Semigroups for flows in networks

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Introduction

Networks have been studied since many years with motivations from and applications to classical natural sciences. As typical examples we mention food-webs, electrical power grids, cellular and metabolic networks, chemical processes, neural networks, telephone call graphs, coauthorship and citation networks of scientists, financial networks, ecological webs, – and, of course, the World-Wide Web. As S. H. Strogatz writes in his review article [Str01]: "*The study of networks pervades all of science, from neurobiology to statistical physics.*"

The main goal of these studies is in most cases to characterize network anatomy – that is, to give an accurate and complete description of complex systems. In this direction much progress has been made and we refer to standard books on graph theory [And91], [Bol98], [KV02] etc., or to M.E.J. Newman [New03] for a survey on recent developments. However, on p. 224 of [New03] he says: *"The next logical step after developing models of network structure, (...) is to look at the behavior of models of physical (or biological or social) processes going on on those networks. Progress on this front has been slower than progress on understanding network structure."*

Clearly, in graph theory, many discrete or combinatorial interactions in networks have been treated. In the monograph [Bol01] an overview is given on an already autonomous discipline, the theory of random graphs which serves for modelling, e.g., gene networks, ecosystems and the spread of infectious diseases or computer viruses. Other important discrete processes traditionally studied in graph theory are Markov processes, see e.g. [Rob03].

In this thesis, we are interested in so called *dynamical graphs*. Here the edges do not only link the vertices but also serve as a transmission media on which time- and space-depending processes take place. Such problems have been first studied by G. Lumer [Lum79, Lum80], who proved well-posedness of second-order problems on ramified spaces. Later J. von Below (see [Bel85] – [BN96]), handled diffusion processes in networks modelled by polygons. F. Ali Mehmeti in [AMe89], [MR03] and other papers investigated wave equations on different types of networks. S. Nicaise also contributed to the study of elliptic operators on networks. We only cite [Nic88.1], [Nic88.2], and the monograph [MBN01] on this topic edited by these three authors. Recently, C. Cattaneo studied in [Cat97] and [Cat99] the spectrum of the Laplacian on networks, connecting it to the discrete Laplacian in graph theory. She used semigroup theory to prove well-posedness of the problem. For aspects of numerical analysis and control theory of dynamic elastic linked structures we refer to the monograph [LLS94].

However, there seems to be no systematic treatment of dynamic processes different from second-order problems. The main goal of the present work is to propose an appropriate functional analytic setting and to investigate linear *transport processes* or *flows* in networks. To do this we use sophisticated semigroup and spectral theoretical methods and refer to [EN00] and [Nag86] as main references. The results are mainly based on the papers [KS04], [MS04] and the preprint [Sik04].

In Chapter 1 we give a short overview on important notations and results from graph theory that will be used during the treatment of the functional analytic problem. We model the network by a *directed graph* where a substance is flowing on the edges in the given directions and redistributed in the vertices.

In Chapter 2 we discuss transport processes in networks with *static ramification nodes*. More precisely, we require for all times in each vertex that the total incoming flow mass equals the total outgoing flow mass (*Kirchhoff law*) and that the outgoing flow is distributed on the outgoing edges according to given proportions. We show that the corresponding system of partial differential equations with appropriate boundary conditions can be rewritten in the form of an abstract Cauchy problem on a (Banach) state space. We prove well-posedness of the system by showing that the underlying operator generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ which gives the solutions of our original system. Using spectral theory and semigroup methods we will be able to describe precisely the asymptotic behavior of $(T(t))_{t\geq 0}$ – that is, of the process in the network. In fact, we prove a dichotomy for the asymptotics of such flows based on a number theoretical condition on the flow velocities on the edges, see Definition 2.3.7. In one case, treated in §2.4, the process converges uniformly towards a periodic flow whose period is determined by the structure of the graph (see Theorems 2.4.8 and 2.4.11). In the other case, see §2.5 we obtain that the flow always converges (in the strong operator topology) to an equilibrium. Most of these results are obtained in collaboration with M. Kramar and T. Mátrai (see [KS04],[MS04]).

We then investigate in Chapter 3 transport processes, where in the ramification nodes a *dynamic* condition is specified. More precisely, the velocity of the total outgoing flow mass is prescribed as a (weighted) sum of incoming flow quantities plus a term depending on the outgoing flow mass in the vertices. This second term can be interpreted as a *feedback-control* of the outgoing flow velocities along “imaginary edges” in the graph – that is, edges having endpoints in our original graph but not necessarily belonging to the original edge set. To handle this problem we modify the semigroup approach to delay differential equations developed by A. Bátkai and S. Piazzera in [BP04]. Again, we can prove well-posedness by rewriting the problem in the form of an abstract Cauchy problem and therefore obtain a semigroup determining the solutions. We then prove that this semigroup has important regularity properties (see Theorems 3.3.2 and 3.3.4) implying the validity of the “Spectral Mapping Theorem”. We show in §3.4 that if the semigroup is positive, the stability of the system depends on the spectral bound of the so-called *adjacency matrix* (see Definition 1.3.6) of the graph obtained by adding the “imaginary edges” to the original graph, along those the feedback-control takes place (see Corollary 3.4.6 and the interpretation below). Finally, we obtain in §3.5 that the semigroup converges towards an equilibrium if the joint structure of the original and the “imaginary” graph is strongly connected (see Theorem 3.5.3).

In the final Chapter 4 we discuss examples on the *Petersen* and *Herschel graph* for the situation studied in Chapter 2 and compute the convergence speed towards the periodic flow. We also investigate how this depends on the distribution weights of the edges.

In this thesis, we suppose the reader to be familiar with large parts of semigroup and spectral theory from [EN00]. Furthermore he needs some experience with the theory of operator matrices from [CENN03] and Greiner’s approach to abstract boundary value problems, developed in [Gre87].

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Chapter 1

Some graph theory

In this section we summarize some graph theoretical notions that we will use frequently. Our terminology is common to graph theory, based mainly on the monograph [And91], but see also [Big93], [Bol98], [CDS95], [God93], or [GR01].

§ 1.1 Main definitions

Definition 1.1.1 — Let $\mathsf{V} = \{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$ and $\mathsf{E} = \{\mathsf{e}_1, \ldots, \mathsf{e}_m\}$ be two disjoint (finite) sets and G a function from E to $\mathsf{V} \times \mathsf{V}$. The triplet $(\mathsf{V}, \mathsf{E}, G)$ is called a *directed graph*. The elements of V are the *vertices* of the graph and the elements of E its *(directed) edges* (or *arcs*). Here G prescribes the two ordered endpoints of the edges and we say that the edge e_j *connects* the vertices v_i and v_p if $G(\mathsf{e}_j) = (\mathsf{v}_i, \mathsf{v}_p)$. For the sake of simplicity, a directed graph will be also denoted only by G. We call the graph G_k a *subgraph* of G induced by the set of vertices $\mathsf{V}_k \subset \mathsf{V}$ if it is obtained from the vertices in V_k and all the edges that connect them in G.

Definition 1.1.2 — In a directed graph, if the edge e_j is associated with the vertex-pair $(\mathsf{v}_i, \mathsf{v}_p)$, v_i is called the *tail* of e_j and v_p is called the *head* of e_j. The edge e_j is called a *loop* if its tail coincides with its head.

We restrict our investigations to the following type of graphs.

Definition 1.1.3 — A directed graph is called *simple* if it contains no loops and no multiple edges (that is, edges connecting the same vertices).

From now on we always assume that G is a simple directed graph.

Remark 1.1.4 — In the subsequent part of the thesis we often use the notion *network*. By that we mean a directed graph on which a dynamical process takes place. Hence we do not only consider the "static" structure of the graph but also some dynamics on it.

§ 1.2 STRONGLY CONNECTED GRAPHS

First we have to introduce the definition of two important graph theoretical notions.

DEFINITION 1.2.1 — A *(directed) path* is a sequence of directed, adjoining edges in G (that is, except the last edge, the head of every edge is the tail of the following edge). It can be uniquely defined by a sequence of vertices $\mathsf{v}_{i_1}, \ldots, \mathsf{v}_{i_l}$, such that $\left(\mathsf{v}_{i_k}, \mathsf{v}_{i_{k+1}}\right)$ are all edges in G for $k = 1, \ldots, l-1$. A *(directed) cycle* is a directed path defined by the vertices $\mathsf{v}_{i_1}, \ldots, \mathsf{v}_{i_l}$ such that $\mathsf{v}_{i_l} = \mathsf{v}_{i_1}$, and $\mathsf{v}_{i_1}, \ldots, \mathsf{v}_{i_{l-1}}$ are all different from each other.

We also can define the *length* of such paths/cycles as the number of their edges. Directed paths/cycles are called *vertex-disjoint* if no two different paths/cycles among them contain a common vertex.

In every directed graph G we can introduce a relation $\mathcal{E}$ on the set V of its vertices which satisfies the following properties: $\mathsf{v}_i \mathcal{E} \mathsf{v}_i$ for each element $\mathsf{v}_i \in \mathsf{V}$; for two distinct elements v_i and v_p of V, $\mathsf{v}_i \mathcal{E} \mathsf{v}_p$ if and only if both v_i can be reached from v_p and v_p can be reached from v_i along directed paths in G.

Obviously, $\mathcal{E}$ is an equivalence relation.

DEFINITION 1.2.2 — Let $\mathsf{V}_1, \mathsf{V}_2, \ldots$ be the partition of the set V of the vertices in G, induced by the equivalence relation $\mathcal{E}$ we have introduced above. The subgraphs of G induced by the sets V_k are called the *strongly connected components* of G.

The following definition is natural.

DEFINITION 1.2.3 — A directed graph is called *strongly connected* if its vertices belong to one single strongly connected component – that is, for every two vertices in the graph there are directed paths connecting them in both directions.

Hence, in strongly connected graphs we can find directed paths and directed cycles.

In networks, where the process on the edges is of high importance, we define a special class of strongly connected components.

DEFINITION 1.2.4 — In a network we call a subgraph G_k of G an *invariant strongly connected component* if it is a strongly connected component, and there are no edges in G having tail in and head outside of G_k.

§ 1.3 GRAPH MATRICES

We now introduce important matrices that can be associated to a directed graph (see [And91, Chapter 3]). Let $\mathsf{V} = \{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$ be the set of the vertices, $\mathsf{E} = \{\mathsf{e}_1, \ldots, \mathsf{e}_m\}$ the set of the edges of our simple directed graph G. We first begin with matrices describing the connection between vertices and edges in G.

DEFINITION 1.3.1 — The *outgoing incidence matrix* $\Phi^- = \left(\phi_{ij}^-\right)_{n \times m}$ of G is defined by

$$\phi_{ij}^- := \begin{cases} 1, & \text{if the tail of } \mathsf{e}_j \text{ is } \mathsf{v}_i, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Accordingly, we call the edge e_j an *outgoing edge for* v_i if $\phi^-_{ij} = 1$ holds. Respectively, we define the *incoming incidence matrix* $\Phi^+ = \left(\phi^+_{ij}\right)_{n\times m}$ by

$$\phi^+_{ij} := \begin{cases} 1, & \text{if the head of } \mathsf{e}_j \text{ is } \mathsf{v}_i, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

and call the edge e_j an *incoming edge for* v_i if $\phi^+_{ij} = 1$ holds.

REMARK 1.3.2 — The matrix

$$\Phi = \Phi^+ - \Phi^-$$

is called the *incidence matrix* of the directed graph G.

We also need matrices having the same zero pattern as the incidence matrices.

DEFINITION 1.3.3 — We define the *weighted outgoing incidence matrix* of the graph G as

$$\Phi^-_w = \left(\omega^-_{ij}\right)_{n\times m} \tag{1.3}$$

with entries

$$0 \le \omega^-_{ij} \le 1$$

satisfying

$$\begin{aligned} \omega^-_{ij} = 0 \quad &\Leftrightarrow \quad \phi^-_{ij} = 0, \text{ and} \\ \sum_{j=1}^{m} \omega^-_{ij} \quad &= \quad 1 \end{aligned} \tag{1.4}$$

for all $i = 1,\dots,n$, $j = 1,\dots,m$.

The name “weighted” comes from the fact that in networks the entries ω^-_{ij} will denote the weights according that the flow mass is distributed to the outgoing edges in the vertices. Condition (1.4) implying the row stochasticity of Φ^-_w will play an important role in our studies.

REMARK 1.3.4 — We have

$$\Phi^- \left(\Phi^-_w\right)^\top = \mathbf{1}, \tag{1.5}$$

where $\mathbf{1}$ denotes the $n \times n$ identity matrix.

PROOF — By a straightforward computation,

$$\left[\Phi^- \left(\Phi^-_w\right)^\top\right]_{ip} = \sum_{j=1}^{m} \phi^-_{ij}\omega^-_{pj}.$$

If $i \neq p$, from the definition of the outgoing incedence matrix, $\phi^-_{ij}\omega^-_{pj} = 0$ for all j, hence $\left[\Phi^- \left(\Phi^-_w\right)^\top\right]_{ip} = 0$. If $i = p$ then by condition (1.4), $\phi^-_{ij}\omega^-_{pj} = \omega^-_{pj}$, and so

$$\left[\Phi^- \left(\Phi^-_w\right)^\top\right]_{ip} = \sum_{j=1}^{m} \omega^-_{pj} = 1.$$

■

DEFINITION 1.3.5 — The *weighted incoming incidence matrix*

$$\Phi_w^+ = \left(\omega_{ij}^+\right)_{n\times m} \tag{1.6}$$

is defined with entries

$$\omega_{ij}^+ \geq 0$$

satisfying

$$\omega_{ij}^+ = 0 \Leftrightarrow \phi_{ij}^+ = 0.$$

Note that in this case we do not require any extra conditions on the sum of the weights ω_{ij}^+.

The following class of graph matrices describes the connections between the vertices.

DEFINITION 1.3.6 — The matrix $\mathbf{A} = (a_{ip})_{n\times n}$ is called the *adjacency matrix* of G if

$$a_{ip} = \begin{cases} 1, & \text{if there exists an edge with tail } \mathsf{v}_i \text{ and head } \mathsf{v}_p, \\ 0, & \text{otherwise.} \end{cases}$$

REMARK 1.3.7 — An easy computation shows that $\mathbf{A}$ can be obtained from the incidence matrices as

$$\mathbf{A} = \Phi^- \left(\Phi^+\right)^\top.$$

DEFINITION 1.3.8 — We call a matrix *weighted adjacency matrix* of G if it has the same zero pattern as the adjacency matrix $\mathbf{A}$.

Let G be a network admitting a weighted adjacency matrix $\mathbf{A}_w = (b_{ip})_{n\times n}$. Then the entry b_{ip} can be regarded as the weight of the edge connecting v_i and v_p. Hence, it is natural to match the weighted incidence matrices of the graph to the weighted adjacency matrix. The next example will play an important role in our setting.

EXAMPLE 1.3.9 — Let $\tilde{\mathbf{A}}$ be the $n \times n$ matrix defined as

$$\left(\tilde{\mathbf{A}}\right)_{ip} = \begin{cases} \omega_{ij}^+\omega_{pj}^-, & \text{if the edge } \mathrm{e}_j \text{ has its tail in } \mathsf{v}_p \text{ and its head in } \mathsf{v}_i, \\ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

Clearly, the matrix $\tilde{\mathbf{A}}$ is a *weighted transposed adjacency matrix* for the graph G and

$$\tilde{\mathbf{A}} = \Phi_w^+ \left(\Phi_w^-\right)^\top$$

holds.

We now cite a result from [And91, Theorem 3.2] that turns out to be very important for the subsequent theory.

PROPOSITION 1.3.10 — *A directed graph is strongly connected if and only if its adjacency matrix is irreducible.*

As usual, a positive matrix $\mathbf{D}$ is called *irreducible* if there is no permutation of the canonical basis such that in this basis the matrix has the form

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_{1,1} & 0 \\ \mathbf{D}_{2,1} & \mathbf{D}_{2,2} \end{pmatrix}.$$

Hence, this property only depends on the zero pattern of the positive matrix. Therefore in the above Proposition 1.3.10 we can substitute the adjacency matrix by any weighted adjacency matrix.

We also need the adjacency matrix of the *line graph* (see [Bol98, Definition III.6.15]), which is roughly the graph obtained from G by exchanging the role of the vertices and edges (maintaining the directions).

DEFINITION 1.3.11 — The $m \times m$ *adjacency matrix* $\mathbf{A}_L$ of the line graph of G is defined by

$$(\mathbf{A}_L)_{jl} = \begin{cases} 1, & \text{if the head of } \mathsf{e}_j \text{ coincides with the tail of } \mathsf{e}_l, \\ 0, & \text{otherwise.} \end{cases}$$

In general, we call a matrix *weighted adjacency matrix* of the line graph if it has the same zero pattern as $\mathbf{A}_L$.

REMARK 1.3.12 — An easy calculation shows that

$$\mathbf{A}_L = \left(\Phi^+\right)^\top \Phi^-. \tag{1.8}$$

EXAMPLE 1.3.13 — As an example for a *weighted transposed adjacency matrix* of the line graph we can take

$$\tilde{\mathbf{A}}_L := \left(\Phi_w^-\right)^\top \Phi_w^+. \tag{1.9}$$

This will be needed in the proof of the well-posedness of our system in Theorem 2.1.5.

Finally, we cite the Sachs Theorem on the determinant of the weighted adjacency matrix that will be used in Section §2.4. We first define a special type of subgraphs.

DEFINITION 1.3.14 — A subgraph $G_{\mathcal{L}}$ of G is called *linear subgraph* if it is a vertex disjoint union of (directed) cycles.

Before stating the theorem we have to introduce some notations.

NOTATION 1.3.15 — Let $G_{\mathcal{L}}$ a linear subgraph of G. We denote by $c\left(G_{\mathcal{L}}\right)$ the number of cycles in $G_{\mathcal{L}}$. Furthermore, if $\mathbf{A}_w$ is a weighted adjacency matrix of G, $W\left(G_{\mathcal{L}}\right)$ denotes the *weight of the linear subgraph* $G_{\mathcal{L}}$, i.e., the product of the weights of the arcs (taken from the corresponding entries of $\mathbf{A}_w$) contained in that subgraph. Let $\mathcal{L}_r$ be the set of linear subgraphs of G having exactly r vertices.

THEOREM 1.3.16 [Sachs Theorem, Theorem 3.1 in [CDGT88]] — *Let $\mathbf{A}_w$ be an $n \times n$ weighted adjacency matrix of the directed graph G. Then*

$$\begin{aligned} \det\left(z\mathbf{1} - \mathbf{A}_w\right) &= z^n + a_1 z^{n-1} + \cdots + a_n \;\; \textit{with} \\ a_r &= \sum_{G_{\mathcal{L}} \in \mathcal{L}_r} (-1)^{c(G_{\mathcal{L}})} W\left(G_{\mathcal{L}}\right), \; r = 1, \ldots, n. \end{aligned} \tag{1.10}$$

§ 1.4 GRAPH THEORY VERSUS FLOWS IN NETWORKS?

In the following two chapters we will discuss several flow processes in networks. People being familiar with graph theory certainly know of flows such as treated, e.g., in the monographs [And91], [Bol98] or [Die00]. There, one can think of a network of roads (modelled by directed edges) where goods are transported from a *source* to a *sink* point. This "flow" is limited by the transmission capabilities, i.e., *capacities* of the arcs. The time consumed by the transportation process (depending on the lengths of the road-sections and the speed of the transporting vehicles) is disregarded. The main goal is to determine the maximal quantity of goods displaceable from the source to the sink, and a corresponding itinerary of the transportation with the quantity of goods nowhere exceeding the capacity limits. The famous "max-flow min-cut" theorem of Ford and Fulkerson from 1962 states that this maximum amount of flow is equal to the minimum of all *cut* capacities (that is, of flow amounts being transportable out of vertex groups containing the source but not the sink). For more recent results on this topic see e.g. [KV02, Mur03, Schr03].

In our setting, the flow can be modelled by any *continuous* material distributed on the directed edges of a network, and no capacities are needed. The material is transported with given (space dependent) velocities and also absorption/inflow along the edges is allowed. The whole transportation process – that is, the distribution of material in the network – can be described at every *continuous time* moment $t \geq 0$ by space variable functions on the (parameterized) edges. The question we are interested in is how the system behaves for t converging to $+\infty$.

Chapter 2

Flows with static ramification nodes

The physical situation motivating our investigation in this chapter is the following. Consider a network (e.g. a closed system of pipe lines or a circuit of wires) in which a substance is flowing with space depending speed. Along the edges absorption or creation of mass may happen, but in each node (vertex) we assume conservation of mass in form of a Kirchhoff law. In mathematical term we describe this system by the equations

$$
(F)\left\{\begin{array}{rcll}
\dfrac{\partial}{\partial t}u_j\left(t,s\right) & = & c_j(s)\dfrac{\partial}{\partial s}u_j\left(t,s\right)+q_j(s)\cdot u_j(t,s),\ s\in(0,1),t\geq 0, & \\
u_j\left(0,s\right) & = & g_j\left(s\right),\ s\in(0,1), & (IC)\\
\phi^-_{ij}u_j\left(t,1\right) & = & \omega^-_{ij}\displaystyle\sum_{k=1}^{m}\phi^+_{ik}u_k\left(t,0\right),\ t\geq 0, & (BC)
\end{array}\right.
$$

for $i=1,\dots,n$, and $j=1,\dots,m$.

The network is modelled by a simple, directed graph G having vertices $\mathsf{v}_1,\dots,\mathsf{v}_n$ and directed edges (arcs) $\mathsf{e}_1,\dots,\mathsf{e}_m$. The arcs are parameterized by the interval $[0,1]$, contrary to the direction of the flow. Therefore we use the notation $\mathsf{e}_j(1)$ for the tail and $\mathsf{e}_j(0)$ for the head of e_j.

The distribution of the material along an edge e_j at time $t\geq 0$ is described by the function $[0,1]\ni s\mapsto u_j\left(t,s\right)$ for $s\in[0,1]$. The functions $c_j(\cdot)$ are the space dependent velocities of the flow on each arc e_j, while the functions $q_j(\cdot)$ describe the absorption along the edges. We arrange them into the diagonal matrices

$$
C(s):=\begin{pmatrix} c_1(s) & & 0\\ & \ddots & \\ 0 & & c_m(s)\end{pmatrix},\ Q(s):=\begin{pmatrix} q_1(s) & & 0\\ & \ddots & \\ 0 & & q_m(s)\end{pmatrix}. \qquad (2.1)
$$

We also assume that the absorption functions q_j and velocities c_j are bounded, that is belong to $L^\infty[0,1]$, and in addition that $c_j(s)\geq\varepsilon>0$ for almost every $s\in[0,1]$, for each $j=1,\dots,m$ and some $\varepsilon>0$.

The boundary conditions (BC) depend on the structure of the network and contain the incidence matrices Φ^+ and Φ^- of the underlying graph, see (1.1) and (1.2). The entries $0 \leq \omega_{ij}^- \leq 1$ of the *weighted outgoing incidence matrix* (defined in (1.3)) express the proportion of the mass leaving the vertex v_i into the edge e_j.

Summing up for $j = 1, ..., m$ the two sides of the equations in the boundary condition (BC), we obtain using the condition (1.4) the *Kirchhoff law*

$$\sum_{j=1}^{m} \phi_{ij}^- u_j\,(t,1) = \sum_{j=1}^{m} \phi_{ij}^+ u_j\,(t,0)\,,\; i = 1,\ldots,n, \tag{2.2}$$

i.e., in each vertex the total outgoing flow equals to the total incoming flow. This condition requires that in every vertex there is at least one outgoing as well as at least one incoming edge.

To treat our problem (F) we rewrite it in the form of an abstract Cauchy problem and prove its well-posedness using semigroup methods. For the basic notions and techniques on semigroups we refer to [EN00]. We then investigate the spectral properties of the generator of the solution semigroup. Finally, we give in §2.3–§2.5 an accurate description for the asymptotic behavior of the solutions. This chapter is an updated version of results obtained in collaboration with M. Kramar and T. Mátrai (see [KS04] and [MS04]).

§ 2.1 WELL-POSEDNESS OF THE PROBLEM

Our first aim is to write the equations (F) in the form of an abstract Cauchy problem on a Banach space (see [EN00, Definition II.6.1]). For this purpose we introduce the *state space* of L^1-functions[1] on the edges

$$X := \left(L^1[0,1]\right)^m \cong \left(L^1[0,1],\mathbb{C}^m\right), \tag{2.3}$$

endowed with the norm

$$\|f\| := \sum_{j=1}^{m} \|f_j\|_{L^1[0,1]} \text{ for } f = (f_1,\ldots,f_m) \in X.$$

Denoting by M_{q_j} the multiplication operator with the function q_j, we define the operator

$$A_w := \begin{pmatrix} c_1(s)\dfrac{d}{ds} + M_{q_1} & & 0 \\ & \ddots & \\ 0 & & c_m(s)\dfrac{d}{ds} + M_{q_m} \end{pmatrix} \tag{2.4}$$

with (dense) domain[1]

$$D\left(A_w\right) := \left\{ f = (f_1,\ldots,f_m) \in \left(W^{1,1}\left[0,1\right]\right)^m : f(1) \in \operatorname{ran}(\Phi_w^-)^\top \right\}.$$

[1] In the following, without further qualification, the point evaluation for functions in $L^1[0,1]$ and the derivative of functions in $W^{1,1}[0,1]$ are understood almost everywhere.

Before proceeding we explain the condition $f(1) \in \operatorname{ran}(\Phi_w^-)^\top$ appearing in the definition of the domain of A_w. The nonzero elements in the i-th row of the matrix Φ_w^- correspond to the arcs with tail v_i, and in each column of Φ_w^- there is exactly one nonzero entry. Therefore, the condition

$$f(1) = (\Phi_w^-)^\top x \text{ for some } x \in \mathbb{C}^n \tag{2.5}$$

implies for fixed j that

$$f_j(1) = \omega_{ij}^- x_i \text{ if } \omega_{ij}^- \neq 0.$$

Note, that the index i is uniquely defined by j and the condition $\omega_{ij}^- \neq 0$. If $\omega_{ik}^- \neq 0$ for some other k, $1 \leq k \leq m$, then (2.5) implies

$$f_k(1) = \omega_{ik}^- x_i,$$

that is,

$$\frac{f_j(1)}{\omega_{ij}^-} = \frac{f_k(1)}{\omega_{ik}^-}.$$

This means that the values of f at the point 1 on the arcs with the same tail are related by the corresponding weights.

