Nonlocal Cauchy Problems and Delay Equations

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Zusammenfassung in deutscher Sprache

In dieser Arbeit untersuchen wir Cauchyprobleme für Differentialgleichungen mit nichtlokalen Anfangsbedingungen und Cauchyprobleme für abstrakten Differentialgleichungen mit unendlicher Verzögerung (siehe, z.B., [13–15, 19, 35, 36, 46, 51, 52, 86, 87] für Motivation und konkrete Anwendungen).

In Kapitel 1 erhalten wir solche Ergebnisse für semilineare Integrodifferentialgleichungen, die bekannte Resultate aus [14, 17, 61, 70] wesentlich verallgemeinern. Dies wird an Beispielen aus der Wärmeleitungsgleichung in Materialien mit Gedächtnis gezeigt.

In Kapitel 2 wird diese Untersuchung für semilineare Evolutionsglei-chungen weitergeführt. Mit Hilfe (C, ω, M_η)-zulässiger Paare erhalten wir neue Existenzresultate für milde und klassische Lösungen.

Im dritten Kapitel untersuchen wir Cauchyprobleme für Funktionaldifferentialgleichungen in Banachräumen mit unendlicher Verzögerung. In Abschnitt 2 diskutieren wir die Gleichung zu einem Cauchyprobleme auf einem Banachraum X der Form

$$\begin{cases} u(t) = g(t) + \int_\sigma^t f(t, s, u(s), u_s)ds \quad (\sigma \le t \le T), \\ u_\sigma = \phi, \end{cases}$$

wobei $0 \le \sigma < T$, $g(t) \in C([\sigma, T], X)$, $f \in C([\sigma, T] \times [\sigma, T] \times X \times \mathcal{P}, X)$ und $\phi \in \mathcal{P}$ (einem Zulässig-Phasenraum). In Abschnitten 3 - 5 untersuchen wir die folgenden Typen von Cauchyproblemen für Funktionaldifferentialgleichungen mit unendlicher Verzögerung:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t), u_t), \quad 0 \le t \le T, \\ u_0 = \phi, \end{cases}$$

$$\begin{cases} u'(t) = A(t)u(t) + f(t, u(t), u_t), \quad 0 \le t \le T \\ u_0 = \phi \end{cases}$$

(nichtautonome Cauchyprobleme), und

$$\begin{cases} u'(t) = A\left[u(t) + \int_0^t F(t-s)u(s)ds\right] + f(t, u(t), u_t), \quad 0 \le t \le T \\ u_0 = \phi \end{cases}$$

(Integrodifferential-Cauchyprobleme), wobei $T > 0$, A und $\{A(t)\}_{t\geq 0}$ lineare Operatoren auf einem Banachraum X sind, $\{F(t)\}_{0\leq t\leq T} \subset \mathbf{L}(X)$, $f \in C([0,T]\times X\times \mathcal{P}, X)$, und $\phi \in \mathcal{P}$. Eine Reihe von neuen Resultaten erhalten wir mit Hilfe Nichtkompaktheitmaßen und Kamke-Funktionen oder Lipschitz-Bedingungen.

In Kapitel 4 beweisen wir Regularitätseigenschaften der Lösungen, falls der Banachraum die Radon-Nikodym Eigenschaft besitzt.

Kapitel 5 enthält eine Untersuchung der Wohlgestelltheit abstrakter Funktionaldifferentialgleichungen und nichtautonomer semilinearer Funktional-Evolutionsgleichungen mit unendlicher Verzögerung in beliebigen Banachräumen. Unter der Annahme, dass der nichtlineare Term Fréchetdifferenzierbar ist, erhalten wir Verallgemeinerungen von Ergebnissen von [3, 8, 13, 22, 23, 35, 36, 45, 46, 48, 51, 58, 59, 71, 77, 78, 84, 86, 87]). Die Wohlgestelltheitresultate für nichtautonomen Cauchyprobleme ist ganz neu.

Contents

Introduction

Nonlocal Cauchy problems

Nonlocal Cauchy problem, namely the Cauchy problem for a differential equation with a nonlocal initial condition $u(t_0) + g(t_1, \ldots, t_p, u) = u_0$ (here $0 \leq t_0 < t_1 < \cdots < t_p \leq t_0 + T$ and g is a given function), is one of the important topics in the study of the analysis theory. Interest in such a problem stems mainly from the better effect of the nonlocal initial condition than the usual one in treating physical problems. Actually, the nonlocal initial condition $u(t_0) + g(t_1, \ldots, t_p, u) = u_0$ models many interesting nature phenomena, with which the normal initial condition $u(0) = u_0$ may not fit in. For instance, the function $g(t_1, \ldots, t_p, u)$ may be given by $g(t_1, \ldots, t_p, u) = \sum_{i=1}^{p} c_i u(t_i)$ (c_i ($i = 1, \ldots, p$) are constants). In this case, we are permitted to have the measurements at $t = 0$, t_1, $\ldots$, t_p, rather than just at $t = 0$. Thus more information is available. More specially, letting $g(t_1, \ldots, t_p, u) = -u(t_p)$ and $u_0 = 0$ yields a periodic problem and letting $g(t_1, \ldots, t_p, u) = -u(t_0) + u(t_p)$ gives a backward problem. From Byszewski [14, 15], L. Byszewski and V. Lakshmikantham [19] and the references given there, one can find other information about the importance of nonlocal initial conditions in applications. There have been many papers concerning this topic (cf., e.g., [5, 9, 14, 15, 17–19, 52, 61, 68] and references therein). However, much of the previous research was done under the condition "$M(K + TL) < 1$" (where M, K, T and L are some internal constants in the related nonlocal Cauchy problem) or its analogues (cf., e.g., [14, 17, 61] or Chapter 1 of this thesis). This condition turns out to be quite restrictive. In particular, limited by it, the results obtained for nonlocal problems can not cover those classical results regarding Cauchy problems with normal initial data. Thus, there naturally arises a question:

> *Can the above condition be relaxed such that the results for nonlocal Cauchy problems cover the corresponding ones for normal Cauchy problems?*

In Chapter 1, we are concerned with the Cauchy problem for semilinear integro-differential equations with nonlocal initial conditions. Under general and natural hypotheses, we establish some new theorems about the existence and uniqueness of solutions for the nonlocal Cauchy problem. As a consequence, we give an affirmative answer to the question above for such a nonlocal Cauchy problem, and we also unify and extend the corresponding theorems given previously for the Cauchy problem for differential equations or integrodifferential equations with nonlocal initial conditions. Moreover, we present two examples, one of which comes from heat conduction in materials with memory, to indicate that, in contrast with ours, the previous results are not applicable to them.

In Chapter 2, we continue our study of the nonlocal Cauchy problems. Our target now is to give some new results about the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems for semilinear evolution equations. We introduce a new notion, called (C, ω, M_η)-*admissible pair*, and carry out our investigation in Banach spaces $W^{B,\omega}_{\eta_1,\eta_2}(T)$ motivated by Jackson [52]. We prove certain nonlinear convolution integral equations in Banach spaces, to which the existing related results did not apply, to possess continuous solutions. As applications, new existence and uniqueness theorems for mild and classical solutions of nonlocal Cauchy problems for semilinear evolution equations are obtained. Moreover, a result on the existence and uniqueness of a classical solution of a semilnear parabolic equation with a boundary condition and a nonlocal initial condition is given as an example. The present results generalize some previous related theorems. Furthermore, even for classical semilinear abstract Cauchy problems, the results here are new.

Delay equations

Equations with delay (i.e., with some of the past states of the systems) are often more realistic mathematical models for practical problems compared with those without delay, and they have been studied for many years (see, e.g., [3, 7, 8, 10–13, 22, 23, 32, 35, 36, 44–48, 51, 53–60, 63, 71, 74–79, 83, 84, 86, 87] and references therein). General references for delay equations are the monographs by Burton [13], Diekmann, van Gils, Verduyn Lunel and Walther [35], Hale and Verduyn Lunel [46], Hino, Murakami and Naito [51], Webb [86], and Wu [87]. From the monograph by Engel and Nagel [36], one can find a very nice treatment of abstract delay equations by the operator semigroup theory.

In this dissertation, we study delay equations in a quite general framework of admissible phase space, which satisfies hypotheses weaker than those required in the previous literature and includes the space $L^p((-\infty,0],X)$. Therefore, our results are extensions of many known results on delay equations for infinite delay as well as for finite delay given in, e.g., [3, 8, 13, 22, 23, 35, 36, 45–48, 51, 53, 54, 58–60, 63, 71, 74–79, 84, 86, 87]).

We would like to mention that the investigation of functional differential equations with infinite delay in an abstract admissible phase space was initiated by Hale and Kato [45] and Schumacher [77] (for $X = R^n$), and that Banks, Burns, Delfour, Herdman and Mitter were among the first who studied equations with finite delay in the state space $X \times L^p([-r,0],X)$ (cf. [7, 10, 32]). The method of using admissible phase spaces has proved to be significant in dealing with infinite delay problems, because in this way one can treat a large class of functional differential equations with infinite delay at the same time and obtain general results. On the other hand, as shown, e.g., in [7, 10–12, 32, 83], the product space $X \times L^p([-r,0],X)$ is well suited for the investigation of certain problems involving control systems governed by delay equations.

In Chapter 3, we consider mainly the solvability of the Cauchy problem for four classes of abstract functional equations with infinite delay. We address first, in Section 2, the Cauchy problem for a functional integral equation with infinite delay in a Banach space X,

$$\begin{cases} u(t) = g(t) + \displaystyle\int_\sigma^t f(t,s,u(s),u_s)ds \quad (\sigma \le t \le T), \\ u_\sigma = \phi, \end{cases}$$

where $0 \le \sigma < T$, $g(t) \in C([\sigma,T],X)$, $u_t(\theta) = u(t+\theta)$ $(\theta \in R^-)$, $f \in C([\sigma,T] \times [\sigma,T] \times X \times \mathcal{P}, X)$ is a given function and $\phi \in \mathcal{P}$ (an admissible phase space). The solvability of the functional integral equation above is investigated under hypotheses based on noncompactness measures and Kamke functions or the Lipschitz condition. The uniqueness and continuous dependence (on initial data) of the solutions are also discussed. Second, in Sections 3 – 5, we consider the Cauchy problem for a semilinear functional differential equation with infinite delay

$$\begin{cases} u'(t) = Au(t) + f(t,u(t),u_t), \quad 0 \le t \le T, \\ u_0 = \phi, \end{cases}$$

the Cauchy problem for a nonautonomous semilinear functional equation with infinite delay

$$\begin{cases} u'(t) = A(t)u(t) + f(t, u(t), u_t), \quad 0 \le t \le T, \\ u_0 = \phi, \end{cases}$$

and the Cauchy problem for a functional integrodifferential equation with infinite delay

$$\begin{cases} u'(t) = A\left[u(t) + \int_0^t F(t-s)u(s)ds\right] + f(t, u(t), u_t), \quad 0 \le t \le T, \\ u_0 = \phi, \end{cases}$$

where $T > 0$, A and $\{A(t)\}_{t \ge 0}$ are given linear operators in X, $\{F(t)\}_{0 \le t \le T} \subset \mathbf{L}(X)$, $f \in C([0,T] \times X \times \mathcal{P}, X)$, and $\phi \in \mathcal{P}$. By applying the given results in Section 2, we obtain some new and basic solvability and wellposedness results for these problems.

In Chapter 4, we investigate the regularity for a functional differential equation with infinite delay in a Banach space X satisfying the Radon-Nikodym property. Some regularity results are established. Theorems 4.2.6 and 4.2.7 in this chapter are entirely new, and others are generalizations of the corresponding results in our papers [57, 59].

In Chapter 5, we are interested in the deep investigation of the wellposedness of the Cauchy problem for abstract functional equations with infinite delay in the general case, i.e., the space X being a general Banach space. Our objective is to establish wellposedness theorems, on the Cauchy problems for a semilinear functional differential equation and a nonautonomous semilinear functional equation with infinite delay, when the nonlinear term f is Fréchet differentiable. In Section 1, we introduce a new concept for a continuously differentiable function $\phi \in \mathcal{P}$, called *one-point-property*. In terms of it, we set up a wellposedness result on the former one (autonomous case), which generalizes the corresponding results in [3, 8, 13, 22, 23, 35, 36, 45, 46, 48, 51, 58, 59, 71, 77, 78, 84, 86, 87]). Section 2 is devoted to the nonautonomous case. The wellposedness result given there is new even for the finite delay case.

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Chapter 1

Semilinear integrodifferential equations with nonlocal initial conditions

1.1 Introduction and Preliminaries

We consider the Cauchy problem for a semilinear integrodifferential equation with a nonlocal initial condition

$$\begin{cases} u'(t) = A\left[u(t) + \displaystyle\int_{t_0}^{t} F(t-s)u(s)ds\right] + f(t, u(t)), \quad t \in [t_0, t_0 + T], \\ u(t_0) + g(t_1, \dots, t_p, u) = u_0, \end{cases} \tag{1.1.1}$$

in a Banach space X, where

- A is the generator of a C_0 semigroup on X;
- $\{F(t)\}_{t\in[0,T]} \subset \mathbf{L}(X)$ (the space of continuous linear operators from X to itself) is a strongly continuously differentiable family such that

$$\begin{cases} F(t)(\mathcal{D}(A)) \subset \mathcal{D}(A), \quad t \in [0, T], \\ AF(\cdot)u(\cdot) \in L^1([0, T], X), \quad u(\cdot) \in C([0, T], [\mathcal{D}(A)]), \\ F(\cdot)u \in C^1([0, T], X), \quad u \in X, \end{cases} \tag{1.1.2}$$

where $\mathcal{D}(A)$ is the domain of A, and $[\mathcal{D}(A)]$ is the space $\mathcal{D}(A)$ with the graph norm;

- $f(\cdot,\cdot) \in C([t_0, t_0+T] \times X, X)$ and

$$\|f(t,x) - f(t,y)\| \le L\|x-y\|, \quad t \in [t_0, t_0+T],\ x,y \in X, \tag{1.1.3}$$

  for a constant $L > 0$;

- $0 \le t_0 < t_1 < \cdots < t_p \le t_0 + T$;

- the X-valued function $g(t_1, \cdots, t_p, \cdot)$ on $C([t_0, t_0+T], X)$ satisfies

$$\begin{aligned} \|g(t_1,\ldots,t_p,\phi) - g(t_1,\ldots,t_p,\psi)\| &\le K \max_{t\in[t_0,t_0+T]} \|\phi(t)-\psi(t)\|, \\ &\qquad \phi,\psi \in C([t_0,t_0+T],X), \end{aligned} \tag{1.1.4}$$

  for a constant $K > 0$.

A typical example of the Cauchy problem for the integrodifferential equation in (1.1.1) with normal initial data is the following mathematical model coming from the study of heat conduction (or viscoelasticity) for materials with memory (see, e.g., [21, 43])

$$\begin{cases} q(t,x) = -cu_x(t,x) - \displaystyle\int_0^t b(t-s)u_x(s,x)ds, \\ u_t(t,x) = -q_x(t,x) + f(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{1.1.5}$$

where q is the heat flux, c a constant, $b : [0,\infty) \to (-\infty,\infty)$, u the temperature of the material, and f the externally supplied heat. The second equation is the balance equation. Assuming $c = 1$, then (1.1.5) can be rewritten as

$$\begin{cases} u_t(t,x) = \dfrac{\partial^2}{\partial x^2}\left[u(t,x) + \displaystyle\int_0^t b(t-s)u(s,x)ds\right] + f(t,x), \\ u(0,x) = u_0(x). \end{cases}$$

This is a form of the normal Cauchy problem for the integrodifferential equation in (1.1.1) with $A = \frac{\partial^2}{\partial x^2}$ by noting that $A = \frac{\partial^2}{\partial x^2}$ with domain $H^2(0,1) \cap H_0^1(0,1)$ generates a C_0 semigroup on $L^2(0,1)$. The integrodifferential equation in (1.1.1) and its analogues have been investigated in many articles. We refer the reader to [5, 33, 34, 40–42, 61, 62] and references cited there.

Interest in the Cauchy problem for differential equations with nonlocal initial conditions stems mainly from the better effect of the nonlocal initial condition than the usual one in treating physical problems. Actually, the nonlocal initial condition

$$u(t_0) + g(t_1, \dots, t_p, u) = u_0 \tag{1.1.6}$$

in (1.1.1) models many interesting nature phenomena, with which the normal initial condition $u(0) = u_0$ may not fit in. For instance, the function $g(t_1, \dots, t_p, u)$ may be given by

$$g(t_1, \dots, t_p, u) = \sum_{i=1}^{p} c_i u(t_i),$$

where c_i $(i = 1, \dots, p)$ are constants. In this case, (1.1.6) allows the measurements at $t = 0$, t_1, $\dots$, t_p, rather than just at $t = 0$. Hence more information is available. More specially, letting $g(t_1, \dots, t_p, u) = -u(t_p)$ and $u_0 = 0$ in (1.1.6) yields a periodic problem and letting $g(t_1, \dots, t_p, u) = -u(t_0)+u(t_p)$ gives a backward problem. From Byszewski [14, 15], L. Byszewski and V. Lakshmikantham [19] and the references given there, one can find other information about the importance of nonlocal initial conditions in applications. There have been many papers concerning this topic (cf., e.g., [5, 9, 14, 15, 17–19, 52, 61, 68] and references therein). However, much of the previous research was done under the condition "$M(K + TL) < 1$" ($M := \max_{t\in[0,T]} \|T(t)\|$ and $\{T(t)\}_{t\geq 0}$ is the C_0 semigroup generated by A) or its analogues (cf., e.g., [14, 17, 61]). This condition turns out to be quite restrictive. In particular, limited by it, the results obtained for nonlocal problems can not cover those classical results regarding the case when $F \equiv 0$ and $g \equiv 0$, i.e., the following differential equations with usual initial conditions

$$u(t) = Au(t) + f(t, u(t)) \ (t_0 \leq t \leq t_0 + T), \quad u(t_0) = u_0 \tag{1.1.7}$$

(cf. [70, Chapter 6]). Thus, there naturally arises a question:

> *Can the above condition be relaxed such that the results for nonlocal problems cover the corresponding ones for* (1.1.7)?

In this chapter, among others we will give an affirmative answer to this question (see Corollary 1.2.2 (1), Theorem 1.2.7, Remark 1.2.3 (c) and Remark 1.2.9 (a)).

In Section 2, we first study the existence and uniqueness of solutions for a general integral equation ((1.2.3) below), and then investigate the corresponding problems for (1.1.1). The theorems formulated are unifications and extensions of those given previously for the Cauchy problem for differential equations or integrodifferential equations with nonlocal initial conditions. As the reader will see, the hypotheses in our theorems are in reasonable weak forms and the proofs provided are concise. Moreover, following every main result, we append a remark with a detailed analysis of how the result extends and improves the known ones. Finally, in Section 3, we apply our theorems to two concrete problems, one of which comes from heat conduction in materials with memory. It is indicated that, in contrast with ours, the previous results are not applicable to them.

To begin with, we recall that there is a strongly continuous family $\{R(t)\}_{t\in[0,T]} \subset \mathbf{L}(X)$ such that

(i) $R(0) = I$, $R(\cdot)y \in C^1([0,T],X) \cap C([0,T],[\mathcal{D}(A)])$ $(y \in \mathcal{D}(A))$.

(ii) for every $t \in [0,T]$, $y \in \mathcal{D}(A)$,

$$\begin{aligned} \frac{d}{dt}R(t)y &= A\left[R(t)y + \int_0^t F(t-s)R(s)yds\right] \\ &= R(t)Ay + \int_0^t R(t-s)AF(s)yds. \end{aligned} \tag{1.1.8}$$

(cf., e.g., [34, 40, 42, 62]).

Definition 1.1.1. A *mild solution* of (1.1.1) is a function $u \in C([t_0, t_0+T], X)$ satisfying

$$u(t) = R(t-t_0)[u_0 - g(t_1,\dots,t_p,u)] + \int_{t_0}^t R(t-s)f(s,u(s))ds, \quad t \in [t_0, t_0+T]. \tag{1.1.9}$$

A *classical solution* of (1.1.1) is a function

$$u \in C^1([t_0,t_0+T],X) \cap C([t_0,t_0+T],[\mathcal{D}(A)])$$

satisfying (1.1.9).

1.2 A general integral equation and an integrodifferential equation with nonlocal initial condition

Assume that

(H1) $\{S(t)\}_{t\in[0,T]} \subset \mathbf{L}(X)$ is a strongly continuous family, and $\|S(t)\| \le Me^{-\omega t}$ ($t \in [0,T]$), where M and $\omega \ge 0$ are constants.

(H2) $h : C([t_0, t_0+T], X) \to X$ and there exists a nonnegative function Φ on $C([t_0, t_0+T],\ [0,\infty))$ satisfying

$$\begin{cases} \Phi(k\mu) \le k\Phi(\mu), & \forall\ k>0,\ \mu \in C([t_0,t_0+T],\ [0,\infty)), \\ \Phi(\mu_1) \le \Phi(\mu_2), & \forall \begin{cases} \mu_1,\ \mu_2 \in C([t_0,t_0+T],\ [0,\infty)) \\ \text{with } \mu_1(t) \le \mu_2(t)\ (t \in [t_0,t_0+T]), \end{cases} \end{cases} \tag{1.2.1}$$

such that

$$\|h(\phi) - h(\psi)\| \le \Phi(\|\phi - \psi\|), \quad \phi, \psi \in C([t_0,t_0+T], X). \tag{1.2.2}$$

We first look at a general integral equation

$$v(t) = S(t-t_0)[u_0 - h(v)] + \int_{t_0}^{t} S(t-s) f(s, v(s)) ds, \quad t \in [t_0, t_0+T]. \tag{1.2.3}$$

Theorem 1.2.1. *Let* (1.1.3), (H1) *and* (H2) *hold and* $M\Phi\left(e^{(ML-\omega)(\bullet - t_0)}\right) < 1$. *Then for all* $u_0 \in X$, (1.2.3) *has a unique solution* $v \in C([t_0, t_0+T], X)$.

Proof. Let $u_1 \in C([t_0,t_0+T], X)$ be fixed and $u_{1,0} := u_0 - h(u_1)$. Define an operator $\mathcal{F}$ on $C([t_0,t_0+T],X)$ by

$$(\mathcal{F}u)(t) = S(t-t_0)u_{1,0} + \int_{t_0}^{t} S(t-s) f(s, u(s)) ds, \quad t \in [t_0, t_0+T]. \tag{1.2.4}$$

Clearly, $\mathcal{F}\left(C([t_0,t_0+T],X)\right) \subset C([t_0,t_0+T],X)$. By a standard argument, we see that $\mathcal{F}$ has a unique fixed point $u_2 \in C([t_0,t_0+T],X)$. Using induction we infer that there exists a sequence $\{u_n\}_{n=2}^{\infty} \subset C([t_0,t_0+T],X)$ such that

$$u_n(t) = S(t-t_0)u_{n-1,0} + \int_{t_0}^{t} S(t-s) f(s, u_n(s)) ds, \quad t \in [t_0, t_0+T],\ n \ge 2, \tag{1.2.5}$$

where

$$u_{n-1,0} = u_0 - h(u_{n-1}). \tag{1.2.6}$$

A combination of (1.1.3), (1.2.1) (1.2.2), (1.2.5), and (1.2.6) shows

$$\begin{aligned} e^{\omega t}\|u_3(t) - u_2(t)\| &\leq e^{\omega t} M\Phi(\|u_2(t) - u_1(t)\|) \\ &\quad + ML\int_{t_0}^{t} e^{\omega s}\|u_3(s) - u_2(s)\| ds, \quad t \in [t_0, t_0 + T]. \end{aligned}$$

By Bellman-Gronwall's inequality,

$$\|u_3(t) - u_2(t)\| \leq Me^{(ML-\omega)(t-t_0)}\Phi(\|u_2(t) - u_1(t)\|), \quad t \in [t_0, t_0 + T].$$

Therefore, by induction again, for each $t \in [t_0, t_0 + T]$,

$$\begin{aligned} &\|u_n(t) - u_{n-1}(t)\| \\ \leq\ & Me^{(ML-\omega)(t-t_0)} \left(M\Phi\left(e^{(ML-\omega)(t-t_0)}\right)\right)^{n-3} \Phi(\|u_2(t) - u_1(t)\|), \quad n \geq 3. \end{aligned}$$

According to the assumption, we obtain for any $m > n \geq 3$

$$\begin{aligned} & \max_{t\in[t_0,t_0+T]} \|u_m(t) - u_n(t)\| \\ \leq\ & \sum_{i=n}^{m-1} \max_{t\in[t_0,t_0+T]} \|u_{i+1}(t) - u_i(t)\| \\ \leq\ & \max\left\{M,\ Me^{(ML-\omega)(T-t_0)}\right\} \Phi(\|u_2(t) - u_1(t)\|) \sum_{i=n}^{m-1} \left(M\Phi\left(e^{(ML-\omega)(t-t_0)}\right)\right)^{i-2} \\ \longrightarrow\ & 0, \quad \text{as } \ n \to \infty; \end{aligned}$$

that is, $\{u_n\}_{n=2}^{\infty}$ is a Cauchy sequence in $C([t_0, t_0 + T], X)$. Therefore, there is a $u \in C([t_0, t_0 + T], X)$ such that

$$\lim_{n\to\infty} u_n(t) = u(t) \qquad \text{uniformly } \text{ for } t \in [t_0, t_0 + T].$$

This together with (1.2.4) – (1.2.6) implies that $u(t)$ is a continuous solution of (1.2.3). The uniqueness of the solution of (1.2.3) is obvious.

□

Corollary 1.2.2. *Let* (1.1.3), (H1) *and one of the following assumptions hold.*

(1) *There is a constant* $K > 0$ *such that*

$$\|h(\phi) - h(\psi)\| \le K \max_{s\in[t_0,t_0+T]} \|\phi(s) - \psi(s)\| \quad (\phi, \psi \in C([t_0, t_0 + T], X)),$$

and $KMe^{T\max\{ML-\omega,0\}} < 1$.

(2) *There are constants* $K > 0$, $t_0 \le q < r \le t_0 + T$ *such that*

$$\|h(\phi) - h(\psi)\| \le K \int_q^r \|\phi(s) - \psi(s)\|ds \quad (\phi, \psi \in C([t_0, t_0 + T], X)),$$

and

$$\begin{cases} KM(r-q) < 1 & \text{if } ML = \omega, \\ \dfrac{KM}{ML-\omega}\left(e^{(ML-\omega)(r-t_0)} - e^{(ML-\omega)(q-t_0)}\right) < 1 & \text{if } ML \ne \omega. \end{cases}$$

(3) *There are* $c_1, \ldots, c_p \in \mathbf{C}$ *such that*

$$\|h(\phi) - h(\psi)\| \le \sum_{i=1}^{p} |c_i| \|\phi(t_i) - \psi(t_i)\| \quad (\phi, \psi \in C([t_0, t_0 + T], X)),$$

and $M \sum_{i=1}^{p} |c_i| e^{(ML-\omega)(t_i-t_0)} < 1$.

Then for all $u_0 \in X$, *equation* (1.2.3) *has a unique solution* $v \in C([t_0, t_0 + T], X)$.

Proof. Applying Theorem 1.2.1 to the functions

$$\Phi(\mu) = K \max_{s\in[t_0,t_0+T]} \mu(s), \qquad \Phi(\mu) = K \int_q^r \mu(s)ds, \qquad \Phi(\mu) = \sum_{i=1}^{p} |c_i|\mu(t_i),$$

respectively, we obtain the desired conclusions.

□

Remark 1.2.3. (a) The proof of Theorem 1.2.1 shows a way to compute the continuous solution of (1.2.3).

(b) Corollary 1.2.2 (1) gives a generalization of [61, Theorem 3.2], because

(1) the operator family $\{S(\cdot)\}$ and the mapping $h(u)$ in Corollary 1.2.2 (1) are more general than the operator family $\{R(\cdot)\}$ and the mapping $g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p))$ respectively;

(2) if we let

$$t_0 = 0, \quad \omega = 0, \quad S(\cdot) = R(\cdot), \quad h(u) = g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)),$$

then Corollary 1.2.2 says that (1.10) – (1.11) in [61] has a unique mild solution for any $u_0 \in X$ provided $MK < e^{-MTL}$. But, Theorem 3.2 in [61] is not applicable for any $K \geq 0$ when $MTL \geq 1$, since then

$$MK + MTL \geq 1.$$

(3) for $M, K, T, L \geq 0$, the inequality $MKe^{MTL} < 1$ does not imply $M(K + TL) < 1$ even if $MTL < 1$ (for example, let $MK = \frac{3}{4}$ and $MTL = \frac{1}{4}$, then $MTL < 1$ and $MKe^{MTL} < 1$, but $M(K + TL) = 1$). However, the converse holds. In fact, for $M, K, T, L \geq 0$ the inequality $M(K+TL) < 1$ implies

$$MKe^{MTL} < MKe^{1-MK} < 1,$$

by noting that the function $\xi \mapsto \xi e^{1-\xi}$ is increasing on $[0, 1]$.

(c) Corollary 1.2.2 (1) covers naturally and directly the "existence and uniqueness" part of [70, p.184, Theorem 6.1.2], because if $h \equiv 0$ then $K \equiv 0$ which means that the assumption $KMe^{T \max\{ML-\omega, 0\}} < 1$ always holds.

Using the idea in the proof of Theorem 1.2.1 we can also obtain the following theorem.

