Conformally Immersed Tori in the 4-Sphere and the Willmore Energy

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Dissertation der Fakultät
für Mathematik und Physik
der Eberhard-Karls-Universität Tübingen
zur Erlangung des Grades eines Doktors der
Naturwissenschaften

2010

.

Tag der mündlichen Prüfung: 14. Juli 2010

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Zusammenfassung

Wir studieren konforme Immersionen $f : T^2 \to S^4$ von Tori T^2 in die konforme 4-Sphäre, welche wir als quaternionisch projektive Gerade $S^4 = \mathbb{HP}^1$ auffassen, mit Hilfe ihrer Spektralkurven. Ein immersierter Torus $f : T^2 \to S^4$ ist dasselbe wie ein holomorphes Linienunterbündel des trivialen $\mathbb{H}^2$-Bündels V über T^2. Konformität der Immersion wird durch die Existenz einer (quaternionisch) holomorphen Struktur D, also eines elliptischen Differentialoperators erster Ordnung, auf Schnitten des Linienbündels V/L beschrieben. Die Spektralkurve des immersierten Torus f ist eine analytische Beschreibung der Menge der Darbouxtransformationen der Immersion f. Falls f triviales Normalbündel hat, ist die Spektralkurve eine 1-dimensionale analytische Varietät. Diese wird durch das Verschwinden der Quillendeterminante einer Familie elliptischer erster Ordnungs Operatoren D_ω, welche über dem Raum der harmonischen 1-Formen $\omega \in \text{Harm}(T^2, \mathbb{C})$ parametriert ist, beschrieben. Wir drücken das Willmorefunktional $W = \int H^2$, das Quadratmittel der mittlern Krümmung der Fläche f, als Krümmungs 2-Form des Quillenzusammenhanges aus. Indem wir die von Feldman, Knörrer und Trubowitz für skalare nicht-lineare Schrödinger Operatoren entwickelten Techniken auf Operatoren D_ω vom Dirac Typ mit Potential verallgemeinern, zeigen wir, daß die Spektralkurve eine Riemmansche Fläche von unendlichem Geschlecht ist, und bestimmen ihr asymptotisches Verhalten.

Conformally Immersed Tori in the 4-Sphere and the Willmore Energy

Abstract

We study conformally immersed tori $f : T^2 \to S^4$ into the 4-sphere, where we view the conformal 4-sphere $S^4 = \mathbb{HP}^1$ as the quaternionic projective line, in terms of their spectral curves. An immersed torus $f : T^2 \to S^4$ gives rise to a holomorphic line subbundle $L \subset V$ of the trivial $\mathbb{H}^2$-bundle V over T^2. Conformality of the immersion is expressed by the existence of a (quaternionic) holomorphic structure D, that is, a first order elliptic operator on sections of the quotient line bundle V/L. The spectral curve of the immersed torus f is an analytic description of the space of Darboux transforms of the immersion f. If f has trivial normal bundle the spectrum is a 1-dimensional analytic variety characterized by the vanishing of the Quillen determinant of a family D_ω of first order elliptic operators associated to D, parameterized over the space of harmonic 1-forms $\omega \in \mathrm{Harm}(T^2, \mathbb{C})$. We express the Willmore energy $W = \int H^2$, the average of the mean curvature squared of the immersion f, as the curvature 2-form of the Quillen connection. Extending techniques developed by Feldman, Knörrer and Trubowitz for scalar non-linear Schrödinger operators to our Dirac type operators D_ω with potential, we show that the spectral curve is a Riemman surface of infinite genus and determine its asymptotic behavior.

Keywords: quaternionic line bundle; conformal immersed tori; Willmore energy; spectral curve; Riemman surfaces of infinite genus.

Introduction

The primary objects of interest in low dimensional differential geometry are surfaces with "good" properties in 3-space (and more generally n-space). For example, minimal surfaces are critical points for the area functional and constant mean curvature surfaces are critical points for the area functional under constrained volume. In conformal geometry, the analog for the area functional is the Willmore energy $W = \int H^2$ given by averaging the mean curvature square over the surface. Willmore surfaces, the critical points for the Willmore energy, include minimal surfaces as special cases.

An important property of the Willmore energy is its invariance under conformal changes of metric in the ambient space. Therefore, it is natural to study conformal maps of surfaces. All the surfaces mentioned above are examples of conformal maps into 3-space. The case of 2-tori, compact surfaces of genus 1, will play a prominent role due to their abelian fundamental groups.

Recently some progress has been made in the description of the space of conformal maps from a torus T^2 into the 3 and 4-sphere. In particular, Martin Schmidt [20] attempts to use this approach to solve the Willmore conjecture: the infimum of the Willmore energy over all immersed 2-tori is $2\pi^2$ and is attained at the Clifford torus $S^1 \times S^1$ in the 3-sphere. In [20] the variational problem of the Willmore functional is rephrased on the space of complex Fermi curves of the generalized Weierstrass potentials corresponding to the conformal immersions. A Fermi curve, or *spectral curve*, is the subset of all complex characters of the fundamental group of T^2 such that the Dirac operator with the generalized Weierstrass potential has a non-trivial kernel on the space of sections on T^2 quasiperiodic with respect to those characters. If the spectral curve has finite genus, algebro-geometric techniques can be used to reconstruct the original immersion [1]. Furthermore, the Willmore energy can be written as a residue of a certain meromorphic differential on the spectral curve. Important examples of conformal tori of finite spectral genus include constant mean curvature [18, 12] and constrained Willmore tori [20, 6].

Some of the above ingredients were already present in Hitchin's work [12] on harmonic 2-tori in the 3-sphere. He obtained a spectral curve from the holonomy representation of a holomorphic family of flat $SL(2,\mathbb{C})$-connections with respect to a chosen base point on the torus. Due to the ellipticity of the harmonic map equation this curve always has finite genus and the energy of the harmonic map can be expressed in terms of the residue of a certain meromorphic differential on the spectral curve. Hitchin also points out that one can view the energy density as the curvature form on the determinant line bundle of a holomorphic family of elliptic operators. Almost being a side remark in Hitchin's paper this point of view is rather central in the theory of conformally immersed surfaces.

In the case of a conformally immersed torus $f : T^2 \to S^4$ in the 4-sphere the spectral curve is given by the divisor of the Quillen determinant [22, 3] of the induced holomorphic family of elliptic operators D_ω parameterized over the space of complex valued harmonic 1-forms $\omega \in \mathrm{Harm}(T^2, \mathbb{C})$, acting on a rank 2 complex bundle. In fact, a section in the kernel of D_ω, i.e. a holomorphic section with monodromy $h = e^{\int \omega}$, gives rise to a Darboux transform of the immersion f, which preserves the Willmore energy if the normal bundle is of zero degree. In this sense the spectral curve characterizes the space of all Darboux transforms of a conformal immersed torus.

In contrast to the case of Hitchen [12], the spectral curve of a conformal immersion from a 2-torus to the 4-sphere generally has infinite genus. An heuristic argument shows that the general spectral curves should be asymptotic to the spectral curve of the operator $\bar\partial$ (the vacuum spectrum) for large ω. In this thesis we will provide a detailed analysis of this statement. Rather than using the description of the spectral curve for a quaternonic holomorphic line bundle by ad hoc methods as in [8], we introduce the methods developed by Feldman, Knörrer and Trubowitz [11], which will provide a more detailed picture and at the same time will give sharp quantitative estimates of the handle asymptotics. The last part of my thesis is devoted to extend the methods of [11], where scalar Schrödinger type operators are considered, to the case of Dirac type operators with potentials in higher rank.

This thesis is organized in the following way: Chapter 1 contains the necessary background material about conformal geometry of surfaces in S^4 via quaternions from [1] and [10]. It is not intended to be read as a whole, but should serve mostly as a reference.

In Chapter 2 a conformal map $f : T^2 \to S^4$ into the 4-sphere (or equivalently, the quaternionic projective line $\mathbb{HP}^1$) is identified with a quaternionic line subbundle L of the trivial bundle $V = M \times \mathbb{H}^2$. The quotient bundle V/L carries an induced complex structure and a quaternionic holomorphic structure

$$D = \bar\partial + Q : \Gamma(V/L) \to \Gamma(\bar K V/L)$$

whose existence express the conformality of the immersion. Geometrically, the complex anti-linear part $Q \in \Gamma(\bar K\,\mathrm{End}(V/L))$ is related to the trace-free second fundamental form of the immersion f. The L^2-norm

$$\mathcal{W}(f) = 2\int < Q \wedge *Q >$$

is the Willmore energy of f. We describe the spectral curve Σ_f of the conformal immersion $f : T^2 \to S^4$ by the holomorphic family of elliptic operators

$$D_\omega = e^{-\int \omega} \circ\ D \circ e^{\int \omega} : \Gamma(V/L) \to \Gamma(\bar K V/L) \tag{0.0.1}$$

parameterized over the space of complex valued harmonic 1-forms $\mathrm{Harm}(T^2, \mathbb{C})$. Since any representation $h : \pi_1(T^2) \to \mathbb{C}\backslash\{0\}$ is given by $h = e^{\int \omega}$ the kernel of D_ω consists of the holomorphic sections on the universal cover $\mathbb{C}$ with monodromy $h = e^{\int \omega}$. The spectral curve Σ_f is the subset of $\mathrm{Harm}(T^2, \mathbb{C})/\Gamma^*$, with Γ^* the lattice of integer valued harmonic forms, for which the operator D_ω has a non-trivial kernel. Geometrically, these sections of V/L with monodromy correspond to Darboux transforms of f. A Darboux transform $f^\sharp$ of f is a conformal map $f^\sharp : T^2 \to S^4$ such that there is a 2-sphere congruence S touching f and left-touching $f^\sharp$ at corresponding points. The Darboux transformation so defined preserves the Willmore energy and the spectral curve of f in case f has zero degree normal bundle.

In Chapter 3, we develop the theory of the determinant line bundle for the family D_ω. We fix a uniformizing coordinate z on the torus $T^2 = \mathbb{C}/\Gamma$ to trivialize the bundle V/L and coordinates $(a, b) \in \mathbb{C}^2$ for $\omega = adz + bd\bar{z} \in \mathrm{Harm}(T^2, \mathbb{C})$. Then the family of operators D_ω takes the form

$$D_{a,b} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}} + b & -\bar{q} \\ q & \frac{\partial}{\partial z} + a \end{pmatrix},$$

where q is a complex valued function on T^2. We adapt the methods of Bismut and Freed [3], which are an extension of Quillen's construction of determinants of Cauchy-Riemann operators, to the operators $D_{a,b}$. We construct the determinant line bundle

$$\mathcal{L} = \lambda(\mathrm{Ker}\, D_{a,b})^* \otimes \lambda(\mathrm{Coker}\, D_{a,b})$$

associated to the family $D_{a,b}$ and calculate the curvature form of the canonical connection ${}^1\nabla$ on $\mathcal{L}$. As in Quillen's case of Cauchy-Riemman operators this turns out to be the Kähler form on $\mathrm{Harm}(T^2, \mathbb{C})$. In order for the Willmore energy of f to appear as curvature, we have to extend the family $D_{a,b}$ by additionally deforming the potential part. We then trivialize the determinant line bundle and define a holomorphic determinant function whose zero locus, a 1-dimensional analytic subvariety, is the spectral curve Σ_f.

In Chapter 4, the vacuum spectrum Σ_0, corresponding to the operator $\bar{\partial}$, is described as a real translate of $\exp(H^0(K)) \cup \exp(\overline{H^0(K)}) \subset \mathrm{Hom}(\Gamma, \mathbb{C})/\Gamma^*$ with double points along the lattice of real representations $\mathrm{Hom}(\Gamma, \mathbb{R}_*)$. For an absolutely continuous potential $q \in W^{1,1}(T^2)$, we get a detailed asymptotic description of the spectral curve as an infinite genus Riemann surface [11]. The main results show how the spectral curve Σ_f approaches the vacuum Σ_0 when a and b tend to infinity. For each sufficiently large $c \in \Gamma^*$ there is a handle on the spectral curve biholomorphic to the model handle

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = t_c \quad \text{and} \quad |z_1|,\ |z_2| \leq \epsilon\}$$

with ϵ sufficiently small and an estimate for the handle size $|t_c| \leq \frac{const}{|c|^2}$. If $t_c = 0$ for some c then the handle degenerates to a double point. If $t_c = 0$ for all but finitely many c then the spectral curve has finite genus and q is called a "finite gap" potential.

Instead of the approach used in [8], we extend the methods developed by Feldman, Knörrer and Trubowitz [11] to a family of Dirac type operators to get a detailed description of the asymptotics of the spectral curve. We only require the smoothness of the potential q as an absolutely continuous function. The precise estimates of the asymptotics make it possible to construct theta functions on the infinite dimensional Jacobian variety corresponding to the spectral curve. These theta functions may be used to describe the original conformal immersion f.

One may wonder whether the above theory can be extended to a conformally immersed compact Riemann surface $f : M \to S^4$ of higher genus $g > 1$. We can define the spectrum Σ_f of the immersion to be the set of abelian representations $h : H_1(M) \to \mathbb{C}_*$ realized by monodromies of non-trivial holomorphic sections of the line bundle $L \subset V$ rather than the quotient bundle V/L. The reason for this is that V/L never has spin bundle degree $g-1$ for $g > 1$ which we need for the construction of the determinant line bundle. This requires to break the conformal symmetry and choose a point at infinity on S^4, i.e. regard $f : M \to \mathbb{R}^4$ as a conformal immersion into Euclidean space $\mathbb{R}^4$. As before, $\Sigma_f \subset \mathrm{Harm}(M, \mathbb{C})/\Gamma^*$, where Γ^* is the rank $2g$ lattice of integer period harmonic forms. We describe the spectrum by the holomorphic family of elliptic operators $D_\omega : \Gamma(L) \to \Gamma(\bar{K}L)$ as in (0.0.1) parameterized over harmonic 1-forms $\mathrm{Harm}(M, \mathbb{C})$: ω lies in the spectrum if and only if D_ω has a non-trivial kernel. In case L has the spin bundle degree $g-1$, which always holds for immersions $f : M \to S^3$ into the round 3-sphere, the Riemann-Roch theorem implies that the family D_ω has index zero. Then we can construct the Quillen determinant line bundle associated to D_ω so that Σ_f is the zero locus of the holomorphic determinant $\det D_\omega$ and thus is an analytic hypersurface in the complex $2g$-dimensional cylinder $\mathrm{Harm}(M, \mathbb{C})/\Gamma^*$. Likewise, we can express the Willmore energy as the curvature of the Quillen connection on the determinant line bundle when we also deform the potential part of D_ω.

The vacuum spectral curve Σ_0 corresponds to the family

$$\bar{\partial}_\omega = \begin{pmatrix} \bar{\partial} + \omega'' & 0 \\ 0 & \partial + \omega' \end{pmatrix}.$$

Here $\bar{\partial} + \omega''$ ranges over the Picard group $\mathrm{Pic}_{g-1}(M)$ of holomorphic structures of line bundles of degree $g-1$ (see Lemma 10.15 in [5]) and has non-trivial kernel along the theta divisor $\Theta \subset \overline{H^0(K)}$ (Theorem 3.16 in [16]). Hence the vacuum spectrum Σ_0 is the hypersurface

$$\exp(H^0(K) \times \Theta \cup \overline{\Theta} \times \overline{H^0(K)}).$$

However, since the degree of L is non-zero for $g > 1$, it is less clear how to adapt the methods in chapter 4 to this case. For one, we can not trivialize the bundles L and KL. This could be overcome by working with respect to a fixed reference line bundle. What is more difficult to understand though is how to extend the family D_ω away from M, or rather its Abel image in $\mathrm{Pic}_{g-1}(M)$, to all of $\mathrm{Pic}_{g-1}(M) \cong \mathbb{C}^g/\Gamma$. This seems necessary to extend the Floquet theory of periodic operators to the higher genus case.

Chapter 1

Preliminaries

In this chapter we give a rapid tour though some of the basic ideas of quaternionic holomorphic geometry. The theory developed in [1] and [10] is the cornerstone of the results presented in this work.

Section 1.1 introduces linear algebra over the quaternions and collects the basic definitions and some fundamental facts of quaternionic holomorphic bundles over Riemann surfaces. Section 1.2 identifies the quaternionic projective line as a model for the conformal 4-sphere and describes the quaternionic holomorphic line bundles related to immersions of Riemann surfaces in Möbius geometry. Section 1.3 discusses the mean curvature sphere congruence condition of a conformal immersion.

1.1 Quaternionic holomorphic bundles

1.1.1 Quaternions

The Hamiltonian quaternions, denoted by $\mathbb{H}$, are the unitary $\mathbb{R}$-algebra generated by symbols $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ with the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Obviously $\mathbb{H}$ is a skew-field and a 4-dimensional division algebra over reals that is associative but not commutative.

The conjugate of a quaternion

$$a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad a_l \in \mathbb{R} \quad \text{for} \quad l = 0, 1, 2, 3$$

is

$$\bar{a} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}.$$

The real and imaginary part of a quaternion a is defined by

$$\operatorname{Re} a := a_0, \quad \operatorname{Im} a = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

In contrast with the complex numbers, $\operatorname{Im} a$ is not a real number and the conjugation obeys

$$\bar{a}\bar{b} = \bar{b}\bar{a} \quad \text{for} \quad a, b \in \mathbb{H}.$$

The subset of real quaternions $\operatorname{Re}\mathbb{H}$ consists of all real multiples of 1 and therefore we have a natural embedding of the real numbers $\mathbb{R} \subset \mathbb{H}$. The subspace of purely imaginary quaternions $\operatorname{Im}\mathbb{H}$ is identified with $\mathbb{R}^3$, the 3-dimensional real vector space with basis $\mathbf{i}, \ \mathbf{j}, \ \mathbf{k}$.

Since the quaternions $\mathbf{i}, \ \mathbf{j}, \ \mathbf{k}$ equally qualify for the complex imaginary unit the embedding of the complex numbers $\mathbb{C}$ is less canonical. However, we shall from now on usually use the subfield $\mathbb{C} \subset \mathbb{H}$ generated by $1, \ \mathbf{i}$.

The Euclidean inner product on $\mathbb{R}^4$ is

$$< a, b >_{\mathbb{R}} = \operatorname{Re}(\bar{a}b) = \operatorname{Re}(a\bar{b}) = \frac{1}{2}(\bar{a}b + b\bar{a}).$$

The norm of a quaternion is defined as

$$|a| := \sqrt{< a, a >_{\mathbb{R}}} = \sqrt{a\bar{a}}.$$

Note that $a^2 = -1$ if and only if $|a| = 1$ and $a = \operatorname{Im} a$. The set of all such a is the usual 2-sphere in $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$:

$$S^2 = \{a \in \mathbb{H} \mid a^2 = -1\} = \{a \in \operatorname{Im}\mathbb{H} \mid |a| = 1\}. \tag{1.1.1}$$

1.1.2 Quaternionic vector spaces

Definition 1.1.1. A *quaternionic vector space* V is a real vector space with a multiplication by quaternions from the right:

$$V \times \mathbb{H} \to V, \quad (v, \lambda) \mapsto v\lambda.$$

The most notions including basis, dimension, subspace and linear map work as in the usual commutative linear algebra. For example, the dual space V^* of a quaternionic vector space V, is given by

$$V^* := \{\omega : V \to \mathbb{H} \mid \omega \quad \mathbb{H} - \text{linear}\}.$$

It can be made into a quaternionic vector space by defining $\omega\lambda := \bar{\lambda}\omega$.

However, due to the non-commutativity of $\mathbb{H}$, tensor products of quaternionic vector spaces, in particular the space of $\mathbb{H}$-linear maps between quaternionic vector spaces, have no natural structure of a quaternionic vector space.

Any quaternionic vector space V is of course a complex vector space, but this structure depends on choosing an imaginary unit as mentioned before and is not canonical. We shall instead fix a quaternionic linear complex structure J on V.

Definition 1.1.2. A *complex quaternionic vector space* (V, J) is a quaternionic vector space V together with a quaternionic linear endomorphism $J \in \operatorname{End}(V)$ such that $J^2 = -\operatorname{Id}$. Then for any $x + iy \in \mathbb{C}$

$$(x + iy)v := vx + (Jv)y.$$

We consider the $\pm\mathbf{i}$-eigenspace $V_\pm$ of J:

$$V_+ := \{v \in V | Jv = v\mathbf{i}\}, \quad V_- := \{v \in V | Jv = -v\mathbf{i}\}.$$

Hence $V = V_+ \oplus V_-$ is a direct sum decomposition. Obviously, V_+ and V_- are J-*complex* subspaces of V, and $V_- \cong V_+$ as J-complex spaces. Hence $V = V_+ \oplus V_+$ as J-complex spaces.

The space of quaternionic linear homomorphisms between two quaternionic complex vector spaces (V, J) and (W, J) splits as a direct sum of the real vector spaces of complex linear and anti-linear homomorphisms

$$\mathrm{Hom}(V, W) = \mathrm{Hom}_+(V, W) \oplus \mathrm{Hom}_-(V, W),$$

where

$$\mathrm{Hom}_\pm(V, W) = \{A \in \mathrm{Hom}(V, W) | AJ = \pm JA\}.$$

In fact, $\mathrm{Hom}(V, W)$ and $\mathrm{Hom}_\pm(V, W)$ are *complex* vector spaces with multiplication given by

$$(x + iy)Av := Avx + (JAv)y \quad \text{for any} \quad x + iy \in \mathbb{C}.$$

1.1.3 Quaternionic holomorphic vector bundles

We shall generalize the concepts of vector spaces to vector bundles over Riemann surfaces.

Definition 1.1.3. A *quaternionic vector bundle* V over a Riemann surface M is a real vector bundle V with a smooth fibre-preserving action of $\mathbb{H}$ from the right such that the fibres become quaternionic vector spaces.

Definition 1.1.4. A *complex quaternionic vector bundle* V is a pair (V, J) consisting of a quaternionic vector bundle V and a section $J \in \Gamma(\mathrm{End}(V))$ with $J^2 = -\mathrm{Id}$.

A quaternionic vector bundle V with complex structure J over a Riemann surface M decomposes into

$$V = V_+ \oplus V_-,$$

where $V_\pm$ are the $\pm\mathbf{i}$-eigenspace of J and $V_+ \cong V_-$ as J-complex bundles. If M is compact, the *degree* of a complex quaternionic vector bundle (V, J) is then defined to be the degree of the underlying complex vector bundle V_+,

$$\deg V := \deg V_+.$$

As in the complex case the degree of a complex quaternionic vector bundle (V, J) over a compact Riemann surface M can be computed by curvature integrals

$$\deg(V) = \frac{1}{2\pi} \int_M < JR^\nabla >, \tag{1.1.2}$$

where ∇ is a connection on V that satisfies $\nabla J = 0$, R^∇ is the curvature 2-form of ∇ and $< B >:= \frac{1}{4n} \operatorname{tr}_\mathbb{R} B$ for a quaternionic endomorphism $B \in \operatorname{End}(W)$ and $\dim_\mathbb{H} W = n$ ($\operatorname{tr}_\mathbb{R}$ the real trace of B seen as a real endomorphism).

Since we have fixed a complex structure on a quaternionic vector bundle V we define holomorphic structures on V analogous to the case of complex vector bundles over a Riemann surface.

Let (V, J) be a complex quaternionic vector bundle over a Riemann surface M, we decompose

$$\operatorname{Hom}_\mathbb{R}(TM, V) = T^*M \otimes_\mathbb{R} V = KV \oplus \bar{K}V,$$

where

$$KV := \{\omega : TM \to V | * \omega = J\omega\} \quad \text{and} \quad \bar{K}V := \{\omega : TM \to V | * \omega = -J\omega\}$$

are the tensor products of the canonical and anti-canonical bundle of M with V. The *type decomposition* of a V-valued 1-form $\omega \in \Omega^1(V)$ is denoted by

$$\omega = \omega^{'} + \omega^{''}, \tag{1.1.3}$$

where $\omega^{'} = \frac{1}{2}(\omega - J * \omega)$ is the K-part and $\omega^{''} = \frac{1}{2}(\omega + J * \omega)$ is $\bar{K}$-part.

Definition 1.1.5. A *holomophic structure* on a complex quaternionic vector bundle (V, J) is a quaternionic linear map

$$D : \Gamma(V) \to \Gamma(\bar{K}V)$$

satisfying the Leibniz rule

$$D(\psi\lambda) = (D\psi)\lambda + (\psi d\lambda)^{''}$$

for any section $\psi \in \Gamma(V)$ and quaternionic function $\lambda : M \to \mathbb{H}$. A section $\psi \in \Gamma(V)$ is called *holomophic* if $D\psi = 0$ and we denote by $H^0(V) = \ker D \subset \Gamma(V)$ the space of holomophic sections.

A complex quaternionic vector bundle with a holomophic structure is called a *holomophic quaternionic vector bundle.*

The decomposition of D into J commuting and anti-commuting parts gives

$$D = \bar{\partial} + Q,$$

where

$$\bar{\partial} = \frac{1}{2}(D - JDJ) \quad \text{and} \quad Q = \frac{1}{2}(D + JDJ).$$

The operator $\bar{\partial}$ is again a quaternionic holomorphic structure on V, actually the double of a complex holomorphic structure on V_+ since we have $V \cong V_+ \oplus V_+$ as J-complex bundles. The tensor field $Q \in \Gamma(\bar{K} \operatorname{End}_-(V))$ is a 1-form of type $\bar{K}$ with values in complex anti-linear endomorphisms of V, which is called the *Hopf field* of D.

For analytic considerations, and to make contact to the theory of Dirac operators with potential, we regard V as a complex vector bundle $(V, \mathbf{i})$ whose complex structure is given by right multiplication by quaternion **i**. Thus $V = V_+ \oplus V_- = V_+ \oplus \bar{V}_+$ as **i**-complex bundles so that Q is **i**-linear and $D = \bar{\partial} + Q$ can be expressed in the form

$$D = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} + \begin{pmatrix} 0 & -\bar{q} \\ q & 0 \end{pmatrix}$$

where $q \in \Gamma(K\,\mathrm{Hom}(V_+, \bar{V}_+))$.

The quaternionic holomophic structure D on V is a zero order perturbation of the operator $\bar{\partial}$ and hence elliptic. Therefore, the space $H^0(V)$ of holomophic sections is finite dimensional for a compact Riemann surface M.
For the potential part Q its L^2-norm is geometrically relevant.

Definition 1.1.6. Let (V, J, D) be a holomophic quaternionic vector bundle over a compact Riemann surface M. Its *Willmore energy* (or Willmore functional) is defined by

$$\mathcal{W}(V) = 2\int_M < Q \wedge *Q >,$$

where Q is Hopf field of D.

1.1.4 Quaternonic holomophic structures and connections

Let ∇ be a quaternionic connection on a complex quaternionic vector bundle V. The type decomposition (1.1.3) $\nabla = \nabla' + \nabla''$ defines naturally a holomorphic structure

$$D = \nabla''$$

and an anti-holomorphic structures ∇' on V, respectively. Of course, every quaternionic holomorphic structure D on V can be augmented to a quaternionic connection in various ways by adding anti-holomorphic structures.
We decompose ∇' and ∇'' further into J-commuting and anti-commuting parts

$$\nabla = \nabla' + \nabla'' = \partial + A + \bar{\partial} + Q. \tag{1.1.4}$$

Note that the J-commuting part $\hat{\nabla} = \partial + \bar{\partial}$ of ∇ is a complex connection, i.e. $\hat{\nabla} J = 0$, so that

$$\nabla J = 2(*Q - *A).$$

Differentiating once more, we obtain

$$[R^{\nabla}, J] = 2d^{\nabla}(*Q - *A),$$

where d^{∇} is the exterior derivative on $\mathrm{End}(V)$-valued forms.

Some explicit formulae. We calculate the Willmore energy and degree of a quaternionic vector bundle in the case of dimension 1. Let L be a complex quaternionic line bundle over a Riemann surface M. We choose a nowhere vanishing section $\psi \in \Gamma(V)$, then

$$J\psi = \psi N$$

defines a map $N : M \to S^2 \subset \mathrm{Im}\,\mathbb{H}$ from M to the 2-sphere defined by (1.1.1). Conversely, since every quaternionic line bundle on a surface admits a nowhere vanishing section, we could actually define every complex structure on a quaternionic line bundle by the above formula for some map $N : M \to S^2$. If L has a holomorphic structure D and ψ is a holomorphic section, i.e. $D\psi = 0$, the Hopf field Q of D is given by

$$Q = \frac{1}{2}(D\psi + JD(J\psi)) = \frac{1}{2}\psi N dN'', \tag{1.1.5}$$

where $dN'' = \frac{1}{2}(dN + N * dN)$. We calculate the Willmore functional of L

$$W(L) = 2\int_M < Q \wedge *Q >= \frac{1}{2}\int_M dN'' \wedge *dN'' = \int_M |dN''|^2$$

(with the identification of 2-forms with quadratic forms via $\sigma(X) := \sigma(X, JX)$ applied to $|dN''|^2$).

We define a quaternionic connection ∇ on L by $\nabla\psi = 0$. Since ∇ is flat, we have

$$R^{\hat{\nabla}} + A \wedge A + Q \wedge Q = 0,$$

where the underlying complex connection $\hat{\nabla}$ is given by $\nabla = \hat{\nabla} + A + Q$. The curvature of $\hat{\nabla}$ satisfies

$$R^{\hat{\nabla}} J = A \wedge *A - Q \wedge *Q.$$

For A we have an analogous formula to (1.1.5)

$$A\psi = \frac{1}{2}\psi N dN',$$

where $dN' = \frac{1}{2}(dN - N * dN))$. By (1.1.2), the degree of L is

$$\begin{aligned}\deg(L) &= \frac{1}{2\pi}\int_M < R^{\hat{\nabla}} J >= \frac{1}{2\pi}\int_M < A \wedge *A > - < Q \wedge *Q > \\ &= \frac{1}{8\pi}\int_M dN' \wedge *dN' - dN'' \wedge *dN''\end{aligned}$$

On the other side the mapping degree of $N : M \to S^2$ is

$$\deg(N) = \frac{1}{4\pi}\int_M N^*\omega_{S^2},$$

where $N^*\omega_{S^2}$ denotes the pull back of the volume form ω_{S^2} on S^2, given by

$$N^*\omega_{S^2} =< *dN, NdN >= -\frac{1}{2}NdN \wedge dN = \frac{1}{2}(dN' \wedge *dN' - dN'' \wedge *dN'').$$

Then the degree of L equals the mapping degree of N, $\deg L = \deg N$.

1.2 Line bundles associated to conformal immersions into $\mathbb{HP}^1$

In this section we first give the description of conformal maps from a Riemann surface M to the quaternionic space $\mathbb{H}$. However, the natural target space for conformal immersions is the projective space $\mathbb{HP}^1$, rather than $\mathbb{H}$, which is exactly the case in complex function theory: the Riemann sphere $\mathbb{CP}^1$ is more convenient as a target space for holomorphic functions than the complex plane. Actually, we can identify a map $f : M \to \mathbb{HP}^1$ with a line subbundle $L \subset V = M \times \mathbb{H}^2$. The conformality of f is provided by the existence of a compatible complex structure on L.

1.2.1 Conformal immersions into $\mathbb{H}$

A linear map $F : V \to W$ between Euclidean vector spaces is called *conformal* if F maps a normalized orthogonal basis of V into a normalized orthogonal basis of $F(V) \subset W$. Here "normalized" means that all vectors have the same length. If $V = W = \mathbb{R}^2 = \mathbb{C}$, and $J \in \mathrm{End}(\mathbb{C})$ denotes multiplication by the imaginary unit, then J is orthogonal. For $x \in \mathbb{C}$, $|x| \neq 0$, the vectors (x, Jx) form a normalized orthogonal basis. The map F is conformal if and only if (Fx, FJx) is again normalized orthogonal. Note that (Fx, JFx) is normalized orthogonal. Hence F is conformal, if and only if

$$FJ = \pm JF,$$

where the sign depends on the orientation of F. From the fact that the conformal condition only involves the complex structure J we could generalize this definition naturally to the quaternionic theory.

Definition 1.2.1. Let M be a Riemann surface, i.e. a 2-dimensional manifold endowed with a complex structure $J_M : TM \to TM, J_M^2 = -\mathrm{Id}$. A map $f : M \to \mathbb{H} = \mathbb{R}^4$ is called *conformal*, if there exist $N, R : M \to S^2 \subset \mathrm{Im}\,\mathbb{H}$ such that with $*df := df \circ J_M$,

$$* \, df = N df = -df R. \tag{1.2.1}$$

Here

$$S^2 = \{N \in \mathrm{Im}\,\mathbb{H} \mid |N| = 1\} = \{N \in \mathbb{H} | N^2 = -1\},$$

N and R are called the *left* and *right normal vector* of f.

Remark 1.2.2. Equation (1.2.1) is an analog of the classical Cauchy-Riemann equations

$$*df = \mathbf{i} df$$

for functions $f : \mathbb{C} \to \mathbb{C}$. In this sense conformal maps into $\mathbb{H}$ are a generalization of complex holomorphic maps. It is known that in complex function theory $\mathbb{CP}^1$ is more convenient as a target space for holomophic functions than $\mathbb{C}$. That is why later we will introduce the quaternionic projective space $\mathbb{HP}^1$ as the target space for conformal immersions.

If $f : M \to \mathbb{H}$ is a conformal immersion, then the image of the tangent space $df(T_pM) \subset \mathbb{H}$ is a 2-dimensional real subspace for any point $p \in M$. According to the Lemma 2 in [1] (Fundamental lemma), there exist left and right vectors $N, R \in S^2$, inducing a complex structure J on the tangent space $T_pM \cong df(T_pM)$, which should coincide with the complex structure J_M already given on T_pM. The tangent and normal bundle along f are (pointwise)

$$T_fM = \{x \in \mathbb{H} \mid Nx = -xR\}$$

$$\perp_f M = \{x \in \mathbb{H} \mid Nx = xR\}$$

The connection d on the trivial $\mathbb{H}$-bundle induce the Levi-Civita connection on T_fM and the normal connection on $\perp_f M$ by orthogonal projection. Both connections are compatible with the respective complex structures, which are both given by left multiplication by N.