The boundary conditions (BC) will now be added using two *boundary operators* L and M (see [CENN03] where this terminology is explained and used in an abstract framework). For that purpose we call

$$\partial X := \mathbb{C}^n \tag{2.6}$$

the *boundary space*. In the physical interpretation, this is the space of flow mass in the vertices. Then we define the *outgoing boundary operator* $L : X \to \partial X$ by

$$L := \Phi^- \otimes \delta_1, \quad D(L) := \left(W^{1,1}[0,1]\right)^m, \tag{2.7}$$

where δ_1 is the point evaluation at 1, hence

$$Lf = \Phi^- f(1) \text{ for } f \in \left(W^{1,1}[0,1]\right)^m.$$

REMARK 2.1.1 — The operator L is surjective from $D(A_w)$ to ∂X.

PROOF — Observe that $D(A_w)$ contains all constant functions f satisfying the boundary condition (2.5), i.e., $f \equiv (\Phi_w^-)^\top x$ for some $x \in \partial X$. The statement now follows from (1.5). ∎

The incoming flow will be taken into account by the *incoming boundary operator* $M : X \to \partial X$,

$$M := \Phi^+ \otimes \delta_0, \quad D(M) := \left(W^{1,1}[0,1]\right)^m, \tag{2.8}$$

where δ_0 is the point evaluation at 0, hence

$$Mf = \Phi^+ f(0) \text{ for } f \in \left(W^{1,1}[0,1]\right)^m.$$

Observe that the equation $Lf = Mf$ expresses the Kirchhoff law (2.2) for each vertex.

After these preparations we are ready to introduce the operator corresponding to problem (F) .

DEFINITION 2.1.2 — On the Banach space X we define the operator $(A, D(A))$ as the restriction of $(A_w, D(A_w))$ to $\ker(L - M)$, i.e.,

$$\begin{aligned} D\left(A\right) &:= \left\{f \in D\left(A_w\right) : Lf = Mf\right\}, \\ Af &:= A_w f. \end{aligned} \tag{2.9}$$

By the definitions of the operators A_w, L and M one can easily see that (BC) implies the conditions in the domain of A. Taking $f \in D(A)$, we obtain by (2.5) that

$$f(1) = \left(\Phi_w^-\right)^\top x \text{ for some } x \in \mathbb{C}^n$$

and

$$\Phi^- f(1) = \Phi^+ f(0).$$

By (1.5),

$$\Phi^- f(1) = x = \Phi^+ f(0),$$

hence

$$f(1) = \left(\Phi_w^-\right)^\top x = \left(\Phi_w^-\right)^\top \Phi^+ f(0)$$

implying (BC). Therefore the conditions in the domain of A are in fact equivalent to (BC). If we write $u(t) = (u_1(t), \ldots, u_m(t)) = (u_1(t, \cdot), \ldots, u_m(t, \cdot)) \in \left(L^1[0, 1]\right)^m$, then the Cauchy problem

$$\begin{cases} \dot{u}\left(t\right) = Au\left(t\right), & t \geq 0, \\ u\left(0\right) = u_0 \end{cases} \tag{2.10}$$

with $u_0 = \left(g_j\right)_{j=1,\ldots,m}$ is an abstract version of our original problem. By standard semigroup theory (see [EN00, Theorem II.6.7]) this problem is well-posed if and only if A generates a strongly continuous semigroup $\left(T\left(t\right)\right)_{t\geq 0}$ on X. In this case, the solutions of (2.10) have the form $u(t) = T(t)u_0$ yielding solutions for (F) as $u\left(t, s\right) := \left(T(t)u_0\right)(s)$ for $s \in [0, 1]$.

To show the generator property we will use the Phillips theorem as in [KS04, Lemma 2.4] and recall from [Nag86, Section C-II.1] the definition of dispersive operators on Banach lattices.

DEFINITION 2.1.3 — An operator A on a Banach lattice X is called *dispersive* if for every $f \in D(A)$ one has $\operatorname{Re}\langle Af, \psi\rangle \leq 0$ for some $\psi \in X'_+$ such that $\|\psi\| \leq 1$ and $\langle f, \psi\rangle = \|f^+\|$.

Clearly, our state space $X = \left(L^1[0, 1]\right)^m$ as well as our boundary space ∂X are Banach lattices and, by physical reasons, we expect the solutions of (F) corresponding to positive initial values g_j to remain positive for all $t \geq 0$. In terms of semigroups this means that the solution semigroup should be positive. We refer to [Nag86] for a systematic treatment, but recall some basic notions in our concrete situation.

DEFINITION 2.1.4 — Let X and ∂X be the spaces defined in (2.3) and (2.6).

1. We call a vector $x \in \partial X$ *positive* and write $x \geq 0$ if $x_i \geq 0$ for all coordinates $i = 1, \dots, n$. We write $x > 0$ if $x \geq 0$ and $x \neq 0$, i.e., x has at least one nonzero coordinate. We call x *strictly positive* and write $x \gg 0$ if $x_i > 0$ for all i. Analogously, we use the same terminology for matrices.

2. A function $f \in X$ is *positive* and we write $f \geq 0$ if $f(s) \geq 0$ for almost all $s \in [0,1]$, and $f > 0$ if $f \geq 0$ but $f \neq 0$ (understood almost everywhere). Furthermore, the function f is called *strictly positive* and denoted by $f \gg 0$ if $f(s) > 0$ for almost every $s \in [0,1]$.

3. An operator T on the Banach lattice X is called *positive* if $0 \leq f \in X$ implies $0 \leq Tf$. A semigroup $(T(t))_{t\geq 0}$ on X is *positive* if the operators $T(t)$ are positive for all $t \geq 0$.

Using the notion of dispersivity and Theorem C-II.1.2 from [Nag86], we can show that the operator A generates a semigroup of *positive* operators on the Banach lattice X.

THEOREM 2.1.5 — *The operator $(A, D(A))$ generates a positive strongly continuous semigroup $(T(t))_{t\geq 0}$ on X.*

PROOF — Our operator A can be written as the sum

$$A = \begin{pmatrix} c_1(s)\dfrac{d}{ds} & & 0 \\ & \ddots & \\ 0 & & c_m(s)\dfrac{d}{ds} \end{pmatrix} + \begin{pmatrix} M_{q_1} & & 0 \\ & \ddots & \\ 0 & & M_{q_m} \end{pmatrix} = A_c + A_q.$$

We show first that A_c generates a positive C_0-semigroup $(T(t))_{t\geq 0}$. For this purpose we introduce a new, but equivalent lattice norm on X defined as

$$\|f\|_c := \sum_{j=1}^{m} \int_0^1 \frac{|f_j(s)|}{c_j(s)} \mathrm{d}s. \tag{2.11}$$

It is clear that

$$\min_{1\leq j\leq m} \|c_j\|_\infty \cdot \|f\|_c \leq \|f\| \leq \max_{1\leq j\leq m} \|c_j\|_\infty \cdot \|f\|_c \,. \tag{2.12}$$

We are going to prove that $(A_c, D(A))$ is dispersive on the Banach lattice $(X, \|\cdot\|_c)$. To verify the required inequality in Definition 2.1.3, it is enough to consider only functions contained in $D(A)$ with values in $\mathbb{R}$, because the operator A_c is real. To each real function $f = (f_j)_{j=1,\dots,m} \in D(A)$ we associate $\psi = (\psi)_{j=,\dots,m} \in (L^\infty[0,1])^m = X'$ defined as

$$\psi_j\,(s) := \begin{cases} \frac{1}{c_j(s)}, & \text{if } f_j\,(s) > 0, \\ 0, & \text{else.} \end{cases}$$

Then ψ satisfies all the conditions in the Definition 2.1.3 for the new norm defined in (2.11). Now it suffices to prove that

$$\langle A_c f, \psi \rangle \leq 0.$$

From the definition of A_c and ψ we obtain

$$\begin{aligned}\langle A_c f, \psi\rangle &= \sum_{k=1}^m \int_0^1 c_k(s) f_k'(s)\psi_k(s)\mathrm{d}s = \sum_{k=1}^m \int_0^1 c_k(s) f_k'(s)\frac{1}{c_k(s)}\chi_{\{f_k>0\}}\mathrm{d}s \\ &= \left\langle [f(1)]^+ - [f(0)]^+ , 1_{\mathbb{R}^m}\right\rangle_{\mathbb{R}^m},\end{aligned}$$

where $1_{\mathbb{R}^m}$ denotes the constant 1 vector in $\mathbb{R}^m$. Furthermore, for $f \in D(A)$ we have $Lf = Mf$ and $f(1) \in \operatorname{ran}(\Phi_w^-)^\top$ which implies

$$\begin{aligned}\Phi^- f(1) &= \Phi^+ f(0), \\ f(1) &= \left(\Phi_w^-\right)^\top x\end{aligned}$$

for some $x \in \partial X = \mathbb{C}^n$. Since $\Phi^-\left(\Phi_w^-\right)^\top = \mathbf{1}$ by (1.5), we have

$$\Phi^-\left(\Phi_w^-\right)^\top x = x = \Phi^+ f(0)$$

and

$$f(1) = \left(\Phi_w^-\right)^\top x = \left(\Phi_w^-\right)^\top \Phi^+ f(0) = \tilde{\mathbf{A}}_L f(0).$$

Here $\tilde{\mathbf{A}}_L$ has entries

$$\left(\tilde{\mathbf{A}}_L\right)_{jl} = \begin{cases} \omega_{ij}^-, & \text{if } \mathsf{e}_l(0) = \mathsf{v}_i = \mathsf{e}_j(1), \\ 0, & \text{otherwise}\end{cases}$$

and is positive column stochastic by (1.4). The matrix $\tilde{\mathbf{A}}_L$ is actually a weighted transposed adjacency matrix of the line graph – see (1.9) – where we take $\Phi_w^+ = \Phi^+$. Continuing the above estimate and using the positivity of $\tilde{\mathbf{A}}_L$ we obtain

$$\begin{aligned}\langle A_c f, \psi\rangle &= \left\langle \left[\tilde{\mathbf{A}}_L f(0)\right]^+ - [f(0)]^+ , 1_{\mathbb{R}^m}\right\rangle_{\mathbb{R}^m} \\ &\leq \left\langle \tilde{\mathbf{A}}_L [f(0)]^+ - [f(0)]^+ , 1_{\mathbb{R}^m}\right\rangle_{\mathbb{R}^m} = \left\langle [f(0)]^+ , \tilde{\mathbf{A}}_L^\top 1_{\mathbb{R}^m} - 1_{\mathbb{R}^m}\right\rangle_{\mathbb{R}^m} = 0\end{aligned}$$

by the column stochasticity of $\tilde{\mathbf{A}}_L$. Hence the operator $(A_c, D(A))$ is dispersive on the Banach lattice $\left((L^1[0,1])^m, \|\cdot\|_c\right)$. Clearly, $(A_c, D(A))$ is closed and densely defined.

For the final step we use Corollary 2.2.15 below, which shows that resolvent set $\rho(A_c)$ is not empty. Therefore we can use the Phillips theorem from [Nag86, Theorem C-II.1.2] and obtain that $(A_c, D(A))$ generates a positive contraction semigroup $(U(t))_{t\geq 0}$ on $(X, \|\cdot\|_c)$, hence it a positive bounded semigroup on $(X, \|\cdot\|_X)$.

By the assumptions on q_j, A_q is a bounded real multiplication operator on X, hence it generates a positive multiplication semigroup $(S(t))_{t\geq 0}$ with $\|S(t)\|_c \leq \mathrm{e}^{\omega t}$ for some $\omega > 0$. Since $(U(t))_{t\geq 0}$ is contractive for $\|\cdot\|_c$, we can apply the *Trotter product formula* (see [EN00, Corollary III.5.8]) to the positive semigroups $(U(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ obtaining

$$T(t)f = \lim_{n\to\infty} \left[U(t/n)S(t/n)\right]^n f, \; f \in X.$$

This formula clearly defines again a positive semigroup $(T(t))_{t\geq 0}$ satisfying the norm estimate $\|T(t)\|_c \leq \mathrm{e}^{\omega t}$, $t \geq 0$. ■

Observe that in the special case when the velocities on the arcs are all constant and equal and there is no absorption, we even obtain a semigroup of contractions for the original norm.

COROLLARY 2.1.6 — *If $c_j \equiv c$ and $q_j \equiv 0$ for all $j = 1, \ldots, m$, then the semigroup $(T(t))_{t\geq 0}$ is contractive on $(X, \|\cdot\|)$.*

PROOF — The estimate (2.12) and the contractivity of $T(t)$ for the norm $\|\cdot\|_c$ implies

$$\|T(t)\| \leq \frac{\max_j \|c_j\|_\infty}{\min_j \|c_j\|_\infty} \|T(t)\|_c \leq 1,$$

and this is what we wanted to prove. ■

We state explicitly our first main result.

COROLLARY 2.1.7 — *The problem (F) is well-posed.*

§ 2.2 SPECTRAL PROPERTIES

In order to obtain qualitative properties of the solutions of (F), or of the semigroup generated by A, we now start with a careful analysis of the spectrum of A. For that purpose we use a perturbation method as proposed in [Nag97] and first consider the operator

$$A_0 := A_w\,|_{\ker L},\ D(A_0) = \{f \in D\left(A_w\right) : Lf = 0\}. \tag{2.13}$$

This means that we consider homogeneous boundary conditions where the right hand side of (BC) is equal to zero. In fact, by (1.5), the domain of A_0 is simply

$$D\left(A_0\right) = \left\{f \in \left(W^{1,1}\left[0,1\right]\right)^m : f\left(1\right) = 0\right\}.$$

To proceed, we will write the resolvent of A_0 explicitly using the following two notations.

DEFINITION 2.2.1 — Take $j = 1, \ldots, m$ and $s_1, s_2 \in [0, 1]$. We set

$$\tau_j(s_1, s_2) := \int_{s_1}^{s_2} \frac{\mathrm{d}s}{c_j(s)} \tag{2.14}$$

and

$$\xi_j(s_1, s_2) := \int_{s_1}^{s_2} \frac{q_j(s)}{c_j(s)} \mathrm{d}s. \tag{2.15}$$

We denote $\tau_+ := \max\limits_{1\leq j\leq m} \tau_j(0, 1)$ and $\tau_- := \min\limits_{1\leq j\leq m} \tau_j(0, 1)$.

INTERPRETATION — The value $\tau_j(s_1, s_2)$ is exactly the time needed to pass on the edge e_j from s_1 to s_2 moving with velocity $c_j(s)$ at every point $s \in [s_1, s_2]$, while $\xi_j(s_1, s_2)$ is the rate of the mass gain or loss on this journey resulting from the factor $q_j(s)$. Note that our assumptions on the flow velocity and the absorption functions imply that the integrals in (2.14) and (2.15) are finite.

With these notations, the resolvent of A_0 – which exists for every $\lambda \in \mathbb{C}$ – can be computed explicitly.

LEMMA 2.2.2 — *For every $\lambda \in \mathbb{C}$ and with the matrices $C(s)$ and $Q(s)$ defined in* (2.1)*, we have*

$$\left(R(\lambda, A_0)f\right)(s) = \int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)d\sigma,\ s \in [0,1]\,,\ f \in X, \tag{2.16}$$

where

$$\epsilon_\lambda(s) := \operatorname{diag}\left(\mathrm{e}^{-\xi_j(0,s)+\lambda\tau_j(0,s)}\right)_{j=1,\dots,m}, \quad s \in [0,1]. \tag{2.17}$$

REMARK 2.2.3 — The expression $\epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}$ occurring in the resolvent is actually

$$\epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1} = \operatorname{diag}\left(\mathrm{e}^{\int_s^\sigma \frac{q_j(u)}{c_j(u)}\mathrm{d}u - \lambda\int_s^\sigma \frac{\mathrm{d}u}{c_j(u)}}\right)_{j=1,\dots,m}.$$

PROOF — An easy calculation shows that

$$\epsilon_\lambda'(s) = \epsilon_\lambda(s)(-Q(s)+\lambda)C(s)^{-1}. \tag{2.18}$$

Clearly, the function

$$[0,1] \ni s \mapsto g(s) := \int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)\mathrm{d}\sigma$$

is contained in $D(A_0)$. By applying $\lambda - A_0$ to it and using (2.18) we obtain

$$\begin{aligned}
\left((\lambda - A_0)\,g\right)(s) &= \lambda g(s) - C(s)g'(s) - Q(s)g(s) \\
&= \lambda g(s) - C(s)\epsilon_\lambda'(s)\int_s^1 \epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)\mathrm{d}\sigma \\
&+ C(s)\epsilon_\lambda(s)\epsilon_\lambda(s)^{-1}C(s)^{-1}f(s) - Q(s)g(s) \\
&= \lambda\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)\mathrm{d}\sigma \\
&- C(s)\epsilon_\lambda(s)\left(-Q(s)+\lambda\right)C(s)^{-1}\int_s^1 \epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)\mathrm{d}\sigma \\
&+ f(s) - Q(s)\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)\mathrm{d}\sigma
\end{aligned}$$

$$\begin{aligned}
&= \lambda\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)\mathrm{d}\sigma \\
&+ Q(s)\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)\mathrm{d}\sigma \\
&- \lambda\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)\mathrm{d}\sigma \\
&+ f(s) - Q(s)\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}f(\sigma)\mathrm{d}\sigma = f(s),
\end{aligned}$$

using the fact that the diagonal matrices commute.

For the other direction take $f \in D(A_0)$ and compute the formula (2.16) for $(\lambda - A_0) f$.

$$\begin{aligned}
& \int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}(\lambda f - A_0 f)(\sigma)\mathrm{d}\sigma \\
= & \int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}C(\sigma)^{-1}(\lambda f(\sigma) - C(\sigma)f'(\sigma) - Q(\sigma)f(\sigma))\mathrm{d}\sigma \\
= & \int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}\epsilon_\lambda(\sigma)^{-1}\epsilon_\lambda'(\sigma)f(\sigma)\mathrm{d}\sigma - \int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1}f'(\sigma)\mathrm{d}\sigma \\
= & -\epsilon_\lambda(s)\int_s^1 \left(\epsilon_\lambda(\sigma)^{-1}\right)' f(\sigma)\mathrm{d}\sigma - \epsilon_\lambda(s)\int_s^1 \epsilon_\lambda(\sigma)^{-1}f'(\sigma)\mathrm{d}\sigma \\
= & -\epsilon_\lambda(s)\left[\epsilon_\lambda(\sigma)^{-1}f(\sigma\right]_{\sigma=s}^{\sigma=1} = f(s),
\end{aligned}$$

where we used (2.18) and $f(1) = 0$. This completes the proof. ■

We also obtain that A_0 generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X which can be expressed explicitly. From this formula we see that $(T_0(t))_{t\geq 0}$ is nilpotent.

LEMMA 2.2.4 — *Let $j \in \{1, \dots, m\}$ fixed. With the notations of Definition 2.2.1, let $\tilde{s}(t) \in [0,1]$ be the location where the flow moves to on the edge e_j from the point s during time $t \leq \tau_j(s,1)$. Hence the function $\tilde{s} \in C[0, \tau_j(s,1)]$ is defined by $\tau_j(s, \tilde{s}(t)) = t$. Then the jth coordinate of the semigroup $(T_0(t))_{t\geq 0}$ generated by $(A_0, D(A_0))$ is*

$$(T_0(t)f)_j(s) = \begin{cases} \mathrm{e}^{\xi_j(s,\tilde{s}(t))} f_j(\tilde{s}(t)), & \textit{if } 0 \leq t \leq \tau_j(s,1), \\ 0, & \textit{otherwise.} \end{cases} \tag{2.19}$$

PROOF — If we write

$$(S_0(t)f)_j(s) = \begin{cases} \mathrm{e}^{\xi_j(s,\tilde{s}(t))} f_j(\tilde{s}(t)), & \text{if } 0 \leq t \leq \tau_j(s,1), \\ 0, & \text{otherwise,} \end{cases} \tag{2.20}$$

we have to prove that $S_0(t) = T_0(t)$ for every $t \geq 0$. Observe that $\xi_j(s,\cdot)$ is continuous with $\xi_j(s,s) = 0$ for every $s \in [0,1]$. Furthermore, by the continuity of $\tau_j(s,\cdot)$,

$$\tilde{s}(0) = s \text{ and } \lim_{t\to 0} \tilde{s}(t) = s. \tag{2.21}$$

From these properties follows that $S_0(0)f = f$ and $(S_0(t))_{t\geq 0}$ is strongly continuous in $t = 0$.
Since

$$\tau_j\left(s, \widetilde{\tilde{s}(u)}(t)\right) = \tau_j(s, \tilde{s}(u)) + \tau_j\left(\tilde{s}(u), \widetilde{\tilde{s}(u)}(t)\right) = u + t = \tau_j(s, \tilde{s}(u+t)),$$

we have

$$\widetilde{\tilde{s}(u)}(t) = \tilde{s}(u+t).$$

So for $0 \leq u + t \leq \tau_j(s,1)$,

$$\begin{aligned}
(S_0(u+t)f)_j(s) &= \mathrm{e}^{\xi_j(s,\tilde{s}(u+t))} f_j(\tilde{s}(u+t)) = \\
&= \mathrm{e}^{\xi_j(s,\tilde{s}(u))}\mathrm{e}^{\xi_j\left(\tilde{s}(u),\widetilde{\tilde{s}(u)}(t)\right)} f_j\left(\widetilde{\tilde{s}(u)}(t)\right) = (S_0(u)(S_0(t)f))_j(s),
\end{aligned}$$

since $u \le \tau_j(s,1)$ and $t \le \tau_j(\tilde{s}(u),1)$. If $u+t > \tau_j(s,1)$, then either $u > \tau_j(s,1)$ or $t > \tau_j(\tilde{s}(u),1)$, hence $\left(S_0(u+t)f\right)_j(s) = \left(S_0(u)\left(S_0(t)f\right)\right)_j(s) = 0$. In the same way we can prove that

$$S_0(u+t) = S_0(t)S_0(u)$$

also holds. Thus $\left(S_0(t)\right)_{t\ge 0}$ is a C_0-semigroup on X. Let B_0 denote its generator. We will show that $B_0 = A_0$ by proving that the resolvents of them coincide. By the well-known formula for the resolvent of a generator (see [EN00, (1.14)] we have for every $f \in X$, $s \in [0,1]$ and $j = 1,\dots,m$,

$$\left(R(\lambda,B_0)f\right)_j(s) = \int_0^\infty \mathrm{e}^{-\lambda t}\left(S_0(t)f\right)_j(s)\mathrm{d}t = \int_0^{\tau_j(s,1)} \mathrm{e}^{-\lambda t}\mathrm{e}^{\xi_j(s,\tilde{s}(t))} f_j\left(\tilde{s}(t)\right)\mathrm{d}t.$$

From (2.16) and (2.17),

$$\left(R(\lambda,A_0)f\right)_j(s) = \int_s^1 \mathrm{e}^{\int_s^\sigma \frac{q_j(u)}{c_j(u)}\mathrm{d}u - \lambda\int_s^\sigma \frac{1}{c_j(u)}\mathrm{d}u}\frac{1}{c_j(\sigma)} f_j(\sigma)\,\mathrm{d}\sigma. \tag{2.22}$$

In the last formula we want to substitute $\sigma = \tilde{s}(t)$. Therefore we compute

$$\begin{aligned}\tilde{s}'(t) &= \lim_{h\to 0}\frac{\tilde{s}(t+h)-\tilde{s}(t)}{h}\\ &= \lim_{h\to 0}\left[\frac{1}{\tilde{s}(t+h)-\tilde{s}(t)}\int_{\tilde{s}(t)}^{\tilde{s}(t+h)}\frac{1}{c_j(u)}\mathrm{d}u\right]^{-1} = c_j(\tilde{s}(t)),\end{aligned}$$

where we used the continuity of $\tilde{s}(\cdot)$. Hence, substituting the variable $\sigma = \tilde{s}(t)$ in (2.22) we have $\mathrm{d}\sigma = c_j(\tilde{s}(t))\mathrm{d}t = c_j(\sigma)\mathrm{d}t$. From this and the definitions of τ_j and ξ_j we obtain

$$\left(R(\lambda,A_0)f\right)_j(s) = \int_0^{\tau_j(s,1)} \mathrm{e}^{\xi_j(s,\tilde{s}(t))-\lambda t} f_j\left(\tilde{s}(t)\right)\mathrm{d}t = \left(R(\lambda,B_0)f\right)_j(s).$$

■

REMARK 2.2.5 — In the case that all the velocities are constant, $\left(T_0(t)\right)_{t\ge0}$ is the weighted translation semigroup

$$(T_0(t)f)_j(s) = \begin{cases} \mathrm{e}^{\int_0^t q_j(s+c_j\tau)\mathrm{d}\tau} f_j(s+c_jt), & s+c_jt \le 1,\\ 0, & \text{otherwise.}\end{cases} \tag{2.23}$$

PROOF — From $\tau_j(s,\tilde{s}(t)) = t$ follows that in this case $\tilde{s}(t) = s + c_jt$ holds. Then Lemma 2.2.4 implies the result. ■

In order to compute the spectrum of the generator A we use operator matrix techniques as developed by K.-J. Engel, R. Nagel, A. Rhandi (see [Eng99], [Nag97], [Rha97]). We extend A to an operator on the product space

$$\mathcal{X} := X \times \partial X.$$

For that purpose we first define the operator matrix

$$\mathcal{A}_0 := \begin{pmatrix} A_w & 0 \\ -L & 0 \end{pmatrix}, \quad D\left(\mathcal{A}_0\right) := D(A_w) \times \{0\}^n,$$

whose part on the closure of its domain

$$\overline{D\left(\mathcal{A}_0\right)} = \overline{D(A_w) \times \{0\}^n} = X \times \{0\}^n =: \mathcal{X}_0 \tag{2.24}$$

can be identified with $(A_0, D(A_0))$.

Using ideas of Greiner [Gre87] we are able to compute the resolvent of $\mathcal{A}_0$. For this we need the so-called *Dirichlet operator* (see e.g. [CENN03]), characterized in the following two lemmas.

LEMMA 2.2.6 — *The operator $L\mid_{\ker(\lambda - A_w)}$ is invertible for any $\lambda \in \rho(A_0) = \mathbb{C}$. We denote its inverse by*

$$D_\lambda := \left(L\mid_{\ker(\lambda - A_w)}\right)^{-1} : \partial X \to \ker\left(\lambda - A_w\right) \tag{2.25}$$

and call it the corresponding Dirichlet operator.