Theorem 1.2.4. *Let A generate a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. Write $\Omega_r := \{u;\ u \in X \text{ and } \|u\| \leq r\}$ $(r > 0)$. Assume the following.*

(i) *There exists a constant $L_0 > 0$ such that*

$$\|f(t, x) - f(t, y)\| \leq L_0\|x - y\|, \quad t \in [t_0, t_0 + T], \quad x, y \in \Omega_r.$$

(ii) *There exists a constant $K_0 > 0$ such that*

$$\|g(t_1, \ldots, t_p, \phi) - g(t_1, \ldots, t_p, \psi)\| \leq K_0 \max_{t\in[t_0,t_0+T]} \|\phi(t) - \psi(t)\|,$$
$$\phi, \psi \in C([t_0, t_0 + T], \Omega_r).$$

(iii) *The inequality* $M_0\left(\|u_0\|+G+T(rL_0+F)\right)\le r$ *holds with*

$$M_0:=\max_{s\in[t_0,t_0+T]}\|T(s)\|,\quad F:=\max_{s\in[t_0,t_0+T]}\|f(s,0)\|,$$

and $G:=\sup_{\phi\in C([t_0,t_0+T],\Omega_r)}\|g(t_1,\dots,t_p,\phi)\|$.

(iv) $M_0K_0e^{M_0TL_0}<1$.

Then

$$\begin{cases} u'(t)=Au(t)+f(t,u(t)), & t_0\le t\le t_0+T,\\ u(t_0)+g(t_1,\dots,t_p,u)=u_0 \end{cases}\tag{1.2.7}$$

has a unique mild solution $u\in C([t_0,t_0+T],\Omega_r)$.

Remark 1.2.5. (a) Theorem 1.2.4 is an extension of [14, Theorem 3.1] for the same reasons as in (1) and (3) of Remark 1.2.3.

(b) The conclusion of Theorem 1.2.4 is also true if replacing the assumption (iii) by the following weaker one.

(iii′) The inequality $M_0\left(\|u_0\|+G+TF_0\right)\le r$ holds with

$$M_0:=\max_{s\in[t_0,t_0+T]}\|T(s)\|,\quad F_0:=\sup_{s\in[t_0,t_0+T],\phi\in C([t_0,t_0+T],\Omega_r)}\|f(s,\phi(s))\|,$$

and $G:=\sup_{\phi\in C([t_0,t_0+T],\Omega_r)}\|g(t_1,\dots,t_p,\phi)\|$.

For the case of $h(\cdot)$ taking the form $h(\phi)=\sum_{i=1}^p c_i\phi(t_i)$ for every $\phi\in C([t_0,t_0+T],X)$, here $c_1,\dots,c_p\in\mathbf{C}$, we present the following Theorem 1.2.6 which is sharper than Corollary 1.2.2 (3). Furthermore, this result unifies and extends both of [61, Theorem 4.3] and [17, Theorem 3.1] (see Remark 1.2.9 below).

Theorem 1.2.6. *Let* (1.1.3) *and* (H1) *hold and for some* $c_1,\dots,c_p\in\mathbf{C}$. *Take*

$$h(\phi):=\sum_{i=1}^p c_i\phi(t_i)\quad(\phi\in C([t_0,t_0+T],X)).$$

Assume that $B:=\left(I+\sum_{i=1}^p c_iS(t_i-t_0)\right)^{-1}\in\mathbf{L}(X)$ *and*

$$\|B\|M\sum_{i=1}^p|c_i|e^{-\omega(t_i-t_0)}\left(e^{ML(t_i-t_0)}-1\right)<1.$$

Then for all $u_0\in X$, *equation* (1.2.3) *has a unique solution* $v\in C([t_0,t_0+T],X)$.

Proof. By the standard arguments, we see that for every $x \in X$, there is a unique $v_x(\cdot) \in C([t_0, t_0 + T], X)$ satisfying

$$v_x(t) = S(t - t_0)x + \int_{t_0}^{t} S(t - s)f(s, v_x(s))ds, \quad t \in [t_0, t_0 + T]. \tag{1.2.8}$$

Hence

$$v_x(t_i) = S(t_i - t_0)x + \int_{t_0}^{t_i} S(t_i - s)f(s, v_x(s))ds, \quad i = 1, \dots, p, \tag{1.2.9}$$

and (1.1.3) implies that for every $x_1,\ x_2 \in X$,

$$e^{\omega t}\|v_{x_1}(t) - v_{x_2}(t)\| \leq e^{\omega t_0}M\|x_1 - x_2\| + ML\int_{t_0}^{t} e^{\omega s}\|v_{x_1}(s) - v_{x_2}(s)\|ds.$$

Thus Gronwall-Bellman's inequality indicates that

$$\|v_{x_1}(t) - v_{x_2}(t)\| \leq Me^{(ML-\omega)(t-t_0)}\|x_1 - x_2\|, \quad x_1,\ x_2 \in X. \tag{1.2.10}$$

Fix $u_0 \in X$ and define an operator $\mathcal{G} : X \to X$ by

$$\mathcal{G}x = Bu_0 - B\sum_{i=1}^{p} c_i \int_{t_0}^{t_i} S(t_i - s)f(s, v_x(s))ds, \quad x \in X. \tag{1.2.11}$$

Then, by virtue of (1.1.3) and (1.2.10) we obtain for every $x_1,\ x_2 \in X$,

$$\begin{aligned} \|\mathcal{G}x_1 - \mathcal{G}x_2\| &\leq \|B\|\sum_{i=1}^{p}|c_i|\int_{t_0}^{t_i} Me^{-\omega(t_i-s)}L\left\|v_{x_1}(s) - v_{x_2}(s)\right\|ds \\ &= \|B\|M\sum_{i=1}^{p}|c_i|e^{-\omega(t_i-t_0)}\left(e^{ML(t_i-t_0)} - 1\right)\|x_1 - x_2\|. \end{aligned}$$

This means that $\mathcal{G}$ is a contractive operator on X. Therefore $\mathcal{G}$ has a unique fixed point $x_* \in X$. Thus, from (1.2.11) and (1.2.9) it follows that

$$\begin{aligned} x_* &= u_0 - \sum_{i=1}^{p} c_iS(t_i - t_0)x_* - \sum_{i=1}^{p} c_i\int_{t_0}^{t_i} S(t_i - s)f\left(s, v_{x_*}(s)\right)ds \\ &= u_0 - \sum_{i=1}^{p} c_iv_{x_*}(t_i). \end{aligned}$$

This together with (1.2.8) shows that $v_{x_*}(t)$ is the solution of (1.2.3) as desired.

□

We now return to the nonlocal Cauchy problem (1.1.1).

Theorem 1.2.7. *Let* (1.1.2) - (1.1.4) *hold. Suppose that* M *and* ω *are constants such that* $\|R(t)\| \leq Me^{-\omega t}$ $(t \in [0,T])$ *and* $\lambda := MKe^{T\max\{ML-\omega,0\}} < 1$. *Then for every* $u_0 \in X$, (1.1.1) *has a unique mild solution* u.

Moreover, (1.1.1) *has a unique classical solution provided*

$$u_0 - g(t_1,\ldots,t_p,u) \in \mathcal{D}(A), \quad f \in C^1([t_0,t_0+T]\times X, X). \tag{1.2.12}$$

Proof. From Corollary 1.2.2 (1) and the fact that a classical solution of (1.1.1) is also a mild solution of (1.1.1), we judge that (1.1.1) has at most one classical solution.

On the other hand, Corollary 1.2.2 (1) says that for every $u_0 \in X$, (1.1.1) has a mild solution $u(t)$. Next, we show that $u(t)$ is continuously differentiable on $[t_0, t_0+T]$. The proof of this fact is almost standard (cf. [70]). We give it here for completeness.

For $s \in [t_0, t_0+T]$ and $x \in X$, denote

$$y_1(s,x) = \frac{\partial}{\partial s} f(s,x), \quad y_2(s,x) = \frac{\partial}{\partial x} f(s,x). \tag{1.2.13}$$

By (1.1.3), we have

$$\max_{s\in[t_0,t_0+T]} \|y_2(s,u(s))\| < \infty, \tag{1.2.14}$$

and

$$\begin{cases} f(s,u(s+\sigma)) - f(s,u(s)) = y_2(s,u(s))(u(s+\sigma)-u(s)) + \omega_1(s,\sigma), \\ f(s+\sigma,u(s+\sigma)) - f(s,u(s+\sigma)) = y_1(s,u(s+\sigma))\sigma + \omega_2(s,\sigma), \end{cases} \tag{1.2.15}$$

where $\lim_{\sigma\to 0} \frac{\|\omega_i(s,\sigma)\|}{\sigma} = 0$ uniformly on $[t_0, t_0+T]$ for $i = 1,2$.

Let (1.2.12) hold. Then

$$\frac{d}{dt}(R(t-t_0)(u_0 - g(t_1,\ldots,t_p,u))) \in C([0,T],X).$$

Thus, by the standard arguments we deduce that the integral equation

$$\begin{aligned} x(t) \;=\; & \left\{ \frac{d}{dt}(R(t-t_0)(u_0 - g(t_1,\ldots,t_p,u))) + R(t-t_0)f(t_0,u(t_0)) \right. \\ & \left. + \int_0^t R(t-s)y_1(s,u(s))ds \right\} + \int_0^t R(t-s)y_2(s,u(s))x(s)ds, \\ & \qquad t \in [t_0, t_0+T] \end{aligned}$$

has a unique solution $x(t) \in C([t_0, t_0 + T], X)$.

Making use of (1.1.9), (1.2.13) - (1.2.15), we obtain

$$
\begin{aligned}
& \frac{u(t+\sigma) - u(t)}{\sigma} - x(t) \\
= \; & \frac{1}{\sigma}[R(t+\sigma - t_0) - R(t - t_0)][u_0 - g(t_1, \ldots, t_p, u)] \\
& + \frac{1}{\sigma}\int_{t_0}^{t} R(t-s)[\omega_1(s,\sigma) + \omega_2(s,\sigma)]ds \\
& + \int_{t_0}^{t} R(t-s)[y_1(s, u(s+\sigma)) - y_1(s, u(s))]ds \\
& + \frac{1}{\sigma}\int_{t_0}^{t_0+\sigma} R(t+\sigma - s) f(s, u(s))ds - R(t - t_0) f(t_0, u(t_0)) \\
& + \int_{t_0}^{t} R(t-s) y_2(s, u(s)) \left[\frac{u(s+\sigma) - u(s)}{\sigma} - x(s)\right] ds.
\end{aligned} \tag{1.2.16}
$$

By virtue of the fact that the norm of each of the four terms on the right-hand side of (1.2.16) tends to 0 as $\sigma \to 0$, in conjunction with the Gronwall-Bellman inequality, we see that $u(t)$ is continuously differentiable on $[t_0, t_0 + T]$ and its derivative is $x(t)$. This implies that $f(t, u(t)) \in C^1([t_0, t_0 + T], X)$. Thus, by (1.1.8) and (1.1.9) we conclude that $u(\cdot)$ satisfies

$$u'(t) = A\left[u(t) + \int_{t_0}^{t} F(t-s)u(s)ds\right] + f(t, u(t)), \quad t \in [t_0, t_0 + T],$$

i.e., $u(\cdot)$ is the unique classical solution of (1.1.1).

□

Likewise, by Corollary 1.2.2 (2)-(3) and Theorem 1.2.6, we have the following result.

Theorem 1.2.8. *Let M and ω be constants such that $\|R(t)\| \le Me^{-\omega t}$ ($t \in [0, T]$), and let one of the following assumptions hold.*

(1) *There are constants $K > 0$, q and r with $t_0 \le q < r \le t_0 + T$ such that*

$$\|g(t_1, \ldots, t_p, \phi) - g(t_1, \ldots, t_p, \psi)\| \le K \int_q^r \|\phi(s) - \psi(s)\| \quad (\phi, \psi \in C([t_0, t_0 + T], X)),$$

and

$$\begin{cases} KM(r-q) < 1 & \text{if} \quad ML = \omega, \\ \dfrac{KM}{ML-\omega}\left(e^{(ML-\omega)(r-t_0)} - e^{(ML-\omega)(q-t_0)}\right) < 1 & \text{if} \quad ML \neq \omega, \end{cases}$$

(2) *For some* $c_1, \ldots, c_p \in \mathbf{C}$,

$$g(t_1, \ldots, t_p, \phi) = \sum_{i=1}^{p} c_i \phi(t_i) \quad (\phi \in C([t_0, t_0+T], X)).$$

Suppose that $B := \left(I + \sum_{i=1}^{p} c_i R(t_i - t_0)\right)^{-1} \in \mathbf{L}(X)$ *and*

$$\|B\| M \sum_{i=1}^{p} |c_i| e^{-\omega(t_i - t_0)} \left(e^{ML(t_i - t_0)} - 1\right) < 1. \tag{1.2.17}$$

Then the conclusions of Theorem 1.2.7 *hold.*

Remark 1.2.9. (a) Theorem 1.2.7 covers naturally and directly [70, p. 187, Theorem 6.1.5].

(b) Theorem 1.2.8 unifies and generalizes [17, Theorems 3.1 and 4.3] and [61, Theorems 4.3 and 4.4]. Let us illustrate this point in detail.

(i) Specialized to the case $F \equiv 0$ and $\omega = 0$, Theorem 1.2.8 (2) extends [17, Theorems 3.1 and 4.3]. Actually, in this case, the inequality (1.2.17) becomes

$$\|B\| M \sum_{i=1}^{p} |c_i| \left(e^{MLt_i} - 1\right) < 1. \tag{1.2.18}$$

Suppose that the hypotheses in [17, Theorems 3.1 and 4.3] hold. Then

$$MLT \left(1 + \|B\| M \sum_{i=1}^{p} |c_i|\right) < 1. \tag{1.2.19}$$

So

$$MLT < 1, \quad \|B\| M \sum_{i=1}^{p} |c_i| < (MLT)^{-1} - 1, \quad \text{if } MLT \neq 0,$$

and hence

$$\begin{aligned}\|B\|M\sum_{i=1}^{p}|c_i|\left(e^{MLt_i}-1\right) &\leq \|B\|M\sum_{i=1}^{p}|c_i|\left(e^{MLT}-1\right)\\ &< \left((MLT)^{-1}-1\right)\left(e^{MLT}-1\right)<1.\end{aligned}$$

Thus (1.2.19) implies (1.2.18).

Clearly the converse is not true.

Moreover, we mention that the assumption on initial data in [17, Theorem 4.3] was

$$Bu_0\in\mathcal{D}(A),\quad B\int_{t_0}^{t_i}R(t_i-s)f(s,u(s))ds\in\mathcal{D}(A),\qquad i=1,2,\dots,p. \tag{1.2.20}$$

Write $w_1:=u_0-\sum_{i=1}^{p}c_iu(t_i)$. Then by

$$u(t)=R(t-t_0)w_1+\int_{t_0}^{t}R(t-s)f(s,u(s))ds\quad (t\in[t_0,t_0+T])$$

and (1.2.20), we have

$$w_1=Bu_0-\sum_{i=1}^{p}c_iB\int_{t_0}^{t_i}R(t_i-s)f(s,u(s))ds\in\mathcal{D}(A).$$

(ii) Taking $t_0=0$ in Theorem 1.2.8 (2), we have

$$\|B\|M\sum_{i=1}^{p}|c_i|e^{-\omega t_i}\left(e^{MLt_i}-1\right)<1. \tag{1.2.21}$$

We say that (1.2.21) is implied in the hypotheses

$$\omega-ML>0,\quad M\sum_{i=1}^{p}|c_i|e^{(ML-\omega)t_i}<1 \tag{1.2.22}$$

given in [61, Theorems 4.3 and 4.4], and (1.2.21) is indeed much weaker than (1.2.22).

In fact, if

$$\alpha:=M\sum_{i=1}^{p}|c_i|e^{(ML-\omega)t_i}<1,$$

then $\beta := M \sum_{i=1}^{p} |c_i| e^{-\omega t_i} < 1$. This implies $\left\| \sum_{i=1}^{p} c_i R(t_i) \right\| < 1$, so that $B \in \mathbf{L}(X)$ and $\|B\| \leq \frac{1}{1-\beta}$. Therefore,

$$\|B\| M \sum_{i=1}^{p} |c_i| e^{-\omega t_i} \left(e^{MLt_i} - 1 \right) \leq \frac{1}{1-\beta}(\alpha - \beta) < \frac{1}{1-\beta}(\alpha - \alpha\beta) = \alpha < 1.$$

This shows that (1.2.22) implies (1.2.21). On the other hand, for

$$\gamma := \left\| \sum_{i=1}^{p} c_i R(t_i) \right\| < \beta < 1, \quad 1 \leq \alpha < 1 + \beta - \gamma,$$

we have

$$\|B\| M \sum_{i=1}^{p} |c_i| e^{-\omega t_i} \left(e^{MLt_i} - 1 \right) \leq \frac{1}{1-\gamma}(\alpha - \beta) < 1,$$

i.e., (1.2.21) holds but not (1.2.22).

In addition, similar to Theorem 1.2.4, we have the following extension of [14, Theorem 5.1].

Theorem 1.2.10. *Let A generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$. Assume the following.*

(i) *The function $f : [t_0, t_0 + T] \times X \to X$ is continuously differentiable and there exists a constant $L_0 > 0$ such that*

$$\|f(t, x) - f(t, y)\| \leq L_0 \|x - y\|, \quad t \in [t_0, t_0 + T], \ x, y \in \Omega_r,$$

where Ω_r is as in Theorem 1.2.4.

(ii) *The function $g : [t_0, t_0 + T]^p \times C([t_0, t_0 + T], X) \to \mathcal{D}(A)$ and there exists a constant $K_0 > 0$ such that*

$$\|g(t_1, \ldots, t_p, \phi) - g(t_1, \ldots, t_p, \psi)\| \leq K_0 \max_{t \in [t_0, t_0+T]} \|\phi(s) - \psi(s)\|, \quad \phi, \psi \in C([t_0, t_0 + T], \Omega_r).$$

(iii) *The vector* $u_0 \in \mathcal{D}(A)$ *and the inequality* $M_0(\|u_0\| + G + TF_0) \leq r$ *is true for*

$$M_0 := \max_{s\in[t_0,t_0+T]} \|T(s)\|, \quad F_0 := \sup_{s\in[t_0,t_0+T],\phi\in C([t_0,t_0+T],\Omega_r)} \|f(s,\phi(s)\|,$$

and $G := \sup_{\phi\in C([t_0,t_0+T],\Omega_r)} \|g(t_1,\ldots,t_p,\phi)\|$.

(iv) $M_0K_0e^{M_0TL_0} < 1$.

Then (1.2.7) *has a unique classical solution.*

1.3 The case concerning compact operator family

Let $\{S(t)\}_{t\geq 0}$ be a family of continuous linear operators from X to X which is strongly continuous on $[0,T]$ and compact on $(0,T]$. Clearly

$$M := \max_{t\in[0,T]} \|S(t)\| < \infty.$$

Denote

$$\overline{B}_r := \{x \in X;\ \|x\| \leq r\}, \quad r > 0,$$

$$Y_r := \{\phi \in C([t_0,t_0+T],X);\ \phi(t) \in \overline{B}_r \text{ for } t \in [t_0,t_0+T]\}, \quad r > 0.$$

Theorem 1.3.1. *Assume that*

(i) $f : [t_0,t_0+T] \times X \to X$ *is continuous in* t *on* $[t_0,t_0+T]$ *and for each* $r > 0$ *there exists a constant* $L(r) > 0$ *such that*

$$\|f(t,u) - f(t,v)\| \leq L(r)\|u - v\|, \quad t \in [t_0,t_0+T], \quad u,v \in \overline{B}_r.$$

(ii) $g(t_1,\ldots,t_p,\cdot) : C([t_0,t_0+T],X) \to X$ *and there is a* $\delta \in (0,T)$ *such that for any* ϕ, $\psi \in Y_r$ *with* $\phi(s) = \psi(s)$ $(s \in [t_0+\delta,t_0+T])$,

$$g(t_1,\ldots,t_p,\phi) = g(t_1,\ldots,t_p,\psi).$$

(iii)

$$\lim_{r\to 0}\left(M \sup_{\phi\in Y_r} \|g(t_1,\ldots,t_p,\phi)\| + MT \sup_{s\in[t_0,t_0+T],\ \phi\in Y_r} \|f(s,\phi(s))\|\right)\frac{1}{r} < 1.$$

Then the integral equation

$$u(t) = S(t-t_0)(u_0 - g(t_1,\dots,t_p,u)) + \int_{t_0}^{t} S(t-s)f(s,u(s))ds \tag{1.3.1}$$

has at least one solution $u \in C([t_0, t_0+T], \overline{B}_r)$.

Proof. Write

$$Y(\delta) := C([t_0+\delta, t_0+T], X),$$

$$Y_r(\delta) := \{\phi \in Y(\delta);\ \phi(t) \in \overline{B}_r \text{ for } t \in [t_0+\delta, t_0+T]\}, \quad r > 0.$$

Fixing $v \in Y_r(\delta)$, we define a mapping $\mathcal{F}_v$ on Y_r by

$$(\mathcal{F}_v\phi)(t) = S(t-t_0)(u_0 - g(t_1,\dots,t_p,\widetilde{v})) + \int_{t_0}^{t} S(t-s)f(s,\phi(s))ds, \quad t \in [t_0, t_0+T],$$

where

$$\widetilde{v}(t) = \begin{cases} v(t) & \text{if} \quad t \in [t_0+\delta, t_0+T], \\ v(t_0+\delta) & \text{if} \quad t \in [t_0, t_0+\delta]. \end{cases}$$

Clearly, by the condition (iii) we know that there is a sufficiently large $r > 0$ such that

$$\begin{aligned} \|(\mathcal{F}_v\phi)(t)\| &\le M\left(\|u_0\| + \sup_{\phi\in Y_r}\|g(t_1,\dots,t_p,\phi)\| + T\sup_{s\in[t_0,t_0+T],\ \phi\in Y_r}\|f(s,\phi(s))\|\right) \\ &\le r, \qquad t \in [t_0+\delta, t_0+T],\ \phi \in Y_r. \end{aligned}$$

Therefore, the mapping $\mathcal{F}_v$ maps Y_r into itself. Moreover, by the definition of $\mathcal{F}_v$ we obtain inductively that for $m \in N$,

$$\begin{aligned} \|(\mathcal{F}_v^m\phi)(t) - (\mathcal{F}_v^m\psi)(t)\| \ &\le \ \frac{(ML(r)(t-t_0))^m}{m!}\max_{s\in[t_0,t]}\|\phi(s)-\psi(s)\|, \\ &\qquad t \in [t_0, t_0+T],\ \phi,\psi \in Y_r. \end{aligned}$$

Hence, we infer that for m large enough, the mapping $\mathcal{F}_v^m$ is a contractive mapping. Thus, by a well known extension of the Banach contraction principle, $\mathcal{F}_v$ has a unique fixed point $\phi_v \in Y_r$, i.e.,

$$\phi_v(t) = S(t-t_0)(u_0 - g(t_1,\dots,t_p,\widetilde{v})) + \int_{t_0}^{t} S(t-s)f(s,\phi_v(s))ds, \quad t \in [t_0, t_0+T]. \tag{1.3.2}$$

Based on this fact, we define a mapping $\mathcal{G}$ from $Y_r(\delta)$ into itself by

$$(\mathcal{G}v)(t) = \phi_v(t), \quad t \in [t_0 + \delta, t_0 + T].$$

From (1.3.2), we deduce that for $t \in [t_0, t_0 + T]$, v_1, $v_2 \in Y_r(\delta)$,

$$\begin{aligned}\|\phi_{v_1}(t) - \phi_{v_2}(t)\| \;&\leq\; \|S(t-t_0)(g(t_1,\dots,t_p,\widetilde{v}_1) - g(t_1,\dots,t_p,\widetilde{v}_2))\| \\ &\quad + ML\int_{t_0}^{t} \|\phi_{v_1}(s) - \phi_{v_2}(s)\|\,ds.\end{aligned}$$

This gives, by Gronwall-Bellman's inequality, that for t, v_1 and v_2 as above

$$\|\phi_{v_1}(t) - \phi_{v_2}(t)\| \leq e^{MLT}\|S(t-t_0)(g(t_1,\dots,t_p,\widetilde{v}_1) - g(t_1,\dots,t_p,\widetilde{v}_2))\|.$$

Therefore

$$\begin{aligned}\|(\mathcal{G}v_1)(t) - (\mathcal{G}v_2)(t)\| \;&\leq\; e^{MLT}\|S(t-t_0)(g(t_1,\dots,t_p,\widetilde{v}_1) - g(t_1,\dots,t_p,\widetilde{v}_2))\|, \\ &\qquad t \in [t_0+\delta, t_0+T],\ v_1,\ v_2 \in Y_r(\delta).\end{aligned} \tag{1.3.3}$$

Next we show that $\mathcal{G}$ maps $Y_r(\delta)$ into a precompact subset of $Y_r(\delta)$. To this end, we recall that $\{S(t)\}_{t\geq 0}$ is a compact semigroup, which means that for each $t \in [t_0+\delta, t_0+T]$,, $S(t-t_0)$ is a compact operator on X and $t \mapsto S(t-t_0)$ is continuous on $[t_0+\delta, t_0+T]$ in the uniform operator topology. Accordingly, we deduce that for each $t \in [t_0+\delta, t_0+T]$, the set

$$\{S(t-t_0)(u_0 - g(t_1,\dots,t_p,\widetilde{v}));\ v \in Y_r(\delta)\} \text{ is precompact in } X,$$

and that the family of functions

$$\{S(\bullet - t_0)(u_0 - g(t_1,\dots,t_p,\widetilde{v}));\ v \in Y_r(\delta)\} \text{ is equicontinuous,} \tag{1.3.4}$$

because the set $\{g(t_1,\dots,t_p,\widetilde{v});\ v \in Y_r(\delta)\}$ is bounded by assumption (iii). Thus for every $t \in [t_0+\delta, t_0+T]$ and every sequence $\{v_n\}_{n\in N} \subset Y_r(\delta)$, there exists $\{n_k\}_{k\in N} \subset \{n\}_{n\in N}$ such that $\{S(\bullet - t_0)(u_0 - g(t_1,\dots,t_p,\widetilde{v}_{n_k}))_{k\in N}$ converges, and therefore $\{(\mathcal{G}v_{n_k})(t)_{k\in N}$ converges by (1.3.3). This implies that for each $t \in [t_0+\delta, t_0+T]$, the set $\{(\mathcal{G}v)(t);\ v \in Y_r(\delta)\}$ is precompact in Y. On the other hand, for each $\varepsilon > 0$, there exists $\sigma > 0$ such that

$$\|(S(t-t_0) - S(s-t_0))g(t_1,\dots,t_p,\widetilde{v})\| < \varepsilon e^{-MLT}$$

valid for all $v \in Y_r(\delta)$, t, $s \in [t_0+\delta, t_0+T]$ with $|t-s| < \sigma$, by assertion (1.3.4). It follows from (1.3.3) that for these v, t, s,

$$\|(\mathcal{G})v(t) - (\mathcal{G}v)(s)\| < \varepsilon,$$

that is, the family of functions $\{(\mathcal{G}v)(\cdot);\ v \in Y_r(\delta)\}$ is equicontinuous. Now an application of Arzela-Ascoli's theorem justifies the precompactness of $\mathcal{G}(Y_r(\delta))$. It is clear that $Y_r(\delta)$ is a bounded closed convex subset of $Y(\delta)$. Therefore we can make use of Schauder's fixed point theorem to conclude that $\mathcal{G}$ has a fixed point $v_* \in Y_r(\delta)$. Put $u = \phi_{v_*}$. Then

$$u(t) = S(t-t_0)(u_0 - g(t_1, \ldots, t_p, \widetilde{v}_*)) + \int_{t_0}^{t} S(t-s)f(s,u(s))ds, \quad t \in [t_0, t_0+T]. \tag{1.3.5}$$

But

$$g(t_1, \ldots, t_p, \widetilde{v}_*)) = g(t_1, \ldots, t_p, u),$$

since

$$v_*(t) = (\mathcal{G}v_*)(t) = \phi_{v_*}(t) = u(t), \quad t \in [t_0+\delta, t_0+T],$$

by the definition of $\mathcal{G}$. This concludes, together with (1.3.5), that $u(t)$ is a solution of (1.3.1). The proof ends then.

□

A direct corollary of Theorem 1.3.1 is the following.

Corollary 1.3.2. *Assume that*

(i) $f : [t_0, t_0+T] \times X \to X$ *is continuous in* t *on* $[t_0, t_0+T]$ *and*

$$\|f(t,u) - f(t,v)\| \le r^{\alpha_1}\|u-v\|, \quad t \in [t_0, t_0+T], \ \ u, v \in \overline{B}_r,$$

for $0 \le \alpha_1 < 1$.

(ii) $g(t_1, \ldots, t_p, \cdot) : C([t_0, t_0+T], X) \to X$ *and there is a* $\delta \in (0, T)$ *such that for any* ϕ, $\psi \in Y_r$ *with* $\phi(s) = \psi(s)$ $(s \in [t_0+\delta, t_0+T])$,

$$g(t_1, \ldots, t_p, \phi) = g(t_1, \ldots, t_p, \psi).$$

(iii)

$$\|g(t_1, \ldots, t_p, \phi)\| < C\left(1 + \|\varphi\|_Y\right)^{\alpha_2}, \quad \varphi \in Y,$$

for $0 \le \alpha_2 < 1$.