We will list some useful formulae for a conformal immersion $f : M \to \mathbb{H}$ omitting the proofs which can be found in [1]. Let N, R denote the left an right normal vector of f, i.e. $*df = Ndf = -dfR$. The second fundamental form $\mathrm{II}(X, Y) = (X \cdot df(Y))^{\perp}$ of f is given by

$$\mathrm{II}(X, Y) = \frac{1}{2}(NdN(X)df(Y) + df(Y)dR(X)R). \tag{1.2.2}$$

The mean curvature vector $\mathcal{H} = \frac{1}{2}tr(\mathrm{II})$ is given by

$$\bar{\mathcal{H}}df = \frac{1}{2}(*dR + RdR), \quad df\bar{\mathcal{H}} = \frac{1}{2}(*dN + NdN). \tag{1.2.3}$$

Using the classical Gauss- and Ricci-equations and the second fundamental form formula (1.2.2) we get the Gaussian curvature $\mathcal{K}$ on T_fM and the normal curvature[1] $\mathcal{K}^{\perp}$ on $\perp_f M$ are given by

$$\mathcal{K}|df|^2 = \frac{1}{2}(N^*\omega_{S^2} + R^*\omega_{S^2}), \tag{1.2.4}$$

$$\mathcal{K}^{\perp}|df|^2 = \frac{1}{2}(N^*\omega_{S^2} - R^*\omega_{S^2}), \tag{1.2.5}$$

where

$$R^*\omega_{S^2} =< *dR, RdR >, \quad N^*\omega_{S^2} =< *dN, NdN >$$

are the pull-backs of the 2-sphere area under R and N.

On a compact Riemann surface, the degree of the tangent bundle T_fM seen as a complex line bundle is

$$\deg(T_fM) = \frac{1}{2\pi}\int_M \mathcal{K}|df|^2 = \deg(N) + \deg(R), \tag{1.2.6}$$

where $\deg(N)$ and $\deg(R)$ denote the mapping degrees of $N, R : M \to S^2$.

[1] In contrast to [1], we use the definition $\mathcal{K}^{\perp}|df|^2 =< R^{\perp}N\xi, \xi >$, where $\xi \in \Gamma(\perp_f M)$ is a section of unit length.

Furthermore, the degree of the normal bundle $\perp_f M$ seen as a complex line bundle is

$$\deg(\perp_f M) = \frac{1}{2\pi}\int_M \mathcal{K}^\perp |df|^2 = \deg(N) - \deg(R). \tag{1.2.7}$$

The classical Willmore functional of the immersion f is

$$W(f) = \int_M |\mathcal{H}|^2|df|^2 = \frac{1}{4}\int_M |*dN + NdN|^2 = \frac{1}{4}\int_M |*dR + RdR|^2.$$

1.2.2 The quaternionic projective line $\mathbb{HP}^1$

Similar to the real and complex projective spaces, the *quaternionic projective space* $\mathbb{HP}^n$ is defined as the set of quaternionic lines in $\mathbb{H}^{n+1}$. The canonical projection is

$$\pi : \mathbb{H}^{n+1}\backslash\{0\} \to \mathbb{HP}^n,\ x \mapsto \pi(x) = [x] = x\mathbb{H}.$$

The manifold structure of $\mathbb{HP}^n$ is defined by the following affine coordinates: for any linear form $\beta \in (\mathbb{H}^{n+1})^*$, $\beta \neq 0$,

$$\pi(x) \mapsto x < \beta, x >^{-1}$$

maps the open set $U_\beta := \{\pi(x) \mid < \beta, x > \neq 0\} \subset \mathbb{HP}^n$ onto the affine hyperplane

$$\{x \in \mathbb{H}^{n+1} \mid < \beta, x >= 1\} \cong \mathbb{H}^n.$$

The complement set of U_β

$$\{\pi(x) \mid < \beta, x >= 0\}$$

is called the *hyperplane at infinity.* In the special case $n = 1$, we choose β as the second coordinate function, the chart gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{HP}^1\backslash\{\infty\} \mapsto \left(x_1 x_2^{-1}\right) \in \mathbb{H},$$

where the hyperplane at infinity is a single point $\infty = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{HP}^1$. This implies the quaternionic projective line $\mathbb{HP}^1$ is the one-point compactification of $\mathbb{R}^4$, hence the 4-sphere S^4.

Before we discuss metrics on $\mathbb{HP}^1$ we should first describe the tangent space $T_l\mathbb{HP}^1$ for $l \in \mathbb{HP}^1$. We consider the differential of the canonical projection $\pi : \mathbb{H}^2\backslash\{0\} \to \mathbb{HP}^1$. In the above chart let h be the composition of the corresponding affine coordinate map and the projection π:

$$h(x) = \begin{pmatrix} x_1 x_2^{-1} \\ 1 \end{pmatrix} : \mathbb{H}^2\backslash\{0\} \to \mathbb{H}^2, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{H}^2\backslash\{0\}.$$

For any $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{H}^2$

$$d_x h(v) = v x_2^{-1} - x x_2^{-1} v_2 x_2^{-1}.$$

Thus

$$\ker d_x h = x\mathbb{H}, \quad d_{x\lambda} h(v\lambda) = d_x h(v)$$

for any $\lambda \in \mathbb{H}\backslash\{0\}$. It is the same for π:

$$\ker d_x \pi = x\mathbb{H}, \quad d\pi_{x\lambda}(v\lambda) = d\pi_x(v)$$

because $\pi(v) = \pi(v\lambda)$. Hence $d_x\pi : \mathbb{H}^2/l \to T_l\mathbb{HP}^1$ is an isomorphism, where $l = [x]$. Eliminating the dependence of the choice of x, we get a well-defined isomorphism:

$$\mathrm{Hom}(l, \mathbb{H}^2/l) \cong T_l\mathbb{HP}^1, \ \xi \mapsto d\pi_x(\xi(x)). \tag{1.2.8}$$

Given a non-degenerate quaternionic hermitian inner product $<,>$ on $\mathbb{H}^2$, we define a (possibly degenerate Pseudo-) Riemannian metric on $\mathbb{HP}^1$ as follows. For $x \in \mathbb{H}^2$ with $< x, x > \neq 0$ and $v, w \in (x\mathbb{H})^\perp$ we define

$$< d_x\pi(v), d_x\pi(w) >= \frac{1}{< x, x >}\mathrm{Re} < v, w > .$$

This is well-defined since, for $\lambda \in \mathbb{H}\backslash\{0\}$, we have

$$< d_{x\lambda}\pi(v\lambda), d_{x\lambda}\pi(w\lambda) >=< d_x\pi(v), d_x\pi(w) > .$$

It extends to arbitrary v, w by

$$\begin{aligned} < d_x\pi(v), d_x\pi(w) > &= \mathrm{Re}\,\frac{< v, w - x < x, w >< x, x >^{-1}>}{< x, x >} \\ &= \mathrm{Re}\,\frac{< v, w > - < x, x > - < v, x >< x, w >}{< x, x >^2}. \end{aligned} \tag{1.2.9}$$

For the standard quaternionic hermitian inner product

$$< x, y >= \bar{x}_1 y_1 + \bar{x}_2 y_2$$

we obtain the standard Riemannian metric on $\mathbb{HP}^1 = S^4$

$$\tau^* < v, w >_x= \frac{1}{(1 + x\bar{x})^2}\mathrm{Re}\,(\bar{v}w) = \frac{1}{(1 + x\bar{x})^2} < v, w >_{\mathbb{R}},$$

via the affine parameter

$$\tau : \mathbb{H} \to \mathbb{HP}^1, \quad x \mapsto \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

Comparing it with the metric on $\mathbb{R}^4$ induced by the stereographic projection from S^4 to $\mathbb{R}^4$

$$\frac{1}{(1 + x\bar{x})^2} < v, w >_{\mathbb{R}},$$

we find that the curvature of the standard Riemannian metric on $\mathbb{HP}^1$ is 4.

Möbius transformations on $\mathbb{HP}^1$. The group $Gl(2, \mathbb{H})$ acts on $\mathbb{HP}^1$ by $G(v\mathbb{H}) := Gv\mathbb{H}$. Analogous to the complex case for $G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2, \mathbb{H})$ in the above affine coordinate the action on $\mathbb{HP}^1$ are given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} ax + b \\ cx + d \end{bmatrix} = \left[(ax + b)(cx + d)^{-1}\right].$$

$x = (ax+b)(cx+d)^{-1}$ holds for any $x \in \mathbb{H}$ if and only if $a = d \in \mathbb{R}$ and $b = c = 0$. This means the kernel of this action is $\{\rho I \mid \rho \in \mathbb{R}\}$.

Under the action G the Riemannian metric given by (1.2.9) satisfies

$$\begin{aligned} |dG(d_x\pi(v)\lambda)|^2 &= \mathrm{Re}\,\frac{< G(v\lambda), G(v\lambda) > - < G(v\lambda), Gx >< Gx, G(v\lambda) >}{< Gx, Gx >^2} \\ &= |\lambda|^2 \mathrm{Re}\,\frac{< Gv, Gv > - < Gv, Gx >< Gx, Gv >}{< Gx, Gx >^2} \\ &= |\lambda|^2 |dG(d_x\pi(v))|^2 \end{aligned}$$

Hence any $G \in GL(2, \mathbb{H})$ is length-preserving up to a constant factor, i.e. is a conformal isomorphism on $\mathbb{HP}^1$. These transformations are called the *Möbius transformations* on $\mathbb{HP}^1$. Actually $GL(2, \mathbb{H})$ is the full group of all orientation preserving conformal diffeomorphisms of S^4 (see [14]).

2-spheres in S^4. Let S be a quaternionic endomorphism of the 2-dimensional quaternonic vector space $\mathbb{H}^2$) with $S^2 = -1$. The fixed point set

$$S' := \{l \in \mathbb{HP}^1 \mid Sl = l\}$$

is a 2-sphere in $\mathbb{HP}^1$, i.e. corresponds to a real 2-plane in $\mathbb{H} = \mathbb{R}^4$ under a suitable affine coordinate. Conversely each 2-sphere can be obtained in this way by an $S \in \mathcal{Z}$ where

$$\mathcal{Z} = \{S \in \mathrm{End}(\mathbb{H}^2) \mid S^2 = -1\}$$

(see Proposition 2 in [1]). Although $-S$ defines the same 2-sphere S' as S, the orientation of S' determined by $-S$ is opposite. Hence the above $\mathcal{Z}$ is the space of oriented 2-spheres in $S^4 = \mathbb{HP}^1$.

1.2.3 Holomorphic curves in $\mathbb{HP}^1$

The points of the projective line $\mathbb{HP}^1$ are the quaternionic lines in $\mathbb{H}^2$. The *tautological bundle*

$$\pi_{\mathcal{T}} : \mathcal{T} \to \mathbb{HP}^1$$

is the bundle with $\mathcal{T}_l = l$ for any $l \in \mathbb{HP}^1$. L is a line subbundle of the trivial bundle $H := \mathbb{HP}^1 \times \mathbb{H}^2$. And it induces a quotient bundle $H/\mathcal{T}$ with fibres $H_l/\mathcal{T}_l$. The *real* vector bundle $\mathrm{Hom}(\mathcal{T}, H/\mathcal{T})$ has the fibres $\mathrm{Hom}(\mathcal{T}_l, H_l/\mathcal{T}_l)$.

It is natural to extend constructions for vector spaces, fibre-wise, to operations in vector bundles. Therefore, according to (1.2.8) we relate real vector bundle of homomorphisms between the tautological bundle and its quotient bundle to the tangent bundle:

$$T\mathbb{HP}^1 \cong \mathrm{Hom}(\mathcal{T}, H/\mathcal{T}). \tag{1.2.10}$$

Using the pullback of tautological bundle, we associate a map $f : M \to \mathbb{HP}^1$ from a surface into the quaternionic projective line to a line bundle

$$L := f^*\mathcal{T}.$$

The fiber L_p over $p \in M$ is the projective point $L_p = f(p)$. L is a quaternionic line subbundle of the trivial product bundle $M \times \mathbb{H}^2$ over M.

Conversely, we can identify any given line subbundle L of $M \times \mathbb{H}^2$ over M with a map $f : M \to \mathbb{HP}^1$ by

$$f(p) := L_p \in \mathbb{HP}^1 \quad \text{for any} \quad p \in M.$$

Derivative of maps and conformality. From now on V always denote the trivial $\mathbb{H}^2$-bundle $M \times \mathbb{H}^2$ over a Riemann surface M.

Let L be a line subbundle of V and $\pi_L : V \to V/L$ the canonical projection from V to the quotient bundle V/L. Any section $\psi \in \Gamma(L) \subset \Gamma(V)$ can be regarded as a map $\psi : M \to \mathbb{H}^2$. For any tangent vector $X \in T_pM$ of M at p,

$$d\psi(X) \in V_p = \mathbb{H}^2 \quad \text{and} \quad \pi_L(d\psi(X)) \in (V/L)_p = \mathbb{H}^2/L_p.$$

Moreover,

$$\pi_L(d(\psi\lambda)(X)) = \pi_L(d\psi(X))\lambda + \psi d\lambda(X)) = \pi_L(d\psi(X))\lambda$$

for any quaternionic function $\lambda : M \to \mathbb{H}$ on M. The above equation shows that $\psi \mapsto \pi_L(d\psi(X))$ is tensorial in ψ. Hence the 1-form

$$\delta := \pi_L d\mid_L \in \Omega^1(\text{Hom}(L, V/L)).$$

on M with values in $\text{Hom}(L, V/L)$ is well-defined. Here d is the trivial connection on V.

We know that $\text{Hom}(\mathcal{T}, H/\mathcal{T})$ is the tangent bundle of $\mathbb{HP}^1$. Since L (respectively V/L) is the pull-back bundle of $\mathcal{T}$ (respectively $H/\mathcal{T}$)

$$\delta : TM \to \text{Hom}(L, V/L)$$

corresponds to the derivative of f. Therefore, we call δ the *derivative* of L. Hence δ is a proper substitute for df in (1.2.1) to describe the conformal condition of L.

Definition 1.2.3. A line subbundle $L \subset V$ over a Riemman surface M is called *conformal* or a *holomophic curve* in $\mathbb{HP}^1$, if there exists a complex structure $J \in \Gamma(\text{End}(L))$ on L such that

$$* \delta = \delta J. \tag{1.2.11}$$

L is called *immersed* if $\delta \in \Omega^1(\text{Hom}(L, V/L))$ is nowhere vanishing.

If L is an immersed holomophic curve in $\mathbb{HP}^1$, then there is also a complex structure $\tilde{J}$ on V/L such that

$$*\delta = \delta J = \tilde{J}\delta.$$

The real 2-plane $\delta(T_pM)$ is given by

$$XJ_p = \tilde{J}_pX, \quad X \in \text{Hom}(L_p, \mathbb{H}^2/L_p).$$

If $f_p = L_p$ takes values in $\mathbb{HP}^1 \backslash \{\infty\} \cong \mathbb{H}$ for any $p \in M$, the bundle L can be written as

$$L = \psi\mathbb{H} \quad \text{with} \quad \psi = \begin{pmatrix} f \\ 1 \end{pmatrix}$$

for $f : M \to \mathbb{H}$. If L is a holomorphic curve with complex structure J on L, we define a map $R : M \to \mathbb{H}$ by

$$J \begin{pmatrix} f \\ 1 \end{pmatrix} = - \begin{pmatrix} f \\ 1 \end{pmatrix} R.$$

Apparently $R^2 = -1$. In this case (1.2.11) holds if and only if

$$*\pi_L \begin{pmatrix} df \\ 0 \end{pmatrix} = -\pi_L \begin{pmatrix} dfR \\ 0 \end{pmatrix},$$

or equivalently

$$\begin{pmatrix} *df \\ 0 \end{pmatrix} \equiv \begin{pmatrix} -dfR \\ 0 \end{pmatrix} \mod \begin{pmatrix} f \\ 1 \end{pmatrix}.$$

Then we get $*df = -dfR$ and hence $f : M \to \mathbb{H}$ is a conformal map with right normal vector R.

Conversely, if $f : M \to \mathbb{H}$ is a conformally map with right normal vector R, then the corresponding line bundle L is given by

$$L_p = \begin{bmatrix} f(p) \\ 1 \end{bmatrix}.$$

We can define a complex structure $J \in \operatorname{End}(L)$ by $J \begin{pmatrix} f \\ 1 \end{pmatrix} = - \begin{pmatrix} f \\ 1 \end{pmatrix} R$. Then $*df = -dfR$ implies $*\delta = \delta J$ and (L, J) is a holomorphic curve.

Clearly, f is immersed if and only if δ is nowhere vanishing, hence an immersed holomorphic curve into $\mathbb{HP}^1$ is the same thing as a conformal immersion into $\mathbb{HP}^1$.

1.3 The mean curvature sphere

As we know the Gauss map is the oriented tangent plane congruence of an immersion in Euclidean surface geometry. As an analog in Möbius geometry the mean curvature sphere congruence of a conformal immersion is oriented tangent to the immersion and satisfies that every 2-sphere of the congruence has the same mean curvature than the immersion at the corresponding point.

1.3.1 Conditions for the mean curvature sphere

Let V be the trivial quaternionic vector bundle $M \times \mathbb{H}^2$ over a Riemann surface M. A complex structure S on V is a family of 2-spheres, a *sphere congruence* $S : M \to \mathcal{Z}$ in classical terms. Here

$$\mathcal{Z} = \{S \in \operatorname{End} \mathbb{H}^2 \mid S^2 = -1\}$$

is the space of oriented 2-spheres in S^4.

Definition 1.3.1. The *mean curvature sphere congruence* of an immersed holomorphic curve $L \subset V$ is a complex structure $S : M \to Z$ on V characterized by the following properties:

(i) $SL = L$,

(ii) $*\delta = S\delta = \delta S$,

(iii) $AV \subset L$, or, equivalently, $Q\,|_L = 0$.

Here the forms $A \in \Gamma(K\,\mathrm{End}_-(V))$ and $Q \in \Gamma(\bar{K}\,\mathrm{End}_-(V))$ are called the *Hopf fields* of the holomorphic curve $L \subset V$ and defined by the canonical decomposition (1.1.4) of the trivial connection $\nabla = \partial + \bar{\partial} + A + Q$ on V with respect to S.

For any point $p \in M$ the above conditions in fact mean: (i) the sphere, the set of fix points defined by $S(p)$, passes through the point $f(p) = L_p \in \mathbb{HP}^1$; (ii) the sphere $S(p)$ is tangent to f at p; (iii) in an affine coordinate system $\begin{bmatrix} f \\ 1 \end{bmatrix} = L$ the sphere $S(p)$ has the same mean curvature vector as $f : M \to \mathbb{H}$ at p.

Theorem 2 of [1] states that every immersed holomorphic curve has a unique mean curvature sphere S. If $L \subset V$ is a holomorphic curve in $\mathbb{HP}^1$ with derivative $\delta \in \Omega^1(\mathrm{Hom}(L, V/L))$, then there exist complex structures J on L and $\tilde{J}$ on V/L such that

$$*\delta = \delta J = \tilde{J}\delta.$$

By proposition 4 of [1] the mean curvature sphere S of an immersed Riemann surface L satisfies

$$< dS, dS >=< *dS, *dS > \text{ and } \quad < dS, *dS >= 0,$$

where the scalar product is defined by

$$< A, B >:=< AB >= \frac{1}{8}\,\mathrm{tr}_{\mathbb{R}} < AB >$$

for any $A, B \in \mathrm{End}\,V$. This implies that $S : M \to \mathcal{Z}$ is conformal.

The mean curvature sphere makes it easy to express the tangent bundle and normal bundle of L in the homomorphism model (1.2.10) of the tangent space of projective space. With the complex structure on L and V/L induced by the mean curvature S, condition (ii) implies that $\delta(TM) \subset \mathrm{Hom}_+(L, V/L)$ which consists of the complex linear homomorphisms from L to V/L. Since δ vanishes nowhere $\mathrm{Hom}_+(L, V/L)$ is exactly the tangent bundle of L, while the normal bundle is $\mathrm{Hom}_-(L, V/L)$, which denotes the complex antilinear homomorphisms.

The fact that $\delta \in \Gamma(K\,\mathrm{Hom}_+(L, V/L))$ is a complex bundle isomorphism gives rise to the degree formula

$$\deg K = \deg L - \deg V/L \qquad (1.3.1)$$

on a compact Riemann surface. The normal bundle degree, which is equal to the degree of V, calculates to

$$\deg \mathrm{Hom}_-(L, V/L) = \deg V/L + \deg L = 2\deg V/L + \deg K = \deg K - 2\deg L. \qquad (1.3.2)$$

In particular, the degree of the normal bundle is 0 if and only if $\deg L = g - 1$, i.e. if and only if L has the degree of a spin bundle.

1.3.2 The mean curvature sphere in affine coordinates

We now describe the mean curvature sphere S of an immersed holomorphic curve L in $\mathbb{HP}^1$ in affine coordinates. We regard L as $L = \psi\mathbb{H}$ with $\begin{pmatrix} f \\ 1 \end{pmatrix}$ for a conformal immersion $f : M \to \mathbb{H}$. With respect to the frame $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} f \\ 1 \end{pmatrix}$ we write $S = GTG^{-1}$, where

$$G = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}.$$

From the property $SL \subset L$ we conclude that S should be expressed in the following form

$$S = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix} \begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}$$

with $N, R, H : M \to \mathbb{H}$. $S^2 = -I$ means

$$N^2 = -1 = R^2 \quad RH = HN.$$

And it is easy to find out that

$$*\delta = S\delta = \delta S \Leftrightarrow \quad *df = Ndf = -dfR.$$

Therefore, N and R turn out to be the left and right normal vectors of f.
Then we compute the Hopf Fields A, Q using the condition $Q|_L = 0$ and the corresponding $\operatorname{Im} A \subset L$:

$$A = \frac{1}{4}(SdS - *dS) = \frac{1}{4}G \begin{pmatrix} 0 & 0 \\ \cdots & RdR - *dR \end{pmatrix} G^{-1},$$

$$Q = \frac{1}{4}(SdS - *dS) = \frac{1}{4}G \begin{pmatrix} NdN - *dN & 0 \\ \cdots & 0 \end{pmatrix} G^{-1}.$$

And H is given by

$$2Hdf = dR - R * dR \quad 2dfH = dN - N * dN.$$

Together with equations (1.2.3) we find that H is closely related to the mean curvature vector $\mathcal{H}$ of the immersion f:

$$H = -\mathcal{H}N = -R\mathcal{H}.$$

The Willmore integrand is given by

$$< Q \wedge *Q > = \frac{1}{8}\operatorname{tr}_{\mathbb{R}}(-Q^2 - (*Q)^2) = -\frac{1}{4}\operatorname{tr}_{\mathbb{R}}(Q^2) = \frac{1}{16}|NdN - *dN|^2$$
$$= \frac{1}{16}| * dN + NdN|^2 - 4 < *dN, NdN >) = \frac{1}{4}(|\mathcal{H}|^2 - K - K^{\perp})|df|^2 \tag{1.3.3}$$

Here we used the formula for the Gaussian curvature $\mathcal{K}$ (1.2.4) on T_fM and the normal curvature $\mathcal{K}^{\perp}$ (1.2.5) on $\perp_f M$. Together with (1.2.6) and (1.2.7) we get

$$4\int_M < Q \wedge *Q > = \mathcal{W}(f) - 4\pi \deg N.$$

This is, up to topological terms, the Willmore energy of the quaternionic holomorphic structure $\bar{\partial} + \pi Q$ on the induced quotient line bundle V/L, which will be studied in the next chapter.

Chapter 2

The spectrum and Darboux transforms of conformal immersions

In this chapter we discuss an important concept of integrable system theory, the spectrum. We define the spectrum of a conformal immersion, i.e. a line subbundle $L \subset V$ of a trivial quaternionic rank 2 bundle with trivial connection d, to be the spectrum of the induced quaternionic holomorphic line bundle $W = V/L$ equipped with the unique holomorphic structure for which all d-parallel sections of V project to holomorphic sections of the quotient V/L.

In the case the underlying surface is a torus we conjugate the quaternionic spectrum to be the complex spectrum due to the abelian fundamental group of the torus. Furthermore the (complex) spectrum is a 1-dimesional analytic subset if the degree of the induced bundle V/L is zero, and hence can be normalized to be a Riemann surface.

On the other hand, the observation that closed Darboux transforms of a conformal immersion $L \subset V$ correspond to nowhere vanishing holomorphic sections with monodromy of V/L implies that geometrically the spectral curve parameterizes the space of Darboux transforms of the conformal immersion.

2.1 The spectrum of a torus immersed in the 4-sphere

In this section, the spectrum of a conformally immersed torus with normal bundle degree 0 is defined using the induced quaternionic holomorphic quotient line bundle.

2.1.1 The holomorphic quotient line bundle induced by the conformal immersion

A conformal immersion $f : M \to S^4$ from a Riemann surface M into the 4-sphere is identified as a line subbundle $L \subset V$ of the trivial $\mathbb{H}^2$- bundle V over M with an unique nowhere vanishing 1-form $\delta \in \Omega^1(\mathrm{Hom}(L, V/L))$ compatible with the complex structures J and $\tilde{J}$ on L and V/L, i.e.

$$*\delta = \delta J = \tilde{J}\delta.$$

The mean curvature sphere congruence S of f satisfies

$$SL = L, \quad S|_L = J, \quad \pi S = \tilde{J}\pi,$$

where $\pi : V \to V/L$ is the the canonical projection. The conformal condition implies that $\delta \in \Gamma(K \operatorname{Hom}(L, V/L))$, i.e.

$$(\pi\nabla|_L)'' = \pi(\bar{\partial}|_L + Q|_L) = 0$$

Therefore, together with the condition $Q|_L = 0$ we get $\pi\bar{\partial}|_L = 0$. Here ∇ denotes the trivial connection on V with the canonical decomposition $\nabla = \partial + A + \bar{\partial} + Q$.

A quaternionic holomorphic structure $D : \Gamma(V/L) \to \Gamma(\bar{K}V/L)$ on the quotient line bundle V/L is well-defined by

$$D\pi = (\pi\nabla)'' \tag{2.1.1}$$

since

$$(\pi\nabla)'' = \pi\bar{\partial} + \pi Q = \pi(\bar{\partial}|_L + \bar{\partial}\pi) + \pi(Q|_L + Q\pi) = \pi(\bar{\partial} + Q)\pi.$$

Therefore, D can be regarded as the "restriction" to the holomorphic structure $\bar{\partial} + Q$ on the quotient line bundle V/L.
We denote J-commuting and anticommuting parts of D again as $\bar{\partial}$ and Q respectively. Then the decomposition gives

$$D = \bar{\partial} + Q.$$

Geometrically, the Hopf field Q is related to the trace-free second fundamental form (see (1.2.2)) of the immersion f. And the Willmore energy of the conformal map f is given by the Willmore energy of the holomorphic line bundle V/L, that is,

$$\mathcal{W}(f) = \mathcal{W}(V/L, D) = 2\int < Q \wedge *Q > .$$

Actually, if we do the same thing as in last chapter (see 1.3.3) [1] we find that in the affine coordinate $2 < Q \wedge *Q >$ coincides with

$$(|\mathcal{H}|^2 - \mathcal{K} - \mathcal{K}^\perp)d\mathcal{A},$$

where $\mathcal{H}$ is the mean curvature vector, $\mathcal{K}$ the Gaussian curvature, $\mathcal{K}^\perp$ the normal bundle curvature, and $d\mathcal{A}$ the induced area of f as a map into $\mathbb{R}^4$, after choosing a point at infinity on S^4.

In the homomorphism model (1.2.10) of the tangent space of projective space, the tangent bundle of f is $\operatorname{Hom}_+(L, V/L)$, while the normal bundle is $N_f = \operatorname{Hom}_-(L, V/L)$, where $\operatorname{Hom}_\pm$ denote the complex linear, respectively complex antilinear, homomorphisms. By (1.3.1) and (1.3.2) the normal bundle degree of the conformal immersion $f : T^2 \to S^4$ calculates to

$$\deg N_f = 2 \deg V/L + \deg K. \tag{2.1.2}$$

[1]Here we notice that $2 < Q \wedge *Q >= 2[\frac{1}{4}\operatorname{tr}_\mathbb{R}(-Q^2 - (*Q)^2)] = -tr_\mathbb{R}(Q^2)$.

Thus, the classical Willmore integral differs from the Willmore energy given by a topological term, that is

$$\int_M |\mathcal{H}|^2 d\mathcal{A} = \mathcal{W}(f) + \pi \deg(V/L).$$

Furthermore the index of the operator D coincides with the index of its first order part $\bar{\partial}$ which, by the Riemann-Roch theorem, calculates to

$$\operatorname{Ind}(D) = \operatorname{Ind}(\bar{\partial}) = 2\deg(V/L) + \frac{r(V/L)}{2}\chi(M), \tag{2.1.3}$$

where $r(V/L)$ denotes the rank of the bundle V/L and $\chi(M) = 2(1-g)$ denotes the Euler characteristic of M.
In particular, the index of the D is 0 if and only if $\deg(V/L) = g - 1$, i.e. if and only if V/L has the degree of a spin bundle.

2.1.2 The quaternionic spectrum of quaternionic holomorphic line bundles

Let W be a quaternionic holomophic line bundle equipped with complex structure $J \in \Gamma(\operatorname{End}(W))$, $J^2 = -\operatorname{Id}$ and quaternionic holomophic structure $D : \Gamma(W) \to \Gamma(\bar{K}W)$. Let $\tilde{M}$ be a universal cover of M with the covering map $\pi_M : \tilde{M} \to M$. Naturally the holomorphic structure D induces an operator on the pullback bundle $\tilde{W} := \pi_M^* W$ over $\tilde{M}$. This operator is periodic with respect to the group Γ of deck transformations of $\pi_M : \tilde{M} \to M$.
The space $H^0(W)$ of holomorphic sections of W are exactly the periodic (i.e. Γ- invariant) sections of $\tilde{W}$ solving $D\psi = 0$. We call a holomorphic section $\psi \in H^0(\tilde{W})$ satisfying

$$\gamma^*\psi = \psi h_\gamma \quad \text{for all} \quad \gamma \in \Gamma,$$

where $h \in \operatorname{Hom}(\Gamma, \mathbb{H}_*)$ is a representation of Γ, *a holomorphic section with monodromy.* The space $H^0_h(\tilde{W})$ of holomorphic sections with monodromy h is a finite dimensional real vector space. Multiplying $\psi \in H^0_h(\tilde{W})$ by some $\lambda \in \mathbb{H}_*$ yields a section $\psi\lambda$ with monodromy $\lambda^{-1}h\lambda$. In particular, unless h is a real representation, $H^0_h(\tilde{W})$ is a real but not a quaternionic vector space.

Definition 2.1.1. Let W be a quaternionic line bundle with holomorphic structure D over a Riemann surface M. The quaternionic spectrum of W is the subspace

$$\operatorname{Spec}_{\mathbb{H}}(W, D) \subset \operatorname{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$$

of conjugacy classes of monodromy representations occurring for holomorphic sections of $\tilde{W}$.

If we set

$$\widehat{\operatorname{Spec}}_{\mathbb{H}}(W, D) = \left\{ h \in \operatorname{Hom}(\Gamma, \mathbb{H}_*) \mid \begin{array}{l} \exists\, \psi \in H^0(\tilde{W}),\ \psi \neq 0 \quad \text{such that} \\ \gamma^*\psi = \psi h_\gamma \quad \text{for all} \quad \gamma \in \Gamma \end{array} \right\},$$

then

$$\operatorname{Spec}_{\mathbb{H}}(W, D) = \widehat{\operatorname{Spec}}_{\mathbb{H}}(W, D)/\mathbb{H}_*.$$

Remark 2.1.2. We will show in the next section that any non-trivial holomorphic section ψ with monodromy h of V/L induces a line subbundle $L^\sharp = \hat{\psi}\mathbb{H}$, where $\hat{\psi}$ is the prolongation of ψ satisfying $\pi\hat{\psi} = \psi$ and $d\psi \in \Omega^1(L)$. The conformal map defined by $L^\sharp$ is exactly a Darboux transform of the conformal immersion L. From this point of view the quaternionic spectrum arises as a parameter space for the space of Darboux transforms.

2.1.3 The spectrum of quaternionic holomorphic line bundles over 2-tori

In this part we study the geometry of $\mathrm{Spec}_{\mathbb{H}}(W, D)$ in the case that the underlying surface is a torus $M = T^2$. The universal covering is $\mathbb{R}^2 \cong \mathbb{C}$ and $T^2 = \mathbb{C}/\Gamma$, where the fundamental group $\pi_1(T^2) = \Gamma$ is abelian.

Lemma 2.1.3. *Every representation in* $\mathrm{Hom}(\Gamma, \mathbb{H}_*)$ *can be conjugated into a complex representation in* $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$.

Proof. We choose generators γ_1, γ_2 of the lattice Γ and then have a map

$$\mathcal{E} : \mathrm{Hom}(\Gamma, \mathbb{H}_*) \to \mathbb{H}_* \times \mathbb{H}_* \quad \text{via} \quad h \mapsto (h_{\gamma_1}, h_{\gamma_2}).$$

Every pair $(h_1, h_2) \in \mathrm{Im}\,\mathcal{E}$ satisfies $h_1 h_2 = h_2 h_1$ due to the fact that Γ is abelian. Without loss of generality we assume that h_1 is not real. Then there exists a $\lambda \in \mathbb{H}_*$ such that $\tilde{h}_1 := \lambda^{-1} h_1 \lambda \in \mathbb{C}_*$. We set $\tilde{h}_2 := \lambda^{-1} h_1 \lambda$. Since $\tilde{h}_1$ is not real,

$$\tilde{h}_1 \tilde{h}_2 = \tilde{h}_2 \tilde{h}_1 \Rightarrow \tilde{h}_2 \mathbf{i} = \mathbf{i} \tilde{h}_2 \Rightarrow \tilde{h}_2 \in \mathbb{C}.$$

We get

$$\lambda^{-1}(h_1, h_2)\lambda \in \mathbb{C}_* \times \mathbb{C}_*.$$

With identification $\mathbb{C}_* \times \mathbb{C}_* \cong \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ we prove the lemma. □

We notice that the complex representation $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ in a conjugacy class in $\mathrm{Hom}(\Gamma, \mathbb{H}_*)$ is uniquely determined up to complex conjugation $h \mapsto \bar{h}$ (which corresponds to conjugation by the quaternion $\mathbf{j}$). In particular, away from the real representations, the map

$$\mathrm{Hom}(\Gamma, \mathbb{C}_*) \to \mathrm{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$$

is 2:1. Taking the lift of the quaternionic spectrum $\mathrm{Spec}_{\mathbb{H}}(W, D)$ of W under this maps leads to the notion of the complex spectrum.