PROOF — Observe that the conditions of [Gre87, Lemma 1.2] are fulfilled (use Remark 2.1.1), hence $L\mid_{\ker(\lambda - A_w)}$ is an isomorphism of $\ker\left(\lambda - A_w\right)$ onto ∂X, and the statement follows. ∎

In order to determine D_λ explicitly we use (2.17) and the notation

$$\begin{aligned} E_\lambda := \epsilon_\lambda(1) = \operatorname{diag}\left(\mathrm{e}^{-\xi_j(0,1)+\lambda\tau_j(0,1)}\right)_{j=1,\dots,m} = \\ = \operatorname{diag}\left(\exp\left(\int_0^1 \frac{-q_j(s)+\lambda}{c_j(s)}\mathrm{d}s\right)\right)_{j=1,\dots,m}. \end{aligned} \tag{2.26}$$

LEMMA 2.2.7 — *The Dirichlet operator D_λ has the form*

$$D_\lambda = \epsilon_\lambda E_\lambda^{-1}(\Phi_w^-)^\top, \tag{2.27}$$

that is

$$\left(D_\lambda x\right)(s) = \epsilon_\lambda(s) \cdot \left[E_\lambda^{-1}(\Phi_w^-)^\top\right] x \text{ for any } x \in \partial X,\ s \in [0,1].$$

PROOF — We set $N_\lambda := \epsilon_\lambda E_\lambda^{-1}(\Phi_w^-)^\top$ and obtain

$$LN_\lambda = \Phi^-(\Phi_w^-)^\top = \mathbf{1}$$

by (1.5). We also need to show that

$$N_\lambda L\mid_{\ker(\lambda - A_w)} = I_{\ker(\lambda - A_w)}.$$

Observe, that the kernel of the operator $\lambda - A_w$ is spanned by the vectors

$$f(s) = \left(\mathrm{e}^{\int_0^s \frac{-q_j(\sigma)+\lambda}{c_j(\sigma)}\mathrm{d}\sigma} \cdot \left(E_\lambda^{-1}\right)_{jj} a_j\right)_{j=1,\dots,m} \quad \text{for some } \left(a_j\right)_{j=1,\dots m} \in \mathbb{C}^m$$

satisfying (2.5). This means that

$$f(1) = (a_j) = (\Phi_w^-)^\top x \text{ for some } x \in \partial X,$$

i.e., by (2.17),

$$f = \epsilon_\lambda E_\lambda^{-1}(\Phi_w^-)^\top x = N_\lambda x \text{ for some } x \in \partial X.$$

Hence,

$$Lf = x \text{ and } N_\lambda L f = N_\lambda x = f,$$

which implies (2.27). ∎

A simple computation now yields a formula for the resolvent of $\mathcal{A}_0$. First we state a lemma which can be found in [Gre87, Lemma 1.2].

LEMMA 2.2.8 — *For every $\lambda \in \rho(A_0) = \mathbb{C}$ we have*

$$D(A_w) = \ker(\lambda - A_w) \oplus D(A_0). \tag{2.28}$$

Furthermore, the corresponding projections in $D(A_w)$ are $D_\lambda L|_{D(A_w)}$ onto $\ker(\lambda - A_w)$, and $R(\lambda, A_0)(\lambda - A_w)$ onto $D(A_0)$.

NOTATION 2.2.9 — In the following, the identity operators on the spaces X resp. $\mathcal{X}$ will be denoted by I_X resp. by $\mathcal{I}$.

PROPOSITION 2.2.10 — *For every $\lambda \in \mathbb{C}$, the resolvent of $\mathcal{A}_0$ is given by*

$$R(\lambda, \mathcal{A}_0) = \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix}. \tag{2.29}$$

PROOF — For $\lambda \in \mathbb{C}$ we show that the operator matrix

$$\mathcal{R}_\lambda := \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix}$$

defines the right inverse of $(\lambda - \mathcal{A}_0, D(A_w) \times \{0\}^n)$ on $\mathcal{X}$. We compute formally

$$\begin{aligned} (\lambda - \mathcal{A}_0) \cdot \mathcal{R}_\lambda &= \begin{pmatrix} \lambda - A_w & 0 \\ L & \lambda \end{pmatrix} \cdot \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (\lambda - A_0) R(\lambda, A_0) & (\lambda - A_w) D_\lambda \\ LR(\lambda, A_0) & LD_\lambda \end{pmatrix}. \end{aligned}$$

From the definition (2.25) of the Dirichlet operator follows that $(\lambda - A_w) D_\lambda = 0$ and $LD_\lambda = \mathbf{1}$. From the definition (2.13) of A_0 follows that $LR(\lambda, A_0) = 0$, hence

$$(\lambda - \mathcal{A}_0) \cdot \mathcal{R}_\lambda = \begin{pmatrix} I_X & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \mathcal{I}.$$

To prove the left-inverse property we compute again formally

$$\begin{aligned} \mathcal{R}_\lambda \cdot (\lambda - \mathcal{A}_0) &= \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda - A_w & 0 \\ L & \lambda \end{pmatrix} |_{D(A_w) \times \{0\}^n} \\ &= \begin{pmatrix} R(\lambda, A_0)(\lambda - A_w) + D_\lambda L|_{D(A_w)} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

By Lemma 2.2.8,

$$R(\lambda, A_0)(\lambda - A_w) + D_\lambda L|_{D(A_w)} = I_{D(A_w)}.$$

Using this we have

$$\mathcal{R}_\lambda \cdot (\lambda - \mathcal{A}_0) = \mathcal{I}_{D(A_w) \times \{0\}^n},$$

and this is what we wanted to prove. ∎

In the next step we add the operator matrix

$$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}, \quad D(\mathcal{B}) := D(M) \times \partial X$$

to $\mathcal{A}_0$ and obtain an operator on $\mathcal{X}$ given by

$$\begin{aligned} D(\mathcal{A}) &:= D(\mathcal{A}_0) = D(A_w) \times \{0\}^n, \\ \mathcal{A} &:= \mathcal{A}_0 + \mathcal{B} = \begin{pmatrix} A_w & 0 \\ M - L & 0 \end{pmatrix}. \end{aligned} \tag{2.30}$$

REMARK 2.2.11 — The part of the operator matrix $\mathcal{A}$ in $\mathcal{X}_0$ (see (2.24)) is

$$\begin{aligned} D(\mathcal{A}\,|_{\mathcal{X}_0}) &= D(A) \times \{0\}^n, \\ \mathcal{A}\,|_{\mathcal{X}_0} &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence it can be identified with the operator A on X.

The extension of the operator A to the operator matrix $\mathcal{A}$ helps to determine the spectrum of A using a simple perturbation argument (see [CENN03] for a systematic exposition of this approach). As a result, $\sigma(A)$ can be determined by a "characteristic equation" in $\partial X = \mathbb{C}^n$. This is based on the fact that for every $\lambda \in \mathbb{C}$, the product MD_λ is well-defined and yields an operator on ∂X – that is a $n \times n$ matrix.

PROPOSITION 2.2.12 — *Let A and $\mathcal{A}$ be the operators defined above on X and $\mathcal{X}$, respectively. Then the following assertions hold.*

1. *For every $\lambda \in \mathbb{C}$ we have*

$$\lambda \in \sigma(\mathcal{A}) \Longleftrightarrow \lambda \in \sigma(A) \Longleftrightarrow 1 \in \sigma(MD_\lambda).$$

2. *For every $\lambda \in \rho(A) = \rho(\mathcal{A})$ the resolvents of A and $\mathcal{A}$ are*

$$R(\lambda, A) = \left(I_X + D_\lambda (\mathbf{1} - MD_\lambda)^{-1} M\right) R(\lambda, A_0) \tag{2.31}$$

and

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A) & D_\lambda (\mathbf{1} - MD_\lambda)^{-1} \\ 0 & 0 \end{pmatrix}. \tag{2.32}$$

PROOF — Since $\lambda \in \rho(\mathcal{A}_0)$ for every $\lambda \in \mathbb{C}$, we can decompose

$$\lambda - \mathcal{A} = \lambda - \mathcal{A}_0 - \mathcal{B} = (\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0))(\lambda - \mathcal{A}_0). \tag{2.33}$$

Observe that $\lambda - \mathcal{A}$ is invertible if and only if $\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0)$ is invertible, and in this case its inverse is

$$R(\lambda, \mathcal{A}) = R(\lambda, \mathcal{A}_0)(\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0))^{-1}.$$

By (2.29), we have

$$\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0) = \begin{pmatrix} I_X & 0 \\ -MR(\lambda, A_0) & \mathbf{1} - MD_\lambda \end{pmatrix}. \tag{2.34}$$

It is easy to see that this operator matrix is invertible if and only if $\mathbf{1} - MD_\lambda$ is invertible, and in this case

$$(\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0))^{-1} = \begin{pmatrix} I_X & 0 \\ (\mathbf{1} - MD_\lambda)^{-1} MR(\lambda, A_0) & (\mathbf{1} - MD_\lambda)^{-1} \end{pmatrix}.$$

Hence, $\lambda \in \sigma(\mathcal{A})$ if and only if $1 \in \sigma(MD_\lambda)$. From these identities we also obtain the formula for the resolvent of $\mathcal{A}$:

$$\begin{aligned} R(\lambda, \mathcal{A}) &= \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} I_X & 0 \\ (\mathbf{1} - MD_\lambda)^{-1} MR(\lambda, A_0) & (\mathbf{1} - MD_\lambda)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} R(\lambda, A_0) + D_\lambda (\mathbf{1} - MD_\lambda)^{-1} MR(\lambda, A_0) & D_\lambda (\mathbf{1} - MD_\lambda)^{-1} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Its upper-left part is obviously the resolvent of A since A is the part of $\mathcal{A}$ on $X \times \{0\}^n$. From our computations follows the form given in (2.31). Thus the assertions are proved. ∎

We now state two consequences for the resolvents and the (point) spectrum of A and $\mathcal{A}$.

COROLLARY 2.2.13 — *The resolvents $R(\lambda, A)$ and $R(\lambda, \mathcal{A})$ are compact for all $\lambda \in \rho(A)$.*

PROOF — Observe that the domain of A is contained in $(W^{1,1}[0,1])^m$ that is compactly imbedded in X. So, by [EN00, Proposition II.4.25], the operators $R(\lambda, A)$ are compact. Since the other entries of $R(\lambda, \mathcal{A})$ have finite range, the corresponding statement for $\mathcal{A}$ follows from the form (2.32). ∎

COROLLARY 2.2.14 — *For $\lambda \in \sigma(A)$ and $x \in \partial X$ the following properties are equivalent.*

(a)	$MD_\lambda x = x$	(b)	$D_\lambda M(D_\lambda x) = D_\lambda x$
(c)	$AD_\lambda x = \lambda D_\lambda x$	(d)	$\mathcal{A}\binom{D_\lambda x}{0} = \lambda \binom{D_\lambda x}{0}$

PROOF — From the above Corollary 2.2.13 follows that the operators A and $\mathcal{A}$ have only point spectrum (see [EN00, Corollary IV.1.19]). Let now $0 \neq \binom{f}{y} \in D(\mathcal{A})$ be an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ. Formula (2.30) obviously implies that $\binom{f}{y}$ is an eigenvector if and only if $y = 0$ and $f \in D(A)$ is the appropriate eigenvector of A, that is

$$Af = \lambda f.$$

By Lemma 2.2.6, this is equivalent to the fact that

$$f = D_\lambda x \text{ for some } x \in \partial X$$

such that

$$Lf = Mf.$$

From this we obtain

$$Lf = LD_\lambda x = x = MD_\lambda x,$$

hence the equivalence of (c) and (d) is proved and they imply (a). If

$$x = MD_\lambda x,$$

then

$$LD_\lambda x = x = MD_\lambda x,$$

therefore $D_\lambda x$ is an eigenfunction of A corresponding to λ, hence (a) implies (c). Applying D_λ to both sides of (a) implies (b) and by applying L to both sides of (b) follows (a). ∎

The operator MD_λ appearing in the characteristic equation is actually an $n \times n$ matrix and will play an important role in the following. We write

$$\mathbf{A}_\lambda := MD_\lambda = \left(\Phi^+ \otimes \delta_0\right)\left(\epsilon_\lambda E_\lambda^{-1} (\Phi_w^-)^\top\right) = \Phi^+ E_\lambda^{-1} (\Phi_w^-)^\top$$

having entries

$$(\mathbf{A}_\lambda)_{ip} = \begin{cases} \omega_{pj}^- \mathrm{e}^{\xi_j(0,1) - \lambda \tau_j(0,1)}, & \text{if } \mathsf{v}_i = \mathsf{e}_j(0) \text{ and } \mathsf{v}_p = \mathsf{e}_j(1), \\ 0, & \text{else,} \end{cases} \tag{2.35}$$

where we used (2.26). It is a *weighted (transposed) adjacency matrix* of G, as defined in Definition 1.3.8.

Let us investigate the matrix $\mathbf{A}_0$ using $\sum_{j=1}^m \omega_{ij}^- = 1$. If $q_l \le 0$ for all l, then by (2.15), the column sums of $\mathbf{A}_0$ are all less than or equal to 1. Therefore in this case $\|\mathbf{A}_\lambda\|_1 < \|\mathbf{A}_0\|_1 \le 1$ for $\operatorname{Re}\lambda > 0$, which implies the following.

COROLLARY 2.2.15 — *For every $\lambda \in \mathbb{C}$ we have*

$$\lambda \in \sigma(A) \Longleftrightarrow \det(\mathbf{1} - \mathbf{A}_\lambda) = 0. \tag{2.36}$$

In particular, if $q_l \le 0$ for all l, this implies

$$\lambda \in \rho(A) \text{ for } \operatorname{Re}\lambda > 0. \tag{2.37}$$

COROLLARY 2.2.16 — *If $c_j \equiv c$ and $q_j \equiv 0$ for $j = 1, \dots, m$, as in Corollary 2.1.6, we have*

$$\mathbf{A}_\lambda = \mathrm{e}^{-\frac{\lambda}{c}} \tilde{\mathbf{A}},$$

which is a scalar multiple of a weighted transposed adjacency matrix $\tilde{\mathbf{A}}$ given as

$$(\tilde{\mathbf{A}})_{ip} := \begin{cases} \omega_{pj}^-, & \text{if } \mathsf{v}_i = \mathsf{e}_j(0) \text{ and } \mathsf{v}_p = \mathsf{e}_j(1), \\ 0, & \text{else.} \end{cases} \tag{2.38}$$

In particular,

$$\lambda \in \sigma(A) \Longleftrightarrow \mathrm{e}^{\frac{\lambda}{c}} \in \sigma\left(\tilde{\mathbf{A}}\right). \tag{2.39}$$

REMARK 2.2.17 — Clearly, this matrix is also obtained as

$$\tilde{\mathbf{A}} = \Phi^+ \left(\Phi_w^-\right)^\top$$

and is column stochastic by (1.4). It coincides with the weighted transposed adjacency matrix (1.7) with $\Phi_w^+ = \Phi^+$.

For the study of the asymptotic behavior of the semigroup generated by A we use the *spectral bound*

$$\tilde{q} := s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \tag{2.40}$$

of A. In the case of positive semigroups on L^p-spaces this number characterizes the exponential growth of the semigroup, see [EN00, Theorem VI.1.15]).

REMARK 2.2.18 — For the *growth bound*

$$\omega_0 = \inf\left\{w \in \mathbb{R} : \exists M_w \geq 1 \text{ such that } \|T(t)\| \leq M_w e^{wt} \text{ for all } t \geq 0\right\}$$

of the semigroup we have

$$\omega_0 = \tilde{q} \in \sigma(A), \tag{2.41}$$

if $\tilde{q} > -\infty$.

This and more subtle spectral properties of A and of $(T(t))_{t\geq 0}$ will allow us to describe the asymptotic behavior of the system.

§ 2.3 ASYMPTOTIC BEHAVIOR

To obtain results on the asymptotic behavior of the semigroup on X, we first restrict ourselves to networks with strongly connected graphs (see Definition 1.2.3). It turns out (see Lemma 2.3.2 below) that for such graphs our semigroup becomes irreducible. This fundamental property of positive semigroups and its consequences are systematically investigated in [Nag86, Chapter C-III.3] and we use the following characterization.

DEFINITION 2.3.1 — A positive semigroup on $L^1(\Omega, \mu)$, μ a σ-finite measure, with generator A is *irreducible* if for all $\lambda > s(A)$ the resolvent $R(\lambda, A)$ maps positive nonzero functions to strictly positive functions.

In the following we will see how the underlying graph structure can be related to this property. By Proposition 1.3.10, the graph G is strongly connected if and only if $\mathbf{A}_\lambda$ from (2.35) is irreducible for one/all $\lambda \in \mathbb{R}$. Using this fact, we can relate the irreducibility of our semigroup on X to the strong connectedness of the underlying graph.

LEMMA 2.3.2 — *Let the graph G be strongly connected. Then the semigroup $(T(t))_{t\geq 0}$ is irreducible.*

PROOF — It suffices to show that for $\lambda > \tilde{q}$ and $f > 0$, the function $R(\lambda, A)f$ is strictly positive. By (2.31) this means that for $0 < f \in X$

$$R(\lambda, A_0) f + D_\lambda (\mathbf{1} - MD_\lambda)^{-1} MR(\lambda, A_0) f \gg 0. \tag{2.42}$$

Take an arbitrary $\lambda > \tilde{q}$. First note that, due to (2.16), $R(\lambda, A_0) f \in X$ is strictly positive everywhere except on the largest interval $(1-\varepsilon, 1]$ for which $f \mid_{(1-\varepsilon,1]} = 0$. Applying M to it we obtain a vector $\mathbb{R}^m \ni d \gg 0$, see (2.8). Observe that under our assumptions the matrix $MD_\lambda = \mathbf{A}_\lambda$ is positive and irreducible. From the form (2.35) of its entries follows by [Sch74, Corollary I.6.4] that its spectral radius $r(\mathbf{A}_\lambda)$ is a strictly monotone decreasing function of λ satisfying $\lim_{\lambda\to+\infty} r(\mathbf{A}_\lambda) = 0$. The positivity yields $r(\mathbf{A}_\lambda) \in \sigma(\mathbf{A}_\lambda)$, see [Sch74, Proposition I.2.3] – thus, being a spectrum point, $r(\mathbf{A}_\lambda)$ is also a continuous function of the entries of $\mathbf{A}_\lambda$ hence of λ. Furthermore, by (2.41), $1 \in \sigma(\mathbf{A}_{\tilde{q}})$ and $1 \notin \sigma(\mathbf{A}_\lambda)$ for $\lambda > \tilde{q}$. These facts imply that

$$r(\mathbf{A}_{\tilde{q}}) = 1 > r(\mathbf{A}_\lambda) \text{ for every } \lambda > \tilde{q}.$$

Now from [Sch74, Proposition I.6.2] follows that $(\mathbf{1} - \mathbf{A}_\lambda)^{-1}$ is strictly positive for $\lambda > \tilde{q}$. Hence $(\mathbf{1} - MD_\lambda)^{-1} d \gg 0$, and applying D_λ to it we obtain a vector of positive multiples of exponential functions – which is also strictly positive, see (2.27). Adding it to the positive function $R(\lambda, A_0) f$, we finally obtain (2.42). ∎

Using the irreducibility of the adjacency matrix, we can prove the following result on the asymptotic behavior of $(T(t))_{t\geq 0}$.

PROPOSITION 2.3.3 — *Assume that G is strongly connected. If $q_l \leq 0$ for all $l = 1, \ldots, m$ and there exists at least one index j such that $q_j \neq 0$ (that is $q_j < 0$ on a set of positive measure), then $\tilde{q} < 0$, hence the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable.*

PROOF — From Corollary 2.2.15 follows that $\tilde{q} \leq 0$ holds, hence we only have to prove that $\tilde{q} \neq 0$. By (2.36) and (2.41) this means $1 \in \rho(\mathbf{A}_0)$. Let $\mathbf{A}_{0,1} := \mathbf{A}_0$ and let $\mathbf{A}_{0,2}$ be the weighted adjacency matrix for $\lambda = 0$ in the case when we replace q_j by 0. From Proposition 1.3.10 follows that both matrices are irreducible. With the notation of [Sch74, Chapter I], from the form (2.15) of ξ_j and from (2.35) follows that $|\mathbf{A}_{0,1}| \leq |\mathbf{A}_{0,2}|$, and there is at least one entry in the first matrix that is strictly less than the same entry in the second one. Using [Sch74, Corollary I.6.4] we obtain that $r(\mathbf{A}_{0,1}) < r(|\mathbf{A}_{0,2}|)$. Since $r(|\mathbf{A}_{0,2}|) \leq \|\mathbf{A}_{0,2}\|_1 \leq 1$, we have that $\det(\mathbf{1} - \mathbf{A}_{0,1}) \neq 0$ and so $1 \in \rho(\mathbf{A}_0)$. ∎

Because of (2.41), the behavior of the semigroup $(T(t))_{t\geq 0}$ is governed by the constant $\tilde{q}$: for $\tilde{q} > 0$ the flow blows up, while for $\tilde{q} < 0$ it decays exponentially. To obtain a finer description, we work with the rescaled semigroup $\tilde{T}(t) := \mathrm{e}^{-\tilde{q}t} T(t)$. In the following lemma we summarize the properties of $(\tilde{T}(t))_{t\geq 0}$ which follow directly from those of $(T(t))_{t\geq 0}$.

LEMMA 2.3.4 — *The rescaled semigroup $(\tilde{T}(t))_{t\geq 0}$ is positive and strongly continuous on X, and its generator $\tilde{A} := A - \tilde{q} I_X$ satisfies $s(\tilde{A}) = 0 \in \sigma(\tilde{A})$. Furthermore, if the graph is strongly connected, $(\tilde{T}(t))_{t\geq 0}$ is irreducible.*

We also obtain that this semigroup is bounded.

THEOREM 2.3.5 — *Assume that the graph is strongly connected. Then the semigroup $(\tilde{T}(t))_{t\geq 0}$ is bounded on X.*

PROOF — From the Banach-Steinhaus theorem follows that it is enough to prove that for all $g \in X,\ f \in X'$ there exists $K_{g,f} > 0$ such that

$$|\langle g, \tilde{T}(t)' f\rangle| \leq K_{g,f},\ t \geq 0, \tag{2.43}$$

where $X' = L^\infty([0,1], \mathbb{C}^m)$ is the dual space of X. Using the positivity and irreducibility of the semigroup $(\tilde{T}(t))_{t\geq 0}$ and the compactness of $R(\lambda, \tilde{A})$ (see Corollary 2.2.13) we obtain that $s(\tilde{A}) = 0$ is a first order pole of the resolvent. By [Nag86, Proposition C-III.3.5] it admits a strictly positive eigenvector $h \in X'$ also for $\tilde{A}'$. From the form of $\tilde{A}$ and $\tilde{A}'$ follows that h is an exponential function, hence we can assume that $h \geq 1$. Since $\tilde{A}'h = 0$, we have that $\tilde{T}'(t)h = h$ for all $t \geq 0$, see [EN00, Proposition IV.2.18] and [EN00, Theorem IV.3.7]. To prove (2.43), take an arbitrary $f \in X'$. Then $|f| \leq \|f\|_\infty \cdot h$. From the positivity of the adjoint semigroup follows that

$$|\tilde{T}(t)'f| \leq \tilde{T}(t)'|f| \leq \|f\|_\infty \cdot (\tilde{T}(t)'h),$$

hence

$$\|\tilde{T}(t)'f\|_\infty \leq \|f\|_\infty \cdot \|\tilde{T}(t)'h\|_\infty = \|f\|_\infty \cdot \|h\|_\infty .$$

From this we obtain

$$|\langle g, \tilde{T}(t)'f\rangle| \leq \|g\|_1 \cdot \|f\|_\infty \cdot \|h\|_\infty \text{ for all } g \in X,\ f \in X',$$

where $\|h\|_\infty$ is fixed. ∎

Having a bounded irreducible semigroup $(\tilde{T}(t))_{t\geq 0}$ with compact resolvent of its generator, we can use the theory developed in [EN00, Section V.2.C] and obtain the following decomposition of the state space X.

PROPOSITION 2.3.6 — *Let the graph be strongly connected. Then the following properties hold for the rescaled semigroup.*

1. *There is a projection $Q : X \to X$, hence a decomposition*

$$X = X_1 \oplus X_2$$

with $X_1 = \operatorname{ran} Q$, $X_2 = \ker Q$ such that

$$X_1 = \overline{\operatorname{lin}} \left\{ f \in D(\tilde{A}) :\ \exists \alpha \in \mathbb{R} : \tilde{A}f = \mathrm{i}\alpha f \right\}$$

and

$$X_2 = \left\{ f \in X :\ \tilde{T}(t)f \to 0,\ t \to +\infty \right\}.$$

2. *Moreover, $X_0 := \ker \tilde{A}$ is one dimensional and is spanned by a strictly positive eigenvector.*

PROOF — Since $\tilde{A}$ has compact resolvent, the first statement is Corollary V.2.15 of [EN00].
Using the fact that $\tilde{A}$ has compact resolvent, we obtain by [EN00, Corollary IV.1.19] that all the elements of the (point) spectrum of $\tilde{A}$ are poles of the resolvent $R(\lambda, \tilde{A})$ with finite algebraic multiplicity. Since $s(\tilde{A}) = 0$ is an element of the spectrum (see (2.41)), the irreducibility of the semigroup $(\tilde{T}(t))_{t\geq 0}$ implies by [Nag86, Proposition C-III.3.5] that 0 is an algebraically simple pole and admits a strictly positive eigenvector for $\tilde{A}$, hence $2.$ holds.

∎

It turns out that the following properties of the velocities are decisive for the spectral properties of $\tilde{A}$ and for the asymptotic behavior of $(\tilde{T}(t))_{t\geq 0}$.

DEFINITION 2.3.7 — We say that $(LD_{\mathbb{Q}})$ $[(LI_{\mathbb{Q}})\,,\ \text{resp.}]$ holds if the numbers

$$\{\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1):\ \mathsf{e}_{j_1},\dots,\mathsf{e}_{j_k}\text{ form a cycle in }G\}$$

are strongly linearly dependent [independent, resp.] over $\mathbb{Q}$.

Here, we call a set of numbers *strongly linearly dependent* over $\mathbb{Q}$ if the quotient of *any* two elements is rational. If this does not hold, the set is (strongly) linearly independent. Strong linear dependency immediately implies the next simple fact.

LEMMA 2.3.8 — *Assume $(LD_{\mathbb{Q}})$. Then there exists a real number c such that*

$$c\big(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\big)\in\mathbb{N}$$

for all $\mathsf{e}_{j_1},\dots,\mathsf{e}_{j_k}$ that form a cycle in G.

We now state an important consequence of the condition $(LI_{\mathbb{Q}})$.

LEMMA 2.3.9 — *Assume $(LI_{\mathbb{Q}})$. Then for every fixed $\delta>0$ there exists $\rho\in\mathbb{R}$ arbitrarily large such that*

$$\big|\rho\big(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\big)-2\pi l\big|<\delta \tag{2.44}$$

for all $\mathsf{e}_{j_1},\mathsf{e}_{j_2},\dots,\mathsf{e}_{j_k}$ forming a cycle in G and for an appropriate $l\in\mathbb{Z}$ depending on the cycle.