Then (1.3.1) *has at least one solution* $u \in C([t_0, t_0+T], \overline{B}_r)$.

1.4 Applications

Example 1.4.1. Let us consider an operator A on a Banach space X generating an analytic semigroup $\{R(t)\}_{t\geq 0}$ on X such that

$$\|R(t)\| \leq e^{-\frac{t}{3}}, \quad \|AR(t)\| \leq \frac{1}{t}e^{-\frac{t}{3}} \quad (t \geq 0).$$

Clearly, the operator $A = \Delta - \frac{1}{3}I$ in the Banach space $X = L^2(R^n)$ with $\mathcal{D}(A) = H^2(R^n)$ is an example. From [24, 36, 37, 39, 70, 73, 85, 88], one can find many other examples.

Suppose that $f : [0,3] \times C([0,3], X) \to C([0,3], X)$ is continuous with

$$\|f(t,x) - f(t,y)\| \leq \frac{1}{3}\|x-y\|, \quad t \in [0,3], \quad x, y \in X,$$

and

$$g(1,2,\phi) = \frac{1}{2}\phi(1) - \frac{1}{2}\phi(2) \quad (\phi \in C([0,3], X)).$$

Set $t_0 = 0$, $T = 3$, $L = \omega = \dfrac{1}{3}$, $M = 1$, $p = 2$, $c_1 = \dfrac{1}{2}$, $c_2 = -\dfrac{1}{2}$, $t_1 = 1$, and $t_2 = 2$. Then

$$\alpha = M\sum_{i=1}^{p} |c_i| e^{(ML-\omega)t_i} = \frac{1}{2}\left(e^{L-\omega} + e^{2(L-\omega)}\right) = 1,$$

$$\beta = M\sum_{i=1}^{p} |c_i| e^{-\omega t_i} = \frac{1}{2}\left(e^{-\frac{1}{3}} + e^{-\frac{2}{3}}\right) < 1,$$

and

$$\gamma = \left\|\sum_{i=1}^{p} c_i R(t_i)\right\| = \frac{1}{2}\|R(2) - R(1)\| = \frac{1}{2}\left\|\int_1^2 AR(s)ds\right\| \leq \frac{1}{2}\int_1^2 \frac{e^{-\frac{s}{3}}}{s}ds \leq \frac{1}{2}e^{-\frac{1}{3}}\ln 2.$$

Hence

$$\beta - \gamma \geq \frac{1}{2}e^{-\frac{1}{3}}\left[\left(1 + e^{-\frac{1}{3}}\right) - \ln 2\right] > 0,$$

and $1 = \alpha < 1 + \beta - \gamma$. By Remark 1.2.9, (1.2.17) holds. So, the nonlocal Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)) \quad (0 \leq t \leq 3), \\ u(0) + g(1,2,u) = u_0 \end{cases}$$

has a unique mild solution $u \in C([0,3],X)$ by Theorem 1.2.6. But [61, Theorem 4.3] is not applicable since $\alpha = 1$; neither is [17, Theorem 3.1] since

$$MTL\left(I + M\|B\|\sum_{i=1}^{p}|c_i|\right) = 1 + \|B\| \geq 1.$$

Example 1.4.2. Let Ω be a bounded open connected subset of R^3 with C^∞-boundary, and let α and β be in $C^2([0,\infty),R)$ with $\alpha(0)$ and $\beta(0)$ positive. We consider an equation arising in the study of heat conduction in materials with memory (cf., e.g., [41, 43]):

$$\begin{aligned}\begin{pmatrix}\theta'(t)\\ \\ \eta'(t)\end{pmatrix} &= \begin{pmatrix}0 & I\\ \\ \alpha(0)\Delta & -\beta(0)I\end{pmatrix}\begin{pmatrix}\theta(t)\\ \\ \eta(t)\end{pmatrix}\\ &+ \int_0^t \begin{pmatrix}0 & I\\ \\ \alpha'(t-s)\Delta & -\beta'(t-s)I\end{pmatrix}\begin{pmatrix}\theta(s)\\ \\ \eta(s)\end{pmatrix}ds + \begin{pmatrix}0\\ \\ a(t,\theta(t))\end{pmatrix}.\end{aligned} \tag{1.4.1}$$

Set $X = H_0^1(\Omega) \times L^2(\Omega)$,

$$A = \begin{pmatrix}0 & I\\ \\ \alpha(0)\Delta & -\beta(0)I\end{pmatrix}, \quad \mathcal{D} = \left(H^2(\Omega)\bigcap H_0^1(\Omega)\right) \times H_0^1(\Omega).$$

From [20], we know that A generates a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on X with $\|T(t)\| \leq Me^{-\gamma t}$ $(t \geq 0)$ for constants $M, \gamma > 0$. For any given $l > 0$ and each $t \in [0,4l]$ set $F(t) = (F_{ij}(t))$, here

$$F_{11}(t) \equiv F_{12}(t) = 0, \quad F_{22}(t) = \frac{\alpha'(t)}{\alpha(0)}I,$$
$$F_{21}(t) = -\beta'(t)I + \beta(0)F_{22}(t).$$

Assume that

$$\|F_{22}(t)\|, \|F_{21}(t)\| \leq \frac{\gamma}{2M}e^{-\gamma t}, \quad t \in [0,4l],$$
$$\|F_{22}'(t)\|, \|F_{21}'(t)\| \leq \frac{\gamma^2}{4M^2}e^{-\gamma t}, \quad t \in [0,4l].$$

Then it follows from [40, p. 344] that the resolvent operator $R(t)$ for (1.4.1) satisfies

$$\|R(t)\| \leq Me^{-\frac{\gamma t}{2}}, \quad t \in [0,4l].$$

Suppose that $a(t,\theta) : [0,\infty) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ satisfies

$$\|a(t,x) - a(t,y)\|_{L^2(\Omega)} \leq \frac{\gamma}{2M}\|x-y\|_{H_0^1(\Omega)}, \quad x,y \in H_0^1(\Omega), \quad t \in [0,4l], \tag{1.4.2}$$

and define $b(\theta) : C([0,4l], H_0^1(\Omega)) \longrightarrow L^2(\Omega)$ by

$$b(\theta) = (Ml)^{-1}\left(\int_{(2-\varepsilon)l}^{2l}(\text{grad }\theta)(s)ds + \int_{(4-\varepsilon)l}^{4l}(\text{grad }\theta)(s)ds\right), \tag{1.4.3}$$

where $\varepsilon < \frac{1}{2}$. Then, by virtue of Theorem 1.2.7, we infer that for each $\theta_0 \in H_0^1(\Omega)$, $\eta_0 \in L^2(\Omega)$, equation (1.4.1) (for $t \in [0,4l]$) together with the nonlocal initial data

$$\begin{aligned} \begin{pmatrix} \theta(0) \\ \\ \eta(0) \end{pmatrix} &+ \begin{pmatrix} 0 \\ \\ (Ml)^{-1}\left(\int_{(2-\varepsilon)l}^{2l}(\text{grad }\theta)(s)ds + \int_{(4-\varepsilon)l}^{4l}(\text{grad }\theta)(s)ds\right) \end{pmatrix} \\ &= \begin{pmatrix} \theta_0 \\ \\ \eta_0 \end{pmatrix} \end{aligned} \tag{1.4.4}$$

has a unique mild solution $\begin{pmatrix} \theta(\cdot) \\ \\ \eta(\cdot) \end{pmatrix} \in C\left([0,4l], H_0^1(\Omega) \times L^2(\Omega)\right)$. In fact, if we write

$$f(t,u) = \begin{pmatrix} 0 \\ \\ a(t,\theta) \end{pmatrix} \quad \text{for } t \in [0,4l], \quad u = \begin{pmatrix} \theta \\ \\ \eta \end{pmatrix} \in X,$$

$$g(2l,4l,\phi) = \begin{pmatrix} 0 \\ \\ b(\theta) \end{pmatrix} \quad \text{for } \phi = \begin{pmatrix} \theta \\ \\ \eta \end{pmatrix} \in C([0,4l],X),$$

then by (1.4.2) and (1.4.3),

$$\|f(t,u) - f(t,v)\| \leq \frac{\gamma}{2M}\|u-v\|, \quad u,v \in X, \quad t \in [0,4l], \tag{1.4.5}$$

$$\|g(2l,4l,\phi) - g(2l,4l,\psi)\| \leq 2\varepsilon M^{-1}\max_{t\in[0,l]}\|\phi(t)-\psi(t)\|, \quad \phi,\psi \in C([0,4l],X).$$

Clearly, λ (in Theorem 1.2.7) $= 2\varepsilon < 1$. Therefore, by using Theorem 1.2.7 we obtain immediately the desired conclusion for any γ and $l > 0$. Nevertheless, [61, Theorem 3.2] is not applicable to the nonlocal Cauchy problem (1.4.1) and (1.4.4) if $\gamma l \geq \frac{1}{2}(1-\varepsilon) > \frac{1}{4}$. From (1.4.5) it is easy to see that the larger γ is, the larger the set of admissible f's becomes.

Chapter 2

Nonlocal Cauchy problems for semilinear evolution equations

2.1 Basic definitions

In this chapter, we will continue our study of the nonlocal Cauchy problems. Our target now is to give some new results about the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems for semilinear evolution equations. We will introduce a new notion, called (C, ω, M_η)-admissible pair (see Definition 2.1.3), and carry out our investigation in Banach spaces $W^{B,\omega}_{\eta_1,\eta_2}(T)$ (see (2.2.1) below) motivated by Jackson [52]. We first, in Section 2, establish an existence and uniqueness theorem for the continuous solution of a general convolution integral equation in a Banach space (equation (2.2.2)), and then in Section 3 apply our main result (Theorem 2.2.1) to yield existence and uniqueness theorems for mild and classical solutions of nonlocal Cauchy problems for semilinear evolution equations. As an example, we give a result on the existence and uniqueness of a classical solution of a semilnear parabolic equation with a boundary condition and a nonlocal initial condition. The results obtained in this chapter are generalizations of related results by Jackson [52] (see Remarks 2.2.2 and 2.3.5). Moreover, even for the corresponding classical abstract Cauchy problems the results here are new.

Let X be a Banach space, and C a bounded and injective linear operator on X.

Definition 2.1.1. (cf., e.g., [25, 30]) A strongly continuous family $\{V(t)\}_{t\geq 0}$ of bounded linear operators on X is called a *C-regularized semigroup* on X, if

(1) $V(0) = C$, and

(2)
$$V(t)V(s) = CV(t+s) \quad \text{for all } s, t \geq 0. \tag{2.1.1}$$

The *generator* G of $\{V(t)\}_{t \geq 0}$ is defined by

$$Gx = C^{-1}\left[\lim_{t \to 0^+} \frac{1}{t}(V(t)x - Cx)\right]$$

with

$$\mathcal{D}(G) = \{x; \ \text{ the limit exists in the range of } C\}.$$

Definition 2.1.2. (Compare, e.g., [36, p. 137]) A closed linear operator B in X is said to have *fractional powers* if there exists a family of closed linear operators $\{B^r\}_{r \geq 0}$ such that

(1) $B^0 = I$ (the identity), $B^1 = B$, and

(2)
$$B^\eta B^\delta \subset B^{\eta+\delta} \quad \text{for all } \eta, \delta \geq 0. \tag{2.1.2}$$

Definition 2.1.3. A pair $\{B, \ \{V(t)\}_{t \geq 0}\}$, comprised of a closed linear operator B and a strongly continuous family $\{V(t)\}_{t \geq 0}$ of bounded linear operators on X, is called a (C, ω, M_η)-*admissible pair* (*admissible pair*, in short) on X if B has fractional powers, $\{V(t)\}_{t \geq 0}$ is a C-regularized semigroup on X, and there exist constants $\omega \in R$ and M_η such that

$$B^\eta V(t)u = V(t)B^\eta u, \quad \eta \geq 0, \ t \in [0, T], \ u \in \mathcal{D}\left(B^\eta\right), \tag{2.1.3}$$

$$\left\|B^\eta V(t)u\right\| \leq M_\eta \frac{e^{-\omega t}}{t^\eta} \left\|u\right\|, \quad 0 \leq \eta \leq 1, \ t \in (0, T], \ u \in X, \tag{2.1.4}$$

Example 2.1.4. Let $\{S(t)\}_{t\geq 0}$ be an analytic semigroup on X generated by $-A$ satisfying

$$\|S(t)\| \leq \text{ const } e^{-\omega t}, \quad t \geq 0, \tag{2.1.5}$$

for some $\omega \in R$. Then $\{A - \omega I, \{S(t)\}_{t\geq 0}\}$ is a (I, ω, M_η)-admissible pair on X for certain constants M_η. In fact, it is clear that $A - \omega I$ has fractional powers (the usual ones, cf., e.g., Balakrishnan [4], Engel and Nagel [36], Henry [49], Pazy [70], van Casteren [85] or Xiao and Liang [88]). Moreover, it is known that (2.1.3) and (2.1.4) hold with $S(\cdot)$, $A-\omega I$ in place of $V(\cdot)$, B, respectively (cf., e.g., [4, 36, 49, 70, 85, 88]).

Example 2.1.5. Let A be an operator of n-type θ ($0 \leq \theta < \frac{\pi}{2}$, $n \in N \cup \{0\}$) (see deLaubenfels, Yao and Wang [31, Definition 1.3]) and the family $\{W(t)\}_{t\geq 0}$ be the analytic A^{-n}-regularized semigroup on X generated by $-A$ (see deLaubenfels [30, Definition 21.3]). Then A has fractional powers $\{A^r\}_{r\geq 0}$ defined as in [31] or in Straub [80]. The formulas (2.1.3) and (2.1.4) can be verified for some constants ω and M_η by a combination of the results and the techniques of [30], [70, Sections 2.5 and 2.6], and [80]. Therefore, $\{A, \ \{W(t)\}_{t\geq 0}\}$ is a (A^{-n}, ω, M_η)-admissible pair on X.

2.2 An integral equation with (C, ω, M_η)-admissible pair

Let $\{B, \ \{V(t)\}_{t\geq 0}\}$ be a (C, ω, M_η)-admissible pair on X, where B has fractional powers $\{B^r\}_{r\geq 0}$. For any two fixed real numbers η_1 and η_2 for which $0 \leq \eta_1 \leq \eta_2$, $\eta_2 - \eta_1 < 1$, and B^{η_1} has a bounded inverse, we define the space $W^{B,\omega}_{\eta_1,\eta_2}(T)$ (see, e.g., Jackson [52]) by

$$\begin{aligned} W^{B,\omega}_{\eta_1,\eta_2}(T) \ := \ & \Big\{ u \in C\left([0,T], \mathcal{D}\left(B^{\eta_1}\right)\right) \cap C\left((0,T], \mathcal{D}\left(B^{\eta_2}\right)\right); \\ & \sup_{0\leq t\leq T} t^{\eta_2-\eta_1} \left\|B^{\eta_2} u(t)\right\| < \infty \Big\} \end{aligned} \tag{2.2.1}$$

equipped with the norm

$$\|u\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} = \sup_{0\leq t\leq T} \left\{ e^{\omega t} \left\|B^{\eta_1} u(t)\right\| + e^{\omega t} t^{\eta_2-\eta_1} \left\|B^{\eta_2} u(t)\right\| \right\}.$$

It is clear that $W^{B,\omega}_{\eta_1,\eta_2}(T)$ is a Banach space.

Consider the nonlinear convolution integral equation

$$u(t) = V(t)[u_0 - h(u)] + \int_0^t V(t-s)f(u)(s)ds, \quad t \in [0,T], \tag{2.2.2}$$

where

(i) $u_0 \in C\left(\mathcal{D}\left(B^{\eta_1}\right)\right),$

(ii) the X-valued function h on $W^{B,\omega}_{\eta_1,\eta_2}(T)$ satisfies

$$h : W^{B,\omega}_{\eta_1,\eta_2}(T) \longrightarrow C\left(\mathcal{D}\left(B^{\eta_1}\right)\right). \tag{2.2.3}$$

(iii) the operator f from $W^{B,\omega}_{\eta_1,\eta_2}(T)$ to the space of X-valued functions on $[0,T]$ satisfies

$$C^{-1}f : W^{B,\omega}_{\eta_1,\eta_2}(T) \longrightarrow L^\infty\left(0,T;\mathcal{D}\left(B^{\mu}\right)\right) \tag{2.2.4}$$

for a constant $\mu \geq 0$ with

$$\eta_2 - 1 < \mu \leq \eta_1. \tag{2.2.5}$$

It is easy to see by $V(0) = C$ and (2.1.3) that

$$C\left(\mathcal{D}\left(B^{\eta_1}\right)\right) \subset \mathcal{D}\left(B^{\eta_1}\right)). \tag{2.2.6}$$

In the sequel, $\beta(\cdot,\cdot)$ denotes the β-function.

In the following we prove existence and uniqueness of continuous solutions to equation (2.2.2). In inequalities (2.2.8) and (2.2.9) of (b) we impose global Lipschitz conditions on h and f, whereas in (2.2.12) of (c) we employ an adapted Lipschitz condition on f and a global one on h.

Theorem 2.2.1. (a) *Fix* $u_0 \in C\left(\mathcal{D}\left(B^{\eta_1}\right)\right)$. *For every* $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$, *the function*

$$t \mapsto V(t)[u_0 - h(u)] + \int_0^t V(t-s)f(u)(s)ds \ \in W^{B,\omega}_{\eta_1,\eta_2}(T). \tag{2.2.7}$$

(b) *Let the function h satisfy*

$$\|B^{\eta_1}h(u) - B^{\eta_1}h(v)\| \leq K\|u - v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)}, \quad u, v \in W^{B,\omega}_{\eta_1,\eta_2}(T) \tag{2.2.8}$$

for some $K > 0$, and let the operator f satisfy

$$\begin{aligned}\|e^{\omega t}t^{\gamma}B^{\mu}C^{-1}[f(u) - f(v)]\|_{L^{\infty}(0,T;X)} \ \leq \ & L_1\|u - v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)}, \\ & u, v \in W^{B,\omega}_{\eta_1,\eta_2}(T),\end{aligned} \tag{2.2.9}$$

for some constants $L_1 > 0$, $\gamma \geq 0$ with $\eta_1 - \mu + \gamma \leq 1$. In addition, assume that the constants in (2.1.4), (2.2.4), (2.2.8) *and* (2.2.9) *satisfy*

$$\begin{aligned}\kappa \ := \ & \Big(M_{\eta_1-\mu}\beta(1 - \eta_1 + \mu, 1 - \gamma) + M_{\eta_2-\mu}\beta(1 - \eta_2 + \mu, 1 - \gamma)\Big) \\ & \times L_1\|C\|T^{1-\eta_1+\mu-\gamma} + (M_0 + M_{\eta_2-\eta_1})\,K \\ < \ & 1,\end{aligned} \tag{2.2.10}$$

Then for any $u_0 \in C(\mathcal{D}(B^{\eta_1}))$, equation (2.2.2) *has a unique solution $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$.*

(c) *Fix $T > 0$, $L_1 \geq 0$. Fix $\mu \geq 0$ and $\gamma \geq 0$ such that*

$$\eta_2 - \mu + \gamma < 1. \tag{2.2.11}$$

Then there exists a constant $K > 0$ such that for all functions h for which (2.2.8) *is valid, and all operators f for which*

$$\begin{aligned} & \|e^{\omega t}t^{\gamma}B^{\mu}C^{-1}[f(u)(t) - f(v)(t)]\| \\ \leq \ & L_1 \sup_{0\leq s\leq t}\Big[e^{\omega s}\,\|B^{\eta_1}(u(s) - v(s))\| + e^{\omega s}s^{\eta_2-\eta_1}\,\|B^{\eta_2}(u(s) - v(s))\|\Big], \\ & u, v \in W^{B,\omega}_{\eta_1,\eta_2}(T),\ t \in [0, T],\end{aligned} \tag{2.2.12}$$

is true, equation (2.2.2) *has a unique solution $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$ for any $u_0 \in C(\mathcal{D}(B^{\eta_1}))$.*

Remark 2.2.2. (1) Theorem 2.2.1 generalizes Lemma 3.1 and Theorem 3.2 in [52] for the case where $R_0 = \infty$, because of the following.

(i) The (C,ω,M_η)-admissible pair $\{B,\ \{V(t)\}_{t\geq 0}\}$ on X is more general than the corresponding ones considered in [12]. One of the reasons is that the regularized operator semigroup is a generalization of the classical strongly continuous operator semigroup (see, e.g., Davies and Pang [25] and deLaubenfels [30]). Therefore, equation (2.2.2) is more general than equation (2.3.1) in [52]. In order to facilitate the comparison with [52], we mention that the number $\frac{1}{2}$ in [52, (2.2) – (2.4), (2.7) – (2.9), (3.3)] can be replaced by 1 without having any influence on the sharpness of the related conditions and the conclusions. It looks as if $M_{s_1/2}$ in [52, (3.3)] should be M_0.

(ii) Assertion (c) of Theorem 2.2.1 reveals that many nonlinear convolution integral equations in Banach spaces of the form (2.2.2), for which

$$\begin{aligned}&(M_{\eta_1-\mu}\beta(1-\eta_1+\mu,1-\gamma)+M_{\eta_2-\mu}\beta(1-\eta_2+\mu,1-\gamma))\\&\qquad\qquad\times L_1\|C\|T^{1-\eta_1+\mu-\gamma}\\&\geq 1,\end{aligned}$$

hence to which the present-day results could not be applied, possess continuous solutions.

(2) Even in the special case where $h=0$, the result is new.

(3) Similar comments apply to Theorems 2.2.4,2.3.1 and 2.3.2 below.

Proof of Theorem 2.2.1. We fix $u_0\in C\left(\mathcal{D}\left(B^{\eta_1}\right)\right)$.

(a). From (2.1.1) – (2.1.4), (2.2.4), (2.2.5) and the strong continuity of $\{V(t)\}_{t\geq 0}$ on $[0,T]$, it follows that for all $0\leq t\leq r\leq T$, $u\in W^{B,\omega}_{\eta_1,\eta_2}(T)$,

$$\begin{aligned}&\left\|B^{\eta_2-\mu}[V(r-s)-V(t-s)]B^{\mu}f(u)(s)\right\|\\=\ &\left\|B^{\eta_2-\mu}V(t-s)[V(r-t)-C]B^{\mu}C^{-1}f(u)(s)\right\|\\\leq\ &M_{\eta_2-\mu}\frac{e^{-\omega(t-s)}}{(t-s)^{\eta_2-\mu}}\left\|[V(r-t)-C]B^{\mu}C^{-1}f(u)(s)\right\|\\&\longrightarrow 0,\quad \text{as } r\to t \text{ for every } s\in[0,t),\end{aligned}$$

and

$$
\begin{aligned}
&\|B^{\eta_2-\mu}[V(r-s)-V(t-s)]B^{\mu}f(u)(s)\| \\
\leq\ & 2M_0M_{\eta_2-\mu}\frac{e^{|\omega|T}}{(t-s)^{\eta_2-\mu}}\|C^{-1}B^{\mu}f(u)\|_{L^{\infty}(0,T;X)}, \quad \text{for a.e. } s\in[0,t].
\end{aligned} \tag{2.2.13}
$$

Since $0 \leq \eta_2 - \mu < 1$ the right-hand side of (2.2.13) is integrable on $[0,t]$. This observation, together with Lebesgue's dominated convergence theorem implies that for every $0 \leq t \leq r \leq T$, $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$,

$$
\begin{aligned}
&\left\|B^{\eta_2}\int_0^r V(r-s)f(u)(s)ds - B^{\eta_2}\int_0^t V(t-s)f(u)(s)ds\right\| \\
\leq\ & \left\|B^{\eta_2}\int_0^t [V(r-s)-V(t-s)]f(u)(s)ds\right\| + \left\|B^{\eta_2}\int_t^r V(r-s)f(u)(s)ds\right\| \\
\leq\ & \left\|B^{\eta_2-\mu}\int_0^t [V(r-s)-V(t-s)]B^{\mu}f(u)(s)ds\right\| \\
&\qquad + \left\|B^{\eta_2-\mu}\int_t^r V(r-s)B^{\mu}f(u)(s)ds\right\| \\
\leq\ & \int_0^t \left\|B^{\eta_2-\mu}[V(r-s)-V(t-s)]B^{\mu}f(u)(s)\right\| ds \\
&\qquad + M_{\eta_2-\mu}\int_t^r \frac{e^{|\omega|T}}{(r-s)^{\eta_2-\mu}}\|B^{\mu}f(u)\|_{L^{\infty}(0,T;X)}ds \\
&\qquad\qquad \longrightarrow \quad 0 \quad \text{as } r\to t.
\end{aligned}
$$

Hence the function $t \mapsto B^{\eta_2}\int_0^t V(t-s)f(u)(s)ds$ is continuous from the right in $[0,T)$. A similar reasoning shows that it is also continuous from the left in $(0,T]$. Therefore the function

$$
t \mapsto B^{\eta_2}\int_0^t V(t-s)f(u)(s)ds \ \in C\left([0,T],X\right). \tag{2.2.14}
$$

Likewise, we obtain the function

$$
t \mapsto B^{\eta_1}\int_0^t V(t-s)f(u)(s)ds \ \in C\left([0,T],X\right). \tag{2.2.15}
$$

As a consequence of (2.2.14) and (2.2.15), the function

$$t \mapsto \int_0^t V(t-s)f(u)(s)ds \ \in W^{B,\omega}_{\eta_1,\eta_2}(T). \tag{2.2.16}$$

In view of (2.2.6) and the strong continuity of $\{V(t)\}_{t\geq 0}$ on $[0,T]$ we see that for all $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$, the function

$$t \mapsto V(t)[u_0 - h(u)] \ \in C\left([0,T], \mathcal{D}\left(B^{\eta_1}\right)\right), \tag{2.2.17}$$

and by (2.1.4),

$$\sup_{0\leq t\leq T}\left\{t^{\eta_2-\eta_1}\left\|B^{\eta_2}V(t)[u_0-h(u)]\right\|\right\} \leq M_{\eta_2-\eta_1}e^{|\omega|T}\left\|B^{\eta_1}[u_0-h(u)]\right\|.$$

Moreover, by (2.1.1) – (2.1.4), (2.2.3) and the strong continuity of $\{V(t)\}_{t\geq 0}$ on $[0,T]$ we deduce that for each $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$, and $0 < t \leq r \leq T$,

$$\begin{aligned}
&\left\|B^{\eta_2}V(r)[u_0-h(u)] - B^{\eta_2}V(t)[u_0-h(u)]\right\| \\
= \ &\left\|B^{\eta_2-\eta_1}V(t)(V(r-t)-C)B^{\eta_1}C^{-1}[u_0-h(u)]\right\| \\
\leq \ &M_{\eta_2-\eta_1}\frac{e^{|\omega|T}}{t^{\eta_2-\eta_1}}\left\|[V(r-t)-C]B^{\eta_1}C^{-1}[u_0-h(u)]\right\| \\
\longrightarrow \ &0 \quad \text{as } r \to t.
\end{aligned}$$

Thus the function $t \mapsto B^{\eta_2}V(t)[u_0 - h(u)]$ is right continuous in $(0,T)$. A similar reasoning shows that it is left continuous in $(0,T]$. So the function

$$t \mapsto V(t)[u_0 - h(u)] \ \in C\left((0,T], \mathcal{D}\left(B^{\eta_2}\right)\right).$$

This together with (2.2.17) gives that the function

$$t \mapsto V(t)[u_0 - h(u)] \ \in W^{B,\omega}_{\eta_1,\eta_2}(T). \tag{2.2.18}$$

According to (2.2.16) and (2.2.18), we infer (2.2.7).

(b). We define an operator $\mathcal{F}$ on $W^{B,\omega}_{\eta_1,\eta_2}(T)$ by

$$(\mathcal{F}u)(t) = V(t)[u_0 - h(u)] + \int_0^t V(t-s)f(u)(s)ds, \quad t \in [0,T],\ u \in W^{B,\omega}_{\eta_1,\eta_2}(T). \tag{2.2.19}$$

Assertion (a) of Theorem 2.2.2 shows

$$\mathcal{F}\left(W^{B,\omega}_{\eta_1,\eta_2}(T)\right) \subset W^{B,\omega}_{\eta_1,\eta_2}(T).$$

By (2.1.2) – (2.1.4), (2.2.6), (2.2.8) and (2.2.9), we deduce that for any $t \in [0,T]$, $u, v \in W^{B,\omega}_{\eta_1,\eta_2}(T)$,

$$\begin{aligned}
& e^{\omega t}\left\|B^{\eta_1}\left[(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\right]\right\| + e^{\omega t}t^{\eta_2-\eta_1}\left\|B^{\eta_2}\left[(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\right]\right\| \\
\leq\ & e^{\omega t}\left\{\left\|V(t)B^{\eta_1}\left[h(u)-h(v)\right]\right\| + t^{\eta_2-\eta_1}\left\|B^{\eta_2-\eta_1}V(t)B^{\eta_1}\left[h(u)-h(v)\right]\right\|\right\} \\
& +e^{\omega t}\left\|\int_0^t B^{\eta_1-\mu}V(t-s)B^{\mu}\left[f(u)(s)-f(v)(s)\right]ds\right\| \\
& +e^{\omega t}t^{\eta_2-\eta_1}\left\|\int_0^t B^{\eta_2-\mu}V(t-s)B^{\mu}\left[f(u)(s)-f(v)(s)\right]ds\right\| \\
\leq\ & M_0\left\|B^{\eta_1}\left[h(u)-h(v)\right]\right\| + M_{\eta_2-\eta_1}\left\|B^{\eta_1}\left[h(y)-h(z)\right]\right\| \\
& +e^{\omega t}M_{\eta_1-\mu}\int_0^t \frac{e^{-\omega(t-s)}}{(t-s)^{\eta_1-\mu}}\left\|B^{\mu}\left[f(u)(s)-f(v)(s)\right]\right\|ds \\
& +e^{\omega t}t^{\eta_2-\eta_1}M_{\eta_2-\mu}\int_0^t \frac{e^{-\omega(t-s)}}{(t-s)^{\eta_2-\mu}}\left\|B^{\mu}\left[f(u)(s)-f(v)(s)\right]\right\|ds \\
\leq\ & \left(M_0+M_{\eta_2-\eta_1}\right)K\|u-v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} \\
& +L_1\|C\|\left[M_{\eta_1-\mu}\int_0^t \frac{1}{(t-s)^{\eta_1-\mu}s^{\gamma}}ds + t^{\eta_2-\eta_1}M_{\eta_2-\mu}\int_0^t \frac{1}{(t-s)^{\eta_2-\mu}s^{\gamma}}ds\right] \\
& \times\|u-v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} \\
\leq\ & \kappa\|u-v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)},
\end{aligned} \tag{2.2.20}$$

and hence

$$\|\mathcal{F}u-\mathcal{F}v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} \leq \kappa\|u-v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)}.$$

Here κ is the constant as defined in (2.2.10). Therefore, $\mathcal{F}$ is a contractive mapping. Thus, $\mathcal{F}$ has a unique fixed point $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$ by the Banach contraction mapping theorem. Clearly, this $u(t)$ is the desired continuous solution of (2.2.2).