Definition 2.1.4. Let W be a quaternionic holomorphic line bundle over the torus with holomorphic structure D. Its *(complex) spectrum* is the subspace

$$\mathrm{Spec}(W, D) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*)$$

of complex monodromies occurring for non-trivial holomorphic sections of $\tilde{W}$. Equivalently,

$$\mathrm{Spec}(W, D) = \left\{ h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*) \mid \begin{array}{l} \exists\, \psi \in H^0(\tilde{W}),\ \psi \neq 0 \quad \text{such that} \\ \gamma^*\psi = \psi h_\gamma \quad \text{for all} \quad \gamma \in \Gamma \end{array} \right\}.$$

By construction, the spectrum is invariant under complex conjugation $\rho(h) = \bar{h}$ and

$$\mathrm{Spec}_{\mathbb{H}}(W, D) = \mathrm{Spec}(W, D)/\rho.$$

The abelian Lie group $G = \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ has Lie algebra $\mathfrak{g} = \mathrm{Hom}(\Gamma, \mathbb{C})$ whose exponential map $\exp : \mathfrak{g} \to G$ is induced by the exponential function $\mathbb{C} \to \mathbb{C}_*$. Furthermore, there is an isomorphism between the space $\mathrm{Harm}(T^2, \mathbb{C})$ of harmonic 1-forms and the Lie algebra $\mathrm{Hom}(\Gamma, \mathbb{C})$ given by $\omega \mapsto \int \omega$. Therefore, the exponential map gives rise to

$$\exp : \mathrm{Harm}(T^2, \mathbb{C}) \to \mathrm{Hom}(\Gamma, \mathbb{C}_*) : \omega \mapsto (\gamma \mapsto h_\gamma = e^{\int_\gamma \omega}).$$

The kernel of the group homomorphism exp is the lattice of harmonic forms $\Gamma^* = \mathrm{Harm}(T^2, 2\pi i\mathbb{Z})$ with integer periods and thus one gets an isomorphism

$$\mathrm{Harm}(T^2, \mathbb{C})/\Gamma^* \cong \mathrm{Hom}(T^2, \mathbb{C}_*).$$

In this way we lift the spectrum to the Γ^*- invariant *logarithmic spectrum*

$$\widetilde{\mathrm{Spec}}(W, D) := \exp^{-1}(\mathrm{Spec}(W, D)) \subset \mathrm{Harm}(T^2, \mathbb{C}),$$

which consists of all harmonic 1-forms ω whose corresponding monodormy representation $e^{\int \omega}$ occur for some holomorphic section ψ of the pullback line bundle $\tilde{W}$.

Because a homomorphism in $\mathrm{Hom}(\Gamma, \mathbb{C})$ uniquely extends to a homomorphism in $\mathrm{Hom}(\mathbb{R}^2, \mathbb{C})$, we can interpret $\omega \in \mathrm{Harm}(T^2, \mathbb{C})$ as $\int \omega \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{C})$. Using $e^{\int \omega} \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{C}_*)$ as a gauge transformation on the universal cover $\mathbb{R}^2$, we can define the operator

$$D_\omega = e^{-\int \omega} \circ D \circ e^{\int \omega} : \; \Gamma(W) \to \Gamma(\bar{K}W), \tag{2.1.4}$$

given by $D_\omega \psi = (D(\psi e^{\int \omega}))e^{-\int \omega}$ where $\psi \in \Gamma(W)$. The Leibniz rule of a quaternionic holomorphic structure implies

$$D_\omega(\psi) = D\psi + (\psi\omega)''.$$

Since D and ω are defined on the torus, D_ω is also well defined on the torus T^2. The operator D_ω is elliptic but due to the term $(\psi\omega)''$ only a complex linear (rather than quaternionic linear) operator between the complex rank 2 bundles W and $\bar{K}W$ where the complex structure I is given by multiplication $I(\psi) = \psi\mathbf{i}$ by the quaternion $\mathbf{i}$. A section $\psi \in \Gamma(W)$ is in the kernel of D_ω if and only if the section $\psi e^{\int \omega} \in \Gamma(\tilde{W})$ is in the kernel of D, that is to say, $\psi e^{\int \omega} \in H^0(\tilde{W})$ is holomorphic. Because the section $\psi e^{\int \omega}$ has monodromy $h = e^{\int \omega}$, we obtain the I-complex linear isomorphism

$$\mathrm{Ker}\, D_\omega \to H^0_h(\tilde{W}) : \quad \psi \mapsto \psi e^{\int \omega}.$$

Therefore, the logarithmic spectrum is the locus of harmonic forms

$$\widetilde{\mathrm{Spec}}(W, D) = \{\omega \in \mathrm{Harm}(T^2, \mathbb{C}) | \, \mathrm{Ker}\, D_\omega \neq 0\} \subset \mathrm{Harm}(T^2, \mathbb{C}).$$

for which D_ω has a non-trivial kernel.

Calculate the index of D_ω in the same way as (2.1.3):

$$\mathrm{Ind}(D_\omega) = \mathrm{Ind}(\bar{\partial}) = 2\deg(W) + \frac{1}{2}r(W)\chi(T^2) = 2\deg W.$$

If $d > 0$, the analytical index formula $\mathrm{Ind}\, D_\omega = \dim(\mathrm{Ker}\, D_\omega) - \dim(\mathrm{Coker}\, D_\omega)$ implies $\dim(\mathrm{Ker}\, D_\omega) \geq 2$. Thus there is always a non-trivial kernel for any D_ω, that is, $\widetilde{\mathrm{Spec}}(W, D) = \mathrm{Harm}(T^2, \mathbb{C})$.

In the case $d < 0$ we use the quternionic Plüker formula with monodormy ((4.2.27) in Appendix A):

$$\mathcal{W}(W, D) \geq 4\pi((n+1)(n(1-g) - d) + \mathrm{ord}H)$$

for every $n+1$-dimensional linear system $H \subset H^0(\tilde{W})$ with monodromy. The estimate

$$(n+1) \leq \frac{1}{4\pi(-d)}\mathcal{W}(W, D)$$

implies only finite number of $h \in \mathrm{Spec}(W, D)$ admit non-trivial holomorphic sections with monodromy. Thus $\dim(\mathrm{Spec}(W, D)) = 0$.

So we may assume that W is of degree zero. In the next chapter we construct the determinant bundle $\mathcal{L} = \lambda(\mathrm{Ker}\, D_\omega)^* \otimes \lambda(\mathrm{Coker}\, D_\omega)$ and the determinant function $\det(D_\omega)$ associated with a family of elliptic first order differential operators by Quillen [22], Bismut and Freed [3]. Then $\mathrm{Spec}(W, D)$ is exactly the divisor of the holomorphic function $\det(D_\omega)$, that is, a complex analytic subset of $\mathrm{Hom}(\Gamma, \mathbb{C}_*) \cong \mathbb{C}_* \times \mathbb{C}_*$.

Note that in the case when $W = V/L$ is the induced bundle from a conformal immersion, $\deg(W) = 0$ is equivalent by (2.1.2) to topological triviality of the normal bundle of the immersion.

Definition 2.1.5. Let $f : T^2 \to S^4$ be a conformal immersion with normal bundle degree 0. The *spectral curve* Σ of f is the Riemann surface normalizing the spectrum $h : \Sigma \to \mathrm{Spec}(V/L, D)$, where $(V/L, D)$ is the associated quotient quaternionic line bundle of degree zero over T^2.

Under the normalization, the involution ρ satisfies $h \circ \rho = \bar{h}$ so that it has no fixed points.

In particular, the spectral curve Σ_0, which normalizes the spectrum $\mathrm{Spec}(\bar{\partial}, V/L)$ of the complex holomorphic structure $\bar{\partial}$, is called the *vacuum spectral curve.*

2.1.4 The coordinate version of the spectrum of immersions of tori with degree zero

Let $L \subset V$ be a conformal immersion of T^2 with normal bundle degree 0 into $\mathbb{HP}^1$. The spectrum of $L \subset V$ is the spectrum of the quaternionic holomorphic line bundle $W := V/L$ of degree 0.

From the identification $T^2 \cong \mathbb{C}/\Gamma$, we have an uniformizing coordinate z on the torus. The degree of W or, equivalently, of the complex line subbundle $W_+ = \{v \in W | Jv = vi\}$ is zero, which implies we can trivialize the bundle W.

There is an unique flat quaternonic connection $\hat{\nabla}$ on W with unitary holonomy, satisfying $\hat{\nabla}'' = \bar{\partial}$ and $\hat{\nabla} J = 0$. The restriction of the connection $\hat{\nabla}$ to W_+, which is still denoted by $\hat{\nabla}$, has unitary holonomy $\hat{h} \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$. We can define a number $b_0 \in \mathbb{C}$ up to the ratios in the lattice Γ' (defined by (2.1.6)) by

$$\hat{h}_\gamma = e^{-\bar{b}_0\gamma + b_0\bar{\gamma}}.$$

Thus $\nabla_+ = \hat{\nabla} - \bar{b}_0 dz + b_0 d\bar{z}$ is the trivial connection on W_+. In order to trivialize W we choose a parallel section $\psi \in \Gamma(W_+)$ with respect to ∇_+. Thus $\bar{\partial}\psi = -\psi b_0 d\bar{z}$ from $\nabla''_+\psi = 0$.

Since $\deg W_+ = 0$, ψ has no zeros. Using ψ we trivialize the quaternionic line bundles W and $\bar{K}W$ as follows:

$$C^\infty(T^2, \mathbb{C}^2) \to \Gamma(W) \quad (u_1, u_2) \mapsto \psi((u_1 + ju_2)$$

and

$$C^\infty(T^2, \mathbb{C}^2) \to \Gamma(\bar{K}W) \quad (u_1, u_2) \mapsto \psi d\bar{z}((u_1 + ju_2).$$

We write the harmonic forms ω in the form of

$$\omega = (a + \bar{b}_0)dz + (b + b_0)d\bar{z} \quad \text{for} \quad (a, b) \in \mathbb{C}^2.$$

Let q denote the complex function defined by $Q\psi = \psi j dz q$. The family of operators D_ω acting on W yields

$$\begin{aligned}
D_\omega(\psi(u_1 + ju_2)) &= \bar{\partial}(\psi(u_1 + ju_2)) + Q(\psi(u_1 + ju_2)) + (\psi(u_1 + ju_2)\omega)'' \\
&= -\psi b_0 d\bar{z}(u_1 + ju_2) + \psi(\frac{\partial u_1}{\partial \bar{z}} + j\frac{\partial u_2}{\partial z}) \\
&\quad + \psi j dz q(u_1 + ju_2) + (\psi(u_1 + ju_2)((a + \bar{b}_0)dz + (b + b_0)d\bar{z}))'' \\
&= \psi d\bar{z}(-b_0 u_1 - ju_2\bar{b}_0) + \psi(\frac{\partial u_1}{\partial \bar{z}} + j\frac{\partial u_2}{\partial z}) \\
&\quad + \psi d\bar{z}(-\bar{q}u_2 + jqu_1) + \psi d\bar{z}(ju_2 a + ju_2\bar{b}_0 + u_1 b + u_1 b_0) \\
&= \psi d\bar{z}(\frac{\partial u_1}{\partial \bar{z}} + bu_1 - \bar{q}u_2 + j(\frac{\partial u_2}{\partial z} + au_2 + qu_1)),
\end{aligned}$$

or equivalently,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial u_1}{\partial \bar{z}} + bu_1 - \bar{q}u_2 \\ \frac{\partial u_2}{\partial z} + au_2 + qu_1 \end{pmatrix}.$$

Under these coordinates z, (u_1, u_2), (a, b), the operator D_ω takes the form

$$D_{a,b} = \bar{\partial}_{a,b} + M$$

with

$$\bar{\partial}_{a,b} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}} + b & 0 \\ & \frac{\partial}{\partial z} + a \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & -\bar{q} \\ q & 0 \end{pmatrix}. \tag{2.1.5}$$

Then the logarithmic spectrum $\widetilde{\mathrm{Spec}}(W, D)$ and logarithmic vacuum spectrum $\widetilde{\mathrm{Spec}}(W, \bar{\partial})$ have their coordinate versions

$$\tilde{\mathcal{S}}(q) = \{(a, b) \in \mathbb{C}^2 | \ker(D_{a,b}) \neq 0\}$$

and
$$\tilde{\mathcal{S}}(0) = \{(a,b) \in \mathbb{C}^2 |\ker(\bar{\partial}_{a,b}) \neq 0\}.$$

We notice that in the above coordinate $\omega_1 - \omega_2 \in \text{Harm}(T^2, 2\pi i\mathbb{Z})$ is equivalent to
$$(a_2, b_2) = (a_1, b_1) + (-\bar{c}, c)$$

for some $c \in \Gamma'$, where the lattice Γ' is
$$\Gamma^{'} = \{c \in \mathbb{C}| - \bar{c}\gamma + c\bar{\gamma} \in 2\pi i\mathbb{Z} \text{ for all } \gamma \in \Gamma\}. \tag{2.1.6}$$

Thus,
$$\mathcal{S}(q) = \tilde{\mathcal{S}}(q)/\Gamma' \quad \text{and} \quad \mathcal{S}(0) = \tilde{\mathcal{S}}(0)/\Gamma'$$

are the coordinate versions of the spectrum $\text{Spec}(W, D)$ and the vacuum spectrum $\text{Spec}(W, \bar{\partial})$.

2.2 Darboux transformations

We introduce Darboux transformations for a conformal immersion $f : M \to S^4$ of a Riemann surface M into the 4-sphere. They are closely related to the spectral curve discussed in the last section.

The classical Darboux transform [9] of a pair of immersed surfaces $f, f^\sharp : M \to \mathbb{R}^3$ in $\mathbb{R}^3$ is that there exists a 2-sphere congruence $S : M \to \{\text{spheres or planes in } \mathbb{R}^3\}$ enveloping f and $f^\sharp$, that is, $f(p), f^\sharp(p) \in S(p)$ and $d_pf(T_pM) = T_{f(p)}S_p = d_pf^\sharp(T_pM)$. Darboux showed that such pair of surfaces are isothermic surfaces.

Similarly, the Darboux transformation of conformal immersions is also given by a non-linear, Möbius geometric touching condition with respect to a 2-sphere congruence. However, we relax the touching condition to avoid both surfaces to be isothermic.

2.2.1 Definition of the Darboux transformation

We recall that given a Riemann surface M a sphere congruence is a complex structure $S \in \Gamma(\text{End}(V))$ on the trivial $\mathbb{H}^2$-bundle V over M.

Definition 2.2.1. Let $f : M \to S^4$ be a conformal immersion with induced line bundle $L = f^*\mathcal{T} \subset V$. A sphere congruence $S \in \Gamma(\text{End}(V))$ *left-envelopes* respectively *right-envelopes* f if
$$SL = L \quad \text{and} \quad * \delta = S\delta \quad \text{respectively} \quad * \delta = \delta S$$

A sphere congruence $S \in \Gamma(\text{End}(V))$ *envelopes* f if S left-envelopes and also right-envelopes f, that is,
$$SL = L \quad \text{and} \quad * \delta = S\delta = \delta S$$

As we discussed in the conditions for the mean curvature sphere, $SL = L$ means that for all points $p \in M$ the spheres $S(p)$ pass through $f(p)$. And $*\delta = S\delta = \delta S$ says that the oriented tangent spaces to f coincide with the oriented tangent spaces to the spheres $S(p)$ at $f(p)$.

The fact that two distinct conformal immersions f and $f^\sharp$ from the same Riemann surface M which are both enveloped by the same sphere congruence S are isothermic surfaces [9, **?**] is too restrictive for us. That is why we relax the enveloping condition to the half-enveloping condition. $*\delta = S\delta$ implies that the oriented tangent space at p to f is left-touching to the oriented tangent space of the sphere $S(p)$ at $f(p)$. Here two oriented planes through the origin in $\mathbb{R}^4$ are left-touching if their associated oriented great circles on S^3 correspond via right translation in the group S^3.

Definition 2.2.2. Let M be a Riemann surface. A conformal map $f^\sharp : M \to S^4$ is called a *Darboux transform* of a conformal immersion $f : M \to S^4$ if $f^\sharp(p)$ is distinct from $f(p)$ at all points $p \in M$, and if there exists a sphere congruence S which envelopes f and left-envelopes $f^\sharp$:

$$V = L \oplus L^\sharp, \quad SL = L, \quad *\delta = S\delta = \delta S \quad \text{and} \quad SL^\sharp = L^\sharp, \quad *\delta^\sharp = S\delta^\sharp$$

where $L = f^*\mathcal{T}$ and $L^\sharp = (f^\sharp)^*\mathcal{T}$ are the induced line subbundles of V.

On the other hand we can also describe Darboux transforms in terms of flat adapted connections and hence there is another definition.

Lemma 2.2.3. *Let $L \subset V$ be a conformal immersion. A subbundle $L^\sharp \subset V$ of V is a* Darboux transform *of L if and only if it satisfies*

$$V = L \oplus L^\sharp \quad \textit{and} \quad \delta \wedge \delta^\sharp = 0$$

where δ and $\delta^\sharp$ are given by the decomposition of the trivial connection d with respect to the splitting $V = L \oplus L^\sharp$:

$$d = \begin{pmatrix} \nabla^L & \delta^\sharp \\ \delta & \nabla^\sharp \end{pmatrix}. \tag{2.2.1}$$

Proof. Notice that $f, f^\sharp : M \to S^4$ maps distinct from each other at all points $p \in M$ is equivalent to $V = L \oplus L^\sharp$ with $L = f^*\mathcal{T}$ and $L^\sharp = (f^\sharp)^*\mathcal{T}$. The conformal condition $*\delta = J\delta = \delta\tilde{J}$ defines the unique sphere congruence S touching f and containing $f^\sharp$, which is expressed in the splitting $V = L \oplus L^\sharp$

$$\begin{pmatrix} \tilde{J} & 0 \\ 0 & J \end{pmatrix}.$$

Then S left-touches $f^\sharp$, i.e. $*\delta^\sharp = S\delta^\sharp$, if and only if

$$*\delta^\sharp = \tilde{J}\delta^\sharp \quad \Leftrightarrow \quad \delta^\sharp \in \Gamma(K\,\mathrm{Hom}(L^\sharp, V/L^\sharp) \quad \Leftrightarrow \quad \delta \wedge \delta^\sharp = 0.$$

$\square$

We identify $V/L = L^\sharp$ and $V/L^\sharp = L$. The trivial connection d on V decomposes as (2.2.1) and the flatness of d becomes

$$0 = \begin{pmatrix} R^{\nabla^L} + \delta^\sharp \wedge \delta & d\delta^\sharp \\ d\delta & R^{\nabla^\sharp} + \delta \wedge \delta^\sharp \end{pmatrix}, \tag{2.2.2}$$

where R^{∇^L} respectively $R^{\nabla^\sharp}$ denotes the curvature of ∇^L respectively $\nabla^\sharp$. The equation $R^{\nabla^\sharp} + \delta \wedge \delta^\sharp = 0$ shows that $L^\sharp$ is a Darboux transform (i.e. $\delta \wedge \delta^\sharp = 0$) if and only if $\nabla^\sharp$ is flat.

2.2.2 Darboux transforms related to the spectrum of a conformal immersion

Let $f^\sharp : M \to S^4$ be a Darboux transform of a conformal immersion $f : M \to S^4$, or equivalently, $L^\sharp \subset V$ be a Darboux transformation of $L \subset V$. The canonical isomorphism $\pi : L^\sharp \to V/L$ pushes forward the connection $\nabla^\sharp$ to a connection ∇ on V/L satisfying

$$\nabla\pi|_{\Gamma(L)} = \pi\nabla^\sharp = \pi d|_{\Gamma(L^\sharp)}.$$

By construction the connection ∇ is *adapted* to the complex structure D on V/L defined in (2.1.1), that is, $\nabla'' = D$.

In this way we find a correspondence between the space of Darboux transforms of f and the space of flat adapted connections on V/L. On the other hand we can compute Darboux transforms from a given flat adapted connections by the following lemmas.

Lemma 2.2.4. *Let $f : M \to S^4$ be a conformal immersion with $*\delta = J\delta = \delta\tilde{J}$ and $f^\sharp : M \to S^4$ be a map so that $V = L \oplus L^\sharp$. Then the map $f^\sharp$ is a Darboux transform of f if and only if there is a non-trivial section $\psi^\sharp \in \Gamma(\widetilde{L^\sharp})$ with monodromy satisfying $d\psi^\sharp \in \Omega^1(\tilde{L})$.*

Proof. For $\psi \in \Gamma(\widetilde{L^\sharp})$, the decomposition (2.2.1) of d implies

$$d\psi = \delta^\sharp\psi + \nabla^\sharp\psi.$$

Therefore, ψ is $\nabla^\sharp$-parallel if and only if $d\psi \in \Omega^1(\tilde{L})$. Since flatness of $\nabla^\sharp$ is equivalent to the existence of a non-trivial parallel section with monodromy of the line bundle $\widetilde{L^\sharp}$, the statement follows. □

Lemma 2.2.5. *Let $f : M \to S^4$ be a conformal immersion. Then the canonical projection $\pi : V \to V/L$ induces a bijective correspondence between sections $\hat{\psi} \in \Gamma(V)$ of V satisfying $d\hat{\psi} \in \Omega^1(L)$ and holomorphic sections $\psi \in H^0(V/L)$.*

Proof. Let $\hat{\psi}_0$ be a lift of $\psi \in H^0(V/L)$, that is, $\pi\hat{\psi}_0 = \psi$. Then we get $\pi d\hat{\psi}_0 \in \Gamma(KV/L)$ from

$$(\pi d\hat{\psi}_0)'' = D\pi\hat{\psi}_0 = D\psi = 0.$$

Since f is an immersion $\delta \in \Gamma(\mathrm{Hom}(L, KV/L))$ is a bundle isomorphism. Therefore, there exist a unique section $\varphi \in \Gamma(L)$ such that $\delta\varphi = \pi d\hat{\psi}_0$. Set $\hat{\psi} := \hat{\psi}_0 - \varphi$, then this is the unique section of V satisfying

$$\pi\hat{\psi} = \pi\hat{\psi}_0 - \pi\varphi = \psi \quad \text{and} \quad \pi d\hat{\psi} = \pi d\hat{\psi}_0 - \delta\varphi = 0.$$

□

Definition 2.2.6. The *prolongation* of a holomorphic section $\psi \in H^0(V/L)$ is the unique section $\hat{\psi} \in \Gamma(V)$ with $\pi\hat{\psi} = \psi$ satisfying $d\hat{\psi} \in \Omega^1(L)$.

Given a flat adapted connection ∇ on V/L, we want to compute the corresponding Darboux transform of f. Firstly there exists a parallel section $\psi \in \Gamma(\widetilde{V/L})$ over the universal cover $\tilde{M}$ of M. ∇ is adapted implies that $D\psi = 0$. So $\psi \in H^0(\widetilde{V/L})$ is actually a holomorphic section with monodromy satisfying $\gamma^*\psi = \psi h_\gamma$ for a representation $h \in \pi_1(M) \to \mathbb{H}_*$. Then the prolongation $\hat{\psi} \in \Gamma(\tilde{V})$ is a section with the same monodromy h. Moreover, as a parallel section ψ has no zeros and neither does $\hat{\psi}$. This defines a line bundle

$$L^\sharp = \hat{\psi}\mathbb{H} \subset V$$

over M which satisfies $V = L \oplus L^\sharp$. The existence of the non-trivial section $\hat{\psi} \in \Gamma(\tilde{L^\sharp})$ with monodromy satisfying $d\hat{\psi} \in \Omega^1(\tilde{L})$ shows by lemma 2.2.4 that the map $f^\sharp : M \to S^4$ corresponding to the line bundle $L^\sharp$ is the Darboux transform of f belonging to the adapted connection ∇.

The parallel sections of flat adapted connections on V/L are precisely the holomorphic sections with monodromy of V/L that are nowhere vanishing. This is stronger than the condition in lemma 2.2.4. A holomorphic section $\psi \in \Gamma(\widetilde{V/L})$ has isolated zeroes [10]. Away from the finite set of zeros there is a unique flat adapted connection ∇ on V/L with $\nabla\psi = 0$. Hence, the prolongation $\hat{\psi}$ of ψ defines a Darboux transform $f^\sharp$ of f away from the zero locus of ψ by $L^\sharp = \hat{\psi}\mathbb{H}$. It follows from [10] that $f^\sharp$ extends continuously across the zeros of ψ where it agrees with f. Such $f^\sharp$ are called *singular* Darboux transforms of f. Therefore, in addition to the two definitions we characterize Darboux transforms in terms of solutions to a linear elliptic equation.

Lemma 2.2.7. *Let $f : M \to S^4$ be a conformal immersion. Then there is a bijective correspondence between the space of (singular) Darboux transforms $f^\sharp : M \to S^4$ of f and the space of non-trivial holomorphic sections with monodromy up to scale of V/L. Under this correspondence, non-singular Darboux transforms get mapped to nowhere vanishing holomorphic sections with monodromy up to scale of V/L.*

When $f : T^2 \to S^4$ is a conformally immersed 2-torus of zero normal bundle degree, the space of non-trivial holomorphic sections with monodromy up to scale of V/L relates exactly to the spectrum we discussed in the last section. So the spectral curve of f, as a Riemann surface, parameterizes the space of Darboux transforms of f.

One may ask about the relation of the spectral curve of a Darboux transform and that of the original conformally immersed torus. We state the result [7] with out proof.

Theorem 2.2.8. *Let $f, f^\sharp : T^2 \to S^4$ be conformal immersions so that $f^\sharp$ is a Darboux transform of f. Then the spectral curves of f and $f^\sharp$ agree, that is, $\Sigma = \Sigma^\sharp$.*

This theorem shows that the Darboux transform is *isospectral.*

2.2.3 Darboux transforms preserve the Willmore energy

In this subsection we compute the Willmore energy of the Darboux transform of a conformal immersion of a compact surface. The result is that the Darboux transform preserves the Willmore energy up to topological quantities. In particular, for a conformally immersed torus with degree 0 normal bundle the Darboux transform preserves the Willmore energy.

Let $f : M \to S^4$ be a conformal immersion of a compact Riemann surface M with $*\delta = J\delta = \delta\tilde{J}$ and $f^\sharp : M \to S^4$ be a Darboux transform of f. With respect to $V = L \oplus L^\sharp$ the trivial connection d splits to

$$\begin{pmatrix} \nabla^L & \delta^\sharp \\ \delta & \nabla^\sharp \end{pmatrix}.$$

It induces a holomorphic structure $(\nabla^L)''$ on L and a flat adapted connection on V/L. With a type argument d^∇ is a holomorphic structure on the line bundle $K(V/L)$. For any holomorphic section $\varphi \in H^0(L)$ of L, the section $\delta\varphi \in H^0(KV/L)$ is holomorphic because

$$d^\nabla(\delta\varphi) = (d\delta)\varphi - \delta \wedge \nabla^L\varphi = -\delta \wedge (\nabla^L)''\varphi = 0.$$

Here we use $d\delta = 0$ in the flat curvature equality (2.2.2) of d. Together with f being an immersion this implies that $\delta \in \Gamma(\mathrm{Hom}_+(L, K(V/L)))$ is a holomorphic bundle isomorphism. Hence $L \cong K(V/L)$. And with the canonical identification $L \cong V/L^\sharp$ induced by $\pi^\sharp : V \to V/L^\sharp$, we get

$$\mathcal{W}(V/L^\sharp) = \mathcal{W}(K(V/L)).$$

The decomposition of the holomorphic structure d^∇ on $K(V/L)$ into J commuting and anticommuting parts gives $d^\nabla = \bar\partial + \tilde{Q}$ and for $\omega \in \Gamma(KV/L)$

$$\tilde{Q}\omega = A \wedge \omega,$$

where $A \in \Gamma(K\,\mathrm{End}_-(V/L)$ is the K-part of the Hopf field of the adapted flat connection ∇ of V/L. Then the Willmore energy of $K(V/L)$ is given by

$$\mathcal{W}(K(V/L)) = 2\int_M < A \wedge *A >, \tag{2.2.3}$$

while the Willmore energy of V/L is given by

$$\mathcal{W}(V/L) = 2\int_M < Q \wedge *Q >, \tag{2.2.4}$$

where $Q \in \Gamma(\bar{K}\,\mathrm{End}_-(V/L)$ is the $\bar{K}$-part of the Hopf field of the adapted flat connection ∇ of V/L since $D = \nabla'' = \bar\partial + Q$.

According to the flatness of ∇, we have

$$R^{\hat{\nabla}} + A \wedge A + Q \wedge Q = 0,$$

where the underlying complex connection $\hat{\nabla}$ is given by $\nabla = \hat{\nabla} + A + Q$. The curvature of $\hat{\nabla}$ satisfies

$$R^{\hat{\nabla}} J = A \wedge *A - Q \wedge *Q. \tag{2.2.5}$$

We summerise (2.2.3), (2.2.4), (2.2.5) and get

$$4\pi \deg(V/L) = \mathcal{W}(V/L^\sharp) - \mathcal{W}(V/L).$$

By the formula (2.1.2) for the normal bundle degree of f, it follows

Theorem 2.2.9. *Let $f : M \to S^4$ be a conformal immersion of a compact Riemann surface M with $*\delta = J\delta = \delta\tilde{J}$ and $f^\sharp : M \to S^4$ be a Darboux transform of f. Then*

$$\mathcal{W}(f^\sharp) = \mathcal{W}(f) + 2\pi(\deg N_f - \deg K), \tag{2.2.6}$$

where $N_f = \mathrm{Hom}_-(L, V/L)$ is the normal bundle of f.
In particular, when $f : T^2 \to S^4$ is a comformally immersed torus with degree 0 normal bundle, then

$$\mathcal{W}(f) = \int_{T^2} |\mathcal{H}|^2 = \mathcal{W}(f^\sharp).$$

Darboux transforms preserve the Willmore energy for comformal tori with degree 0 normal bundle.

Chapter 3

The determinant line bundle

In the last chapter we have gotten the coordinate version of the logarithmic (vacuum) spectrum of the conformal immersion $f : T^2 \to S^4$. The spectrum corresponds to the operators with non-trivial kernel.

$$\tilde{\mathcal{S}} = \{(a,b) \in \mathbb{C}^2 |\, \mathrm{Ker}(D_{a,b}) \neq 0\},$$

$$\tilde{\mathcal{S}}_0 = \{(a,b) \in \mathbb{C}^2 |\, \mathrm{Ker}(\bar{\partial}_{a,b}) \neq 0\}.$$

We would like to note that the operators $D_{a,b}$ are Dirac operators with potential part Q, and in the case the potential part $Q = 0$, they are actually direct sums of two Cauchy-Riemman operators. That is why we will introduce in the next sections the construction of the determinant line bundle associated to a family of first order differential operators by Bismut and Freed [3]. It is an extension of Quillen's construction of determinants of Cauchy-Riemann operators.

3.1 Determinant line bundles associated to Fredholm operators

First we introduce the determinant line associated to a Fredholm operator $\mathcal{H}_1 \to \mathcal{H}_2$ with index zero, where $\mathcal{H}_1$, $\mathcal{H}_2$ are separable Hilbert spaces. Then we describe local smooth identifications of the determinant line bundle over the space of Fredholm operators with index zero.

3.1.1 Determinant lines

In the case of a linear transformation $T : V_1 \to V_2$ between two complex vector space with $\dim_{\mathbb{C}} V_1 = \dim_{\mathbb{C}} V_2 < \infty$, the kernel of the transformation T is non-trivial if and only if the determinant $\det(T)$ of the linear transformation is zero. There is another way to look at the determinant of operators. T induces a mapping on $\lambda(V_1) \to \lambda(V_2)$, where $\lambda(V)$ denotes the top exterior power of the vector space V (defined to be the trivial line $\mathbb{C}$, when $V = 0$). In other words the determinant number $\det(T)$ can be regarded as an element of the complex line

$$\mathrm{Det}_T = \lambda(V_1)^* \otimes \lambda(V_2).$$

If this diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_1 & \longrightarrow & V_1' & \longrightarrow & V_1'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T'} & & \downarrow{\scriptstyle T''} & & \\
0 & \longrightarrow & V_2 & \longrightarrow & V_2' & \longrightarrow & V_2'' & \longrightarrow & 0
\end{array}$$

satisfies that both of the rows are exact, then

$$\mathrm{Det}_{T'} \cong \mathrm{Det}_T \otimes \mathrm{Det}_{T''}.$$

One of the key properties of the finite dimensional determinant function is that $\det(T)$ is holomorphic in T since it is a polynomial in the entries of the matrix. And in the special case that $V_1 = V_2$, we have eigenvalues of T and $\det(T)$ is given by the product of the eigenvalues of T.

In the case of infinite dimensions there is an analogue of the complex line $\lambda(V_1)^* \otimes \lambda(V_2)$, the determinant line for a certain class of operators, Fredholm operators of index zero. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be separable Hilbert spaces.

Definition 3.1.1. An operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is Fredholm if and only if the range of T is closed in $\mathcal{H}_2$ and $\mathrm{Ker}(T)$ and $\mathrm{Coker}(T)$ are finite dimensional. Each Fredholm operator T has an index, defined as $\mathrm{Ind}\, T = \dim \mathrm{Ker}(T) - \dim \mathrm{Coker}(T)$.

Definition 3.1.2. The determinant line Det associated to each Fredholm operator of index zero, T, is defined to be

$$\mathrm{Det}_T = \lambda(\mathrm{Ker}\, T)^* \otimes \lambda(\mathrm{Coker}\, T).$$

In the case that T is invertible this determinant line is canonically isomorphic to the trivial line $\mathbb{C}$ and we set an element $\sigma_T = 1$ under this isomorphism. If T is not invertible we set $\sigma_T = 0$. We shall show that σ is a holomorphic section of the forthcoming determinant line bundle.

3.1.2 Smoothing determinant line bundles

Let $\mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of Fredholm operators mapping $\mathcal{H}_1$ to $\mathcal{H}_2$. As an open subset of the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, $\mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2)$ is an infinite dimensional complex Banach manifold. The index defined above is a continuous integer valued function $\mathrm{Ind} : \mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{Z}$. This surjective function indicates that $\mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2)$ consists of a series of connected components. The components of index zero is denoted by $\mathrm{Fred}_0(\mathcal{H}_1, \mathcal{H}_2)$.

Proposition 3.1.3. *The family of lines*

$$\coprod_{T \in \mathrm{Fred}_0(\mathcal{H}_1, \mathcal{H}_2)} \mathrm{Det}_T$$

forms a holomorphic line bundle over the space $\mathrm{Fred}_0(\mathcal{H}_1, \mathcal{H}_2)$*, called the* ***determinant line bundle****.*

Proof. To show the holomorphic structure of $\mathrm{Det} \to \mathrm{Fred}_0(\mathcal{H}_1, \mathcal{H}_2)$ we trivialize the line bundle locally on the open sets of the space $\mathrm{Fred}_0(\mathcal{H}_1, \mathcal{H}_2)$. For each finite dimensional subspace F of $\mathcal{H}_1$ we define open subset $U_F = \{T \in | \operatorname{Im}(T) + F = \mathcal{H}_2\} \subset \mathrm{Fred}_0(\mathcal{H}_1, \mathcal{H}_2)$, which contains all the T with the range transversal to F. Then we have an exact sequence for each $T \in U_F$

$$0 \longrightarrow \operatorname{Ker} T \longrightarrow T^{-1}F \xrightarrow{T} F \longrightarrow \operatorname{Coker} T \longrightarrow 0 \tag{3.1.1}$$

$\dim \operatorname{Ker} T = \dim \operatorname{Coker} T$ implies that $\dim T^{-1}F = \dim F$ for all T. There exists a canonical isomorphism $\mathrm{Det}_T \cong \lambda(T^{-1}F)^* \otimes \lambda(F)$. Since the finite dimensional vector bundle $T^{-1}F : T \in U_F$ is holomorphic over U_F, $\lambda(T^{-1}F)^* \otimes \lambda(F)$ is a holomorphic line bundle over U_F.

