}(\mathcal{R}) \neq 0$, which implies $\tau(\mathrm{W}(\mathcal{R})) \neq 0$. Now we can construct equilibrium positions $\mathcal{R}$, with nonvanishing irreducible components. For example, take a spherical curvature operator $\mathcal{S}$ on $\bigwedge^2 V$ and a Weyl curvature operator $\mathcal{W}$ of $\bigwedge^2 V$ with $\Phi(\mathcal{W}) = \tau(\mathcal{W})\mathcal{W}$ and $\tau(\mathcal{W}) > 0$. Now adjust the lengths of $\mathcal{S}$ and $\mathcal{W}$, such that $\mathcal{S} \oplus \mathcal{W}$ is an equilibrium of $\overline{\Phi}$ in $\mathrm{LC}\left(\bigwedge^2(\mathrm{V} \oplus \mathrm{V})\right)$.
These are the only possibilities, as the following two propositions show.

Proposition 4.4.0.22. $\overline{\Phi}$ *has no fixed points* $\mathcal{R} \in \mathrm{RIC}_0$ *other than* $\mathcal{R} = 0$.

Beweis. Suppose that $\mathcal{R} \in RIC_0$ with $\|\mathcal{R}\| = 1$ satisfies

$$\Phi(\mathcal{R}) = \tau(\mathcal{R})\mathcal{R}.$$

Then $\Phi(\mathcal{R})$ is of traceless Ricci-type as well. Together with theorem 3.3.1.3 we get

$$0 = tr(\Phi(\mathcal{R})) = \|Ric(\mathcal{R})\|^2 = \|Ric_0(\mathcal{R})\|^2 = \|\mathcal{R}\|^2,$$

so $\mathcal{R} = 0$.

□

Proposition 4.4.0.23. *If* $\mathcal{R} \in RIC_0 \oplus WEYL$ *with norm equal to one is a fixed point of* $\overline{\Phi}$*, then* $Ric_0(\mathcal{R}) = 0$

Beweis. Theorem 3.3.1.3 implies

$$\mathrm{tr}(\overline{\Phi}(\mathcal{R})) = \|\mathrm{Ric}(\mathcal{R})\|^2 - \tau(\mathcal{R})\mathrm{tr}(\mathcal{R})$$

for every algebraic curvature operator $\mathcal{R}$. Hence, we have

$$\frac{d}{dt}tr(\mathcal{R}(t)) = \|Ric(\mathcal{R}(t))\|^2 - \tau(\mathcal{R}(t))tr(\mathcal{R}(t))$$

along the flow of $\overline{\Phi}$. If $\mathcal{R} \in \mathrm{RIC}_0 \oplus \mathrm{WEYL}$, then the right hand side equals $\|\mathrm{Ric}_0(\mathcal{R})\|^2$, which implies that the trace of the $\overline{\Phi}$-trajectory through $\mathcal{R}$ becomes instantaneously positive unless $Ric_0(\mathcal{R}) = 0$ and the claim follows.

□

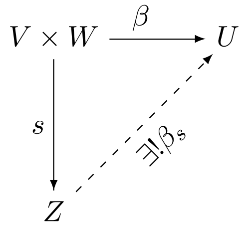
Anhang A

Multilinear Algebra and Basics of Representation Theory

This section gives a rough introduction to Euclidean multilinear algebra with special focus on the bilinear case and we provide the basic material of representation theory which is used in the text.
Let $(V, \langle \cdot\ , \cdot \rangle)$ be an Euclidean vector space of dimension n.

A.1 The Tensor Algebra

In this subsection we recall some fundamental properties of the tensor algebra of V. We start with the general construction of tensor products of (real) vector spaces. Let W be another vector space. Up to canonical isomorphisms, the tensor product $V \otimes W$ of V and W is completely characterized by the universal property: Let Z be a vector space and $s : V \times W \to Z$ bilinear. We say that Z is the tensor product of V and W, if the following situation occurs. Whenever U is any other vector space, and $\beta : V \times W \to U$ is any bilinear map, then there exists precisely one linear map $\beta_s : Z \to U$, such that the following diagram commutes



To see that the universal property determines the tensor product up to canonical isomorphism, let $s' : V \times W \to Z'$ be another bilinear map satisfying the universal property. Then, replacing U by Z' in the diagram from above, the universal property yields two canonically (and uniquely) defined linear maps $s^{'}_{s} : Z \to Z'$ and $s_{s'} : Z^{'} \to Z$, each of them making the diagram commute. These maps are inverse to each other: First, we convince ourselves, that the restrictions of these maps to the images of s and $s^{'}$ are inverse to each other. Observing that the images of s and s' span Z and $Z^{'}$ respectively, which is a consequence of the uniqueness statement in the universal property, the claim follows.

Now, there are two possibilities to generalize the definition of the tensor product to finite families of vector spaces. One by induction, the other by writing down the appropriate universal property. But, up to canonical isomorphisms, the tensor product turns out to be a commutative and associative operation. So it doesn't really matter, which way we choose. Since the main ideas concerning existence and functoriality of tensor products are the same, no matter how many factors we want to include, we choose the inductive way and keep our minds focused on the easiest non trivial case, where we are left with only two factors V and W.

Let F be the vector space with basis $\{(v,w) : v \in V, w \in W\}$ and let $\mathrm{E} \subseteq \mathrm{F}$ be the subspace generated by elements of the form

$$\lambda(v_1, w) + \mu(v_2, w) - (\lambda v_1 + \mu v_2, w)$$

or

$$\lambda(v, w_1) + \mu(v, w_2) - (v, \lambda w_1 + \mu w_2)$$

where $\lambda, \mu \in \mathbb{R}$ and all the vectors in play belong to the appropriate spaces. Then the quotient space F/E fulfills the universal property from above and therefore it is the desired tensor product of V and W. Given $v \in V$ and $w \in W$, we write $v \otimes w$ rather than $(v,w) + \mathrm{E} \in \mathrm{F}/\mathrm{E}$.

One can show, that if $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ are bases of V and W respectively, then $\{v_i \otimes w_j\}_{(i,j)\in I\times J}$ is a basis of $V \otimes W$. Thus, if V and W are both finite dimensional, then the dimension of $V \otimes W$ is $\dim V \cdot \dim W$.

What about the functoriality of this construction? Suppose that we are given linear maps $f_i : V \to W$, $i = 1, 2$. Then, by the universal property, we get a linear map $f_1 \otimes f_2 : V \otimes V \to W \otimes W$, such that $f_1 \otimes f_2(v \otimes w) = f_1(v) \otimes f_2(w)$ for all $v \in V$, $w \in W$. With this definition, the tensor product becomes a covariant functor on the category of vector spaces.

Now let us see, how we can turn the tensor product into a functor on the category of Euclidean vector spaces. First of all, we have to define a scalar product on the tensor product of two given Euclidean vector spaces $(V, \langle\cdot\ ,\cdot\rangle_V)$ and $(W, \langle\cdot\ ,\cdot\rangle_W)$ in a canonical way. To do so, we simply define

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle := \langle v_1, v_2\rangle_V \langle w_1, w_2\rangle_W .$$

The universal property assures that this definition gives a bilinear map on $V \otimes W$. The symmetry of this map is obvious and so is its positive definiteness. It is also clear that the map $f \otimes g : V \otimes W \to V \otimes W$ is a linear

isometry provided that f and $g : V \to W$ where linear isometries. Thus, the tensor product is indeed a functor on the category of Euclidean vector spaces.
Now we take a look at the covariant tensor algebra $\mathrm{T(V)} := \bigoplus_{\mathrm{p}\in\mathbb{Z}} \mathrm{T^pV}$ associated to the vector space V. Here, $\mathrm{T^pV} = \underbrace{V \otimes ... \otimes V}_{\text{p times}}$ denotes the p-fold tensor product of V provided that $p \geq 0$. Otherwise it is zero by definition. As we have mentioned above, the tensor product operation is associative and commutative. Thus, the multiplication $(x, y) \mapsto x \otimes y$, $x, y \in \mathrm{T(V)}$, turns $\mathrm{T(V)}$ into a commutative graduate algebra.
As each linear map $f : V \to W$ gives rise to linear maps $f_p : \mathrm{T^pV} \to \mathrm{T^pW}$ for any $p \in \mathbb{Z}$ in the obvious way, we get a linear map $f_{\mathrm{T}} : \mathrm{T(V)} \to \mathrm{T(W)}$, which is simply the direct sum of the maps f_p. f_{T} is a graduate algebra homomorphism and one sees immediately, that this construction is functorial. Any scalar product on V induces scalar products on $\mathrm{T^pV}$ for any $p \in \mathbb{Z}$. The direct sum of these yields a scalar product on the tensor algebra $\mathrm{T(V)}$. It is obvious that for any linear isometry $g : V \to W$, the induced map $g_{\mathrm{T}} : \mathrm{T(V)} \to \mathrm{T(W)}$ will also be isometric. Thus, we may view T as a functor from the category of Euclidean vector spaces to the category of Euclidean commutative graduate algebras as well.
Next to the covariant tensor algebra $\mathrm{T(V)}$ there is the contravariant tensor algebra $\mathrm{T^*(V)}$, which is defined by $\mathrm{T^*(V)} := \mathrm{T(V^*)}$. Clearly, $\mathrm{T^*}$ is a functor. But it is contravariant. As we have restricted our considerations to the case of finite dimension, we get that $\mathrm{T^*(V)}$ is canonically isomorphic to $\bigoplus_{p\in\mathbb{Z}} (\mathrm{T^pV})^*$. If $\langle \cdot\, , \cdot \rangle$ is a scalar product on V, then we can identify V^* and V canonically via the map $* : V \to V^*$,

$$v \mapsto (w \mapsto v^*(w) := \langle v, w \rangle).$$

Letting

$$\langle v^*, w^* \rangle := \langle v, w \rangle\,,$$

for $v, w \in V$, we turn V^* into a Euclidean vector space and the isomorphism $*$ into a linear isometry. Moreover, one can show that the $*$-isomorphisms are compatible with building tensor products, as we have the identity

$$(v \otimes w)^*(x \otimes y) = v^* \otimes w^*(x, y)$$

for all $x, y, v, w \in V$. This means that the canonical linear isometry $* : V \otimes V \to (V \otimes V)^*$ is simply the tensor product $* \otimes *$ of the canonical linear isometry $* : V \to V^*$ with itself. If we consider an arbitrary finite number k of factors, then we get the same results.

Obviously, it is also possible to mix co- and contravariant tensors. Doing so, we obtain some fundamental, canonical identifications of certain vector spaces with tensor spaces. We have, for example,

$$V^* \otimes W \cong \mathrm{Hom(V, W)},$$

where $\mathrm{Hom(V,W)}$ is the space of linear maps $F : V \to W$. Using the universal property, the desired isomorphism may be defined canonically by

$$\alpha \otimes w \mapsto (v \mapsto \alpha(v) \cdot w).$$

Analogously, one can show that for each $k \in \mathbb{N}$ $\mathrm{Mult_k(V_1, ..., V_k; W)}$, the space of k-multilinear maps on $V_1 \times ... \times V_k$ with values in W, is canonically isomorphic to the tensor space

$$V_1^* \otimes ... \otimes V_k^* \otimes W.$$

Here, we define the desired isomorphism by

$$\alpha_1 \otimes ... \otimes \alpha_k \otimes w \mapsto ((v_1, ..., v_k) \mapsto \alpha_1(v_1) \cdot ... \cdot \alpha_k(v_k) \cdot w).$$

This means that Euclidean structures on vector spaces induce Euclidean structures on the corresponding spaces of multilinear maps in a natural way. To us, it is important to know how these scalar products look like and how they can be computed explicitly in terms of the underlying Euclidean vector spaces.