(c). Fix operator f for which (2.2.12) is valid. For each $z \in W^{B,\omega}_{\eta_1,\eta_2}(T)$ and each function h on $W^{B,\omega}_{\eta_1,\eta_2}(T)$ satisfying (2.2.3), we define an operator $\mathcal{F}_{z,h}$ on $W^{B,\omega}_{\eta_1,\eta_2}(T)$ by

$$(\mathcal{F}_{z,h}u)(t) = V(t)[u_0-h(z)] + \int_0^t V(t-s)f(u)(s)ds, \quad t \in [0,T]. \tag{2.2.21}$$

Assertion (a) of Theorem 2.2.2 indicates

$$\mathcal{F}_{z,h}\left(W^{B,\omega}_{\eta_1,\eta_2}(T)\right) \subset W^{B,\omega}_{\eta_1,\eta_2}(T), \quad z \in W^{B,\omega}_{\eta_1,\eta_2}(T),$$

for every function h defined on $W^{B,\omega}_{\eta_1,\eta_2}(T)$ satisfying (2.2.3). In the same way as we got (2.2.20), by (2.1.2) – (2.1.4), (2.2.6) and (2.2.12) we now obtain, for any $t \in [0,T]$, $u, v \in W^{B,\omega}_{\eta_1,\eta_2}(T)$,

$$\begin{aligned} & e^{\omega t}\left\|B^{\eta_1}\left[(\mathcal{F}_{z,h}u)(t) - (\mathcal{F}_{z,h}v)(t)\right]\right\| + e^{\omega t}t^{\eta_2-\eta_1}\left\|B^{\eta_2}\left[(\mathcal{F}_{z,h}u)(t) - (\mathcal{F}_{z,h}v)(t)\right]\right\| \\ \leq\ & L_1\|C\|\left[M_{\eta_1-\mu}\int_0^t \frac{1}{(t-s)^{\eta_1-\mu}s^\gamma}ds + t^{\eta_2-\eta_1}M_{\eta_2-\mu}\int_0^t \frac{1}{(t-s)^{\eta_2-\mu}s^\gamma}ds\right] \\ & \times \sup_{0\leq s\leq t}\left[e^{\omega s}\left\|B^{\eta_1}(u(s)-v(s))\right\| + e^{\omega s}s^{\eta_2-\eta_1}\left\|B^{\eta_2}(u(s)-v(s))\right\|\right]. \end{aligned} \tag{2.2.22}$$

By (2.2.11) we can choose a constant ε such that

$$\eta_2 - \mu + \gamma < \varepsilon < 1.$$

This means

$$\frac{1}{\varepsilon}(\eta_1-\mu) < 1, \quad \frac{1}{\varepsilon}\gamma < 1, \quad \frac{1}{\varepsilon}(\eta_2-\mu) < 1, \quad \varepsilon - \eta_1 + \mu - \gamma > 0. \tag{2.2.23}$$

By virtue of (2.2.22), (2.2.23) and the Hölder inequality, we deduce that for any $t \in [0,T]$,

$$\begin{aligned} & e^{\omega t}\left\|B^{\eta_1}((\mathcal{F}_{z,h}u)(t) - (\mathcal{F}_{z,h}v)(t))\right\| + e^{\omega t}t^{\eta_2-\eta_1}\left\|B^{\eta_2}((\mathcal{F}_{z,h}u)(t) - (\mathcal{F}_{z,h}v)(t))\right\| \\ \leq\ & L_1\|C\|\left[M_{\eta_1-\mu}\left(\int_0^t\left(\frac{1}{(t-s)^{\eta_1-\mu}s^\gamma}\right)^{\frac{1}{\varepsilon}}ds\right)^{\varepsilon}\left(\int_0^t ds\right)^{1-\varepsilon}\right. \\ & \left. + t^{\eta_2-\eta_1}M_{\eta_2-\mu}\left(\int_0^t\left(\frac{1}{(t-s)^{\eta_2-\mu}s^\gamma}\right)^{\frac{1}{\varepsilon}}ds\right)^{\varepsilon}\left(\int_0^t ds\right)^{1-\varepsilon}\right] \\ & \times \sup_{0\leq s\leq t}\left[e^{\omega s}\left\|B^{\eta_1}(u(s)-v(s))\right\| + e^{\omega s}s^{\eta_2-\eta_1}\left\|B^{\eta_2}(u(s)-v(s))\right\|\right] \end{aligned}$$

$$
\begin{aligned}
&\leq L_1\|C\|t^{1-\varepsilon}\Bigg[M_{\eta_1-\mu}t^{\varepsilon-\eta_1+\mu-\gamma}\beta^{\varepsilon}\left(1-\frac{1}{\varepsilon}(\eta_1-\mu),1-\frac{1}{\varepsilon}\gamma\right) \\
&\quad +t^{\eta_2-\eta_1}M_{\eta_2-\mu}t^{\varepsilon-(\eta_2-\mu+\gamma)}\beta^{\varepsilon}\left(1-\frac{1}{\varepsilon}(\eta_2-\mu),1-\frac{1}{\varepsilon}\gamma\right)\Bigg] \\
&\quad \times\sup_{0\leq s\leq t}\Big[e^{\omega s}\left\|B^{\eta_1}(u(s)-v(s))\right\|+e^{\omega s}s^{\eta_2-\eta_1}\left\|B^{\eta_2}(u(s)-v(s))\right\|\Big] \\
&\leq L_1\|C\|t^{1-\varepsilon}T^{\varepsilon-\eta_1+\mu-\gamma}\Bigg[M_{\eta_1-\mu}\beta^{\varepsilon}\left(1-\frac{1}{\varepsilon}(\eta_1-\mu),1-\frac{1}{\varepsilon}\gamma\right) \\
&\quad +M_{\eta_2-\mu}\beta^{\varepsilon}\left(1-\frac{1}{\varepsilon}(\eta_2-\mu),1-\frac{1}{\varepsilon}\gamma\right)\Bigg] \\
&\quad \times\sup_{0\leq s\leq t}\Big[e^{\omega s}\left\|B^{\eta_1}(u(s)-v(s))\right\|+e^{\omega s}s^{\eta_2-\eta_1}\left\|B^{\eta_2}(u(s)-v(s))\right\|\Big].
\end{aligned} \tag{2.2.24}
$$

Therefore, for any $t \in [0,T]$, we obtain

$$
\begin{aligned}
&e^{\omega t}\left\|B^{\eta_1}((\mathcal{F}^2_{z,h}u)(t)-(\mathcal{F}^2_{z,h}v)(t))\right\|+e^{\omega t}t^{\eta_2-\eta_1}\left\|B^{\eta_2}((\mathcal{F}^2_{z,h}u)(t)-(\mathcal{F}^2_{z,h}v)(t))\right\| \\
&\leq L_1\|C\|\int_0^t\left[M_{\eta_1-\mu}\frac{1}{(t-s)^{\eta_1-\mu}s^{\gamma}}+t^{\eta_2-\eta_1}M_{\eta_2-\mu}\frac{1}{(t-s)^{\eta_2-\mu}s^{\gamma}}\right] \\
&\quad \times\sup_{0\leq r\leq s}\Big[e^{\omega r}\left\|B^{\eta_1}((\mathcal{F}_{z,h})u(r)-(\mathcal{F}_{z,h}v)(r))\right\| \\
&\qquad +e^{\omega r}r^{\eta_2-\eta_1}\left\|B^{\eta_2}((\mathcal{F}_{z,h})u(r)-(\mathcal{F}_{z,h}v)(r))\right\|\Big]ds \\
&\leq (L_1\|C\|)^2\left[M_{\eta_1-\mu}\int_0^t\frac{s^{1-\varepsilon}}{(t-s)^{\eta_1-\mu}s^{\gamma}}ds+t^{\eta_2-\eta_1}M_{\eta_2-\mu}\int_0^t\frac{s^{1-\varepsilon}}{(t-s)^{\eta_2-\mu}s^{\gamma}}ds\right] \\
&\quad \times T^{\varepsilon-\eta_1+\mu-\gamma}\Bigg[M_{\eta_1-\mu}\beta^{\varepsilon}\left(1-\frac{1}{\varepsilon}(\eta_1-\mu),1-\frac{1}{\varepsilon}\gamma\right) \\
&\quad +M_{\eta_2-\mu}\beta^{\varepsilon}\left(1-\frac{1}{\varepsilon}(\eta_2-\mu),1-\frac{1}{\varepsilon}\gamma\right)\Bigg] \\
&\quad \times\sup_{0\leq r\leq t}\Big[e^{\omega r}\left\|B^{\eta_1}(u(r)-v(r))\right\|+e^{\omega r}r^{\eta_2-\eta_1}\left\|B^{\eta_2}(u(r)-v(r))\right\|\Big] \\
&\leq Q(L_1\|C\|)^2\Bigg[M_{\eta_1-\mu}\left(\int_0^t\left(\frac{1}{(t-s)^{\eta_1-\mu}s^{\gamma}}\right)^{\frac{1}{\varepsilon}}ds\right)^{\varepsilon}\left(\int_0^t\left(s^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}ds\right)^{1-\varepsilon} \\
&\quad +t^{\eta_2-\eta_1}M_{\eta_2-\mu}\left(\int_0^t\left(\frac{1}{(t-s)^{\eta_2-\mu}s^{\gamma}}\right)^{\frac{1}{\varepsilon}}ds\right)^{\varepsilon}\left(\int_0^t\left(s^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}ds\right)^{1-\varepsilon}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
& \times \sup_{0\le r\le t}\Big[e^{\omega r}\left\|B^{\eta_1}(u(r)-v(r))\right\| + e^{\omega r} r^{\eta_2-\eta_1}\left\|B^{\eta_2}(u(r)-v(r))\right\|\Big] \\
\le\ & (QL_1\|C\|)^2\left(\frac{t^2}{2!}\right)^{1-\varepsilon}\sup_{0\le r\le t}\Big[e^{\omega r}\left\|B^{\eta_1}(u(r)-v(r))\right\| \\
& \qquad + e^{\omega r} r^{\eta_2-\eta_1}\left\|B^{\eta_2}(u(r)-v(r))\right\|\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
Q \ :=\ & T^{\varepsilon-\eta_1+\mu-\gamma}\Bigg[M_{\eta_1-\mu}\beta^{\varepsilon}\left(1-\frac{1}{\varepsilon}(\eta_1-\mu), 1-\frac{1}{\varepsilon}\gamma\right) \\
& \qquad + M_{\eta_2-\mu}\beta^{\varepsilon}\left(1-\frac{1}{\varepsilon}(\eta_2-\mu), 1-\frac{1}{\varepsilon}\gamma\right)\Bigg].
\end{aligned}
$$

Using induction we infer that for any $t\in[0,T]$, $n\in N$,

$$
\begin{aligned}
& \left\|B^{\eta_1}((\mathcal{F}^n_{z,h}u)(t)-(\mathcal{F}^n_{z,h}v)(t))\right\| + t^{\eta_2-\eta_1}\left\|B^{\eta_2}((\mathcal{F}^n_{z,h}u)(t)-(\mathcal{F}^n_{z,h}v)(t))\right\| \\
\le\ & (QL_1\|C\|)^n\left(\frac{t^n}{n!}\right)^{1-\varepsilon}\sup_{0\le s\le t}\Big[e^{\omega s}\left\|B^{\eta_1}(u(s)-v(s))\right\| \\
& \qquad + e^{\omega s} s^{\eta_2-\eta_1}\left\|B^{\eta_2}(u(s)-v(s))\right\|\Big]
\end{aligned}
\tag{2.2.25}
$$

Let n be a positive integer so large that

$$
(QL_1\|C\|)^n\left(\frac{T^n}{n!}\right)^{1-\varepsilon} < 1.
$$

Then (2.2.25) shows that $\mathcal{F}^n_{z,h}$ is a contractive mappings from $W^{B,\omega}_{\eta_1,\eta_2}(T)$ to $W^{B,\omega}_{\eta_1,\eta_2}(T)$. By the well known extension of the Banach contraction principle $\mathcal{F}_{z,h}$ has a unique fixed point $u_{z,h}\in W^{B,\omega}_{\eta_1,\eta_2}(T)$.

Accordingly, we see that for every $z\in W^{B,\omega}_{\eta_1,\eta_2}(T)$ and function h on $W^{B,\omega}_{\eta_1,\eta_2}(T)$ satisfying (2.2.3), there exists a unique $u_{z,h}\in W^{B,\omega}_{\eta_1,\eta_2}(T)$ such that

$$
u_{z,h}(t) = V(t)[u_0-h(z)] + \int_0^t V(t-s)f\left(u_{z,h}\right)(s)ds, \quad t\in[0,T]. \tag{2.2.26}
$$

For each function h on $W^{B,\omega}_{\eta_1,\eta_2}(T)$ satisfying (2.2.3), define an operator $\mathcal{G}_h$ on $W^{B,\omega}_{\eta_1,\eta_2}(T)$ by

$$
\mathcal{G}_h z = u_{z,h}, \quad \text{for every } \ z\in W^{B,\omega}_{\eta_1,\eta_2}(T).
$$

Clearly, $\mathcal{G}_h$ is well defined and

$$\mathcal{G}_h\left(W^{B,\omega}_{\eta_1,\eta_2}(T)\right) \subset W^{B,\omega}_{\eta_1,\eta_2}(T).$$

A combination of the definition of $\mathcal{G}_h$, (2.2.26), (2.1.1) – (2.1.4), (2.2.3), (2.2.6), (2.2.11) and (2.2.12) shows that for all $y, z \in W^{B,\omega}_{\eta_1,\eta_2}(T)$ and $t \in [0,T]$,

$$\begin{aligned}
& e^{\omega t}\left\|B^{\eta_1}\left[(\mathcal{G}_h y)(t) - (\mathcal{G}_h z)(t)\right]\right\| + e^{\omega t} t^{(\eta_2-\eta_1)}\left\|B^{\eta_2}\left[(\mathcal{G}_h y)(t) - (\mathcal{G}_h z)(t)\right]\right\| \\
\leq\ & e^{\omega t}\Bigg\{ \left\|B^{\eta_1} V(t)\left[h(y) - h(z)\right]\right\| \\
& + \left\|\int_0^t B^{\eta_1-\mu} V(t-s) B^{\mu}\left[f(\mathcal{G}_h y)(s) - f(\mathcal{G}_h z)(s)\right] ds\right\| \\
& + \left\|t^{\eta_2-\eta_1} B^{\eta_2-\eta_1} V(t) B^{\eta_1}\Big[h(y) - h(z)\Big]\right\| \\
& + t^{\eta_2-\eta_1}\left\|\int_0^t B^{\eta_2-\mu} V(t-s) B^{\mu}\left[f(\mathcal{G}_h y)(s) - f(\mathcal{G}_h z)(s)\right] ds\right\| \Bigg\} \\
\leq\ & M_0\left\|B^{\eta_1}\left[h(y) - h(z)\right]\right\| + M_{\eta_2-\eta_1}\left\|B^{\eta_1}\left[h(y) - h(z)\right]\right\| \\
& + L_1\|C\| \int_0^t \left[M_{\eta_1-\mu}\frac{1}{(t-s)^{\eta_1-\mu}s^{\gamma}} + t^{\eta_2-\eta_1} M_{\eta_2-\mu}\frac{1}{(t-s)^{\eta_2-\mu}s^{\gamma}}\right] \\
& \times \sup_{0\leq r\leq s}\Bigg\{ e^{\omega r}\Big[\left\|B^{\eta_1}\left[(\mathcal{G}_h y)(r) - (\mathcal{G}_h z)(r)\right]\right\| \\
& \qquad + r^{(\eta_2-\eta_1)}\left\|B^{\eta_2}\left[(\mathcal{G}_h y)(r) - (\mathcal{G}_h z)(r)\right]\right\| \Big]\Bigg\} ds \\
\leq\ & \left[M_0 + M_{\eta_2-\eta_1}\right]\left\|B^{\eta_1}\left[h(y) - h(z)\right]\right\| \\
& + L_1\|C\|\left[M_{\eta_1-\mu} + M_{\eta_2-\mu}\right] T^{\eta_2-\eta_1}\int_0^t \frac{1}{(t-s)^{\eta_2-\mu}s^{\gamma}} \\
& \times \sup_{0\leq r\leq s}\Bigg\{ e^{\omega r}\Big[\left\|B^{\eta_1}\left[(\mathcal{G}_h y)(r) - (\mathcal{G}_h z)(r)\right]\right\| \\
& \qquad + r^{(\eta_2-\eta_1)}\left\|B^{\eta_2}\left[(\mathcal{G}_h y)(r) - (\mathcal{G}_h z)(r)\right]\right\| \Big]\Bigg\} ds.
\end{aligned}$$

Thus, thanks to [49, p. 189, Lemma 7.1.2], we obtain by (2.2.11) that there exists a positive constant M independent of u_0 such that

$$\left\|(\mathcal{G}_h y) - (\mathcal{G}_h z)\right\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} \leq M\left\|B^{\eta_1}\left[h(y) - h(z)\right]\right\|, \quad y, z \in W^{B,\omega}_{\eta_1,\eta_2}(T).$$

We conclude that if h satisfies (2.2.8) for $K = \frac{1}{2M}$, then

$$\|(\mathcal{G}_h y) - (\mathcal{G}_h z)\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} \leq \frac{1}{2}\|y - z\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)}, \quad y, z \in W^{B,\omega}_{\eta_1,\eta_2}(T). \tag{2.2.27}$$

This means that the operator $\mathcal{G}_h$ is contractive operator on $W^{B,\omega}_{\eta_1,\eta_2}(T)$. Therefore by the Banach contraction principle, $\mathcal{G}_h$ has a unique fixed point $z_h \in W^{B,\omega}_{\eta_1,\eta_2}(T)$, and this $z_h(t)$ is precisely the desired continuous solution of (2.2.2).

The proof is then complete.

□

The following result, which is an extension of [52, Theorem 3.2], is about the case where the function h and the operator f only satisfy local conditions in u. Write

$$U(R_0) = \left\{u \in W^{B,\omega}_{\eta_1,\eta_2}(T);\ \|u\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} \leq R_0\right\}, \quad R_0 > 0.$$

Theorem 2.2.3. *Let the function h satisfy*

$$h : U(R_0) \longrightarrow C\left(\mathcal{D}\left(B^{\eta_1}\right)\right), \tag{2.2.28}$$

$h(0) = 0$, *and suppose*

$$\|B^{\eta_1} h(u) - B^{\eta_1} h(v)\| \leq K\|u - v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)}, \quad u, v \in U(R_0), \tag{2.2.29}$$

for some constant $K > 0$. Let the operator f satisfy

$$C^{-1} f : U(R_0) \longrightarrow L^{\infty}\left(0, T; \mathcal{D}\left(B^{\mu}\right)\right) \tag{2.2.30}$$

for some constant $\mu \geq 0$ with $\eta_2 - 1 < \mu \leq \eta_1$.

(a) *Assume that*

$$\left\|e^{\omega t} t^{\gamma} B^{\mu} C^{-1}[f(u) - f(v)]\right\|_{L^{\infty}(0,T;X)} \leq L_1\|u - v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)}, \quad u, v \in U(R_0), \tag{2.2.31}$$

for constants $L_1 > 0$, $\gamma \geq 0$ with $\eta_1 - \mu + \gamma \leq 1$. Let the constants in (2.1.4), (2.2.29), (2.2.30) *and* (2.2.31) *satisfy* (2.2.10). *Then for any $u_0 \in C\left(\mathcal{D}\left(B^{\eta_1}\right)\right)$ for which*

$$\|V(t)u_0 + \int_0^t V(t - s) f(0) ds\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} \leq (1 - \kappa) R_0, \tag{2.2.32}$$

where κ is as in (2.2.10), *equation* (2.2.2) *has a unique solution $u \in U(R_0)$.*

(b) *Assume that*

$$\begin{aligned} & \|e^{\omega t}t^{\gamma}B^{\mu}C^{-1}[f(u)(t)-f(v)(t)]\| \\ \leq \ & L_1 \sup_{0\leq s\leq t}\Big[e^{\omega s}\left\|B^{\eta_1}(u(s)-v(s))\right\| + e^{\omega s}s^{\eta_2-\eta_1}\left\|B^{\eta_2}(u(s)-v(s))\right\|\Big], \\ & \qquad u,v\in U(R_0),\ t\in[0,T], \end{aligned} \tag{2.2.33}$$

and

$$\begin{aligned} & \|V(\cdot)u_0\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} + (M_0+M_{\eta_2-\eta_1})C_h \\ & \quad + \left(\frac{M_{\eta_1-\mu}}{1-\eta_1+\mu}+\frac{M_{\eta_2-\mu}}{1-\eta_2+\mu}\right)T^{1-\eta_1+\mu}C_f \leq R_0, \end{aligned} \tag{2.2.34}$$

where

$$C_f := \sup_{u\in U(R_0)}\|B^{\mu}f(u)\|_{L^{\infty}(0,T;X)}, \quad C_h := \sup_{u\in U(R_0)}\|B^{\eta_1}h(u)\|.$$

Then if the constant K in (2.2.29) *is small enough, equation* (2.2.2) *admits a unique solution $u\in U(R_0)$.*

Proof. (a) Fix $u_0\in C\left(\mathcal{D}\left(B^{\eta_1}\right)\right)$ such that (2.2.32) holds. For any $z\in U(R_0)$, define (as in (2.2.19)) an operator $\mathcal{F}$ on $U(R_0)$ by

$$(\mathcal{F}u)(t) = V(t)[u_0-h(u)] + \int_0^t V(t-s)f(u)(s)ds, \quad t\in[0,T].$$

Then by the same arguments as in the proof of Theorem 2.2.2 (b), we see that $\mathcal{F}$ is a contractive mapping. Also, by (2.2.32) we have for any $u\in U(R_0)$,

$$\begin{aligned} \|\mathcal{F}(u)\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} & \leq \|\mathcal{F}(u)-\mathcal{F}(0)\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} + \|\mathcal{F}(0)\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} \\ & \leq R_0, \end{aligned}$$

that is,

$$\mathcal{F}(U(R_0))\subset U(R_0).$$

The Banach contraction principle yields the desired conclusion.

(b) For every $z\in U(R_0)$, define an operator $\mathcal{F}_z$ on $z\in U(R_0)$ by

$$(\mathcal{F}_z u)(t) = V(t)[u_0-h(z)] + \int_0^t V(t-s)f(u)(s)ds, \quad t\in[0,T].$$

Then by (2.1.2) – (2.1.4), (2.2.6), (2.2.28) – (2.2.29), and (2.2.34), we obtain for any $u \in U(R_0)$,

$$
\begin{aligned}
& e^{\omega t}\left\|B^{\eta_1}(\mathcal{F}_z u)(t)\right\| + e^{\omega t} t^{\eta_2-\eta_1}\left\|B^{\eta_2}(\mathcal{F}_z u)(t)\right\| \\
\leq\ & \|V(t)u_0\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} + e^{\omega t}\left\|V(t)B^{\eta_1}h(z)\right\| + e^{\omega t} t^{\eta_2-\eta_1}\left\|B^{\eta_2-\eta_1}V(t)B^{\eta_1}h(z)\right\| \\
& \qquad + e^{\omega t}\int_0^t \left\|B^{\eta_1-\mu}V(t-s)B^{\mu}f(u)(s)\right\| ds \\
& \qquad\qquad + e^{\omega t} t^{\eta_2-\eta_1}\int_0^t \left\|B^{\eta_2-\mu}V(t-s)B^{\mu}f(u)(s)\right\| ds \\
\leq\ & \|V(\cdot)u_0\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} + (M_0 + M_{\eta_2-\eta_1})\left\|B^{\eta_1}h(z)\right\| \\
& \qquad + M_{\eta_1-\mu}\int_0^t \frac{1}{(t-s)^{\eta_1-\mu}}\left\|B^{\mu}f(u)(s)\right\| ds \\
& \qquad\qquad + M_{\eta_2-\mu} t^{\eta_2-\eta_1}\int_0^t \frac{1}{(t-s)^{\eta_2-\mu}}\left\|B^{\mu}f(u)(s)\right\| ds \\
\leq\ & \|V(\cdot)u_0\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)} + (M_0 + M_{\eta_2-\eta_1})C_h \\
& \qquad\qquad + \left(\frac{M_{\eta_1-\mu}}{1-\eta_1+\mu} + \frac{M_{\eta_2-\mu}}{1-\eta_2+\mu}\right) T^{1-\eta_1+\mu} C_f \\
\leq\ & R_0,
\end{aligned}
$$

that is,

$$
\mathcal{F}_z(U(R_0)) \subset U(R_0), \quad z \in U(R_0).
$$

The same reasons as in the proof of Theorem 2.2.2 (c) give the desired conclusion.

This completes the proof.

□

2.3 Nonlocal Cauchy problems for evolution equations

In this section we apply Theorem 2.2.1 to give some existence and uniqueness theorems for mild and classical solutions of nonlocal Cauchy problems for semilinear evolution equations.

Let $\{B,\ \{V(t)\}_{t\geq 0}\}$ be a (C,ω,M_η)-admissible pair on X, let B have fractional powers $\{B^r\}_{r\geq 0}$ and $W^{B,\omega}_{\eta_1,\eta_2}(T)$ be the space defined in Section 2. Suppose that G is the generator of $\{V(t)\}_{t\geq 0}$, and $0 < t_1 < t_2 < \cdots < t_p \leq T$ $(p \in N)$.

By virtue of Theorem 2.2.1, we have the following results. The first two is about the mild solutions and the third one is about classical solutions.

Corollary 2.3.1. *Assume that the function g satisfies*

$$g : [0,T]^p \times W^{B,\omega}_{\eta_1,\eta_2}(T) \longrightarrow C\left(\mathcal{D}\left(B^{\eta_1}\right)\right) \tag{2.3.1}$$

and

$$\begin{aligned} &\|B^{\eta_1} g(t_1,\cdots,t_p,u) - B^{\eta_1} g(t_1,\cdots,t_p,v)\| \\ \leq\ \ & K\|u-v\|_{W^{B,\omega}_{\eta_1,\eta_2}(T)}, \quad u,v \in W^{B,\omega}_{\eta_1,\eta_2}(T) \end{aligned} \tag{2.3.2}$$

for some constant $K > 0$, and the operator f satisfies (2.2.4), (2.2.5) *and* (2.2.9). *Let the constants in* (2.1.4), (2.2.4), (2.2.9) *and* (2.3.2) *satisfy* (2.2.10). *Then for any $u_0 \in C\left(\mathcal{D}\left(B^{\eta_1}\right)\right)$, the nonlocal Cauchy problem for semilinear evolution equation*

$$\begin{cases} u'(t) = Gu(t) + Cf(u)(t), \quad 0 < t \leq T, \\ u(0) + Cg(t_1,\ldots,t_p,u) = Cu_0, \end{cases} \tag{2.3.3}$$

has a unique mild solution $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$, i.e., there exists a unique function $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$ which satisfies

$$u(t) = V(t)[u_0 - g(t_1,\ldots,t_p,u)] + \int_0^t V(t-s)f(u)(s)ds, \quad t \in [0,T]. \tag{2.3.4}$$

Corollary 2.3.2. *Fix $T > 0$, $L_1 \geq 0$. Fix $\mu \geq 0$ and $\gamma \geq 0$ such that* (2.2.5) *and* (2.2.11) *hold. Then there exists a constant $K > 0$ such that for all functions g satisfying* (2.3.1) *and* (2.3.2), *and for all operators f satisfying* (2.2.4) *and* (2.2.12), *the problem* (2.3.3) *has a unique mild solution $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$ for any $u_0 \in C\left(\mathcal{D}\left(B^{\eta_1}\right)\right)$.*

Theorem 2.3.3. *Assume that* $\{V(t)\}_{t\geq 0}$ *is an analytic semigroup on* X. *Let* $\omega_0 < \omega$ *and* $B = -(G + \omega_0 I)$. *Fix* $T > 0$, $L_1 \geq 0$. *Fix* $\mu \geq 0$ *and* $\gamma \geq 0$ *such that* $\eta_2 - 1 < \mu < \eta_1$ *and* $\eta_2 - \mu - \gamma < 1$. *Suppose that the operator* f *from* $W^{B,\omega}_{\eta_1,\eta_2}(T)$ *to* $L^\infty(0, T; \mathcal{D}(B^\mu))$ *satisfies* (2.2.12) *with* $C = I$, *and suppose in addition that* $f(u)(\cdot)$ *is locally Hölder continuous on* $(0, T]$ *whenever* $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$ *and* $B^{\eta_2}u(\cdot)$ *is locally Hölder continuous on* $(0, T]$. *Then there exists a constant* $K > 0$ *such that for any* $u_0 \in \mathcal{D}(B^{\eta_1})$ *and any function* g *from* $[0, T]^p \times W^{B,\omega}_{\eta_1,\eta_2}(T)$ *to* $\mathcal{D}(B^{\eta_1})$ *satisfying* (2.3.2), *the nonlocal Cauchy problem for semilinear evolution equation*

$$\begin{cases} u'(t) = Gu(t) + f(u)(t), \quad 0 < t \leq T, \\ u(0) + g(t_1, \dots, t_p, u) = u_0, \end{cases} \tag{2.3.5}$$

has a unique classical solution $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$, *i.e., there exists a unique function* $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$ *which is continuously differentiable on* $(0, T]$ *and satisfies* (2.3.5).