If G is another finite dimensional subspace of $\mathcal{H}_2$ and we need to show that the trivialization over U_G and U_F agree on the overlap. Without loss of generality we assume that G is a subspace of F, then $U_G \subset U_F$ and we have an exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow F/G \longrightarrow 0\,.$$

Hence $\lambda(F) \cong \lambda(G) \otimes \lambda(F/G)$. Further, the exact sequence

$$0 \longrightarrow T^{-1}G \longrightarrow T^{-1}F \longrightarrow T^{-1}F/T^{-1}G \longrightarrow 0\,.$$

implies the canonical isomorphism $\lambda(T^{-1}F) \cong \lambda(T^{-1}G) \otimes \lambda(T^{-1}F/T^{-1}G)$. The operator T induces an isomorphism $T^{-1}F/T^{-1}G \cong F/G$. Thus we get the trivializations agree on the overlap:

$$\begin{aligned}
\mathrm{Det}_T &\cong \lambda(T^{-1}F) \otimes \lambda(F) \\
&\cong (\lambda(T^{-1}G) \otimes \lambda(T^{-1}F/T^{-1}G))^* \otimes (\lambda(G) \otimes \lambda(F/G)) \\
&\cong \lambda(T^{-1}G) \otimes \lambda(G).
\end{aligned}$$

Since $\bigcup_{F \subset \mathcal{H}_2 finite-dim} U_F = \mathrm{Fred}_0(\mathcal{H}_1, \mathcal{H}_2)$, the above argue shows that Det_T is a holomorphic line bundle over the space $\mathrm{Fred}_0(\mathcal{H}_1, \mathcal{H}_2)$. □

3.2 Quillen determinant line bundles

We repeat the construction by Bismut and Freed [3] of the Quillen metric and a unitary connection on the determinant bundle associated with a family of elliptic first order differential operators as an extension of the earlier results by Quillen [22] for the case of a family of Cauchy-Riemann operators on a Riemann surface. The curvature of the unitary connection on the determinant line bundle will be computed in terms of asymptotic expansions of certain heat kernels.

3.2.1 Description of the setting

We now return to the case of the elliptic first order differential operators D acting in a infinite dimensional bundle over a Riemann surface. Hence we first describe the fibered manifolds $N \xrightarrow{X} B$ and the connections on the corresponding infinite dimensional bundle H^∞.

Let B be a connected manifold of dimension m endowed with a smooth Riemannian metric g_B and let X be a connected oriented spin manifold of even dimension $2l$ endowed with a metric g_X. Let N be an $m+2l$ dimensional connected manifold fibering $\pi : N \to B$ over B, satisfying that there is an open covering $\mathfrak{U}$ of B such that if $U \in \mathfrak{U}$, $\pi^{-1}(U)$ is diffeomorphic to $U \times X$. For $w \in B$, we denote the fiber by $X_w := \pi^{-1}(\{w\})$.

The tangent bundle TX of X is the $2l$ dimensional subbundle of the tangent bundle TN of N whose fiber at $y \in N$ is $T_y X_{\pi(y)}$. Let $T^H M$ be a smooth subbundle of TN such that

$$TN = T^H N \oplus TX.$$

$T^H N$ is called the horizontal part of TN, and TX the vertical part of TN.

Let O be the $SO(2l)$ bundle of oriented orthonormal frames in TX, O' a $Spin(2l)$ bundle which lifts O such that $O' \to O$ induces the covering projection $Spin(2l) \to SO(2l)$ on each fiber.

$Spin(2l)$ acts unitarily on the vector space of spinors $S = S_+ \oplus S_-$. F, $F_\pm$ denoted the Hermitian bundles of spinors

$$F = O' \times_{spin(2l)} S, \quad F_\pm = O' \times_{spin(2l)} S_\pm.$$

Under π_*, $T^H_y N$ and $T_{\pi(y)} B$ are isomorphic. We lift the scalar product of TB in $T^H N$. Then TN is naturally endowed with a metric $g_B \oplus g_X$. Let ∇^L be the Levi-Civita connection on TN.

Definition 3.2.1. ∇ denote the connection on TX, $\nabla = P_X \nabla^L$, where P_X is the projection operator of TN on TX.

∇ lifts naturally with a unitary connection on $F_\pm$. Let ξ be a Hermitian bundle on N, which is endowed with a unitary connection ∇^ξ. The Hermitian bundles $F_\pm \otimes \xi$ are then naturally endowed with a unitary connection which we also denote ∇.

Denote by $H^\infty = H^\infty_+ \oplus H^\infty_-$ the set of C^∞-sections of $F \otimes \xi = F_+ \otimes \xi \oplus F_- \otimes \xi$ over N. We will regard H^∞, $H^\infty_\pm$ as being the sets of C^∞-sections over B of infinite dimensional bundles which still note H^∞, $H^\infty_\pm$. For $w \in B$, H^∞_w, $H^\infty_{\pm,w}$ are the sets of C^∞-sections over X_w of $F \otimes \xi$, $F_\pm \otimes \xi$.

Let dx be the Riemannian volume element of in the fibers X. H^∞_w is naturally endowed with the Hermitian product

$$< h, h' >_w = \int_{X_w} < h, h' > (x) dx.$$

For $Y \in TB$, let Y^H be the lift of Y in $T^H N$, so that

$$Y^H \in T^H N, \quad \pi_* Y^H = Y.$$

We now define a connection on H^∞, denoted by $\tilde{\nabla}$, which is characterized by

$$\tilde{\nabla}_Y h = \nabla_{Y^H} h$$

for any $h \in H^\infty$.
However, $\tilde{\nabla}$ is in general not unitary on H^∞, since the volume element dx generally is not invariant under the holonomy group of the connection $\tilde{\nabla}$. If Y is a smooth vector field on B, Y^H acts on the volume element dx of X. For any $y \in N$, the divergence $div_X Y^H(y)$ of Y^H is well defined.
Let k denote the smooth vector field in $T^H N$ such that for any $Y \in TB$

$$div_X Y^H(y) = 2 < k, Y^H > (y).$$

Definition 3.2.2. $\tilde{\nabla}^u$ is the connection on H^∞ defined by the relation

$$\tilde{\nabla}^u_Y = \tilde{\nabla}_Y + < k, Y^H > . \tag{3.2.1}$$

After this modification, we have:

Proposition 3.2.3. *The connection $\tilde{\nabla}^u$ is unitary on H^∞.*

Proof. : For any $h, h' \in H^\infty$ we have the relation

$$\begin{aligned} Y \int_X < h, h' > (x)dx &= \int_X (< \tilde{\nabla}_Y h, h' > + < h, \tilde{\nabla}_Y h' > + div_X(Y)^H < h, h' >)dx \\ &= \int_X (< (\tilde{\nabla}_Y + < k, Y^H >)h, h' > + < h, (\tilde{\nabla}_Y + < k, Y^H >)h' >)dx \\ &= \int_X (< \tilde{\nabla}^u_Y h, h' > + < h, \tilde{\nabla}^u_Y h' >)dx. \end{aligned}$$

□

For our problem we consider the case that $B = \mathrm{Harm}(T^2, \mathbb{C})$ is the space of harmonic forms on a torus $X = T^2 = \mathbb{C}/\Gamma$ and $N = B \times T^2$ is the direct product of the two above manifolds. Since T^2 is a Kähler manifold, the spinors bundle is $S = \bigwedge^* T_{\mathbb{C}} T^2 \cong \bigwedge^* \bar{T}^*_{\mathbb{C}} T^2$. The spinor bundle has a Hermitian metric and connection which we will introduce later. The space of all smooth sections is $C^\infty(S) \cong \Omega^{0,0}(T^2) \oplus \Omega^{0,1}(T^2)$. The $(0,0)$-forms and $(0,1)$-forms on T^2 form a cochain complex under the operator $\bar{\partial}$, called the *Dolbeault complex*:

$$\Omega^{0,0}(T^2) \xrightarrow{\bar{\partial}} \Omega^{0,1}(T^2) \xrightarrow{\bar{\partial}} 0$$

Since T^2 is a Kähler manifold, the Dirac operator of S is exactly the 'Dolbeault operator' $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ (see propositon 3.27 in [19]).
The direct product $N = B \times T^2$ implies that $O' \to O$ is trivial and the Hermitian bundle of spinors F is exactly the pull-back of the spin bundle S on T^2, that is

$$F = \pi_1^* S$$

where $\pi_1 : N \to T^2$ is the canonical projection. F has fibers $F_{z,\omega} = S_z$. The Levi-Civita connection on TN is trivial, $\nabla^L = d_N = d_B + d_{T^2}$. The projection of the connection ∇^L to TX is also trivial $\nabla = P_{T^2}\nabla^L = d_{T^2}$. ∇ lifts into a unitary connection on F in the trivial form $\nabla^F = d_N$.

The Hermitian bundle ξ on N is the pull-back of a Hermitian bundle W on T^2

$$\xi = \pi_1^* W.$$

ξ is endowed with a unitary connection, denoted by ∇^ξ, of the form

$$\nabla^\xi = d + \begin{pmatrix} \omega'' - \bar{\omega}'' & 0 \\ 0 & \omega' - \bar{\omega}' \end{pmatrix}.$$

Then the induced unitary connection ∇ on $F \otimes \xi = \pi_1^*(S \otimes W)$ is given by

$$\nabla = \nabla^F \otimes \nabla^\xi = d_N + \begin{pmatrix} \omega'' - \bar{\omega}'' & 0 \\ 0 & \omega' - \bar{\omega}' \end{pmatrix}.$$

The set of C^∞-sections of the Hermitian bundle $F \otimes \xi$ is $H^\infty_\omega = H^\infty_{+,\omega} \oplus H^\infty_{-,\omega} = \Gamma(W) \oplus \Gamma(\bar{K}W)$ over T^2. ∇ induces the trivial connection $\tilde{\nabla} = \nabla \mid_{TB} = d_B$ on H^∞.

Since the horizontal lift of any smooth tangent vector field Y on B is

$$Y^H = (Y, 0) \in \Gamma(TN),$$

the modification vector field $k = 0$. Thus the unitary connection $\tilde{\nabla}^u$ induced by $\tilde{\nabla}$ on H^∞ equals to $\tilde{\nabla} = d_B$.

It is easy to see that the family of operators D_ω defined in last chapter maps $H^\infty_{+,\omega} = \Gamma(W)$ to $H^\infty_{-,\omega} = \Gamma(\bar{K}W)$.

3.2.2 Quillen's superconnections and asymptotic expansions

In this part we describe the family of super-connections corresponding to a family of first order elliptic differential operators and the asymptotic expansions of their traces and super-traces, whose finite parts will play an important role in expressing the canonical connection on the determinant line bundle.

Let $D_{+,w}$ be a smooth family of first order elliptic differential operators acting fiberwise on X which sends H^∞_+ into H^∞_-. $D_{-,w}$ denotes the formal adjoint operator of $D_{+,w}$ with respect to the Hermitian product. Consider the operator

$$D_w = \begin{pmatrix} 0 & D_{-,w} \\ D_{+,w} & 0 \end{pmatrix}$$

acting on H^∞_w. Then D_w is a smooth family of elliptic self-adjoint first order differential operators.

$H^\infty = H^\infty_+ \oplus H^\infty_-$ is a $\mathbb{Z}_2$-graded vector bundle over B. Let τ be the involution of H^∞ satisfying $\tau = \pm 1$ on $H^\infty_\pm$. Then $\mathrm{End}(H^\infty)$ is naturally $\mathbb{Z}_2$-graded as well. For a given $w \in B$, $\mathrm{End}(H^\infty_w) \hat{\otimes} \Lambda(T^*_w B)$ is also $\mathbb{Z}_2$-graded.

Definition 3.2.4. If A is trace class operator acting on H^∞_w, then

$$\mathrm{Tr}_s A = \mathrm{Tr}(\tau A)$$

is called the *supertrace* of A.

Furthermore, $\mathrm{End}(H^\infty)\hat{\otimes}\Lambda(T^*B)$ is also $\mathbb{Z}_2$-graded so that we could extend the trace and supertrace to trace class elements A in $\mathrm{End}(H^\infty)\hat{\otimes}\Lambda(T^*B)$. Here $\mathrm{Tr}\, A$ and $\mathrm{Tr}_s\, A$ are elements of $\Lambda(T^*B)$.
If $\omega \in \Lambda(T^*B)$, we use the convention

$$\mathrm{Tr}(\omega A) = \omega\, \mathrm{Tr}\, A, \quad \mathrm{Tr}_s(\omega A) = \omega\, \mathrm{Tr}_s\, A.$$

For any $t > 0$, $\tilde{\nabla}^u + \sqrt{t}D$ is a superconnection on H^∞ in sense of Quillen [21]. By [2], $(\tilde{\nabla}^u + \sqrt{t}D)^2$ is an elliptic second order differential even operator acting in $\mathrm{End}\, H^\infty\hat{\otimes}\Lambda(T^*B)$. The corresponding exponential operator $\exp[-(\tilde{\nabla}^u + \sqrt{t}D)^2]$ is then even in $\mathrm{End}\, H^\infty\hat{\otimes}\Lambda(T^*B)$ and is given by a C^∞-kernel $T_t(x, x')$ along the fibers X. Since $T_t(x, x)$ is even in $\mathrm{End}(H^\infty)\hat{\otimes}\Lambda(T^*B)$, its supertrace $\mathrm{Tr}_s\, T_t(x, x)$ is also an even element of $\Lambda(T^*B)$.

An extension of Atiyah-Bott-Patodi Theorem is proven in [3]:

Theorem 3.2.5. *For any $t > 0$ the C^∞-differential form over B*

$$\mathrm{Tr}_s \exp[-(\tilde{\nabla}^u + \sqrt{t}D)^2] = \int_X \mathrm{Tr}_s[T_t(x, x)]dx \tag{3.2.2}$$

is closed and its cohomology class does not depend on t. If B is compact, its cohomology class is $ch_1(\mathrm{Ker}\, D_+ - \mathrm{Ker}\, D_-)$[1]*, which is a scaled representative of the Chern character of the bundle $\mathrm{Ker}\, D_+ - \mathrm{Ker}\, D_-$.*
As $t \downdownarrows 0$[2]*, for any $k \in \mathbb{N}$, we have the asymptotic expansion*

$$\mathrm{Tr}_s \exp[-(\tilde{\nabla}^u + \sqrt{t}D)^2)] = \sum_{-l-[\frac{m}{2}]}^{k} a_j(w)t^j + o(t^k, w), \tag{3.2.3}$$

where (a_j) are C^∞-differential forms on B, and $o(t^k, w)$ is uniform on compact sets in B. For p even (respectively odd) $a_j^{(2p)}$ is real (respectively purely imaginary). For $j \neq 0$, a_j is exact.
a_0 is closed and represents the same cohomology class as (3.2.2) so that if B is compact a_0 represents $ch_1(\mathrm{Ker}\, D_+ - \mathrm{Ker}\, D_-)$. For $0 \leq p \leq [m/2]$, $j < -l - p$, $a_j^{(2p)} = 0$.
Here $\omega^{(i)}$ denotes the component of ω in $\Lambda^{(i)}(T^*B)$. All the asymptotic expansions which we will consider are uniform on compact subsets of B.

Remark 3.2.6. Although we have omitted the proof of the above theorem, we need to highlight an important homomorphism φ_t used in [3]. Its definition is given by

$$\varphi_t : \Lambda(T^*B) \to \Lambda(T^*B), \quad \omega \mapsto \omega/\sqrt{t}$$

Via φ_t, the differential form on the left-hand side of (3.2.2) can be transformed in the following way

$$\mathrm{Tr}_s \exp[-(\tilde{\nabla}^u + \sqrt{t}D)^2] = \varphi_t[\mathrm{Tr}_s \exp -t(\tilde{\nabla}^u + D)^2]$$

The right-hand side of the above equation can be regarded as the asymptotic heat kernel expansion.

---

[1]Here $ch_1 E := \mathrm{Tr} \exp(-L)$ where L is the curvature of the endowed connection of a complex vector bundle E.

[2]Here $t \downdownarrows 0$ denotes that $t > 0$ goes to zero.

In order to construct a metric and a connection on the determinant line bundle associated with the family of operators D_w we need some results on the asymptotic expansion of the trace and supertrace of the exponential operator $\exp(-tD^2)$. As $t \downdownarrows 0$, we have

$$\frac{1}{2}\operatorname{Tr}[\exp(-tD^2)] = \Sigma_{-l}^{k} A_j t^j + o(t^k, w), \tag{3.2.4}$$

where the A_j are real C^∞-functions on B. The differential with respect to variables in B of the left-hand side of the above equation is given by

$$d[\frac{1}{2}\operatorname{Tr}\exp(-tD^2)] = -t\operatorname{Tr}[\exp(-tD^2)\tilde{\nabla}^u DD].$$

Thus, as $t \downdownarrows 0$ we have

$$\operatorname{Tr}[\exp(-tD^2)\tilde{\nabla}^u DD] = -\Sigma_{-l}^{k} dA_j t^{j-1} + o(t^{k-1}, w).$$

Similarly, for the supertrace we have as $t \downdownarrows 0$

$$\operatorname{Tr}_s[\exp(-tD^2)\tilde{\nabla}^u DD] = -\Sigma_{-1}^{k} B_j t^{j-1} + o(t^{k-1}, \omega).$$

We have the following relation among the B_j and a_j (from Theorem 1.7 in [3]) which we will use later.

Proposition 3.2.7. *The B_j are smooth purely imaginary 1-forms on B and satisfy $dB_j = -2ja_j^{(2)}$. In particular B_0 is closed.*

3.2.3 The Quillen metric on the determinant line bundle

We assume from now on that the index of the family of elliptic first order differential operator D_w is zero. With respect to the metric on N and that in ξ, the spaces $H_\pm^\infty$ can be completed to Hilbert spaces: $\mathcal{H}_1 = W^{1,2}(H_+^\infty)$ is the Sobolev space of sections of $F_+ \otimes \xi$ with square-integrable first derivative, and $\mathcal{H}_2 = L^2(H_-^\infty)$ is the space of square-integrable sections of $F_\pm \otimes \xi$. The (bounded) linear operator $D_+ : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ induced by D_+ is Fredholm. Hence we could regard the family of operators $D_{+,w}$ as a subset of $\operatorname{Fred}_0(\mathcal{H}_1, \mathcal{H}_2)$.

Definition 3.2.8. The *determinant line bundle* of the family of elliptic first order differential operators $D_{+,w}$ is the complex line bundle over B whose fiber $\mathcal{L}_w$ at $w \in B$ is

$$\mathcal{L}_w = \lambda(\operatorname{Ker} D_{+,w})^* \otimes \lambda(\operatorname{Coker} D_{+,w}). \tag{3.2.5}$$

As shown in the last section, $\mathcal{L}$ is a well-defined bundle on B, smooth and holomorphic, even if the dimension of $\operatorname{Ker} D_{+,w}$ may jump as w varies in B.

We will give a canonical local trivialization of the determinate line bundle $\mathcal{L}$ to construct a canonical smooth metric, the Quillen metric [3], and a unitary holomorphic connection ${}^1\nabla$ on $\mathcal{L}$ over B.

The operator

$$D^2 = \begin{pmatrix} D_- D_+ & 0 \\ 0 & D_+ D_- \end{pmatrix} \tag{3.2.6}$$

is a non-negative, self adjoint, second order, elliptic,(unbounded) differential operator acting on the Hilbert space $H^2(H^\infty) := W^{2,2}(H^\infty)$, which is dense in

$\mathcal{H}_1 \oplus \mathcal{H}_2$. Then the spectrum of D^2 is real and discrete, consisting of eigenvalues which accumulate only at $+\infty$. The non-zero eigenvalues of D_+D_- and D_-D_+ agree, and the corresponding eigenspaces are mapped isomorphically by D.

Take $w_0 \in B$ and $c > 0$ not in the discrete spectrum of $D^2_{w_0}$. Then c is not an eigenvalue of D^2 on a neighborhood U of w_0, as long as U is small enough. Let K^c_w be the sum of the eigenspaces of D_w for eigenvalues less than c. Then K^c is a smooth subbundle of H^∞ on U, and is stable under D. K^c splits into

$$K^c = K^c_+ \oplus K^c_-.$$

The exact sequence corresponding to (3.1.1) in the last section is now

$$0 \longrightarrow \operatorname{Ker} D_+ \longrightarrow K^c_+ \xrightarrow{D_+} K^c_- \longrightarrow \operatorname{Ker} D_- \longrightarrow 0$$

We denote

$$\mathcal{L}^c = \lambda(K^c_+)^* \otimes \lambda(K^c_-).$$

The above exact sequence shows that $\mathcal{L}^c$ is locally identified with $\mathcal{L}$ on U. Obviously, $\mathcal{L}^c$ is a smooth line bundle on U, and also $\mathcal{L}$ is a smooth line bundle on B. Since K^c inherits the Hermitian product from H^∞, $\mathcal{L}^c$ is naturally endowed with a metric, denoted by $|\ \ |^c$. However, for another $c' > 0$ not in the spectrum of D^2, we will get another local identification $\mathcal{L}^{c'}$ over a neighborhood U'. Similarly, $\mathcal{L}^{c'}$ is endowed with a metric $|\ \ |^{c'}$. We must modify the metrics $|\ \ |^c$ and $|\ \ |^{c'}$. Without loss of generality we assume $c < c'$ and let $K^{(c,c')}_\pm$ be the sum of the eigenspaces corresponding to eigenvalues μ between c and c'. Then $K^{(c,c')}_+$ and $K^{(c,c')}_+$ are smooth subbundles of H^∞ over $U \cap U'$ and so is

$$\mathcal{L}^{(c,c')} = \lambda(K^{(c,c')}_+)^* \otimes \lambda(K^{(c,c')}_-).$$

Let $D^{(c,c')}_+$ be the restriction of D_+ to $K^{(c,c')}$. Then $D^{(c,c')}_+$ is a finite dimensional linear invertible operator mapping $K^{(c,c')}_+$ into $K^{(c,c')}_-$. Its determinant $\det(D^{(c,c')}_+)$ is an element of the line $\mathcal{L}^{(c,c')}$. Obviously, on the overlaps $U \cap U'$ we have

$$\mathcal{L}^c \otimes \mathcal{L}^{(c,c')} = \mathcal{L}^{c'},\ s \in \mathcal{L}^c \mapsto s \otimes \det(D^{(c,c')}_+) \in \mathcal{L}^{c'}.$$

Since the $K^{(c)}_\pm$ is orthogonal to $K^{(c,c')}_\pm$, the metrics satisfy

$$|s \otimes \det(D^{(c,c')}_+)|^{c'} = |s|^c |\det(D^{(c,c')}_+)|^{(c,c')} \tag{3.2.7}$$

In order to identify $\mathcal{L}$ with $\mathcal{L}^{c'}$ or $\mathcal{L}^c$, we modify the metrics $|\ \ |^{c'}$ and $|\ \ |^c$ by a zêta function regularization of $|\det D_+|$.

Let P^c be the orthogonal projection operator on K^c. P^c is a smooth family of regularizing operators on U. Set $Q^c = I - P^c$.

Definition 3.2.9. For $s \in \mathbb{C}$, the zêta function $\zeta(s)$ is defined by

$$\zeta^c(s) = \frac{1}{2\Gamma(s)} \int_0^{+\infty} t^{s-1} \operatorname{Tr}[e^{-tD^2} Q^c] dt.$$

Equivalently,

$$\zeta^c(s) = \frac{1}{2} \operatorname{Tr}[(D^2)^{-s} Q^c].$$

$\zeta^c(s)$ is the zêta function of the operator D_-D_+ restricted to the sum of eigenspaces corresponding to eigenvalues larger than c. For each eigenvalue λ of D_-D_+ (or D_+D_-) with $\lambda > c > 0$, the Mellin transform of the function λ^{-s} is given by

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-t\lambda} dt.$$

There is an analogue for the operator

$$(D_-D+)^{-s} Q^c = \frac{1}{2}(D^2)^{-s} Q^c = \frac{1}{2\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-tD^2} Q^c dt,$$

where e^{-tD^2} is the heat kernel of the operator D^2. Thus

$$\sum_{\lambda > c} \lambda^{-s} = \frac{1}{2} \operatorname{Tr}[(D^2)^{-s} Q^c] = \frac{1}{2\Gamma(s)} \int_0^{+\infty} t^{s-1} \operatorname{Tr}[e^{-tD^2} Q^c] dt. \tag{3.2.8}$$

Since P^c is trace class we get as a conclusion of (3.2.4) for $t \downdownarrows 0$

$$\frac{1}{2} \operatorname{Tr}[e^{-tD^2} Q^c] = \sum_{-l}^{-1} A_j t^j + O(1, w).$$

The above asymptotics of the trace of the heat kernel provide an analytic continuation of $\zeta^c(s)$ to a meromorphic function on $\mathbb{C}$. Moreover, $\zeta^c(s)$ is holomorphic in an open neighborhood of zero and $\zeta^c_w(0)$ and $\frac{\partial \zeta^c_w}{\partial s}(0)$ are smooth in $w \in U_c$. For $0 < c < c' < +\infty$, we define

$$\zeta^{(c,c')} := \zeta^{c'} - \zeta^c.$$

Then the definition of ζ and (3.2.8) implies

$$|\det D^{(c,c')}_{+,w}|^{(c,c')} = \Pi_{c<\lambda<c'} \lambda = \exp\{-\frac{1}{2} \frac{\partial \zeta^{(c,c')}_w}{\partial s}(0)\}.$$

Definition 3.2.10. Let $\| \|^c$ denote the metric on $\mathcal{L}^c$ given by

$$\|l\|^c = | l |^c \exp\{-\frac{1}{2} \frac{\partial \zeta^c}{\partial s}(0)\}$$

for any $l \in \mathcal{L}^c$.

Then the relation (3.2.7) implies that we have a well-defined metric on $\mathcal{L}$.

Theorem 3.2.11. *Under the canonical identification of $\mathcal{L}$ with $\mathcal{L}^c$ over U_c, the metrics $\| \|^c$ patch to a smooth metric $\| \|$ on $\mathcal{L}$ over B.*

3.2.4 A unitary connection for the Quillen metric

Definition 3.2.12. Let ${}^0\nabla^c$ denote the unitary connection on the bundle K^c over U_c given by

$$ {}^0\nabla^c k = P^c \tilde{\nabla}^u k $$

for a section k of K^c. The fact that $\tilde{\nabla}^u$ is unitary on H^∞ implies that ${}^0\nabla^c$ is a unitary connection on $\mathcal{L}^c$ for the metric $|\ \ |^c$.

In order to construct a well-defined unitary connection for the metric $\|\ \ \|^c$ on $\mathcal{L}^c$, we need to modify the connection ${}^0\nabla^c$ by a 1-form on U_c.

Definition 3.2.13. For $t > 0$ define C^∞ 1-forms over U_c via

$$ \gamma_t^c = \int_t^{+\infty} \mathrm{Tr}[\exp(-sD^2)(\tilde{\nabla}^u D)DQ^c]ds, $$

$$ \delta_t^c = \int_0^{+\infty} \mathrm{Tr}_s[\exp(-tD^2)(\tilde{\nabla}^u D)DQ^c]ds. $$

Note that γ_t^c and δ_t^c are real and purely imaginary respectively. We state here the key properties of these differential forms which are proven in [3]:

Proposition 3.2.14. *As $t \Downarrow 0$ we have the expansions*

$$ \gamma_t^c = \sum_{-l}^{-1} dA_j \frac{t^j}{j} + dA_0 \log(t) + \gamma_0^c + O(t,w), \tag{3.2.9} $$

$$ \delta_t^c = \sum_{-l}^{-1} B_j \frac{t^j}{j} + B_0 \log(t) + \delta_0^c + O(t,w), \tag{3.2.10} $$

where γ_0^c, δ_0^c are C^∞-1-forms on U_c, which are real and purely imaginary respectively.

The following identities hold:

$$ d\zeta^c(0) = dA_0, $$

$$ \gamma_0^c + \Gamma'(1)dA_0 = -d[\frac{\partial}{\partial s}\zeta^c(0)], \tag{3.2.11} $$

$$ \gamma_0^c + \Gamma'(1)dA_0 = -(s\,\mathrm{Tr}[(D^2)^{-s}D^{-1}\tilde{\nabla}^u DQ^c])'(0), $$

$$ \delta_0^c + \Gamma'(1)B_0 = -(s\,\mathrm{Tr}_s[(D^2)^{-s}D^{-1}\tilde{\nabla}^u DQ^c])'(0) $$

$$ \frac{1}{2}[(\gamma_0^c - \delta_0^c) + \Gamma'(1)(dA_0 - B_0)] = -(s\,\mathrm{Tr}_s[(D_-D_+)^{-s}(D_+)^{-1}\tilde{\nabla}^u D_+ Q^c])'(0) \tag{3.2.12} $$

Here dA_0 (respectively $-B_0$) is the residue at $s = 0$ of the meromorphic function $\mathrm{Tr}[(D^2)^{-s}D^{-1}\tilde{\nabla}^u DQ^c]$ *(respectively* $\mathrm{Tr}_s[(D^2)^{-s}D^{-1}\tilde{\nabla}^u DQ^c]$*).*

If $0 < c < c' < \infty$, we have

$$ \gamma_0^c = \gamma_0^{(c,c')} + \gamma^{c'},\ \ \delta_0^c = \delta_0^{(c,c')} + \delta_0^{c'}, \tag{3.2.13} $$

$$ \frac{1}{2}(\gamma_0^{(c,c')} - \delta_0^{(c,c')}) = \frac{{}^0\nabla^{(c,c')} \det D_+^{(c,c')}}{\det D_+^{(c,c')}}. \tag{3.2.14} $$

We now define a unitary connection on the determinant line bundle $\mathcal{L}$ via local identifications.

Theorem 3.2.15. *Let ${}^1\nabla^c$ be the connection on $\mathcal{L}^c$ given by*

$$ {}^1\nabla^c = {}^0\nabla^c + \frac{1}{2}(\gamma_0^c - \delta_0^c) + \frac{1}{2}\Gamma'(1)(dA_0 - B_0). \tag{3.2.15}$$

Then using the canonical identification of $\mathcal{L}^c$ with $\mathcal{L}$ over U_c, the connections ${}^1\nabla^c$ patch to a smooth connection ${}^1\nabla$ on $\mathcal{L}$ over B, which is unitary for the metric $\|\ \|$. The curvature of ${}^1\nabla$ is the purely imaginary 2-form $a_0^{(2)}$ which is given by the asymptotic expansion of $\mathrm{Tr}_s \exp[-(\tilde{\nabla}^u + \sqrt{t}D)^2)]$ *in (3.2.3).*

Proof. Since δ_0^c and B_0 are purely imaginary 1-forms on U_c, we can ignore them when we check that ${}^1\nabla^c$ is unitary on $(\mathcal{L}^c, \|\ \|^c)$. So the fact ${}^0\nabla^c$ is unitary on $(\mathcal{L}^c, |\ \ |^c)$ and the equality (3.2.11) imply that ${}^1\nabla^c$ is unitary with respect to $\|\ \|^c$. Let $0 < c < c' < \infty$ and l be a smooth section of $\mathcal{L}^c$ over $U_c \cap U_{c'}$. We get by equality (3.2.14)

$$\begin{aligned} {}^0\nabla^{c'}(l \otimes \det D_+^{(c,c')}) &= {}^0\nabla^c l \otimes \det D_+^{(c,c')} + l \otimes {}^0\nabla^{(c,c')} \det D_+^{(c,c')} \\ &= ({}^0\nabla^c + \frac{1}{2}(\gamma_0^{(c,c')} - \delta^{(c,c')}))l \otimes \det D_+^{(c,c')}. \end{aligned}$$

Inserting the two equalities (3.2.13) into the above equation, we have

$$ {}^1\nabla^{c'}(l \otimes \det D_+^{(c,c')}) = ({}^1\nabla^c l) \otimes \det D_+^{(c,c')}. $$

This shows that the connections ${}^1\nabla^c$ patch together.
Now we calculate the curvature of ${}^1\nabla^c$:

$$\begin{aligned} -\mathrm{Tr}_s[{}^1\nabla^c]^2 &= -\mathrm{Tr}_s[{}^0\nabla^c]^2 + \frac{1}{2}(d\gamma_0^c - d\delta_0^c) - \frac{1}{2}\Gamma'(1)(dB_0) \\ &= -\mathrm{Tr}_s[{}^0\nabla^c]^2 - \frac{1}{2}d\delta_0^c. \end{aligned}$$

Here we have used the fact that $d\gamma_0^c = 0$ (by (3.2.11)) and $dB_0 = 0$ (by Proposition 3.2.7). Now $-\mathrm{Tr}_s[{}^0\nabla^c]^2$ is exactly the curvature of $\mathcal{L}^c$ for the connection ${}^0\nabla^c$. On the finite dimensional subbundle K^c we have

$$ \lim_{t \downarrow\downarrow 0} [\mathrm{Tr}_s \exp -({}^0\nabla^c + \sqrt{t}D)^2]^{(2)} = [\mathrm{Tr}_s \exp -({}^0\nabla^c)^2]^{(2)} = -\mathrm{Tr}_s[{}^0\nabla^c]^2. $$

On the other hand, the following fundamental transgression formula has been proven in [3] for $0 < t < T < +\infty$:

$$\begin{aligned} &\mathrm{Tr}_s \exp -({}^0\nabla^{(c,+\infty)} + \sqrt{t}D^{(c,+\infty)})^2 - \mathrm{Tr}_s \exp -({}^0\nabla^{(c,+\infty)} + \sqrt{T}D^{(c,+\infty)})^2 \\ &\quad = -\frac{d}{2}\int_t^T \mathrm{Tr}_s[\exp -(-sD^2)\tilde{\nabla}^u DDQ^c]^2. \end{aligned}$$

As $T \to +\infty$, $\mathrm{Tr}_s \exp -({}^0\nabla^{(c,+\infty)} + \sqrt{T}D^{(c,+\infty)})^2$ decays exponentially. Hence

$$ \mathrm{Tr}_s \exp -({}^0\nabla^{(c,+\infty)} + \sqrt{t}D^{(c,+\infty)})^2 = -\frac{d}{2}\delta_t^c $$

By differentiating the asymptotic expansion of δ^c_t (3.2.10), we find that as $t \downdownarrows 0$,

$$d\delta^c_t = \sum_{-l}^{-1} dB_j \frac{t^j}{j} + dB_0 Log(t) + d\delta^c_0 + O(t,w).$$

At last we us the additivity property of Quillen's superconnection [3]:

$$\begin{aligned}[\mathrm{Tr}_s \exp -(\tilde{\nabla}^u + \sqrt{t}D)^2]^{(2)} =& \mathrm{Tr}_s[\exp -({}^0\nabla^c + \sqrt{t}D^c)^2]^{(2)} \\ &+ \mathrm{Tr}_s[\exp -({}^0\nabla^{(c,+\infty)} + \sqrt{t}D^{(c,+\infty)})^2]^{(2)}.\end{aligned}$$

Identifying the coefficients in the expansions on both sides of the above equality implies that

$$a_0^{(2)} = -\,\mathrm{Tr}_s[\ {}^0\nabla^c]^2 - \frac{1}{2} d\delta^c_0.$$

This is exactly the curvature of the connection ${}^1\nabla^c$ on $\mathcal{L}^c$. □

Bismut and Freed also prove that if D_+ depends holomorphically on $w \in B$, the connection ${}^1\nabla$ is the unique holomorphic connection on $\mathcal{L}$ preserving $\| \ \|$. The special case considered by Quilen [22] is the Cauchy-Riemann operator $D = \bar{\partial}$, which can depend holomorphically on a parameter.