PROOF — By simultaneous Diophantine approximation, see e.g. [Per60, Proposition V. 55], for any set of real numbers $\zeta_1,\dots,\zeta_k$ containing at least one irrational number and for any $\varepsilon>0$, there exists q arbitrarily large such that

$$|q\cdot\zeta_\nu-l_\nu|<\varepsilon \tag{2.45}$$

for all $\nu=1,\dots,k$ and for appropriate $l_\nu\in\mathbb{Z}$. Taking the quotients

$$\frac{\tau_{j_1}(0,1)+\cdots+\tau_{j_i}(0,1)}{\tau_{l_1}(0,1)+\cdots+\tau_{l_k}(0,1)}$$

for each pair of cycles $\{\mathsf{e}_{j_1},\dots,\mathsf{e}_{j_i}\}$ and $\{\mathsf{e}_{l_1},\dots,\mathsf{e}_{l_k}\}$, by condition $(LI_{\mathbb{Q}})$ we find at least one irrational number among them, say

$$\zeta_1:=\frac{\tau_{j_1}(0,1)+\cdots+\tau_{j_i}(0,1)}{\tau_{l_1}(0,1)+\cdots+\tau_{l_k}(0,1)}$$

for the cycles $\{\mathsf{e}_{j_1},\dots,\mathsf{e}_{j_i}\}$ and $\{\mathsf{e}_{l_1},\dots,\mathsf{e}_{l_k}\}$. Then define

$$\zeta_\nu:=\frac{\tau_{p_1}(0,1)+\cdots+\tau_{p_\nu}(0,1)}{\tau_{l_1}(0,1)+\cdots+\tau_{l_k}(0,1)}$$

for all cycles $Z_\nu=\{\mathsf{e}_{p_1},\dots,\mathsf{e}_{p_\nu}\}\,,\ \nu=2,\dots,N$ in G. Let us fix $\delta>0$. The inequality (2.45) for $\zeta_1,\zeta_2,\dots,\zeta_N$ and $\varepsilon=\frac{\delta}{2\pi}>0$ implies that we can find q arbitrarily large such that

$$\left|\frac{q}{\tau_{l_1}(0,1)+\cdots+\tau_{l_k}(0,1)}\big(\tau_{p_1}(0,1)+\cdots+\tau_{p_\nu}(0,1)\big)-l_\nu\right|<\frac{\delta}{2\pi}$$

for every cycle $Z_\nu=\{\mathsf{e}_{p_1},\dots,\mathsf{e}_{p_\nu}\}\,,\ \nu=2,\dots,N$ and appropriate integer l_ν. Multiplying both sides by 2π we obtain the desired result. ■

In the sequel we will treat the above two alternatives separately.

§ 2.4 THE $(LD_{\mathbb{Q}})$ CASE

Let us first investigate the characteristic equation (2.36) in the case when condition $(LD_{\mathbb{Q}})$ holds. Using the Sachs Theorem (see Theorem 1.3.16) and the form of the entries of $\mathbf{A}_\lambda$ given in (2.35), we can easily prove the following.

PROPOSITION 2.4.1 — *The determinant* $\det(\mathbf{1}-\mathbf{A}_\lambda)$ *has the form*

$$\det(\mathbf{1}-\mathbf{A}_\lambda) = 1 + a_1(\lambda) + \cdots + a_n(\lambda) \tag{2.46}$$

with

$$a_r(\lambda) = \sum_{p=1}^{r} (-1)^p \sum_{\substack{k_1+\cdots+k_p=r\\ Z_1,\ldots,Z_p}} \prod_{j=1}^{p} \Big(w_j \cdot \mathrm{e}^{-\lambda \sum_{\mathsf{e}_l \in Z_j} \tau_l(0,1)} \Big).$$

Here the second sum runs over all positive integers $k_1,\ldots,k_p$ *having sum* r *such that there exist vertex disjoint cycles* $Z_1,\ldots,Z_p$ *in the graph* G *having* $k_1,\ldots,k_p$ *vertices, respectively. The numbers* w_j *are defined as*

$$w_j := \prod_{\mathsf{e}_k \in Z_j} \omega_{ik}^- \cdot \ \mathrm{e}^{\xi_k(0,1)},$$

where $\omega_{ik}^- \neq 0$ *is uniquely determined by* e_k *in the cycle* Z_j.

PROOF — We have to apply Theorem 1.3.16 in the case $z=1$. To compute the coefficients $a_r = a_r(\lambda)$ from (1.10) we have to take the sum over sets $G_{\mathcal{L}}$ containing vertex disjoint unions of cycles such that the sum of the vertices in these cycles is at most r. The weight $W(G_{\mathcal{L}})$ is the product of the entries of $\mathbf{A}_\lambda$ corresponding to the edges in the cycles in $G_{\mathcal{L}}$, see (2.35). ∎

In the case condition $(LD_{\mathbb{Q}})$ holds we even have a simpler form for this determinant occurring in the characteristic equation. Let us fix a number c obtained in Lemma 2.3.8. We can take the greatest common divisor

$$l(c) := \gcd\Big\{c\big(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\big); \mathsf{e}_{j_1},\ldots,\mathsf{e}_{j_k} \text{ form a cycle in } G\Big\}$$

and observe that the fraction $\frac{l(c)}{c}$ does not depend on the special choice of c. Therefore the number

$$\gamma := \frac{l(c)}{c} \tag{2.47}$$

is well-defined. This leads to the following expression for the terms $a_r(\lambda)$ in (2.46):

$$a_r(\lambda) = \sum_{p=1}^{r} (-1)^p \sum_{\substack{k_1+\cdots+k_p=r\\ Z_1,\ldots,Z_p}} \prod_{j=1}^{p} w_j \cdot \left(\mathrm{e}^{-\lambda\gamma}\right)^{l_j} \tag{2.48}$$

with

$$l_j := \frac{1}{\gamma} \sum_{\mathsf{e}_l \in Z_j} \tau_l(0,1) \in \mathbb{N}.$$

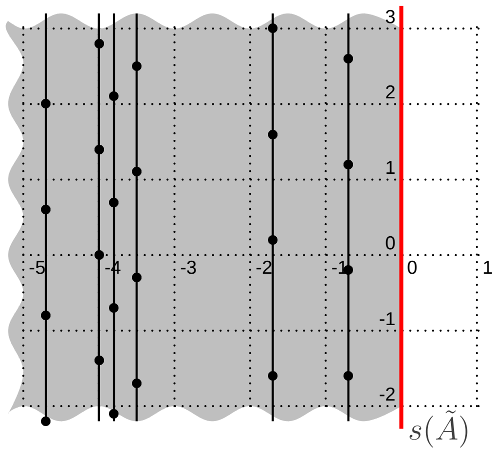

Figure 2.1: The spectrum of $\tilde{A}$ in the $(LD_{\mathbb{Q}})$ case

The form (2.48) implies that $\det(\mathbf{1}-\mathbf{A}_\lambda)$ can be written as

$$\det(\mathbf{1}-\mathbf{A}_\lambda) = p\left(\mathrm{e}^{-\lambda\gamma}\right) \tag{2.49}$$

with a polynomial p. This immediately leads to the following result on the spectrum of $\tilde{A}$.

PROPOSITION 2.4.2 — *Suppose that the condition $(LD_{\mathbb{Q}})$ is fulfilled. Then the eigenvalues of A, hence that of $\tilde{A}$, lie on finitely many vertical lines.*

PROOF — By (2.36) and (2.49) the zeros of $p\left(\mathrm{e}^{-\lambda\gamma}\right)$ are exactly the eigenvalues of A, hence the statement follows for A and $\tilde{A}$ since $\sigma(\tilde{A}) = \sigma(A) - \tilde{q}$. ∎

We are now able to relate the spectral properties of the generator to those of the semigroup as already shown in [KS04, Proposition 3.8].

PROPOSITION 2.4.3 [Circular Spectral Mapping Theorem] — *Suppose that the condition $(LD_{\mathbb{Q}})$ is satisfied. Then the semigroup $(\tilde{T}(t))_{t\geq 0}$ satisfies the so called* circular spectral mapping theorem*, that is*

$$\Gamma \cdot \mathrm{e}^{t\sigma(\tilde{A})} = \Gamma \cdot \sigma(\tilde{T}(t)) \setminus \{0\} \text{ for every } t \geq 0,$$

where Γ denotes the unit circle.

The subsequent proof is based on a result of Greiner and Schwarz [GS91, Corollary 1.2] and the following result on almost periodic functions (see [Cor68] and [Pit37]).

LEMMA 2.4.4 — *Let h be an analytic almost periodic function in the vertical strip $S_{(a,b)} = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ and $h(z) \neq 0$. Then $1/h(z)$ is analytic and almost periodic in any strip $S_{[a_1,b_1]} \subset S_{(a,b)}$. Moreover, if $h(z) = \sum_{j=1}^{\infty} a_j e^{zr_j}$, $r_j \in \mathbb{R}$, then $1/h(z) = \sum_{l=0}^{\infty} b_l e^{zs_l}$ for suitable $b_l, s_l \in \mathbb{R}$ in any strip $S_{[a_1,b_1]}$.*

We prove the statement for the original semigroup $(T(t))_{t\geq 0}$. The result then follows by $\sigma(\tilde{A}) = \sigma(A) - \tilde{q}$ and $\sigma(\tilde{T}(t)) = \mathrm{e}^{-\tilde{q}t}\sigma(T(t))$.

PROOF — The inclusion

$$\Gamma \cdot e^{t\sigma(A)} \subseteq \Gamma \cdot \sigma(T(t)) \setminus \{0\}$$

is the spectral inclusion theorem (see [EN00, Theorem IV.3.6]) that holds for all C_0-semigroups. Clearly, for $t = 0$ the opposite inclusion also holds. If $t > 0$, we have to prove that for the elements $\mu \in \rho(A)$ for which the entire vertical line $\operatorname{Re}\mu + i\mathbb{R}$ is contained in $\rho(A)$, we also have

$$e^{t(\operatorname{Re}\mu + i\mathbb{R})} = \Gamma \cdot e^{t\mu} \subseteq \rho(T(t)) \cup \{0\}.$$

In order to show this we use Greiner's criterion from [GS91, Corollary 1.2]. Take an element $e^{t\lambda} \in \Gamma \cdot e^{t\mu}$. We then have to prove that $\lambda + i(2\pi/t)\mathbb{Z} \subseteq \rho(A)$ and that the sequence

$$S_N := \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} R(\lambda + i(2\pi/t)k, A), \; N \in \mathbb{N}, \tag{2.50}$$

is bounded in $\mathcal{L}(X)$. The first fact is obvious from the assumption. To prove the boundedness of $(S_N)_{N\in\mathbb{N}}$, we use ideas from the proof of [GS91, Theorem 3.1]. By (2.27) and (2.31), the resolvent of A is

$$\begin{aligned} R(\lambda, A) &= D_\lambda (\mathbf{1} - MD_\lambda)^{-1} MR(\lambda, A_0) + R(\lambda, A_0) = \\ &= \epsilon_\lambda E_\lambda^{-1} \left(\Phi_w^-\right)^\top (\mathbf{1} - \mathbf{A}_\lambda)^{-1} MR(\lambda, A_0) + R(\lambda, A_0). \end{aligned}$$

For the sake of simplicity, we write

$$\begin{aligned} R_\lambda &:= R(\lambda, A_0), \\ \lambda_k &:= \lambda + i(2\pi/t)k. \end{aligned}$$

So, S_N has the form

$$S_N = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} \left(\epsilon_{\lambda_k} E_{\lambda_k}^{-1} \left(\Phi_w^-\right)^\top (\mathbf{1} - \mathbf{A}_{\lambda_k})^{-1} MR_{\lambda_k} + R_{\lambda_k} \right).$$

We can now estimate its L^1-norm by

$$\begin{aligned} \|S_N\|_1 \leq & \left\| \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} \epsilon_{\lambda_k} E_{\lambda_k}^{-1} \left(\Phi_w^-\right)^\top (\mathbf{1} - \mathbf{A}_{\lambda_k})^{-1} MR_{\lambda_k} \right\|_1 \\ & + \left\| \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} R_{\lambda_k} \right\|_1 := \|U_N\|_1 + \|V_N\|_1. \end{aligned}$$

For the estimate of the term $\|V_N\|_1$ observe first that R_λ is the resolvent of the generator of a strongly continuous nilpotent semigroup $(T_0(t))_{t\geq 0}$, as we have seen in Lemma 2.2.4. For any semigroup $(T(t))_{t\geq 0}$ and its generator A the formula

$$R(\mu, A)\left(1 - \mathrm{e}^{-\mu u} T(u)\right) = \int_0^u \mathrm{e}^{-\mu s} T(s) \mathrm{d}s \text{ for } \mu \in \rho(A), \; u \geq 0,$$

holds. Take any $u > 0$ such that $u > \tau_j(0,1)$ for all j. Then from Lemma 2.2.4 follows that $T_0(u) = 0$. Using the above formula we obtain that

$$\begin{aligned} V_N &= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j} R\left(\lambda + i(2\pi/t)k, A_0\right) \\ &= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j}\int_0^u \mathrm{e}^{-i(2\pi/t)ks}e^{-\lambda s}T_0(s)\mathrm{d}s = \int_0^u \sigma_N(2\pi s/t)\mathrm{e}^{-\lambda s}T_0(s)\mathrm{d}s \\ &= \frac{t}{2\pi}\int_0^{\frac{2\pi u}{t}} \sigma_N(v)\mathrm{e}^{\frac{-\lambda t v}{2\pi}}T_0(\frac{tv}{2\pi})\mathrm{d}v, \end{aligned}$$

where

$$\sigma_N\left(v\right) = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j}\mathrm{e}^{-ivk} \text{ for } v \in \mathbb{R}.$$

An elementary computation shows that

$$\sigma_N\left(v\right) = \frac{1}{N}\frac{1-\cos Nv}{1-\cos v}, \tag{2.51}$$

hence σ_N is periodic with period 2π, and

$$\sigma_N\left(v\right) \geq 0 \text{ and } \int_0^{2\pi}\sigma_N\left(v\right)\mathrm{d}v = 2\pi. \tag{2.52}$$

Choosing $u = l \cdot t$ for an appropriate $1 \leq l \in \mathbb{N}$, it follows that

$$\left\|V_N\right\|_1 \leq l \cdot t \cdot C$$

with $C := \sup\left\{\left\|e^{-\lambda s}T_0(s)\right\| : 0 \leq s \leq u\right\}$. This estimate is independent of N, hence we only have to continue with $\left\|U_N\right\|_1$.
According to Proposition 2.4.2, our assumption implies that the zeros of the analytic function $h\left(\lambda\right) := \det\left(\mathbf{1} - \mathbf{A}_\lambda\right)$ lie on finitely many vertical lines, hence $h\left(\lambda\right) \neq 0$ on a strip $S_{(\alpha,\beta)}$ containing λ. In the case when $(LD_{\mathbb{Q}})$ holds, $h(\lambda)$ has the form (2.49), hence is a finite linear combination of exponential functions. Therefore we can apply Lemma 2.4.4 for $h(\lambda)$, and using the well-known formula for the entries of the inverse matrix, we have that for $\lambda \in S_{(\alpha,\beta)}$

$$\left(\mathbf{1} - \mathbf{A}_\lambda\right)^{-1} = \sum_{l=0}^{\infty} B_l e^{\lambda s_l}$$

for suitable $B_l \in M_n(\mathbb{R}), s_l \in \mathbb{R}$, and this series converges absolutely. Continuing the estimate of $\left\|U_N f\right\|_1$, we obtain by using (2.8), (2.16), and (2.17)

$$\begin{aligned} &\left\|U_N f\right\|_1 = \\ &= \sum_{p=1}^{m}\int_0^1\left|\frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j}\left(\epsilon_{\lambda_k}\left(s\right)E_{\lambda_k}^{-1}\left(\Phi_w^-\right)^\top\sum_{l=0}^{\infty}B_l e^{\lambda_k s_l}\Phi^+\int_0^1\epsilon_{\lambda_k}\left(u\right)^{-1}C(u)^{-1}f\left(u\right)\mathrm{d}u\right)_p\right|\mathrm{d}s \\ &= \sum_{p=1}^{m}\int_0^1\left|\sum_{l=0}^{\infty}\frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{j}\mathrm{e}^{\xi_p(s,1)+\lambda_k(s_l-\tau_p(s,1))}\int_0^1\left(\left(\Phi_w^-\right)^\top B_l\Phi^+\epsilon_{\lambda_k}\left(u\right)^{-1}C(u)^{-1}f\left(u\right)\right)_p\mathrm{d}u\right|\mathrm{d}s. \end{aligned}$$

In order to proceed we introduce the $m \times m$ matrix

$$\Psi_l := \left(\Phi_w^-\right)^\top B_l \Phi^+ = \left(\psi_{ij}^l\right)_{m\times m}$$

and obtain for the p-th coordinate

$$\begin{aligned}
&\left\|\left(U_N f\right)_p\right\|_1 = \\
&= \int_0^1 \left| \sum_{l=0}^{\infty} \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} \int_0^1 \mathrm{e}^{\xi_p(s,1)+\lambda_k(s_l-\tau_p(s,1))} \sum_{h=1}^{m} \psi_{p,h}^l \mathrm{e}^{\xi_h(0,u)-\lambda_k\tau_h(0,u)} \frac{1}{c_h(u)} f_h(u)\,\mathrm{d}u \right| \mathrm{d}s \\
&= \int_0^1 \left| \sum_{l=0}^{\infty} \sum_{h=1}^{m} \psi_{p,h}^l \mathrm{e}^{\xi_p(s,1)+\lambda_k(s_l-\tau_p(s,1))} \int_0^1 \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=-j}^{j} \frac{1}{c_h(u)} \mathrm{e}^{\xi_h(0,u)-\lambda_k\tau_h(0,u)} f_h(u)\,\mathrm{d}u \right| \mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 \sum_{l=0}^{\infty} \sum_{h=1}^{m} \left|\psi_{p,h}^l \mathrm{e}^{\xi_p(s,1)+\lambda(s_l-\tau_p(s,1))}\right| \times \\
&\times \int_0^1 \left| \frac{1}{c_h(u)} \mathrm{e}^{\xi_h(0,u)-\lambda\tau_h(0,u)} \sigma_N\left(\frac{2\pi\left(\tau_p(s,1)-s_l+\tau_h(0,u)\right)}{t}\right) f_h(u)\,\mathrm{d}u \right| \mathrm{d}s \\
&\leq \sum_{l=0}^{\infty} \sum_{h=1}^{m} \left|c_{\lambda,p}\psi_{p,h}^l \mathrm{e}^{\lambda s_l}\right| \int_0^1 \left| f_h(u) \frac{1}{c_h(u)} \mathrm{e}^{\xi_h(0,u)-\lambda\tau_h(0,u)} \right| \times \\
&\times \int_0^1 \sigma_N\left(\frac{2\pi\left(\tau_p(s,1)-s_l+\tau_h(0,u)\right)}{t}\right) \mathrm{d}s\mathrm{d}u
\end{aligned}$$

with

$$c_{\lambda,p} := \max_{s\in[0,1]} \mathrm{e}^{\xi_p(s,1)-\lambda\tau_p(s,1)}.$$

Using the above properties (2.51) and (2.52) of the function $\sigma_N(v)$, by an appropriate variable substitution we obtain that

$$\left\|\left(U_N f\right)_p\right\|_1 \leq C_{\lambda,p} \cdot 2\pi k_{p,t} \cdot \sum_{l=0}^{\infty} \sum_{h=1}^{m} \left|\psi_{p,h}^l \mathrm{e}^{\lambda s_l}\right| \|f\|_1$$

with $k_{p,t} \in \mathbb{N}$ and

$$C_{\lambda,p} := c_{\lambda,p} \cdot \max_{1\leq h\leq m} \left\{ \sup_{u\in[0,1]} \left| \frac{1}{c_h(u)} \mathrm{e}^{\xi_h(0,u)-\lambda\tau_h(0,u)} \right| \right\}.$$

Summing up for $p = 1, ..., m$ and using the definition of $\psi_{p,h}^l$ we obtain

$$\begin{aligned}
\|U_N f\|_1 &\leq C_\lambda \cdot 2\pi \cdot \sum_{l=0}^{\infty} \sum_{h=1}^{m} \sum_{p=1}^{m} \left| \left(\left(\Phi_w^-\right)^\top B_l \Phi^+\right)_{p,h} \mathrm{e}^{\lambda s_l} \right| \|f\|_1 \\
&\leq C_\lambda \cdot 2\pi \cdot m \sum_{l=0}^{\infty} \left\| \left(\Phi_w^-\right)^\top B_l \Phi^+ \right\| \left|\mathrm{e}^{\lambda s_l}\right| \|f\|_1 \\
&\leq C_\lambda \cdot 2\pi \cdot m \left\| \left(\Phi_w^-\right)^\top \right\| \left\|\Phi^+\right\| \left(\sum_{l=0}^{\infty} \|B_l\| \left|\mathrm{e}^{\lambda s_l}\right| \right) \|f\|_1,
\end{aligned}$$

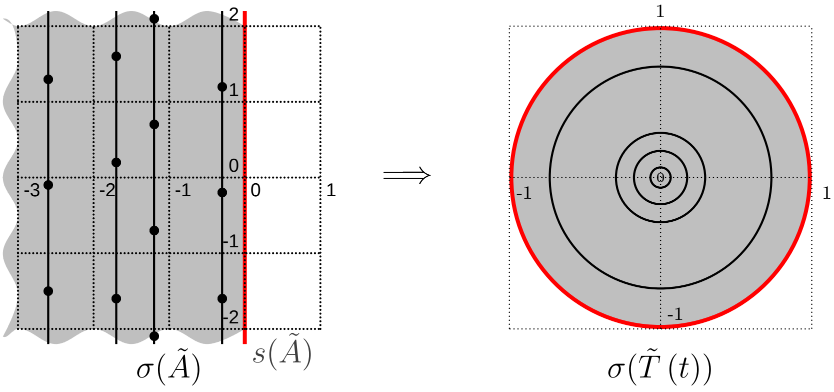

Figure 2.2: Circular Spectral Mapping Theorem

with

$$C_\lambda := \max_{1\le p\le m} C_{\lambda,p} k_{p,t}.$$

This completes the proof since the estimate is independent of $N \in \mathbb{N}$. ∎

The Circular Spectral Mapping Theorem and the above Lemma 2.4.2 imply that the spectrum $\sigma(\tilde{T}(t))$ lies on finitely many circles, where the largest one is the unit circle Γ (see Lemma 2.3.4). This immediately allows the following decomposition of hyperbolic type of the semigroup.

PROPOSITION 2.4.5 — *Suppose that condition $(LD_{\mathbb{Q}})$ holds and the graph is strongly connected. Then for the decomposition in Proposition 2.3.6.1 the following assertions are true.*

1. *X_1 and X_2 are (closed) $\tilde{T}(t)$-invariant subspaces.*
2. *The operators $S(t) := \tilde{T}(t)\mid_{X_1}$ form a bounded C_0-group on X_1.*
3. *The semigroup $\left(\tilde{T}(t)\mid_{X_2}\right)_{t\ge 0}$ is uniformly exponentially stable, hence*

$$\|\tilde{T}(t) - S(t)\|_X \le M e^{-\varepsilon t}$$

for some constants $M \ge 1$, $\varepsilon > 0$.

PROOF — Using Theorem 2.4.3, denote the second largest circle in $\sigma(\tilde{T}(t))$ by $\Gamma\cdot \mathrm{e}^{-t\cdot\eta}$ with $\eta > 0$. Take any $\delta > 0$ such that $\alpha := -\eta + \delta < 0$, then the spectrum of the rescaled semigroup $(\tilde{T}_\alpha(t)) := (\mathrm{e}^{-\alpha t}\tilde{T}(t))$ does not intersect the unit circle. Hence we can use [EN00, Theorem V.1.17] for $(\tilde{T}_\alpha(t))$ and obtain a decomposition that has the desired properties for the original semigroup. ∎

If the graph is strongly connected, our (rescaled) semigroup is irreducible (see Lemma 2.3.2). The Perron-Frobenius theory for positive irreducible semigroups (and the compactness of $R(\lambda, \tilde{A})$) imply that

$$\sigma(\tilde{A}) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z} \text{ for some } \alpha \ge 0,$$

where each $\mathrm{i}\alpha k$ is a simple pole of the resolvent (see [EN00, Theorem VI.1.12] or [Nag86, Section C-III]). In the following we want to identify α. The statement (2.41) and the form (2.49) of the characteristic equation (2.36) imply that $\lambda = \tilde{q}$ is a zero of $p(\mathrm{e}^{-\lambda\gamma}) = \det(\mathbf{1} - \mathbf{A}_\lambda)$, therefore all the numbers $\lambda = \tilde{q} + \mathrm{i}2\pi\frac{1}{\gamma}k$, $k \in \mathbb{Z}$, are also zeros of $\det(\mathbf{1} - \mathbf{A}_\lambda)$ – hence eigenvalues of A. So we obtain

$$\mathrm{i}2\pi\frac{1}{\gamma}\mathbb{Z} \subseteq \sigma_b(\tilde{A}), \tag{2.53}$$

where $\sigma_b(\tilde{A}) = \sigma(\tilde{A}) \cap \mathrm{i}\mathbb{R}$ denotes the boundary spectrum of $\tilde{A}$. Indeed, we show now that equality holds in (2.53). For this purpose we need the following lemma.

LEMMA 2.4.6 — *Let $\mathbf{B}_0 = (b_{i,p})_{i,p=1}^n$ be a positive irreducible matrix having a strictly positive vector x satisfying*

$$\mathbf{B}_0 x = x.$$

Let $\mathbf{B}$ denote any matrix obtained from $\mathbf{B}_0$ by multiplying each of its entries by a complex number having absolute value 1, that is

$$(\mathbf{B})_{i,p} = \mathrm{e}^{\mathrm{i}\vartheta_{i,p}} b_{i,p}.$$

Then $\det(\mathbf{1} - \mathbf{B}) = 0$ if and only if

$$\prod_{l=1}^{s} \mathrm{e}^{\mathrm{i}\vartheta_{i_l,i_{l+1}}} = 1$$

for every sequence $i_1, i_2, \ldots, i_s, i_{s+1} = i_1$.