Proof. As in Example 2.1.4, $\{B, \{V(t)\}_{t\geq 0}\}$ is a (I, ω, M_η)-admissible pair on X for certain constants M_η, associated with the usual fractional powers $\{B^r\}_{r\geq 0}$. By Theorem 2.3.2 with $C = I$, there exist a constant K such that for any $u_0 \in \mathcal{D}(B^{\eta_1})$ and any function g satisfying (2.3.2), equation (2.3.5) has a unique mild solution $u \in W^{B,\omega}_{\eta_1,\eta_2}(T)$.

Fix $u_0 \in \mathcal{D}(B^{\eta_1})$ and g with (2.3.2). We will show that the mild solution u of (2.3.5) is a classical solution. To this end, we observe

$$u(t) = V(t)[u_0 - g(t_1, \dots, t_p, u)] + \int_0^t V(t-s) f(u)(s) ds, \quad t \in [0, T]. \tag{2.3.6}$$

In view of Pazy [70, p. 113, Corollary 4.3.3], it suffices to prove $f(u)(\cdot)$ is locally Hölder continuous on $(0, T]$.

From (2.3.6) we have for $0 < t < t + \sigma \leq T$,

$$\begin{aligned} & \|B^{\eta_2}(u(t+\sigma) - u(t))\| \\ \leq \; & \|B^{\eta_2-\eta_1}(V(t+\sigma) - V(t))\| \, \|B^{\eta_1}[u_0 - g(t_1, \dots, t_p, u)]\| \\ & + \int_0^t \left\|B^{\eta_2-\mu}(V(t-s+\sigma) - V(t-s))\right\| \, \|B^\mu f(u)(s)\| \, ds \\ & + \int_t^{t+\sigma} \left\|B^{\eta_2-\mu} V(t-s+\sigma)\right\| \, \|B^\mu f(u)(s)\| \, ds. \end{aligned} \tag{2.3.7}$$

Note that for $0 \le s < t < t+\sigma \le T$,

$$\begin{aligned} V(t-s+\sigma) - V(t-s) &= \int_0^\sigma GV(t-s+\tau)d\tau \\ &= -\int_0^\sigma \omega_0 V(t-s+\tau)d\tau - \left(\int_0^\sigma B^{\frac{1}{2}}V(\tau)d\tau\right) B^{\frac{1}{2}}V(t-s), \end{aligned}$$

and so

$$\begin{aligned} \|B^{\eta_2-\eta_1}(V(t+\sigma)-V(t))\| &\le |\omega_0| \int_0^\sigma \left\|B^{\eta_2-\eta_1}V(t+\tau)\right\| d\tau \\ &\quad + \left(\int_0^\sigma \left\|B^{\frac{\eta_2-\eta_1+1}{2}}V(\tau)\right\| d\tau\right) \left\|B^{\frac{\eta_2-\eta_1+1}{2}}V(t-s)\right\|, \end{aligned}$$

$$\begin{aligned} \|B^{\eta_2-\mu}(V(t-s+\sigma)-V(t-s))\| &\le |\omega_0| \int_0^\sigma \left\|B^{\eta_2-\mu}V(t-s+\tau)\right\| d\tau \\ &\quad + \left(\int_0^\sigma \left\|B^{\frac{\eta_2-\mu+1}{2}}V(\tau)\right\| d\tau\right) \left\|B^{\frac{\eta_2-\mu+1}{2}}V(t-s)\right\|. \end{aligned}$$

Hence, we see by (2.3.7) and (2.1.4) that $B^{\eta_2}u(\cdot)$ is locally Hölder continuous on $(0,T]$. So is $f(u)(\cdot)$ by the assumption on f. This ends the proof.

□

We now discuss an example.

Example 2.3.4. Suppose that Ω is a bounded open subset of R^n with a smooth boundary Γ, and A is a strongly elliptic operator in $X = L^2(\Omega)$, defined by

$$\begin{cases} Au = \displaystyle\sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \\ \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \end{cases}$$

where $a_{ij}(x)$, $b_j(x)$, $c(x)$ are sufficiently smooth real-valued functions of x in $\overline{\Omega}$, $H^2(\Omega)$ and $H_0^1(\Omega)$ are Sobolev spaces (see, e.g. Lions and Magenes [64], Adams [2], or Pazy [70] for more information on the Sobolev spaces $H_0^m(\Omega)$ $(m \in N)$). It is known that $-A$ is the generator of an analytic semigroup $\{S(t)\}_{t\ge0}$ on $L^2(\Omega)$. Take $\omega \in R$ such that

$$\|S(t)\| \le \text{ const } e^{-\omega t}, \quad t \ge 0.$$

Let A be as above. Let $\omega_0 < \omega$ and $B = A - \omega_0 I$. Assume that the continuous function $F : \overline{\Omega} \times [0,T] \times R \times R^n \to R$ satisfies

$$t^\gamma |F(x,t,\xi,y) - F(x,t,\xi',y')| \leq l\left(|\xi - \xi'| + |y - y'|\right), \tag{2.3.8}$$

for some constants $l > 0$ and $0 \leq \gamma < \frac{1}{2}$. Then there exists a constant $K > 0$ such that for all functions $g : [0,T] \times W^{B,\omega}_{\frac{1}{2},\frac{1}{2}}(T) \to X$ satisfying

$$\|g(t_1,\ldots,t_p,u) - g(t_1,\ldots,t_p,v)\| \leq K\|u - v\|_{W^{B,\omega}_{\frac{1}{2},\frac{1}{2}}(T)}, \quad u,v \in W^{B,\omega}_{\frac{1}{2},\frac{1}{2}}(T),$$

the semilinear nonlocal Cauchy problem

$$\begin{cases} u_t + Au = F(x,t,u,\nabla u), & \text{on } \Omega \times (0,T) \\ u\big|_\Gamma = 0, & \\ u(x,0) + g(t_1,\ldots,t_p,u) = u_0(x), & \end{cases}$$

has a unique classical solution $u \in W^{B,\omega}_{\frac{1}{2},\frac{1}{2}}(T)$ for any $u_0 \in H^1_0(\Omega)$.

Proof. First we note that

$$\mathcal{D}\left(B^{\frac{1}{2}}\right) = H^1_0(\Omega).$$

Define the operator f from $W^{B,\omega}_{\frac{1}{2},\frac{1}{2}}(T)$ to $L^\infty(0,T;X)$ by

$$f(v)(t) := F(x,t,v,\nabla v), \quad v \in W^{B,\omega}_{\frac{1}{2},\frac{1}{2}}(T),\ t \in [0,T].$$

From (2.3.8) we deduce that

$$\|t^\gamma\left[f(u)(t) - f(v)(t)\right]\| \leq l_0 \left\|B^{\frac{1}{2}}(u - v)\right\|, \quad t \in [0,T],\ u,v \in W^{B,\omega}_{\frac{1}{2},\frac{1}{2}}(T) \tag{2.3.9}$$

for some constant $l_0 > 0$. Thus applying Theorem 2.3.3 with

$$\eta_1 = \eta_2 = \frac{1}{2}, \quad \mu = 0$$

gives rise to the conclusion. The proof is complete then.

□

Remark 2.3.5. Example 2.3.4 shows that a number of nonlocal problems, for which previous results are not applicable, do have unique classical solutions. For instance, when the constants T or l, for a nonlocal Cauchy problem as in Example 2.3.4, is large enough such that

$$M_{\frac{1}{2}}\beta(\frac{1}{2}, 1-\gamma)l_0 T^{\frac{1}{2}-\gamma} \geq 1$$

where l_0 is the constant in (2.3.9), then the condition (2.3.3) in [52] fails to be satisfied (in the case of Example 2.3.4, the associated constants in [52] read $s_1 = s_2 = 1$, $\delta = 1$, $k = K$, $L_1 = l_0$) no matter how small is the constant K.

Chapter 3

Solvability of the Cauchy problem for abstract functional equations with infinite delay

3.1 Introduction

Equations with delay (i.e., with some of the past states of the systems) are often more realistic to describe natural phenomena compared with those without delay, and they have been studied for many years (see, e.g., [3, 7, 8, 10–13, 22, 23, 32, 35, 36, 44–48, 51, 53–60, 63, 71, 74–79, 83, 84, 86, 87] and references therein). In the present chapter, we will consider mainly the solvability of the Cauchy problem for four classes of abstract functional equations with infinite delay.

We will address first, in Section 2, the Cauchy problem for a functional integral equation with infinite delay in a Banach space X,

$$\begin{cases} u(t) = g(t) + \displaystyle\int_{\sigma}^{t} f(t, s, u(s), u_s)ds \quad (\sigma \leq t \leq T), \\ u_{\sigma} = \phi, \end{cases} \tag{3.1.1}$$

where $0 \leq \sigma < T$, $g(t) \in C([\sigma, T], X)$, $u_t(\theta) = u(t + \theta)$ $(\theta \in R^-)$, $f \in C([\sigma, T] \times [\sigma, T] \times X \times \mathcal{P}, X)$ is a given function and $\phi \in \mathcal{P}$ (cf. the definitions of notations below). The solvability of (3.1.1) is investigated under hypotheses based on noncompactness measures and Kamke functions or the Lipschitz condition. The uniqueness and continuous dependence (on initial data) of the solutions are also discussed.

Second, in Sections 3 – 5 , we consider the following Cauchy problems for the semilinear functional differential equations, nonautonomus functional equations and functional integrodifferential equations with infinite delay in Banach spaces

$$\begin{cases} u'(t) = Au(t) + f(t, u(t), u_t), \quad 0 \le t \le T, \\ u_0 = \phi, \end{cases} \tag{3.1.2}$$

$$\begin{cases} u'(t) = A(t)u(t) + f(t, u(t), u_t), \quad 0 \le t \le T, \\ u_0 = \phi, \end{cases} \tag{3.1.3}$$

and

$$\begin{cases} u'(t) = A\left[u(t) + \int_0^t F(t-s)u(s)ds\right] + f(t, u(t), u_t), \quad 0 \le t \le T, \\ u_0 = \phi, \end{cases} \tag{3.1.4}$$

where $T > 0$, A and $\{A(t)\}_{t\ge 0}$ are given linear operators in X, $\{F(t)\}_{0\le t\le T} \subset \mathbf{L}(X)$, $f \in C([0,T] \times X \times \mathcal{P}, X)$, and $\phi \in \mathcal{P}$. By applying the given results on (3.1.1), we obtain some new and basic solvability and wellposedness results for (3.1.2) – (3.1.4)

We undertake our study in a quite general framework of admissible phase space, which satisfies hypotheses weaker than those required in the previous literature and includes the space $L^p((-\infty, 0], X)$. Therefore, our results are extensions of many known results on delay equations for infinite delay as well as for finite delay given in, e.g., [3, 8, 13, 22, 23, 35, 36, 45–48, 51, 53, 54, 58–60, 63, 71, 74–79, 84, 86, 87]).

We would like to mention that the investigation of functional differential equations with infinite delay in an abstract admissible phase space was initiated by Hale and Kato [45] and Schumacher [77] (for $X = R^n$), and that Banks, Burns, Delfour, Herdman and Mitter were among the first who studied equations with finite delay in the state space $X \times L^p([-r, 0], X)$ (cf. [7, 10, 32]). The method of using admissible phase spaces has proved to be significant in dealing with infinite delay problems, because in this way one can treat a large class of functional differential equations with infinite delay at the same time and obtain general results. On the other hand, as shown, e.g., in [7, 10–12, 32, 83], the product space $X \times L^p([-r, 0], X)$ is well suited for the investigation of certain problems involving control systems governed by delay equations.

Definition 3.1.1. A Banach space $(\mathcal{P}, \|\cdot\|_{\mathcal{P}})$, consisting of functions from R^- into X, is called an *admissible phase space* if $\mathcal{P}$ has the following properties.

(H1) For any $t_0 \in R$ and $a > 0$, if $x : (-\infty, t_0 + a] \to X$ is continuous on $[t_0, t_0 + a]$ and $x_{t_0} \in \mathcal{P}$, then $x_t \in \mathcal{P}$ and x_t is continuous in $t \in [t_0, t_0 + a]$.

(H2) There exist a continuous $K(t) > 0$ and a locally bounded function $M(t) \geq 0$ of $t \geq 0$ such that

$$\|x_t\|_{\mathcal{P}} \leq K(t - t_0) \max_{s \in [t_0, t]} \|x(s)\| + M(t - t_0)\|x_{t_0}\|_{\mathcal{P}}$$

for $t \in [t_0, t_0 + a]$ and x as in (H1).

The following are three typical examples of admissible phase spaces.

Example 3.1.2. Let $1 \leq p < \infty$. Then $\mathcal{P} = L^p(R^-, X)$, consisting of X-valued p-Bochner integrable functions on R^-, is an admissible phase space.

Example 3.1.3. Let $r > 0$, $1 \leq p < \infty$ and $q : (-\infty, -r] \to R^+$ be a nondecreasing function. Let

$$\mathcal{P} := \Big\{ \phi(\theta);\ \phi : R^- \to X \text{ strongly measurable,} \\ \text{continuous on } [-r, 0], \text{ and } \textstyle\int_{-\infty}^{-r} q(\theta)\|\phi(\theta)\|^p d\theta < \infty \Big\},$$

with norm

$$\|\phi\|_{\mathcal{P}} = \left\{ \int_{-\infty}^{-r} q(\theta)\|\phi(\theta)\|^p d\theta \right\}^{\frac{1}{p}} + \max_{-r \leq \theta \leq 0} \|\phi(\theta)\|.$$

Then $\mathcal{P}$ is an admissible phase space satisfying $\|\phi(0)\| \leq K\|\phi\|_{\mathcal{P}}$ (for all $\phi \in \mathcal{P}$) for a constant K.

Example 3.1.4. Let $q : R^- \to R^+$ be a nondecreasing continuous function such that

$$q(0) = 1, \quad q(-\infty) = \infty, \quad \sup_{\theta \in [-\infty, -t]} \frac{q(t + \theta)}{q(t)} \text{ is locally bounded for } t \geq 0.$$

Let

$$\mathcal{P} := \left\{ \phi(\theta);\ \phi : R^{-} \to X \text{ continuous and } \lim_{\theta \to -\infty} \frac{\|\phi(\theta)\|}{q(\theta)} = 0 \right\},$$

with norm

$$\|\phi\|_{\mathcal{P}} = \sup_{-\infty \le \theta \le 0} \frac{\|\phi(\theta)\|}{q(\theta)}.$$

Then $\mathcal{P}$ is an admissible phase space satisfying $\|\phi(0)\| \le K\|\phi\|_{\mathcal{P}}$ (for all $\phi \in \mathcal{P}$) for a constant K.

For the convenience of the reader we recall the following definitions and lemma.

Definition 3.1.5. (cf., e.g., [6, 26–28, 50]) Let B be a bounded subset of a Banach space X. The *Kuratowski measure of noncompactness* of B is defined as

$$\alpha(B) := \inf\{\gamma > 0;\ \ B \text{ admits a finite cover by sets of diameter } \le \gamma\}.$$

Lemma 3.1.6. (cf., e.g., [6, 26–28, 50]) *Let X be a Banach space with $\dim X = \infty$, B and G bounded sets of X. Then*

(1) $\alpha(B) = 0$ *if and only if B is relatively compact.*

(2) $\alpha(\lambda B) = |\lambda|\alpha(B)$ *for every $\lambda \in R$.*

(3) $\alpha(B + G) \le \alpha(B) + \alpha(G)$.

(4) $\alpha(B \cup G) \le \max\{\alpha(B), \alpha(G)\}$.

(5) $B \subset G$ *implies* $\alpha(B) \le \alpha(G)$.

(6) α *is continuous with respect to the Hausdorff distance ϱ_H defined by*

$$\varrho_H(B, G) = \max\left\{ \sup_B d(x, G),\ \sup_G d(x, B) \right\}.$$

Definition 3.1.7. (compare, e.g., [6, p. 70]) Let a, $b \in R$, c and $\overline{c} \in R^{+}$. A real nonnegative function $\mathcal{K}(t, \mu, \nu)$ on $(a, b) \times [0, c) \times [0, \overline{c})$ is called a Kamke function if

(i) it is Lebesgue measurable in t for every $(\mu,\nu)\in[0,c)\times[0,\overline{c})$ and continuous in (μ,ν) for a.e. $t\in(a,b)$, and $\mathcal{K}(\cdot,0,0)=0$;

(ii) for all $0\le\mu\le\overline{\mu}\le c$, $0\le\nu\le\overline{\nu}\le\overline{c}$ and a.e. $t\in(a,b)$,

$$\mathcal{K}(t,\mu,\nu)\le\mathcal{K}(t,\overline{\mu},\overline{\nu})\le k_{(\overline{\mu},\overline{\nu})}(t), \tag{3.1.5}$$

where $k_{(\overline{\mu},\overline{\nu})}(t)$ is a locally integrable function on (a,b) for each $\overline{\mu}$, $\overline{\nu}$.

3.2 Functional integral equations with infinite delay

In this section we are concerned with the solvability of the Cauchy problem for the functional integral equations with infinite delay (3.1.1). We first give a general local solvability result for (3.1.1).

Theorem 3.2.1. *Let $0\le\sigma<T$ and $\mathcal{P}$ be an admissible phase space. Let $f\in C([\sigma,T]\times[\sigma,T]\times X\times\mathcal{P},X)$ and $f(\cdot,s,x,\phi)$ be uniformly continuous in $(s,x,\phi)\in[\sigma,T]\times X\times\mathcal{P}$. Suppose that there is a Kamke function $\mathcal{K}(\cdot,\cdot,\cdot)$ on $[\sigma,T]\times[0,a]\times[0,\max_{t\in[0,T-\sigma]}K(t)a]$ for some $a>0$ such that*

(i) *for every bounded set $B\subset X$ and $\Omega\subset\mathcal{P}$,*

$$\alpha(f([\sigma,T]\times\{s\}\times B\times\Omega))\le\mathcal{K}(s,\alpha(B),\alpha(\Omega)),\quad \text{a.e. } s\in[\sigma,T]; \tag{3.2.1}$$

(ii) *$\varpi(t)\equiv 0$ is the unique nonnegative absolutely continuous solution to the differential equation*

$$\varpi'(t)=2\mathcal{K}(t,\varpi(t),K(t-\sigma)\varpi(t)),\quad \text{a.e. } t\in(\sigma,T]$$

satisfying

$$\lim_{t\uparrow\sigma}\frac{\varpi(t)}{t-\sigma}=\varpi(\sigma)=0,$$

where $K(\cdot)$ is the function as in (H2).

Then for every $\phi \in \mathcal{P}$ and $g(t) \in C([\sigma, T], X)$ with $g(\sigma) = \phi(0)$, there exists a real number $\tau(\sigma, \phi, g, f)$ such that (3.1.1) *has a solution $u(t)$ on $(-\infty, \tau(\sigma, \phi, g, f)]$. In this case, we also say that* (3.1.1) *has a solution $u(t)$ on $[\sigma, \tau(\sigma, \phi, g, f)]$.*

Proof. For each $\tau > \sigma$, $b > 0$, $\phi \in \mathcal{P}$, we introduce the following notation, which will be used throughout this chapter,

$$\mathcal{P}^{[\sigma,\tau]} := \left\{ u : (-\infty, \tau] \to X; \quad u\Big|_{[\sigma,\tau]} \in C([\sigma, \tau], X) \text{ and } u_\sigma \in \mathcal{P} \right\},$$

and

$$\mathcal{P}^{[\sigma,\tau]}_{\phi,g}(b) := \left\{ u \in \mathcal{P}^{[\sigma,\tau]}; \quad \max_{t\in[\sigma,\tau]} \|u(t) - g(t)\| \le b, \ u_\sigma = \phi \right\}.$$

Then the space $\mathcal{P}^{[\sigma,\tau]}$ is a Banach space under the norm

$$\|u\|_{\mathcal{P}^{[\sigma,\tau]}} := \max_{t\in[\sigma,\tau]} \|u(t)\| + \|u_\sigma\|_{\mathcal{P}},$$

and the set $\mathcal{P}^{[\sigma,\tau]}_{\phi,g}(b)$ is nonempty, closed and convex.

Since $f \in C([\sigma, T] \times [\sigma, T] \times X \times \mathcal{P}, X)$, we see that for every $\phi \in \mathcal{P}$ there exists a real number $\delta(\phi, f) > 0$ such that

$$\|f(t, s, x, \psi)\| \le \|f(t, s, \phi(0), \phi)\| + 1, \quad \text{for all } t, s \in [\sigma, T], \tag{3.2.2}$$

if

$$\|x - \phi(0)\| \le \delta(\phi, f), \quad \|\psi - \phi\|_{\mathcal{P}} \le \delta(\phi, f). \tag{3.2.3}$$

Moreover, it follows from $g(t) \in C([\sigma, T], X)$ that there exists a real number $\tau(\sigma, g) \in [\sigma, T]$ such that

$$\|g(t) - g(\sigma)\| \le \frac{\delta(\phi, f)}{2\left[\max\limits_{t\in[\sigma,T-\sigma]} K(t) + 1\right]} \quad \text{if } t \in [\sigma, \tau(\sigma, g)]. \tag{3.2.4}$$

For every $\phi \in \mathcal{P}$ with $\phi(0) = g(\sigma)$, we let

$$u^0(t) := \begin{cases} g(t), & t \in [\sigma, \tau(\sigma, g)], \\ \phi(t - \sigma), & t \in (-\infty, \sigma]. \end{cases} \tag{3.2.5}$$

Then by (3.2.4),

$$\|u^0(t) - \phi(0)\| \le \delta(\phi, f), \quad \text{if } t \in [\sigma, \tau(\sigma, g)], \tag{3.2.6}$$

and by (H1), there exists a real number $\tau(\sigma, \phi, g) \leq \tau(\sigma, g)$ such that

$$\|u_t^0 - \phi\|_{\mathcal{P}} = \|u_t^0 - u_\sigma^0\|_{\mathcal{P}} \leq \frac{1}{2}\delta(\phi, f), \quad \text{ if } \ t \in [\sigma, \tau(\sigma, \phi, g)]. \tag{3.2.7}$$

For any $b > 0$, set

$$\begin{aligned} & \tau(\sigma, \phi, g, f, b) \\ := \ & \min\left\{ \frac{b}{2\left[\max\limits_{t,s\in[\sigma,T]}(\|f(t, s, \phi(0), \phi)\| + 1)\right]} + \sigma, \right. \\ & \left. \frac{\delta(\phi, f)}{2\left[\max\limits_{t,s\in[\sigma,T]}(\|f(t, s, \phi(0), \phi)\| + 1)\right]\left[\max\limits_{t\in[\sigma,T-\sigma]} K(t) + 1\right]} + \sigma, \quad \tau(\sigma, \phi, g) \right\} \end{aligned} \tag{3.2.8}$$

and for each $n \in N$, define

$$u^n(t) := \begin{cases} g(t) + \displaystyle\int_\sigma^t f(t, s, u^{n-1}(s), u_s^{n-1})ds, & t \in [\sigma, \tau(\sigma, \phi, g, f, b)], \\ \phi(t - \sigma), & t \in (-\infty, \sigma]. \end{cases} \tag{3.2.9}$$

Then

$$u_\sigma^n = \phi, \quad n \in N. \tag{3.2.10}$$

Moreover, by (3.2.9), (3.2.6), (3.2.7), (3.2.2) (with (3.2.3)) and (3.2.8),

$$\|u^1(t) - g(t)\| \leq b, \quad t \in [\sigma, \tau(\sigma, \phi, g, f, b)];$$

and by (3.2.9), (3.2.6), (3.2.7), (3.2.2) (with (3.2.3)) and (3.2.4),

$$\begin{aligned} \|u^1(t) - \phi(0)\| &= \|u^1(t) - g(\sigma)\| \\ &\leq \|g(t) - g(\sigma)\| + \int_\sigma^t \|f(t, s, u^0(s), u_s^0)\|ds \\ &\leq \frac{1}{2}\delta(\phi, f) + \frac{1}{2}\delta(\phi, f) \\ &= \delta(\phi, f), \quad \text{ for } \ t \in [\sigma, \tau(\sigma, \phi, g, f, b)]; \end{aligned} \tag{3.2.11}$$

and by (3.2.10), (H2), (3.2.9), (3.2.5), (3.2.8), (3.2.6), (3.2.7) and (3.2.2) (with

(3.2.3)),

$$\begin{aligned} & \|u_t^1 - \phi\|_{\mathcal{P}} \\ = \; & \|u_t^1 - u_\sigma^0\|_{\mathcal{P}} \\ \le \; & \|u_t^1 - u_t^0\|_{\mathcal{P}} + \|u_t^0 - u_\sigma^0\|_{\mathcal{P}} \\ \le \; & \max_{t\in[\sigma,T-\sigma]} K(t)[\tau(\sigma,\phi,g,f,b)-\sigma] \max_{t,s\in[\sigma,\tau(\sigma,\phi,g,f,b)]} \|f(t,s,u^0(s),u_s^0)\| \\ & \qquad\qquad + \|u_t^0 - u_\sigma^0\|_{\mathcal{P}} \\ \le \; & \frac{1}{2}\delta(\phi,f) + \frac{1}{2}\delta(\phi,f) \\ = \; & \delta(\phi,f), \quad \text{ for } \; t\in[\sigma,\tau(\sigma,\phi,g,f,b)]. \end{aligned} \tag{3.2.12}$$

By induction and noting that

$$\begin{aligned} & \|u_t^2 - \phi\|_{\mathcal{P}} \\ \le \; & \|u_t^2 - u_t^0\|_{\mathcal{P}} + \|u_t^0 - u_\sigma^0\|_{\mathcal{P}} \\ \le \; & \max_{t\in[\sigma,T-\sigma]} K(t)[\tau(\sigma,\phi,g,f,b)-\sigma] \max_{t,s\in[\sigma,\tau(\sigma,\phi,g,f,b)]} \|f(t,s,u^1(s),u_s^1)\| \\ & \qquad\qquad + \|u_t^0 - u_\sigma^0\|_{\mathcal{P}} \quad \text{ for } \; t\in[\sigma,\tau(\sigma,\phi,g,f,b)], \end{aligned} \tag{3.2.13}$$

it will now be verified that

$$\|u^n(t) - g(t)\| \le b, \quad t\in[\sigma,\tau(\sigma,\phi,g,f,b)], \; n\in N, \tag{3.2.14}$$

$$\|u^n(t) - \phi(0)\| \le \delta(\phi,f), \quad \text{ for } \; t\in[\sigma,\tau(\sigma,\phi,g,f,b)], \; n\in N, \tag{3.2.15}$$

$$\|u_t^n - \phi\|_{\mathcal{P}} \le \delta(\phi,f), \quad \text{ for } \; t\in[\sigma,\tau(\sigma,\phi,g,f,b)], \; n\in N. \tag{3.2.16}$$

(3.2.10) and (3.2.14) imply that

$$u^n(\cdot) \in \mathcal{P}_{\phi,g}^{[\sigma,\tau(\sigma,\phi,g,f,b)]}(b), \quad n\in N. \tag{3.2.17}$$

(3.2.15), (3.2.16) and (3.2.2) (with (3.2.3)) imply that

$$\max_{t,s\in[\sigma,\tau(\sigma,\phi,g,f,b)]} \|f(t,s,u^n(s),u_s^n)\| \;\le\; \max_{t,s\in[\sigma,T]} \|f(t,s,\phi(0),\phi)\| + 1, \quad n\in N. \tag{3.2.18}$$