We start with $\mathrm{Hom(V,W)}$. Let $\{e_1, ...e_n\}$ and $\{d_1, ..., d_m\}$ be orthonormal bases of V and W, respectively. Given linear maps $F, G : V \to W$, we can describe them as follows:

$$F = \sum_{i,j} f_i^j e_i^* \otimes d_j, f_i^j \in \mathbb{R},$$

and

$$G = \sum_{i,j} g_i^j e_i^* \otimes d_j, g_i^j \in \mathbb{R}.$$

Now, we have

$$\langle e_i^* \otimes d_j, e_k^* \otimes d_l \rangle = \langle e_i^*, e_k^* \rangle \cdot \langle d_j, d_l \rangle = \langle e_i, e_k \rangle \cdot \langle d_j, d_l \rangle,$$

which implies

$$\begin{aligned} \langle F, G \rangle &= \sum_{i,j,k,l} f_i^j g_k^l \langle e_i, e_k \rangle \langle d_j, d_l \rangle \\ &= \sum_{i,j} f_i^j g_i^j \\ &= \mathrm{tr(G^* \circ F)}, \end{aligned}$$

since we have

$$G^* \circ F = \sum_{i,j,k,l} \left(g_i^j d_j^* \otimes e_i \right) \circ \left(f_k^l e_k^* \otimes d_l \right) = \sum_{i,j,k} g_i^j f_k^j e_k^* \otimes e_i.$$

Note carefully that

$$\langle F, G \rangle = \langle G^* \circ F, \mathrm{id_V} \rangle .$$

Now we describe scalar products on spaces of multilinear maps $\mathrm{Mult_k(V_1, ..., V_k; W)}$, $k \in \mathbb{N}$.
Let $\left\{ e_1^j, ...e_{n_j}^j \right\}$, $j = 1, ..., k$, and $\{d_1, ..., d_m\}$ be orthonormal bases of V_j and W, respectively. Take k-multilinear maps $\mu, \nu \in \mathrm{Mult_k(V_1, ..., V_k; W)}$. Then we can write

$$\mu = \sum_{i_1,...,i_k,l} \mu^l_{i_1...ik} (e^1_{i_1})^* \otimes ... \otimes (e^k_{i_k})^* \otimes d_l$$

and

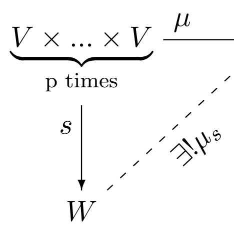
$$\nu = \sum_{i_1,...,i_k,l} \nu^l_{i_1...ik} (e^1_{i_1})^* \otimes ... \otimes (e^k_{i_k})^* \otimes d_l.$$

Arguing as above, we get

$$\langle \mu, \nu \rangle = \sum_{i_1,...,i_k,l} \mu^l_{i_1...ik} \cdot \nu^l_{i_1...ik}.$$

A.2 Exterior Powers

Besides the tensor algebra $T(V) = \bigoplus_p T^p V$ with its induced Euclidean structure $\langle \cdot \ , \cdot \rangle$, there is another functorial construction of a graduate Euclidean algebra, namely the antisymmetric algebra $\bigwedge V = \bigoplus \bigwedge^p V$, which is of interest in our studies. $\bigwedge V$ is also called the algebra of exterior powers of V. We may characterize $\bigwedge^p V$ up to canonical isomorphisms by the following universal property: Let W be a vector space and $s : \underbrace{V \times ... \times V}_{\text{p times}} \to W$ an alternating p-multilinear map. We say that W is an exterior power of V of order p if the tuple (W, s) satisfies the following condition: Given a vector space U and an alternating p-multilinear map $\mu : \underbrace{V \times ... \times V}_{\text{p times}} \to U$ there is one and only one linear map $\mu_s : W \to U$ such that the following diagram is commutative



The universal property from above characterizes W up to canonical isomorphisms. To see this, let (W', s') be another tuple satisfying the universal property. Then the universal property yields two canonically defined linear

maps $s'_s : W \to W'$ and $s_{s'} : W' \to W$ and these maps will be inverse to each other. (It is easy to see that the restrictions of these maps to the images of s and s' are inverse to each other. Then use that the images of s and s' span W and W', which follows from the uniqueness statement about the induced linear maps in the universal property.)
Now we come to the construction of $\bigwedge^p V$: Each permutation $\sigma \in S_p$ defines a linear map $\lambda_\sigma : T^pV \mapsto T^pV$ with $\lambda_\sigma(v_1 \otimes ... \otimes v_p) = v_{\sigma(1)} \otimes ... \otimes v_{\sigma(p)}$ by the universal property of the tensor product.
Now define $E^p := \{t - \mathrm{sgn}(\sigma)\lambda_\sigma(\mathrm{t}) : \mathrm{t} \in \mathrm{T^pV}, \sigma \in \mathrm{S_p}\}$, set $\bigwedge^p V := T^pV/E^p$ and let $\pi : T^pV \to \bigwedge^p V$ the canonical projection. Further, let $\wedge := \pi \circ \otimes$. Then the universal property from above is easily established for $(\bigwedge^p V, \wedge)$ using the universal properties of tensor products and quotient spaces.
$\bigwedge^p V$ is generated by elements of the form $v_1 \wedge ... \wedge v_p := \wedge(v_1, ..., v_p)$, $v_1, ..., v_p \in V$ and one can show that if $(e_1, ..., e_n)$ is a basis of V, then $(e_{i_1} \wedge ... \wedge e_{i_p})_{1 \le i_1 < ... < i_p \le n}$ is a basis of $\bigwedge^p V$. By the way, this shows that the dimension of $\bigwedge^p V$ equals $\binom{n}{p}$.
If W is another vector space and $F : V \to W$ is linear, then we can define a linear map $\bigwedge^p F : \bigwedge^p V \to \bigwedge^p W$, first letting

$$\bigwedge^p F(v_1 \wedge ... \wedge v_p) := F(v_1) \wedge ... \wedge F(v_p)$$

on the generators of $\bigwedge^p V$ and then extending this map linearly to the whole vector space using the universal property. It is easy to see that $\bigwedge^p \mathrm{id_V} = \mathrm{id}_{\bigwedge^p \mathrm{V}}$ and $\bigwedge^p G \circ F = \bigwedge^p G \circ \bigwedge^p F$, showing that $\bigwedge^p$ is actually a covariant functor on the category of vector spaces.
It is also possible to define a linear map $F_1 \wedge ... \wedge F_p : \bigwedge^p V \to \bigwedge^p W$, if we are given linear maps $F_1, ..., F_p : V \to W$:
To do so, first define $\mu(v_1, ..., v_p) := \frac{1}{p!} \sum_{\sigma \in S_p} F_{\sigma(1)}(v_1) \wedge ... \wedge F_{\sigma(p)}(v_p)$. This map is clearly p-multilinear and alternating, so we can use the universal property to define

$$F_1 \wedge .... \wedge F_p := \mu_\wedge.$$

Note that if $F_1, ..., F_p = F$, then $F_1 \wedge ... \wedge F_p = \bigwedge^p F$.
Now let us see how we can turn $\bigwedge^p V$ into a Euclidean vector space in a natural way, such that $\bigwedge^p$ becomes a functor on the category of Euclidean vector spaces: Fix $w_1, ..., w_p \in V$.Then the p-multilinear alternating map $(v_1, ..., v_p) \mapsto \det(\langle v_i, w_j \rangle)_{ij}$ induces a linear map $\mu_w : \bigwedge^p V \to \mathbb{R}$ $(w = (w_1, ..., w_p))$ with

$$\mu_w(v_1 \wedge ... \wedge v_p) = \det(\langle v_i, w_j \rangle)_{ij}.$$

This gives us a p-multilinear alternating map $\mu : \underbrace{V \times ... \times V}_{\text{p times}} \to (\bigwedge^p V)^* :$ $w \mapsto \mu_w$ which descends to a linear map $\mu_\wedge$ on $\bigwedge^p V$. Now we simply define a symmetric bilinear map $\langle \cdot \, , \cdot \rangle$ on $\bigwedge^p V$, letting

$$\langle \varepsilon, \delta \rangle := \mu_\wedge(\varepsilon)(\delta),$$

for arbitrarily chosen $\varepsilon, \delta \in \bigwedge^p V$.
It is clear that if $(e_1, ..., e_n)$ is an orthonormal basis of $(V, \langle \cdot\ , \cdot \rangle)$, then $(e_{i_1} \wedge ... \wedge e_{i_p})_{1 \leq i_1 < ... < i_p \leq n}$ is an orthonormal basis of $(\bigwedge^p V, \langle \cdot\ , \cdot \rangle)$ and that for any linear isometry F of V, $\bigwedge^p F$ is a linear isometry of $\bigwedge^p V$, so we may view $\bigwedge^p$ as a covariant functor on the category of Euclidean vector spaces as well.

Finally, we consider the direct sum $\bigwedge V := \bigoplus_p \bigwedge^p V$ together with the anticommutative multiplication

$$\varepsilon \wedge \delta = (-1)^{pq} \delta \wedge \varepsilon,$$

for $\varepsilon \in \bigwedge^p V$ and $\delta \in \bigwedge^q V$, turning $\bigwedge V$ into a graduate anticommutative algebra.

A.3 The Algebra Tensor Product

An algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathcal{A}$ together with a bilinear map $\beta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. Let $(\mathcal{A}, \beta)$ and $(\mathcal{A}', \beta')$ be $\mathbb{K}$-algebras. Then we can turn the tensor product $\mathcal{A} \otimes \mathcal{A}'$ into a $\mathbb{K}$-algebra in the following way:
Consider the map $\mathcal{A} \times \mathcal{A}' \times \mathcal{A} \times \mathcal{A}' \to \mathcal{A} \otimes \mathcal{A}'$,

$$(a, a', b, b') \mapsto \beta(a, b) \otimes \beta'(a', b').$$

This map is 4-linear and therefore induces a bilinear map

$$\otimes_{Alg} = \beta \otimes_{Alg} \beta' : \mathcal{A} \otimes \mathcal{A}' \times \mathcal{A} \otimes \mathcal{A}' \to \mathcal{A} \otimes \mathcal{A}'$$

with

$$(a \otimes a', b \otimes b') \mapsto \beta(a, b) \otimes \beta'(a', b')$$

by the universal property of tensor products. Thus, we have constructed a new $\mathbb{K}$-algebra $(\mathcal{A} \otimes \mathcal{A}', \otimes_{Alg})$. This algebra is called the algebra tensor product of the algebras $\mathcal{A}$ and $\mathcal{A}'$.

A.4 Group Actions on Vector spaces and Induced Actions

Let G be a group and X a set. A map $\varphi : G \times X \to X$ is called a G-action on X, if for all $g, h \in G$ and $x \in X$ holds

$$\varphi(g, \varphi(h, x)) = \varphi(gh, x)$$

and

$$\varphi(e, x) = x,$$

where $e \in G$ the neutral element. Sometimes we write $g.x$ instead of $\varphi(g,x)$ and sometimes, if we want to be more precise, we write φ^X rather than φ. To each $g \in G$ belongs an associated map $\varphi_g : X \to X$,

$$x \mapsto \varphi(g,x)$$

It is clear from the definition that for every $g \in G$ holds

$$\varphi_{g^{-1}} \circ \varphi_g = \varphi_g \circ \varphi_{g^{-1}} = \mathrm{id_X}.$$

Thus, the association $g \mapsto \varphi_g$ defines a group homomorphism $\rho : G \to \mathrm{S(X)}$, $\mathrm{S(X)}$ the group formed by the bijective maps $X \to X$. Conversely, each group homomorphism $\rho : G \to \mathrm{S(X)}$ gives rise to an action of G on X. Simply define $\varphi(g,x) := \rho(g)(x)$ for $g \in G$ and $x \in X$.
Since the inclusion $\iota : H \to G$ of an arbitrary subgroup $H \subseteq G$ is a group homomorphism, the restriction of φ to H induces an action of H on X.
Moreover, each subgroup $H \subseteq G$ acts on G by left translation, $\varphi : H \times G \to G$,

$$(h,g) \mapsto l_g(h) = gh.$$

A subset $Y \subseteq X$ is called G-invariant, or simply invariant, if $y \in Y$ implies $\varphi(g,y) \in Y$ for each $g \in G$. As above, the restriction of a G-action on X to a G-invariant subset Y yields a G-action on Y.
Clearly, the G-orbits $G(x) := \{\varphi(g,y) : g \in G\}$ are the smallest nontrivial invariant subsets of X and X itself is the biggest. Further, X decomposes as the disjoint union of G-orbits. We say that a G-action on X is transitive, if X has no invariant subsets other that X. Given $x \in X$, we can look at all elements $g \in G$ which fix x. These elements build a subgroup

$$G_x = \{g \in G : \varphi(g,x) = x\}.$$

G_x is called the stabilizer of x. If G is acting transitively on X, we get an bijection

$$G/G_x \to X$$

for each $x \in X$, which is induced by the map $g \mapsto \varphi(g,x)$.
Now suppose that G is acting on the sets X and Y. A map $f : X \to Y$ is called G-equivariant (or simply equivariant), if it is compatible with the actions of G on X and Y. More precisely, we require $f \circ \varphi_g^X = \varphi_g^Y \circ f$ for each $g \in G$.