PROOF — Observe first that by similarity transformation, we can assume $x = \underline{1} := (1, 1, \ldots, 1)$, hence

$$\mathbf{B}_0 \underline{1} = \underline{1}. \tag{2.54}$$

By definition, the determinant of any $n \times n$ matrix $\mathbf{D} = (d_{i,p})_{i,p=1}^n$ can be written as

$$\det \mathbf{D} = \sum_{\pi \in \mathcal{P}} (-1)^{\mathrm{sign}\pi} d_{1,\pi(1)} \cdot d_{2,\pi(2)} \cdot \cdots \cdot d_{n,\pi(n)}, \tag{2.55}$$

where $\mathcal{P}$ denotes the set of all n-permutations. Furthermore, any permutation defined as $\pi = \{(1,\pi(1)), (2,\pi(2)), \ldots, (n,\pi(n))\}$ can be (unambiguously) written as a product of disjoint cycles of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_s, i_1)$. The "if" part now follows by applying (2.55) to the matrix $\mathbf{1} - \mathbf{B}$ and using $\det(\mathbf{1} - \mathbf{B}_0) = 0$.
Suppose now that $\det(\mathbf{1} - \mathbf{B}) = 0$, that is there exists $y = (y_1, \ldots, y_n)$ such that $\mathbf{B}y = y$. Since $|\mathbf{B}| = \mathbf{B}_0$ and $\mathbf{B}_0 \underline{1} = \underline{1}$, we can apply [Sch74, Proposition V.7.4] and obtain

$$\mathbf{B} = \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \cdot \mathbf{B}_0 \cdot \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix}^{-1}. \tag{2.56}$$

That means that

$$b_{i,p}\mathrm{e}^{\mathrm{i}\vartheta_{i,p}} = b_{i,p}\frac{y_i}{y_p}$$

for all $i, p = 1, \ldots, n$. Computing

$$\prod_{l=1}^{s} b_{i_l,i_{l+1}} \mathrm{e}^{\mathrm{i}\vartheta_{i_l,i_{l+1}}} = \prod_{l=1}^{s} b_{i_l,i_{l+1}} \frac{y_l}{y_{l+1}} = \prod_{l=1}^{s} b_{i_l,i_{l+1}},$$

we obtain

$$\prod_{l=1}^{s} \mathrm{e}^{\mathrm{i}\vartheta_{i_l,i_{l+1}}} = 1$$

for every sequence $i_1, i_2, \ldots, i_s, i_{s+1} = i_1$, and this is what we wanted to prove.

■

COROLLARY 2.4.7 — *Suppose that $(LD_{\mathbb{Q}})$ is satisfied and that the graph is strongly connected. Then the boundary spectrum $\sigma_b(\tilde{A})$ is given by*

$$\sigma_b(\tilde{A}) = \mathrm{i}2\pi \frac{1}{\gamma} \mathbb{Z}, \tag{2.57}$$

where γ is defined in (2.47).

PROOF — By (2.53) we only have to prove that if $\mathrm{i}\beta \in \sigma_b(\tilde{A})$, then $\beta \in \frac{2\pi}{\gamma}\mathbb{Z}$. By the characteristic equation (2.36) we have $1 \in \sigma(\mathbf{A}_{\tilde{q}+\mathrm{i}\beta})$, and by (2.41), $1 \in \sigma(\mathbf{A}_{\tilde{q}})$. Applying the above Lemma 2.4.6 for $\mathbf{B}_0 = \mathbf{A}_{\tilde{q}}$ and $\mathbf{B} = \mathbf{A}_{\tilde{q}+\mathrm{i}\beta}$, and using (2.35) this can happen only if

$$\prod_{l=1}^{s} \mathrm{e}^{\mathrm{i}\beta\tau_{j_l}(0,1)} = 1,$$

that is

$$\beta \sum_{l=1}^{s} \tau_{j_l}(0,1) = 2\pi \cdot k_{Z_s} \text{ for some } k_{Z_s} \in \mathbb{Z}$$

for all $Z_s = \{\mathsf{e}_{j_1}, \ldots, \mathsf{e}_{j_s}\}$ forming a cycle in the graph. Taking c from the definition (2.47) of γ we have

$$c \cdot \frac{2\pi}{\beta} \cdot k_{Z_s} = c \cdot \sum_{l=1}^{s} \tau_{j_l}(0,1) \in \mathbb{Z}$$

for all cycle Z_s, hence

$$c \cdot \frac{2\pi}{\beta} \cdot \gcd \{k_{Z_s} : Z_s \text{ cycle } \} = c \cdot \gamma,$$

which implies the desired result. ■

Applying now the result of Nagel [Nag84, Theorem 4.3] generalized in [Nag86, C-IV, Theorem 2.14] we obtain that under $(LD_{\mathbb{Q}})$ and the strong connectedness of the graph the rescaled semigroup $(\tilde{T}(t))_{t\geq 0}$ behaves asymptotically as a periodic rotation group on $L^1(\Gamma)$ where Γ is the unit circle.

THEOREM 2.4.8 — *Suppose that the condition $(LD_{\mathbb{Q}})$ holds and that the graph G is strongly connected. Then the decomposition $X = X_1 \oplus X_2$ from Proposition 2.4.5 has the following additional properties.*

1. *X_1 is a closed sublattice of X isomorphic to $L^1(\Gamma)$.*
2. *The group $(S(t))_{t\geq 0}$ is isomorphic to the rotation group on $L^1(\Gamma)$ with period*

$$\begin{aligned}\tau &= \gamma \qquad (2.58)\\ &= \frac{1}{c}\gcd\left\{c\big(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\big) : \mathsf{e}_{j_1},\ldots,\mathsf{e}_{j_k} \text{ form a cycle in } G\right\},\end{aligned}$$

where c is any number such that $c\left(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\right) \in \mathbb{N}$ for all edges $\mathsf{e}_{j_1},\ldots,\mathsf{e}_{j_k}$ forming a cycle in G.

PROOF — By Lemma 2.3.4, the semigroup $(\tilde{T}(t))_{t\geq 0}$ is irreducible, positive and by Theorem 2.3.5 bounded. Since $s(\tilde{A}) = 0$ and because of the compactness of the resolvent, 0 is a pole of $R(\lambda, \tilde{A})$. By the above corollary, we also know that there are nonzero spectral points on the imaginary axis. So, all the hypotheses of [Nag86, C-IV, Theorem 2.14] are fulfilled, and we obtain the statements 1. and the first half of 2.
By [Nag86, C-IV, Lemma 2.12 (c)] the period τ equals $\frac{2\pi}{\alpha}$, where $\alpha \in \mathbb{R}$ is determined by

$$\sigma(\tilde{A}) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z}.$$

Due to (2.57), formula (2.58) holds. ∎

In less technical terms the above result can be expressed as follows.

COROLLARY 2.4.9 — *Under the assumptions of the above Theorem 2.4.8, the rescaled semigroup $(\tilde{T}(t))_{t\geq 0}$ is* asymptotically periodic *with period*

$$\tau = \frac{1}{c}\gcd\left\{c\big(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\big) : \mathsf{e}_{j_1},\ldots,\mathsf{e}_{j_k} \text{ form a cycle in } G\right\},$$

where c is any number such that $c\big(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\big) \in \mathbb{N}$ for all $\mathsf{e}_{j_1},\ldots,\mathsf{e}_{j_k}$ that form a cycle in G.

REMARK 2.4.10 — Observe that the period does not depend on the weights on the edges.

We extend our description of the asymptotic behavior when $(LD_{\mathbb{Q}})$ holds to the case when the underlying graph is not strongly connected. For the sake of simplicity, we assume that all $q_j = 0$, implying that the adjacency matrix $\mathbf{A}_0$ is column stochastic, hence $1 \in \sigma(\mathbf{A}_0)$ and by (2.37), $\tilde{q} = 0$. Using invariant strongly connected components of our directed graph, defined in Definition 1.2.4, we obtain the following result for the asymptotics.

THEOREM 2.4.11 — *Consider a flow in an arbitrary network modelled by the directed graph G, and assume that $(LD_{\mathbb{Q}})$ holds and $q_j = 0,\ j = 1,\ldots,m$. Then the corresponding rescaled semigroup behaves asymptotically as a direct sum of rotation groups. The period of these rotation groups are given by the modification of the formula* (2.58) *for each invariant strongly connected component of G.*

PROOF — By Proposition 2.4.5 we have a spectral decomposition $X = X_1 \oplus X_2$ of the state space such that $(S(t))_{t\geq 0} := (\tilde{T}(t)\mid_{X_1})_{t\geq 0}$ is a bounded C_0-group and $(\tilde{T}(t)\mid_{X_2})_{t\geq 0}$ is uniformly exponentially stable. By Proposition 2.3.6,

$$X_1 = \overline{\mathrm{lin}}\left\{f \in D(A) : \exists\beta \in \mathbb{R} \text{ such that } \tilde{A}f = \mathrm{i}\beta f\right\}.$$

Therefore, if $t \to +\infty$, the semigroup converges (exponentially) in norm to the bounded group $(S(t))_{t\geq 0}$ acting on the closed subspace generated by the eigenvectors that belong to the imaginary (that is, the boundary) spectrum of $\tilde{A}$. We want to prove that this limit is isomorphic to a direct sum of rotation groups with the proper periods.
For this purpose we first characterize the spectral values $\mathrm{i}\beta \in \sigma(\tilde{A})$, $\beta \in \mathbb{R}$. Taking into account the characteristic equation (2.36), we have to investigate in which case

$$1 \in \sigma(\mathbf{A}_{\mathrm{i}\beta})$$

for some $\beta \in \mathbb{R}$ holds.
Observe, that the positive matrix $\mathbf{A}_0$ is similar (via a permutation P of the canonical basis) to a block-triangular matrix, i.e.,

$$P^{-1}\mathbf{A}_0 P = \begin{pmatrix} Q_0^0 & 0 & \dots & 0 \\ B_1^0 & Q_1^0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ B_q^0 & 0 & \dots & Q_q^0 \end{pmatrix}, \tag{2.59}$$

where the diagonal blocks $Q_1^0, \dots, Q_q^0$ are irreducible and if the $k_0 \times k_0$ matrix Q_0^0 is non-empty, then at least one B_i^0 is nonzero (see [Sch74, Proposition I.8.8]). This form is unique up to permutations of the coordinates within each diagonal block and up to the order of $Q_1^0, \dots, Q_q^0$. It is easy to see that, since the zero-patterns of the matrices $\mathbf{A}_\lambda$ coincide for every λ, the same permutation matrix P yields an analogous block-form

$$P^{-1}\mathbf{A}_\lambda P = \begin{pmatrix} Q_0^\lambda & 0 & \dots & 0 \\ B_1^\lambda & Q_1^\lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ B_q^\lambda & 0 & \dots & Q_q^\lambda \end{pmatrix} \text{ for every } \lambda \in \mathbb{C}. \tag{2.60}$$

We now renumber the vertices of G such that the adjacency matrices $\mathbf{A}_\lambda$ have the above block-triangular form (2.60). Clearly, this does not change the spectral properties we need. From the block-triangular form (2.60) follows that

$$\sigma(\mathbf{A}_\lambda) = \bigcup_{p=0}^{q} \sigma\left(Q_p^\lambda\right) \text{ for every } \lambda \in \mathbb{C}.$$

We will show that the $k_0 \times k_0$ matrices $Q_0^{\mathrm{i}\beta}$ for $\beta \in \mathbb{R}$ do not contribute to the boundary spectrum of $\tilde{A}$. This means that if $\mathbf{A}_{\mathrm{i}\beta}x = x$, then the first k_0 coordinates of x have to be equal to 0. Let us first investigate the case $\beta = 0$, hence we assume $\mathbf{A}_0 x = x$. To the column stochastic matrix $\mathbf{A}_0$ we can apply [Sch74, Corollary of I.8.4] and obtain that x is contained in the direct sum of the minimal $\mathbf{A}_0$-invariant ideals in $\mathbb{C}^n$. By the proof of [Sch74, Proposition I.8.8] this direct sum is exactly the direct sum of the ideals spanned by the basis vectors that correspond to the blocks $Q_1^0, \dots, Q_q^0$. Hence we obtain that the first k_0 coordinates of x are 0. It means that 1 is not in the spectrum of the positive matrix Q_0^0. Furthermore, because of the column stochasticity of $\mathbf{A}_0$, all the column sums of Q_0^0 are less than or equal to 1, hence $r\left(Q_0^0\right) \leq 1$. Since at least one $B_i \neq 0$, the Perron-Frobenius theorem yields (see, e.g., [Sch74, Proposition I.2.3]) that $r\left(Q_0^0\right) < 1$. From the form (2.35) of the entries of $\mathbf{A}_\lambda$ follows that $\left|Q_0^{\mathrm{i}\beta}\right| = Q_0^0$ and using, e.g., the column-sum norm we obtain that $r\left(Q_0^{\mathrm{i}\beta}\right) = r\left(Q_0^0\right) < 1$. Hence $1 \notin \sigma\left(Q_0^{\mathrm{i}\beta}\right)$ for all $\beta \in \mathbb{R}$ and therefore

$$1 \in \sigma\left(\mathbf{A}_{\mathrm{i}\beta}\right) \Longleftrightarrow 1 \in \bigcup_{p=1}^{q} \sigma\left(Q_p^{\mathrm{i}\beta}\right). \tag{2.61}$$

For each $p \in \{1, \dots, q\}$, the irreducible block Q_p^{λ} is the weighted (transposed) adjacency matrix of a subgraph G_p of G, which is by [And91, Theorem 3.2] strongly connected. From the form (2.60) of the adjacency matrix of the whole graph G follows that that there are no outgoing edges of G_p. Hence G_p is an *invariant* strongly connected component, see Definition 1.2.4. This implies that the subspace $X^p \subset X$ of all functions having their support on the edges of G_p is invariant under the semigroup $(\tilde{T}(t))_{t\geq 0}$. We can apply Theorem 2.4.8 to the restricted positive *irreducible* semigroup $(\tilde{T}_p(t))_{t\geq 0} := (\tilde{T}(t)\mid_{X^p})_{t\geq 0}$. Hence its generator $\tilde{A}_p$ – which is the part of $\tilde{A}$ in X^p – satisfies

$$\sigma(\tilde{A}_p) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha_p\mathbb{Z} \text{ for some } \alpha_p \in \mathbb{R}.$$

By (2.36) we obtain the equivalences

$$1 \in \sigma\left(Q_p^{\mathrm{i}\beta}\right) \Longleftrightarrow \mathrm{i}\beta \in \sigma(\tilde{A}_p) \Longleftrightarrow \beta = \alpha_p k \text{ for some } k \in \mathbb{Z} \tag{2.62}$$

for each $\beta \in \mathbb{R}$. By Theorem 2.4.8 the semigroup $(\tilde{T}_p(t))_{t\geq 0}$ converges exponentially to a rotation group on a subspace X_1^p of X^p having the form

$$X_1^p = \overline{\mathrm{lin}}\left\{f \in D(\tilde{A}_p) : \tilde{A}_p f = \mathrm{i}\alpha_p k f \text{ for some } k \in \mathbb{Z}\right\}. \tag{2.63}$$

The period of the rotation is given by the formula (2.58) for the cycles in G_p. Since $X_1^p \subset X_1$ for each p, we conclude that

$$Y := X_1^1 \oplus \cdots \oplus X_1^q \subseteq X_1.$$

We will show that equality holds, in this way proving that the semigroup converges to a direct sum of rotation groups with the appropriate periods. For this purpose it suffices to show that if for some $\beta \in \mathbb{R}$ and $f \neq 0$ we have $\tilde{A}f = \mathrm{i}\beta f$, then $f \in Y$. By Corollary 2.2.14 we know that

$$\tilde{A}f = \mathrm{i}\beta f \Longleftrightarrow \mathbf{A}_{\mathrm{i}\beta}(Lf) = Lf.$$

Let $L^{(p)} := \Phi_p^- \otimes \delta_1$, $p = 0, \dots, q$, where Φ_p^- denotes the matrix obtained from the rows of Φ^- belonging to the vertices that correspond to the block Q_p^{λ} in the adjacency matrix. Similarly, let $D_{\lambda}^{(p)} := \epsilon_{\lambda} E_{\lambda}^{-1}(\Phi_{w,p}^-)^{\top}$ where $\Phi_{w,p}^-$ is obtained from Φ_w^- in the same way. Since $r(Q_0^{\mathrm{i}\beta}) < 1$, clearly

$$L^{(0)} f = 0.$$

Hence there exists $p \in \{1, \dots, q\}$ such that $L^{(p)} f \neq 0$ and

$$Q_p^{\mathrm{i}\beta} L^{(p)} f = L^{(p)} f.$$

Again by Corollary 2.2.14, this is equivalent to the fact that the identity

$$\tilde{A}_p\left(D_{\mathrm{i}\beta}^{(p)} L^{(p)} f\right) = \mathrm{i}\beta\left(D_{\mathrm{i}\beta}^{(p)} L^{(p)} f\right) \tag{2.64}$$

holds, hence $\beta = \alpha_p k$ for some $k \in \mathbb{Z}$. For $p = 1, \dots, q$ we denote

$$k_p := \begin{cases} l, & \text{if } \beta = \alpha_p l \text{ for some } l \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

A simple calculation shows that

$$f = D_{\mathrm{i}\beta} L f = \sum_{p=0}^{q} D_{\mathrm{i}\beta}^{(p)} L^{(p)} f = \sum_{p=1}^{q} D_{\mathrm{i}\beta}^{(p)} L^{(p)} f.$$

By (2.64),

$$\tilde{A}_p \left(D_{\mathrm{i}\beta}^{(p)} L^{(p)} f \right) = \mathrm{i}\alpha_p k_p \left(D_{\mathrm{i}\beta}^{(p)} L^{(p)} f \right),$$

and using (2.63) we obtain that $f \in Y$. ∎

§ 2.5 THE $(LI_{\mathbb{Q}})$ CASE

We have seen that condition $(LD_{\mathbb{Q}})$ yields periodic limit semigroups. As we will show now, this condition is also necessary for the existence of such a nontrivial limit flow. First we prove that under the condition $(LI_{\mathbb{Q}})$ the eigenvalue structure of $\tilde{A}$ is completely different from the $(LD_{\mathbb{Q}})$ case.

THEOREM 2.5.1 — *Suppose that $(LI_{\mathbb{Q}})$ holds and that the graph G is strongly connected.*

1. *With the notations of Proposition 2.3.6 we have $X_0 = X_1$. In particular,*

$$\sigma(\tilde{A}) \cap \mathrm{i}\mathbb{R} = \{0\}.$$

2. *On the other hand,*

$$\sup \left\{ \operatorname{Re} \lambda : \lambda \in \sigma(\tilde{A}),\ \lambda \neq 0 \right\} = 0.$$

PROOF — Let $\mathrm{i}\beta \in \sigma(\tilde{A})$. This means, by the characteristic equation (2.36), that $1 \in \sigma(\mathbf{A}_{\tilde{q}+\mathrm{i}\beta})$. Since $1 \in \sigma(\mathbf{A}_{\tilde{q}})$, from Lemma 2.4.6 follows that this happens only if

$$\prod_{l=1}^{s} \mathrm{e}^{\mathrm{i}\beta\tau_{j_l}(0,1)} = 1,$$

hence

$$\beta \sum_{l=1}^{s} \tau_{j_l}(0,1) = 2\pi \cdot k_{Z_s} \text{ for some } k_{Z_s} \in \mathbb{Z}$$

for all $Z_s = \{\mathsf{e}_{j_1}, \dots, \mathsf{e}_{j_s}\}$ forming a cycle in the graph. If $\beta \neq 0$, this contradicts $(LI_{\mathbb{Q}})$.

We now prove the second statement by showing that for every "infinite" rectangle

$$R_{\varepsilon,K} = \{z \in \mathbb{C} :\ -\varepsilon \leq \operatorname{Re} z \leq \varepsilon,\ K \leq \operatorname{Im} z\} \quad (\varepsilon, K > 0)$$

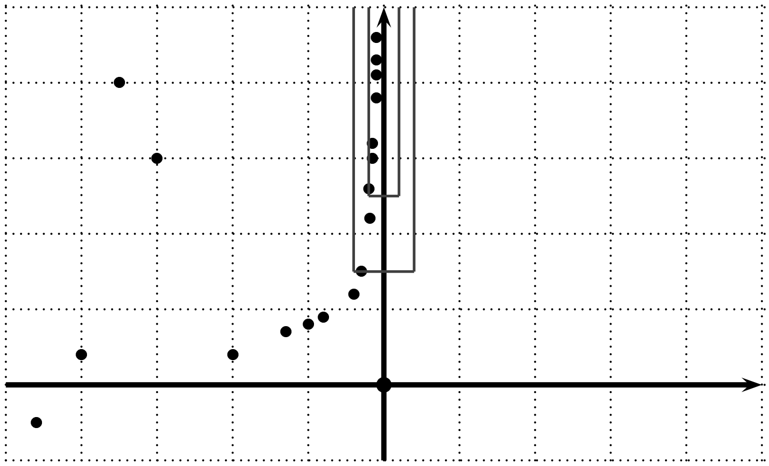

Figure 2.3: The spectrum of $\tilde{A}$ in the $(LI_{\mathbb{Q}})$ case

we have $\sigma(\tilde{A}) \cap R_{\varepsilon,K} \neq \emptyset$ (see Figure 2.3). Consider the holomorphic function $h(\lambda) := \det(\mathbf{1} - \mathbf{A}_\lambda)$. By (2.36), $\lambda \in \sigma(\tilde{A})$ if and only if $h(\lambda) = 0$. For every rectangle $R_{\varepsilon,K}$ we construct a rectangle $R' \subset R_{\varepsilon,K}$ such that the curve $h(\partial R')$ goes around zero with the multiplicity of the root 0. This clearly proves the existence of a root of h in R', hence in $R_{\varepsilon,K}$.

Fix now $\varepsilon, K > 0$. From condition $(LI_{\mathbb{Q}})$ follows that also the set

$$\{\tau_{j_1}(0,1) + \cdots + \tau_{j_k}(0,1) : \mathsf{e}_{j_1}, \ldots, \mathsf{e}_{j_k} \text{ form vertex disjoint cycles in } G\}$$

is strongly linearly independent over $\mathbb{Q}$. Hence, according to Lemma 2.3.9, for any $\delta > 0$ we can find $\rho > K$ such that

$$|\rho\left(\tau_{j_1}(0,1) + \cdots + \tau_{j_k}(0,1)\right) - 2\pi l| < \delta \tag{2.65}$$

for all $\mathsf{e}_{j_1}, \mathsf{e}_{j_2}, \ldots, \mathsf{e}_{j_k}$ forming vertex disjoint cycles in G, and for an appropriate $l \in \mathbb{Z}$ depending on the edge indices. As we have seen in (2.46), the function $h(z)$ is a sum of terms of the form

$$c\mathrm{e}^{z\left(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\right)},$$

where $\mathsf{e}_{j_1}, \ldots, \mathsf{e}_{j_k}$ form vertex disjoint cycles in G. For $-\varepsilon \leq \operatorname{Re} z \leq \varepsilon$ we have

$$\begin{aligned} &\left| c\mathrm{e}^{(z+\mathrm{i}\rho)\left(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\right)} - c\mathrm{e}^{z\left(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\right)} \right| \leq \\ &\leq |c| \mathrm{e}^{\varepsilon\left(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\right)} \left| \mathrm{e}^{\mathrm{i}\rho\left(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\right)} - 1 \right|. \end{aligned}$$

By (2.65) and using that the exponential function is holomorphic on $\mathbb{C}$, there exists $\tilde{C} > 0$ such that

$$\left| \mathrm{e}^{\mathrm{i}\rho\left(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\right)} - 1 \right| = \left| \mathrm{e}^{\mathrm{i}\rho\left(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)\right)} - \mathrm{e}^{\mathrm{i}2\pi l} \right| \leq \tilde{C} \cdot \delta$$

for every appropriate indices $j_1, \ldots, j_k$. So, there is a constant C for which

$$|h(z + \mathrm{i}\rho) - h(z)| < C \cdot \delta \tag{2.66}$$

holds whenever $-\varepsilon \leq \mathrm{Re}\ z \leq \varepsilon$.
Since $\sigma(\tilde{A})$ is discrete, one can find a rectangle

$$R'' \subset \{-\varepsilon \leq \mathrm{Re}\ z \leq \varepsilon\}$$

such that $R'' \cap \sigma(\tilde{A}) = \{0\}$. Since 0 is a root of h, the curve $h(\partial R'')$ goes around zero with the multiplicity of 0. If δ is small enough, (2.66) for $R' := R'' + \mathrm{i}\rho \subset R_{\varepsilon,K}$ implies that the curve $h(\partial R')$ also goes around zero with the multiplicity of 0, which completes the proof. ■

In case $(LI_{\mathbb{Q}})$, not only the periodic limit but also the uniform convergence is lost. The following theorem is the counterpart to Theorem 2.4.8.

THEOREM 2.5.2 — *Assume $(LI_{\mathbb{Q}})$ and that the graph G is strongly connected. Using the notation of Proposition 2.3.6, $\tilde{T}(t)$ converges strongly but not uniformly to Pr_{X_0} – the projection to X_0 – on X.*

PROOF — The strong convergence on X means that

$$\|\tilde{T}(t)f - Pr_{X_0}f\|_X \to 0 \tag{2.67}$$

for every $f \in X$. By Theorem 2.5.1.1, for the projection Q of Proposition 2.3.6.1 we have $\mathrm{ran}Q = X_0$, so $Q = Pr_{X_0}$ which gives (2.67). By Theorem 2.5.1.2 this convergence can not be uniform. ■

Chapter 3

Flows with dynamic ramification nodes

In this chapter we still consider flows in a network, but change the transmission process in the nodes. Instead of conservation of mass as in Chapter 2, we assume that the *velocity* of the outgoing flow mass in the vertices is determined by the total incoming flow mass. In addition, we take into consideration a control process in each vertex, depending on the outgoing flow mass in the other vertices. However, for the sake of simplicity, we assume now that the flow velocities are constant on each edge and there is no absorption/inflow. This problem can be described in the following way. On the edges we choose the same transport equations (with adequate initial conditions) as in the previous chapter.

$$\begin{cases} \frac{\partial}{\partial t} u_j(t,s) &= c_j \frac{\partial}{\partial s} u_j(t,s), \ s \in (0,1), t > 0, \\ u_j(0,s) &= f_j(s), \ s \in (0,1), \end{cases}$$

where $f_j \in L^1[0,1]$ for $j = 1, \ldots, m$.

The boundary conditions in the vertices contain again the distribution weights of the outgoing flow – that is, using the notation $u(t,s) = (u_1(t,s), \ldots, u_m(t,s))$,

$$u(t,1) \in \operatorname{ran} \left(\Phi_w^-\right)^\top, \ t \geq 0,$$

where $\Phi_w^- = \left(\omega_{ij}^-\right)_{n \times m}$ the *weighted outgoing incidence matrix* as defined in (1.3). The new boundary condition

$$\frac{\partial}{\partial t} \Phi^- u(t,1) = \Phi_w^+ u(t,0)$$

means that the (sum of the) outgoing flow *velocities* – and not the total outgoing flow *mass*, as in (BC) in Chapter 2 – is given by the incoming flow mass in each vertex v_i. We assume that different edges have different effects on the outgoing velocities, therefore we take a weighted sum of the incoming flow mass on the right-hand side, using the *weighted incoming incidence matrix* $\Phi_w^+ = \left(\omega_{ij}^+\right)_{n \times m}$ defined in (1.6).