Furthermore, by (3.2.9) and (3.2.18)

$$\begin{aligned}
&\max_{z\in[\sigma,t]} \left\|u^n(z) - u^0(z)\right\| \\
= &\max_{z\in[\sigma,t]} \|u^n(z) - g(z)\| \\
\leq &(t-\sigma) \max_{t,s\in[\sigma,T]} (\|f(t,s,\phi(0),\phi)\| + 1), \quad t \in [\sigma, \tau(\sigma,\phi,g,f,b)], \ n \in N,
\end{aligned} \tag{3.2.19}$$

and for every $n \in N$, $\sigma \leq w \leq z \leq t \leq \tau(\sigma,\phi,g,f,b)$,

$$\begin{aligned}
&\left\|\int_\sigma^z f(z,s,u^n(s),u_s^n)ds - \int_\sigma^w f(w,s,u^n(s),u_s^n)ds\right\| \\
\leq &\int_\sigma^\tau \|f(z,s,u^n(s),u_s^n) - f(w,s,u^n(s),u_s^n)\|ds \\
&\qquad + \left[\max_{t,s\in[\sigma,T]} \|f(t,s,\phi(0),\phi)\| + 1\right](z-w).
\end{aligned} \tag{3.2.20}$$

Since $f(\cdot,s,x,\phi)$ is uniformly continuous with respect to $(s,x,\phi) \in [\sigma,T] \times X \times \mathcal{P}$, it follows from (3.2.20) that for each $t \in [\sigma, \tau(\sigma,\phi,g,f,b)]$,

$$\text{the set } \left\{\int_\sigma^\cdot f(\cdot,s,u^n(s),u_s^n)ds\Big|_{[\sigma,t]}\right\}_{n\in N} \text{ is equicontinuous.}$$

Therefore by virtue of [27, Proposition 7.3, p. 43] we have

$$\begin{aligned}
&\alpha\left(\left\{\int_\sigma^\cdot f(\cdot,s,u^n(s),u_s^n)ds\Big|_{[\sigma,t]}\right\}_{n\in N}\right) \\
\leq &\sup_{z\in[\sigma,t]} \alpha\left(\left\{\int_\sigma^z f(z,s,u^n(s),u_s^n)ds\right\}_{n\in N}\right), \quad t \in [\sigma, \tau(\sigma,\phi,g,f,b)].
\end{aligned}$$

Thus, thanks to Heinz's theorem ([50, Theorem 2.1]) (see also [28]) we obtain

$$\begin{aligned}
&\alpha\left(\left\{u^n(\cdot)\Big|_{[\sigma,t]}\right\}_{n\in N}\right) \\
\leq &\alpha\left(\left\{g(\cdot)\Big|_{[\sigma,t]}\right\}\right) + \sup_{z\in[\sigma,t]} \alpha\left(\left\{\int_\sigma^z f(z,s,u^{n-1}(s),u_s^{n-1})ds\right\}_{n\in N\setminus\{1\}}\right) \\
\leq &2\int_\sigma^t \sup_{z\in[\sigma,t]} \alpha\left(\{f(z,s,u^n(s),u_s^n)\}_{n\in N}\right) ds \\
\leq &2\int_\sigma^t \sup_{z\in[\sigma,\tau]} \alpha\left(\{f(z,s,u^n(s),u_s^n)\}_{n\in N}\right) ds, \quad t \in [\sigma, \tau(\sigma,\phi,g,f,b)].
\end{aligned} \tag{3.2.21}$$

Let $b = \frac{a}{2}$ (the constant given in the hypotheses) and $\tau(\sigma, \phi, g, f) := \tau\left(\sigma, \phi, g, f, \frac{a}{2}\right)$. Then by (3.2.19) and (H2), we have

$$\max_{z\in[\sigma,s]} \left\|u_z^n - u_z^0\right\|_{\mathcal{P}} \leq (s-\sigma) \max_{t,s\in[\sigma,T]} (\|f(t,s,\phi(0),\phi)\| + 1) \max_{t\in[\sigma,T-\sigma]} K(t)a,$$
$$s \in [\sigma, \tau(\sigma,\phi,g,f)], \ n \in N. \tag{3.2.22}$$

By (3.2.19) and (3.2.22), we see that for every $\varepsilon > 0$, there is a $0 < \eta \leq \tau(\sigma, \phi, g, f) - \sigma$ such that for all $s \in [\sigma, \sigma + \eta]$,

$$\left\|f(t,s,u^n(s),u_s^n) - f(t,s,u^0(s),u_s^0)\right\| < \frac{\varepsilon}{2}, \quad t \in [\sigma,T], \ n \in N.$$

Consequently,

$$\begin{aligned} & \alpha\left(\{f(t,s,u^n(s),u_s^n)\}_{t\in[\sigma,T],\ s\in[\sigma,\sigma+\delta],\ n\in N}\right) \\ \leq\ & \alpha\left(\{f(t,s,u^0(s),u_s^0)\}_{t\in[\sigma,T],\ s\in[\sigma,\sigma+\delta]}\right) \\ & +\alpha\left(\{f(t,s,u^n(s),u_s^n) - f(t,s,u^0(s),u_s^0)\}_{t\in[\sigma,T],\ s\in[\sigma,\sigma+\delta],\ n\in N}\right) \\ \leq\ & \frac{\varepsilon}{2}. \end{aligned}$$

Thus, if we define

$$\varsigma(t) := 2\int_\sigma^t \sup_{z\in[\sigma,\tau]} \alpha\left(\{f(z,s,u^n(s),u_s^n)\}_{n\in N}\right) ds, \quad t \in [\sigma, \tau(\sigma,\phi,g,f)],$$

then the nonnegative function $\varsigma(t)$ is absolutely continuous on $[\sigma, \tau(\sigma, \phi, g, f)]$ and $\varsigma(\sigma) = 0$. Moreover, (2.9) implies that

$$\lim_{t\uparrow\sigma} \frac{\varsigma(t)}{t-\sigma} = 0.$$

A combination of (3.1.5), (3.2.1), (3.2.9), (H2) and (3.2.21) gives that

$$\begin{aligned}\varsigma(t) &\leq 2\int_\sigma^t \mathcal{K}\left(s, \alpha\left(\{u^n(s)\}_{n\in N}\right), \alpha\left(\{u^n_s\}_{n\in N}\right)\right) ds \\ &\leq 2\int_\sigma^t \mathcal{K}\Bigg(s, \alpha\left(\left\{\int_\sigma^s f(s,\nu,u^{n-1}(\nu),u^{n-1}_\nu)d\nu\right\}_{n\in N\setminus\{1\}}\right), \\ &\qquad\qquad K(s-\sigma)\alpha\left(\left\{u^n(\cdot)\Big|_{[\sigma,s]}\right\}_{n\in N}\right)\Bigg) ds \\ &\leq 2\int_\sigma^t \mathcal{K}\left(s, \varsigma(s), K(s-\sigma)\varsigma(s)\right) ds, \quad t\in(\sigma,T].\end{aligned}$$

This, together with hypothesis (ii) and the comparison theorem, yields that $\varsigma(t)\equiv 0$. Hence, by (3.2.21) we have

$$\alpha\left(\left\{u^n(\cdot)\Big|_{[\sigma,\tau(\sigma,\phi,g,f)]}\right\}_{n\in N}\right) = 0.$$

So the set $\left\{u^n(\cdot)\Big|_{[\sigma,\tau(\sigma,\phi,g,f)]}\right\}_{n\in N}$ is relatively compact in $C([\sigma,\tau(\sigma,\phi,g,f)],X)$. Therefore, by noting (3.2.17), there exist a sequence $\{n_i\}\subset N$ and a function

$$u(t)\in \mathcal{P}^{[\sigma,\tau(\sigma,\phi,g,f)]}_{\phi,g}(b)$$

such that

$$\lim_{i\to\infty}\max_{t\in[\sigma,\tau]}\|u^{n_i}(t)-u(t)\| = 0.$$

Moreover, (H2) implies that

$$\lim_{i\to\infty}\max_{t\in[\sigma,\tau]}\|u^{n_i}_t-u_t\|_{\mathcal{P}} = 0.$$

Thus thanks to Lebesgue's dominated convergence theorem, we obtain $u(t)$ is a solution of (3.1.1) on $[\sigma,\tau(\sigma,\phi,g,f)]$.

□

The following theorem concerns the situation when f is compact.

Theorem 3.2.2. *Let $0 \le \sigma < T$ and $\mathcal{P}$ be an admissible phase space. Let $f \in C([\sigma,T]\times[\sigma,T]\times X\times\mathcal{P}, X)$ being compact. Then for every $\phi \in \mathcal{P}$ and $g(t) \in C([\sigma,T],X)$ with $g(\sigma)=\phi(0)$, there exists a real number $\tau(\sigma,\phi,g,f)$ such that* (3.1.1) *has a solution $u(t)$ on $[\sigma,\tau(\sigma,\phi,g,f)]$.*

Proof. Let $\phi \in \mathcal{P}$ and $g(t) \in C([\sigma,T],X)$ with $g(\sigma)=\phi(0)$. From the proof of Theorem 3.2.1, we know that there is a sequence $\{u^n(\cdot)\}_{n\in N}$ such that (3.2.15), (3.2.16), (3.2.17) and (3.2.18) hold for any given $b>0$. Fix $b>0$ and write $\tau(\sigma,\phi,g,f) := \tau(\sigma,\phi,g,f,b)$. The compactness of f and $[\sigma,\tau(\sigma,\phi,g,f)]$ implies that there exists a subsequence $\{n_k\}\subset N$ and a continuous function $h(t,s)$ of (t,s) such that

$$f\left(t,s,u^{n_k}(s),u_s^{n_k}\right) \to h(t,s), \qquad \text{as } \ k\to\infty$$

uniformly for $t\in[\sigma,\tau(\sigma,\phi,g,f)]$ and $s\in[\sigma,\tau(\sigma,\phi,g,f)]$. So for any $\varepsilon>0$, there is a $\overline{k}\in N$ such that for all $k\ge\overline{k}$,

$$\|f\left(t,s,u^{n_k}(s),u_s^{n_k}\right)-h(t,s)\|\le\varepsilon, \qquad \text{for all } \ t,s\in[\sigma,\tau(\sigma,\phi,g,f)].$$

Therefore, for every $k\ge\overline{k}$, $\sigma\le w\le z\le t\le\tau(\sigma,\phi,g,f)$,

$$\begin{aligned}
&\left\|\int_\sigma^z f(z,s,u^{n_k}(s),u_s^{n_k})ds-\int_\sigma^w f(w,s,u^{n_k}(s),u_s^{n_k})ds\right\| \\
\le\;& \int_\sigma^z \|f(z,s,u^{n_k}(s),u_s^{n_k})-h(z,s)\|ds \\
&+\int_\sigma^z \|f(w,s,u^{n_k}(s),u_s^{n_k})-h(w,s)\|ds \\
&+\int_\sigma^z \|h(z,s)-h(w,s)\|ds+\left(\max_{t,s\in[\sigma,T]}\|f(t,s,\phi(0),\phi)\|+1\right)(z-w) \\
\le\;& (2\varepsilon+\|h(z,\overline{s})-h(w,\overline{s})\|)T+\left(\max_{t,s\in[\sigma,T]}\|f(t,s,\phi(0),\phi)\|+1\right)(z-w),
\end{aligned}$$

where $\overline{s}\in[\sigma,\tau(\sigma,\phi,g,f)]$. This implies that for each $t\in[\sigma,\tau(\sigma,\phi,g,f)]$,

$$\text{the set}\left\{\int_\sigma^{\cdot} f\left(\cdot,s,u^{n_k}(s),u_s^{n_k}\right)ds\bigg|_{[\sigma,t]}\right\}_{k\in N,k\ge\overline{k}} \qquad \text{is equicontinuous.}$$

Hence

$$\alpha\left(\left\{\int_\sigma^\cdot f\left(\cdot,s,u^{n_k}(s),u_s^{n_k}\right)ds\Big|_{[\sigma,t]}\right\}_{k\in N,k\geq\overline{k}}\right)$$

$$\leq \sup_{z\in[\sigma,t]}\alpha\left(\left\{\int_\sigma^z f\left(z,s,u^{n_k}(s),u_s^{n_k}\right)ds\right\}_{k\in N,k\geq\overline{k}}\right),\quad t\in[\sigma,\tau(\sigma,\phi,g,f)],$$

and then

$$\alpha\left(\left\{u^{n_k}(\cdot)\Big|_{[\sigma,t]}\right\}_{k\in N,k\geq\overline{k}}\right)$$

$$\leq 2\int_\sigma^t \sup_{z\in[\sigma,\tau]}\alpha\left(\left\{f\left(z,s,u^{n_k}(s),u_s^{n_k}\right)\right\}_{k\in N,k\geq\overline{k}}\right)ds,\quad t\in[\sigma,\tau(\sigma,\phi,g,f)].$$

This means there is a sequence $n_{k_i}\subset N$ and a function $u(t)\in\mathcal{P}_{\phi,g}^{[\sigma,\tau(\sigma,\phi,g,f)]}(b)$ such that

$$\lim_{i\to\infty}\max_{t\in[\sigma,\tau]}\|u^{n_i}(t)-u(t)\|=0.$$

Consequently, $u(t)$ is a solution of (3.1.1) on $[\sigma,\tau(\sigma,\phi,g,f)]$.

□

When f has a local Lipschitz continuity in third and forth component, we have the following local existence, uniqueness and continuous dependence theorem for (3.1.1).

Theorem 3.2.3. *Let $0\leq\sigma<T$ and $\mathcal{P}$ be an admissible phase space. Let $f\in C([\sigma,T]\times[\sigma,T]\times X\times\mathcal{P},X)$ and for every $r>0$, there exist a constant $H(r)$ such that for each $t,s\in[\sigma,T]$,*

$$\|f(t,s,x,\phi)-f(t,s,y,\psi)\|\leq H(r)\left(\|x-y\|+\|\phi-\psi\|_{\mathcal{P}}\right),$$
$$\text{for all } x,y\in X,\ \phi,\psi\in\mathcal{P} \text{ with } \max\{\|x\|,\ \|y\|,\ \|\phi\|_{\mathcal{P}},\ \|\psi\|_{\mathcal{P}}\}\leq r. \tag{3.2.23}$$

Then for every $\phi\in\mathcal{P}$ and $g(t)\in C([\sigma,T],X)$ with $g(\sigma)=\phi(0)$, there exists a real number $\tau(\sigma,\phi,g,f)$ such that (3.1.1) has a unique solution $u(t)$ on $[\sigma,\tau(\sigma,\phi,g,f)]$.

Moreover, define

$$T_{\sup}(\sigma,\phi,g,f) := \sup\{\tau > \sigma;\quad (3.1.1) \text{ has a unique solution } u(\cdot) \text{ on } [\sigma,\tau)\}, \tag{3.2.24}$$

and let $u(t)$ *(resp.* $\widehat{u}(t)$*) be the solution of* (3.1.1) *on* $[\sigma, T_{\sup}(\sigma,\phi,g,f)]$ *(resp.* $[\sigma, T_{\sup}(\sigma,\widehat{\phi},\widehat{g},f)]$*) with respect to* $\phi \in \mathcal{P}$ *(resp.* $\widehat{\phi} \in \mathcal{P}$*) and* $g(t)$ *(resp.* $\widehat{g}(t)$*). Then there is a constant* $\widetilde{L}(u,\widehat{u},\tau_0)$ *such that*

$$\|u-\widehat{u}\|_{\mathcal{P}[\sigma,\tau_0]} \le \widetilde{L}(u,\widehat{u},\tau_0)\left(\max_{t\in[\sigma,\tau_0]}\|g(t)-\widehat{g}(t)\| + \|\phi-\widehat{\phi}\|_{\mathcal{P}}\right)$$

for each

$$\tau_0 < \min\{T_{\sup}(\sigma,\phi,g,f), T_{\sup}(\sigma,\widehat{\phi},\widehat{g},f)\}.$$

Proof. As in the proof of Theorem 3.2.1, we can define, for every $b > 0$, $\phi \in \mathcal{P}$ and $g(t) \in C([\sigma,T],X)$ with $g(\sigma) = \phi(0)$, a real number $\tau(\sigma,\phi,g,f,b)$ by (3.2.8) and a sequence $\{u^n(\cdot)\}_{n\in N}$ by (3.2.9) such that (3.2.15), (3.2.16), and (3.2.17) hold. Fix $b > 0$ and write $\tau(\sigma,\phi,g,f) := \tau(\sigma,\phi,g,f,b)$. By (3.2.23) and (H2) we have

$$\begin{aligned}
&\|u^n(t)-u^{n-1}(t)\| \\
\le\ & (t-\sigma)H\left(\max\{\|\phi(0)\|+\delta(\phi,f), \|\phi\|_{\mathcal{P}}+\delta(\phi,f)\}\right) \\
&\qquad \left(\|u^{n-1}(t)-u^{n-2}(t)\| + K(t-\sigma)\max_{s\in[0,t]}\|u^{n-1}(s)-u^{n-2}(s)\|\right), \\
&\qquad\qquad t\in[\sigma,\tau(\sigma,\phi,g,f)],\ n\in N\setminus\{1\},
\end{aligned}$$

that is,

$$\begin{aligned}
&\max_{s\in[0,t]}\|u^n(s)-u^{n-1}(s)\| \\
\le\ & (t-\sigma)H\left(\max\{\|\phi(0)\|+\delta(\phi,f), \|\phi\|_{\mathcal{P}}+\delta(\phi,f)\}\right) \\
&\qquad \left(1+\max_{t\in[0,T-\sigma]}K(t)\right)\max_{s\in[0,t]}\|u^{n-1}(s)-u^{n-2}(s)\|, \\
&\qquad\qquad t\in[\sigma,\tau(\sigma,\phi,g,f)],\ n\in N\setminus\{1\}.
\end{aligned}$$

Then by using (H2) and a standard argument based on the generalized Banach contractive mapping principle (by replacing $\tau(\sigma,\phi,g,f)$ with a smaller one if necessary), we verify the existence of a solution of (3.1.1).

The uniqueness of the solution of (3.1.1) is implied by (H2), (3.2.23) and Gronwall-Bellman's inequality. So $T_{\sup}(\sigma, \phi, g, f)$ exists. Let $u(t)$ (resp. $\widehat{u}(t)$) be the solution of (3.1.1) on $[\sigma, T_{\sup}(\sigma, \phi, g, f)]$ (resp. $[\sigma, T_{\sup}(\sigma, \widehat{\phi}, \widehat{g}, f)]$) with respect to $\phi \in \mathcal{P}$ (resp. $\widehat{\phi} \in \mathcal{P}$) and $g(t)$ (resp. $\widehat{g}(t)$), and fix

$$\tau_0 < \min\{T_{\sup}(\sigma, \phi, g, f), T_{\sup}(\sigma, \widehat{\phi}, \widehat{g}, f)\}.$$

Then by (3.1.1), (3.2.23) and (H2) we have

$$\begin{aligned} & \|u(t) - \widehat{u}(t)\| \\ \leq\ & \|g(t) - \widehat{g}(t)\| + H\left(\max_{t\in[\sigma,\tau_0]}\{\|u(t)\|, \|\widehat{u}(t)\|, \|u_t\|, \|\widehat{u}_t\|\}\right) \\ & \int_0^t \left[\left(1 + \max_{t\in[0,T-\sigma]} K(t)\right) \sup_{\eta\in[0,s]} \|u(\eta) - \widehat{u}(\eta)\| + \sup_{t\in[0,T-\sigma]} M(t)\|\phi - \widehat{\phi}\|_{\mathcal{P}}\right] ds. \end{aligned}$$

Hence,

$$\begin{aligned} & \max_{\eta\in[\sigma,t]} \|u(\eta) - \widehat{u}(\eta)\| \\ \leq\ & \max_{t\in[\sigma,\tau_0]} \|g(t) - \widehat{g}(t)\| \\ & + TH\left(\max_{t\in[\sigma,\tau_0]}\{\|u(t)\|, \|\widehat{u}(t)\|, \|u_t\|, \|\widehat{u}_t\|\}\right) \sup_{t\in[0,T-\sigma]} M(t)\|\phi - \widehat{\phi}\|_{\mathcal{P}} \\ & + H\left(\max_{t\in[\sigma,\tau_0]}\{\|u(t)\|, \|\widehat{u}(t)\|, \|u_t\|, \|\widehat{u}_t\|\}\right) \\ & \times \left(1 + \max_{t\in[0,T-\sigma]} K(t)\right) \int_0^t \max_{\eta\in[0,s]} \|u(\eta) - \widehat{u}(\eta)\| ds. \end{aligned}$$

By the Gronwall-Bellman's inequality, there is a constant $\widetilde{L}(u, \widehat{u}, \tau_0)$ such that

$$\|u(\eta) - \widehat{u}(\eta)\|_{\mathcal{P}[0,T]} \leq \widetilde{L}(u, \widehat{u}, \tau_0)\left(\max_{t\in[\sigma,\tau_0]} \|g(t) - \widehat{g}(t)\| + \|\phi - \widehat{\phi}\|_{\mathcal{P}}\right).$$

□

Now we turn to the global existence of solutions for (3.1.1). One will find that further assumptions must be made since the global existence of the solutions for

(3.1.1) fails quite often, more precisely the solutions of (3.1.1) having finite maximal intervals of existence blow up (in some sense).

Theorem 3.2.4. *Let $\sigma \geq 0$ and $\mathcal{P}$ be an admissible phase space. Suppose the continuous function $f : [\sigma, \infty) \times [\sigma, \infty) \times X \times \mathcal{P} \to X$ satisfies one of the following conditions:*

(1) *the hypotheses of Theorem* 3.2.1 *holds for every $T > 0$, and for every T, $r > 0$, there exists a constant $H(T, r)$ such that*

$$\|f(t, s, x(s), x_s)\| \leq H(T, r), \quad \text{for all } t, s \in [\sigma, T], x(\cdot) \in \mathcal{P}^{[\sigma,T]} \text{ with } \max_{s\in[\sigma,T]} \{\|x(s)\|, \|x_s\|_{\mathcal{P}}\} \leq r; \tag{3.2.25}$$

(2) $f\Big|_{[\sigma,T]\times[\sigma,T]\times X\times\mathcal{P}}$ *is compact for every $T > 0$;*

(3) *for every $T > 0$ and $r > 0$, there exists a constant $H(T, r)$ such that*

$$\|f(t, s, x(s), x_s) - f(t, s, y(s), y_s)\| \leq H(T, r) \left(\|x(s) - y(s)\| + \|x_s - y_s\|_{\mathcal{P}}\right), \text{ for all } t, s \in [\sigma, T], x(\cdot), y(\cdot) \in \mathcal{P}^{[\sigma,T]} \text{ with } \max_{s\in[\sigma,T]} \{\|x(s)\|, \|y(s)\|, \|x_s\|_{\mathcal{P}}, \|y_s\|_{\mathcal{P}}\} \leq r. \tag{3.2.26}$$

Then for every $\phi \in \mathcal{P}$ and $g(t) \in C([\sigma, T], X)$ with $g(\sigma) = \phi(0)$,

$$\overline{\lim}_{t\uparrow T_{\sup}(\sigma,\phi,g,f)}\|u(t)\| := \limsup_{t\uparrow T_{\sup}(\sigma,\phi,g,f)} \|u(t)\| = \infty, \tag{3.2.27}$$

$$\lim_{t\uparrow T_{\sup}(\sigma,\phi,g,f)} (\|u(t)\| + \|u_t\|_{\mathcal{P}}) = \infty, \tag{3.2.28}$$

provided that $T_{\sup}(\sigma, \phi, g, f) < \infty$, where $T_{\sup}(\sigma, \phi, g, f)$ is the number as in (3.2.24).

Proof. The proof of case (1).

Given $\phi \in \mathcal{P}$ and $g(t) \in C([\sigma, \infty), X)$ with $g(\sigma) = \phi(0)$. The existence of $T_{\sup}(\sigma, \phi, g, f)$ is ensured by Theorem 3.2.1. Let $u(t)$ be the solution of (3.1.1) with respect to σ, ϕ, g, and f on $[\sigma, T_{\sup}(\sigma, \phi, g, f))$, and suppose that

$$T_{\sup}(\sigma, \phi, g, f) < \infty \text{ and } \overline{\lim}_{t\uparrow T_{\sup}(\sigma,\phi,g,f)}\|u(t)\| < \infty.$$

Then there exists a constant b_1 such that

$$\left.\begin{array}{l} \sup\limits_{t\in[0,T_{\sup}(\sigma,\phi,g,f)+1-\sigma]} \{K(t),\ M(t)\} \\ \max\limits_{t\in[\sigma,T_{\sup}(\sigma,\phi,g,f)+1]} \|g(t)\| \\ \sup\limits_{t\in[\sigma,T_{\sup}(\sigma,\phi,g,f))} \|u(t)\| \end{array}\right\} \le b_1. \tag{3.2.29}$$

For each $b > 0$, $t \in (\sigma, T_{\sup}(\sigma, \phi, g, f))$ and $\eta \in (0, 1)$, we define

$$\begin{aligned} \mathcal{P}_{\phi,g,u}^{[t,t+\eta]}(b) &= \Big\{ x : (-\infty, t+\eta] \to X;\ \ x\Big|_{[t,t+\eta]} \in C([t, t+\eta], X), \\ &\qquad \max_{\tau\in[t,t+\eta]} \|x(\tau) - g(\tau) + g(t) - u(t)\| \le b,\ x\Big|_{(-\infty,t]} = u\Big|_{(-\infty,t]} \Big\}. \end{aligned}$$

Then $\mathcal{P}_{\phi,g,u}^{[t,t+\eta]}(b)$ is a closed convex subset of $\mathcal{P}^{[\sigma,t+\eta]}$. From (H2) and (3.2.29) it follows that

$$\max_{\tau\in[\sigma,t+\eta]} \{\|x(\tau)\|, \|x_\tau\|\} \le b_2, \quad x \in \mathcal{P}_{\phi,g,u}^{[t,t+\eta]}(b),$$

where $b_2 = \max\{b + 3b_1, b_1(b + 3b_1 + \|\phi\|_{\mathcal{P}})\}$.

Let $b = \frac{a}{2}$ (the constant given in the hypotheses) and for every $x \in \mathcal{P}_{\phi,g,u}^{[t,t+\eta]}(b)$, define

$$(\mathbf{F}x)(s) = \begin{cases} g(s) - g(t) + u(t) + \displaystyle\int_t^s f(s, \mu, x(\mu), x_\mu) d\mu, & t \le s \le t+\eta, \\ u(s), \quad s \in (-\infty, t]. \end{cases}$$

Then $\mathbf{F}x \in \mathcal{P}^{[t,t+\eta]}$ by (H1). Moreover, by (3.2.25) we have for each $x \in \mathcal{P}_{\phi,g,u}^{[t,t+\eta]}(b)$,

$$\max_{s\in[t,t+\eta]} \|(\mathbf{F}x)(s) - g(s) + g(t) - u(t)\| \le H(T_{\sup}(\sigma, \phi, g, f) + 1, b_2)(\tau - \sigma). \tag{3.2.30}$$

Hence there exists a real number

$$\widetilde{\tau}(\sigma, \phi, g, f, b) \in (0, 1) \ \text{ being independent of } \ t \in [\sigma, T_{\sup}(\sigma, \phi, g, f)), \tag{3.2.31}$$

such that

$$\max_{s\in[t,t+\widetilde{\tau}(\sigma,\phi,g,f,b)]} \|(\mathbf{F}x)(s) - g(s) + g(t) - u(t)\| \le b, \quad x \in \mathcal{P}_{\phi,g,u}^{[t,t+\widetilde{\tau}(\sigma,\phi,g,f,b)]}(b). \tag{3.2.32}$$

That means that

$$\mathbf{F}x \in \mathcal{P}_{\phi,g,u}^{[t,t+\widetilde{\tau}(\sigma,\phi,g,f,b)]}(b). \tag{3.2.33}$$

Let

$$x^0(s) = \begin{cases} g(s) - g(t) + u(t), & s \in [t, t+\widetilde{\tau}(\sigma,\phi,g,f,b)], \\ u(s-t), & s \in (-\infty, t]. \end{cases}$$

Clearly $x^0(\cdot) \in \mathcal{P}_{\phi,g,u}^{[t,t+\widetilde{\tau}(\sigma,\phi,g,f,b)]}(b)$. Define

$$x^n(s) := \mathbf{F}x^{n-1}(s), \quad s \in (-\infty, t+\widetilde{\tau}(\sigma,\phi,g,f,b)],\ n \in N. \tag{3.2.34}$$

Then $x^n(\cdot) \in \mathcal{P}_{\phi,g,u}^{[t,t+\widetilde{\tau}(\sigma,\phi,g,f,b)]}(b)$ for all $n \in N$, and (3.2.30) says that

$$\begin{aligned} & \max_{z\in[t,s]} \|x^n(z) - x^0(z)\| \\ = & \max_{z\in[\sigma,t]} \|x^n(z) - g(z) + g(t) - u(t)\| \\ \leq & H(T_{\sup}(\sigma,\phi,g,f)+1, b_2)\widetilde{\tau}(\sigma,\phi,g,f,b), \quad t \in [t, t+\widetilde{\tau}(\sigma,\phi,g,f,b)],\ n \in N. \end{aligned} \tag{3.2.35}$$

Observing that for every $t \leq w \leq z \leq s \leq t+\widetilde{\tau}(\sigma,\phi,g,f,b)$,

$$\begin{aligned} & \left\| \int_t^z f(z,\mu,x^n(\mu),x^n_\mu)d\mu - \int_t^w f(w,\mu,x^n(\mu),x^n_\mu)d\mu \right\| \\ \leq & \int_t^\tau \|f(z,\mu,x^n(\mu),x^n_\mu) - f(w,\mu,x^n(\mu),x^n_\mu)\| d\mu \\ & +H(T_{\sup}(\sigma,\phi,g,f)+1, b_2)(z-w), \end{aligned}$$

and using the similar arguments as in the proof of Theorem 3.2.1, we deduce that (3.1.1) has a solution $x(\cdot)$ in $\mathcal{P}_{\phi,g,u}^{[t,t+\widetilde{\tau}(\sigma,\phi,g,f,b)]}(b)$. (3.2.31) allows us to take a $t \in [\sigma, T_{\sup}(\sigma,\phi,g,f))$ such that

$$0 < T_{\sup}(\sigma,\phi,g,f) - t < \widetilde{\tau}(\sigma,\phi,g,f,b),$$

that is,

$$t + \widetilde{\tau}(\sigma,\phi,g,f,b) > T_{\sup}(\sigma,\phi,g,f).$$

This is in contradiction with the definition of $T_{\sup}(\sigma,\phi,g,f)$. As a consequence we get (3.2.27).