In the text we are mostly concerned with linear group actions on vector spaces. In these cases X is replaced by a vector space and we additionally require that the associated maps φ_g are linear for every $g \in G$, i.e. we want $\rho(G) \subseteq \mathrm{GL(V)} \subseteq \mathrm{End(V)}$. In the text, G is usually given by a subgroup of $O(V, \langle \cdot\,,\cdot \rangle)$, where $\langle \cdot\,,\cdot \rangle$ is a scalar product on V. In this case the action is

called orthogonal.
If a group G is acting linearly on the vector spaces $V_1, ..., V_k$, $k \in \mathbb{N}$, then G is also acting on the tensor product $V_1 \otimes ... \otimes V_k$. To see this, let $\rho_i : G \to \mathrm{GL(V_i)}$, $i = 1, ..., k$, the group homomorphism corresponding to the action of G on V_i, and define $\rho : G \to \mathrm{GL(V_1 \otimes ... \otimes V_k)}$,

$$\rho(g) := \rho_1(g) \otimes ... \otimes \rho_k(g).$$

The functoriality of the tensor product assures that ρ is indeed a homomorphism of groups. Moreover, if we assume the V_i to be Euclidean vector spaces and G acting orthogonally on each factor V_i, then G is acting orthogonally on the tensor product $V_1 \otimes ... \otimes V_k$ w.r.t. the induced Euclidean structure.
If we are given vector spaces $W_1, ..., W_k$ together with linear G-actions and equivariant linear maps $F_i : V_i \to W_i$, the induced map

$$F_1 \otimes ... \otimes F_k : V_1 \otimes ... \otimes V_k \to W_1 \otimes ... \otimes W_k$$

is equivariant either.
Analogously, we can define induced actions of G on exterior powers, since

each $g \in G$, induces a map

$$\underbrace{\rho(g) \wedge ... \wedge \rho(g)}_{k-times}$$

on $\bigwedge^k V$ for every $k \in \mathbb{N}$. Again, if the action of G on V is orthogonal w.r.t. some given Euclidean structure, then the induced G-action on $\bigwedge^k V$ will be orthogonal w.r.t. induced scalar product as well. And obviously, equivariant linear maps $F_1, ..., F_k : V \to W$ give rise to an equivariant linear map

$$F_1 \wedge ... \wedge F_k : \bigwedge^k V \to \bigwedge^k W$$

Now we explain some special examples of linear group actions, which are frequently used in the text.

For example, each subgroup G of $\mathrm{GL(V)}$ acts linearly on V by multiplication

$$g.v := g(v), v \in V.$$

Further, we see that the linear G-action on V gives rise to linear G-actions on V^* and $\mathrm{End(V)}$ in a canonical way: simply define

$$g.\alpha := \alpha \circ g^{-1}, \text{ for } \alpha \in V^*,$$

and

$$g.F := g \circ F \circ g^{-1}, \text{ for } F \in \mathrm{End(V)}.$$

On the other hand, as we have seen above, we can use the linear actions of G on V and V^* to define a linear action on the tensor product $V^* \otimes V$, letting

$$g.(\alpha \otimes v) := (g.\alpha) \otimes (g.v)$$

on generators $\alpha \otimes v$ of $V^* \otimes V$. A sharp look makes sure that the canonical isomorphism $V^* \otimes V \to \mathrm{End(V)}$, mapping $\alpha \otimes V$ to the endomorphism $w \mapsto \alpha(w)v$ preserves the corresponding G-actions, in other words, it is equivariant.
O(V) embeds into $\mathrm{O}(\bigwedge^2 \mathrm{V})$ canonically via

$$G \mapsto G \wedge G.$$

As this map is a homomorphism of groups, it induces orthogonal actions of O(V) on $\bigwedge^2 V$,

$$(G, \varepsilon) \mapsto G \wedge G(\varepsilon)$$

and on $\mathrm{End}(\bigwedge^2 \mathrm{V})$,

$$(G, \mathcal{F}) \mapsto G \wedge G \circ \mathcal{F} \circ G^{-1} \wedge G^{-1}.$$

The action of O(V) on $\mathrm{End}\bigwedge^2 \mathrm{V}$ preserves the subspaces $\mathrm{Sym}\left(\bigwedge^2 \mathrm{V}\right)$, the space of self-adjoint linear maps $\bigwedge^2 V \to \bigwedge^2 V$, and the subspace $\mathrm{Skew}\left(\bigwedge^2 \mathrm{V}\right)$ consisting out of skew-adjoint linear maps $\bigwedge^2 V \to \bigwedge^2 V$. Therefore, O(V) acts on these spaces orthogonally either.

We say that G is acting linearly on the $\mathbb{R}$-algebra (V, β), provided that G is acting linearly on V and the bilinear map β is equivariant (this is the case, if and only if the induced linear map $\beta_\otimes$ is equivariant). Now suppose that a group G is acting on the real algebras (V, β) and (V', β'). Then G also acts on the algebra tensor product

$$(V \otimes V', \beta \otimes_{Alg} \beta'),$$

as one easily sees.

A.5 Representation Theory

Let $(V, \langle \cdot\, , \cdot \rangle)$ be a Euclidean vector space of finite dimension and $O(V) = O(V, \langle \cdot\, , \cdot \rangle)$ the orthogonal group with respect to $\langle \cdot\, , \cdot \rangle$.

Definition A.5.0.24. *A group homomorphism $\rho : G \to GL(V)$ is called a representation of G. It is called an orthogonal representation, if the image of ρ lies in $O(V)$.*

Any (orthogonal) representation $\rho : G \to GL(V)$ gives rise to a (an isometric) left-action of G and vice versa: $\varphi : G \times V \to V : (g, v) \mapsto g.v := \rho(g)v$.

A subspace of U of V is called G-invariant, or simply invariant, if it is preserved by the action of G on V. In this case we get a new homomorphism $\rho : G \to GL(U)$, i.e. a new (orthogonal) representation of of G. Note that if U is invariant, then its orthogonal complement $U^\perp$ is invariant either, if the given representation was orthogonal.

$\rho : G \to O(V)$ is called irreducible, if every invariant subspace of V is either 0 or V. Irreducible orthogonal representations are the building blocks of orthogonal representations:

Theorem A.5.0.25. *Every orthogonal representation is the direct (orthogonal) sum of irreducible orthogonal representations*

Now let $\sigma : G \to O(W)$ be another orthogonal representation. A linear map $F : V \to W$ is called G-equivariant (or simply equivariant), if $F \circ \rho(g) = \sigma(g) \circ F$ for all $g \in G$. If $V = W$, then F is called an intertwining map. One can take the orthogonal projections onto - and inclusions of - invariant subspaces as examples for intertwining maps.

Now we get new examples of invariant subspaces: if $F : V \to W$ is linear and equivariant, then $\ker F$ and $\operatorname{im} F$ are invariant subspaces. Moreover, if λ is an eigenvalue of F, then the corresponding eigenspace $\mathrm{E}_\lambda(\mathrm{F})$ is invariant. As an immediate consequence we get the obtain theorem.

Theorem A.5.0.26 (Schur)**.** *Let $\rho : G \to O(V)$ and $\sigma : G \to W$ be irreducible representations. Then every linear equivariant map $F : V \to W$ is either* 0 *or an isomorphism.*

Beweis. There is nothing left to do.

□

Anhang B

Riemannian Geometry

In this second part of the appendix we summarize some facts about Riemannian manifolds, which play a certain role in our context, such as (induced) connections on vector bundles, Lie groups and Lie algebras, the relation between curvature and holonomy, Killing fields and isometry groups of Riemannian manifolds, curvature description of Riemannian homogeneous manifolds and symmetric spaces and finally, convergence of Riemannian manifolds in the sense of Cheeger and Gromov. The reader is assumed to be familiar with the basic concepts of differential geometry.

B.1 Connections on Vector Bundles

Let $\pi : \mathrm{E} \to \mathrm{M}$ be a smooth vector bundle of rank r. As usual, the space of sections of E will be denoted be $\Gamma(E)$.

Definition B.1.0.27. *A connection on E is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E) : (X, s) \mapsto \nabla_X s$ which is tensorial in X and derivative in s, meaning that for all $X \in \Gamma(TM)$, $s \in \Gamma(E)$ and all smooth functions f on M holds*

1. $\nabla_{fX} s = f \nabla_X s$ *and*
2. $\nabla_X fs = (Xf)s + f\nabla_X s$

Alternatively, we could have said that a connection ∇ on E assigns to each section s of E a 1-form ∇s on M with values in E, such that for any function f on M holds

$$\nabla fs = f\nabla s + \mathrm{Df} \otimes \mathrm{s}.$$

The space of all connections on E is obviously an affine space (of infinite dimension, however): the difference of any two connections is a tensor, or, more precisely, a 1-form on M with values in End(E). This implies that, starting from any connection ∇^0 on E, we obtain all the other possible connections by subtracting 1-forms A on M with values in End(E),

$$\nabla = \nabla^0 - A.$$

What about the existence of connections? If $\mathrm{E} = \mathrm{M} \times \mathbb{R}^\mathrm{n}$ is a trivial bundle we can take the ordinary differential operator D as a connection on E. However, this doesn't work if E is not trivial, for the differential $\mathrm{D_X s}$ of a sections s of E in direction of a vector field X on M won't be a section of E any longer. How is this to be repaired?
Take a countable collection $\mathcal{U} = \{U_i : i \in I\}$ of open subsets of M which are covering M, together with trivializing diffeomorphisms $\varphi_i : \mathrm{E}|_{\mathrm{U_i}} \to \mathrm{U_i} \times \mathbb{R}^\mathrm{r}$ and a subordinated partition of unity $\{\rho_i : M \to [0,1]\}_{i \in I}$. Now, for each i, we define a (local) connection $\nabla^i : \Gamma(\mathrm{TU_i}) \times \Gamma(\mathrm{E}|_{\mathrm{U_i}}) \to \Gamma(\mathrm{E}|_{\mathrm{U_i}})$ by

$$\nabla^i_X s(q) := \varphi_i^{-1} \left(\mathrm{D(s \circ \varphi_i)_q(X(q))} \right).$$

Finally, we define

$$\nabla := \sum_i \rho_i \nabla^i,$$

which is the desired connection on M.

Now we come to the concept of parallel transport:
Let $p, q \in M$ and $c : [0,1] \to M$ be a smooth curve joining p with q. Then, for any prescribed vector $v \in \mathrm{T_pM}$ we can find a vector field X_v along c, satisfying

$$\nabla_{\dot{c}} X_v = 0$$

along c. Proofs of this fact can be found in almost any book about differential or Riemannian geometry, for example in [18]. The parallel transport along c is then defined by

$$P_c : \mathrm{T_pM} \to \mathrm{T_qM} : \mathrm{v} \mapsto \mathrm{X_v(1)}.$$

This definition also applies to piecewise smooth curves in the obvious way.