3.3 Application to Dirac operators with potentials

In this section we apply the concepts introduced above to the family of Dirac type operators induced by a holomophic structure on a quaternionic line bundle over a torus. We construct the associated determinant line bundle and calculate its curvature. The curvature is related to the Willmore energy of the quaternionic line bundle if we deform the potential part of the operators. Furthermore, we trivialize the determinant line bundle and define a holomorphic determinant function whose zero locus defines the spectral curve.

3.3.1 The Dirac operator over a Riemann surface

We consider the Dirac operator over a compact Riemann surface M. The tangent space TM carries an almost complex structure J and we choose a compatible Hermitian metric $\mathbf{h}$. The spinor bundle is $S = \wedge^* \bar{T}^*_{\mathbb{C}} M$ and via the usual constructions of complex geometry, S carries a canonical Hermitian metric and connection, denoted by $<,>$ and ∇.
The set of smooth sections of the spinor bundle is $C^\infty(S) = \Omega^{0,0}(M) \oplus \Omega^{0,1}(M)$. The induced L^2-inner product $<,>$ on $C^\infty(S)$ is given by

$$< f, g >= \frac{1}{\pi} \int f \bar{g} dV_M$$

for smooth functions $f, g \in \Omega^{0,0}(M) = C^\infty(M)$ and

$$< \alpha, \beta >= \frac{1}{2i\pi} \int \alpha \wedge \overline{\beta} = \frac{i}{2\pi} \int \overline{\beta} \wedge \alpha$$

for smooth $(0,1)$-forms $\alpha, \beta \in \Omega^{0,1}(M)$. We now compute the formal adjoint operator of $\bar{\partial}$.

Proposition 3.3.1. *The adjoint operator $\bar{\partial}^*$ satisfies*

$$\bar{\partial}^*(\alpha) = -\frac{i}{2} * (d\alpha) \tag{3.3.1}$$

for any $\alpha \in \Omega^{(0,1)}(M)$. Here $*$ are Hodge star operator given by $*\alpha(X) = \alpha(JX)$, for any $X \in TM$.

Proof. Using the Stokes Theorem, for any smooth function f and $(0,1)$-form α on no-boundary M we have

$$\begin{aligned} < f, \bar{\partial}^*(\alpha) > &=< \bar{\partial} f, \alpha > \\ &= \frac{1}{2i\pi}\int_M \bar{\partial} f \wedge \bar{\alpha} = \frac{1}{2i\pi}\int_M df \wedge \bar{\alpha} \\ &= \frac{1}{2i\pi}\int_M d(f\bar{\alpha}) - \frac{1}{2i\pi}\int_M f d\bar{\alpha} \\ &= \frac{1}{2i\pi}\int_M f\overline{(*d\alpha)} dV_M. \end{aligned}$$

According to $< f, \bar{\partial}^*(\alpha) >:= \frac{1}{\pi}\int f\overline{(\bar{\partial}^*(\alpha))} dV_M$ the proof is done. □

Since M with $\mathbf{h}$ is a Kähler manifold, the Dirac operator of S is $D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ ([19]). This identity does not hold for general complex manifolds, but the difference between the two sides is always an operator of 'zero order', that is an endomorphism of S.

In our case the set of operators of zero order mapping from $\Omega^{(0,0)}(M)$ to $\Omega^{(0,1)}(M)$ is exactly the infinite dimensional vector space $\Omega^{(0,1)}(M)$. For each element $\alpha \in \Omega^{(0,1)}(M)$, the action is

$$\alpha(f) = f\alpha \quad \text{for} \quad f \in \Omega^0(M).$$

Proposition 3.3.2. *The adjoint operator α^* of $\alpha \in \Omega^{(0,1)}(M)$ satisfies*

$$\alpha^*(\beta) = \frac{i}{2} * (\bar{\alpha} \wedge \beta).$$

Proof. From the definition of the adjoint operator

$$< \alpha^*(\beta), f >=< \beta, \alpha(f) >$$

we get

$$\frac{1}{\pi}\int_M \alpha^*(\beta)\bar{f} dV_M = \frac{1}{2i\pi}\int_M \beta \wedge \overline{(f\alpha)}.$$

Comparing the two sides of the above equation, we find

$$\alpha^*(\beta) dV_M = \frac{1}{2i}(\beta \wedge \bar{\alpha}).$$

Using the Hodge star operator on the last equality we have

$$\alpha^*(\beta) = \frac{1}{2i} * (\beta \wedge \bar{\alpha}) = \frac{i}{2} * (\bar{\alpha} \wedge \beta).$$

□

Let W be the rank 2 complex vector bundle $\mathbb{C}\oplus\bar{\mathbb{C}}$ over M with Hermitian product given by

$$< (u_1, v_1), (u_2, v_2) >= \int_M (u_1\bar{u}_2 + \bar{v}_1 v_2) dV_M$$

for any smooth sections $(u_1, v_1), (u_2, v_2) \in \Gamma(W)$.

Next we endow the Hermitian bundle W with the family of connections

$$\nabla = d + \begin{pmatrix} \omega'' - \bar{\omega''} & 0 \\ 0 & \omega' - \bar{\omega'} \end{pmatrix} \tag{3.3.2}$$

for $\omega \in \mathrm{Harm}(M, \mathbb{C})$. Since the 1-forms $\omega'' - \bar{\omega''}$ and $\omega' - \bar{\omega'}$ are purely imaginary, the family of connections ∇ is always unitary with respect to $<,>$ on W.

We consider the infinite dimensional bundles $H_+^\infty = C^\infty(S_+ \otimes W) = \Gamma(W)$ and $H_-^\infty = C^\infty(S_- \otimes W) = \Gamma(\bar{K}W)$ on $M \times \mathrm{Harm}(M, \mathbb{C})$. The $\bar{\partial}$-operator corresponding to the unitary connection ∇ is given by

$$\bar{\partial} = \begin{pmatrix} \bar{\partial} + \omega'' & 0 \\ 0 & \partial + \omega' \end{pmatrix}.$$

Notice that the trivial Cauchy-Riemann operator on the vector bundle $\bar{\mathbb{C}}$ is ∂.

The formal adjoint operator is

$$\bar{\partial}^* = \begin{pmatrix} \bar{\partial}^* + (\omega'')^* & 0 \\ 0 & \partial^* + (\omega')^* \end{pmatrix},$$

where ∂^*, $(\omega')^*$ are calculated analogously as in the last two propositions. The Hermitian product $<,>$ on $(1,0)$-forms α, β is given by

$$< \alpha, \beta >= \frac{i}{2\pi} \int \alpha \wedge \overline{\beta} = -\frac{i}{2\pi} \int \overline{\beta} \wedge \alpha$$

and we have

$$\partial^*(\beta) = \frac{i}{2} * (d\beta)$$

and

$$\alpha^*(\beta) = -\frac{i}{2} * (\bar{\alpha} \wedge \beta)$$

for any $\beta \in \Omega^{(1,0)}(M)$.

If M is a torus, we can fix an uniformizing complex coordinate z and set the Riemannian metric

$$g = \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz). \tag{3.3.3}$$

The family of unitary connections (3.3.2) for

$$\omega = a dz + b d\bar{z} \in \mathrm{Harm}(T^2, \mathbb{C})$$

has the associated family of $\bar{\partial}$ operators

$$\bar{\partial}_{a,b} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}} + b & 0 \\ 0 & \frac{\partial}{\partial z} + a \end{pmatrix}.$$

for $(a,b) \in \mathbb{C}^2$. From the above calculations its formal adjoint operator is

$$\bar{\partial}^*_{a,b} = \begin{pmatrix} -\frac{\partial}{\partial z} + \bar{b} & 0 \\ 0 & -\frac{\partial}{\partial \bar{z}} + \bar{a} \end{pmatrix}.$$

By Proposition 3.27 in [19] $\sqrt{2}(\bar{\partial}_{a,b} + \bar{\partial}^*_{a,b})$ is a family of Dirac operators on sections of the spin bundle $S \otimes W$. The difference between the family of first order operators $D_{a,b}$ in last chapter and $\bar{\partial}_{a,b}$ has only the potential part Q.

3.3.2 The determinant line bundle

We now return to the case of the family of Dirac operators with potential D_ω which we defined in the second chapter. We adapt these operators to the setting introduced in this chapter as follows: the base space $B = \mathrm{Harm}(T^2, \mathbb{C}) \cong \mathbb{C}^2$ is the space of harmonic forms over T^2 and the oriented compact spin connected manifold $X = T^2 = \mathbb{C}/\Gamma$ is the torus. The fibered manifold is the direct product $N = B \times T^2$. Thus, the spin bundle F on N is the pull-back of the spin bundle $S = \bigwedge^* \bar{T}^*_{\mathbb{C}} T^2$ on T^2. The Hermitian bundle $\xi = \pi_1^* W$ on N is the pull-back of the bundle W on T^2. We endow ξ with a unitary connection $\nabla^\xi = d + \begin{pmatrix} \omega'' - \bar{\omega}'' & 0 \\ 0 & \omega' - \bar{\omega}' \end{pmatrix}$. Since the Livi-Civita connection on TN is trivial, the naturally endowed unitary connection on the Hermitian bundle $(F \otimes \xi)_\omega$ is of the form ∇. The induced unitary connection $\tilde{\nabla}^u$ on the set of smooth sections $H^\infty_\omega = C^\infty(W) \oplus \Gamma(\bar{K}W)$ is trivial.
As before we denote by

$$D_{a,b} = \bar{\partial}_{a,b} + Q$$

where

$$\bar{\partial}_{a,b} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}} + b & 0 \\ 0 & \frac{\partial}{\partial z} + a \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -\bar{q} \\ q & 0 \end{pmatrix}.$$

The family of elliptic first order operators $D_+ = \bar{\partial}_{a,b} + Q$ maps $\Gamma(W)$ to $\Gamma(\bar{K}W)$. Let D_- be the formal adjoint

$$D_- = D_+^* = \begin{pmatrix} -\frac{\partial}{\partial z} + \bar{b} & \bar{q} \\ -q & -\frac{\partial}{\partial \bar{z}} + \bar{a} \end{pmatrix} \tag{3.3.4}$$

of D_+ with respect to the canonical Hermitian product in $C^\infty(T^2, \mathbb{C}^2)$ and let

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}, \tag{3.3.5}$$

be the operator acting on $H^\infty = \Gamma(W) \oplus \Gamma(\bar{K}W) \cong C^\infty(T^2, \mathbb{C}^2) \oplus C^\infty(T^2, \mathbb{C}^2)$.
H^∞ is a $\mathbb{Z}_2$-graded vector bundle over B and so are $\mathrm{End}\, H^\infty$ and $\mathrm{End}\, H^\infty \hat{\otimes} \Lambda(T^* B)$, for which the supertrace is well defined. For any $t > 0$, $\nabla + \sqrt{t} D$ is a super connection on H^∞ in the sense of Quillen [21].
We get the determinant bundle $\mathcal{L}$ of the elliptic family $D_{+,w}$ over B, whose fiber at $w \in B$ is

$$\mathcal{L}_w = \lambda(\mathrm{Ker}\, D_{+,w})^* \otimes \lambda(\mathrm{Coker}\, D_{+,w}). \tag{3.3.6}$$

In the last section, we have explained how to turn $\mathcal{L}_w$ into a smooth line bundle using local identifications. Moreover, we defined the Quillen metric $\| \ \|$ and a unitary connection ${}^1\nabla$ with respect to $\| \ \|$ on $\mathcal{L}$.

Theorem 3.2.15 tells us that the curvature form of ${}^1\nabla$ on the determinant line bundle is given by $a_0^{(2)}$. This 2-form on B is the degree 2 term in the differential form a_0, which is the constant coefficient in the asymptotic expansion

$$\operatorname{Tr}_s \exp[-(\nabla + \sqrt{t}D)^2)] = \sum_{-l-[\frac{m}{2}]}^{k} a_j(w)t^j + o(t^k, w).$$

We will now calculate the curvature form of the canonical connection on the determinant line bundle. In the special case $Q = 0$, as we have pointed out before, $D = \bar{\partial} + \bar{\partial}^*$ is a family of Dirac operators acting on H^∞. Hence we can apply the result of Theorem 1.21 in [4] directly to the operators $D = \bar{\partial} + \bar{\partial}^*$ over the torus T^2:

Theorem 3.3.3. *Let ${}^1\nabla$ be the canonical unitary connection on the determinant line bundle $\mathcal{L}$ of the family of Dirac operators D over B. The curvature form of ${}^1\nabla$ is the term of degree 2 in the differential form*

$$2i\pi \int_{T^2} \hat{A}(\frac{R^{T^2}}{2\pi}) \operatorname{Tr}[\exp -\frac{\Omega}{2i\pi}]), \tag{3.3.7}$$

where R^{T^2} is the curvature of the tangent bundle of the torus. Ω is the curvature tensor of the Hermitian vector bundle ξ over $T^2 \times B$.

Definition 3.3.4. Let $\hat{A}$ be the $adO(n)$ invariant polynomial on $\mathscr{A}(n)$[3] given by

$$\hat{A}(B) = \prod_1^l \frac{\frac{x_i}{2}}{\sinh \frac{x_i}{2}}$$

for $B \in \mathscr{A}(n)$ has diagonal entries $\begin{pmatrix} 0 & x_i \\ -x_i & 0 \end{pmatrix}$.

The connection ∇^ξ on the graded vector bundle $\xi = \pi_1^* W$ over $T^2 \times B$ is given by (3.3.2). The curvature tensor 2-form of ∇^ξ calculates to

$$\begin{aligned}\Omega &= d\begin{pmatrix} \omega'' - \bar{\omega}'' & 0 \\ 0 & \omega' - \bar{\omega}' \end{pmatrix} + \begin{pmatrix} \omega'' - \bar{\omega}'' & 0 \\ 0 & \omega' - \bar{\omega}' \end{pmatrix} \wedge \begin{pmatrix} \omega'' - \bar{\omega}'' & 0 \\ 0 & \omega' - \bar{\omega}' \end{pmatrix} \\ &= \begin{pmatrix} db \wedge d\bar{z} - d\bar{b} \wedge dz & 0 \\ 0 & da \wedge dz - d\bar{a} \wedge d\bar{z} \end{pmatrix},\end{aligned}$$

which can be viewed as a 1-form on B with values in $\Omega^1(T^2)$. Since $T^2 = \mathbb{C}/\Gamma$ is equipped with the trivial metric (3.3.3) and the trivial connection, we have $R^{T^2} = 0$. Therefore, we get

$$\hat{A}(\frac{R^{T^2}}{2\pi}) = \lim_{x \to 0} \frac{x}{\sinh x} = 1.$$

[3] $\mathscr{A}(n) :== \left\{ diag[\begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n \\ -x_n & 0 \end{pmatrix}] \mid x_i \in \mathbb{R} \right\}$

Since W is a $\mathbb{Z}_2$-graded vector bundle on T^2 we modify the trace part of the curvature form in (3.3.7) to be Tr_{grad}. The curvature form $a_0^{(2)}$ is

$$\begin{aligned}
a_0^{(2)} =& 2i\pi[\int_{T^2} \hat{A}(\frac{R^{T^2}}{2\pi}) \mathrm{Tr}_{grad}[\exp - \frac{\Omega}{2i\pi}])]^{(2)} \\
=& 2i\pi \int_{T^2} \frac{1}{2}\frac{1}{(2i\pi)^2} \mathrm{Tr}_{grad}\, \Omega \wedge_{T^2} \Omega \\
=& \frac{1}{4i\pi}\int_{T^2} \mathrm{Tr}_{grad} \begin{pmatrix} -2(db\wedge d\bar{b})(d\bar{z}\wedge dz) & 0 \\ 0 & 2(da\wedge d\bar{a})(d\bar{z}\wedge dz) \end{pmatrix} \\
=& \frac{-1}{2i\pi}(da\wedge d\bar{a} + db\wedge d\bar{b})\int_{T^2} d\bar{z}\wedge dz \\
=& \frac{-1}{\pi}(da\wedge d\bar{a} + db\wedge d\bar{b})\int_{T^2} dV.
\end{aligned} \tag{3.3.8}$$

Here we used

$$\begin{aligned}
&(db\wedge d\bar{z} - d\bar{b}\wedge dz)\wedge_{T^2}(db\wedge d\bar{z} - d\bar{b}\wedge_{T^2} dz) \\
&= -(db\otimes d\bar{b})(d\bar{z}\wedge_{T^2} dz) - (d\bar{b}\otimes db)(dz\wedge_{T^2} d\bar{z}) \\
&= -2(db\wedge d\bar{b})(d\bar{z}\wedge dz)
\end{aligned}$$

and

$$\begin{aligned}
&(da\wedge dz - d\bar{a}\wedge d\bar{z})\wedge_{T^2}(da\wedge dz - d\bar{a}\wedge d\bar{z}) \\
&= -(da\otimes d\bar{a})(dz\wedge_{T^2} d\bar{z}) - (d\bar{a}\otimes da)(d\bar{z}\wedge_{T^2} dz) \\
&= 2(da\wedge d\bar{a})(d\bar{z}\wedge_{T^2} dz).
\end{aligned}$$

Note that for the harmonic form $adz + bd\bar{z} \in \mathrm{Harm}(T^2, \mathbb{C})$ the Hermitian metric is

$$\mathbf{h} = \frac{1}{\pi}(da\otimes d\bar{a} + db\otimes d\bar{b})\int_{T^2} dV$$

and its Kähler form is

$$k = \frac{i}{2}(\mathbf{h} - \bar{\mathbf{h}}) = \frac{i}{2\pi}(da\wedge d\bar{a} + db\wedge d\bar{b})\int_{T^2} dV.$$

Therefore, the Kähler form on $\mathrm{Harm}(T^2, \mathbb{C})$ multiplied by $2i$ is exactly the curvature form of the determinant line bundle $\mathcal{L}$. This is in accordance with the case of a family of Cauchy-Riemann operators over a Riemann surface [22].

When the potential part is non-trivial, that is, $Q \neq 0$, we consider instead of (3.2.3) another asymptotic expansion

$$\mathrm{Tr}_s \exp[-t(\nabla^u + D)^2] = \sum_{-\frac{1}{2}-3}^{k'} a_j'(w)t^j + o(t^{k'}, (w)). \tag{3.3.9}$$

As we have mentioned in remark 3.2.6: we let φ_t be the homomorphism from ΛT^*B into itself associating ω to $\omega/\sqrt{t}$. Clearly

$$\mathrm{Tr}_s \exp[-(\nabla^u + \sqrt{t}D)^2] = \varphi[\mathrm{Tr}_s \exp[-t(\nabla^u + D)^2].$$

Comparing the coefficients of the asymptotic expansions for the two sides of the above equality, we get $a_0^{(2)} = a_1^{'(2)}$.

We now introduce the main result in [13] in order to compute $a_1^{'(2)}$ in the heat kernel expansion (3.3.9).

Theorem 3.3.5. *(Gilkey) Let M be a compact Riemannian manifold of dimension m with the metric*

$$ds^2 = g_{\nu\mu}dx^\nu dx^\mu.$$

Let V be a smooth vector bundle over M and D a second order partial differential operator on $C^\infty(V)$ of the form

$$D = -(g^{\nu\mu}I \cdot \partial_\nu \partial_\mu) + A^\nu \partial_\nu + B,$$

where A^ν and B are endomorphisms of V.
Then there exists a unique connection ∇ and a unique endomorphism E of V so that

$$D = D(g, \nabla, E) = -(\sum_{\nu,\mu} g^{\nu\mu}\nabla_\nu \nabla_\mu + E).$$

If ω_ν is the connection 1-form of ∇, then

$$\omega_\nu = \frac{1}{2} g_{\nu\mu}(A^\mu + g^{\sigma\epsilon}\Gamma^\mu_{\sigma\epsilon}), \tag{3.3.10}$$

$$E = B - g^{\mu\sigma}(\partial_\sigma \omega_\mu + \omega_\mu \omega_\sigma - \omega_\epsilon \Gamma^\epsilon_{\mu\sigma}, \tag{3.3.11}$$

where $\Gamma^\epsilon_{\mu\sigma}$ are the Chirstoffel symbols of the Levi-Civita connection.

The exponential operator e^{-tD} has a smooth kernel function $K(t, x, y, D)$ such that

$$e^{-tD}u(x) = \int K(t, x, y, D)u(y)dy.$$

Moreover, we have the asymptotic expansion

$$\operatorname{Tr} K(t, x, x, D) \sim \sum_n a_n(D)(x) t^{\frac{n-m}{2}}, \quad as \quad t \to 0^+$$

of the heat kernel and its integral

$$\operatorname{Tr} e^{-tD} = \int_M \operatorname{Tr} K(t, x, x, D) \sim \sum_n a_n(D) t^{\frac{n-m}{2}},$$

$$a_n(D) = \int_M a_n(D)(x).$$

Here the coefficients $a_n(D)(x)$ vanish if n is odd since $\partial M = \emptyset$. The $a_n(D)$ are locally computable:

$$a_0(D)(x) = (4\pi)^{-m/2} \operatorname{Tr}(1). \tag{3.3.12}$$

$$a_2(D)(x) = (4\pi)^{-m/2} \frac{1}{6} \operatorname{Tr}(6E + \tau). \tag{3.3.13}$$

$$a_4(D)(x) = (4\pi)^{-m/2}\frac{1}{360}\,\mathrm{Tr}\{(60E_{;kk} + 60\tau E + 180E^2 + 30\Omega_{ij}\Omega_{ij} \\ + 12\tau_{;kk} + 5\tau^2 - a\rho^2 + 2R^2)\}. \tag{3.3.14}$$

In the above equality R_{ijkl} is the curvature tensor of the Levi-Civita connection of M with sign convention $R_{1212} = -1$ on the standard sphere.

$$\tau = -R_{ijij},\ \rho_{ij} = -R_{ikjk},\ \rho^2 = R_{ikjk}R_{iljl},\ R^2 = R_{ijkl}R_{ijkl}$$

and Ω is the curvature of the connection ∇ on V.

We shall apply this theorem to our problem of computing $a_1^{'(2)}$ in (3.3.9). For any section h of the bundle H^∞ over B, there exists a section ψ of the bundle $S \otimes W$ over T^2 such that $h = \pi_1^*\psi$. This implies that $\tilde{\nabla}^u h = 0$. Then we have

$$(\tilde{\nabla}^u + D)^2 h = (\tilde{\nabla}^u + D)(Dh) = (D^2)h + \tilde{\nabla}^u(D)h,$$

where

$$\tilde{\nabla}^u(D) = \begin{pmatrix} 0 & d_B D_- \\ d_B D_+ & 0 \end{pmatrix},$$

with

$$d_B D_- = \begin{pmatrix} d\bar{b} & 0 \\ 0 & d\bar{a} \end{pmatrix},\quad d_B D_+ = \begin{pmatrix} db & 0 \\ 0 & da \end{pmatrix}.$$

For the second-order differential operator

$$D^2 = \begin{pmatrix} D_- D_+ & 0 \\ 0 & D_+ D_- \end{pmatrix}$$

we find that the leading symbol of $D_- D_+$ is

$$\begin{pmatrix} -\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} & 0 \\ 0 & -\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} \end{pmatrix} = \frac{-1}{4}\begin{pmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} \end{pmatrix},$$

which is also the leading symbol of the operator $D_+ D_-$. The Hermitian metric on $T^2 = \mathbb{C}/\Gamma$ is trivial and given by

$$ds^2 = dx^2 + dy^2$$

in the coordinate $z = x + iy$ so that

$$g_{\nu\mu} = g^{\nu\mu} = \delta_{\nu\mu},\quad \Gamma^{\epsilon}_{\nu\mu} = 0.$$

If we ignore the part $d_B D$ with values in differential 1-forms on B, the operator

$$\hat{D} = 4(\tilde{\nabla}^u + D)^2 = 4(D^2 + d_B D)$$

can be regarded as a second-order partial differential operator on the $\mathbb{Z}_2$-graded bundle $C^\infty(S \otimes W)$ with leading symbol given by the metric tensor, i.e.,

$$D^2 = -(I_4(\partial\bar{\partial}) + \begin{pmatrix} A_+^z & 0 \\ 0 & A_-^z \end{pmatrix}\partial + \begin{pmatrix} A_+^{\bar{z}} & 0 \\ 0 & A_-^{\bar{z}} \end{pmatrix}\bar{\partial} + \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}).$$

The endomorphisms in the above equality are given by

$$A_+^z = -\begin{pmatrix} -b & 2\bar{q} \\ 0 & \bar{a} \end{pmatrix}, \quad A_+^{\bar{z}} = -\begin{pmatrix} \bar{b} & 0 \\ -2q & -a \end{pmatrix},$$

$$A_-^z = -\begin{pmatrix} -b & 0 \\ -2q & \bar{a} \end{pmatrix}, \quad A_+^{\bar{z}} = -\begin{pmatrix} \bar{b} & 2\bar{q} \\ 0 & -a \end{pmatrix},$$

$$B_+ = -\begin{pmatrix} |b|^2 + |q|^2 & -\bar{b}\bar{q} + \frac{\partial \bar{q}}{\partial z} + a\bar{q} \\ -bq - \frac{\partial q}{\partial \bar{z}} + \bar{a}q & |q|^2 + |a|^2 \end{pmatrix},$$

and

$$B_- = -\begin{pmatrix} |b|^2 + |q|^2 & b\bar{q} + \frac{\partial \bar{q}}{\partial \bar{z}} - \bar{a}\bar{q} \\ \bar{b}q - \frac{\partial q}{\partial z} - aq & |q|^2 + |a|^2 \end{pmatrix}.$$

From $A^1\partial_x + A^2\partial_y = 4(A^z\partial + A^{\bar{z}}\bar{\partial})$ we have

$$A^1 = 2(A^z + A^{\bar{z}}), \quad A^2 = 2i(-A^z + A^{\bar{z}}).$$

From Theorem 3.3.5 there exists a unique connection $\hat{\nabla}$ on $S \otimes W$ and an unique endomorphism E of $S \otimes W$ so that

$$\hat{D} = -(\hat{\nabla}_1\hat{\nabla}_1 + \hat{\nabla}_2\hat{\nabla}_2 + E).$$

According to (3.3.10) $\hat{\nabla}$ has connection form

$$\begin{aligned} \omega_1 dx + \omega_2 dy &= \frac{1}{2}(A^1 dx + A^2 dy) \\ &= (A^z + A^{\bar{z}})dx + (-A^z + A^{\bar{z}})dy \\ &= A^z d\bar{z} + A^{\bar{z}} dz, \end{aligned}$$

so that

$$\hat{\nabla} = d + \begin{pmatrix} \begin{pmatrix} -\bar{b} & 0 \\ 2q & a \end{pmatrix} dz + \begin{pmatrix} b & -2\bar{q} \\ 0 & -\bar{a} \end{pmatrix} d\bar{z} & 0 \\ 0 & \begin{pmatrix} -\bar{b} & -2\bar{q} \\ 0 & a \end{pmatrix} dz + \begin{pmatrix} b & 0 \\ 2q & -\bar{a} \end{pmatrix} d\bar{z} \end{pmatrix}.$$

The endomorphism part from (3.3.11) is given by

$$\begin{aligned} \hat{E} &= 4B - 4(\partial_x\omega_1 + \partial_y\omega_2 + \omega_1^2 + \omega_2^2) - 4d_B D \\ &= 4B - 2(\partial A^z + \bar{\partial} A^{\bar{z}}) - 2(A^z A^{\bar{z}} + A^{\bar{z}} A^z) - 4d_B D \\ &= 4|q|^2 I_4 - 4d_B D. \end{aligned}$$

Obviously, $\hat{E}$ has a part values in differential forms over B and the leading part of $\hat{E}$ is

$$\hat{E}^{(1)} = -4(d_B D)^{(1)}.$$

We can easily extend Gilkey's theorem to the case of an operator $\hat{D}$ acting on the $\mathbb{Z}_2$-graded vector bundle $S \otimes W$:

$$\operatorname{Tr}_s \hat{D} = a_0(\hat{D})t^{-1} + a_2(\hat{D}) + a_4(\hat{D})t + O(t),$$

where a_n are given by (3.3.12-3.3.14) using, however, Tr_s instead of Tr. By comparing the above expansion with (3.3.9) we claim

$$4a_1' = a_4(\hat{D}).$$

Now we calculate the degree 2 term $a_1^{'(2)}$ in the heat kernel asymptotics by using the equality (3.3.14). The part contributing to the differential forms on B is only $\hat{E}^2$:

$$\begin{aligned}
a_1^{'(2)} &= \frac{1}{4}\frac{1}{4\pi}360^{-1}[\int_{T^2} \mathrm{Tr}_s(\cdots + 180\hat{E}^2 + \cdots)dV]^{(2)} \\
&= \frac{1}{2\pi}[\int_{T^2} \mathrm{Tr}_s(dD^2)dV]^{(2)} \\
&= \frac{1}{2\pi}\int_{T^2} \mathrm{Tr}_s \begin{pmatrix} dD_- \otimes dD_+ & 0 \\ 0 & dD_+ \otimes dD_- \end{pmatrix} dV \\
&= \frac{1}{\pi}\int_{T^2} \mathrm{Tr}[dD_- \wedge dD_+]dV \\
&= \frac{1}{\pi}\int_{T^2} \mathrm{Tr} \begin{pmatrix} d\bar{b} \wedge db & \cdots \\ \cdots & d\bar{a} \wedge da \end{pmatrix} dV \\
&= \frac{-1}{\pi}(da \wedge d\bar{a} + db \wedge d\bar{b}) \int_{T^2} dV.
\end{aligned} \tag{3.3.15}$$

Here we have used the diagonal entries of $dD_- \otimes dD_+$ and $dD_+ \otimes dD_-$, namely

$$dD_- \otimes dD_+ = \begin{pmatrix} d\bar{b} \otimes db & \cdots \\ \cdots & d\bar{a} \otimes da \end{pmatrix}, \quad dD_+ \otimes dD_- = \begin{pmatrix} db \otimes d\bar{b} & \cdots \\ \cdots & da \otimes d\bar{a} \end{pmatrix}.$$

To summarize, we get the same result (3.3.8) as the case $q = 0$. If we fix a base point

$$D_0 = \bar{\partial} + Q = \begin{pmatrix} \bar{\partial} & -\bar{q} \\ q & \partial \end{pmatrix},$$

the inner product on D_ω for any $\omega \in \mathrm{Harm}(T^2, \mathbb{C})$ is given by

$$\begin{aligned}
\|D_\omega - D_0\|^2 &= \frac{i}{2\pi}\int_{T^2} \mathrm{Tr}_{graded}[(D_\omega - D_0)^*(D_\omega - D_0)] \\
&= < bd\bar{z}, bd\bar{z} > + < adz, adz > \\
&= \frac{1}{\pi}(|a|^2 + |b|^2)\int_{T^2} dV.
\end{aligned}$$

The Kähler form is minus the imaginary part of the Hermitian metric

$$\frac{i}{2\pi}(da \wedge d\bar{a} + db \wedge d\bar{b})\int_{T^2} dV.$$

We should point out that the above form is independent of the choice of the base point D_0. Therefore, we conclude

Theorem 3.3.6. *Let D_ω be the holomorphic family of elliptic operator given by (2.1.4) for $\omega \in \mathrm{Harm}(T^2, \mathbb{C})$ and let $\mathcal{L}$ (3.2.5) be its determinant line bundle endowed with the Quillen metric $\| \ \|$. The curvature of the canonical unitary connection ${}^1\nabla$ (3.2.15) of $\mathcal{L}$ is $2i$ times the Kähler form on $\mathrm{Harm}(T^2, \mathbb{C})$.*

3.3.3 Construction of $\det : B \to \mathbb{C}$

According to Theorem 3.3.6 we are able to construct a holomorphic determinant function in the same way as Quillen [22].
First we define a canonical section σ of $\mathcal{L}$ satisfying

$$\sigma(D) = \begin{cases} 1 \in \mathbb{C}, & D_+ \text{ is invertible}, \\ 0 \in \lambda(\mathrm{Ker}\, D_+)^* \otimes \lambda(\mathrm{Coker}\, D_+), & \text{otherwise}. \end{cases}$$

Moreover, by definition of the Quillen metric the norm $\|\sigma\|^2$ of the canonical section is exactly the zêta function determinant

$$\det_\zeta(D_- D_+) = \begin{cases} \exp\{-\frac{\partial \zeta^0}{\partial s}(0)\}, & D_+ \text{ is invertible}, \\ 0, & \text{otherwise}. \end{cases}$$

of the opetator $D_- D_+$. Here $\exp\{-\frac{\partial \zeta^0}{\partial s}(0)\}$ can be interpreted as the zêta function determinant of the operator $D_- D+$ restricted to the orthogonal complement of its kernel. We have the following property:

Theorem 3.3.7. *The function* $\det_\zeta(D_- D_+)$ *is smooth as a function on* B.

Proof. Let $D_{w_0}, w_0 \in B$, be a non-invertible first order elliptic differential operator. We have to show that $\det_\zeta(D_- D_+)$ is smooth in a neighborhood of w_0. The eigenvalues of $D_- D_+$ are smooth as functions of w and thus we can find a small open neighborhood U of w_0 such that there exists $c > 0$ not in the discrete spectrum of any $(D_- D_+)_w$ in U. Write $\zeta^0(s) = \zeta^{(0,c)}(s) + \zeta^c(s)$ where $\zeta^{(0,c)}(s) = \sum_{0<\lambda<c} \lambda^{-s}$ and $\zeta^c(s) = \zeta_{>c}(s) = \sum_{\lambda > c} \lambda^{-s}$. Then for each w in U the zeta function determinant of $D_- D_+$ factors as follows:

$$e^{-\frac{\partial \zeta}{\partial s}(0)} = e^{-\frac{\partial \zeta^{(0,c)}}{\partial s}(0)} e^{-\frac{\partial \zeta^c}{\partial s}(0)} = \det((D_- D_+)|_{K^c}) e^{-\frac{\partial \zeta^c}{\partial s}(0)},$$

where K^c is a finite dimensional bundle on U. Now $-\frac{\partial \zeta^c}{\partial s}(0)$ depends smoothly on w and the finite dimensional determinant, $\det(D_- D_+ |_{K^c})$ is smooth for all w in U. We see that, off the divisor of non-invertible operators, $\det_\zeta(D_- D_+)$ can be represented as a product of two smooth functions, one of which vanishes as D_+ tends to a non-invertible operator, while the other remains bounded. □

Remark 3.3.8. Note that the zêta function determinant $\det_\zeta(D_- D_+)$ vanishes precisely when D_+ is not invertible.