Now let us show (3.2.28). If this is false then there is a sequence $\{t_n\}_{n\in N} \subset [\sigma, T_{\sup}(\sigma, \phi, g, f))$ and a constant b_3 such that

$$\lim_{n\to\infty} t_n = T_{\sup}(\sigma, \phi, g, f), \tag{3.2.36}$$

$$\|u(t_n)\| + \|u_{t_n}\| \leq b_3, \quad n \in N. \tag{3.2.37}$$

From (3.2.27) and the fact that $\|u(\cdot)\|$ is a continuous function, it follows that there exists a sequence $\{\eta_n\}_{n\in N}$ such that

$$\lim_{n\to\infty} \eta_n = 0, \quad \|u(t_n + \eta_n)\| = b_3 + 2 \max_{t\in[\sigma, T_{\sup}(\sigma,\phi,g,f)]} \|g(t)\| + 1,$$

and

$$\max_{t\in[t_n, t_n+\eta_n]} \|u(t)\| \leq \|u(t_n + \eta_n)\|, \tag{3.2.38}$$

by noting that if necessary, we can replace the sequence $\{t_n\}_{n\in N}$ with another one satisfying (3.2.36) and (3.2.37).

On the other hand, it is clear that

$$\begin{aligned}
& \|u(t_n + \eta_n)\| \\
\leq\ & \|u(t_n)\| + \|g(t_n + \eta_n) - g(t_n)\| + \int_{t_n}^{t_n+\eta_n} \|f(t_n + \eta_n, z, u(z), u_z)\| dz \\
\leq\ & b_3 + 2 \max_{t\in[\sigma, T_{\sup}(\sigma,\phi,g,f)]} \|g(t)\| + \eta_n H(T_{\sup}(\sigma, \phi, g, f), b_4),
\end{aligned}$$

where

$$b_4 = \left(b_3 + 2 \max_{t\in[\sigma, T_{\sup}(\sigma,\phi,g,f)]} \|g(t)\| + 1 \right) \left(\max_{t\in[\sigma, T-\sigma]} K(t) + 1 \right) + \sup_{t\in[\sigma, T-\sigma]} M(t) b_3.$$

Letting $n \to \infty$ yields a contraction with (3.2.38). This implies that (3.2.28) is true.

The proof of case (2).

In this case, there is certainly a constant $\overline{H}(T_{\sup}(\sigma, \phi, g, f), b_2)$ such that

$$\begin{aligned}
& \max\nolimits_{s\in[t,t+\eta]} \|(\mathbf{F}x)(s) - g(s) + g(t) - u(t)\| \\
\leq\ & \overline{H}(T_{\sup}(\sigma, \phi, g, f), b_2)(\tau - \sigma), \quad x \in \mathcal{P}_{\phi,g,u}^{[t,t+\eta]}(b).
\end{aligned}$$

A similar argument in the proof of (1), combined with the techniques in the proof of Theorem 3.2.2, leads to the conclusion .

The proof of case (3).

A combination of the condition (3.2.26) with the proof of case (1) and the proof of Theorem 3.2.3 gives the desired conclusion.

□

The following result presents a sufficient condition for the existence of the global solution of (3.1.1).

Theorem 3.2.5. *Let* $\sigma \geq 0$, $\mathcal{P}$ *and* $f \in C([\sigma,\infty) \times X \times \mathcal{P}, X)$ *be as in Theorem* 3.2.3 *and*

$$\|f(t,s,x,\phi)\| \leq h_0(t)[h_1(s)\|x\| + h_2(s)\|\phi\|_{\mathcal{P}} + h_3(s)], \quad t,s \in [\sigma,\infty),\ x \in X,\ \phi \in \mathcal{P}, \tag{3.2.39}$$

where $h_0 \geq 0$ *is a locally bounded function on* $[\sigma,\infty)$ *and* $h_i \geq 0$ $(i = 1,2,3)$ *are locally integrable functions on* $[\sigma,\infty)$*. Then*

$$T_{\sup}(\sigma,\phi,g,f) = \infty$$

for any $\phi \in \mathcal{P}$ *and* $g(t) \in C([\sigma,\infty),X)$ *with* $g(\sigma) = \phi(0)$*.*

Proof. Take $\sigma \geq 0$, $\phi \in \mathcal{P}$, and $g(t) \in C([\sigma,\infty),X)$ with $g(\sigma) = \phi(0)$. Let $u(t)$ be the corresponding solution of (3.1.1) on $[\sigma, T_{\sup}(\sigma,\phi,g,f))$. Then by (3.2.26) and (H2) we have for any $t \in [\sigma, T_{\sup}(\sigma,\phi,g,f))$,

$$\begin{aligned}\|u(t)\| \;\leq\; & \|g(t)\| + \int_\sigma^t h_0(t)\Big[h_1(s)\|u(s)\| + h_2(s)K(s-\sigma)\max_{\tau\in[\sigma,s]}\|u(\tau)\| \\ & +h_2(s)M(s-\sigma)\|\phi\|_{\mathcal{P}} + h_3(s)\Big]ds.\end{aligned}$$

Therefore for each $t \in [\sigma, T_{\sup}(\sigma,\phi,g,f))$,

$$\begin{aligned}&\max_{\tau\in[\sigma,t]}\|u(\tau)\| \\ \leq\; & \max_{\tau\in[\sigma,T_{\sup}(\sigma,\phi,g,f)]}\|g(\tau)\| + \sup_{\tau\in[\sigma,T_{\sup}(\sigma,\phi,g,f)]} h_0(\tau) \\ & \left[\sup_{\tau\in[0,T_{\sup}(\sigma,\phi,g,f)-\sigma]}\|M(\tau)\|\|\phi\|_{\mathcal{P}}\int_\sigma^{T_{\sup}(\sigma,\phi,g,f)} h_2(s)ds + \int_\sigma^{T_{\sup}(\sigma,\phi,g,f)} h_3(s)ds\right] \\ & + \sup_{t\in[0,T_{\sup}(\sigma,\phi,g,f)]} h_0(\tau)\int_\sigma^t [h_1(s) + h_2(s)K(s-\sigma)]\max_{\tau\in[\sigma,s]}\|u(\tau)\|ds.\end{aligned}$$

Thus by Gronwall-Bellman's inequality, for any $t \in [\sigma, T_{\sup}(\sigma, \phi, g, f))$,

$$\max_{\tau \in [\sigma,t]} \|u(\tau)\| \leq \text{const},$$

where the constant is independent of t. This, together with Theorem 3.2.4, shows that $T_{\sup}(\sigma, \phi, g, f) = \infty$.

$\square$

The next result shows that the solution of (3.1.1) exists uniquely on the whole interval $[\sigma, T]$ and depends continuously on g and ϕ in the normal sense, provided that f has the uniform Lipschitz continuity.

Theorem 3.2.6. *Let $0 \leq \sigma < T$, $f \in C([\sigma, T] \times [\sigma, T] \times X \times \mathcal{P}, X)$ satisfying the "Uniform Lipschitz Condition", i.e., there exists a constant $L_f > 0$ such that for each $t, s \in [\sigma, T]$,*

$$\|f(t, s, x, \phi) - f(t, s, y, \psi)\| \leq L_f \left(\|x - y\| + \|\phi - \psi\|_{\mathcal{P}}\right), \quad \text{for } \; x, y \in X, \; \phi, \psi \in \mathcal{P}.$$

Then for every $\phi \in \mathcal{P}$ and $g(t) \in C([\sigma, T], X)$ with $g(\sigma) = \phi(0)$, (3.1.1) has a unique solution $u(t)$ on $[\sigma, T]$.

Moreover, let $u(t)$ and $\widehat{u}(t)$ be the solutions of (3.1.1) on $[\sigma, T]$ with respect to $\phi \in \mathcal{P}$ and $g(t)$ and to $\widehat{\phi} \in \mathcal{P}$ and $\widehat{g}(t)$ respectively. Then there is a constant $\breve{L}$ such that

$$\|u - \widehat{u}\|_{\mathcal{P}[0,T]} \leq \breve{L} \left(\max_{t \in [0,T]} \|g(t) - \widehat{g}(t)\| + \|\phi - \widehat{\phi}\|_{\mathcal{P}} \right).$$

Proof. A combination of the related arguments in the proof of Theorems 3.2.3 and 3.2.4 yields the result. Another approach of proving this theorem is to employ the generalized Banach contractive mapping principle.

$\square$

Remark 3.2.7. Similarly, there exists a corresponding result to every related theorem in Sections 3 – 5.

3.3 Applications to the functional differential equation

From now on, we concentrate on the case of σ being 0. It is not so hard to modify our results below to the case of σ being not 0 .

Definition 3.3.1. ([82]) Let C be an injective operator in $\mathbf{L}(X)$ and $\tau > 0$. An operator family $\{E(t)\}_{t\in[0,\tau]} \subset \mathbf{L}(X)$ is called a *local C-regularized semigroup* on X if

(i) $E(0) = C$ and $E(t+s)C = E(t)E(s)$ for $s,\ t,\ s+t \in [0,\tau]$,

(ii) $\{E(t)\}_{t\in[0,\tau]}$ is strongly continuous.

The operator A defined by

$$\mathcal{D}(A) = \{x \in X : \lim_{t\to 0^+} \frac{1}{t}(E(t)x - Cx) \text{ exists and is in } \mathcal{R}(C)\}$$

and

$$Ax = C^{-1} \lim_{t\to 0^+} \frac{1}{t}(E(t)x - Cx), \quad \text{for each } x \in \mathcal{D}(A),$$

is called the *generator* of $\{E(t)\}_{t\in[0,\tau]}$. We also say that A *generates* $\{E(t)\}_{t\in[0,\tau]}$.

Definition 3.3.2. Let $E \in \mathbf{L}(X)$, A a closed operator in X and $\tau > 0$. An operator family $\{E(t)\}_{t\in[0,\tau]} \subset \mathbf{L}(X)$ is called a *local E-existence family* for A if

(i) $\{E(t)\}_{t\in[0,\tau]}$ is strongly continuous,

(ii) $\int_0^t E(s)xds \in \mathcal{D}(A)$ and

$$A\left(\int_0^t E(s)xds\right) = E(t)x - Ex, \quad \text{for every } x \in X,\ t \in [0,\tau]. \tag{3.3.1}$$

We also say that the operator A *has* a local E-existence family $\{E(t)\}_{t\in[0,\tau]}$.

Remark 3.3.3. It is easy to see that a local C-regularized semigroup generated by A is also a local C-existence family for A. When $\tau = \infty$, the local C-regularized semigroup coincides with the C-regularized semigroup. It is known that the concept of local C-regularized semigroups or C-regularized semigroups is really a generalization of classical C_0 semigroups as well as integrated semigroups since there are many examples of operators which generate C-regularized semigroup or local C-regularized semigroups but C_0 semigroups or integrated semigroups (cf., e.g., [29, 30, 82, 88] and references cited there). On the other hand, the concept of local existence families is an extension of local C-regularized semigroups. When $\tau = \infty$, the existence family in Definition 3.2 was called the mild existence family (cf., [29, 30]). For the sufficient conditions for A having an E-existence family $\{E(t)\}_{t\in[0,T]}$ and other information on existence families, please refer to [29, 30]. For the local one, please refer to, e.g., [82].

In what follows, it is supposed that

$$\text{the zero function is the unique continuous solution of } x(t) = A\int_0^t x(s)ds \ (t \geq 0), \tag{3.3.2}$$

where the operator A is the coefficient operator in (3.1.2).

Remark 3.3.4. It is easy to see that (3.3.2) holds automatically for the generator A of a local C-regularized semigroup.

Definition 3.3.5. A function $u : (-\infty, a) \to X$ is called a *mild solution* of (3.1.2) on $[0, a)$ if $u \in C([0, a), X)$ satisfying

$$u(t) = \begin{cases} E(t)z + \displaystyle\int_0^t E(t-s)\widetilde{f}(s, u(s), u_s)ds, & t \in [0, a), \\ \phi(t), & t \in (-\infty, 0], \end{cases} \tag{3.3.3}$$

where $z \in X$ with $Ez = \phi(0)$, and $\widetilde{f} \in C([0, T] \times X \times \mathcal{P}, X)$ with $E\widetilde{f} = f$.

Remark 3.3.6. The integral equation (3.3.3) is independent of the choices of z and $\widetilde{f}$. This can be seen by (3.3.2), which implies that for every $x, y \in X$ with $Ex = Ey$,

$$E(t)x = E(t)y, \quad t \geq 0.$$

This indicates

$$\max_{t\in[0,T]} \|E(t)z\| \le \widetilde{L}\|\phi(0)\|_{[\mathcal{R}(E)]}, \tag{3.3.4}$$

where $\widetilde{L}$ is a constant, and $\|\phi(0)\|_{[\mathcal{R}(E)]} := \inf\{\|z\|;\ Ez = \phi(0)\}$.

Definition 3.3.7. A function $u : (-\infty, a) \to X$ is called a *classical solution* of (3.1.2) if

$$u \in C^1([0,a), X) \cap C([0,a), [\mathcal{D}(A)])$$

satisfying (3.1.2) on $[0,a)$.

Now we are in a position to give the solvability and wellposedness results for (3.1.2) by applying the obtained results on (3.1.1) in Section 2.

Theorem 3.3.8. *Assume that $T > 0$, A has a local E-existence family $\{E(t)\}_{t\in[0,T]}$. Let $\mathcal{P}$ be an admissible phase space and $\widetilde{f} \in C([0,T] \times X \times \mathcal{P}, X)$.*

(1) *Suppose one of the following conditions*

(1i) *$\widetilde{f}$ is compact;*

(1ii) *$E(t)$ is compact for $0 < t \le T$;*

(1iii) *$\{E(t)\}_{t\in[0,T]}$ is norm continuous for $t > 0$, and there is a Kamke function $\mathcal{K}(\cdot,\cdot,\cdot)$ on $[0,T] \times [0,a] \times [0, \max_{t\in[0,T]} K(t)a]$ for some $a > 0$ such that for every bounded set $B \in X$ and $\Omega \in \mathcal{P}$,*

$$\alpha(\widetilde{f}(\{s\} \times B \times \Omega)) \le \mathcal{K}(s, \alpha(B), \alpha(\Omega)), \quad a.e.\ s \in [0,T],$$

and $\varpi(t) \equiv 0$ is the unique nonnegative absolutely continuous solution to the differential equation

$$\varpi'(t) = 2\overline{\lim}_{\delta\uparrow 0}\|E(\delta)\| \sup_{t\in[0,T]} \beta(E(t))\mathcal{K}(t, \varpi(t), K(t)\varpi(t)), \quad t \in (0,T] \tag{3.3.5}$$

satisfying

$$\lim_{t\uparrow 0} \frac{\varpi(t)}{t} = \varpi(0) = 0, \tag{3.3.6}$$

where $K(\cdot)$ is the function as in (H2), and for each $t \in [0,T]$,

$$\beta(E(t)) = \inf\{\gamma \in R^+;\ \alpha(E(t)B) \le \gamma\alpha(B) \text{ for all bounded countable sets } B \subset X\}.$$

Then for each $\phi \in \mathcal{P}$ with $\phi(0) \in \mathcal{R}(E)$, there exists a real number $T_{\sup}(\phi, E(\cdot), \widetilde{f})$ such that (3.1.2) has a mild solution $u(t)$ on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f})]$.

(2) *Suppose that for every $r > 0$ there is a constant $H(r)$ such that for all $t \in [0,T]$,*

$$\|\widetilde{f}(t,x,\phi) - \widetilde{f}(t,y,\psi)\| \le H(r)\left(\|x-y\| + \|\phi - \psi\|_{\mathcal{P}}\right),$$

$$\text{for every } x, y \in X,\ \phi, \psi \in \mathcal{P} \text{ with } \max\{\|x\|,\ \|y\|,\ \|\phi\|_{\mathcal{P}},\ \|\psi\|_{\mathcal{P}}\} \le r. \tag{3.3.7}$$

Then for each $\phi \in \mathcal{P}$ with $\phi(0) \in \mathcal{R}(E)$, there exists a real number $T_{\sup}(\phi, E(\cdot), \widetilde{f})$ such that (3.1.2) has a unique mild solution $u(t)$ on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f})]$. Moreover, if $u(t)$ and $\widehat{u}(t)$ are the mild solutions of (3.1.2) on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f})]$ with respect to $\phi \in \mathcal{P}$ and on $[0, T_{\sup}(\widehat{\phi}, E(\cdot), \widetilde{f})]$ to $\widehat{\phi} \in \mathcal{P}$ respectively, then there is a constant $\overline{L}(u, \widehat{u}, \tau_0)$ such that

$$\|u(t) - \widehat{u}(t)\|_{\mathcal{P}[0,\tau_0]} \le \overline{L}(u, \widehat{u}, \tau_0)\left(\|\phi(0) - \widehat{\phi}(0)\|_{[\mathcal{R}(E)]} + \|\phi - \widehat{\phi}\|_{\mathcal{P}}\right),$$

for each $\tau_0 < \min\{T_{\sup}(\phi, E(\cdot), \widetilde{f}), T_{\sup}(\widehat{\phi}, E(\cdot), \widetilde{f})\}$.

Proof. Let $g(t) = E(t)z$ $(Ez = \phi(0))$ and $f = E(t-s)\widetilde{f}(s,\cdot,\cdot)$. Then the conclusions, except that under the condition (1iii), come from Theorems 3.2.2 and 3.2.3.

Now we prove that (1iii) implies also that for each $\phi \in \mathcal{P}$ with $\phi(0) \in \mathcal{R}(E)$, there exists a real number $T_{\sup}(\phi, E(\cdot), \widetilde{f})$ such that (3.1.2) has a mild solution $u(t)$ on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f})]$.

After a repetition of the first part of the proof of Theorem 3.2.1, we get a sequence $\{u^n(\cdot)\}_{n\in N}$ such that (3.2.15), (3.2.16), (3.2.17) and (3.2.18) hold for any given $b > 0$.

By (3.2.18), we have for every $n \in N$, $0 \leq \eta \leq w \leq z \leq t \leq \tau(0, \phi, g, f, b)$,

$$\begin{aligned}
&\left\| \int_0^z f(z, s, u^n(s), u^n_s) ds - \int_0^w f(w, s, u^n(s), u^n_s) ds \right\| \\
\leq\ & \int_0^\eta \|f(z, s, u^n(s), u^n_s) - f(w, s, u^n(s), u^n_s)\| ds \\
&+ \int_\eta^w \|E(z) - E(w)\| \|\widetilde{f}(s, u^n(s), u^n_s)\| ds + \int_w^z \|f(w, s, u^n(s), u^n_s)\| ds \\
\leq\ & 2 \left[\max_{t,s \in [0,T]} \|f(t, s, \phi(0), \phi)\| + 1 \right] (\eta + z - w) \\
&\qquad + \int_\eta^w \|E(z) - E(w)\| \|\widetilde{f}(s, u^n(s), u^n_s)\| ds.
\end{aligned}$$

Hence, by the norm continuity of $\{E(t)\}_{t \in [0,T]}$ for $t > 0$, we get for each $t \in [0, \tau(0, \phi, g,\ f, b)]$, the set $\left\{ \int_0^\cdot f(\cdot, s, u^n(s), u^n_s) ds \Big|_{[0,t]} \right\}_{n \in N}$ is equicontinuous. Thus proceeding as in the second part of the proof of Theorem 3.2.1, we obtain the desired result.

□

When the family $\{E(t)\}_{t \in [0,T]}$ is a local C-regularized semigroup on X, which means it has the semigroup property (i.e., (i) of Definition 3.3.1 holds), we can obtain the following result without the compactness or norm continuity of $\{E(t)\}_{t \in [0,T]}$ as required in Theorem 3.3.8.

Theorem 3.3.9. *Let $T > 0$ and A generate a local C-regularized semigroup $\{E(t)\}_{t \in [0,T]}$. Let $\mathcal{P}$ be an admissible phase space and $C^{-1}\widetilde{f} \in C([0,T] \times X \times \mathcal{P}, X)$. Suppose that there is a Kamke function $\mathcal{K}(\cdot, \cdot, \cdot)$ on $[0,T] \times [0,a] \times [0, \max_{t \in [0,T]} K(t) a]$ for some $a > 0$ such that for every bounded set $B \in X$ and $\Omega \in \mathcal{P}$,*

$$\alpha(C^{-1}\widetilde{f}(\{s\} \times B \times \Omega)) \leq \mathcal{K}(s, \alpha(B), \alpha(\Omega)), \quad a.e.\ s \in [0,T],$$

and that $\varpi(t) \equiv 0$ is the unique nonnegative absolutely continuous solution to the differential equation (3.3.5) *satisfying* (3.3.6). *Then for each $\phi \in \mathcal{P}$ with $\phi(0) \in \mathcal{R}(C)$, there exists a real number $T_{\sup}(\phi, E(\cdot), \widetilde{f})$ such that* (3.1.2) *has a mild solution $u(t)$ on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f})]$.*

Proof. Let $\phi \in \mathcal{P}$ with $\phi(0) \in \mathcal{R}(E)$, and let $g(t) = E(t)z$ $(Cz = \phi(0))$ and $f = E(t-s)\widetilde{f}(s,\cdot,\cdot)$. Then from the proof of Theorem 3.2.1, we have a sequence $\{u^n(\cdot)\}_{n\in N}$ such that (3.2.15), (3.2.16), (3.2.17) and (3.2.18) hold for any given $b > 0$.

By Definition 1.5, we know that for every $0 \le t < \tau(0,\phi,g,f,b)$ and $\varepsilon > 0$, there are sets $B_1(t), \ldots, B_m(t)$ $(m \in N)$ such that

$$B_k(t) \subset \left\{\int_0^t E(t-s)C^{-1}\widetilde{f}(s,u^n(s),u^n_s)ds\right\}_{n\in N}, \quad k = 1,\ldots,m,$$

$$\left\{\int_0^t E(t-s)C^{-1}\widetilde{f}(s,u^n(s),u^n_s)ds\right\}_{n\in N} = \bigcup_{k=1}^m B_k(t),$$

$$\begin{aligned} & diameter(B_k(t)) \\ \le\ & \alpha\left(\left\{\int_0^t E(t-s)C^{-1}\widetilde{f}(s,u^n(s),u^n_s)ds\right\}_{n\in N}\right) + \varepsilon, \quad k = 1,\ldots,m, \end{aligned}$$

where $diameter(B_k(t))$ means the diameter of the set $B_k(t)$. Thus, letting

$$N_k(t) = \left\{n \in N;\ \int_0^t E(t-s)C^{-1}\widetilde{f}(s,u^n(s),u^n_s)ds \in B_k(t)\right\}, \quad k = 1,\ldots,m,$$

gives $N = \cup_{k=1}^m N_k(t)$. Fix t and $\eta \in [t,\tau(0,\phi,g,f,b)]$, and define

$$\widetilde{B}_k(t,\eta) := \left\{\int_0^{\cdot} f(\cdot,s,u^n(s),u^n_s)ds\Big|_{[t,\eta]}\right\}_{n\in N_k(t)}, \quad k = 1,\ldots,m.$$

Then

$$\left\{\int_0^{\cdot} f(\cdot,s,u^n(s),u^n_s)ds\Big|_{[t,\eta]}\right\}_{n\in N} = \bigcup_{k=1}^m \widetilde{B}_k(t,\eta).$$

Choose arbitrarily two elements

$$\int_0^{\cdot} f(\cdot,s,u^{i_k}(s),u^{i_k}_s)ds\Big|_{[t,\eta]}, \quad \int_0^{\cdot} f(\cdot,s,u^{j_k}(s),u^{j_k}_s)ds\Big|_{[t,\eta]}$$

from every $\widetilde{B}_k(t,\eta)$ $(k=1,\ldots,m)$. Then for every $t \le z \le w \le \eta$,

$$
\begin{aligned}
& \left\| \int_0^z f(z,s,u^{i_k}(s),u_s^{i_k})ds - \int_0^z f(z,s,u^{j_k}(s),u_s^{j_k})ds \right\| \\
\le\ & \left\| \int_0^t E(z-s)\widetilde{f}(s,u^{i_k}(s),u_s^{i_k})ds - \int_0^t E(z-s)\widetilde{f}(s,u^{j_k}(s),u_s^{j_k})ds \right\| \\
& + \left\| \int_t^z f(z,s,u^{i_k}(s),u_s^{i_k})ds - \int_t^z f(z,s,u^{j_k}(s),u_s^{j_k})ds \right\| \\
\le\ & \|E(z-t)\| \left\| \int_0^t E(t-s)C^{-1}\left[\widetilde{f}(s,u^{i_k}(s),u_s^{i_k})ds - \widetilde{f}(s,u^{j_k}(s),u_s^{j_k})\right] ds \right\| \\
& + 2 \max_{\nu,s\in[0,T]} (\|f(\nu,s,\phi(0),\phi)\|+1)(z-t), \quad k=1,\ldots,m,\ i,j\in N.
\end{aligned}
$$

Accordingly, when $\eta - t$ is small enough, we obtain by $i_k \in N_k(t)$ and $j_k \in N_k(t)$,

$$
diameter(\widetilde{B}_k(t,\eta)) \le \max_{\nu\in[0,\delta]} \|E(\nu)\| diameter(B_k(t)) + 2\varepsilon.
$$

Therefore,

$$
\begin{aligned}
& \alpha\left(\left\{ \int_0^{\cdot} f(\cdot,s,u^n(s),u_s^n)ds \Big|_{[t,\eta]} \right\}_{n\in N}\right) \\
\le\ & diameter(B_k(t,\eta)) \\
\le\ & \max_{\nu\in[0,\delta]} \|E(\nu)\| \alpha\left(\left\{ \int_0^t E(t-s)C^{-1}\widetilde{f}(s,u^n(s),u_s^n)ds \right\}_{n\in N}\right) + 2\varepsilon.
\end{aligned}
$$

Thus, thanks to Heinz's theorem ([50, Theorem 2.1]) and Nussbaum's Lemma ([69, Lemma 1]), we get for each $\bar{t} \in [0, \tau(0, \phi, g, f, b))$,

$$\alpha \left(\left\{ \int_0^{\cdot} f(\cdot, s, u^n(s), u_s^n) ds \Big|_{[0,\bar{t}]} \right\}_{n \in N} \right)$$

$$\leq \sup_{t \in [0,\bar{t}]} \lim_{\eta \uparrow t} \alpha \left(\left\{ \int_0^{\cdot} f(\cdot, s, u^n(s), u_s^n) ds \Big|_{[t,\eta]} \right\}_{n \in N} \right)$$

$$\leq \overline{\lim}_{\delta \uparrow 0} \|E(\delta)\| \sup_{t \in [0,\bar{t}]} \alpha \left(\left\{ \int_0^t E(t-s) C^{-1} \widetilde{f}(s, u^n(s), u_s^n) ds \right\}_{n \in N} \right)$$

$$\leq \overline{\lim}_{\delta \uparrow 0} \|E(\delta)\| \sup_{t \in [0,T]} \beta(E(t)) \sup_{t \in [0,\bar{t}]} \int_0^t \alpha \left(\left\{ C^{-1} \widetilde{f}(s, u^n(s), u_s^n) ds \right\}_{n \in N} \right).$$

Let $b = \frac{a}{2}$ (the constant given in the hypotheses) and $\tau(\sigma, \phi, g, f) := \tau\left(\sigma, \phi, g, f, \frac{a}{2}\right)$. Then the similar arguments as in the last part of the proof of Theorem 3.2.1 leads to our conclusion.

□

Remark 3.3.10. Suppose that $C^{-1}\widetilde{f} \in C([0,T] \times X \times \mathcal{P}, X)$ and $C^{-1}\widetilde{f} = f_{(1)} + f_{(2)}$ where $f_{(1)}$ is Lipschitz continuous and $f_{(2)}$ is compact. Then $\widetilde{f}$ satisfies the related assumption in Theorem 3.3.9

As shown in the following theorem, a (or unique) classical solution can be obtained with one more condition on $\widetilde{f}$.