If $\mathrm{E}' \to \mathrm{M}$ is another vector bundle over M with connection ∇', then we get an induced connection $\nabla^\otimes$ on $\mathrm{E} \otimes \mathrm{E}'$ requiring this new connection to satisfy the product rule

$$\nabla^\otimes s \otimes s' = \nabla s \otimes s' + s \otimes \nabla s'.$$

Thus, we get induced connections on any tensor bundle $\mathrm{T^{(r,s)}E}$. Further, ∇ induces connections on the exterior powers $\bigwedge^r E$ with

$$\nabla_X(s_1 \wedge ... \wedge s_r) = \sum_i s_1 \wedge ... \wedge \nabla_X s_i \wedge .... \wedge s_r.$$

It is somehow clear how the induced parallel transports will look like. Nevertheless, we will describe the parallel transport on $\bigwedge^2 \mathrm{E}$, since this case is quite important to us:
If P denotes the parallel transport in E along a curve $c : [0,1] \to M$, then the parallel transport in $\bigwedge^2 \mathrm{E}$ along c is given by $P \wedge P$, as one easily shows.

If E is equipped with an inner product $\langle \cdot\, , \cdot \rangle$, there is a special class of connections on E: the $\langle \cdot\, , \cdot \rangle$-metric connections. A connection ∇ is called metric w.r.t. $\langle \cdot\, , \cdot \rangle$, if $\langle \cdot\, , \cdot \rangle$ is parallel w.r.t. ∇, i.e. if $\nabla \langle \cdot\, , \cdot \rangle \equiv 0$, or equivalently, if $X \langle s, t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle$ holds for all vector fields X on M and all smooth sections s, t of E. It is easy to see that the induced connections from above will be metric w.r.t. the corresponding induced inner products, provided that the connection on E itself is metric. Further, the parallel transport corresponding to metric connections is always isometric.
What about the existence of metric connections on Euclidean vector bundles? We already know that, staring from any connection ∇^0 on E, we get all the other possible connections by subtracting 1-forms on M with values in End(E). In particular, we get the metric connections in this way. So let ∇^0 be an arbitrary connection. Now we are going to define a 1-form A on M with values in End(E), such that the connection $\nabla := \nabla^0 - A$ will be metric.
Consider the map

$$(X, s, t) \mapsto \frac{1}{2} \left(\left\langle \nabla^0_X s, t \right\rangle + \left\langle s, \nabla^0_X t \right\rangle - X \langle s, t \rangle \right),$$

$X \in \Gamma(\mathrm{TM})$, $s, t \in \Gamma(\mathrm{E})$. As this map is a tensor and symmetric in s and t, we may use it to define the desired 1-form through

$$\langle A_X s, t \rangle := \frac{1}{2} \left(\left\langle \nabla^0_X s, t \right\rangle + \left\langle s, \nabla^0_X t \right\rangle - X \langle s, t \rangle \right).$$

Note that, for any vector field X on M, A_X is by definition self-adjoint w.r.t. $\langle \cdot\, , \cdot \rangle_p$ within each fiber of E. Using this, a short computation shows that

$$\nabla = \nabla^0 - A$$

is indeed a metric connection.
The space of metric connections on a Euclidean vector bundle E is an affine subspace of the space of all connections: The difference $\nabla - \nabla'$ of the two metric connections ∇ and ∇' is a 1-form on M with values in $\mathfrak{so}(\mathrm{E})$, the space of skew-adjoint bundle endomorphisms of E. Therefore, starting from any metric connection ∇ on E, we obtain all the other possible metric connections, by subtracting 1-forms on M with values in $\mathfrak{so}(\mathrm{E})$.

In order to define higher order covariant derivatives, we need a connection ∇ on TM as well. So let ∇ be a connection on M. We proceed as follows: If s is a section of E, then ∇s is a section of $\mathrm{T^*M} \otimes \mathrm{E}$ - and this is actually the reason why one needs a connection on M to define higher order covariant derivatives on E! We define

$$\nabla^2_{X,Y} s := (\nabla_X \nabla s)(Y) = \nabla_X \nabla_Y s - \nabla_{\nabla_X Y} s$$

for all vector fields X, Y on M and all sections s of E to assure that $\nabla^2 s$ will be tensorial in X and Y.
The k-th covariant derivative of s shall be a section of $\underbrace{\mathrm{TM^*} \otimes ... \otimes \mathrm{TM^*}}_{\text{k times}} \otimes \mathrm{E}$.

Therefore we define

$$\left(\nabla_X \nabla^k s\right)(X_1, ..., X_k) := \nabla_X \nabla_{X_1,...,X_k} s - \sum_{i=1}^{k} \nabla s(X_1, ..., \nabla_X X_i, ..., X_k)$$

to guaranty that $\nabla^{k+1} s$ will actually be a tensor. For short, the inductive definition of higher order covariant derivatives is this:

$$\nabla^0 s = s \text{ and } \nabla^{k+1} s = \nabla(\nabla^k s)$$

B.2 Basic Concepts of Riemannian Geometry

A Riemannian manifold (M, g) is a smooth n-dimensional manifold M together with a smooth $(0, 2)$-tensor field g which restricts to a scalar product g_p on each tangent space $\mathrm{T_pM}$, $p \in M$. Such a tensor field g is called a Riemannian metric on M. Riemannian metrics do always exist. They are constructed easily, using partitions of unity and the fact that the space of scalar products on a vector space is a convex cone.
Riemannian metrics allow us to carry over the basic concepts of classic geometry (on linear spaces) to manifolds (non-linear spaces), such as angles between tangent vectors and their lengths, the length of smooth curves and distances between points: If $p, q \in M$ are points and $c : [0, 1] \mapsto M$ is a curve connection p and q, then the length $\mathcal{L}(c)$ of c is defined to be

$$\mathcal{L}(c) := \int_0^1 \sqrt{g_{c(t)}(\dot{c}, \dot{c})}\mathrm{dt},$$

just like in $\mathbb{R}^n$, and the distance between p and q is defined to be the infimum over the lengths of all curves connecting these points,

$$\mathrm{dist(p, q)} := \inf \left\{\mathcal{L}(\mathrm{c}) : \mathrm{c \text{ connects } p \text{ and } q}\right\}.$$

It is worthwhile to mention that this function turns M into a metric space and that the derived (metric) topology is the same as the underlying manifold topology. As a consequence, which is indeed not very surprising, a big part of the theory of Riemannian manifolds is devoted to the fruitful study of the interplay between (algebraic) topology and Riemannian geometry.
To each Riemannian metric g belongs a uniquely determined connection ∇ on the tangent bundle of M, the so called Levi-Civita connection of g. This statement is known as the fundamental theorem of Riemannian geometry. ∇ is determined by the requirements that it shall be metric, i.e.

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

and torsion free, i.e.

$$\nabla_X Y - \nabla_Y X = [X,Y]\,,$$

where X, Y and Z are arbitrary vector fields on M. Both, existence and uniqueness of such a connection, follow from the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & + g([X,Y]\,, Z) - g([Y,Z]\,, X) + g([Z,X]\,, Y). \end{aligned}$$

which is easily established for metric and torsion free connections.
Now we are ready to introduce another fundamental concept of Riemannian geometry: Curvature. First of all, there is the curvature tensor R. It is defined by

$$R(X,Y)Z := \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z$$

for all smooth vector fields X, Y and Z. The curvature tensor measures to which extend the covariant derivatives of vector fields fail to be commutative. It has the following properties: The curvature tensor

- is tensorial in X, Y and Z,
- skew-symmetric in X and Y,
- satisfies the 1. Bianchi identity:
  $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
- and the 2. Bianchi identity:
  $\nabla_X R(Y,Z)U + \nabla_Y R(Z,X)U + \nabla_{Z,} R(X,Y)U = 0.$

By the theorem of Cartan, Ambrose and Hicks (see [8] for the precise statement and the proof), the curvature tensor R carries the whole information about the geometry of (M, g) - at least if (M, g) is metrically complete and simply connected.
Apart from this rather global statement, there is an infinitesimal one pointing in the same direction. The curvature tensor R_p, $p \in M$, gives rise to a

Riemannian metric $\tilde{g}$, defined on a neighborhood U of $0 \in \mathrm{T_pM} \cong \mathbb{R}^{\mathrm{n}}$, such that $\tilde{R}_0$ equals R_p. The metric $\tilde{g}$ is given by

$$\tilde{g}_q(x,y) = g_p(x,y) - \frac{1}{3} g_p(R_p(x,q)q,y).$$

Furthermore, the Taylor expansions of $\tilde{g}$ and $g|_U$ (or rather the pullback of g under the exponential map) agree up to second order. But this is something which is actually true for any locally defined Riemannian metric on $\mathrm{T_pM}$, whose curvature tensor in 0 agrees with R_p (see [8] for the details, where the Taylor expansion of g around p is computed by means of Jacoby fields). So from this perspective, the curvature tensor R may be viewed as a canonical infinitesimal representative of g up to second order. And the algebraic curvature operators on $\mathrm{T_pM}$ as a whole, namely the (1,3) tensors on $\mathrm{T_pM}$ sharing the algebraic properties of the curvature tensor R_p, may be viewed as the 2-jets of Riemannian metrics in p.
Further, there is the Riemannian curvature tensor, defined by

$$R(X,Y,Z,U) = g(R(X,Y)Z,U).$$

Clearly, the Riemannian curvature tensor R inherits the symmetries of the curvature tensor R in the first three arguments. In addition, it is skew-symmetric in Z and U and fulfills

$$R(X,Y,Z,U) = R(Z,U,X,Y).$$

It is clear, that the curvature tensor and the Riemannian curvature tensor are algebraically equivalent.
Next to these constructions, there is a third, which is algebraically equivalent to the others: The curvature operator $\mathcal{R} : \bigwedge^2 \mathrm{TM} \to \bigwedge^2 \mathrm{TM}$. It is defined (uniquely) by the equation

$$g(\mathcal{R} X \wedge Y, Z \wedge U) := R(X,Y,U,Z),$$

where the g on the left hand side is the induced inner product on $\bigwedge^2 \mathrm{TM}$ and X, Y, Z and U are vector fields on M. It is the universal property of exterior powers that guaranties $\mathcal{R}$ to be an endomorphism of $\bigwedge^2 \mathrm{TM}$. One can easily show that $\mathcal{R}$ is self-adjoint w.r.t. to the induced metric g on $\bigwedge^2 \mathrm{TM}$.
From a geometrical point of view, $\mathcal{R}$ is a linear self-adjoint operator on formal linear combinations of 2-dimensional subspaces of TM.
For sake of completeness we also introduce the concept of sectional curvature (we do not need it in our text). Let $\mathrm{E} \subseteq \mathrm{T_pM}$ a 2 dimensional subspace, generated by elements $x, y \in V$. Then the sectional curvature of E is defined to be

$$\mathrm{sec(E)} := \frac{\mathrm{R(x,y,y,x)}}{\|\mathrm{x} \wedge \mathrm{y}\|^2}.$$

Actually, this definition is independent of the choice of the vectors generating E. Moreover, the sectional curvature carries the full geometric information about the underlying Riemannian manifold (M, g). One can recover the curvature tensor R from the knowledge of the sectional curvatures. A quite long formula, describing the curvature tensor in terms of the sectional curvatures can be found in [8] , for example.
Beside the curvature concepts from above, there are two other curvature constructions of fundamental importance, which we would like to mention here: Ricci curvature and scalar curvature.
The Ricci tensor ric is a symmetric bilinear tensor flied on the tangent bundle TM of M. For each point $p \in M$, the Ricci curvature $\mathrm{ric_p(x, y)}$ of $x, y \in \mathrm{T_pM}$ is defined to be the trace of the map $z \mapsto R_p(z, x)y$, i.e.

$$\mathrm{ric_p(x,y) = \sum_i g_p(R_p(e_i, x)y, e_i)},$$

where $\{e_i\}$ is an orthonormal basis of $\mathrm{T_pM}$. As $\mathrm{ric_p}$ is symmetric in x and y for any $p \in M$, there exists a uniquely determined self-adjoint endomorphism field Ric on TM, satisfying $g(\mathrm{Ric(X), Y}) = \mathrm{ric(X, Y)}$ for all vector fields X and Y and we have

$$\mathrm{Ric_p(x) = \sum_i R_p(x, e_i)e_i},$$

where $x \in \mathrm{T_pM}$ and $\{e_i\}$ is as above.
The scalar curvature scal is a function on M. It is the trace of the Ricci curvature,

$$\mathrm{scal(p) := trRic_p}.$$

There are numerous applications of these curvature concepts.