Choose an arbitrary holomorphic section η of $\mathcal{L}$. Over an arbitrary holomorphic parameter w of B the curvature takes the form $\bar{\partial}_w \partial_w \log \|\eta\|^2$. From Theorem 3.3.6 we know it equals the Kähler form multiplied by $2i$ on B:

$$\bar{\partial}_w \partial_w \log \|\eta\|^2 = \frac{-1}{\pi} dw \wedge d\bar{w} (|\frac{\partial a}{\partial w}|^2 + |\frac{\partial b}{\partial w}|^2) \int_{T^2} dV = -\partial_w \bar{\partial}_w \|D_+(w) - D_0\|^2.$$

Analogous to the case of a family of Cauchy-Riemann operators, we modify the metric $\| \ \|$ in determinant bundle $\mathcal{L}$ by multiplication by $e^{-\|D_+ - D_0\|^2}$. Then the canonical connection for the new metric has curvature zero. Indeed, for an arbitrary parameter w,

$$\begin{aligned}\bar{\partial}_w \partial_w log \|\eta\|_{new}^2 &= \bar{\partial}_w \partial_w \log e^{-\|D_+ - D_0\|^2} \|\sigma\|^2 \\ &= -\bar{\partial}_w \partial_w \|D_+ - D_0\|^2 - \partial_w \bar{\partial}_w \|D_+ - D_0\|^2 \\ &= 0\end{aligned}$$

The last equality is trivial since $\bar{\partial}_w \partial_w + \partial_w \bar{\partial}_w = 0$.

Because the curvature is zero we can trivialize $\mathcal{L}$. There exists a holomorphic everywhere flat section with norm 1 with respect to this new metric, that is

$$\eta \in \Gamma(\mathcal{L}), \quad \|\eta\|_{new} = 1.$$

The canonical section σ is then identified with a holomorphic function $\det(D_+)$ on B defined by

$$\sigma(D_+) = \det(D_+)\eta(D_+).$$

Moreover, from the norm of the canonical section σ we get

$$|\det(D_+)|^2 = e^{-\|D_+ - D_0\|^2} \det_\zeta(D_- D_+).$$

Notice that $\det(D_+)$ vanishes precisely when D_+ is not invertible.

Theorem 3.3.9. *The holomorphic family (2.1.4) of first order elliptic operators D_ω, parameterized over the space of harmonic 1-forms $\omega \in \mathrm{Harm}(T^2, \mathbb{C})$, has a determinant line bundle and determinant function* $\det : \mathrm{Harm}(T^2, \mathbb{C}) \to \mathbb{C}$. *The 1-dimensional analytic subvariety of its zero locus is exactly the spectral variety of the holomorphic structure on the line bundle V/L over T^2.*

3.3.4 The Willmore energy as curvature of the determinant line bundle

In Hitchen's paper [12] the energy of the harmonic map can be expressed in terms of the residue of a certain meromorphic differential on the spectral curve. He also points out that one can view the energy density as the curvature form on the determinant line bundle of a holomorphic family of elliptic operators. This suggests that in our setup the Willmore energy should be expressed as the curvature form of the determinant line bundle for the family D_ω of elliptic operators. However, as we have seen, we only get the area functional which corresponds to the vacuum case of no potential $Q = 0$.

On the other hand, as we will show in the next chapter, the spectral curve is asymptotic to the vacuum spectral curve of the operator $\bar{\partial}$ for large $\omega \in \mathrm{Harm}(T^2, \mathbb{C})$. Conceptually this fact could be understood by showing that the determinant of D_ω converges to that of the operator $\bar{\partial}_\omega$ as we deform the potential to zero.

This suggests that the Willmore energy will appear as curvature in the direction of the deformation of the potential part Q in the family of Dirac-type operators $D = \bar{\partial} + Q$. Consider a new family of elliptic first order differential operators

$$\tilde{D}_{+,(a,b,\mu)} = \bar{\partial}_{a,b} + Q_\mu \tag{3.3.16}$$

with the same $\bar{\partial}_{a,b}$ as in (2.1.5) but scaled potential part

$$Q_\mu = \begin{pmatrix} 0 & -\mu\bar{q} \\ \mu q & 0 \end{pmatrix},$$

where the parameter $\mu \in \mathbb{C}$ is introduced so that $\tilde{D}$ depends holomorphically on μ.

Compared to the setting before, the orientable compact spin manifold and the hermitian bundle are still $X = T^2$ and $\xi = \pi_1^* W$. We change the base space to be $\tilde{B} = \mathrm{Harm}(T^2, \mathbb{C}) \oplus \mathbb{C}$ and the fibered manifold to be $\tilde{N} = \tilde{B} \times T^2$.

Again we construct the determinant line bundle $\tilde{\mathcal{L}}$ over $\tilde{B}$, whose fiber at $\tilde{w} = (a, b, \mu) \in \tilde{B}$ is

$$\mathcal{L}_{\tilde{w}} = \lambda(\mathrm{Ker}\, \tilde{D}_{+,\tilde{w}})^* \otimes \lambda(\mathrm{Coker}\, \tilde{D}_{+,\tilde{w}}).$$

Choosing the base point $\tilde{D}_{+,0} = \bar{\partial}$ in $\tilde{B}$, we define the hermitian metric $<,>$ on $\tilde{B}$ via the norm

$$\begin{aligned}
\|\tilde{D}_{+,a,b,\mu} - \tilde{D}_{+,0}\|^2 &= \frac{1}{\pi}\int_{T^2} \mathrm{Tr}[(\tilde{D}_+ - \tilde{D}_{+,0})^*(\tilde{D}_+ - \tilde{D}_{+,0})]dV \\
&= \frac{1}{\pi}\int_{T^2} \mathrm{Tr}(\begin{pmatrix} \bar{b} & \overline{\mu q} \\ -\bar{\mu}q & \bar{a} \end{pmatrix}\begin{pmatrix} b & -\mu\bar{q} \\ \mu q & a \end{pmatrix})dV \\
&= \frac{1}{\pi}\int_{T^2} \mathrm{Tr}\begin{pmatrix} |b|^2 + |\mu|^2\bar{q}q & \cdots \\ \cdots & |a|^2 + |\mu|^2 q\bar{q} \end{pmatrix} dV \\
&= \frac{1}{\pi}[(|a|^2 + |b|^2)\int_{T^2} dV + |\mu|^2 \int_{T^2} 2|q|^2 dV].
\end{aligned}$$

The canonical unitary connection, denoted by ${}^1\tilde{\nabla}$, is constructed as before on the determinant line bundle $\tilde{\mathcal{L}}$. In order to calculate the curvature form using the theorem by Gilkey, we follow our above calculations as in the case of $\mathcal{L}$. Instead of $\hat{E}$ we have

$$\tilde{E}^{(1)} = -4 d_{\tilde{B}}\tilde{D}$$

with

$$d_{\tilde{B}}\tilde{D}_+ = \begin{pmatrix} db & -\bar{q}d\mu \\ qd\mu & da \end{pmatrix}, \quad d_{\tilde{B}}\tilde{D}_- = \begin{pmatrix} d\bar{b} & \bar{q}d\bar{\mu} \\ -qd\bar{\mu} & d\bar{a} \end{pmatrix}.$$

Their wedge product is given by

$$d_{\tilde{B}}\tilde{D}_- \wedge d_{\tilde{B}}\tilde{D}_+ = \begin{pmatrix} d\bar{b}\wedge db + |q|^2 d\bar{\mu}\wedge d\mu & \cdots \\ \cdots & d\bar{a}\wedge da + |q|^2 d\bar{\mu}\wedge d\mu \end{pmatrix}$$

and after similar computations as (3.3.15) the curvature form turns out to be

$$-[\frac{1}{\pi}(da\wedge d\bar{a} + db\wedge d\bar{b})\int_{T^2} dV + (d\mu\wedge d\bar{\mu})\int_{T^2} 2|q|^2 dV)]. \tag{3.3.17}$$

Remark 3.3.10. If we ignore the a, b parts, the curvature will be exactly the Willmore energy (up to a factor)

$$\mathcal{W} = \frac{1}{\pi}\int_{T^2} 2|q|^2 dV.$$

This is analogous to the Hitchen's setup in case of CMC tori.

We find again that (3.3.17) is the Kähler form multiplied by $2i$ on the base space $\tilde{B}$. Again, we multiply the metric by $-e^{\|D-D_0\|^2_{\tilde{B}}}$, then the canonical connection for the modified metric has curvature zero. Thus, we can trivialize $\tilde{\mathcal{L}}$ and define a holomorphic function $\det(\tilde{D}_{+,a,b,\mu})$ on $\tilde{B}$ which vanishes precisely when $\tilde{D}$ is not invertible, satisfying

$$|\det(\tilde{D}_+)|^2 = e^{-\|\tilde{D}_+ - \tilde{D}_0\|^2_{\tilde{B}}} \det_\zeta(\tilde{D}_- \tilde{D}_+). \qquad (3.3.18)$$

Here $\det_\zeta(\tilde{D}_-\tilde{D}_+)$ is the zêta function determinant of the operator $\tilde{D}_-\tilde{D}_+$ restricted to the orthogonal complement of its kernel. It is a smooth function on $\tilde{B}$ (see Theorem 3.3.7). As an analog to Theorem 3.3.9 we have

Theorem 3.3.11. *The holomorphic family (3.3.16) of first order elliptic operators $D_{a,b,\mu}$, parameterized over the space* $\mathrm{Harm}(T^2, \mathbb{C}) \oplus \mathbb{C}$*, has a determinant line bundle and determinant function* $\det : \mathrm{Harm}(T^2, \mathbb{C}) \oplus \mathbb{C} \to \mathbb{C}$*. Its zero locus defines a 2-dimensional analytic subvariety, the spectral variety, in a, b, μ space.*

Particularly, the spectral variety induced by $\det(a, b, \mu)$ gives rise to the spectral curve $\mathrm{Spec}(V/L, D)$ as $\mu \to 1$ and the vacuum spectrum $\mathrm{Spec}(V/L, \bar{\partial})$ as $\mu \to 0$. But we point out here that one could not use this yet to get the asymptotics.

Chapter 4

Asymptotic geometry of the spectral curve

The spectral curve Σ_f of a conformal immersed torus $f : T^2 \to S^4$ is a 1-dimensional analytic subvariety in $\mathbb{C}_*^2$ consisting of the zero locus of a holomorphic function. In this chapter we give a more detailed picture of the spectral curve. We show that the vacuum spectrum $\operatorname{Spec}(V/L, \bar\partial)$ is a real translate of

$$\exp(H^0(K)) \cup \exp(\overline{H^0(K)})$$

with double points along the lattice of real representations $\operatorname{Hom}(\Gamma, \mathbb{R}_*)$ (see also [7]). Then we study the asymptotic geometry of the spectrum in the case the induced holomorphic structure D has a potential $Q \in W^{1,1}(T^2, \mathbb{C}^2)$. We give a detailed asymptotic description of the spectral curve as an infinite genus Riemann surface [11]. The spectral curve Σ_f approaches the vacuum spectrum Σ_0 asymptotically.

4.1 The spectrum for a constant potential

We assume that (W, D) is a quaternionic holomorphic line bundle of degree zero over a torus T^2. We will show that the vacuum spectrum in the case $D = \bar\partial$ consists of two planes and the spectral curve in the case of a non-zero constant potential is geometrically a handle with two planar ends.

4.1.1 The vacuum spectrum

We fix a uniformizing coordinate z on the torus $T^2 = \mathbb{C}/\Gamma$. We trivialize the quaternionic holomorphic line bundle W of degree zero over T^2 with a nowwhere vanishing section $\psi \in \Gamma(W_+)$ of the complex line subbundle W_+ like in subsection 2.1.4. Then we have isomorphisms

$$W \cong \underline{\mathbb{C}}^2, \quad \bar K W \cong \underline{\mathbb{C}}^2.$$

In addition, we choose coordinates (a, b) on the space $\operatorname{Harm}(T^2, \mathbb{C})$ so that the family of operators D_ω takes form

$$D_{a,b} = \bar\partial_{a,b} + Q$$

for any $\omega = adz + bd\bar{z} \in \text{Harm}(T^2, \mathbb{C})$ with

$$\bar{\partial}_{a,b} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}} + b & 0 \\ 0 & \frac{\partial}{\partial z} + a \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & -\bar{q} \\ q & 0 \end{pmatrix},$$

where q denotes the complex function defined by $Q\psi = \psi j dz q$.

The coordinate versions of the logarithmic $\widetilde{\text{Spec}}(W, D)$ and vacuum spectrum $\widetilde{\text{Spec}}(W, \bar{\partial})$ are

$$\tilde{\mathcal{S}}(q) = \{(a,b) \in \mathbb{C}^2 | ker(D_{a,b}) \neq 0\}$$

and

$$\tilde{\mathcal{S}}(0) = \{(a,b) \in \mathbb{C}^2 | ker(\bar{\partial}_{a,b}) \neq 0\}.$$

The induced Γ'−action on $\mathbb{C}^2$ is given by

$$(a,b) \to (a - \bar{c}, b + c) \quad \text{with} \quad c \in \Gamma',$$

where

$$\Gamma' = \{c \in \mathbb{C} | -\bar{c}\gamma + c\bar{\gamma} \in 2\pi i \mathbb{Z} \quad \text{for all} \quad \gamma \in \Gamma\}. \tag{4.1.1}$$

Thus, the spectrum $\text{Spec}(W, D)$ and the vacuum spectrum $\text{Spec}(W, \bar{\partial})$ have the following coordinate versions

$$\mathcal{S}(q) = \tilde{\mathcal{S}}(q)/\Gamma'$$

and

$$\mathcal{S}(0) = \tilde{\mathcal{S}}(0)/\Gamma'.$$

We expand the sections of the trivial bundle $\underline{\mathbb{C}}^2$ over T^2, i.e. the functions on T^2, in Fourier series. Smooth funcitons $C^\infty(T^2, \mathbb{C}^2)$ are a dense subspace of the Banach space $l^1(T^2, \mathbb{C}^2)$ of continuous functions with absolutely convergent Fourier series with Wiener norm

$$\|u\| = \sum_{c \in \Gamma'} |u_{1,c}| + |u_{2,c}|.$$

Here Γ' is given by (4.1.1) and $u_{1,c}, u_{2,c} \in \mathbb{C}$ are the Fourier coefficients of

$$u(z) = (\sum_{c \in \Gamma'} u_{1,c} e^{-\bar{c}z + c\bar{z}}, \sum_{c \in \Gamma'} u_{2,c} e^{-\bar{c}z + c\bar{z}}) \in l^1(T, \mathbb{C}^2).$$

The extension of $D_{a,b} : C^\infty \subset L^2 \to L^2$ to the Sobolev space H^1 of functions whose first derivatives have finite L^2-norms, is closed and also denoted by $D_{a,b} : H^1 \subset L^2 \to L^2$. By ellipticity this extension has the same kernel as the original operator [23].

Lemma 4.1.1. *The curve $\tilde{\mathcal{S}}(0)$ for $q = 0$ is the locally finite union $\bigcup_{c \in \Gamma', \nu = 1,2} \mathcal{N}_\nu(c)$ of lines*

$$\mathcal{N}_\nu(c) = \{(a,b) \in \mathbb{C}^2 \mid N_{c,\nu}(a,b) = 0\}, \tag{4.1.2}$$

where $N_{c,\nu}$ are complex functions

$$N_{c,1}(a,b) = b + c, \quad N_{c,2}(a,b) = a - \bar{c}. \tag{4.1.3}$$

In particular, the vacuum spectrum $\mathcal{S}(0)$ is a complex analytic curve in $\mathbb{C}^2$.

Proof. For all $(a,b)\in\mathbb{C}^2$ the Fourier monomials $\begin{pmatrix} e^{-\bar{c}z+c\bar{z}} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e^{-\bar{c}z+c\bar{z}} \end{pmatrix}$ are a complete set of eigenfunctions for $\bar\partial_{a,b}$ in L^2 satisfying

$$\bar\partial_{a,b}\begin{pmatrix} e^{-\bar{c}z+c\bar{z}} \\ 0 \end{pmatrix} = N_{c,1}(a,b)\begin{pmatrix} e^{-\bar{c}z+c\bar{z}} \\ 0 \end{pmatrix}, \quad \bar\partial_{a,b}\begin{pmatrix} 0 \\ e^{-\bar{c}z+c\bar{z}} \end{pmatrix} = N_{c,2}(a,b)\begin{pmatrix} 0 \\ e^{-\bar{c}z+c\bar{z}} \end{pmatrix}.$$

Then clearly

$$\tilde{\mathcal{S}}(0) = \bigcup_{c\in\Gamma',\ \nu=1,2} \mathcal{N}_\nu(c).$$

Observe that only a finite number of the line pairs $\mathcal{N}_c$ can intersect any bounded subset of $\mathbb{C}^2$. Thus, the union is locally finite.

We get $\bar\partial_{a,b}$ is not invertible if and only if $b\in\Gamma'$ or $\bar{a}\in\Gamma'$. The space of (a,b) for which $\bar\partial_{a,b}$ has non-trivial kernel is

$$\tilde{\mathcal{S}}(0) = (\mathbb{C}\times\Gamma')\cup(\bar\Gamma'\times\mathbb{C}).$$

Notice that for points $(a,b)\in\bar\Gamma'\times\Gamma'$ the kernel of $\bar\partial_{a,b}$ is 2-dimensional. Away from these double points the kernel for $(a,b)\in\tilde{\mathcal{S}}_0$ is 1-dimensional.

The vacuum spectrum $\mathrm{Spec}(W,\bar\partial)$ is a real translate of

$$\exp(H^0(K))\cup\exp(\overline{H^0(K)}).$$

The double points are $\mathrm{Spec}(W,\bar\partial)\cap\mathrm{Hom}(\Gamma,\mathbb{R}_*)$. $\square$

We describe the spectrum in somewhat more detail. Notice that

$$\mathcal{N}_\nu(c)\cap\mathcal{N}_\nu(d)=\emptyset \quad \text{if} \quad c\neq d,$$

$$\mathcal{N}_1(0)\cap\mathcal{N}_2(c)=\{(\bar{c},0)\},\quad \mathcal{N}_1(-c)\cap\mathcal{N}_2(0)=\{(0,c)\}$$

and the map $(a,b)\mapsto(a-\bar{c},b+c)$ maps $\mathcal{N}_1(0)\cap\mathcal{N}_2(c)$ to $\mathcal{N}_1(-c)\cap\mathcal{N}_2(0)$. If we change the coordinate to be

$$u=\frac{1}{2}(\bar{a}+b),\quad v=\frac{1}{2}(-\bar{a}+b),$$

the induced Γ'-action turns to be

$$(u,v)\mapsto(u,v+c),$$

where u is invariant under the Γ'-action.

If $\Gamma'\cap\mathbb{R}\neq\emptyset$, the real part of the curve $\tilde{\mathcal{S}}(0)$ is the union of lines

$$\bigcup_{c\in\Gamma'\cap\mathbb{R}}\{u+v+c=0\}\cup\{u-v-c=0\}.$$

Modulo the Γ'-action, we should identify the lines $v=-\frac{c_0}{2}$ and $v=\frac{c_0}{2}$, where $c_0 = min_{\Gamma'\cap\mathbb{R}}\{c\neq 0\}$, to get two "helices" (see the left and middle pictures in Figure 4.1) on a circular cylinder of the same constant slope. The two helices intersect each other twice on each cycle of the cylinder-once on the front half of the cylinder and once on the back half. So the real part of the vacuum curve is just two copies of $\mathbb{R}$ with pairs of points that correspond to the intersections of two helices.

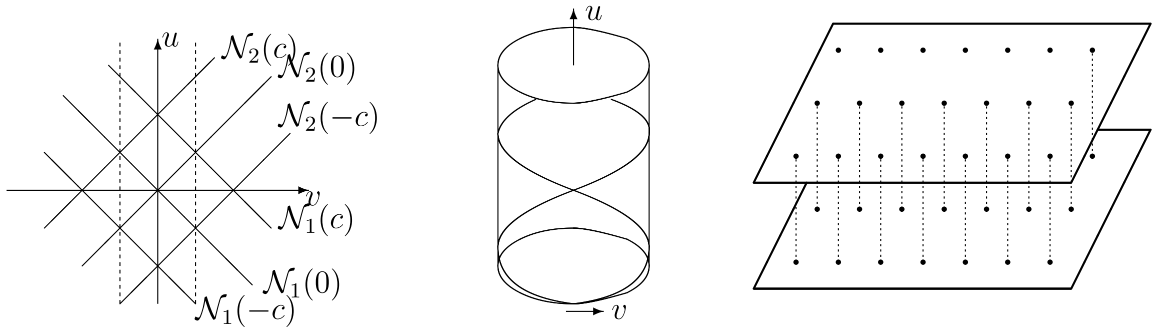

Figure 4.1: Vacuum Spectrum

The full $\mathcal{S}(0)$ (see the right picture in Figure 4.1) is just two copies of $\mathbb{C}$, with $u = \frac{1}{2}(\bar{a} + b)$ as coordinates, provided that we identify $\frac{c}{2}$ on each of the planes for all $c \in \Gamma'$.

4.1.2 The case q is a non-zero constant

In the same setting as in the last subsection, but now $q \neq 0$ is a non-zero constant, the operators act on the basis by

$$D_{(a,b)} \begin{pmatrix} e^{-\bar{c}z+c\bar{z}} \\ 0 \end{pmatrix} = \begin{pmatrix} (b+c)e^{-\bar{c}z+c\bar{z}} \\ qe^{-\bar{c}z+c\bar{z}} \end{pmatrix}, \quad D_{a,b} \begin{pmatrix} 0 \\ e^{-\bar{c}z+c\bar{z}} \end{pmatrix} = \begin{pmatrix} -\bar{q}e^{-\bar{c}z+c\bar{z}} \\ (a-\bar{c})e^{-\bar{c}z+c\bar{z}} \end{pmatrix}.$$

The eigenvector of $D_{(a,b)}$ with eigenvalue λ are of form $\begin{pmatrix} e^{-\bar{c}z+c\bar{z}} \\ \mu e^{-\bar{c}z+c\bar{z}} \end{pmatrix}$ satisfying

$$\begin{cases} (b+c) - \bar{q}\mu = \lambda \\ \mu(a-\bar{c}) + q = \lambda\mu. \end{cases}$$

For $(b - a + c + \bar{c})^2 \neq 4|q|^2$ and for any $c \in \Gamma'$ there are always two different solutions to the above equations hence all eigenvectors generate the basis. Then the space of (a, b) for which $D_{a,b}$ has non-trivial kernel is

$$\tilde{\mathcal{S}}(q) = \bigcup_{c\in\Gamma'} \{(a, b) \in \mathbb{C}^2 \mid (a-\bar{c})(b+c) = -|q|^2\}.$$

For any $c \in \Gamma'$, the kernel vector is $\begin{pmatrix} e^{-\bar{c}z+c\bar{z}} \\ \mu e^{-\bar{c}z+c\bar{z}} \end{pmatrix}$ with $\mu = (b + c)/\bar{q}$.

Obviously, the curve $\{(a, b) \in \mathbb{C}^2|(a-\bar{c})(b+c) = -|q|^2\}$ is topologically a cylinder and asymptotic to the complex lines $\mathcal{N}_1(c)$ and $\mathcal{N}_2(c)$ (4.1.2) when a or b is sufficiently large.

Since

$$\mathrm{Spec}(W, D) = \tilde{\mathcal{S}}(q)/\Gamma' = \{(a, b) \in \mathbb{C}^2 \mid ab = -|q|^2\}/\Gamma',$$

the double points on the cylinder are exactly the roots of the equations

$$\begin{cases} ab = -|q|^2 \\ (a-\bar{c})(b+c) = -|q|^2 \end{cases}$$

or

$$b^2 + cb + \frac{c}{\bar{c}}|q|^2 = 0.$$

If $\Delta := 1 - 4|q|^2/|c|^2 \neq 0$, there are two roots $b = \frac{-c}{2}(1 \pm \sqrt{1-4|q|^2/|c|^2})$, corresponding to the identified points

$$(\frac{\bar{c}}{2}(1-\sqrt{\Delta}), \frac{-c}{2}(1+\sqrt{\Delta})) \sim (\frac{-\bar{c}}{2}(1+\sqrt{\Delta}), \frac{c}{2}(1-\sqrt{\Delta})).$$

Here $\sqrt{\Delta}$ denote one of the (complex) square roots of Δ.
If $\Delta = 0$, $b = \frac{-c}{2}$, corresponding to the identified points

$$(\frac{\bar{c}}{2}, \frac{-c}{2}) \sim (\frac{-\bar{c}}{2}, \frac{c}{2}).$$

Since

$$c\sqrt{1-4|q|^2/|c|^2} \to c \quad \text{as} \quad |c| \to \infty,$$

the double points approach the double points of the vacuum spectrum.
If q is constant, $\mathrm{Spec}(W, D)$ is a genus 1 annulus asymptotic to the vacuum spectrum.

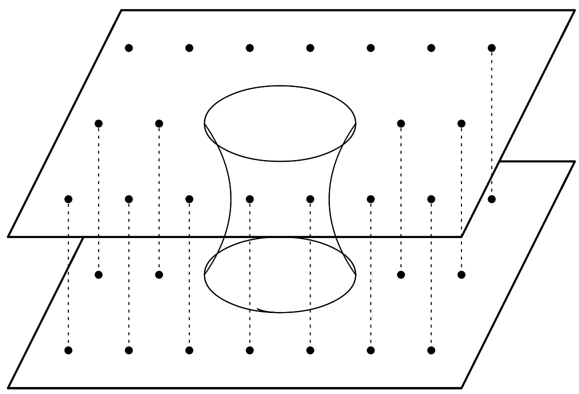

Figure 4.2: Spectrum in the case of Constant q

Remark 4.1.2. Similarly, we study the case $q = \hat{q}_{c_0} e^{-\bar{c}_0 z + c_0 \bar{z}}$, for $c_0 \in \Gamma'$ and $\hat{q}_{c_0} \in \mathbb{C}$,

$$\mathrm{Spec}(W, D) = \{(a,b) \in \mathbb{C}^2 \mid (a - \bar{c}_0)b = -|q_{c_0}|^2\}/\sim.$$

This just is a translation of

$$\{(a,b) \in \mathbb{C}^2 \mid ab = -|q_{c_0}|^2\}/\sim$$

moving the origin to $(\bar{c}_0, 0)$. Again $\mathrm{Spec}(W, D)$ is geometrically a handle with two planar ends.

4.2 Asymptotic analysis in the general case

We study the asymptotic geometry of the spectrum $\mathrm{Spec}(W, D)$ of the holomorphic quaternonic line bundle W of degree zero over T^2 in the case D has absolutely continuous potential $Q \in W^{1,1}(T^2, \mathbb{C}^2)$. Generally, the spectrum $\mathrm{Spec}(W, D)$ is close to the vacuum spectrum $\mathrm{Spec}(W, \bar{\partial})$ when a and b are large. Actually, a detailed asymptotic analysis shows the spectral curve Σ is bi-holomorphic to a pair of planes joined by at most countable many handles.

Here we use the notion of a Riemann surface of infinite genus discussed by Feldmann, Knörrer and Trubowitz in [11]. We shall apply their methods, developed for Schrödinger type operators, to the family of Dirac type operators $D_{a,b}$.

4.2.1 The main theorems

To facilitate the discussion write

$$a = \bar{u} - \bar{v}, \quad b = u + v,$$

so that

$$N_{c,1}(a, b) = u + v + c, \quad N_{c,2}(a, b) = \bar{u} - \bar{v} - \bar{c},$$

and

$$|N_{c,\nu}| = |u - (-1)^\nu (v + c)|.$$

Let 2Λ be the length of the shortest nonzero vector in Γ'. Then there is at most one $c \in \Gamma'$ with $|u + (v + c)| < \Lambda$ and at most one $c \in \Gamma'$ with $|u - (v + c)| < \Lambda$. For $0 < \epsilon < \frac{\Lambda}{2}$ and $c \in \Gamma'$ define the ϵ-tube around the complex line $\mathcal{N}_\nu(c)$ (4.1.2) by

$$T_\nu(c) = \{(a, b) \in \mathbb{C}^2 \mid |N_\nu(c)| = |u - (-1)^\nu (v + c)| < \epsilon\}. \qquad (4.2.1)$$

From

$$|u - (v + c)| + |u + (v + c)| \geq |[u - (v + c)] + [u + (v + c)]| = 2|u|$$

we get at least one of the factors in

$$|N(c)| := |N_1(c)| \cdot |N_2(c)|$$

must always be at least $|u|$.

The pairwise intersection $\overline{T}(c) \bigcap \overline{T}(c')$ is compact for each pair $c \neq c'$. Obviously, $\overline{T}_\nu(c) \bigcap \overline{T}_\nu(c') = \emptyset$ if $c \neq c'$ since

$$|N_\nu(c)| + |N_\nu(c')| \geq |c - c'| \geq 2\Lambda.$$

If $(a, b) \in \overline{T}_1(c) \bigcap \overline{T}_2(c')$, then we have

$$|u + \frac{c' - c}{2}| = \frac{1}{2}|u - (v + c) + u + (v + c')| \geq \epsilon$$

and

$$|v + \frac{c' + c}{2}| = \frac{1}{2}| - u + (v + c) + u + (v + c')| \geq \epsilon.$$

We also have $\overline{T}(c)\bigcap\overline{T}(c')\bigcap\overline{T}(c'')=\emptyset$ for all distinct elements $c,c',c''\in\Gamma'$. We shall asymptotically confine $(a,b)\in\widetilde{\mathcal{S}}(q)$ to the union of the tubes $T(c)$, $c\in\Gamma'$.

We will apply the decomposition of a Riemann surface of infinite genus [11] to the spectrum $\mathcal{S}(q)$. Thus $\mathcal{S}(q)$ splits into three parts

$$\mathcal{S}(q)=\mathcal{S}^{com}\cup\mathcal{S}^{reg}\cup\mathcal{S}^{han}. \tag{4.2.2}$$

Here $\mathcal{S}^{com}$ is a compact, connected submanifold of $\mathcal{S}(q)$ with smooth boundary and finite genus; $\mathcal{S}^{reg}=\mathcal{S}^{reg}_1\cup\mathcal{S}^{reg}_2$ is the union of two open "regular pieces" asymptotic to the vacuum spectrum $\mathcal{S}(0)$; and $\mathcal{S}^{han}$ consists of an infinite number of closed "handles" at the double points of the the vacuum.

For $\rho>0$ define a closed subset of $\mathbb{C}^2$

$$\mathcal{K}_\rho=\{(a,b)\in\mathbb{C}^2\mid |u|\le\rho\}. \tag{4.2.3}$$

Since the induced Γ'-action on $(u,v)\in\mathbb{C}^2$ with $c\in\Gamma'$ is $(u,v)\mapsto(u,v+c)$, the subset $\mathcal{K}_\rho$ is invariant under the Γ'-action and $\mathcal{K}_\rho/\Gamma'$ is compact. The image of $\widetilde{\mathcal{S}}(q)\cap\mathcal{K}_\rho$ under the exponential map $\exp:\widetilde{\mathcal{S}}(q)\to\mathcal{S}(q)$ is compact in $\mathcal{S}(q)$. It is a compact submanifold with smooth boundary and finite genus and will play the role of $\mathcal{S}^{com}$ in the above decomposition (4.2.2).

The following theorem describes the regular pieces $\mathcal{S}^{reg}$.

Theorem 4.2.1. *Let the potential $q\in W^{1,1}(\mathbb{C}/\Gamma)$ be an absolutely continuous function with Fourier series $\sum_{c\in\Gamma'}\hat{q}(c)e^{-\bar{c}z+c\bar{z}}$ so that the first-order derivative is absolutely convergent,*

$$\|q'\|:=\sum_{c\in\Gamma'}|c\hat{q}(c)|<\infty.$$

Let $0<\epsilon<\Lambda$. Then there is a constant ρ, which depends on $\|q'\|$, Λ, and ϵ, such that

a) the complement of $\mathcal{K}_\rho$ in $\widetilde{\mathcal{S}}(q)$ is contained in the union of the tubes

$$\widetilde{\mathcal{S}}(q)\backslash\mathcal{K}_\rho\subset\cup_{c\in\Gamma}T(c).$$

b) For $\nu=1,2$ the map $(a,b)\mapsto\frac{a+b}{2}$ induces a bi-holomorphic map between

$$(\widetilde{\mathcal{S}}(q)\cap T_\nu(0))\backslash(\mathcal{K}_\rho\cup\bigcup_{c\in\Gamma'}T_{(3-\nu)}(c))$$

and its image in $\mathbb{C}$. This image component contains

$$\{z\in\mathbb{C}\mid |z|>2\rho \text{ and } |z-z_\nu(c)|>\epsilon \text{ for all } c\in\Gamma' \text{ with } c\neq0\}$$

and is contained in

$$\{z\in\mathbb{C}\mid |z-z_\nu(c)|>\frac{\epsilon}{4} \text{ for all } c\in\Gamma' \text{ with } c\neq0\}$$

where $z_1(c)=\frac{\bar{c}}{2}$, $z_2(c)=-\frac{c}{2}$.