In the next step we add a *boundary control* in the vertices. For this purpose we choose a control space W and a linear control operator $C : W \to \partial X = \mathbb{C}^n$. Then our boundary control problem becomes

$$\frac{\partial}{\partial t}\Phi^- u\,(t,1) = \Phi_w^+ u\,(t,0) + Cw(t).$$

In the final step we assume that the control function $w(\cdot)$ is given by a feedback from the values of the flow in the vertices. More precisely, we take $w(t) = D\Phi^- u(t,1)$. This leads to the following system.

$$(DE)\begin{cases} \frac{\partial}{\partial t}u_j\,(t,s) &= c_j \frac{\partial}{\partial s}u_j\,(t,s)\,,\ s\in(0,1)\,,\ t>0, \\ u_j\,(0,s) &= f_j\,(s)\,,\ s\in(0,1)\,, \\ u\,(t,1) &\in \operatorname{ran}(\Phi_w^-)^\top\,,\ t\geq 0, \\ \frac{\partial}{\partial t}\Phi^- u\,(t,1) &= \Phi_w^+ u\,(t,0) + \mathbf{B}\Phi^- u\,(t,1)\,,\ t\geq 0, \quad (BC) \\ \Phi^- u\,(0,1) &= x\in\mathbb{C}^n, \end{cases}$$

where $\mathbf{B}$ is an $n\times n$ matrix.

We can look at the boundary condition (BC) as a *delay equation* for the process in the tails of the edges, that is for $\Phi^- u\,(t,1)$. The time-derivative of this process – that is the total outgoing flow velocities in the vertices – is determined by the outgoing flow mass values at *different* earlier times. This earlier moment depends on the flow velocities c_j, and can be computed for large t's as t minus the time needed to pass from one endpoint of the edge to the other. Hence, for the edge e_j this yields $t - 1/c_j$. (If t is small, then instead of the outgoing flow values, the flow mass on the edges at time 0 at place tc_j is taken into consideration). The total outgoing flow mass at these earlier times (resp., the flow mass on the edges at place tc_j) is the same as the mass at time t coming into the heads of the edges, hence $\Phi^+ u\,(t,0)$. Actually, instead of the simple sum of the incoming flow masses we take a weighted sum of the incoming mass values on the edges written as $\Phi_w^+ u\,(t,0)$. It turns out that this approach leads to a systematic semigroup treatment of the problem.

§ 3.1 WELL-POSEDNESS OF THE PROBLEM

We are going to use techniques as developed for partial differential equations with delay by A. Bátkai and S. Piazzera in [BP04]. For this purpose we use again the space of functions on the edges

$$X := L^1\left([0,1]\right)^m \cong L^1\left([0,1],\mathbb{C}^m\right),$$

and the (boundary) space of the values in the vertices

$$\partial X := \mathbb{C}^n.$$

The most significant difference to the situation in [BP04] is that here ∂X does not coincide with the "boundary" of X, that is with $\mathbb{C}^{2m}$ – but has (in most cases) smaller dimension. As in (2.7), we introduce a "boundary operator" $L : X \to \partial X$ by

$$\begin{aligned} L &:= \Phi^-\otimes\delta_1, \\ D\,(L) &:= W^{1,1}\left([0,1],\mathbb{C}^m\right), \end{aligned}$$

and the "delay operator"

$$\begin{aligned} M &:= \Phi_w^+ \otimes \delta_0 \\ D(M) &:= W^{1,1}([0,1], \mathbb{C}^m), \end{aligned}$$

in analogy to (2.8). Though we call M the "delay operator" as in [BP04], it does not act on the "history function" (depending on time), but on the spatial distribution along the edges. However, since the flow has finite velocity on every edge, the incoming flow is always delayed with respect to the outgoing flow.

If we take the operator

$$\begin{aligned} A_w &:= \begin{pmatrix} c_1 \frac{d}{ds} & & 0 \\ & \ddots & \\ 0 & & c_m \frac{d}{ds} \end{pmatrix}, \\ D(A_w) &:= \left\{ f \in \left(W^{1,1}[0,1]\right)^m : f(1) \in \operatorname{ran}\left(\Phi_w^-\right)^\top \right\}, \end{aligned}$$

as in (2.4), then the problem (DE) can be written as an abstract Cauchy problem for the operator

$$\begin{aligned} \mathcal{A} &:= \begin{pmatrix} A_w & 0 \\ M & \mathbf{B} \end{pmatrix}, \\ D(\mathcal{A}) &:= \left\{ \begin{pmatrix} f \\ x \end{pmatrix} \in D(A_w) \times \mathbb{C}^n : Lf = x \right\} \end{aligned} \tag{3.1}$$

on the space

$$\mathcal{X} := X \times \partial X.$$

Indeed, (DE) is equivalent to

$$(ACP) \begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \begin{pmatrix} f \\ x \end{pmatrix} \end{cases}$$

in the sense of the following result, proved in [BP04, Corollary 3.1.4] and [BP04, Proposition 3.1.8].

THEOREM 3.1.1 — *The system (DE) admits a solution u with*

1. $u \in C^1([0,+\infty), X)$ *and*
2. $u(t,\cdot) \in W^{1,1}([0,1], \mathbb{C}^m)$ *for all* $t \geq 0$

if and only if (ACP) admits a continuously differentiable solution $\mathcal{U} : \mathbb{R}_+ \to \mathcal{X}$. In this case

$$\mathcal{U}(t) = \begin{pmatrix} u(t,\cdot) \\ \Phi^- u(t,1) \end{pmatrix}.$$

By standard semigroup theory (see [EN00, Section II.6]) follows that (ACP) is well-posed if and only if $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup on $\mathcal{X}$. For the well-posedness of (DE) we therefore show that the above operator (3.1) is a generator.

In the spirit of Greiner's approach to abstract boundary value problems (see also in [CENN03], [KS04], [Nick04.1], [Nick04.2]), we first introduce the so-called *Dirichlet operator*

$$D_\lambda := \left(L\mid_{\ker(\lambda-A_w)}\right)^{-1}$$

from ∂X to $\ker(\lambda - A_w)$ (compare with (2.27)). In our situation this becomes

$$(D_\lambda x)(s) = \epsilon_\lambda(s)\cdot\left(\left(\Phi_w^-\right)^\top x\right), x\in\partial X, s\in[0,1],$$

with

$$\epsilon_\lambda(s) = \begin{pmatrix} \exp(\frac{\lambda}{c_1}(s-1)) & & 0 \\ & \ddots & \\ 0 & & \exp(\frac{\lambda}{c_m}(s-1)) \end{pmatrix},$$

corresponding to (2.17). Again we consider the restriction of A_w to $\ker L$, i.e.,

$$\begin{aligned} A_0 &:= A_w\mid_{\ker L\cap D(A_w)}, \\ D(A_0) &:= \{f\in D(A_w) : Lf = 0\} = \left\{f\in W^{1,1}([0,1],\mathbb{C}^m) : f(1)=0\right\}. \end{aligned}$$

This operator $(A_0, D(A_0))$ generates the nilpotent left shift semigroup $(T_0(t))_{t\geq 0}$ on X defined by

$$(T_0(t)f)_j(s) = \begin{cases} f_j(s+c_jt), & s+c_jt\leq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

see (2.23). We also know that the resolvent of A_0 exists for every $\lambda\in\mathbb{C}$ – as proved in §2.2.

We now give a decomposition of $\lambda - \mathcal{A}$ that turns out to be very useful. In the following we denote by $\mathbf{1}$ the $n\times n$ identity matrix.

LEMMA 3.1.2 — *For any $\lambda\in\mathbb{C}$ one has*

$$\lambda - \mathcal{A} = \tag{3.3}$$
$$\begin{pmatrix} I_X & 0 \\ -MR(\lambda,A_0) & \mathbf{1}\end{pmatrix}\begin{pmatrix}\lambda - A_0 & 0 \\ 0 & \lambda-\mathbf{B}-\mathbf{A}_\lambda\end{pmatrix}\begin{pmatrix} I_X & -D_\lambda \\ 0 & \mathbf{1}\end{pmatrix}$$

with $\mathbf{A}_\lambda := MD_\lambda$ an $n\times n$ matrix.

PROOF — Let us denote the operator on the right-hand side of (3.3) by $\mathcal{B}$ and write

$$S := \begin{pmatrix} I_X & -D_\lambda \\ 0 & \mathbf{1}\end{pmatrix}.$$

Then the condition $\binom{f}{x}\in D(\mathcal{B})$ is equivalent to the fact that $S\binom{f}{x}\in D(A_0)\times\mathbb{C}^n$, which means that $f - D_\lambda x\in\ker L\cap D(A_w)$. This is again equivalent to $Lf = x$, so to

$\binom{f}{x} \in D(\mathcal{A})$, hence the two domains coincide. Let $\binom{f}{x} \in D(\mathcal{A})$. Then

$$\begin{aligned}
\mathcal{B}\begin{pmatrix} f \\ x \end{pmatrix} &= \begin{pmatrix} I_X & 0 \\ -MR(\lambda, A_0) & \mathbf{1} \end{pmatrix} \begin{pmatrix} \lambda - A_0 & 0 \\ 0 & \lambda - \mathbf{B} - \mathbf{A}_\lambda \end{pmatrix} \begin{pmatrix} f - D_\lambda x \\ x \end{pmatrix} \\
&= \begin{pmatrix} I_X & 0 \\ -MR(\lambda, A_0) & \mathbf{1} \end{pmatrix} \begin{pmatrix} (\lambda - A_0)(f - D_\lambda x) \\ (\lambda - \mathbf{B} - \mathbf{A}_\lambda) x \end{pmatrix} \\
&= \begin{pmatrix} (\lambda - A_0)(f - D_\lambda x) \\ -M(f - D_\lambda x) + (\lambda - \mathbf{B} - \mathbf{A}_\lambda) x \end{pmatrix} \\
&= \begin{pmatrix} (\lambda - A_w) f \\ -Mf + (\lambda - \mathbf{B}) x \end{pmatrix} = (\lambda - \mathcal{A}) \begin{pmatrix} f \\ x \end{pmatrix},
\end{aligned}$$

where in the last equality we used $D_\lambda x \in \ker(\lambda - A_w)$. Now the proof is complete. ■

Using the above decomposition, we obtain the desired well-posedness for (ACP), hence for (DE).

THEOREM 3.1.3 — *The operator* $(\mathcal{A}, D(\mathcal{A}))$ *defined in* (3.1) *generates a strongly continuous semigroup* $(\mathcal{T}(t))_{t \geq 0}$ *on the space* $\mathcal{X} = X \times \partial X$. *Hence, the system* (DE) *is well-posed.*

PROOF — Again, we proceed as in [BP04, Theorem 3.3.1]. Since $\mathbf{B}$ is bounded, using the bounded perturbation theorem (see [EN00, Theorem III.1.3]) for the sum

$$\mathcal{A} = \begin{pmatrix} A_w & 0 \\ M & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{B} \end{pmatrix},$$

we may assume that $\mathbf{B} = 0$. From the decomposition (3.3),

$$\mathcal{A} = \begin{pmatrix} I_X & 0 \\ -MA_0^{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & \mathbf{A}_0 \end{pmatrix} \begin{pmatrix} I_X & -D_\lambda \\ 0 & \mathbf{1} \end{pmatrix}$$

with

$$\begin{pmatrix} I_X & -D_\lambda \\ 0 & \mathbf{1} \end{pmatrix}$$

being an invertible operator with inverse

$$\begin{pmatrix} I_X & D_\lambda \\ 0 & \mathbf{1} \end{pmatrix}.$$

By similarity, it is enough to prove that the operator

$$\begin{aligned}
\mathcal{C} &= \begin{pmatrix} I_X & -D_\lambda \\ 0 & \mathbf{1} \end{pmatrix} \mathcal{A} \begin{pmatrix} I_X & D_\lambda \\ 0 & \mathbf{1} \end{pmatrix} \\
&= \begin{pmatrix} I_X & -D_\lambda \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} I_X & 0 \\ -MA_0^{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & \mathbf{A}_0 \end{pmatrix}
\end{aligned}$$

with domain $D(\mathcal{C}) = D(A_0) \times \partial X$ is a generator.
To proceed we compute

$$\begin{aligned}
\begin{pmatrix} I_X & -D_\lambda \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} I_X & 0 \\ -MA_0^{-1} & \mathbf{1} \end{pmatrix} &= \begin{pmatrix} I_X + D_\lambda MA_0^{-1} & -D_\lambda \\ -MA_0^{-1} & \mathbf{1} \end{pmatrix} \\
&:= \mathcal{I} + \mathcal{D},
\end{aligned}$$

where

$$\mathcal{D} := \begin{pmatrix} D_\lambda M A_0^{-1} & -D_\lambda \\ -MA_0^{-1} & 0 \end{pmatrix}.$$

Using now that $M = \Phi_w^+ \otimes \delta_0 : W^{1,1}([0,1], \mathbb{C}^m) \to \partial X$ is a bounded operator, we obtain that $MA_0^{-1} : X \to \partial X$ is bounded, hence $\mathcal{D}$ is a bounded operator on $\mathcal{X}$. So we have

$$\mathcal{C} = (\mathcal{I} + \mathcal{D}) \begin{pmatrix} A_0 & 0 \\ 0 & \mathbf{A}_0 \end{pmatrix}. \tag{3.4}$$

The matrix

$$\begin{pmatrix} A_0 & 0 \\ 0 & \mathbf{A}_0 \end{pmatrix}$$

with domain $D(\mathcal{C}) = D(A_0) \times \partial X$ generates the strongly continuous semigroup

$$\mathcal{S}(t) := \begin{pmatrix} T_0(t) & 0 \\ 0 & \mathrm{e}^{t\mathbf{A}_0} \end{pmatrix},\ t \geq 0.$$

We now use a multiplicative version of the Desch-Schappacher Perturbation Theorem (see [EN00, Theorem III.3.1] and [EN00, Corollary III.3.4]) as stated in [BP04, Theorem 1.4.4] for the operator $\mathcal{C}$ in (3.4). For this purpose we take $\binom{f_1}{f_2} \in L^p([0,1], \mathcal{X})$ and compute

$$\begin{aligned}
&\int_0^1 \mathcal{S}(1-r)\mathcal{D}\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix}\mathrm{d}r \\
= &\int_0^1 \begin{pmatrix} T_0(1-r) & 0 \\ 0 & \mathrm{e}^{(1-r)\mathbf{A}_0} \end{pmatrix} \begin{pmatrix} D_\lambda M A_0^{-1} & -D_\lambda \\ -MA_0^{-1} & 0 \end{pmatrix} \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix}\mathrm{d}r \\
= &\int_0^1 \begin{pmatrix} T_0(1-r) & 0 \\ 0 & \mathrm{e}^{(1-r)\mathbf{A}_0} \end{pmatrix} \begin{pmatrix} D_\lambda M A_0^{-1} f_1(r) - D_\lambda f_2(r) \\ -MA_0^{-1} f_1(r) \end{pmatrix}\mathrm{d}r \\
= &\begin{pmatrix} \int_0^1 T_0(1-r) D_\lambda \left[MA_0^{-1} f_1(r) - f_2(r)\right] \mathrm{d}r \\ -\int_0^1 \mathrm{e}^{(1-r)\mathbf{A}_0} MA_0^{-1} f_1(r)\mathrm{d}r \end{pmatrix}.
\end{aligned}$$

If we can show that the vector so obtained belongs to $D(\mathcal{C})$, we have that $\mathcal{C}$ – hence $\mathcal{A}$ – is a generator by [BP04, Theorem 1.4.4]. Using the boundedness of $MA_0^{-1} : X \to \partial X$ we have

$$g := MA_0^{-1} f_1 - f_2 \in L^p([0,1], \partial X).$$

From (3.2) the jth coordinate can be computed as

$$\left[\int_0^1 T_0(1-r) D_\lambda g(r)\mathrm{d}r\right]_j (\cdot) = \omega_{ij}^- \int_{\frac{\cdot-1}{c_j}+1}^1 \mathrm{e}^{\frac{\lambda}{c_j}(\cdot-1+c_j(1-r))} g_i(r)\mathrm{d}r$$

with $\omega_{ij}^- \neq 0$ uniquely defined by j. From this

$$\left(\left[\int_0^1 T_0(1-r) D_\lambda g(r)\mathrm{d}r\right]_j\right)_{j=1,\ldots,m} \in D(A_0).$$

Clearly it follows that

$$\int_0^1 \mathcal{S}(1-r)\mathcal{D}\binom{f_1(r)}{f_2(r)}\mathrm{d}r \in D(A_0)\times\partial X = D(\mathcal{C}),$$

hence the proof is complete. ∎

§ 3.2 SPECTRAL PROPERTIES

To prove asymptotic properties for the semigroup $(\mathcal{T}(t))_{t\geq 0}$ we now describe the spectrum of $(\mathcal{A}, D(\mathcal{A}))$ and determine its resolvent.

PROPOSITION 3.2.1 — *For $\lambda\in\mathbb{C}$ the following characteristic equation holds:*

$$\lambda\in\sigma(\mathcal{A}) \Longleftrightarrow \lambda\in\sigma(\mathbf{B}+\mathbf{A}_\lambda).$$

Here,

$$\mathbf{A}_\lambda = MD_\lambda = \Phi_w^+\epsilon_\lambda(0)\left(\Phi_w^-\right)^\top$$

is a weighted (transposed) adjacency matrix *(see Definition 1.3.8) with entries*

$$(\mathbf{A}_\lambda)_{ip} = \begin{cases} \omega_{ij}^+ \mathrm{e}^{-\frac{\lambda}{c_j}}\omega_{pj}^-, & \textit{if } \mathsf{v}_i = \mathsf{e}_j(0) \textit{ and } \mathsf{v}_p = \mathsf{e}_j(1), \\ \textit{0}, & \textit{otherwise.} \end{cases} \tag{3.5}$$

Moreover, for any $\lambda\in\rho(\mathcal{A})$ the resolvent $R(\lambda,\mathcal{A})$ is given by

$$R(\lambda,\mathcal{A}) = \begin{pmatrix} [D_\lambda R(\lambda,\mathbf{B}+\mathbf{A}_\lambda)M + I_X]R(\lambda,A_0) & D_\lambda R(\lambda,\mathbf{B}+\mathbf{A}_\lambda) \\ R(\lambda,\mathbf{B}+\mathbf{A}_\lambda)MR(\lambda,A_0) & R(\lambda,\mathbf{B}+\mathbf{A}_\lambda) \end{pmatrix}. \tag{3.6}$$

PROOF — We follow [BP04, Proposition 3.2.1]. To compute the resolvent in λ, we have to find for $\binom{g}{y}\in\mathcal{X}$ a unique $\binom{f}{x}\in D(\mathcal{A})$ such that

$$(\lambda-\mathcal{A})\binom{f}{x} = \begin{pmatrix} \lambda f - A_w f \\ -Mf + (\lambda-\mathbf{B})x \end{pmatrix} = \binom{g}{y}.$$

Using Lemma 2.2.8 and $(\lambda - A_w)f = g$ we obtain that

$$f = D_\lambda Lf + R(\lambda,A_0)g = D_\lambda x + R(\lambda,A_0)g, \tag{3.7}$$

since $Lf = x$. Therefore x has to satisfy the equation

$$(\lambda-\mathbf{B}-\mathbf{A}_\lambda)x = MR(\lambda,A_0)g + y,$$

where $\mathbf{A}_\lambda = MD_\lambda = \Phi_w^+\epsilon_\lambda(0)(\Phi_w^-)^\top$. Furthermore, if $R(\lambda,\mathbf{B}+\mathbf{A}_\lambda)$ exists, from this follows

$$x = R(\lambda,\mathbf{B}+\mathbf{A}_\lambda)MR(\lambda,A_0)g + R(\lambda,\mathbf{B}+\mathbf{A}_\lambda)y. \tag{3.8}$$

Using this and (3.7) we obtain

$$f = D_\lambda R(\lambda,\mathbf{B}+\mathbf{A}_\lambda)MR(\lambda,A_0)g + D_\lambda R(\lambda,\mathbf{B}+\mathbf{A}_\lambda)y + R(\lambda,A_0)g. \tag{3.9}$$

Equalities (3.8) and (3.9) now imply (3.6) and

$$\lambda\in\rho(\mathcal{A}) \Longleftrightarrow \lambda\in\rho(\mathbf{B}+\mathbf{A}_\lambda),$$

which is equivalent to the desired characteristic equation. ∎

From this form of the resolvent we obtain the following property.

REMARK 3.2.2 — For any $\lambda \in \rho(\mathcal{A})$, the resolvent $R(\lambda, \mathcal{A})$ is compact.

PROOF — It is enough to prove that the entries of the operator matrix (3.6) are compact operators. In the second row this is clear since the entries have range in $\mathbb{C}^n$. In the first row, the second entry also has finite dimensional range contained in the span of exponential functions. The first entry is the sum of an operator with finite dimensional range and the resolvent of an operator having domain contained in $W^{1,1}([0,1], \mathbb{C}^m)$ – hence being compact by [EN00, II.4.30 (4)]. ∎

COROLLARY 3.2.3 — *The operator $(\mathcal{A}, D(\mathcal{A}))$ has only point spectrum.*

Since $\mathbf{B}$ and $\mathbf{A}_\lambda$ are matrices, we can reformulate the above *characteristic equation* as

$$\lambda \in P\sigma(\mathcal{A}) = \sigma(\mathcal{A}) \Longleftrightarrow \det(\lambda - \mathbf{B} - \mathbf{A}_\lambda) = 0. \tag{3.10}$$

In the next section we will prove useful qualitative properties for the semigroup that lead to a finer description of the spectrum of $\mathcal{A}$, particularly to a spectral decomposition. Using the regularity of the semigroup, we will obtain a corresponding spectral decomposition of the semigroup.

§ 3.3 ASYMPTOTIC BEHAVIOR

In order to study the qualitative (and, in particular, asymptotic) behavior of the solutions of (DE), we first prove a regularity property of the solution semigroup. We show that this semigroup is eventually differentiable, that is, the orbits $t \mapsto \mathcal{T}(t)\binom{f}{x}$ are differentiable for t large enough for every $\binom{f}{x} \in \mathcal{X}$ (see [EN00, Definition II.4.13]). For this purpose we first show how the first coordinate of $\mathcal{T}(t)\binom{f}{x}$ can be obtained from the second one.

LEMMA 3.3.1 — *Denoting by π_1 and π_2, resp., the projections from $\mathcal{X}$ to X and to ∂X, resp., we have*

$$\left[\pi_1 \mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix}\right]_j (r) = \begin{cases} \left[(\Phi_w^-)^\top \pi_2 \mathcal{T}(t - \frac{1-r}{c_j})\binom{f}{x}\right]_j, & \text{if } 1 - tc_j \le r \le 1, \\ f_j(r + tc_j), & \text{if } 0 \le r < 1 - tc_j, \end{cases}$$

for $j = 1, \dots, m$, and almost all r.

PROOF — If $\binom{f}{x} \in D(\mathcal{A})$, then $\mathcal{T}(t)\binom{f}{x}$ defines a classical solution for (ACP), and, by Theorem 3.1.1, the function $\pi_1 \mathcal{T}(t)\binom{f}{x}$ is a solution for (DE) with $L\pi_1 \mathcal{T}(t)\binom{f}{x} = \pi_2 \mathcal{T}(t)\binom{f}{x}$. It is easy to check that given formula for $\pi_1 \mathcal{T}(t)\binom{f}{x}$ satisfies these requirements. Heuristically this means that $\left[\pi_1 \mathcal{T}(t)\binom{f}{x}\right]_j (r)$ is the distribution of flow mass on the edges e_j at point r. If $1 - tc_j \le r \le 1$ that is $t \ge \frac{1-r}{c_j}$, this flow mass is equal to the flow mass that has been at the tail of e_j at time $t - \frac{1-r}{c_j}$. This is exactly the expression in the first part of the above formula, where we have used the condition $\pi_1 \mathcal{T}(t)\binom{f}{x}(1) \in \operatorname{ran}(\Phi_w^-)^\top$. If $0 \le r < 1 - tc_j$, that is $t < \frac{1-r}{c_j}$, the flow mass at point r is equal to the initial flow mass at $r + tc_j$.

If $\binom{f}{x} \notin D(\mathcal{A})$, we can choose a sequence $\binom{f_n}{x_n}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ with $\binom{f_n}{x_n} \to \binom{f}{x}$ as $n \to +\infty$. From this follows for every $j = 1, \dots, m$ that

$$\left[\pi_1 \mathcal{T}(t)\begin{pmatrix} f_n \\ x_n \end{pmatrix}\right]_j \Bigg|_{[0,1-tc_j)} = [f_n]_j \;|_{[0,1-tc_j)} \to [f]_j \;|_{[0,1-tc_j)}, \text{ as } n \to +\infty.$$

We have

$$\lim_{n\to\infty} \mathcal{T}(t)\begin{pmatrix} f_n \\ x_n \end{pmatrix} = \mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix}, \tag{3.11}$$

hence, by the continuity of π_1,

$$\left[\pi_1 \mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix}\right]_j \Bigg|_{[0,1-tc_j)} = [f]_j \;|_{[0,1-tc_j)} .$$

Since the convergence in (3.11) is uniform for t in compact intervals, and since π_2 is continuous, we obtain

$$\left[\left(\Phi_w^-\right)^\top \pi_2 \mathcal{T}(t - \frac{1-r}{c_j})\begin{pmatrix} f_n \\ x_n \end{pmatrix}\right]_j \to \left[\left(\Phi_w^-\right)^\top \pi_2 \mathcal{T}(t - \frac{1-r}{c_j})\begin{pmatrix} f \\ x \end{pmatrix}\right]_j$$

uniformly for $r \in [1 - tc_j, 1]$. This implies that

$$\left[\pi_1 \mathcal{T}(t)\begin{pmatrix} f_n \\ x_n \end{pmatrix}\right]_j \;|_{[1-tc_j,1]} \to \left[\left(\Phi_w^-\right)^\top \pi_2 \mathcal{T}(t - \frac{1-\cdot}{c_j})\begin{pmatrix} f \\ x \end{pmatrix}\right]_j \;|_{[1-tc_j,1]}$$

uniformly, hence in L^1-norm on $[1 - tc_j, 1]$. However, by (3.11) this limit is equal to

$$\left[\pi_1 \mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix}\right]_j \;|_{[1-tc_j,1]},$$

and this completes the proof. ■

We can now prove the differentiability of the semigroup.