Riemannian metrics provide a way to translate the classical differential operators, such as the gradient, the Hessian, the divergence and the Laplacian, to the manifold setting.
Using the induced isomorphism $\mathrm{TM \to T^*M, x \mapsto (y \mapsto g(x, y))}$, we define the gradient ∇f of a smooth function $f : M \to \mathbb{R}$ requiring

$$g(\nabla f, X) = \mathrm{Df(X)}$$

for any smooth vector field X on M.
The Hessian Hess(f) of f is simply the covariant derivative of the ordinary differential Df of f,

$$\mathrm{Hess(f)(X, Y) = \nabla_X Df(Y) = g(\nabla_X \nabla f, Y)}.$$

It is a tensor and symmetric in X and Y and reflects the qualitative behavior of f near extremal points just like its relative on the flat Euclidean space.
The divergence of a vector field X is given by

$$\mathrm{div(X) = tr(Z \mapsto \nabla_Z X)}.$$

W.r.t. an orthonormal basis $\{e_i\}$ of $\mathrm{T_pM}$, $p \in M$, the divergence of X in p may be written as follows:

$$\mathrm{div(X)(p)} = \sum_{\mathrm{i}} \mathrm{g_p}(\nabla_{\mathrm{e_i}}\mathrm{X(p)}, \mathrm{e_i}).$$

Just like the divergence of vector fields on $\mathbb{R}^n$, the Riemannian divergence $\mathrm{div(X)(p)}$, $p \in M$, measures the infinitesimal flux of the vector field X within p.
As in $\mathbb{R}^n$, the Laplacian Δf of f is defined to be the divergence of the gradient of f,

$$\Delta f = \mathrm{div}(\nabla \mathrm{f}).$$

Both, the divergence and the Laplacian, have their own canonical generalizations to differential operators on sections of Euclidean vector bundles over Riemannian manifolds:
Let $(\mathrm{E}, \langle \cdot\, , \cdot \rangle) \to \mathrm{M}$ be a Euclidean vector bundle over the Riemannian manifold (M, g) and ∇ a metric connection on E. If s is a smooth section of E and $\{e_i\}$ is an orthonormal basis of $\mathrm{T_pM}$, then we define

$$\Delta s(p) := \sum_i \nabla^2_{e_i, e_i} s.$$

The definition of the divergence of s is somehow more sophisticated. ∇s is a 1-form on M with values in E. So there is a priori no possibility to perform contractions w.r.t. variables coming from the tangent bundle of M. Thus, we have to restrict ourselves to the case where $\mathrm{E} \cong \mathrm{T^*M} \otimes \mathrm{E}'$. Then we can define the divergence div(s) of s as the contraction of ∇s w.r.t. the first two variables,

$$\mathrm{div(s)(p)} := \sum_{\mathrm{i}} \nabla_{\mathrm{e_i}} \mathrm{s}(\mathrm{e_i}),$$

where $p \in M$ and $\{e_i\}$ is as above. Note that, using this definition of the divergence, we still get

$$\Delta s = \mathrm{div} \nabla \mathrm{s}$$

for all sections of Euclidean vector bundles over Riemannian manifolds, which are equipped with a compatible connections.

B.3 Lie Groups and Lie Algebras

Definition B.3.0.28. *A Lie Group is a differentiable manifold endowed with a smooth group structure. This means, we have a smooth manifold G together with a binary operation*

$$\mu : G \times G \to G$$

turning G into a group, such that the map

$$(g,h) \mapsto \mu(g,h^{-1})$$

is smooth.
A subgroup H of G is called a Lie subgroup of G, if it is closed, i.e if it is not only a subgroup but also a submanifold of G.
And finally, the smooth group homomorphisms between Lie groups are called the Lie group homomorphisms.

There are many examples of Lie groups. For example, every matrix group is a Lie group. But also the discrete groups $\mathbb{Z}$ and $\mathbb{Z}_p$, $p \in \mathbb{N}$, are Lie groups. The Lie groups, which are of most importance in our context, are the orthogonal groups O(n) and their compact subgroups.

Definition B.3.0.29. *A Lie algebra is a vector space $\mathfrak{g}$ together with a skew-symmetric bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity*

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$$

for all $x,y,z \in \mathfrak{g}$. $[\cdot,\cdot]$ is called the Lie bracket of $\mathfrak{g}$.
A linear subspace $\mathfrak{g}'$ of $\mathfrak{g}$ is called a Lie subalgebra if it is closed under $[\cdot,\cdot]$, i.e. if $x,y \in \mathfrak{g}'$ implies $[x,y] \in \mathfrak{g}'$.
A linear subspace I of $\mathfrak{g}$ is called an ideal, if $[x,y] \in I$ for all $x \in I$ and $y \in \mathfrak{g}$
And finally, a linear map between Lie algebras is called a Lie algebra homomorphism, if it respects the Lie brackets of the underlying Lie algebras. In the following we will only consider real Lie algebras, i.e. Lie algebras over $\mathbb{R}$.

For example, the matrix ring $\mathrm{Mat(n,\mathbb{R})}$ together with the commutator $[A,B] = AB - BA$ is a Lie algebra, or $\mathbb{R}^3$ together with the cross product. The Lie algebra $\mathfrak{so}(n)$, $n \in \mathbb{N}$ is one of the most important examples in our context. By definition

$$\mathfrak{so}(n) := \left\{A \in \mathrm{Mat(n,\mathbb{R}) : A^t = -A}\right\}.$$

It is easy to see that $\mathfrak{so}(n)$ is a Lie subalgebra of $\mathrm{Mat(n,\mathbb{R})}$ and that the set of matrices $\{\mathrm{E_{i,j}}\}_{1\leq i<j\leq n}$, $(\mathrm{E_{i,j}})_{ij} = 1$, $(\mathrm{E_{i,j}})_{ji} = -1$ and $(\mathrm{E_{i,j}})_{kl} = 0$ otherwise, forms a basis of $\mathfrak{so}(n)$, which gives $\dim \mathfrak{so}(n) = \frac{n(n-1)}{2}$. This is the standard model of $\mathfrak{so}(n)$.
It is possible to give coordinate free definitions of the spaces from above: First, we replace $\mathrm{Mat(n,\mathbb{R})}$ by the space of endomorphisms End(V), where V is an n-dimensional real vector space. The Lie bracket will be given by the commutator of endomorphisms, $[F,G] = F \circ G - G \circ F = FG - GF$. Now we fix a scalar product $\langle \cdot\,,\cdot\rangle$ on V and define

$$\mathfrak{so}(V, \langle \cdot\,,\cdot\rangle) := \{F \in \mathrm{End(V) : F^* = -F}\},$$

where F^* is the adjoint of F w.r.t. $\langle \cdot\ , \cdot \rangle$. Again, it is easy to see that $\mathfrak{so}(V, \langle \cdot\ , \cdot \rangle)$ is a Lie subalgebra of End(V). Now we wish to see that $\mathfrak{so}(V, \langle \cdot\ , \cdot \rangle)$ and $\mathfrak{so}(n)$ are isomorphic as Lie algebras.
Let $\langle \cdot\ , \cdot \rangle_{can}$ be the standard scalar product on $\mathbb{R}^n$. Using coordinate representations w.r.t. the standard basis of $\mathbb{R}^n$, one sees quickly that $\mathfrak{so}(\mathbb{R}^n, \langle \cdot\ , \cdot \rangle_{can})$ is isomorphic to $\mathfrak{so}(n)$ as a Lie algebra. Now pick an orthonormal basis $\{e_i\}$ of $(V, \langle \cdot\ , \cdot \rangle)$ and define a linear isometry $\varphi : (V, \langle \cdot\ , \cdot \rangle) \to (\mathbb{R}^n, \langle \cdot\ , \cdot \rangle_{can})$ by sending e_i to the i-th member of the standard basis of $\mathbb{R}^n$. φ induces an isomorphism of vector spaces $\mathfrak{so}(V, \langle \cdot\ , \cdot \rangle) \to \mathfrak{so}(\mathbb{R}^n, \langle \cdot\ , \cdot \rangle_{can})$,

$$F \mapsto \varphi \circ F \circ \varphi^{-1}.$$

This isomorphism is easily seen to be an isomorphism of Lie algebras as well. Thus, we have shown that $\mathfrak{so}(V, \langle \cdot\ , \cdot \rangle)$ and $\mathfrak{so}(n)$ are isomorphic as Lie algebras either.
There is a third possibility to define $\mathfrak{so}(n)$ using exterior powers of Euclidean vector spaces. These representations of $\mathfrak{so}(n)$ are discussed in detail in section 1.

Further, Lie algebras arise naturally in differential geometry:
If we are given a smooth manifold M, the space $\Gamma(\mathrm{TM})$ of smooth vector fields on M together with the Lie bracket of $[\cdot, \cdot]$ of vector fields, defined by

$$[X, Y]f := X(Yf) - Y(Xf)$$

for all smooth functions $f : M \to \mathbb{R}$, is a Lie algebra. However, it has infinite dimension.
If we are given a Lie group G, there is a special type of vector fields. The left invariant vector fields: A vector field X on M is called left invariant, if it satisfies

$$X \circ L_g = \mathrm{DL_g X}$$

for all $g \in G$. One can show easily that left invariant vector fields are smooth and that they form a finite dimensional Lie subalgebra of the space of all smooth vector fields. This Lie algebra is called Lie algebra of G and labeled by $\mathfrak{g}$ or Lie(G). As a vector space, $\mathfrak{g}$ is canonically isomorphic to $\mathrm{T_e G}$, the tangent space of G the identity. The isomorphism $\mathfrak{g} \to \mathrm{T_e G}$ is given by $X \mapsto X(e)$.
Any left invariant vector field X on G defines a homogeneous 1-parameter family $\alpha_X : \mathbb{R} \to G$ with $\dot{\alpha}_X = X \circ \alpha$ and $\alpha_X(0) = e$. The map $\exp : \mathfrak{g} \to G : X \mapsto \alpha_X(1)$ is called the exponential map of G. exp is a local diffeomorphism at $0 \in \mathfrak{g}$.
G acts on itself by conjugation, $(g, h) \mapsto c_g(h) := ghg^{-1}$. This map provides a representation $\mathrm{Ad : G \to GL}(\mathfrak{g})$ through

$$g \mapsto (\mathrm{Dc_g})_\mathrm{e}.$$

We call it the adjoint representation of G. Taking the differential DAd_0 of adjoint representation of G at $0 \in \mathfrak{g}$ gives a representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : \mathrm{x} \mapsto \mathrm{ad_x} := \mathrm{DAd_0(x)}$, the so called adjoint representation of $\mathfrak{g}$. Here, $\mathfrak{gl}(\mathfrak{g})$ denotes the Lie algebra of $\mathrm{GL}(\mathfrak{g})$. It turns out that we have $\mathrm{ad_x(y) = [x, y]}$ for all $x, y \in \mathfrak{g}$. Using the Jacoby identity, we get

$$\mathrm{ad_{[x,y]} = [ad_x, ad_y]}$$

and

$$\mathrm{ad_x([y, z]) = [ad_x(y), z] + [y, ad_x(z)]}$$

for all $x, y, z \in \mathfrak{g}$.