Clearly under the Γ'-action $d \cdot T_\nu(c) = T_\nu(c-d)$ for every $c, d \in \Gamma', \nu = 1, 2$, the complement of $\exp(\widetilde{\mathcal{S}}(q) \cap \mathcal{K}_\rho)$ in $\mathcal{S}(q)$ is the disjoint union of

$$\exp(\widetilde{\mathcal{S}}(q) \cap T_1(0) \backslash (\mathcal{K}_\rho \cup \bigcup_{c \in \Gamma'} T_2(c))), \quad \exp(\widetilde{\mathcal{S}}(q) \cap T_2(0) \backslash (\mathcal{K}_\rho \cup \bigcup_{c \in \Gamma'} T_1(c)))$$

and

$$\bigcup_{c \in \Gamma',\ c \neq 0} \exp(\widetilde{\mathcal{S}}(q) \cap T(0) \cap T(c)).$$

We are going to show that the first two sets will be the regular pieces of $\mathcal{S}(q)$, which are asymptotic to a pair of complex planes. The third set will be the infinitely many handles, which join the regular pieces. The composition of exp with the inverse of the map discussed in part (b) of Theorem 4.2.1. parametrises the regular part. For the handle part we have

Theorem 4.2.2. *Let $0 < \epsilon < \Lambda$ be sufficiently small. Assume that $q \in W^{1,1}(\mathbb{C}/\Gamma)$. There are constants such that for every sufficiently large $c \in \Gamma' \setminus \{0\}$ there are maps*

$$\phi_{c,1} : \{(z_1, z_2) \in \mathbb{C}^2 \mid \ |z_1| \leq \frac{\epsilon}{2},\ |z_2| \leq \frac{\epsilon}{2}\} \to T_1(0) \cap T_2(c),$$

$$\phi_{c,2} : \{(z_1, z_2) \in \mathbb{C}^2 \mid \ |z_1| \leq \frac{\epsilon}{2},\ |z_2| \leq \frac{\epsilon}{2}\} \to T_1(c) \cap T_2(0)$$

and a complex number t_c with $|t_c| \leq \frac{const}{|c|^2}$ such that

(i) $\phi_{c,\nu}$ is bi-holomorphic to its image. The image of $\phi_{c,1}$ contains

$$\{(a, b) \in \mathbb{C}^2 \mid |a - \bar{c}| \leq \frac{\epsilon}{8},\ |b| \leq \frac{\epsilon}{8}\}.$$

The image of $\phi_{c,2}$ contains

$$\{(a, b) \in \mathbb{C}^2 \mid |a| \leq \frac{\epsilon}{8},\ |b + c| \leq \frac{\epsilon}{8}\}.$$

The Jacobian of the map $\phi_{c,\nu}$ satisfies

$$D\phi_{c,\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \{1 + O(\frac{1}{|c|^2})\}.$$

Furthermore

$$|\phi_{c,1}(0) - (\bar{c}, 0)| \leq \frac{const}{|c|^2}, \quad |\phi_{c,2}(0) - (0, -c)| \leq \frac{const}{|c|^2}.$$

(ii) The pre-images of the maps are

$$\phi_{c,1}^{-1}(T_1(0) \cap T_2(c) \cap \widetilde{S}(q)) = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = t_c,\ |z_1| \leq \frac{\epsilon}{2}, |z_2| \leq \frac{\epsilon}{2}\},$$

$$\phi_{c,2}^{-1}(T_1(-c) \cap T_2(0) \cap \widetilde{S}(q)) = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = t_c,\ |z_1| \leq \frac{\epsilon}{2}, |z_2| \leq \frac{\epsilon}{2}\}.$$

(iii) The relationship between the two maps is given by

$$\phi_{c,2}(z_1, z_2) = \phi_{c,1}(z_2, z_1) - c.$$

The maps $\phi_{c,\nu}$ in the above theorem give us the parametrization of the handles of the spectrum in the small neighborhoods of the double points of the vacuum. The Jacobian $D\phi_{c,\nu}$ is almost the identity, which keeps the size of the corresponding handle controlled. The images of the origin, the center of the handles, are close to the double points of the vacuum. Part (ii) of Theorem 4.2.2 shows that the part of the spectrum in the intersection of two non-parallel tubes is a handle. Since the handle size t_c is controlled by $\frac{1}{|c|^2}$, the handles will get closer to the double points when a or b go to infinity.

4.2.2 Technical lemmas and propositions

In order to prove the main theorems stated in the last subsection, we first introduce some notation. Recall that (a,b) is in $\widetilde{\mathcal{S}}(q))$ if and only if $D = \bar{\partial}_{a,b} + Q$ has a non-trivial kernel in $L^2(T^2, \mathbb{C}^2)$. To study the part of the spectrum in the intersection of $\cup_{d_1 \in G_1} T_1(d_1) \bigcup \cup_{d_2 \in G_2} T_2(d_2)$ and $\mathbb{C}^2 \backslash (\cup_{c_1 \notin G_1} T_1(c_1) \bigcup \cup_{c_2 \notin G_2} T_2(c_2)$ for some finite subsets G_1 and G_2 of the dual lattice Γ' it is natural to look for a nontrivial solution of

$$(\bar{\partial}_{a,b} + Q) \begin{pmatrix} \psi_{G_1} \\ \psi_{G_2} \end{pmatrix} + (\bar{\partial}_{a,b} + Q) \begin{pmatrix} \psi_{G_1'} \\ \psi_{G_2'} \end{pmatrix} = 0.$$

This is equivalent to

$$(\bar{\partial}_{a,b} + Q) \begin{pmatrix} \phi_{G_1} \\ \phi_{G_2} \end{pmatrix} + (1 + Q\bar{\partial}_{a,b}^{-1}) \begin{pmatrix} \phi_{G_1'} \\ \phi_{G_2'} \end{pmatrix} = 0, \tag{4.2.4}$$

where

$$\psi_{G_\nu}, \phi_{G_\nu} \in L^2_{G_\nu} := \operatorname{Span}\{e^{-\bar{c}z + c\bar{z}} \mid c \in G_\nu\},$$

$$\psi_{G_\nu'}, \phi_{G_\nu'} \in L^2_{G_\nu'} := \operatorname{Span}\{e^{-\bar{c}z + c\bar{z}} \mid c \in \Gamma' \backslash G_\nu\}.$$

We shall prove later in Lemma 4.2.4 that, for (a,b) in the region under consideration, the restriction of $1 + Q\bar{\partial}_{a,b}^{-1}$ to $L^2_{G_1'} \oplus L^2_{G_2'}$, denoted as R, has a bounded inverse. Then the projection of (4.2.4) on $L^2_{G_1'} \oplus L^2_{G_2'}$ is equivalent to

$$\begin{pmatrix} \phi_{G_1'} \\ \phi_{G_2'} \end{pmatrix} = -R^{-1}Q \begin{pmatrix} \phi_{G_1} \\ \phi_{G_2} \end{pmatrix},$$

since $L^2_{G_\nu}$ is invariant under the operator $\bar{\partial}_{a,b}^{-1}$. Substituting this into the projection on $L^2_{G_1} \oplus L^2_{G_2}$ yields

$$\pi_G(\bar{\partial}_{a,b} + Q - Q\bar{\partial}_{a,b}^{-1} R^{-1} Q) \begin{pmatrix} \phi_{G_1} \\ \phi_{G_2} \end{pmatrix} = 0. \tag{4.2.5}$$

Here $G = G_1 \oplus G_2$ and π_G is the projection operator. Obviously, (4.2.5) has a nontrivial solution if and only if the $|G| \times |G|$ determinant

$$\det[\pi_G(\bar{\partial}_{a,b} + Q - Q\bar{\partial}_{a,b}^{-1} R^{-1} Q)\pi_G] = 0. \tag{4.2.6}$$

We write all operators in matrix-form in the basis $\{\begin{pmatrix} e^{-\bar{c}z+c\bar{z}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^{-\bar{c}z+c\bar{z}} \end{pmatrix} \mid c \in \Gamma'\}$, thus

$$\bar{\partial}_{a,b} = \begin{bmatrix} (b+c_1)\delta_{d_1,c_1} & 0 \\ 0 & (a-\bar{c}_2)\delta_{d_2,c_2} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & -\bar{\hat{q}}(c_2-d_1) \\ \hat{q}(d_2-c_1) & 0 \end{bmatrix}.$$

In general, we define the operators R^1_{DC}, R^2_{DC} to have matrix elements

$$(R^1_{DC})_{d,c} = \left[\frac{\hat{q}(d-c)}{N_1(c)}\right]_{d\in D, c\in C}, \quad (R^2_{DC})_{d,c} = \left[-\frac{\bar{\hat{q}}(c-d)}{N_2(c)}\right]_{d\in D, c\in C},$$

then

$$R_{G'G'} = \begin{pmatrix} \pi_{G'_1} & R^2_{G'_1G'_2} \\ R^1_{G'_2G'_1} & \pi_{G'_2} \end{pmatrix}.$$

We assume that the inverse of R has the form

$$R^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \tag{4.2.7}$$

and $Q\bar{\partial}^{-1}_{a,b}R^{-1}Q$ has the form

$$Q\bar{\partial}^{-1}_{a,b}R^{-1}Q = \begin{pmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{D}_{21} & \mathcal{D}_{22} \end{pmatrix}. \tag{4.2.8}$$

Then we have relations

$$\begin{aligned} (\mathcal{D}_{11})_{d_1,c_1} &= -\sum_{f,g\in G'_2} \frac{1}{a-\bar{f}}\bar{\hat{q}}(f-d_1)(R_{22})_{fg}\hat{q}(g-c_1), \\ (\mathcal{D}_{12})_{d_1,c_2} &= \sum_{f\in G'_2, g\in G'_1} \frac{1}{a-\bar{f}}\bar{\hat{q}}(f-d_1)(R_{21})_{fg}\bar{\hat{q}}(c_2-g), \\ (\mathcal{D}_{21})_{d_2,c_1} &= \sum_{f\in G'_1, g\in G'_2} \frac{1}{b+f}\hat{q}(d_2-f)(R_{12})_{fg}\hat{q}(g-c_1), \\ (\mathcal{D}_{22})_{d_2,c_2} &= -\sum_{f\in G'_1, g\in G'_1} \frac{1}{b+f}\hat{q}(d_2-f)(R_{11})_{fg}\bar{\hat{q}}(c_2-g). \end{aligned} \tag{4.2.9}$$

The determinant equation (4.2.6) can be expressed as

$$\det \begin{bmatrix} (b+c_1)\delta_{d_1,c_1} - (\mathcal{D}_{11})_{d_1,c_1} & -\bar{\hat{q}}(c_2-d_1) - (\mathcal{D}_{12})_{d_1,c_2} \\ \hat{q}(d_2-c_1) - (\mathcal{D}_{21})_{d_2,c_1} & (a-\bar{c}_2)\delta_{d_2,c_2} - (\mathcal{D}_{22})_{d_2,c_2} \end{bmatrix}_{d_\nu,c_\nu\in G_\nu} = 0. \tag{4.2.10}$$

Our analysis of the above finite determinants is based on two lemmas. The first lemma estimates the sums of certain entries of the matrices R^1 and R^2.

Lemma 4.2.3. *(a) There is a constant such that, for all $(a,b) \in \mathbb{C}^2$*

$$\sum_{c\in\Gamma', |N_\nu(c)|\geq\epsilon} \frac{1}{|N_c(a,b)|^2} \leq const\frac{\ln|u|}{|u|^2}.$$

(b) If $d, c \in \Gamma'$ obey $|N_\nu(d)| < \frac{|u|}{2}$ and $|N_{3-\nu}(c)| < \frac{|u|}{2}$, then

$$|c - d| > |u|.$$

(c) If $(a, b) \in T_1(0) \backslash \cup_{c \in \Gamma'} T_2(c)$ with $|u| > 2\epsilon$, then there is a constant depending on ϵ, $\|q\|$ and $\|q'\|$ such that

$$|\sum_{f,g \in \Gamma'} \frac{\bar{\hat{q}}(f)\hat{q}(g)}{N_{f,2}(a,b)}| \leq \frac{const}{|u|}.$$

If $(a, b) \in T_2(0) \backslash \cup_{c \in \Gamma'} T_1(c)$ with $|u| > 2\epsilon$, then there is a constant depending on ϵ, $\|q\|$ and $\|q'\|$ such that

$$|\sum_{f,g \in \Gamma'} \frac{\bar{\hat{q}}(-f)\hat{q}(-g)}{N_{g,1}(a,b)}| \leq \frac{const}{|u|}.$$

(d) If $(a, b) \in T_\nu(d_\nu) \cap T_{3-\nu}(d_\nu)$ with $|u| > 2\epsilon$, then there is a constant depending on ϵ, $\|q\|$ and $\|q'\|$ such that

$$\sum_{f \neq d_\nu, g \neq d_{3-\nu}} \frac{|\hat{q}((-1)^{3-\nu}(d_{3-\nu} - f))||\hat{q}((-1)^\nu(d_\nu - g))|}{|N_{f,\nu}(a,b)|} \leq \frac{const}{|u|}.$$

Proof. (a) The sum is bounded by

$$\sum_{\substack{c \in \Gamma' \\ |N_1(c)| \geq \epsilon \\ |N_1(c)| \geq \epsilon}} \frac{\pi\epsilon^2}{2|N_c(a,b)|^2} \leq \int_{\substack{|u + (v + x)| > \epsilon \\ |u - (v + x)| > \epsilon}} \frac{dx_1 dx_2}{|u + (v + c)|^2 |u - (v + c)|^2}$$

$$= \int_{\substack{|x| > \epsilon \\ |x - 2u| > \epsilon}} \frac{dx_1 dx_2}{|x|^2 |x - 2u|^2}$$

$$= \int_{\epsilon < |x| < |u|} \frac{dx_1 dx_2}{|x|^2 |x - 2u|^2} + \int_{\epsilon < |x - 2u| < |u|} \frac{dx_1 dx_2}{|x|^2 |x - 2u|^2}$$

$$+ \int_{\substack{|u| \leq |x| < 4|u| \\ |x - 2u| \geq |u|}} \frac{dx_1 dx_2}{|x|^2 |x - 2u|^2} + \int_{\substack{|x| \geq 4|u| \\ |x - 2u| \geq |u|}} \frac{dx_1 dx_2}{|x|^2 |x - 2u|^2}$$

$$\leq \frac{2}{|u|^2} \int_{\epsilon < |x| < |u|} \frac{dx_1 dx_2}{|x|^2} + \frac{1}{|u|^2} \int_{|u| \leq |x| < 4|u|} \frac{dx_1 dx_2}{|x|^2}$$

$$+ \int_{|x| \geq 4|u|} \frac{4 dx_1 dx_2}{|x|^4} (\text{ here } |x - 2u| \geq |x| - |2u| \geq \frac{|x|}{2})$$

$$= \int_0^{2\pi} d\theta [\frac{2}{|u|^2} \int_\epsilon^{|u|} \frac{r dr}{r^2} + \frac{1}{|u|^2} \int_{|u|}^{4|u|} \frac{r dr}{r^2} + \int_{4|u|}^\infty \frac{4 r dr}{r^4}]$$

$$= 2\pi [\frac{2(\ln|u| - \ln\epsilon)}{|u|^2} + \frac{\ln 4}{|u|^2} + \frac{1}{8|u|^2}].$$

The computation of the integral has been divided into four parts bounding the norms $|x|$ and $|x-2u|$ by ϵ, $|u|$ and $4|u|$. At last we choose a constant depending only on ϵ for sufficient large $|u|$ to control the sum.

(b) Since $|N_\nu(c)|+|N_{3-\nu}(c)|\geq 2|u|$ we easily get from $|N_{3-\nu}(c)|<\frac{|u|}{2}$ that

$$|N_\nu(c)|\geq 2|u|-|N_{3-\nu}(c)|>\frac{3}{2}|u|.$$

Together with $|N_\nu(d)|<\frac{|u|}{2}$, we have

$$|c-d|=|N_\nu(c)-N_\nu(d)|>|u|.$$

(c) When $(a,b)\in T_1(0)\backslash\cup_{c\in\Gamma'}T_2(c)$, we estimate the sum:

$$\begin{aligned}
|\sum_{f,g\in\Gamma'}\frac{1}{N_2(f)}\bar{\hat{q}}(f)\hat{q}(g)| &\leq |\sum_{f\in\Gamma'}\frac{1}{N_2(f)}\bar{\hat{q}}(f)|\sum_{g\in\Gamma'}|\hat{q}(g)| \\
&\leq [\sum_{\epsilon\leq|N_2(f)|\leq\frac{|u|}{2}}\frac{1}{|N_2(f)|}|\hat{q}(f)|+\sum_{|N_2(f)|>\frac{|u|}{2}}\frac{1}{|N_2(f)|}|\hat{q}(f)|]\|q\| \\
&\leq [\sum_{\epsilon\leq|N_2(f)|\leq\frac{|u|}{2}}\frac{|f\hat{q}(f)|}{\epsilon|u|}+\sum_{|N_2(f)|\geq\frac{|u|}{2}}\frac{|2\hat{q}(f)|}{|u|}]\|q\| \\
&\leq \frac{\|q\|}{|u|}(\frac{\|q'\|}{\epsilon}+2\|q\|).
\end{aligned}$$

First we separate f and g to cancel g. The case $|N_{c,2}(a,b)|\geq\frac{|u|}{2}$ obviously includes $f=0$, since $|N_{0,2}|\geq 2|u|-\epsilon$. Using the result (b) with condition $|N_1(0)|<\epsilon$ and $\epsilon\leq|N_{c,2}(a,b)|\leq\frac{|u|}{2}$ we get the last inequality.

(d) For the most part it suffices to repeat the argument in (c) and notice that $f\neq d_\nu$ and $(a,b)\in T_\nu(d_\nu)$ implies

$$|N_{f,\nu}|\geq|f-d_\nu|-|N_{d_\nu,\nu}|>2\Lambda-\epsilon>\epsilon.$$

Using the result (b) with condition $|N_{(3-\nu)}(d_{3-\nu})|<\epsilon$ and $|N_{f,\nu}|\leq\frac{|u|}{2}$, we get

$$\begin{aligned}
&\sum_{f\neq d_\nu,g\neq d_{3-\nu}}\frac{|\hat{q}[(-1)^{3-\nu}(d_{3-\nu}-f)]||\hat{q}[(-1)^i(d_i-g)]|}{|N_{f,\nu}(a,b)|} \\
&\leq \sum_{f\neq d_\nu}\frac{|\hat{q}[(-1)^{3-\nu}(d_{3-\nu}-f)]|}{|N_{f,\nu}(a,b)|}\|q\| \\
&\leq [\sum_{\epsilon<|N_\nu(f)|\leq\frac{|u|}{2}}\frac{|\hat{q}[(-1)^{3-\nu}(d_{3-\nu}-f)](d_{3-\nu}-f)|}{\epsilon|u|}+\sum_{|N_\nu(f)|>\frac{|u|}{2}}\frac{2|\hat{q}[(-1)^{3-\nu}(d_{3-\nu}-f)]|}{|u|}]\|q\| \\
&\leq \frac{\|q\|}{|u|}(\frac{\|q'\|}{\epsilon}+2\|q\|).
\end{aligned}$$

□

Lemma 4.2.4. *Let $(a,b) \in \mathbb{C}^2$ with $|u| > 2\epsilon$ and define the following subsets of the dual lattice Γ':*

$$B_1 \subset \{c \in \Gamma' \mid \epsilon \leq |N_1(c)| < \frac{|u|}{2}\}, \quad B_2 \subset \{c \in \Gamma' \mid |N_1(c)| \geq \frac{|u|}{2}\},$$

$$S_1 \subset \{c \in \Gamma' \mid \epsilon \leq |N_2(c)| < \frac{|u|}{2}\}, \quad S_2 \subset \{c \in \Gamma' \mid |N_2(c)| \geq \frac{|u|}{2}\}.$$

(a) We divide the resolvent operators R^1 and R^2 into 4 parts according to the subsets B_ν, S_ν, $\nu = 1, 2$. The l^1-norms of the operators satisfy

$$\|R^1_{S_2,B_1}\|,\ \|R^2_{B_2,S_1}\| \leq \frac{\|q\|}{\epsilon}; \quad \|R^1_{S_\nu,B_2}\|,\ \|R^2_{B_\nu,S_2}\| \leq \frac{2\|q\|}{|u|}, \quad \nu = 1, 2; \tag{4.2.11}$$

$$\|R^1_{S_1,B_1}\|,\ \|R^2_{B_1,S_1}\| \leq \frac{\|q'|}{\epsilon|u|}. \tag{4.2.12}$$

(b) Let the operator $R_{G'G'} = \begin{pmatrix} \pi_{G'_1} & R^2_{G'_1G'_2} \\ R^1_{G'_2G'_1} & \pi_{G'_2} \end{pmatrix}$ satisfy $c \in G'_\nu \Rightarrow |N_\nu| \geq \epsilon$, then R^1R^2 and R^2R^1 have bounded l^1-norm:

$$\|R^1R^2\|, \quad \|R^2R^1\| \leq \max \left\{ \begin{array}{l} \frac{2\|q\|^2}{\epsilon|u|} + \frac{8\|q\|^2}{|u|^2} + \frac{2\|q'\|\|q\|}{\epsilon|u|^2}, \\ \frac{4\|q\|^2}{\epsilon|u|} + \frac{\|q'\|\|q\|}{\epsilon^2|u|} + \frac{\|q'\|^2}{\epsilon^2|u|^2} \end{array} \right\}. \tag{4.2.13}$$

(c) R^1R^2 and R^2R^1 are Hilbert-Schmidt operators with finite Hilbert-Schmidt norms

$$\|R^1R^2\|_{HS},\ \|R^2R^1\|_{HS} \leq const \frac{\sqrt{\ln|u|}\|q\|}{|u|}(\|q\| + \frac{\|q'\|}{\epsilon}).$$

(d) Let

$$|u| \geq \max\{\frac{6\|q\|^2}{\epsilon} + \frac{\|q\|\|cq_c\|}{\epsilon^2}, \quad \frac{4\|q\|^2}{\epsilon} + \frac{\|q\|\|q'\|}{\epsilon^2} + \frac{\|q'\|^2}{2\epsilon^3}\}.$$

The operator $R_{G'G'}$ has a bounded inverse. The norm obeys

$$\| \begin{pmatrix} \pi_{G'_1} & R^2 \\ R^1 & \pi_{G'_2} \end{pmatrix}^{-1} - \begin{pmatrix} \pi_{G'_1} & -R^2 \\ -R^1 & \pi_{G'_2} \end{pmatrix} \| \leq const \frac{1 + 2\|q\|/\epsilon}{|u|}. \tag{4.2.14}$$

(e) Under the assumption of (d), the determinant of the operator $R_{G'G'}$ satisfies

$$|\ det_2 R_{G'G'} - 1\ | \leq const \frac{\sqrt{\ln|u|}\|q\|}{|u|}(\|q\| + \frac{\|q'\|}{\epsilon}).$$

Proof. (a) For line (4.2.11) it suffices to observe that

$$c \in B_1 \Rightarrow \frac{1}{|N_1(c)|} \leq \frac{1}{\epsilon}, \quad c \in B_2 \Rightarrow \frac{1}{|N_1(c)|} \leq \frac{2}{|u|},$$

$$c \in S_1 \Rightarrow \frac{1}{|N_2(c)|} \leq \frac{1}{\epsilon}, \quad c \in S_2 \Rightarrow \frac{1}{|N_2(c)|} \leq \frac{2}{|u|},$$

and that the convolution operator $\hat{q}(d-c)$ (or $-\bar{\hat{q}}(c-d)$) has operator norm bounded by $\|q\|$.

For line (4.2.12), by part (c) in Lemma 4.2.3 for example

$$\|R^1_{S_1,B_1}\| \leq \sup_{c\in B_1} \sum_{d\in S_1} \frac{|d-c||\hat{q}(d-c)|}{|u||N_1(c)|} \leq \frac{\|q'\|}{\epsilon|u|}. \tag{4.2.15}$$

(b) We choose

$$B_1 = \{c \in G_1' \mid \epsilon \leq |N_1(c)| < \frac{|u|}{2}\}, \quad B_2 =\subset \{c \in G_1' \mid |N_1(c)| \geq \frac{|u|}{2}\},$$

$$S_1 = \{c \in G_2' \mid \epsilon \leq |N_2(c)| < \frac{|u|}{2}\}, \quad S_2 = \{c \in G_2' \mid |N_2(c)| \geq \frac{|u|}{2}\}.$$

Use

$$\|(R^1R^2)_{S_iS_j}\| \leq \sum_{\nu=1}^{2} \|(R^1)_{S_iB_\nu}\|\|(R^2)_{B_\nu S_j}\|,$$

for example

$$\|(R^1R^2)_{S_1S_1}\| \leq (\frac{\|q'\|}{\epsilon|u|})^2 + \frac{2\|q\|}{|u|}\frac{\|q\|}{\epsilon}.$$

Then we can get the bound (4.2.13) from

$$\|R^1R^2\| \leq \max_{1\leq j\leq 2}\{\sum_{i=1}^{2} \|(R^1R^2)_{S_iS_j}\|\},$$

where

$$\sum_{i=1}^{2} \|(R^1R^2)_{S_iS_1}\| \leq \frac{4\|q\|^2}{\epsilon|u|} + \frac{\|q'\|\|q\|}{\epsilon^2|u|} + \frac{\|q'\|^2}{\epsilon^2|u|^2}$$

and

$$\sum_{i=1}^{2} \|(R^1R^2)_{S_iS_2}\| \leq \frac{2\|q\|^2}{\epsilon|u|} + \frac{8\|q\|^2}{|u|^2} + \frac{2\|q'\|\|q\|}{\epsilon|u|^2}.$$

We proceed similarly for R^2R^1.

(c) Writing the entries of the matrix R^1R^2,

$$(R^1R^2)_{d,c} = \sum_{f\in G_1'} \frac{-\hat{q}(d-f)\bar{\hat{q}}(c-f)}{N_1(f)N_2(c)} = \frac{-1}{N(c)} \sum_{f\in G_1'} \frac{N_1(c)}{N_1(f)}\hat{q}(d-f)\bar{\hat{q}}(c-f),$$

and noticing that

$$f \in G_\nu' \Rightarrow |N_\nu(f)| \geq \epsilon$$

we have

$$\begin{aligned}
\|R^1R^2\|_{HS}^2 &= \sum_{d,c\in G_2'} \frac{1}{|N(c)|^2} |\sum_{f\in G_1'} \hat{q}(d-f)\bar{\hat{q}}(c-f)\frac{N_1(c)}{N_1(f)}|^2 \\
&= \sum_{c\in G_2'} \frac{1}{|N(c)|^2} \{ \sum_{f,g\in G_1'} [\bar{\hat{q}}(c-f)\frac{N_1(c)}{N_1(f)}\hat{q}(c-g)\overline{\frac{N_1(c)}{N_1(g)}}(\sum_{d\in G_2'} \hat{q}(d-f)\bar{\hat{q}}(d-g))]\} \\
&\le \|q\|^2[\|q\| + \frac{\|q'\|}{\epsilon}]^2 \sum_{c\in G_2'} \frac{1}{|N(c)|^2} \\
&\le \|q\|^2[\|q\| + \frac{\|q'\|}{\epsilon}]^2 \frac{\text{const} \ln |u|}{|u|^2}
\end{aligned}$$

Here we estimated the sum $\sum_{d\in G_2'} \hat{q}(d-f)\bar{\hat{q}}(d-g)$ by $\|q\|^2$, then wrote $\frac{N_1(c)}{N_1(f)}$ and $\frac{N_1(c)}{N_1(g)}$ in the form of $1 + \frac{c-f}{N_1(g)}$ and $1 + \frac{c-g}{N_1(g)}$, which are bounded by $\frac{\|q'\|}{\epsilon}$, and lastly we used (a) in Lemma 4.2.3 to get the inequality.

(d) We calculate the product of the two operators

$$R_{G'G'} \begin{pmatrix} \pi_{G_1'} & -R^2 \\ -R^1 & \pi_{G_2'} \end{pmatrix} = \begin{pmatrix} \pi_{G_1'} - R^2R^1 & \\ 0 & \pi_{G_2'} - R^1R^2 \end{pmatrix} = 1 - E,$$

where

$$E = \begin{pmatrix} R^2R^1 & 0 \\ 0 & R^1R^2 \end{pmatrix}$$

is bounded and $(1-E)^{-1}$ could be expanded as a geometric series in E, which uniformly converges since $\|E\| = \max\{\|R^2R^1\|, \|R^1R^2\|\} < 1$ under the bound of $|u|$ in the assumption:

$$\begin{aligned}
R^{-1} &= \begin{pmatrix} \pi_{G_1'} & -R^2 \\ -R^1 & \pi_{G_2'} \end{pmatrix} (1-E)^{-1} \\
\Longrightarrow R^{-1} - \begin{pmatrix} \pi_{G_1'} & -R^2 \\ -R^1 & \pi_{G_2'} \end{pmatrix} &= \begin{pmatrix} \pi_{G_1'} & -R^2 \\ -R^1 & \pi_{G_2'} \end{pmatrix} \sum_{k=1}^{\infty} E^k \\
\Longrightarrow \|R^{-1} - \begin{pmatrix} \pi_{G_1'} & -R^2 \\ -R^1 & \pi_{G_2'} \end{pmatrix}\| &\le (1 + \max\{\|R^1\|, \|R^2\|\})\|E\|.
\end{aligned}$$

By the estimate in (a) and (b), there exists a constant, depending only on ϵ and the norms $\|q\|$, $\|q'\|$, such that the inequality (4.2.14) holds.

(e) We write the matrix

$$\begin{pmatrix} \pi_{G_1'} & R^2 \\ R^1 & \pi_{G_2'} \end{pmatrix} = \begin{pmatrix} \pi_{G_1'} & R^2 \\ 0 & \pi_{G_2'} \end{pmatrix} \begin{pmatrix} \pi_{G_1'} - R^2R^1 & 0 \\ R^1 & \pi_{G_2'} \end{pmatrix}. \tag{4.2.16}$$

First consider the case that G_1' and G_2' are finite sets. Then

$$\det{}_2 \begin{pmatrix} \pi_{G_1'} & R^2 \\ R^1 & \pi_{G_2'} \end{pmatrix} = \det \begin{pmatrix} \pi_{G_1'} & R^2 \\ R^1 & \pi_{G_2'} \end{pmatrix}$$

since π_{G_ν} both agree exactly with 1 on the diagonal:

$$\begin{aligned}\det{}_2\begin{pmatrix}\pi_{G_1'} & R^2\\ R^1 & \pi_{G_2'}\end{pmatrix} &= \det(1-R^2R^1) = \det{}_2(1-R^2R^1)e^{\mathrm{Tr}(-R^2R^1)}\\ &= 1+O(\|R^2R^1\|_{HS}+\mathrm{Tr}\,R^2R^1)\\ &= 1+O(\|R^2R^1\|_{HS})\\ &= 1+O(\frac{\sqrt{\ln|u|}\|q\|}{|u|}(\|q\|+\frac{\|q'\|}{\epsilon})).\end{aligned} \tag{4.2.17}$$

The bound holds by taking limits when G_1' and G_2' are infinite. □

From now on we choose

$$\rho = \max\{2\Lambda, \frac{6\|q\|^2}{\epsilon}+\frac{\|q\|\|q'\|}{\epsilon^2},\quad \frac{4\|q\|^2}{\epsilon}+\frac{\|q\|\|q'\|}{\epsilon^2}+\frac{\|q'\|^2}{2\epsilon^3}\}$$

and the closed subset $\mathcal{K}_\rho = \{(a,b)\in\mathbb{C}^2 \mid |u|\le\rho\}$. Then we can apply the last two lemmas to analyse the asymptotic behavior of the spectrum. Outside the compact part $\mathcal{K}_\rho/\Gamma'$, the inverse of R exists and so dose the matrix $(\mathcal{D}_{ij})$ by (4.2.7) and (4.2.9). The following proposition gives the concrete representations of the matrix $(\mathcal{D}_{ij})$ whose entries involve the finite determinant equation (4.2.10) in the two simplest cases.

Proposition 4.2.5. *Let $(a,b)\in\mathbb{C}^2\backslash\mathcal{K}_\rho$.*
(a) If $(a,b)\in T_\nu(0)\backslash\cup_{c\in\Gamma'}T_{3-\nu}(c)$, then $(a,b)\in\tilde{S}_q$ if and only if

$$N_{0,\nu}(a,b) = \mathcal{A}_\nu(a,b). \tag{4.2.18}$$

Here

$$\mathcal{A}_1 = -\sum_{f,g\in\Gamma'}\frac{1}{N_2(f)}\bar{\hat{q}}(f)(A_1)_{fg}\hat{q}(g),\ \mathcal{A}_2 = -\sum_{f,g\in\Gamma'}\frac{1}{N_1(f)}\hat{q}(-f)(A_2)_{fg}\bar{\hat{q}}(-g). \tag{4.2.19}$$

A_ν is the $R_{(3-\nu)(3-\nu)}$ part of R^{-1} in case $G_\nu=\{0\}$ and $G_{3-\nu}=\emptyset$, and obeys

$$|A_\nu|\le\frac{const}{|u|},$$

where const depends only on ϵ and the norms $\|q\|,\|q'\|$.
(b) If $(a,b)\in T_1(d_1)\cap T_2(d_2)$, then $(a,b)\in\tilde{S}_q$ if and only if

$$(N_1(d_1)-\mathcal{D}_{1,1})(N_2(d_2)-\mathcal{D}_{2,2}) = (q_{(d_2-d_1)}-\mathcal{D}_{2,1})(-\bar{q}_{(d_2-d_1)}-\mathcal{D}_{1,2}), \tag{4.2.20}$$

where

$$\mathcal{D}_{11} = -\sum_{f,g\in G_2'}\frac{1}{a-f}\bar{\hat{q}}(f-d_1)(R_{22})_{fg}\hat{q}(g-d_1),$$

$$\mathcal{D}_{12} = \sum_{f\in G_2',g\in G_1'}\frac{1}{a-f}\bar{\hat{q}}(f-d_1)(R_{21})_{fg}\bar{\hat{q}}(d_2-g),$$

$$\mathcal{D}_{21} = \sum_{f\in G_1',g\in G_2'} \frac{1}{b+f}\hat{q}(d_2-f)(R_{12})_{fg}\hat{q}(g-d_1),$$

$$\mathcal{D}_{22} = -\sum_{f\in G_1',g\in G_1'} \frac{1}{b+f}\hat{q}(d_2-f)(R_{11})_{fg}\bar{\hat{q}}(d_2-g).$$

And $\mathcal{D}_{i,j}$ obeys

$$|\mathcal{D}_{i,j}| \le \frac{const}{|u|},$$

where again const depends only on ϵ and the norms $\|q\|, \|q'\|$.