THEOREM 3.3.2 — *The semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ is differentiable for $t > 2c$ with $c = \frac{1}{\min_j c_j}$.*

PROOF — We have to prove the differentiability of the orbits $t \mapsto \mathcal{T}(t)\binom{f}{x}$ for $t > 2c$ and every $\binom{f}{x} \in \mathcal{X}$. For this purpose fix a vector $\binom{f}{x} \in \mathcal{X}$. We will show that both coordinates of $\mathcal{T}(t)\binom{f}{x}$ are differentiable for $t > 2c$. The formula

$$\mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix} = \begin{pmatrix} f \\ x \end{pmatrix} + \mathcal{A}\int_0^t \mathcal{T}(s)\begin{pmatrix} f \\ x \end{pmatrix} \mathrm{d}s \tag{3.12}$$

holds for any C_0-semigroup, see [EN00, Lemma II.1.9]. From the form (3.1) of $\mathcal{A}$ we obtain, applying π_2 to both sides of (3.12),

$$\pi_2 \mathcal{T}(t)\begin{pmatrix} f \\ x \end{pmatrix} = x + \mathbf{B}\int_0^t \pi_2 \mathcal{T}(r)\begin{pmatrix} f \\ x \end{pmatrix} \mathrm{d}r + M\int_0^t \pi_1 \mathcal{T}(r)\begin{pmatrix} f \\ x \end{pmatrix} \mathrm{d}r.$$

Denoting

$$v(t) := \pi_2 \mathcal{T}(t)\binom{f}{x},$$

this becomes

$$v(t) = x + \mathbf{B}\int_0^t v(r)\mathrm{d}r + M\int_0^t \pi_1\mathcal{T}(r)\binom{f}{x}\mathrm{d}r.$$

If $t > c$, then for every $j = 1, \dots, m$ and $s \in [0,1]$ the relation $1 - tc_j < s \le 1$ holds. Using Lemma 3.3.1, we obtain

$$\begin{aligned} v(t) &= v(c) + \mathbf{B}\int_c^t v(r)\mathrm{d}r + M\int_c^t \left(\left[\left(\Phi_w^-\right)^\top v\left(r - \frac{1-\cdot}{c_j}\right)\right]_j\right)_{j=1,\dots,m}\mathrm{d}r \\ &= v(c) + \mathbf{B}\int_c^t v(r)\mathrm{d}r + \Phi_w^+\int_c^t \left(\left[\left(\Phi_w^-\right)^\top v\left(r - \frac{1}{c_j}\right)\right]_j\right)_{j=1,\dots,m}\mathrm{d}r. \end{aligned}$$

This formula and the continuity of $\mathbb{R}_+ \ni t \mapsto v(t) \in \partial X$ imply that the map $(c, +\infty) \ni t \mapsto v(t)$ is even continuously differentiable. Hence, the statement holds for $\pi_2\mathcal{T}(t)\binom{f}{x}$. For the first coordinate we apply Lemma 3.3.1 again and obtain

$$w(t) := \pi_1\mathcal{T}(t)\binom{f}{x} = \left(\left[\left(\Phi_w^-\right)^\top v\left(t - \frac{1-\cdot}{c_j}\right)\right]_j\right)_{j=1,\dots,m} \quad \text{for } t > c.$$

Observe that for every $s \in [0,1]$, the function $(2c, +\infty) \ni t \mapsto w(t)(s)$ is continuously differentiable. We denote its derivative by

$$\dot{w}(t)(s) := \frac{d}{dt}w(t)(s).$$

We have to show that the vector-valued function $(2c, +\infty) \ni t \mapsto w(t) \in (L^1[0,1])^m$ is differentiable. Let $t \in (2c, +\infty)$ be fixed and take a sequence $h_n \downarrow 0$. Then

$$\left|\frac{w(t+h_n)(s) - w(t)(s)}{h_n} - \dot{w}(t)(s)\right| \to 0 \text{ for every } s \in [0,1]. \tag{3.13}$$

Clearly,

$$\dot{w}(t)(s) = \left(\left[\left(\Phi_w^-\right)^\top \dot{v}\left(t - \frac{1-s}{c_j}\right)\right]_j\right)_{j=1,\dots,m}, \tag{3.14}$$

and this function is continuous in t (and in s), because $t - \frac{1-s}{c_j} > t - c > c$. Thus, for every $s \in [0,1]$ there exist $0 \le \vartheta_{n,j}(s) \le h_n$, $j = 1, \dots, m$, such that

$$\left[\frac{w(t+h_n)(s) - w(t)(s)}{h_n}\right]_j = \left[\dot{w}(t + \vartheta_{n,j}(s))(s)\right]_j.$$

Rewriting (3.13), we obtain

$$\left|\left[\dot{w}(t + \vartheta_{n,j}(s))(s)\right]_j - \left[\dot{w}(t)(s)\right]_j\right| \to 0 \text{ for every } s \in [0,1],$$

$j = 1, \ldots, m$. To apply the Lebesgue dominated convergence theorem, observe that from (3.14),

$$\left|[\dot{w}(t + \vartheta_{n,j}(s))(s)]_j - [\dot{w}(t)(s)]_j\right| \le 2 \sup_{r\in[t-c,t+1]} \left|\left[\left(\Phi_w^-\right)^\top \dot{v}(r)\right]_j\right| \text{ for every } s \in [0,1],$$

if n is large enough. We therefore obtain that

$$\left\|\left[\frac{w(t+h_n) - w(t)}{h_n} - \frac{d}{dt}w(t)\right]_j\right\|_{L^1[0,1]} \to 0,$$

$j = 1, \ldots, m$, and this is what we wanted to prove. ∎

As a consequence we have norm continuity of the semigroup for large t. It can be found in many books on semigroup theory, but in our main reference it is only an exercise, see [EN00, Exercise II.4.21(1)]. Therefore we give a short proof.

COROLLARY 3.3.3 — *The semigroup $(\mathcal{T}(t))_{t\ge0}$ is eventually norm continuous.*

PROOF — We will show that $t \to \mathcal{T}(t) \in \mathcal{L}(\mathcal{X})$ is norm continuous for $t > 2c$. Since for every $\mathrm{x} \in \mathcal{X}$ the function $(2c, +\infty) \ni t \to \mathcal{T}(t)\mathrm{x}$ is differentiable, we have that $\mathcal{T}(t)\mathrm{x} \in D(\mathcal{A})$, $t > 2c$, that is $D(\mathcal{A}\mathcal{T}(t)) = \mathcal{X}$, $t > 2c$. The operators $\mathcal{A}\mathcal{T}(t)$ are closed, so by the closed graph theorem we have that they are also bounded on $\mathcal{X}$ for $t > 2c$. Let now $t > 2c$ be fixed and take $t_n \downarrow t$. Then

$$\begin{aligned}
\|\mathcal{T}(t_n) - \mathcal{T}(t)\|_{\mathcal{L}(\mathcal{X})} &= \sup_{\|\mathrm{x}\|\le1} \|\mathcal{T}(t_n)\mathrm{x} - \mathcal{T}(t)\mathrm{x}\|_{\mathcal{X}} \\
&= \sup_{\|\mathrm{x}\|\le1} \left\|\int_t^{t_n} \mathcal{A}\mathcal{T}(s)\mathrm{x}\,\mathrm{d}s\right\|_{\mathcal{X}} \\
&= \sup_{\|\mathrm{x}\|\le1} \left\|\int_t^{t_n} \mathcal{T}(s-t)\mathcal{A}\mathcal{T}(t)\mathrm{x}\,\mathrm{d}s\right\|_{\mathcal{X}} \\
&\le \sup_{\|\mathrm{x}\|\le1} \|\mathcal{A}\mathcal{T}(t)\mathrm{x}\|_{\mathcal{X}} \cdot \sup_{s\in[0,1]} \|\mathcal{T}(s)\|_{\mathcal{L}(\mathcal{X})} \cdot |t_n - t| \\
&= K \cdot |t_n - t|,
\end{aligned}$$

if n is large enough. From this we obtain the statement. ∎

We even obtain that the operators of the semigroup are compact for large t.

THEOREM 3.3.4 — *The semigroup $(\mathcal{T}(t))_{t\ge0}$ in eventually compact.*

PROOF — Since $R(\lambda, \mathcal{A})$ is compact by Remark 3.2.2 and $t \mapsto \mathcal{T}(t)$ is norm continuous for $t > 2c = 2\frac{1}{\min_j c_j}$ by the above theorem, we obtain from [EN00, Lemma II.4.28] that $\mathcal{T}(t)$ is compact for $t > 2c$. ∎

As a first consequence of the above result we observe that the Spectral Mapping Theorem from [EN00, Theorem IV.3.10] holds, hence the spectral bound and the growth bound of the semigroup coincide.

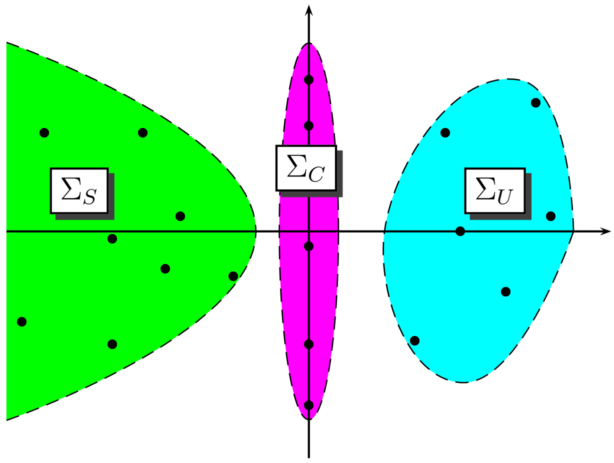

Figure 3.1: The spectrum of $\mathcal{A}$

PROPOSITION 3.3.5 — *For the semigroup $(\mathcal{T}(t))_{t\geq 0}$ we have*

$$\sigma(\mathcal{T}(t)) \setminus \{0\} = \mathrm{e}^{t\sigma(\mathcal{A})}, \ t \geq 0,$$

and

$$s(\mathcal{A}) = \omega_0(\mathcal{T}).$$

In particular, the semigroup is uniformly exponentially stable ($\omega_0(\mathcal{T}) < 0$) if and only if the following implication holds:

$$\lambda x - \mathbf{B}x - \mathbf{A}_\lambda x = 0 \text{ for some } 0 \neq x \in \mathbb{C}^n \Rightarrow \mathrm{Re}\lambda < 0.$$

PROOF — The first equalities follow by the eventually norm continuity of the semigroup, see [EN00, Theorem IV.3.10] and [EN00, Corollary IV.3.11]. The second statement follows from the characteristic equation (3.10) and the fact that the spectrum of the generator of an eventually norm continuous semigroup is bounded on halfplanes $\{\lambda : \mathrm{Re}\lambda \geq b\}$ (cf. [EN00, Theorem II.4.18]). ∎

From the eventually compactness of the semigroup we obtain first a spectral decomposition for the generator.

PROPOSITION 3.3.6 — *For the spectrum of $\mathcal{A}$ the decomposition*

$$\sigma(\mathcal{A}) = \Sigma_U \cup \Sigma_C \cup \Sigma_S$$

into closed subsets holds with

$$\begin{aligned} \Sigma_U &:= \sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}, \\ \Sigma_C &:= \sigma(\mathcal{A}) \cap \mathrm{i}\mathbb{R}, \\ \Sigma_S &:= \sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}. \end{aligned}$$

Here, Σ_U and Σ_C are finite. Furthermore, all spectral points are eigenvalues with finite dimensional spectral projections.

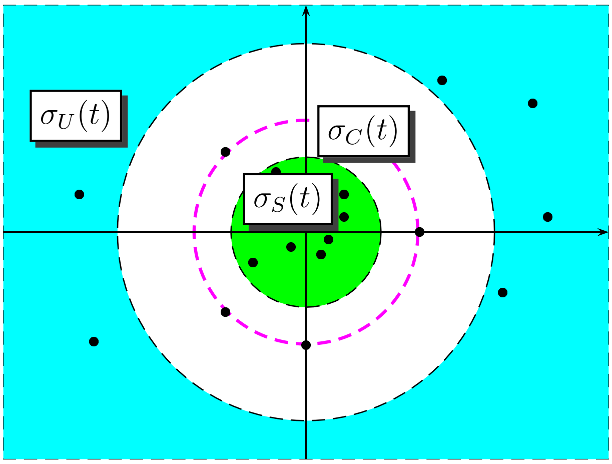

Figure 3.2: The spectrum of $\mathcal{T}(t)$

PROOF — The statement is (i) and (ii) of [EN00, Corollary V.3.2]. ∎

Using the spectral mapping theorem from Proposition 3.3.5 this yields a corresponding decomposition of the spectrum of $\mathcal{T}(t)$.

COROLLARY 3.3.7 — *For the spectrum of the semigroup operators the decomposition*

$$\sigma(\mathcal{T}(t)) = \sigma_U(t) \cup \sigma_C(t) \cup \sigma_S(t)$$

holds with

$$\begin{aligned} |\sigma_U(t)| &> 1, \\ |\sigma_S(t)| &< 1, \\ |\sigma_C(t)| &= 1 \end{aligned}$$

for all $t \geq 0$.

Finally, this spectral decomposition implies a decomposition of the semigroup with the following asymptotic properties.

PROPOSITION 3.3.8 — *There exist subspaces* $\mathcal{X}_S, \mathcal{X}_U$ *and* $\mathcal{X}_C$ *which are invariant under the semigroup such that* $\mathcal{X} = \mathcal{X}_S \oplus \mathcal{X}_U \oplus \mathcal{X}_C$, $\dim \mathcal{X}_C < \infty$, $\dim \mathcal{X}_U < \infty$, *and*

- *the semigroup* $\mathcal{T}_S(t) = \mathcal{T}(t)|_{\mathcal{X}_S}$ *is uniformly exponentially stable,*
- *the semigroup* $\mathcal{T}_U(t) = \mathcal{T}(t)|_{\mathcal{X}_U}$ *is invertible and the semigroup* $\left(\mathcal{T}_U^{-1}(t)\right)$ *is uniformly exponentially stable,*
- *the semigroup* $\mathcal{T}_C(t) = \mathcal{T}(t)|_{\mathcal{X}_C}$ *is a polynomially bounded group, hence has growth bound* 0 *in both time directions.*

PROOF — Using [EN00, Corollary V.3.2(iii)] we obtain the statement. ∎

§ 3.4 POSITIVITY

By the physical interpretation of our system as a flow of certain substance the semigroup should be positive. As in Chapter 2 this fact will then have important consequences for the asymptotic behavior. We investigate this aspect in our situation. We first cite from [EN00, Theorem VI.1.8] the basic characterization for operators generating positive semigroups (see Definition 2.1.4).

PROPOSITION 3.4.1 — *Let $(A, D(A))$ be the generator of a strongly continuous semigroup on a Banach lattice X.*

(i) The semigroup is positive if and only if the resolvent $R(\lambda, A)$ is a positive operator for all λ large enough.

(ii) In the finite dimensional case, the semigroup is positive if and only if the matrix A is real and positive off-diagonal.

Based on the above criteria we can characterize the positivity of our semigroup.

THEOREM 3.4.2 — *If $\mathbf{B} \in \mathcal{M}_n(\mathbb{C})$ is real and positive off-diagonal, then the semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ is positive.*

PROOF — We will use Proposition 3.4.1 and show that $R(\lambda, \mathcal{A})$ is positive for λ large enough. For this purpose we have to prove that the entries of the operator matrix (see (3.6))

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} [D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) M + I_X] R(\lambda, A_0) & D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \\ R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) M R(\lambda, A_0) & R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \end{pmatrix}$$

are all positive for large λ. As can be seen from the form (2.16) of $R(\lambda, A_0)$,

$$(R(\lambda, A_0) f)(s) = \int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\sigma)^{-1} C(\sigma)^{-1} f(\sigma) \mathrm{d}\sigma, \; s \in [0,1], \; f \in X,$$

it is positive for every real λ. Observe that the operator $M = \Phi_w^+ \otimes \delta_0$ is positive because Φ_w^+ is a positive matrix. Hence $MR(\lambda, A_0)$ is also positive for real λ.
Under the above assumptions, $\mathbf{B}$ generates a positive (matrix)semigroup, hence $R(\lambda, \mathbf{B})$ is positive for λ large enough. Using the equality

$$\lambda - \mathbf{B} - \mathbf{A}_\lambda = (\mathbf{1} - \mathbf{A}_\lambda R(\lambda, \mathbf{B}))(\lambda - \mathbf{B})$$

and the Neumann series

$$R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) = R(\lambda, \mathbf{B}) \sum_{n=0}^{\infty} (\mathbf{A}_\lambda R(\lambda, \mathbf{B}))^n,$$

by $\mathbf{A}_\lambda \geq 0$ we obtain that $R(\lambda, \mathbf{B} + \mathbf{A}_\lambda)$ is also positive for large λ. Combining all these facts with the positivity of $D_\lambda = \epsilon_\lambda (\Phi_w^-)^\top$, we have that all the entries of (3.6) are positive for large λ. ∎

Combining the positivity and the eventually norm continuity of the semigroup $(\mathcal{T}(t))_{t\geq 0}$, we obtain that the generator $\mathcal{A}$ has a dominant eigenvalue (see [EN00, Theorem VI.1.10]). More precisely, the following holds.

PROPOSITION 3.4.3 — *If* $\mathbf{B}$ *be is real and positive off-diagonal, then there exists* $\varepsilon > 0$ *such that*

$$\sigma(\mathcal{A}) = \{s(\mathcal{A})\} \cup \{\lambda \in \sigma(\mathcal{A}) : \operatorname{Re}\lambda \leq s(\mathcal{A}) - \varepsilon\}.$$

In order to determine the dominant eigenvalue $s(\mathcal{A})$ we first state an important property of the spectral bound function

$$s(\lambda) := s(\mathbf{B} + \mathbf{A}_\lambda),$$

which can be found in [BP04, Proposition 6.2.5].

LEMMA 3.4.4 — *Let* $\mathbf{B}$ *be real and positive off-diagonal. Then the spectral bound function* $\mathbb{R} \ni \lambda \mapsto s(\lambda)$ *is decreasing and continuous.*

PROPOSITION 3.4.5 — *Let* $\mathbf{B} \in \mathcal{M}_n(\mathbb{C})$ *be real and positive off-diagonal. Then* $s(\mathcal{A})$ *is the unique real solution of the characteristic equation*

$$\lambda = s(\lambda) \tag{3.15}$$

and for the spectral bound $s(\mathcal{A})$ *the following equivalences hold:*

$$s(\lambda) \lesseqqgtr \lambda \Longleftrightarrow s(\mathcal{A}) \lesseqqgtr \lambda.$$

PROOF — From the assumption follows that $\mathbf{B}$, hence $(\mathcal{A}, D(\mathcal{A}))$ generate positive semigroups, see Theorem 3.4.2. Clearly $\sigma(\mathbf{B}) \neq \emptyset$, hence $-\infty < s(\mathbf{B}) \leq s(\lambda)$ for all $\lambda \in \mathbb{R}$ by the positivity of $\mathbf{A}_\lambda$, and using that for positive matrices the spectral bound equals the spectral radius. By the above Lemma 3.4.4 the equation (3.15) has a unique solution λ_0. Since $\mathbf{B} + \mathbf{A}_\lambda$ generates a positive semigroup, we can again use [EN00, Theorem VI.1.10] and obtain $\lambda_0 = s(\lambda_0) \in \sigma(\mathbf{B} + \mathbf{A}_{\lambda_0})$, hence $\lambda_0 \in \sigma(\mathcal{A})$ by (3.10). However, for all $\mu > \lambda_0$, using Lemma 3.4.4, we have

$$\mu > \lambda_0 = s(\lambda_0) \geq s(\mu),$$

hence $\mu \notin \sigma(\mathbf{B} + \mathbf{A}_\mu)$ and so $\mu \in \rho(\mathcal{A})$ by (3.10). Therefore $\lambda_0 = s(\mathcal{A})$. The estimates on $s(\mathcal{A})$ follow from these considerations. ∎

From Proposition 3.3.5 and 3.4.5 we obtain a simple necessary and sufficient condition for the uniform exponential stability of the semigroup.

COROLLARY 3.4.6 — *If* $\mathbf{B} \in \mathcal{M}_n(\mathbb{C})$ *is real and positive off-diagonal, then the semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *is uniformly exponentially stable if and only if the spectral bound* $s(\mathbf{B} + \mathbf{A}_0) < 0$.

INTERPRETATION — The above characterization of uniform exponential stability depends on the spectral bound of $\mathbf{B} + \mathbf{A}_0$. Here $\mathbf{A}_0$ is the usual (weighted) adjacency matrix of our graph. The matrix $\mathbf{B}$ can be interpreted as the (weighted) adjacency matrix of an "imaginary" graph, whose vertices belong to the original graph but the edges do not. Its (directed) edges are those along which we control the outgoing flow velocities, depending on the outgoing flow mass in the vertices. This control happens in every time moment immediately. Hence we can say that on these "imaginary edges" the information passes with infinite velocity. Observe that from Proposition 3.4.5 follows that

$$\begin{aligned} s(\mathcal{A}) &< 0 \Longleftrightarrow s(\mathbf{B}+\mathbf{A}_0) < 0, \\ s(\mathcal{A}) &> 0 \Longleftrightarrow s(\mathbf{B}+\mathbf{A}_0) > 0 \\ \text{and } s(\mathcal{A}) &= 0 \Longleftrightarrow s(\mathbf{B}+\mathbf{A}_0) = 0, \end{aligned}$$

that is only the *joint* structure of the original graph and the "imaginary graph" determines the asymptotic behavior of the system. That means, we can change "real" edges to "imaginary" edges and vice versa without changing the stability of the whole system. In other words:

"Stability is independent of the velocity of transportation."

In the case in which $\mathbf{B}$ is a (real) diagonal matrix – i.e., in each vertex, the effect of the outgoing flow mass on its velocity is independent of the other vertices –, $\mathbf{B}$ always generates a positive semigroup and we obtain the following simple criteria.

COROLLARY 3.4.7 — *If $\mathbf{B} \in \mathcal{M}_n(\mathbb{C})$ is diagonal and $\mathbf{B} \leq -\alpha \cdot \mathbf{1}$ with $\alpha > s(\mathbf{A}_0)$, then the semigroup is uniformly exponentially stable.*

PROOF — Under the assumptions $\mathbf{B}+\mathbf{A}_0 \leq -\alpha\cdot\mathbf{1}+\mathbf{A}_0$, hence, using positivity, $s(\mathbf{B}+\mathbf{A}_0) \leq s(-\alpha\cdot\mathbf{1}+\mathbf{A}_0) < 0$. By Corollary 3.4.6 we obtain uniform exponential stability for $(\mathcal{T}(t))_{t\geq 0}$. ∎

§ 3.5 IRREDUCIBILITY

In this section we always assume that $\mathbf{B} \in \mathcal{M}_n(\mathbb{C})$ is real and positive off-diagonal, hence the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is positive. We are now going to investigate when our semigroup becomes *irreducible* (see Definition 2.3.1). In this case $s(\mathcal{A})$ is an algebraically simple pole of the resolvent of the generator $\mathcal{A}$, admitting a one dimensional spectral projection P, and we can describe precisely the asymptotic behavior of $(\mathcal{T}(t))_{t\geq 0}$.

PROPOSITION 3.5.1 — *If the matrix $\mathbf{B}+\mathbf{A}_0$ is irreducible, then $(\mathcal{T}(t))_{t\geq 0}$ is irreducible on $\mathcal{X}$.*

PROOF — From Proposition 3.4.5 follows that $\lambda > s(\mathcal{A})$ holds if and only if $\lambda > s(\mathbf{B}+\mathbf{A}_\lambda)$. Since the zero patterns of $\mathbf{B}+\mathbf{A}_0$ and $\mathbf{B}+\mathbf{A}_\lambda$ coincide for every $\lambda\in\mathbb{C}$, the assumptions imply that $\mathbf{B}+\mathbf{A}_\lambda$ is irreducible for every $\lambda\in\mathbb{R}$. Using [Sch74, Proposition I.6.2] we obtain that for $\lambda > s(\mathcal{A})$ the matrix $R(\lambda, \mathbf{B}+\mathbf{A}_\lambda)$ is strictly positive. Take now a vector $L^1([0,1],\mathbb{C}^m)\times\mathbb{C}^n \ni \binom{f}{x} \gneqq 0$, and investigate $R(\lambda,\mathcal{A})\binom{f}{x}$ using

$$R(\lambda,\mathcal{A}) = \begin{pmatrix} [D_\lambda R(\lambda,\mathbf{B}+\mathbf{A}_\lambda)M + I_X]R(\lambda,A_0) & D_\lambda R(\lambda,\mathbf{B}+\mathbf{A}_\lambda) \\ R(\lambda,\mathbf{B}+\mathbf{A}_\lambda)MR(\lambda,A_0) & R(\lambda,\mathbf{B}+\mathbf{A}_\lambda) \end{pmatrix}$$

from (3.6). In the second coordinate we obtain

$$R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \, M R(\lambda, A_0) \, f + R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \, x,$$

where the second term is strictly positive by the above consideration. From the form of $R(\lambda, A_0)$ follows that the function $R(\lambda, A_0) \, f$ is strictly positive except on the largest interval $(1-\varepsilon, 1]$ for which $f \mid_{(1-\varepsilon,1]} = 0$. Applying $M = \Phi_w^+ \otimes \delta_0$ to it we obtain a vector y of positive numbers, hence $R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \, y$ yields a strictly positive vector. For the first coordinate we have

$$D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \, M f + R(\lambda, A_0) \, f + D_\lambda R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \, x.$$

As before, we have that $R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \, M f$ and $R(\lambda, \mathbf{B} + \mathbf{A}_\lambda) \, x$ are strictly positive vectors of numbers. Using the strict positivity of exponential functions and the positivity of $(\Phi_w^-)^\top$, we obtain that $D_\lambda = \epsilon_\lambda \, (\Phi_w^-)^\top$ is strictly positive, hence the first and third terms are vectors of (everywhere) strictly positive functions. The second term is again positive, hence the sum yields a strictly positive vector of $L^1[0,1]$-functions. ∎

The irreducibility of $\mathbf{B} + \mathbf{A}_0$ can again be related to the strong connectedness of the graph G, see Proposition 1.3.10. If G is already strongly connected, then $\mathbf{A}_0$ is irreducible, hence for any positive off-diagonal $\mathbf{B}$, the matrix $\mathbf{B} + \mathbf{A}_0$ is irreducible and the assumption in the above theorem is satisfied. If the graph is not strongly connected, we can describe the irreducibility of $\mathbf{B} + \mathbf{A}_0$ in the following way. Let us assume that $\mathbf{B}$ has positive entries $b_{ip} > 0$ for index pairs (ip) such that adding edges to G pointing from v_p to v_i we obtain a strongly connected graph. In this case $\mathbf{B} + \mathbf{A}_0$ becomes irreducible, and we again have the result above. The condition on the entries of $\mathbf{B}$ means that the outgoing flow is controlled along such “imaginary” edges that make the graph strongly connected. Hence, here also only the *joint* structure of the “real” and the “imaginary” graph determines the irreducibility.