At the end of this section we mention the famous theorem of Ado-Iwasawa, which states that every finite dimensional Lie algebra is the Lie algebra of a uniquely defined simply connected Lie group.

Theorem B.3.0.30 (Ado-Iwasawa)**.** *Given a Lie algebra $\mathfrak{g}$ there exists a simply connected Lie group G whose Lie algebra is $\mathfrak{g}$. The isomorphy type of G is uniquely determined by $\mathfrak{g}$.*

B.3.1 Lie Groups and Riemannian geometry

Let G be a Lie group. A Riemannian metric $\langle \cdot \, , \cdot \rangle$ on G is called left-invariant, if every left translation $L_g : G \to G$,

$$h \mapsto gh$$

is an isometry of $(G, \langle \cdot \, , \cdot \rangle)$, i.e. if for every $g \in G$ and every pair of vector fields X and Y holds

$$\langle DL_g(X), DL_g(Y) \rangle = \langle X, Y \rangle \, .$$

$\langle \cdot \, , \cdot \rangle$ is called right-invariant if the right translations are all isometries and it is called biinvariant if it is both left- and right-invariant.
Each scalar product $\langle \cdot \, , \cdot \rangle'$ on the Lie algebra $\mathfrak{g}$ of G gives rise to a left-invariant Riemannian metric $\langle \cdot \, , \cdot \rangle$ on G. Simply define

$$\langle \cdot \, , \cdot \rangle_g := (DL_{g^{-1}})_g^* \langle \cdot \, , \cdot \rangle'$$

for each $g \in G$. Thus, there is a natural one to one correspondence between the set of left-invariant Riemannian metrics on G and the set of scalar products on $\mathfrak{g}$.
Now let H be a Lie subgroup of G and consider the homogeneous space G/H. G acts on G/H by

$$(g_1, g_2 H) \mapsto L_{g_1}(g_2 H) := (g_1 g_2) H.$$

We say that a Riemannian metric on G/H is G-invariant, if G acts by isometries. We have the following theorem (compare proposition 3.16 in [8]):

Theorem B.3.1.1. *1. The set of G-invariant Riemannian metrics on G/H is naturally isomorphic to the set of scalar products on $\mathfrak{g}/\mathfrak{h}$, which are invariant under the action of $\mathrm{Ad_H}$ on $\mathfrak{g}/\mathfrak{h}$.*

2. If G acts effectively on G/H, then G/H admits a G-invariant Riemannian metric if and only if the closure of the group $\mathrm{Ad_H(G)}$ in $\mathrm{GL}(\mathfrak{g})$ is compact.

B.3.2 Simple, Semi Simple and Compact Lie algebras

Every real Lie algebra $\mathfrak{g}$ comes with a bilinear map $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, defined by

$$\kappa(\varepsilon, \delta) := \mathrm{tr}(\mathrm{ad}_\varepsilon \circ \mathrm{ad}_\delta).$$

κ is called the Killing form of $\mathfrak{g}$. It is obviously symmetric in ε and δ. But it has another property which proved to be quite useful:
For all $x, y, z \in \mathfrak{g}$ holds:

$$\kappa([x, y], z) = \kappa([y, z], x) = \kappa([z, x], y).$$

For example, this formula can be used to proof that the nullspace of κ, also known as the radical of $\mathfrak{g}$, always forms an ideal in $\mathfrak{g}$. We write

$$\mathrm{Rad}(\mathfrak{g}) := \{\varepsilon \in \mathfrak{g} : \kappa(\varepsilon, \delta) = 0 \text{ for all } \delta \in \mathfrak{g}\}.$$

Another important observation is the following one: If $\mathfrak{g}$ is the Lie algebra of the Lie group G, the Killing form is Ad-invariant. It turned out in the past that κ is closely related to the structure of $\mathfrak{g}$ in many important cases. These observations led to the following definitions:

Definition B.3.2.1. *Let $\mathfrak{g}$ be a real Lie algebra with Killing form κ. $\mathfrak{g}$ is called*

1. *semi-simple, if the Killing for is nondegenerate,*
2. *simple, if it is semi-simple and has no ideals other that $\{0\}$ and $\mathfrak{g}$,*
3. *compact (reductive), if the Killing form is negative semi-definite.*

Lemma B.3.2.2. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra with Killing form κ and I an ideal in $\mathfrak{g}$.*

1. *I is a Lie algebra with Killing form equal to $\kappa|_{I\times I}$.*
2. *If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then $I \cap \mathfrak{h}$ is an ideal in $\mathfrak{h}$.*
3. *If $\mathfrak{g}$ is semisimple, then $I^\perp$, the orthogonal complement of I w.r.t. the Killing form, is also an ideal in $\mathfrak{g}$. Moreover, I is semisimple and $\mathfrak{g}$ is the direct sum of I and $I^\perp$.*

Beweis.

1. Clear.
2. Clear.
3. The first and the second part follows using that the map $(x, y, z) \mapsto \kappa([x, y], z)$ is invariant under cyclic permutations of the arguments. Let $x \in I^{\perp}$. We have to show $[x, y] \in I^{\perp}$ for all $y \in \mathfrak{g}$. So let $y \in \mathfrak{g}$ be arbitrary and $z \in I$. Then

$$\kappa([x, y], z) = \kappa([y, z], x).$$

But z lies in I, which implies that $[y, z]$ lies in I as well. Thus, we get

$$\kappa([y, z], x) = 0,$$

since x is a member of $I^{\perp}$ and therefore $[x, y]$ lies in $I^{\perp}$ as claimed. The semisimplicity of I follows using 1.
We are left to show that the intersection $I \cap I^{\perp}$ is trivial.
We have $\dim I + \dim I^{\perp} = \dim \mathfrak{g}$, since κ is nondegenerate. If x and y lie in $I \cap I^{\perp}$ and $z \in \mathfrak{g}$ is arbitrary, then

$$\kappa([x, y], z) = \kappa([y, z], x) = 0$$

by arguments similar to the arguments from above. This tells us that $[x, y] = 0$, so $I \cap I^{\perp}$ is an abelian ideal in $\mathfrak{g}$. Now let $x \in I \cap I^{\perp}$. We are done, if we can show $\kappa(x, y) = 0$ for all $y \in \mathfrak{g}$.

   - Let $z \in I \cap I^{\perp}$. Then we get $[y, z] \in I \cap I^{\perp}$ and hence $[[y, z], x] = 0$.
   - Let z be κ-perpendicular to $I \cap I^{\perp}$, then $[[y, z], x] \in I \cap I^{\perp}$.

   Thus, we get $\kappa(x, y) = 0$ for all $y \in \mathfrak{g}$ and we are done.

□

Corollary B.3.2.3. *A semisimple real Lie algebra $\mathfrak{g}$ has no abelian ideals other than $\{0\}$.*

Beweis. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$ and κ the Killing form of $\mathfrak{g}$. We have $\mathrm{Rad}(\mathfrak{g}) = \{0\}$ by definition of semisimplicity. Hence, it is sufficient to show, that any abelian ideal of $\mathfrak{g}$ lies completely in $\mathrm{Rad}(\mathfrak{g})$.
Let I be an abelian ideal in $\mathfrak{g}$. By lemma B.3.2.2, we get that $I^{\perp}$, the orthogonal complement of I w.r.t. κ, is an ideal and that $\mathfrak{g} = I \oplus I^{\perp}$.
Let $\varepsilon \in I$. It is clear that

$$\kappa(\varepsilon, \delta) = 0$$

for all $\delta \in I$, since I is an abelian ideal and it is also clear that

$$\kappa(\varepsilon, \delta) = 0$$

for all $\delta \in I^{\perp}$. This shows $I \subseteq \mathrm{Rad}(\mathfrak{g})$ and the claim follows. □

Corollary B.3.2.4. *Simple Lie algebras are non-abelian.*

Beweis. Clear. □

Lemma B.3.2.2 has certain consequence concerning the structure of compact and semi-simple Lie algebras.

Corollary B.3.2.5. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. Then, $\mathfrak{g}$ is semi-simple if and only if $\mathfrak{g}$ decomposes as a direct sum of simple ideals.*

Beweis. It is obvious that direct sums of simple Lie algebras are semisimple. The other direction follows from lemma B.3.2.2. □

B.4 Curvature and Holonomy

In the following we consider an n-dimensional Riemannian manifold (M, g). As usual, the Levi-Civita connection of g will be denoted by ∇, the curvature tensor by R. Let $p \in M$. Any piecewise smooth loop $c : [0, 1] \to M$, which is based at p, gives rise to a linear map $P_c : T_pM \to T_pM$, obtained by parallel translating tangent vectors along c. Since ∇ is a metric connection P_c lies in $O(T_pM)$ for each given loop c. The holonomy group $\mathrm{Hol_p} = \mathrm{Hol_p(M, g)}$ is defined to be the group of all transformations P_c of T_pM. One can show that $\mathrm{Hol_p}$ is a Lie group which is usually a closed subgroup of $O(T_pM)$. We also have the restricted holonomy group Hol^0_p, which is the connected normal subgroup of $\mathrm{Hol_p}$ which comes from using only contractible loops. The restricted holonomy group is always compact, so it is a Lie subgroup of $O(T_pM)$ in any case. Now we state two elementary properties of holonomy (there are many of elementary properties of holonomy, but these two are the most important for our purpose):

- $\mathrm{Hol_p}$ is conjugate to $\mathrm{Hol_q}$ via parallel translation along any smooth curve connecting p and q.
- A tensor field on M is parallel if and only if it is invariant under the action of the restricted holonomy group.

Now we take a look at the Lie algebra $\mathfrak{hol}_p$ of the restricted holonomy group $\mathrm{Hol^0_p}$. It is clear that $\mathfrak{hol}_p$ is a sub algebra of $\mathfrak{so}(T_pM)$. Note that the curvature transformations $z \mapsto R(x, y)z$ are skew-symmetric transformations of T_pM as well. How are they related to holonomy? Let X and Y be smooth vector fields on M commuting in p and c_t the loop at p obtained by following the flow of X for time t, then the flow of Y for time t, then the flow of X for time -t and then the flow of y for time -t again. Define P_t to be the parallel transport along c_t. Then one can proof that

$$R(X(p), Y(p)) = \lim_{t \mapsto 0} \frac{1}{t}(P_t - id_{T_pM}),$$

showing that $R(X(p), Y(p))$ is actually an element of $\mathfrak{hol}_p$. But then we get that $P^{-1} \circ R(Px, Py) \circ P$ is also an element of $\mathfrak{hol}_p$, P the parallel transport along any curve emanating in p and $x, y \in T_pM$. The theorem of Ambrose and Singer [1] states that this is all, meaning that $\mathfrak{hol}_p$ is generated by elements of the form

$$P^{-1} \circ R(Px, Py) \circ P,$$

where x, y and P are as above. For a proof of this theorem we also refer to [18]. Thus, it is at least possible to recover the reduced holonomy from curvature, but that's a hard task in general. One can say much more if the underlying space provides more geometric structure, for example if it is a symmetric space.

B.5 The DeRham Decomposition Theorem

Let (M, g) be a Riemannian manifold and $\mathrm{E} \subseteq \mathrm{TM}$ a parallel subbundle. Since the parallel transport along any curve in M is isometric, we conclude that the orthogonal complement $\mathrm{E}^{\perp}$ in TM is parallel either. Thus, the whole tangent bundle decomposes orthogonally into parallel subbundles $\mathrm{E_i}$, each of which is irreducible, i.e. does not contain any further proper parallel subbundles,

$$\mathrm{TM} = \mathrm{E_1} \oplus ... \oplus \mathrm{E_r},$$

$r \in \mathbb{N}$. Such a decomposition of the tangent bundle is called holonomy-irreducible. The reason for this is that the $\mathrm{E_i}$ are the parallel translates of the holonomy-irreducible subspaces $V_1, ..., V_r \subseteq \mathrm{T_pM}$, for any $p \in M$.
Since the Levi-Civita connection is torsionfree, it follows, that each E_i an involutive distribution on the tangent bundle. Thanks to Frobenius' theorem each $\mathrm{E_i}$ yields a foliation of M. The leaves of all these foliations are totally geodesic. As each point $p \in M$ has a totally convex neighborhood, this implies that M is locally isometric to the product of (local) leaves $L_1, ..., L_r$ tangent to $\mathrm{E_1}, ..., \mathrm{E_r}$ respectively, meeting in some point and carrying the induced Riemannian metrics. Thus, we have shown the local version of DeRham's decomposition theorem.