Proof. (a) We discuss only the case $\nu = 1$. The another case is the same. For the region in question $\tilde{\mathcal{S}}(q)$ is given by the determinant equation (4.2.10) with $G_1 = \{0\}$ and $G_2 = \emptyset$. Then $N_{0,1}(a,b) = \mathcal{A}_\nu(a,b)$ since $\pi_G Q = 0$. Lemma 4.2.3(c) and Lemma 4.2.4(d) give the desired bound for the contribution from $R_{\nu\nu}^{-1} - \pi_{G_\nu'} = \sum_{k\ge1}(R^2R^1)^k$:

$$|\mathcal{A}_1 - \sum_{f\in\Gamma'} \frac{1}{N_2(f)}\bar{\hat{q}}(f)\hat{q}(f)| \le (\frac{2\|q\|^2}{|u|} + \frac{\|q\|\|q'\|}{\epsilon|u|})\|R^2R^1\| \le \frac{const}{|u|^2}$$

and

$$|\mathcal{A}_1| \le |\sum_{f\in\Gamma'} \frac{1}{N_2(f)}\bar{\hat{q}}(f)\hat{q}(f)| + \frac{const}{|u|^2} \le \frac{const}{|u|}.$$

(b) We repeat the construction of $\mathcal{D}_{ij}$ using (4.2.9) when

$$G_1 = \{d_1\}, \quad G_2 = \{d_2\}.$$

The determinant equation (4.2.10) corresponding to a 2×2 matrix is exact given by (4.2.20). $\|R^{-1}\|$ is bounded by $1 + 2\frac{\|q\|}{\epsilon}$, and remaining parts are bounded by $\frac{\|q\|}{|u|}(\frac{\|q'\|}{\epsilon} + 2\|q\|)$ by Lemma 4.2.3 (d). □

In the next lemma we consider the derivatives of $\mathcal{A}$ and $\mathcal{D}$ and estimate their bounds to show that the spectral curve does not wiggle too much. The main result is

Lemma 4.2.6. *Under the hypotheses of Proposition 4.2.5, if $m+n \ge 1$, then*

$$|\frac{\partial^{m+n}\mathcal{A}_\nu(a,b)}{\partial a^m \partial b^n}| \le \frac{const}{|u|^2} \quad \text{for} \quad \nu = 1,2$$

and

$$|\frac{\partial^{m+n}\mathcal{D}(a,b)_{ij}}{\partial a^m \partial b^n}| \le \frac{const}{|u|^2} \quad \text{for} \quad i,j = 1,2,$$

where the constants depend only on ϵ and the norms $\|q\|, \|q'\|$.

Proof. Let Q^1_{DC}, Q^2_{DC} denote the matrix $[q(d-c)]_{d\in D, c\in C}$ and $[-\bar{q}(c-d)]_{d\in D, c\in C}$ respectively. We define

$$Q_{GG'} := \begin{pmatrix} 0 & Q^2_{G_1 G'_2} \\ Q^1_{G_2 G'_1} & 0 \end{pmatrix}$$

for $G = (G_1, G_2)$ and $G' = (G'_1, G'_2)$. Then $\mathcal{A}_\nu$ and $\mathcal{D}$ are given by

$$Q_{GG'} \bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} Q_{G'G}$$

with $G_\nu = \{0\}$, $G_{3-\nu} = \emptyset$ and $G_1 = \{d_1\}$, $G_2 = \{d_2\}$ respectively. Using $R_{G'G'} = 1 + Q_{G'G'} \bar{\partial}^{-1}_{a,b}$, we get

$$\begin{aligned} \frac{\partial \mathcal{A}_\nu}{\partial a}, \frac{\partial \mathcal{D}}{\partial a} &= -Q_{GG'} \bar{\partial}^{-1}_{a,b} \frac{\partial}{\partial a}(\bar{\partial}_{a,b}) \bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} Q_{G'G} \\ &\quad + Q_{GG'} \bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} Q_{G'G'} \bar{\partial}^{-1}_{a,b} \frac{\partial}{\partial a}(\bar{\partial}_{a,b}) \bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} Q_{G'G} \\ &= -Q_{GG'} \bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} \frac{\partial}{\partial a}(\bar{\partial}_{a,b}) \bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} Q_{G'G}. \end{aligned}$$

Differentials of the matrix form are

$$\frac{\partial}{\partial a}(\bar{\partial}_{a,b}) \mid_{G'} = \begin{pmatrix} 0 & 0 \\ 0 & \pi_{G'_2} \end{pmatrix}, \quad \frac{\partial}{\partial b}(\bar{\partial}_{a,b}) \mid_{G'} = \begin{pmatrix} \pi_{G'_1} & 0 \\ 0 & 0 \end{pmatrix}$$

and then the higher derivatives of $Q_{G'G'} + \bar{\partial}_{a,b}$ are zero.
Since $(\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1})(Q_{G'G'} + \bar{\partial}_{a,b}) = 1$ taking the m-order derivative ($m \geq 1$) on both sides

$$\frac{\partial^m}{\partial a^m}(\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1}) + m \frac{\partial^{m-1}}{\partial a^{m-1}}(\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1}) \frac{\partial}{\partial a}(\bar{\partial}_{a,b}) \mid_{G'} = 0,$$

or equivalently,

$$\frac{\partial^m}{\partial a^m}(\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1}) = -m \frac{\partial^{m-1}}{\partial a^{m-1}}(\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1}) \frac{\partial}{\partial a}(\bar{\partial}_{a,b}) \mid_{G'} .$$

Through recursive induction for $m \geq 1$, we get the result

$$\frac{\partial^m}{\partial a^m}[\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1}] = (-1)^m (m!) [\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} \frac{\partial}{\partial a}(\bar{\partial}_{a,b})]^m [\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1}].$$

For the other variable b, we similarly have

$$\begin{aligned} &\frac{\partial^n}{\partial b^n}[\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} \frac{\partial}{\partial a}(\bar{\partial}_{a,b})] \\ &= (-1)^n (n!) [\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} \frac{\partial}{\partial b}(\bar{\partial}_{a,b})]^n [\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} \frac{\partial}{\partial a}(\bar{\partial}_{a,b})]. \end{aligned}$$

All higher derivatives are given by finite linear combinations of terms of the form

$$-Q_{GG'} \prod_j [\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} \frac{\partial}{\partial a}(\bar{\partial}_{a,b})]^{m_j} [\bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} \frac{\partial}{\partial b}(\bar{\partial}_{a,b})]^{n_j} \bar{\partial}^{-1}_{a,b} (R_{G'G'})^{-1} Q_{G'G},$$

where $\sum m_j = m$ and $\sum n_j = n$.
From Lemma 4.2.3(b) we get

$$
\begin{aligned}
\|(Q\bar{\partial}_{a,b}^{-1})_{G_{(3-\nu)}G'_\nu}\| &\leq \sup_{c\in G'_\nu} \sum_{d\in G_{3-\nu}} \frac{|\hat{q}((-1)^\nu(c-d))|}{|N_\nu(c)|} \\
&\leq \max\{ \sup_{c\in G'_\nu, |N_\nu(c)|\geq \frac{|u|}{2}} \sum_{d\in G_{3-\nu}} \frac{|\hat{q}((-1)^\nu(c-d))|}{|N_\nu(c)|}, \\
&\qquad \sup_{c\in G'_\nu, \epsilon\leq |N_\nu(c)|<\frac{|u|}{2}} \sum_{d\in G_{3-\nu}} \frac{|\hat{q}((-1)^\nu(c-d))|}{|N_\nu(c)|}\} \\
&\leq \max\{ \sup_{c\in G'_\nu, |N_\nu(c)|\geq \frac{|u|}{2}} \frac{2\|q\|}{|u|}, \sup_{c\in G'_\nu, \epsilon\leq |N_\nu(c)|\leq\frac{|u|}{2}} \frac{\|q'\|}{\epsilon|u|}\} \\
&\leq \max\{\frac{2\|q\|}{|u|}, \frac{\|q'\|}{\epsilon|u|}\}
\end{aligned}
$$

and

$$
\begin{aligned}
\|(Q\bar{\partial}_{a,b}^{-1})_{GG'}\| &\leq \max\{\|(Q\bar{\partial}_{a,b}^{-1})_{G_1\times G'_2}\|, \quad \|(Q\bar{\partial}_{a,b}^{-1})_{G_2\times G'_1}\|\} \\
&\leq \max\{\frac{2\|q\|}{|u|}, \frac{\|q'\|}{\epsilon|u|}\}.
\end{aligned}
$$

Similarly we get the same bounds for $\|(\bar{\partial}_{a,b}^{-1}Q)_{G'G}\|$, which is written in matrix form as

$$
(\bar{\partial}_{a,b}^{-1}Q)_{G'G} = \begin{bmatrix} 0 & \frac{-\hat{\bar{q}}(c_2-d_1)}{b+d_1} \\ \frac{\hat{q}(d_2-c_1)}{a-\bar{d}_2} & 0 \end{bmatrix} \begin{matrix} d_1\in G'_1, d_2\in G'_2 \\ c_1\in G'_1, c_2\in G_2 \end{matrix}.
$$

Since

$$
\|(R_{G'G'})^{-1}\| \leq (1+\frac{2\|q\|}{\epsilon}),\ \|\frac{\partial}{\partial a}(\bar{\partial}_{a,b})\| = \|\frac{\partial}{\partial a}(\bar{\partial}_{a,b})\| = 1,\ \|(\bar{\partial}_{a,b})_{G'G'}^{-1}\| \leq \frac{1}{\epsilon},
$$

we get

$$
\begin{aligned}
&|\frac{\partial^{m+n}\mathcal{A}_\nu}{\partial a^m \partial b^n}|, |\frac{\partial^{m+n}\mathcal{D}}{\partial a^m \partial b^n}| \\
&\leq \text{const}\|(Q\bar{\partial}_{a,b}^{-1})_{GG'}\|\|(\bar{\partial}_{a,b})_{G'G'}^{-1}\|^{m+n-1}\|R_{G'G'}^{-1}\|^{m+n+1}\|(\bar{\partial}_{a,b}^{-1}Q)_{G'G}\| \\
&\leq \text{const}[\max\{\frac{2\|q\|}{|u|}, \frac{\|q'\|}{\epsilon|u|}\}]^2(1+\frac{2\|q\|}{\epsilon})^{m+n+1}(\frac{1}{\epsilon})^{m+n-1} \\
&\leq \frac{\text{const}}{|u|^2}.
\end{aligned}
$$

□

4.2.3 Proof of the main result

We are now going to prove the main theorems by using the estimates in the above lemmas. We first prove Theorem 4.2.1.

Proof. (a) Let $(a,b) \in \mathbb{C}^2 \backslash (\mathcal{K}_\rho \cup \bigcup_{c\in\Gamma'} T_c)$. This is exactly the case $G_\nu = \emptyset$ for $\nu = 1,2$. Assume that $(a,b) \in \tilde{S}_q$. Lemma 4.2.4 (d) with the condition $G_1' = G_2' = \Gamma'$ implies that $1 - Q\bar{\partial}_{a,b}$ is invertible. Hence the solution $\phi_{G_\nu'}$ to (4.2.4) must be zero. Hence the operator $\bar{\partial}_{a,b} + Q$ has trivial kernel for any $(a,b) \in \mathbb{C}^2 \backslash (\mathcal{K}_\rho \cup \bigcup_{c\in\Gamma'} T_c)$. Then we have

$$\tilde{\mathcal{S}}(q) \subset (\mathcal{K}_\rho \cup \bigcup_{c\in\Gamma'} T_c).$$

(b) We just need to prove the case $\nu = 1$. We write $(a,b) \in \mathbb{C}^2$ as

$$a = k_1 - k_2 \quad \text{and} \quad b = k_1 + k_2.$$

Obviously the map $\psi : (a,b) \mapsto (k_1,k_2)$ is a biholomorphic isomorphism. Let pr denote the projection $(k_1,k_2) \mapsto k_1$, then the map $(a,b) \mapsto \frac{a+b}{2}$ is exactly $\mathrm{pr}\circ\psi$. Since

$$(a,b) \in T_1(0) \Leftrightarrow |N_{1,0}| = |b| = |k_1 + k_2| = |u + v| < \epsilon,$$

we have that in $T_1(0)$

$$|\,|u| - |k_1|\,| \leq |\bar{u} - k_1| = |\frac{\bar{b} - b}{2}| < \epsilon.$$

If $(a,b) \in T_1(0) \cap T_2(d)$ then

$$|k_1 + k_2| < \epsilon \quad \text{and} \quad |k_1 - \frac{\bar{d}}{2}| < \epsilon.$$

Conversely, if

$$|k_1 + k_2| < \frac{\epsilon}{2} \quad \text{and} \quad |k_1 - \frac{\bar{d}}{2}| < \frac{\epsilon}{4}$$

then $(a,b) = \psi^{-1}(k_1,k_2) \in T_1(0) \cap T_2(d)$.
For $z \in \mathbb{C}$ let F_z be the subset of $\mathbb{C}^2$

$$F_z = \mathrm{pr}^{-1}(z) \cap \psi(T_1(0) \backslash \bigcup_{c\in\Gamma'} T_2(c)).$$

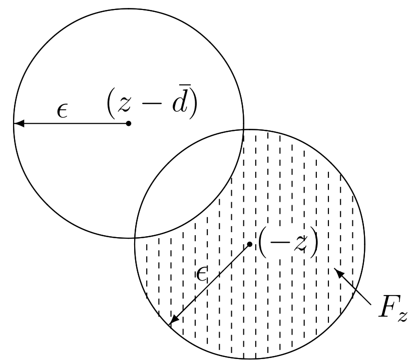

Figure 4.3: F_z

If $|z| > 2\epsilon$ and $|z - \frac{\bar{c}}{2}| \geq \epsilon$ for all $c \in \Gamma'$ then (see Figure 4.3)

$$F_z = \{(z, k_2) \in \mathbb{C}^2 \mid |k_2 + z| < \epsilon\},$$

whereas if there exists some $d \in \Gamma'$ such that $|z - \frac{\bar{d}}{2}| < \epsilon$, then

$$F_z = \{(z, k_2) \in \mathbb{C}^2 \mid |k_2 + z| < \epsilon, \text{ and } |k_2 - (z - \bar{d})| \geq \epsilon\}.$$

$\psi^{-1}(z, k_2)$ lies in $\tilde{\mathcal{S}}(q)$ by Proposition 4.2.5 if and only if

$$z + k_2 = \mathcal{A}_1(z, k_2) \tag{4.2.21}$$

for an analytic function $\mathcal{A}_1$. By Lemma 4.2.6 we have

$$|\mathcal{A}_1(z, k_2)| \leq \frac{\text{const}}{|u|} \text{ and } |\frac{\partial}{\partial k_2}\mathcal{A}_1(z, k_2)| = |\frac{\partial}{\partial a}\mathcal{A}_1 - \frac{\partial}{\partial b}\mathcal{A}_1| \leq \frac{\text{const}}{|u|^2}.$$

Since $|u| > |z| - \epsilon$, the estimate above shows that for z big enough, the equation has at most one solution in F_z, and the solution is isolated. And any such solution satisfies

$$|z + k_2| \leq \frac{const}{|u|}.$$

For this reason, there is no solution in F_z, if for some $d \in \Gamma'$

$$\{k_2 \in \mathbb{C} \mid |z + k_2| \leq \frac{\text{const}}{|u|}\} \subset \{k_2 \in \mathbb{C} \mid |k_2 - (z - \bar{d})| \leq \epsilon\},$$

that is, if

$$\frac{\text{const}}{|u|} + |2z - \bar{d}| \leq \epsilon \text{ or equivalently } |z - \frac{\bar{d}}{2}| \leq \frac{\epsilon}{2} - \frac{\text{const}}{2|u|}.$$

Let $z_1(c) = \frac{\bar{c}}{2}$. Thus

$$(\bigcup_{c\in\Gamma'}\{z \in \mathbb{C} \mid |z - z_1(c)| \leq \frac{\epsilon}{4}\}) \cap \text{pr} \circ \psi(\tilde{\mathcal{S}}(q) \cap T_1(0) \backslash (\mathcal{K}_\rho \cup \bigcup_{c\in\Gamma'} T_2(c)) = \emptyset,$$

or equivalently

$$\text{pr} \circ \psi(\tilde{\mathcal{S}}_q \cap T_1(0) \backslash (\mathcal{K}_\rho \cup \bigcup_{c\in\Gamma'} T_2(c)) \subset \{z \in \mathbb{C} \mid |z - z_1(c)| > \frac{\epsilon}{4} \text{ for all } c \in \Gamma'\}.$$

Similarly equation (4.2.21) has a solution in F_z if for all $d \in \Gamma'$

$$\{k_2 \in \mathbb{C} \mid |z + k_2| \leq \frac{\text{const}}{|u|}\} \cap \{k_2 \in \mathbb{C} \mid |k_2 - (z - \bar{d})| \leq \epsilon\} = \emptyset,$$

that is for all $d \in \Gamma'$,

$$|z - \frac{\bar{d}}{2}| > \frac{\epsilon}{2} + \frac{\text{const}}{2|u|}.$$

Then we get

$$\{z \in \mathbb{C} \mid |z| > 2\rho \text{ and } |z - \frac{\bar{c}}{2}| > \epsilon \text{ for all } c \in \Gamma'\} \subset \text{pr} \circ \psi(\tilde{S}(q) \cap T_1(0) \backslash (\mathcal{K}_\rho \cup \bigcup_{c\in\Gamma'} T_2(c)).$$

□

Now we prove Theorem 4.2.2 which describes the asymptotics of the handles of the spectral curve.

Proof. First we construct the map $\phi_{c,1}$. We change the coordinate as below

$$x_1 = N_2(c) - \mathcal{D}_{22} = a - \bar{c} - \mathcal{D}_{22}, \quad x_2 = N_1(0) - \mathcal{D}_{11} = b - \mathcal{D}_{11},$$

or conversely

$$a = x_1 + \bar{c} + \mathcal{D}_{22}, \quad b = x_2 + \mathcal{D}_{11}. \tag{4.2.22}$$

where $\mathcal{D}_{ij}$ are given by Proposition 4.2.9 in the case $d_1 = 0$ and $d_2 = c$. From Lemma 4.2.6 we know the Jacobian of this map satisfies

$$\begin{pmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} \end{pmatrix} = \begin{pmatrix} 1 + O(\frac{1}{|c|^2}) & O(\frac{1}{|c|^2}) \\ O(\frac{1}{|c|^2}) & 1 + O(\frac{1}{|c|^2}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (1 + O(\frac{1}{|c|^2}))$$

since $|u - \frac{c}{2}| < \epsilon$ for $(a, b) \in T_1(0) \cap T_2(c)$. Its inverse is

$$\begin{pmatrix} \frac{\partial a}{\partial x_1} & \frac{\partial a}{\partial x_2} \\ \frac{\partial b}{\partial x_1} & \frac{\partial b}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (1 + O(\frac{1}{|c|^2})),$$

which means for $i = 1, 2$

$$\frac{\partial D_{22}}{\partial x_i} = O(\frac{1}{|c|^2}), \quad \frac{\partial D_{11}}{\partial x_i} = O(\frac{1}{|c|^2}).$$

We calculate the 2-order derivatives of $\mathcal{D}$:

$$\begin{aligned} \frac{\partial^2 D_{22}}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i}[\frac{\partial D_{22}}{\partial a}(\delta_{1j} + \frac{\partial D_{22}}{\partial x_j}) + \frac{\partial D_{22}}{\partial b}(\delta_{2j} + \frac{\partial D_{11}}{\partial x_j})] \\ &= \cdots \\ &= O(\frac{1}{|c|^2}) + O(\frac{1}{|c|^2})\frac{\partial^2 D_{22}}{\partial x_i \partial x_j} + O(\frac{1}{|c|^2})\frac{\partial^2 D_{11}}{\partial x_i \partial x_j}. \end{aligned} \tag{4.2.23}$$

Similarly,

$$\frac{\partial^2 D_{11}}{\partial x_i \partial x_j} = O(\frac{1}{|c|^2}) + O(\frac{1}{|c|^2})\frac{\partial^2 D_{11}}{\partial x_i \partial x_j} + O(\frac{1}{|c|^2})\frac{\partial^2 D_{22}}{\partial x_i \partial x_j}. \tag{4.2.24}$$

We conclude from (4.2.23) and (4.2.24)

$$\frac{\partial^2 a}{\partial x_i \partial x_j} = \frac{\partial^2 D_{22}}{\partial x_i \partial x_j} = O(\frac{1}{|c|^2}), \quad \frac{\partial^2 b}{\partial x_i \partial x_j} = \frac{\partial^2 D_{11}}{\partial x_i \partial x_j} = O(\frac{1}{|c|^2}).$$

If $r \neq s$ then

$$\frac{\partial D_{rs}}{\partial x_i} = \frac{\partial D_{rs}}{\partial a}(\delta_{1i} + \frac{\partial D_{22}}{\partial x_i}) + \frac{\partial D_{rs}}{\partial b}(\delta_{2i} + \frac{\partial D_{11}}{\partial x_i}) = O(\frac{1}{|c|^2}),$$

$$\frac{\partial^2 D_{rs}}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}[\frac{\partial D_{rs}}{\partial a}(\delta_{1j} + \frac{\partial D_{22}}{\partial x_j}) + \frac{\partial D_{rs}}{\partial b}(\delta_{2j} + \frac{\partial D_{11}}{\partial x_j})] = O(\frac{1}{|c|^2}).$$

Let

$$(N_1(0) - \mathcal{D}_{1,1})(N_2(c) - \mathcal{D}_{2,2}) - (\hat{q}(c) - \mathcal{D}_{2,1})(-\bar{\hat{q}}(c) - \mathcal{D}_{1,2}) = x_1 x_2 + h(x_1, x_2),$$

where

$$h(x_1, x_2) = -(\hat{q}(c) - \mathcal{D}_{2,1})(-\bar{\hat{q}}(c) - \mathcal{D}_{1,2}).$$

Since

$$\sum_{c \in \Gamma'} |c\hat{q}(c)| < \infty \Rightarrow \lim_{|c| \to \infty} |c||\hat{q}(c)| = 0,$$

we have $\hat{q}(c) = o(\frac{1}{|c|})$. We conclude from $\mathcal{D}_{r,s} = O(\frac{1}{|c|})$

$$|h(0,0)| \le \frac{const}{|c|^2}.$$

The first and second derivatives of h satisfy

$$|\frac{\partial h}{\partial x_i}(x_1, x_2)| = | - \frac{\partial \mathcal{D}_{21}}{\partial x_i}(\bar{\hat{q}}(c) + \mathcal{D}_{12}) + \frac{\partial \mathcal{D}_{12}}{\partial x_i}(\hat{q}(c) - \mathcal{D}_{21})| \le \frac{const}{|c|^2},$$

$$|\frac{\partial h}{\partial x_i \partial x_j}(x_1, x_2)| \le \frac{const}{|c|^2}.$$

By the quantitative Morse Lemma [11] in Appendix B, with $A = \frac{const}{|c|^2}$, $B = \frac{const}{|c|^2}$, there exists a biholomorphism Φ defined on

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \le \frac{\epsilon}{2},\ |z_2| \le \frac{\epsilon}{2}\}$$

with range containing

$$\{(x_1, x_2) \in \mathbb{C}^2 \mid |x_1| \le \frac{\epsilon}{4},\ |x_2| \le \frac{\epsilon}{4}\}.$$

Φ satisfies

$$\|D\Phi - 1\| \le \frac{\text{const}}{|c|^2}, \quad (x_1 x_2 + h) \circ \Phi = z_1 z_2 - t_c,$$

$$|t_c - h(0,0)| \le \frac{\text{const}}{|c|^4} \Rightarrow |t_c| \le \frac{\text{const}}{|c|^2}, \quad \|\Phi(0)\| \le \frac{\text{const}}{|c|^2}.$$

Now we get the composition

$$\phi_c(z_1, z_2) = (a(\Phi(z_1, z_2)), b(\Phi(z_1, z_2)))$$

where $a(x), b(x)$ are given by (4.2.22). The Jacobian

$$D\phi_c = \begin{pmatrix} \frac{\partial a}{\partial x_1} & \frac{\partial a}{\partial x_2} \\ \frac{\partial b}{\partial x_1} & \frac{\partial b}{\partial x_2} \end{pmatrix} \circ D\Phi = (1 + O(\frac{1}{|c|^2}))(1 + O(\frac{1}{|c|^2})) = 1 + O(\frac{1}{|c|^2})$$

and the center satisfies

$$\begin{aligned} |\phi_{c,1}(0) - (\bar{c}, 0)| &\le |\phi_c(0) - (a \circ \Phi(0), b \circ \Phi(0))| + |(a \circ \Phi(0), b \circ \Phi(0)) - (a(0), b(0))| \\ &\le \frac{const}{|c|^2}. \end{aligned}$$

Since $T_1(0) \cap T_2(c) \cap \tilde{\mathcal{S}}(q)$ is mapped to $T_1(c) \cap T_2(0) \cap \tilde{\mathcal{S}}(q)$ by Γ'-translation by $-c$, we define

$$\phi_{c,2}(z_1, z_2) = \phi_{c,1}(z_2, z_1) - c$$

and check that the theorem holds. □

4.2.4 The geometric picture

Theorem 4.2.7. *Let $q \in W^{1,1}(\mathbb{C}/\Gamma)$. Then $\mathcal{S}(q)$ is a one-dimensional complex analytic variety, which consists of at most two components. If Σ_q is smooth then it is irreducible.*

Proof. Let $\epsilon > 0$ be small and choose ρ such that Theorem 4.2.1 holds. By part (b) of this Theorem and part (iii) of Theorem 4.2.2 there are two components C_1, C_2 of $\tilde{\mathcal{S}}(q)$ such that

$$\tilde{\mathcal{S}}(q) \cap T(0) \backslash (\mathcal{K}_\rho \cup \bigcup_{c \in \Gamma', c \neq 0} T(c)) = ((C_1 \cap C_2) \cap T(0)) \backslash (\mathcal{K}_\rho \cup \bigcup_{c \in \Gamma', c \neq 0} T(c)).$$

Clearly $\exp(C_1)$ and $\exp(C_2)$ are components of $\mathcal{S}(q)$. Assume that $\mathcal{S}(q)$ has a component K not contained in $\exp(C_1) \cup \exp(C_2)$. Then every component C' of $\exp^{-1}(K)$ lies in

$$\mathcal{K}_\rho \cup \bigcup_{c,d \in \Gamma'} T(c) \cap T(d).$$

In particular the complement of $\mathrm{pr} \circ \varphi(C')$ contains an open subset of $\mathbb{C}$.

We point out without proof that $\mathcal{S}(q)$ is the zero-set of an entire function of finite order(the proof is almost same as Theorem 13.8 [11]). Therefore, the indicator of growth of C' is also of finite order, and hence by the solution of the "Cousin problem with finite order" ([15] 3.30) C' is also the zero set of an entire function of finite order. Therefore, by [15] 3.44 the set $\{z \in \mathbb{C} \mid (\mathrm{pr} \circ \varphi)^{-1}(z) \cap C' = \emptyset\}$ is either $\mathbb{C}$ itself or discrete. Since its complement contains an open set, it is in fact discrete. As C' is irreducible it follows that this set consists of one point z_0. So $\varphi(C') \subset z_0 \times \mathbb{C}$. If we now apply the same argument with the projection $(k_1, k_2) \to k_2$ we conclude that $\varphi(C')$ is a point, and hence so is C', which is impossible. Thus $\mathcal{S}(q) = \exp(C_1) \cup \exp(C_2)$ consists of at most two components.

If $\mathcal{S}(q)$ is smooth then the constants t_c of Theorem 4.2.2 are all different from zero. Therefore, C_1 and C_2 can be connected by an arc inside the set of smooth points of $\mathcal{S}(q)$. Thus $C_1 = C_2$ and $\mathcal{S}(q)$ is irreducible. □

For general q, we get the following detailed picture of these infinite genus spectral curve $\mathcal{S}(q)$ (see Figure 4.4). Take a pair of complex planes, the k_1-plane and k_2-plane. Cut out a compact simply connected neighborhood of the origin on each plane. Glue in each place a compact Riemann surface with boundary. This is the part of $\det(D_{a,b}) = 0$ with a, b too small for us to be able to do an accurate analysis. Also cut out of the k_1-plane a small disk around each $\frac{\bar{c}}{2}$ and respectively the k_2-plane a small disk around each $\frac{c}{2}$ with $c \in \Gamma'$ outside the simply connected neighborhood of the origin. Glue in a handle joining each matching pair of the disks. The handles are bi-holomorphic to the model handles

$$H(t_c) = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = t_c \text{ and } |z_1|, \ |z_2| \leq \epsilon\}$$

satisfying $|t_c| \leq \frac{const}{|c|^2}$, which control the size of the handles joining the two planes.

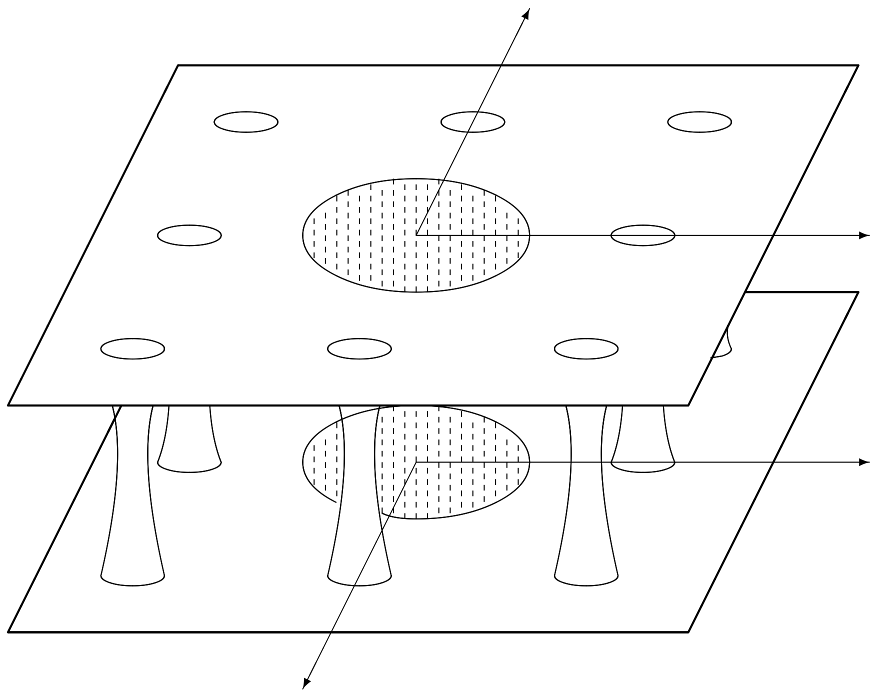

Figure 4.4: Spectral Curve of infinite genus

Remark 4.2.8. If $t_c = 0$ for some c then the handle degenerates to a double point. If $t_c = 0$ for all but finitely many c then the normalization of Σ_q has finite genus and q is a "finite gap" potential.

Appendix

A. Quaternonic Plücker formula with monodromy

Given a linear system of a holomorphic bundle W, that is, a linear subspace $H \subset H^0(W)$, we have the Kodaira map

$$\mathrm{ev}^* : W^* \to H^*. \tag{4.2.25}$$

Here ev denotes the bundle map which, for $p \in M$, evaluates holomorphic sections at p. In case the Kodaira map is injective, its image $W^* \subset H^*$ defines a map into the Grassmannian $f : M \to Gr(r, H^*)$, where $\operatorname{rank} W = r$. The derivative $\delta = \pi d|_{W^*}$ of this map satisfies $*\delta = \delta J$, where d denotes the trivial connection on H^* and $\pi : H^* \to H^*/W^*$ denotes the canonical projection. In other words, f is a holomorphic curve [10]. If M is a compact Riemann surface of genus g and L is a holomorphic line bundle over M, the Plücker formula [10] gives a lower bound for the Willmore energy $\mathcal{W}(L)$ of L in terms of the genus g of M, the degree of L, and vanishing orders of holomorphic sections of L. For the purposes of this paper, we need a more general version of the Plücker formula which includes linear systems with monodromy. If $\tilde{M}$ is the universal cover of M, we denote by $\tilde{L}$ the pullback to $\tilde{M}$ of L by the covering map. The holomorphic structure of L lifts to make $\tilde{L}$ into a holomorphic line bundle. A linear system with monodromy is a linear subspace $H \subset H^0(\tilde{L})$ of holomorphic sections of $\tilde{L}$ such that

$$\gamma^* H = H \quad \text{for all} \quad \gamma \in \pi_1(M), \tag{4.2.26}$$

where $\pi_1(M)$ acts via deck transformations. Adapting the proof in [10] by replacing the trivial connection with a flat connection on H, we obtain the Plücker formula for an n-dimensional linear system H with monodromy

$$\frac{1}{4\pi}\mathcal{W}(L) \geq n((n-1)(1-g) - \deg L) + \operatorname{ord} H. \tag{4.2.27}$$

The order $\operatorname{ord} H$ of the linear system $H \subset H^0(\tilde{L})$ with monodromy is computed as follows: to a point $x \in \tilde{M}$ we assign the Weierstrass gap sequence $n_0(x) < \cdots < n_{n-1}(x)$ inductively by letting $n_k(x)$ be the minimal vanishing order strictly greater than $n_{k-1}(x)$ of holomorphic sections in H. Away from isolated points this sequence is $n_k(x) = k$ and

$$\operatorname{ord}_x H = \sum_{k=0}^{n-1} (n_k - k)$$

measures the deviation from the generic sequence. Since H is a linear system with monodromy the Weierstrass gap sequence is invariant under deck transformations. Therefore, $\operatorname{ord}_p H$ for $p \in M$ is well-defined, zero away from finitely many points, and the order of the linear system H is given by

$$\operatorname{ord} H = \sum_{p \in M} \operatorname{ord}_p H. \tag{4.2.28}$$

For a linear system $H \subset H^0(L)$ without monodromy the formula (4.2.27) is the usual Plücker relation as discussed in [10].

B. Quantitative Morse lemma

Lemma 4.2.9. *Let $f(x_1, x_2) = x_1 x_2 + h(x_1, x_2)$ be a holomorphic function on*

$$D_\delta = \{(x_1, x_2) \in \mathbb{C}^2 \mid |x_1| \leq \delta,\ |x_2| \leq \delta\},$$

for some $\delta < 1$. Assume the function h fulfills the estimates

$$|\frac{\partial h}{\partial x_i}(x_1, x_2)| \leq A, \quad \|\frac{\partial^2 h}{\partial x_i \partial x_j}(x_1, x_2)\| \leq B$$

on D_δ with constants $A, B > 0$ such that $A < \delta$ and $B < \frac{1}{30}$. Then f has unique critical point $\xi = (\xi_1, \xi_2)$ in D_δ, and $|\xi_1| < A$, $|\xi_2| < A$.
Put $s = max\{|\xi_1|, |\xi_2|\}$. Then there is a biholomorphic map Φ from $D_{(\delta - s)(1-10B)}$ to a neighborhood of ξ in D_δ that contains

$$\{(x_1, x_2) \in \mathbb{C}^2 \mid |x_i - \xi_i| < (\delta - s)(1 - 30B)\}$$

such that

$$\|\Phi(x) - x\| \leq 5B(|x_1| + |x_2|) \leq 10B\delta$$

and

$$f \circ \Phi(z_1, z_2) = z_1 z_2 + c$$

with a constant $c \in \mathbb{C}$ fulfilling $|c - h(0,0)| \leq A^2$. The differential $D\Phi$ satisfies

$$\|D\Phi - 1\| \leq 12B.$$

In the case that $\frac{\partial h}{\partial x_1}(0,0) = \frac{\partial h}{\partial x_2}(0,0) = 0$ one has $\xi = 0$ and $s = 0$.

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