COROLLARY 3.5.2 — *Assume that after adding edges to G from v_p to v_i where the corresponding entry of $\mathbf{B} = (b_{ip})_{n \times n}$ is different from 0, the graph G becomes strongly connected. Then the semigroup $(\mathcal{T}(t))_{t \geq 0}$ is irreducible on $\mathcal{X}$.*

The following result on the asymptotics of the semigroup now follows from the general theory of positive semigroups (see [Nag86, Chapter C-IV] and [EN00, Section V.3]).

THEOREM 3.5.3 — *Under the conditions of Corollary 3.5.2 there exists a strictly positive one-dimensional projection $\mathcal{P} = \mu \otimes y$, $\mu \in \mathcal{X}'$ with $\mathcal{P}x = \mu(x) \cdot y$ for all $x \in \mathcal{X}$, such that*

$$\lim_{t \to +\infty} \left\| \mathrm{e}^{-s(\mathcal{A})t} \mathcal{T}(t) - \mathcal{P} \right\| = 0.$$

If $s(\mathcal{A}) = 0$ (e.g. $\mathbf{B} = -s(\mathbf{A}_0) \cdot \mathbf{1}$), then $(\mathcal{T}(t))_{t \geq 0}$ converges to the projection $\mathcal{P}$.

PROOF — By Proposition 3.4.3 we know that the spectral bound $s(\mathcal{A})$ is a dominant eigenvalue of $\mathcal{A}$. Using the irreducibility and [Nag86, Proposition C-III.3.5], $s(\mathcal{A})$ is a first-order pole of the resolvent and the corresponding residue has the form $\mathcal{P} = \mu \otimes \mathrm{y}$, where μ and y are strictly positive eigenvectors of $\mathcal{A}'$ and $\mathcal{A}$, respectively. By [EN00, Corollary V.3.3] we now have the desired result. ∎

The property above is called *balanced exponential growth* (or *asynchronous exponential growth*) and plays an important role in applications, e.g., to population equations (see [DBW02]).

Chapter 4

Examples

[1]As a first concrete example we investigate an orientation of the well-known Petersen graph. On this graph we consider the process described by the system (F) (see Chapter 2) with constant velocities (normalized to 1) and no absorption – that is, $c_j \equiv 1$ and $q_j \equiv 0$ on all edges.

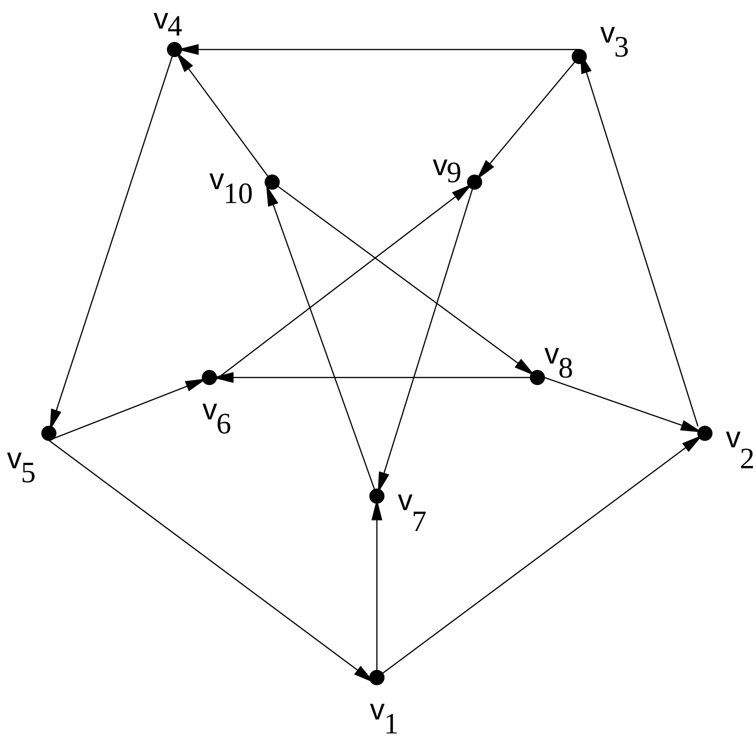

Figure 4.1: An oriented Petersen graph

By Theorem 2.4.8, the system is asymptotically periodic with period equal to $\gcd\{$cycle lengths$\}$, hence in our case equal to 1. We are now interested in the velocity of the

[1]Computations made with Maple

convergence, that is the (optimal) value of ε in Proposition 2.4.5 for which

$$\|T_2(t)\| \leq M \cdot \mathrm{e}^{-\varepsilon t}.$$

More precisely we investigate how ε is related to the weights on the edges. From the characteristic equation (2.39) and the Circular Spectral Mapping Theorem 2.4.3 follows that

$$\varepsilon = -\log r, \tag{4.1}$$

where r is the second largest absolute value – that is, the largest absolute value different from 1 – of the spectral points of $\tilde{\mathbf{A}}$ in (2.38).

1. To start we choose identical outgoing flow mass proportions for each ramification node. Then the weighted adjacency matrix is

$$\tilde{\mathbf{A}}_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

   Its eigenvalues are (approx.):

   $1.0,$

   $0.35944 + 0.81346\mathrm{i},\ 0.35944 - 0.81346\mathrm{i},$

   $-0.57462 + 0.6659\mathrm{i},\ -0.57462 - 0.6659\mathrm{i},$

   $0.2135 + 0.5791\mathrm{i},\ 0.2135 - 0.5791\mathrm{i},$

   $-0.6478 + 0.16971\mathrm{i},\ -0.6478 - 0.16971\mathrm{i},$

   $0.29897.$

   The corresponding absolute values (different from 1) are:

   $r_1 = \sqrt{0.35944^2 + 0.81346^2} = 0.88933$

   $r_2 = \sqrt{0.57462^2 + 0.6659^2} = 0.87955,$

   $r_3 = \sqrt{0.2135^2 + 0.5791^2} = 0.6172,$

   $r_4 = \sqrt{0.6478^2 + 0.16971^2} = 0.66966.$

   Hence, in this case $\varepsilon \approx -\ln \sqrt{0.35944^2 + 0.81346^2} \approx -\ln 0.88933 \approx 0.11728.$

2. We now change the proportions in the vertex v_1 such that 80% part of the mass flows into the "inner part" of the graph, hence to the vertex v_7, i.e.,

$$\tilde{\mathbf{A}}_2 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Computing the greatest non-1 absolute value of the spectrum points we have

$$r_1 = \sqrt{0.35505^2 + 0.83163^2} = 0.90425.$$

Hence, in this case the convergence speed is smaller.

3. We next require that more flow mass from v_1 remains in the "outer part" of the graph, e.g., only 20% of the flows goes into the vertex v_7, hence

$$\tilde{\mathbf{A}}_3 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since there is an eigenvalue with modulus $r_1 = \sqrt{0.58224^2 + 0.67469^2} = 0.89118$, the convergence speed is again smaller than in case 1 but greater than in the previous case.

4. We now set equal outgoing flow proportions in the "outer part" of the graph, and change the weights in v_8 in the following way.

$$\tilde{\mathbf{A}}_4 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.8 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Here the absolute value $r_3 = \sqrt{0.35375^2 + 0.85231^2} = 0.92281$ of the 3^d eigenvalue is greater than any absolute value in the first example –, the convergence speed will be smaller than in the case of equal proportions.

5. In the next step we change the proportions for v_8 in the opposite direction and obtain

$$\tilde{\mathbf{A}}_5 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.2 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since the modulus of the first spectral point is $r_1 = \sqrt{0.53892^2 + 0.71154^2} = 0.89259$, the convergence speed is smaller than in the first case, but it is greater than in case 4.

6. We now investigate in general how the weight in v_1 as parameter effects the convergence speed to the periodic semigroup. Let

$$\tilde{\mathbf{A}}_6 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

for $0 < a < 1$. The characteristic polynomial becomes $p(z) = z^{10} - 0.5z^5 - 0.375z^4 - 0.0625(1+a)\cdot z^2 - 0.0625z + 0.0625a$. Clearly, $z_1 = 1$ is a root of $p(z)$. Dividing $p(z)$ by $z-1$ yields a polynomial $p_1(z)$ whose root with the greatest absolute value – depending on a – is the value r occuring in (4.1). Actually, r is the greatest absolute value of the roots of the polynomial $\tilde{p}_1(z) = 16z^9 + 16z^8 + 16z^7 + 16z^6 + 16z^5 + 8z^4 + 2z^3 + 2z^2 + (1-a)\,z - a$. Figure 4.2 shows how r depends on a.

The value of r has a minimum at approximately $a = 0.6085 \pm 0.0003$, which means that in this case the convergence speed is maximal.

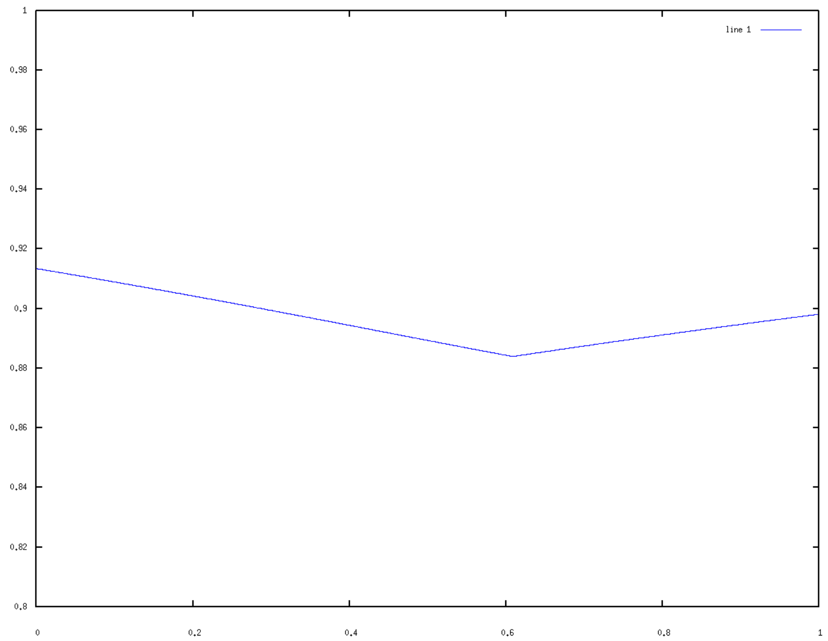

Figure 4.2: $r = \mathrm{e}^{-\varepsilon}$ (ε: convergence speed) depending on the weight in v_1

7. We now take equal outgoing flow proportions in v_1 and investigate the effect of different proportions in an “inner” vertex.

$$\tilde{\mathbf{A}}_7 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 1-a & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We obtain a relationship as shown on Figure 4.3 between the parameter a and the second greatest absolute value of the spectral points r.

The maximal convergence speed is attained at $a \approx 0.56295 \pm 0.00006$.

We now investigate an orientation of the Herschel graph, see Figure 4.4.

On this graph we again consider the process described by the system (F) (see Chapter 2) with constant velocities (normalized to 1) and no absorption – that is, $c_j \equiv 1$ and $q_j \equiv 0$ on all edges.

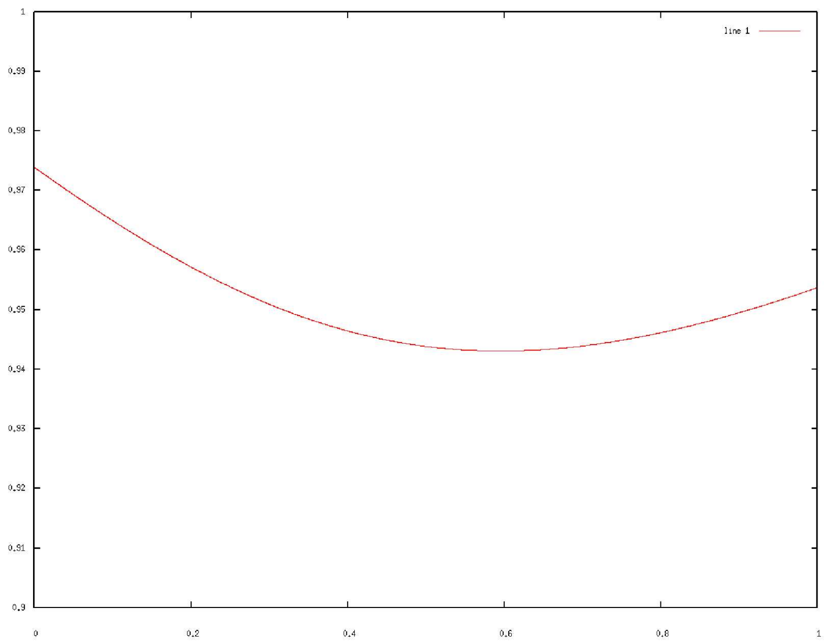

Figure 4.3: $r = \mathrm{e}^{-\varepsilon}$ (ε: convergence speed) depending on the weight in v_8

By Theorem 2.4.8, the system is asymptotically periodic with period equal to $\gcd\{$cycle lengths$\}$, hence in our case equal to 2. We are again interested in the velocity of the convergence, that is the (optimal) value of ε in Proposition 2.4.5 for which

$$\|T_2(t)\| \leq M \cdot \mathrm{e}^{-\varepsilon t}.$$

1. We set at the vertex v_1 outgoing weights a resp. $1-a$ into the vertex v_5 resp. v_9. Then the weighted adjacency matrix becomes

$$\tilde{\mathbf{A}}_1 := \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}$$

with $0 < a < 1$. Computing the characteristic polynomial, we obtain $p(z) = z^{11} - 0.75z^7 - 0.25z^5$ – hence, it does not depend on the weights in v_1. As a consequence, the eigenvalues and the convergence speed are also independent of a.

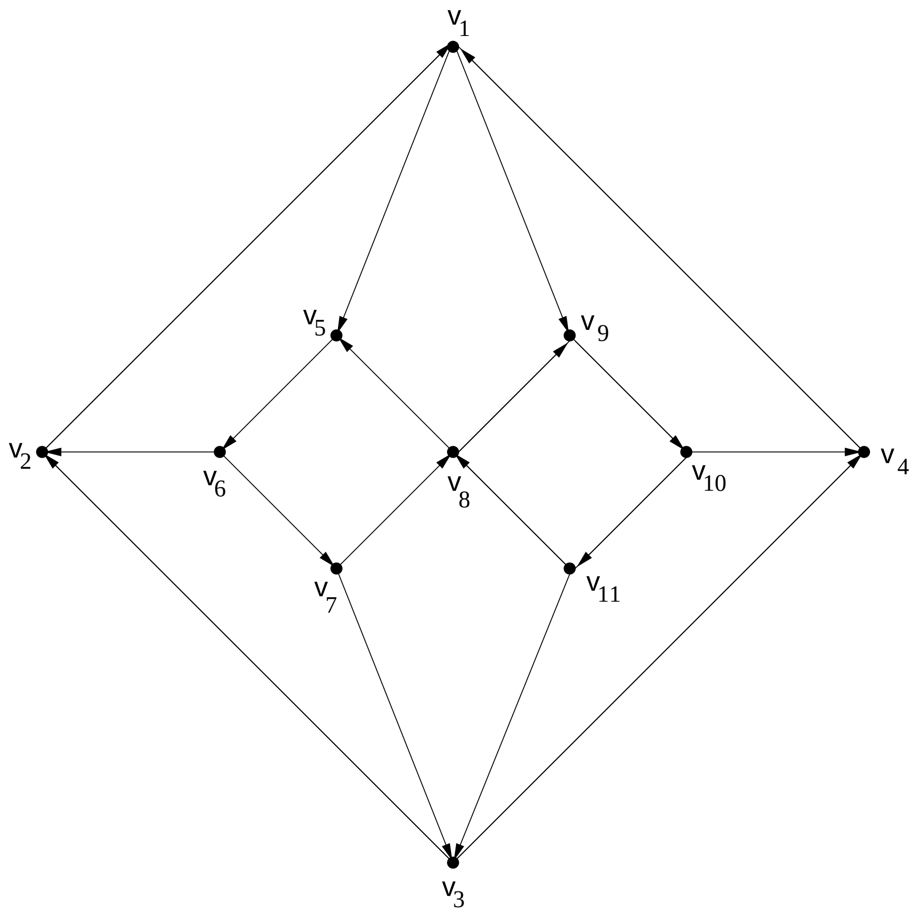

Figure 4.4: An oriented Herschel graph

For the eigenvalues we obtain $-1, 1, 0$ and approximately $0,707107\mathrm{i}, -0,707107\mathrm{i}$. Hence, the convergence speed is

$$\varepsilon \approx -\log 0,707107 \approx 0.346574.$$

2. If we now require equal distribution weights in v_1 and change them in the vertex v_8, we obtain as adjacency matrix

$$\tilde{\mathbf{A}}_2 := \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 1-a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}.$$

Surprisingly, the characteristic polynomial – hence also the eigenvalues – is the same as in the previous case. Therefore the convergence speed again does not depend on the distribution weights and it has the same value as before.

3. We now change the weights in the vertex v_7 and obtain

$$\tilde{\mathbf{A}}_3 := \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-a & 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}.$$

The eigenvalues are: 0 with multiplicity 5, further 1 and -1 with multiplicity 1, and

$$\begin{aligned} z_{1,2} &= \pm\frac{1}{2}\sqrt{-2+\sqrt{2-4a}}, \\ z_{3,4} &= \pm\frac{1}{2}\sqrt{-2-\sqrt{2-4a}}. \end{aligned}$$

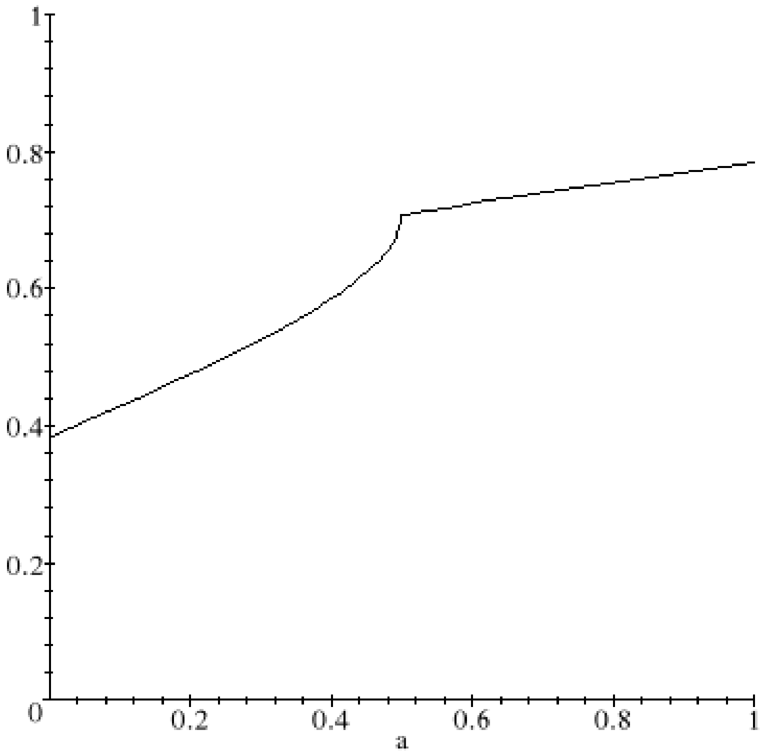

Figure 4.5: $|z_{1,2}|$ depending on a

Figures 4.5 and 4.6 illustrate how $|z_{1,2}|$ resp. $|z_{3,4}|$ depend on the parameter a.

We are interested in the root with the largest absolute value different from 1. If $0 < a \leq 0.5$, then $|z_{3,4}| \geq |z_{1,2}|$ and if $0.5 \leq a < 1$, then the absolute values of $z_{1,2}$ and $z_{3,4}$ all coincide. It is easy to see that for $a = 0.5$ the minimum of the (second) largest absolute value, that is $|z_{3,4}|$, is attained and is equal to $r = \frac{1}{2}\sqrt{2}$. Hence, the convergence speed is

$$\varepsilon = -\log \frac{1}{2}\sqrt{2} \approx 0.34657.$$

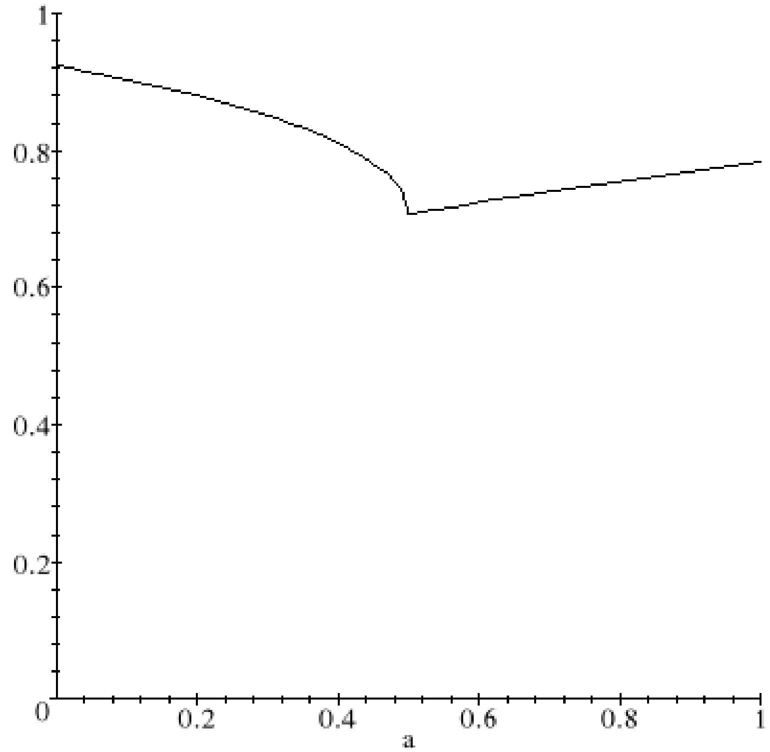

Figure 4.6: $|z_{3,4}|$ depending on a

Final remark. A systematic analysis of the dependence of the convergence speed on the weights of the graph edges is, at this stage, not in sight.

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Appendix A

Zusammenfassung in deutscher Sprache

Das Ziel der vorliegenden Arbeit ist, lineare Transportprozesse (oder Flüsse) in Netzwerken zu untersuchen. Zu diesem Zweck stellen wir den funktionalanalytischen Rahmen auf und benutzen halbgruppen- und spektraltheoretische Methoden. Die Hauptresultate stammen aus den Artikeln [KS04], [MS04] und dem Preprint [Sik04].

Im ersten Kapitel geben wir eine kurze Übersicht über die wichtigsten Notationen und Sätze aus der Graphentheorie, die zur Behandlung des funktionalanalytischen Modells nötig sind. Wir modellieren das Netzwerk durch einen gerichteten Graphen, auf dessen Kanten eine Substanz in die angegebenen Richtungen fliesst und in den Ecken neu verteilt wird.

Im zweiten Kapitel betrachten wir Transportprozesse in Netzwerken mit statischen Verzweigungsknoten. Das bedeutet, dass wir in jedem Zeitpunkt in den Ecken Bedingungen für die Menge der Flussmasse vorschreiben. Insbesondere verlangen wir, dass die gesamte einkommende gleich der gesamten ausgehenden Flussmasse ist (*Kirchhoffsche Regeln*) und dass die ausgehende Flussmasse in die ausgehenden Kanten nach angegebenen Proportionen weiterfliesst. Wir zeigen, dass das zugehörige System von partiellen Differentialgleichungen und entsprechenden Randbedingungen als ein abstraktes Cauchy Problem auf einem Banachraum umgeschrieben werden kann. Wir zeigen die Wohlgestelltheit dadurch, dass der unterliegende Operator eine starkstetige Halbgruppeerzeugt, die die Lösungen für das originelle System angibt. Mit Hilfe von spektral- und halbgruppentheoretischen Methoden können wir präzise das asymptotische Verhalten von der Halbgruppe – d.h., von dem Prozess in dem Netzwerk – beschreiben. In der Qualität der Asymptotik zeigt sich eine Dichotomie, abhängig davon ob eine zahlentheoretische Bedingung für die Flussgeschwindigkeiten auf den Kanten – siehe Definition 2.3.7 – besteht oder nicht. Den ersten Fall untersuchen wir in §2.4, und hier konvergiert der Prozess gleichmässig gegen einen periodischen Fluss, dessen Periode von der Graphenstruktur bestimmt ist, siehe Theoreme 2.4.8 und 2.4.11. Im zweiten Fall – betrachtet in §2.5 – konvergiert der Fluss in der starken Operatortopologie gegen ein Gleichgewicht.

Wir untersuchen dann im Kapitel 3 Transportprozesse, bei denen in den Verzweigungsknoten dynamische Bedingungen vorgeschrieben sind. Das heisst, die Geschwindigkeit der gesamten ausgehenden Flussmasse ist bestimmt als eine gewichtete Summe von einkommenden Flussmengen plus ein Term, der von den ausgehenden Flussmasswerten in den Ecken abhängt. Der zweite Term kann auch als eine *Feedback-Kontrolle* der ausgehenden Flussgeschwindigkeiten interpretiert werden, wo sich die Kontrolle entlang “imaginären Kanten” abspielt, deren Endpunkte Ecken in dem originellen Graphen sind, die aber nicht unbedingt zu der originellen Kantenmenge gehören. Zur Behandlung dieses Problems benutzen wir einen entsprechend modifizierten halbgruppentheoretischen Ansatz für retardierte Differentialgleichungen, der in [BP04] entwickelt wurde. Wir können die Wohlgestelltheit wieder dadurch beweisen, dass wir das Problem in die Form eines abstrakten Cauchy Problems umschreiben und so eine starkstetige Halbgruppe bekommen, die die Lösungen angibt. Wir zeigen, dass diese Halbgruppe wichtige Regularitätseigenschaften hat (siehe Theoreme 3.3.2 und 3.3.4), die den “Spektralen Abbildungssatz” implizieren. Bei der Untersuchung der Asymptotik stellt sich in §3.4 heraus, dass wenn die Halbgruppe positiv ist, deren Stabilität an der Negativität der Spektralschranke der sog. *Adjazenzmatrix* (siehe Definition 1.3.6) von einem Graphen liegt. Dieser Graph entsteht dadurch, dass wir zu dem originellen Graphen die “imaginären Kanten” addieren, entlang denen die Feedback-Kontrolle wirkt (siehe Corollary 3.4.6 und die nachfolgende Interpretation). In dem Fall von Irreduzibilität (§3.5) konvergiert die Halbgruppe gegen ein Gleichgewicht, wenn die gemeinsame Struktur des originellen und des “imaginären” Graphen stark zusammenhängend ist (siehe Theorem 3.5.3).

Im letzten Kapitel erörtern wir Beispiele für die Situation von Kapitel 2 auf dem *Petersen* und *Herschel Graph*, und wir berechnen die Konvergenzgeschwindigkeit gegen den periodischen Fluss. Wir untersuchen auch, wie diese von den Gewichten auf den Kanten abhängt.

Appendix B

Lebenslauf

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