Theorem B.5.0.6 (DeRham). *Let (M, g) be a Riemannian manifold and $\mathrm{TM} = \mathrm{E_1} \oplus ... \oplus \mathrm{E_r}$ a holonomy-irreducible splitting of the tangent bundle. Then M is locally isometric to a product $(U_1, g_1) \times, ..., \times (U_r, g_r)$ of simply connected Riemannian submanifolds manifolds (U_i, g_i) of (M, g) with $\mathrm{TU_i} = \mathrm{E_i}|_{\mathrm{U_i}}$ and $\mathrm{Hol_0(M, g)} = \mathrm{Hol(U_1, g_1)} \times ... \times \mathrm{Hol(U_r, g_r)}$. Moreover, if M is simply connected and complete, then the splitting is global.*

Beweis. See [23]

□

B.6 Killing Fields and Isometry Groups

Definition B.6.0.7. *A smooth vector field X on M is called a Killing field if $\mathcal{L}_X g = 0$ (where $\mathcal{L}_X g$ is the Lie derivative of g in w.r.t X), i.e. if the local flows of X are isometric. Equivalently, one could say that X is a Killing field if and only if the endomorphism field ∇X is skew-symmetric w.r.t. g. The set of Killing fields on (M, g) will be denoted by $\mathfrak{kill} = \mathfrak{kill}(M, g)$*

The equation $\mathcal{L}_X g = 0$ is linear in X, so the set of Killing fields is a vector space. Moreover, it is a Lie algebra, since $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ holds for any two vector fields X and Y, which implies that $[X, Y]$ is a Killing field, whenever X and Y are Killing fields. Further, one can show that the dimension of $\mathfrak{kill}$ is finite. The reason for this is, that, given $p \in M$, any Killing field X is uniquely determined by its values $X(p)$ and $\nabla X(p)$, which implies that the linear map $\mathfrak{kill} \to T_pM \times \mathfrak{so}(T_pM) : X \mapsto (X(p), \nabla X(p))$ is injective, so that the dimension of $\mathfrak{kill}$ will not be greater than $\frac{n(n+1)}{2}$.
A famous Theorem of Myers and Steenrod [19] says that the group $\mathrm{Isom} = \mathrm{Isom(M, g)}$ of isometries of any connected, complete Riemannian manifold (M, g) is a Lie group whose Lie algebra $\mathfrak{iso} = \mathfrak{iso}(M, g)$ is the space of complete Killing fields on (M, g). The exponential map is given by $\exp(X) = \Phi_1^X$, $\Phi^X : M \times \mathbb{R} \to M$ the flow of X. Let us take a closer look on the Lie algebra structure of $\mathfrak{iso}$. For $p \in M$, let $\mathrm{Isom_p} = \mathrm{Isom_p(M, g)} \subseteq \mathrm{Isom(M, g)}$ the isotropy group of (M, g) at p and $\mathfrak{iso}_p = \mathfrak{iso}_p(M, g)$ its Lie algebra. It is clear that the flow of any element $X \in \mathfrak{iso}_p$ fixes p, showing that $\mathfrak{iso}_p$ consists of complete Killing fields on M vanishing in p and that $\mathfrak{iso}_p$ is a subalgebra of $\mathfrak{so}(T_pM)$. Let $\mathfrak{t}_p := \{X \in \mathfrak{iso} : \nabla X(p) = 0\}$. Then we have:

- If $X, Y \in \mathfrak{iso}_p$, then $[X, Y] \in \mathfrak{iso}$ is identified with $-\left[\nabla X, \nabla Y\right](p) \in \mathfrak{so}(T_pM)$.
- If $X \in \mathfrak{t}_p$ and $Y \in \mathfrak{iso}$, then $[X, Y] = (\nabla Y)(X(p))$
- If $X, Y \in \mathfrak{t}_p$, then $[X, Y] \in \mathfrak{iso}_p$

Thus, we can write $\mathfrak{iso}$ as the direct sum $\mathfrak{t}_p \oplus \mathfrak{iso}_p$. Its Lie algebra structure is given by

- $[F, G] = -(FG - GF) \in \mathfrak{iso}_p$, if $F, G \in \mathfrak{iso}_p$.
- $[F, x] = -\left[x, F\right] = F(x)$ if $x \in \mathfrak{t}_p$ and $F \in \mathfrak{iso}_p$
- $[x, y] \in \mathfrak{iso}_p$ if $x, y \in \mathfrak{t}_p$

Lemma B.6.0.8. *Let X, Y and $Z \in \mathfrak{t}_p$, then $R(X, Y)Z = [Z, [X, Y]] \in \mathfrak{t}_p$.*

Beweis. □

Lemma B.6.0.9. *If K is a Killing field on M, then $R(K, X)Y = -\nabla^2_{X,Y} K$. In particular we have* $\mathrm{Ric(K)} = -\Delta\mathrm{K}$.

B.7 Symmetric Spaces

There is lots of theory concerned with symmetric spaces. The reader who wants to learn more about this beautiful subject is referred to the book of Helgason [16]: "Differential Geometry and Symmetric Spaces" Here, the symmetric spaces only appear as class of Riemannian manifolds representing a certain class of algebraic curvature operators, namely the algebraic symmetric curvature operators, which are examined in section 3.4.
The intention of this section is to provide the knowledge about symmetric spaces which is necessary to understand what we are doing in chapter 3.4.

There are several popular perspectives, from which symmetric spaces maybe defined. The geometric viewpoint, the analytic and the algebraic viewpoint. And there is fourth one, the curvature description of symmetric spaces, which will be treated in detail in section 3.4. Now let (M, g) be a Riemannian manifold.

B.7.1 The Geometric Viewpoint

Form the geometric point of view, we wish to call (M, g) symmetric, provided that for each point $p \in M$ there is a globally defined isometry $\varphi : M \to M$, which fixes p and whose differential at p is the reflection in the origin of the tangent space based at p, i.e.

$$\varphi(p) = p \text{ and } \mathrm{D}\varphi_{\mathrm{p}} = -\mathrm{id}_{\mathrm{T_pM}}.$$

From this definition follows immediately that any symmetric space M

1. has parallel curvature tensor,
2. is metrically complete and
3. a homogeneous space, since the group of isometries of M is always acting transitively on M.

B.7.2 The Analytic Viewpoint

From the analytic viewpoint we wish to call (M, g) symmetric if it is simply connected and has parallel curvature tensor. One can show with little work that simply connected Riemannian manifolds with parallel curvature tensors are actually symmetric spaces in the geometric sense from above. If we drop the simply connectedness, then we get only the locally symmetric spaces, i.e. the isometries φ from the definition of a symmetric space are only defined locally.

B.7.3 The Algebraic Viewpoint

Now let G be a connected Lie group and H a closed subgroup. Following Helgason [16], the pair (G,H) is called a symmetric pair, if there exists an involutive automorphism σ of G, such that $(H_\sigma)_0 \subseteq H \subseteq H_\sigma$, where H_σ is the set of fixed points of σ and $(H_\sigma)_0$ is the identity component of H_σ. If $\mathrm{Ad_G(H)} \subseteq \mathrm{Ad_G(G)}$ is compact, we call the symmetric pair (G, H) a Riemannian symmetric pair. One can show the following: If we are given a Riemannian symmetric pair (G, H) together with the involutive automorphism σ from the definition, then any $G-$invariant Riemannian metric g the quotient space $M = G/H$ turns M into a symmetric space in the geometric sense. The proof of this theorem can be found [16].
On the other hand, a symmetric space (M, g) gives rise to a symmetric pair (G, H). Take $G = \mathrm{Isom(M, g)}$ and $H = \mathfrak{iso}_p(M, g)$, $p \in M$. This gives $M = G/H$. We are left finding the involutive automorphism σ from the definition of a symmetric pair. We know that there exists an isometry $\varphi \in \mathfrak{iso}_p(M, g)$, whose differential in p is simply the reflection in the origin of the tangent space $\mathrm{T_pM}$. We use φ to define σ by $\sigma(g) := \varphi \circ g \circ \varphi$ for all $g \in G$. Clearly, φ is an automorphism of G. To prove $(H_\sigma)_0 \subseteq H \subseteq H_\sigma$ requires a little more work than we wish to do here, but the proof is not too hard and may also be found in [16]. Now we take a look at symmetric spaces at the infinitesimal level of the Lie algebras $\mathfrak{h} \leq \mathfrak{g}$ belonging to the Lie groups H and G, respectively: The differential $s := \mathrm{D}\sigma_\mathrm{e}$ of σ $e \in G$ is an involutive automorphism of $\mathfrak{g}$. Therefore, the tangent space $\mathfrak{g} \cong \mathrm{T_eG}$ decomposes orthogonally as the direct sum of the ± 1-eigenspaces of s, $\mathfrak{g} = \mathrm{E_{-1}(s)} \oplus \mathrm{E_1(s)}$. With little extra work, one can show that $+1$-eigenspace of s coincides with $\mathfrak{h}$. Thus, we have $\mathfrak{g} = \mathfrak{h} \oplus \mathrm{E_{-1}(s)}$. Further, it follows that the center $\mathfrak{z}$ of $\mathfrak{g}$ and $\mathfrak{h}$ intersect trivially.
In other words, the pair $(\mathfrak{g}, \mathfrak{h})$ is an effective orthogonal symmetric pair of Lie algebras.
Now, one could ask whether the converse is also true. Given an effective orthogonal symmetric pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras together with an involutive Lie algebra homomorphism $s : \mathfrak{g} \to \mathfrak{g}$, is there a Riemannian symmetric pair (G, H) together with corresponding involutive Lie group homomorphism $\sigma : G \to G$, such that the Lie algebras Lie(G) and Lie(H) of G and H agree with $\mathfrak{g}$ and $\mathfrak{h}$ and the differential $\mathrm{D}\sigma_\mathrm{e}$ of σ at the identity of G equals s,

$$\mathfrak{g} = \mathrm{Lie(G)}, \mathfrak{h} = \mathrm{Lie(H)} \text{ and } \mathrm{D}\sigma_\mathrm{e} = \mathrm{s}?$$

The answer is yes. The proofs can also be found in [16]. We will use this fact in chapter 3.4, where we construct symmetric spaces from algebraic algebraic curvature operators sharing a special property.
The following statements about curvature tensors of symmetric spaces are important in this context:

Proposition B.7.3.1. *Suppose that R is the curvature tensor of a symmetric space $M = G/H$ and let $p = [H] \in M$. Then we have $R_p(X,Y)Z = -[[X,Y],Z](p)$ for all vector fields X, Y, Z on M with $X(p), Y(p), Z(p) \in \mathrm{E}_{-1}(\mathrm{s})$.*

Beweis. See [16] or [8], for example. □

Theorem B.7.3.2. *Let (M, g) be a symmetric space and $p \in M$. Then we have*

$$\mathfrak{hol}_p \subseteq \mathfrak{iso}_p.$$

Beweis. The proof can be found in [23].

□

Corollary B.7.3.3. *Let (M, g) be a symmetric space and $\mathcal{R}$ its curvature operator. Then we have*

$$[\mathcal{R}_p, \mathrm{id} \wedge \mathrm{h}] = 0$$

for all $p \in M$ and $h \in \mathfrak{hol}_p$.

Beweis. Let $p \in M$ and pick some $h \in \mathfrak{hol}_p$. Following theorem B.7.3.2, there exists a smooth 1-parameter family φ_t of isometries of M with $\varphi_t(p) = p$ for all t and $\frac{d}{dt}\big|_{t=0} \varphi_t = h$. Thus, we get

$$[\mathcal{R}_p, \mathrm{id} \wedge \mathrm{h}] = \frac{1}{2} \frac{d}{dt}\bigg|_{t=0} \varphi_t^{-1} \wedge \varphi_t^{-1} \circ \mathcal{R}_p \circ \varphi_t \wedge \varphi_t = 0$$

□

B.8 Parabolic Partial Differential Equations on Vector Bundles and Tensor Maximum Principles

Let M be smooth manifold, connected and possibly with boundary, $T > 0$, $(g_t)_{t\in[0,T)}$ a smooth family of Riemannian metrics on M and $\pi : \mathrm{E} \to \mathrm{M} \times [0, \mathrm{T})$ a smooth vector bundle, equipped with bundle metric $\langle \cdot\,, \cdot\rangle$ and a metric connection D. We wish to study parabolic partial differential equation of the form

$$\text{(PDE)} \quad \mathrm{D}_{\frac{\partial}{\partial \mathrm{t}}} \sigma = \Delta\sigma + \Phi(\sigma, \mathrm{t}),$$

where $\sigma \in \Gamma(E)$ is a section, Δ is the Laplacian in spatial direction w.r.t. the bundle metrics $\langle \cdot\,, \cdot\rangle$ and $g = (g_t)$ on E and $\pi_1^*\mathrm{TM}$ ($\pi_1 : M \times [0, T) \to M$ the projection onto the first factor), respectively, and Φ is a time dependent vertical vector field on E, i.e. a time dependent vector field on E which is tangent to the fibers of E.
It turns out that the qualitative properties of solutions of (PDE) are strongly

influenced by the boundary data and the properties of solutions of the (non-autonomous) ordinary differential equation

$$\text{(ODE)} \quad \dot{\sigma} = \left(\phi + \frac{\partial}{\partial \mathrm{t}}\right) \circ \sigma.$$

In order to explain the connections between solutions of (PDE), (ODE) and the boundary data, we have to introduce some further notation and make some additional assumptions:
Throughout this section let

$$\partial_{par} E := \partial(M \times [0,T)) = M \cup \partial M \times [0,T)$$

the parabolic boundary of E. Further, let

$$C = (C_{p,t})_{(p,t)\in M\times[0,T)} \subseteq E$$

a smooth subbundle of E with boundary which is closed in E, parallel in spatial direction and fiberwise convex.
Suppose in addition that the flow of $\Phi + \frac{\partial}{\partial t}$ preserves C.
Under these assumptions the following theorems hold:

Theorem B.8.0.4 (Weak Tensor Maximum Principle)**.** *Let M be compact. If σ solves the (PDE) and takes values in C on the parabolic boundary of E then σ takes values in C everywhere on $M \times [0,T)$.*

Beweis. See Hamilton's paper on 4-manifolds with nonnegative curvature operator [11]. □

Theorem B.8.0.5 (Tensor Maximum Principle)**.** *Suppose that C is also parallel in time direction. Let σ be a solution of (PDE) and assume that σ takes values in ∂C in p at time $t > o$. Then σ takes values in ∂C on $M \times [0,t)$.*

Beweis. See Hamilton's paper on 4-manifolds with nonnegative curvature operator [11]. □

Anhang C

Ricci Flow

This section gives a rough introduction to Ricci flow and the methods being used within this subject with special focus on the tensor maximum principles and the evolution equations of geometric quantities. At the end we present some results of this subject.

Let M be smooth manifold of finite dimension and $(g_t)_{t\in[0,T)}$, $T > 0$ a smooth family of Riemannian metrics on M. (g_t) is called a Ricci flow on M if

$$\frac{d}{dt}g_t = -2\mathrm{Ric(g_t)}$$

holds on $[0, T)$.
There is another version of the Ricci flow which is equivalent to the Ricci flow modulo rescaling of space and time: The Volume normalized Ricci flow. Its solutions satisfy the equation

$$\frac{d}{dt}g_t = -2\mathrm{Ric(g_t)} + \frac{2}{\mathrm{n}}\mathrm{r(g_t)g_t}$$

on n-dimensional manifolds. Here

$$\mathrm{r(g_t)} := \frac{1}{\mathrm{vol(M, g_t)}}\int_{\mathrm{M}} \mathrm{scal(g_t)dvol(g_t)}$$

denotes the average scalar curvature of the Riemannian manifold (M, g_t).

C.1 Ricci Flow Basics

One property of the Ricci flow which is of fundamental importance is the invariance of set of solutions under the action of the full group of diffeomeorphisms of a given manifold. This means that, whenever (g_t) is a solution of the Ricci flow on a given manifold M and $\Phi : M \to M$ is a diffeomorphism, so is the family $(\Phi^* g_t)$ of pullbacks under under Φ. This is an immediate

consequence of the fact that the curvature tensor, and therefore also the Ricci tensor, behaves naturally under pullbacks via diffeomorphisms.
Thus, the Ricci flow descends to the moduli space $\mathcal{M}/\mathrm{Diff(M)}$, and therefore has a purely geometric meaning. Another important property is the following: the Ricci flow preserves products. This means that, if (g_t) is a Ricci flow on a manifold M and (h_t) is a Ricci flow on N than the product metric $(g_t \times h_t)$ is a Ricci flow on $M \times N$. This is not very surprising, since the Ricci tensor of a Riemannian product is simply the direct sum of the Ricci tensors of the factors in play, $\mathrm{Ric(g \times h) = Ric(g) \times Ric(h)}$.
The third fundamental property is: isometries remain isometries under the Ricci flow. This means that any isometry $\Phi : (M, g_{t_0}) \to (M, g_{t_0})$ will be an isometry of (M, g_t) for all times $t \geq t_0$. This is a consequence of the naturality of the Ricci tensor again together with the fact that solutions of the Ricci flow are uniquely determined by the initial metric. This third property guaranties that the Ricci flow preserves certain classes of Riemannian manifolds, for example, the class of symmetric spaces and the class of Riemannian homogeneous manifolds.

C.2 Evolution Of Geometric Quantities

It is clear that the whole geometry of M is varying with g. As it takes a long time to compute the evolution equations of the geometric quantities in play, we only present the results here. For the proofs, we refer to the diploma thesis of the author, where all these calculations have been done in detail and in a coordinate free manner.

Proposition C.2.0.6. *Under the Ricci flow, the volume vol(M, g_t) changes like* $\frac{d}{dt}\mathrm{vol} = -\mathrm{rvol}$

□

Another easy example is given by the evolution of the gradient operator.

Proposition C.2.0.7. *The gradient operator ∇ evolves like*

$$\frac{d}{dt}\nabla = 2\mathrm{Ric} \circ \nabla$$

□

Proposition C.2.0.8. *Under the Ricci flow, the Levi-Civita connection ∇ evolves like*

$$\frac{d}{dt}\nabla = \mathrm{divR} - \nabla\mathrm{Ric}$$

□

In order to describe the evolution of the family of associated Riemannian curvature operators in a compact and coordinate free way and apply the maximum principle to it, it is useful to consider the Ricci flow as an inner product on the pullback π^*TM of the tangent bundle of M, $\pi : M \times [0,T)$ the projection onto the first factor. To do so, simply define $g_{(p,t)} := (g_t)_p$ and you are done. Now let us see, how the associated family of Levi-Civita connections appears in this new setting. Taking covariant derivatives in spatial directions shall give the same results as before, so we define $\nabla^\pi_X Y(p,t) := \nabla^t_{X(p,t)} Y(p,t)$ for all smooth sections of π^*TM. But what about the covariant derivatives in time direction? The first idea is to define $\nabla^\pi_{\frac{\partial}{\partial t}} X := \frac{d}{dt} X$. But this will not give a metric connection. We repair this by subtracting the term Ric(X), i.e. we define

$$\nabla^\pi_{\frac{\partial}{\partial t}} X := \frac{d}{dt} X - \mathrm{Ric(X)}$$

for smooth sections X of π^*TM. Now we can write

$$\nabla = \nabla^\pi - dt \otimes \mathrm{Ric}.$$

Theorem C.2.0.9. *Under the Ricci flow, the curvature operator evolves like*

$$\nabla_{\frac{\partial}{\partial t}} \mathcal{R} = \Delta \mathcal{R} + \Phi(\mathcal{R}),$$

where Δ is the time dependent Laplacian with respect to the metric g_t and Φ is the Ricci vector field.

□

Having computed the evolution equation of the curvature operator it is easy to obtain the evolution equation of the Ricci and scalar curvature

Corollary C.2.0.10.

- $\nabla_{\frac{\partial}{\partial t}} \mathrm{Ric} = \Delta \mathrm{Ric} + 2 \sum_\mathrm{i} \mathrm{R}(\cdot, \mathrm{e_i})\mathrm{Ric}(\mathrm{e_i})$, *$e_i$ an arbitrarily chosen orthonormal frame.*
- $\frac{d}{dt}\mathrm{scal} = \Delta \mathrm{scal} + 2 \left\| \mathrm{Ric} \right\|^2.$

□

Literaturverzeichnis

[1] W. Ambrose and I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. **79** (1953), 428–443.

[2] M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d'une variété riemanniene*, Lecture Notes in Math. **194** (1971).

[3] A. L. Besse, *Einstein Manifolds*, Berlin-Heidelberg: Springer-Verlag, 1978.

[4] ______, *Géométrie riemannienne en dimension 4*, Cedic-Fernand Nathan, 1981.

[5] C. Böhm and B. Wilking, *Manifolds with positive curvature operator are space forms*, Annals of Math. **2** (2008), no. 3, 1079–1097. http://www.arxiv.org/math.DG/0606187.

[6] S. Brendle and R. Schoen, *Manifolds with $\frac{1}{4}$-pinched curvature are space forms.*, Journal of the AMS **22** (2009), 287.

[7] É Cartan, *Sur une classe remarquable d'espace de Riemann*, Bull. Soc. Math. France **54** (1926), 214–264.

[8] J. Cheeger and D. G. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland Publishing Company, 1975.

[9] M. P. do Carmo, *Riemannian Geometry*, Boston: Birkhäuser, 1993.

[10] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.

[11] ______, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153–179.

[12] ______, *The Harnack Estimate for the Ricci flow*, J. Differential Geom. **37** (1993), 225–243.

[13] ______, *A compactness property for solutions of the Ricci flow*, Amer. J. Math. **117** (1995), 545–572.

[14] ______, *Formation of singularities in the Ricci flow*, Surveys in Differential Geom. **2** (1995), 7–136.

[15] ______, *Non-singular solutions of the Ricci flow on three-manifolds*, Comm. Anal. Geom. **7** (1999), no. 4, 695–729.

[16] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, New York-London: Academic Press, 1962.

[17] G. Huisken, *Ricci deformation of the metric on a Riemannian manifold*, J. Differential Geom. **17** (1985), 47–62.

[18] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I,II, New York: Wiley-Interscience, 1963.

[19] S. B. Myers and N. Steenrod, *The group of isometries of a Riemannian manifold*, Annals of Math. **40** (1939).

[20] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, 2002. http://www.arxiv.org/math.DG/0211159.

[21] ______, *Finite extiction time for the solutions to the Ricci flow on certain three-manifolds*, 2003. http://www.arxiv.org/math.DG/0307245.

[22] ______, *Ricci flow with surgery on three-manifolds*, 2003. http://www.arxiv.org/math.DG/0303109.

[23] P. Petersen, *Riemannian Geometry*, New York: Springer-Verlag, 1998.

[24] F. Schmidt, *On the longtime behavior of solutions to the Ricci flow on compact three-manifolds,* Diploma thesis, 2006, München http://www.mathematik.uni-tuebingen.de/∼loose/Diplomarbeiten/Dipl.Schmidt.pdf.