Theorem 3.3.11. *Let $T > 0$, A generate a local C-regularized semigroup $\{E(t)\}_{t \in [0,T]}$ and $\mathcal{P}$ be an admissible phase space. Let $\widetilde{f} \in C([0,T] \times X \times \mathcal{P}, X)$, and for all $u(\cdot) \in \mathcal{P}^{[0,T]}$ with $u(0) \in C(\mathcal{D}(A))$,*

$$\begin{cases} \widetilde{f}(s, u(s), u_s) \in \mathcal{D}(A), \quad \text{for } s \in [0,T], \\ \displaystyle\int_0^T \|A\widetilde{f}(s, u(s), u_s)\| ds < \infty. \end{cases} \tag{3.3.8}$$

(1) *If $C^{-1}\widetilde{f}$ satisfies the conditions in Theorem 3.3.9, then for each $\phi \in \mathcal{P}$ with $\phi(0) \in C(\mathcal{D}(A))$, there exists a real number $T_{\sup}(\phi, E(\cdot), \widetilde{f})$ such that (3.1.2) has a classical solution $u(t)$ on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f}))$.*

(2) *If $\widetilde{f}$ satisfies (3.3.7), then for each $\phi \in \mathcal{P}$ with $\phi(0) \in C(\mathcal{D}(A))$, there exists a real number $T_{\sup}(\phi, E(\cdot), \widetilde{f})$ such that (3.1.2) has a unique classical solution $u(t)$ on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f}))$. Moreover, let $u(t)$ and $\widehat{u}(t)$ be the classical solutions of (3.1.2) on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f}))$ with respect to $\phi \in \mathcal{P}$ and on $[0, T_{\sup}(\widehat{\phi}, E(\cdot), \widetilde{f}))$ to $\widehat{\phi} \in \mathcal{P}$ respectively. Then there is a constant $\underline{L}(u, \widehat{u}, \tau_0)$ such that*

$$\|u(t) - \widehat{u}(t)\|_{\mathcal{P}^{[0,\tau_0]}} \leq \underline{L}(u, \widehat{u}, \tau_0) \left(\|\phi(0) - \widehat{\phi}(0)\| + \|\phi - \widehat{\phi}\|_{\mathcal{P}} \right), \tag{3.3.9}$$

for each $\tau_0 < \min\{T_{\sup}(\phi, E(\cdot), \widetilde{f}), T_{\sup}(\widehat{\phi}, E(\cdot), \widetilde{f})\}$.

Proof. The proof of (1).

By Theorem 3.3.9, for each $\phi \in \mathcal{P}$ with $\phi(0) \in C(\mathcal{D}(A))$, there exists a real number $T_{\sup}(\phi, E(\cdot), \widetilde{f})$ such that (3.1.2) has a mild solution $u(t)$ on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f}))$ given by

$$u(t) = \begin{cases} E(t)z + \displaystyle\int_0^t E(t-s)\widetilde{f}(s, u(s), u_s)ds, \quad t \in [0, T_{\sup}(\phi, E(\cdot), \widetilde{f})), \\ \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{3.3.10}$$

where $z \in \mathcal{D}(A)$ and $Cz = \phi(0)$. Fix $t \in [0, T_{\sup}(\phi, E(\cdot), \widetilde{f}))$ and set

$$\overline{u}^t(s) = \begin{cases} u(t), \quad s \in [t, T], \\ u(s), \quad s \in (-\infty, t]. \end{cases}$$

Then $\overline{u}^t(\cdot) \in \mathcal{P}^{[0,T]}$. Thus (3.3.8) implies that

$$\widetilde{f}(s, u(s), u_s) = \widetilde{f}(s, \overline{u}^t(s), \overline{u}^t_s) \in \mathcal{D}(A), \quad \text{for} \ \ s \in [0, t], \tag{3.3.11}$$

and

$$\int_0^t \|A\widetilde{f}(s, u(s), u_s)\|ds \leq \int_0^T \|A\widetilde{f}(s, \overline{u}^t(s), \overline{u}^t_s)\|ds \leq \infty. \tag{3.3.12}$$

On the other hand, from [40] it follows that for $x \in \mathcal{D}(A)$, $t \in [0, \tau]$,

$$E(t)x \in \mathcal{D}(A), \quad AE(t)x = E(t)Ax, \tag{3.10}$$

and

$$\int_0^t E(s)Axds = A\int_0^t E(s)xds = E(t)x - Cx. \tag{3.3.13}$$

Therefore, by (3.3.11) we have

$$\frac{d}{dt}E(t-s)\widetilde{f}(s,u(s),u_s) = E(t-s)A\widetilde{f}(s,u(s),u_s), \quad 0 \le s \le t < T_{\sup}(\phi, E(\cdot), \widetilde{f}). \tag{3.3.14}$$

Taking a derivative in t of the first equality of (3.3.10), we get, by (3.3.11) – (3.3.14), $u(t)$ is a classical solution of (3.1.2).

The proof of (2).

Suppose that $u(t)$ is a classical solution of (3.1.2) on $[0, T_{\sup}(\phi, E(\cdot), \widetilde{f}))$. Then $u_0 = \phi$, and

$$\begin{aligned} & \frac{d}{dt}E(t-s)C^{-1}u(s) \\ = \quad & -AE(t-s)C^{-1}u(s) + E(t-s)C^{-1}Au(s) + E(t-s)\widetilde{f}(s,u(s),u_s), \\ & \qquad \text{for } \ 0 \le s \le t < T_{\sup}(\phi, E(\cdot), \widetilde{f}), \end{aligned}$$

i.e.,

$$u(t) - E(t)C^{-1}\phi(0) = \int_0^t E(t-s)\widetilde{f}(s,u(s),u_s), \quad 0 \le t < T_{\sup}(\phi, E(\cdot), \widetilde{f}).$$

This means that the classical solution of (3.1.2) must be the mild solution of (3.1.2). Thus conclusion (2) is a consequence of Theorem 3.3.8 (2) and the arguments in the proof of (1) above.

□

3.4 Applications to the nonautonomous functional differential equations

In this section, we consider the Cauchy problem for nonautonomous functional differential equations (3.1.3). We first recall the following notion.

Definition 3.4.1. An operator family $\{U(t,s)\}_{0 \le s \le t \le T} \subset \mathbf{L}(X)$ is called a (strongly continuous) evolution system if

(1) $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$ for $0 \le s \le r \le t \le T$.

(2) $(t,s) \to U(t,s)$ is strongly continuous for $0 \le s \le t \le T$.

Remark 3.4.2. "Evolution system" is also called evolution family, evolution operators, evolution process, propagator, or fundamental solution. Please refer to, e.g., [1, 36–38, 70, 81, 90] for more information about this system.

Now we give a general result about the "mild solution" of (3.1.3).

Theorem 3.4.3. *Let $T > 0$, $\{U(t,s)\}_{0\le s\le t\le T} \subset \mathbf{L}(X)$ being an evolution system, $\mathcal{P}$ an admissible phase space, and $f \in C([0,T]\times X\times\mathcal{P}, X)$.*

(1) *Suppose that there is a Kamke function $\mathcal{K}(\cdot,\cdot,\cdot)$ on $[0,T]\times[0,a]\times[0,\max_{t\in[0,T]} K(t)a]$ for some $a>0$ such that*

(1i) *for every $t\in[0,T]$ and for every bounded set $B\subset X$ and $\Omega\subset\mathcal{P}$,*

$$\alpha(F(\{t\}\times\{s\}\times B\times\Omega)) \le \mathcal{K}(s,\alpha(B),\alpha(\Omega)), \quad a.e.\ s\in[0,t],$$

where $F(t,s,\cdot,\cdot) = U(t,s)f(s,\cdot,\cdot)$.

(1ii) *$\varpi(t)\equiv 0$ is the unique nonnegative absolutely continuous solution to the differential equation*

$$\varpi'(t) = 2\max_{\eta\in[0,T]} \overline{\lim}_{z\uparrow\eta}\|U(z,\eta)\|\mathcal{K}(t,\varpi(t),K(t)\varpi(t)), \quad a.e.\ t\in(\sigma,T]$$

satisfying (3.3.6).

Then for each $\phi\in\mathcal{P}$, there exists a real number $T_{\sup}(\phi,U(\cdot,\cdot),f)$ and a

$$u : [-\infty, T_{\sup}(\phi,U(\cdot,\cdot),f)) \to X$$

such that

$$u(t) = \begin{cases} U(t,0)\phi(0) + \displaystyle\int_0^t U(t,s)f(s,u(s),u_s)ds, & t\in[0,T_{\sup}(\phi,U(\cdot,\cdot),f)), \\ \phi(t), & t\in(-\infty,0]. \end{cases} \tag{3.4.1}$$

(2) *Suppose that for every $r>0$ there is a constant $H(r)$ such that for all $t\in[0,T]$,*

$$\|f(t,x,\phi) - f(t,y,\psi)\| \le H(r)\left(\|x-y\| + \|\phi-\psi\|_{\mathcal{P}}\right),$$

for every $x,y\in X$, $\phi,\psi\in\mathcal{P}$ with $\max\{\|x\|,\ \|y\|,\ \|\phi\|_{\mathcal{P}},\ \|\psi\|_{\mathcal{P}}\} \le r$.

Then for each $\phi \in \mathcal{P}$, there exists a real number $T_{\sup}(\phi, U(\cdot,\cdot), f)$ such that (3.4.1) *has a unique solution $u(t)$ on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$. Moreover, let $u(t)$ and $\widehat{u}(t)$ be the solution of* (3.4.1) *on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$ with respect to $\phi \in \mathcal{P}$ and on $[0, T_{\sup}(\widehat{\phi}, U(\cdot,\cdot), f))$ to $\widehat{\phi} \in \mathcal{P}$ respectively. Then* (3.3.9) *holds for a constant $\widehat{L}(u, \widehat{u}, \tau_0)$.*

Proof. Applying Theorem 3.2.3 to $g(t) = U(t,0)\phi(0)$ and $F(t,s,\cdot,\cdot)$, we get the conclusion (2).

Now we prove the conclusion (1). From the proof of Theorem 3.2.1, we know that for each $\phi \in \mathcal{P}$ and $b > 0$, there is a sequence $\{u^n(\cdot)\}_{n\in N}$ such that (3.2.15), (3.2.16), (3.2.17) and (3.2.18) hold.

Since for every $0 < t \leq \tau(0, \phi, g, f, b)$ and $\varepsilon > 0$, there are subsets $C_1(t), \ldots, C_l(t)$ $(l \in N)$ of $\left\{\int_0^t F(t,s,u^n(s),u^n_s)ds\right\}_{n\in N}$, such that

$$\left\{\int_0^t F(t,s,u^n(s),u^n_s)ds\right\}_{n\in N} = \bigcup_{k=1}^{l} C_k(t),$$

$$diameter(C_k(t)) \leq \alpha\left(\left\{\int_0^t F(t,s,u^n(s),u^n_s)ds\right\}_{n\in N}\right) + \varepsilon.$$

For any $\eta \in [t, t(0,\phi,g,f,b))$, if we define

$$\widetilde{C}_k(t,\eta) := \left\{\int_0^{\cdot} F(\cdot,s,u^n(s),u^n_s)ds\bigg|_{[t,\eta]};\ \int_0^t F(t,s,u^n(s),u^n_s)ds \in C_k(\eta)\right\},$$

$$k = 1, \ldots, l,$$

then

$$\left\{\int_0^{\cdot} F(\cdot,s,u^n(s),u^n_s)ds\bigg|_{[t,\eta]}\right\}_{n\in N} = \bigcup_{k=1}^{m} \widetilde{C}_k(t,\eta).$$

Observing that for every $t \le z \le w \le \eta$,

$$\begin{aligned}
&\left\| \int_0^z F(z,s,u^{i_k}(s),u_s^{i_k})ds - \int_0^z F(z,s,u^{j_k}(s),u_s^{j_k})ds \right\| \\
\le\ & \left\| \int_0^t U(z,s)F(s,u^{i_k}(s),u_s^{i_k})ds - \int_0^t U(z,s)F(s,u^{j_k}(s),u_s^{j_k})ds \right\| \\
&+ \left\| \int_t^z F(z,s,u^{i_k}(s),u_s^{i_k})ds - \int_t^z F(z,s,u^{j_k}(s),u_s^{j_k})ds \right\| \\
\le\ & \|U(z,t)\| \left\| \int_0^t U(t,s)\left[F(s,u^{i_k}(s),u_s^{i_k})ds - F(s,u^{j_k}(s),u_s^{j_k})\right] ds \right\| \\
&+2 \max_{\nu,s\in[0,T]} (\|F(\nu,s,\phi(0),\phi)\|+1)(z-t), \quad k=1,\ldots,l,
\end{aligned}$$

where $\int_0^{\cdot} F(\cdot,s,u^{i_k}(s),u_s^{i_k})ds\Big|_{[t,\eta]}$ and $\int_0^{\cdot} F(\cdot,s,u^{j_k}(s),u_s^{j_k})ds\Big|_{[t,\eta]}$ are two elements of $\widetilde{C}_k(t,\eta)$ $(k=1,\ldots,l)$. Therefore, for each $\bar{t} \in (0,\tau(0,\phi,g,f,b)]$,

$$\begin{aligned}
&\alpha\left(\left\{ \int_0^{\cdot} F(\cdot,s,u^n(s),u_s^n)ds \Big|_{[0,\bar{t}]} \right\}_{n\in N}\right) \\
\le\ & \sup_{t\in[0,\bar{t}]} \lim_{\eta\uparrow t} \alpha\left(\left\{ \int_0^{\cdot} F(\cdot,s,u^n(s),u_s^n)ds \Big|_{[t,\eta]} \right\}_{n\in N}\right) \\
\le\ & \overline{\lim}_{z\uparrow t}\|U(z,t)\| \sup_{t\in[0,\bar{t}]} \alpha\left(\left\{ \int_0^t F(t,s,u^n(s),u_s^n)ds \right\}_{n\in N}\right).
\end{aligned}$$

Let $b=\frac{a}{2}$ (the constant given in the hypotheses) and $\tau(\sigma,\phi,g,f) := \tau\left(\sigma,\phi,g,f,\frac{a}{2}\right)$. Then the similar arguments as in the last part of the proof of Theorem 3.2.1 leads to our conclusion.

□

Next we present results about the "Y-valued solution" of (3.1.3) (under Hyperbolicity assumption) and the "classical solution" of (3.1.3) (under Parobolicity assumption). We start with the following definitions of the "Y-valued solution" and the "classical solution" of (3.1.3).

Definition 3.4.4. (1) Let Y be a Banach space and be densely and continuously imbedded in X. A function $u : (-\infty, a) \to X$ with $u \in C([0,a), Y]$ is called a *Y-valued solution* of (3.1.3) on $[0,a)$ if $u \in C^1((0,a), X)$ satisfying (3.1.3) in X.

(2) A function $u : (-\infty, a) \to X$ is called a *classical solution* of (3.1.3) on $[0,a)$ if $u \in C^1([0,a), X)$, $u(t) \in \mathcal{D}(A(t))$ for $t \in [0,a)$ and $u(t)$ satisfies (3.1.3) on $[0,a)$.

Hyperbolicity Assumption (cf., e.g., [70, 81]): For each $t \in [0,T]$, $A(t)$ is the generator of a strongly continuous semigroup $\{S^t(s)\}_{s\geq 0}$ on X, and there exist constants W and ω such that

(HA1) $(\omega, \infty) \subset \rho(A(t))$ for all $t \in [0,T]$, and for every nondecreasing sequence $\{t_n\}_1^k \subset [0,T]$,

$$\left\| \prod_{n=1}^{k} (\lambda - A(t_n))^{-1} \right\| \leq \frac{W}{(\lambda - \omega)^k}, \quad \lambda > \omega.$$

There is a Banach space Y which is densely and continuously imbedded in X and satisfies

(HA2) For each $t \in [0,T]$, $S^t(s)Y \subset Y$ $(s \geq 0)$, $\left\{ S^t(s)\Big|_Y \right\}_{s\geq 0}$ is a strongly continuous semigroup on Y, $A(t)\Big|_Y$ is the generator of $\left\{ S^t(s)\Big|_Y \right\}_{s\geq 0}$ on Y, and $A(t)\Big|_Y$ satisfies (HA1) for some constants $\widetilde{W}$ and $\widetilde{\omega}$,

(HA3) For each $t \in [0,T]$, $Y \subset \mathcal{D}(A(t))$, $A(t) \in \mathbf{L}(Y,X)$ and $t \to A(\cdot)$ is continuous in the $\mathbf{L}(Y,X)$ norm.

Theorem 3.4.5. (Hyperbolic case) *Assume that the "Hyperbolicity Assumption" holds and $\{U(t,s)\}_{0\leq s\leq t\leq T}$ is the evolution system associated with the family $\{A(t)\}_{t\in[0,T]}$. Let $U(t,s)Y \subset Y$ $(0 \leq s \leq t \leq T)$ and for each $y \in Y$, $U(t,s)y$ be continuous in Y for $0 \leq s \leq t \leq T$, and let $\mathcal{P}$ be an admissible phase space and $f \in C([0,T] \times X \times \mathcal{P}, Y)$.*

(1) *If f is as in* (1) *of Theorem* 3.4.3*, then for each $\phi \in \mathcal{P}$ with $\phi(0) \in Y$, there exists a real number $T_{\sup}(\phi, U(\cdot,\cdot), f)$ such that* (3.1.3) *has a Y-valued solution $u(t)$ on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$.*

(2) *If f is as in* (2) *of Theorem* 3.4.3*, then for each $\phi \in \mathcal{P}$ with $\phi(0) \in Y$, there exists a real number $T_{\sup}(\phi, U(\cdot,\cdot), f)$ such that* (3.1.3) *has a unique Y-valued solution $u(t)$ on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$. Moreover, there is a constant $\widehat{L}(u, \widehat{u}, \tau_0)$ such that* (3.3.9) *holds for every $u(t)$ and $\widehat{u}(t)$ being the Y-valued solution of* (3.1.3) *on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$ with respect to $\phi \in \mathcal{P}$ and on $[0, T_{\sup}(\widehat{\phi}, U(\cdot,\cdot), f))$ to $\widehat{\phi} \in \mathcal{P}$ respectively.*

Proof. A combination of Theorem 3.4.3 and [81, Theorem 4.5.2] yields the existence of the number $T_{\sup}(\phi, U(\cdot,\cdot), f)$ ensuring that (3.1.3) has a (resp. unique) Y-valued solution $u(t)$ on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$ under the hypotheses in (1) (resp. (2)).

On the other hand, let $u(\cdot)$ be a Y valued solution of (3.1.3) on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$. Then by

$$\frac{\partial}{\partial s} U(t,s)v = -U(t,s)A(s)v, \qquad \text{for } v \in Y,\ 0 \le s \le t \le T$$

(cf. [81]) and the hypotheses, we have

$$\frac{\partial^+}{\partial s} U(t,s)u(s) = U(t,s)f(s, u(s), u_s), \quad 0 \le s \le t < T_{\sup}(\phi, U(\cdot,\cdot), f),$$

that is, $u(\cdot)$ satisfies (4.2). Hence, for each $\phi \in \mathcal{P}$ with $\phi(0) \in Y$, the Y-valued solution of (3.1.3) on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$ is unique under the hypotheses in (2). Moreover, by Theorem 3.4.3 (2), we get the conclusion (2).

□

Parabolicity Assumption (cf., e.g., [70, 81]):

(PA1) For all $t \in [0, T]$, $\mathcal{D}(A(t)) = D$ being dense in X.

(PA2) For every $t \in [0, T]$ and complex number λ with $\mathrm{Re}\lambda \le 0$, $(\lambda + A(t))^{-1}$ exists and satisfies

$$\left\|(\lambda + A(t))^{-1}\right\| \le \frac{\overline{W}}{1 + |\lambda|}, \quad \mathrm{Re}\lambda \le 0,\ t \in [0, T],$$

for a constant $\overline{W}$.

(PA3) There are constants $\alpha \in (0,1]$ and $\underline{W}$ such that

$$\left\|(A(t) - A(s))A(r)^{-1}\right\| \leq \underline{W}\ |t-s|^{\alpha}, \quad t, s, r \in [0,T].$$

Theorem 3.4.6. (Parabolic case) *Let the "Parabolicity Assumption" hold and $\{U(t,s)\}_{0\leq s\leq t\leq T}$ be the evolution system associated with the family $\{A(t)\}_{t\in[0,T]}$. Let $\mathcal{P}$ be an admissible phase space, $f \in C([0,T]\times X\times\mathcal{P}, X)$, and for all $u(\cdot) \in \mathcal{P}^{[0,T]}$, $f(s, u(s), u_s) \in D$ $(s \in [0,T])$ and*

$$\int_0^T \|A(t_0)f(s, u(s), u_s)\| ds < \infty \tag{3.4.2}$$

for some $t_0 \in [0,T]$.

(1) *If f is as in* (1) *of Theorem* 3.4.3*, then for each $\phi \in \mathcal{P}$, there exists a real number $T_{\sup}(\phi, U(\cdot,\cdot), f)$ such that* (3.1.3) *has a classical solution $u(t)$ on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$.*

(2) *If f is as in* (2) *of Theorem* 3.4.3*, then for each $\phi \in \mathcal{P}$, there exists a real number $T_{\sup}(\phi, U(\cdot,\cdot), f)$ such that* (3.1.3) *has a unique classical solution $u(t)$ on $[0,\ T_{\sup}(\phi, U(\cdot,\cdot), f))$. Moreover, let $u(t)$ and $\widehat{u}(t)$ be the classical solutions of* (3.1.3) *on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$ with respect to $\phi \in \mathcal{P}$ and on $[0, T_{\sup}(\widehat{\phi}, U(\cdot,\cdot), f))$ to $\widehat{\phi} \in \mathcal{P}$ respectively. Then* (3.3.9) *holds for a constant $\widehat{L}(u, \widehat{u}, \tau_0)$.*

Proof. From [81, Section 5.2] we know that the evolution system $\{U(t,s)\}_{0\leq s\leq t\leq T}$ satisfies

(i)′ For all $0 \leq s \leq t \leq T$, $U(t,s) : X \to D := \mathcal{D}(A(t))$ (for all $t \in [0,T]$), $t \to U(t,s)$ is strongly differentiable, and $\frac{\partial}{\partial t}U(t,s) \in \mathbf{L}(X)$ being strongly continuous on $0 \leq s < t \leq T$.

(ii)′ For all $0 \leq s \leq t \leq T$,

$$\begin{cases} \dfrac{\partial}{\partial t}U(t,s) = A(t)U(t,s), \\ \|A(t)U(t,s)A^{-1}(s)\| \leq \widetilde{M}, \end{cases} \tag{3.4.3}$$

where $\widetilde{M}$ is a constant.

(iii)′ For every $z \in D$ and $t \in (0, T]$, $U(t, s)z$ is differentiable in s on $0 \leq s \leq t \leq T$, and

$$\frac{\partial}{\partial s} U(t, s)z = -U(t, s)A(s)z. \tag{3.4.4}$$

Thus from Theorem 3.4.3 it follows that for each $\phi \in \mathcal{P}$, there exists a real number $T_{\sup}(\phi, U(\cdot, \cdot), f)$ such that (3.4.1) has a (resp. unique) solution $u(t)$ on $[0, T_{\sup}(\phi, U(\cdot, \cdot), f))$ under the hypotheses in (1) (resp. (2)). We have by (3.4.3),

$$\begin{aligned} \frac{\partial}{\partial t} U(t, s) f(s, u(s), u_s) &= A(t)U(t, s)A(s)^{-1}A(s)A(t_0)^{-1}A(t_0) f(s, u(s), u_s), \\ & \qquad 0 \leq s \leq t \leq T_{\sup}(\phi, U(\cdot, \cdot), f), \end{aligned}$$

and by "Parabolic Assumption", there is a constant $\overline{M}$ such that

$$\|A(t)A(t_0)^{-1}\| \leq \overline{M}, \qquad \text{for each } t \in [0, T_{\sup}(\phi, U(\cdot, \cdot), f)).$$

Therefore, by (3.4.2), we get

$$\begin{aligned} \int_0^t \left\| \frac{\partial}{\partial t} U(t, s) f(s, u(s), u_s) \right\| ds &\leq \int_0^T \left\| \frac{\partial}{\partial t} U(t, s) f(s, \overline{u}^t(s), \overline{u}_s^t) \right\| ds \\ &< \infty, \qquad t \in [0, T_{\sup}(\phi, U(\cdot, \cdot), f)), \end{aligned}$$

where

$$\overline{u}^t(s) = \begin{cases} u(t), & s \in [t, T], \\ u(s), & s \in (-\infty, t]. \end{cases}$$

Hence,

$$\begin{aligned} u'(t) &= A(t)U(t, 0)\phi(0) + f(t, u(t), u_t) + \int_0^t A(t)U(t, s) f(s, u(s), u_s) ds \\ &= A(t)u(t) + f(t, u(t), u_t), \quad t \in [0, T_{\sup}(\phi, U(\cdot, \cdot), f)), \end{aligned}$$

i.e., $u(t)$ is a classical solution of (3.1.3) on $[0, T_{\sup}(\phi, U(\cdot, \cdot), f))$.

Moreover, (i)′ and (3.4.4) imply that a classical solution of (3.1.3) is also a mild solution of (3.4.1). This means (3.1.3) has a unique classical solution for each $\phi \in \mathcal{P}$ under the hypotheses in (2). Another direct consequence of this fact and Theorem 3.4.3 (2) is the conclusion (2).

□

3.5 Applications to the functional integrodifferential equations

In this section, we assume that

(A1) A generates a strongly continuous semigroup on X, and (1.1.2) holds.

(A2) $\mathcal{P}$ is an admissible phase space and $f \in C([0,T] \times X \times \mathcal{P}, X)$.

As known in Chapter 1, there is a strongly continuous operator family $\{R(t)\}_{t\in[0,T]} \subset \mathbf{L}(X)$ such that $R(0) = I$, $R(\cdot)y \in C^1([0,T], X) \cap C([0,T], [\mathcal{D}(A)])$ ($y \in \mathcal{D}(A)$), and (1.1.8) holds. $\{R(t)\}_{t\in[0,T]}$ is called the resolvent family for (3.1.4). See, e.g., [34, 40, 42, 62, 72] and references given there for more information about the resolvent family or the integrodifferential equations without delay.

Definition 3.5.1. (1) A function $u : (-\infty, a) \to X$ is called a *mild solution* of (3.1.4) on $[0, a)$ if it satisfies

$$u(t) = \begin{cases} R(t)\phi(0) + \displaystyle\int_0^t R(t-s)f(s, u(s), u_s)ds, \quad t \in [0, a), \\ \phi(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.5.1}$$

(2) A function $u : (-\infty, a) \to X$ is called a classical solution of (3.1.4) on $[0, a)$ if

$$u \in C^1([0, a), X) \cap C([0, a), [\mathcal{D}(A)])$$

satisfying (3.1.4).

Theorem 3.5.2. *Assume that for every $r > 0$, there exists a constant $H(r)$ such that for each $t \in [\sigma, T]$,*

$$\|f(t, x, \phi) - f(t, y, \psi)\| \le H(r) \left(\|x - y\| + \|\phi - \psi\|_{\mathcal{P}}\right),$$

$$\text{for all } \; x, y \in X, \; \phi, \psi \in \mathcal{P} \; \text{with} \; \max\{\|x\|, \; \|y\|, \; \|\phi\|_{\mathcal{P}}, \; \|\psi\|_{\mathcal{P}}\} \le r. \tag{3.5.2}$$

Then for each $\phi \in \mathcal{P}$, there exists a real number $T_{\sup}(\phi, E(\cdot), f)$ such that (3.1.4) *has a unique mild solution $u(t)$ on $[0, T_{\sup}(\phi, E(\cdot), f))$. Moreover, there is a constant*

$\overline{N}(u,\widehat{u},\tau_0)$ *such that* (3.3.9) *holds for every* $u(t)$ *and* $\widehat{u}(t)$ *being mild solutions of* (3.1.4) *on* $[0,T_{\sup}(\phi,E(\cdot),f))$ *with respect to* $\phi\in\mathcal{P}$ *and on* $[0,T_{\sup}(\widehat{\phi},E(\cdot),f))$ *to* $\widehat{\phi}\in\mathcal{P}$ *respectively.*

Proof. Apply Theorem 3.2.3 to $g(t)=R(t)\phi(0)$ and

$$f(t,s,\cdot,\cdot)=\begin{cases} R(t-s)f(s,\cdot,\cdot), & t\geq s,\\ R(s-t)f(s,\cdot,\cdot), & t<s.\end{cases}$$

□

Theorem 3.5.3. *Let* (3.5.2) *hold and let* $f\in C([0,T]\times X\times\mathcal{P},[\mathcal{D}(A)])$*. Then for each* $\phi\in\mathcal{P}$*, there exists a real number* $T_{\sup}(\phi,E(\cdot),f)$ *such that* (3.1.4) *has a unique classical solution* $u(t)$ *on* $[0,T_{\sup}(\phi,E(\cdot),f))$*. Moreover, let* $u(t)$ *and* $\widehat{u}(t)$ *be classical solutions of* (3.1.4) *on* $[0,T_{\sup}(\phi,E(\cdot),f))$ *with respect to* $\phi\in\mathcal{P}$ *and on* $[0,T_{\sup}(\widehat{\phi},E(\cdot),f))$ *to* $\widehat{\phi}\in\mathcal{P}$ *respectively. Then* (3.3.9) *holds for a constant* $\overline{N}(u,\widehat{u},\tau_0)$.

Proof. By Theorem 3.5.2, we know that for each $\phi\in\mathcal{P}$ with $\phi(0)\in\mathcal{D}(A)$, there exists a real number $T_{\sup}(\phi,E(\cdot),f)$ such that (3.1.4) has a unique mild solution $u(t)$ on $[0,T_{\sup}(\phi,E(\cdot),f))$. Thus by (1.1.8) and (3.5.1), we have, for $t\in[0,T_{\sup}(\phi,E(\cdot),f))$,

$$\begin{aligned}
\frac{du(t)}{dt} &= A\left[R(t)\phi(0)+\int_0^t F(t-s)R(s)\phi(0)ds\right]+f(t,u(t),u_t)\\
&\quad +\int_0^t AR(t-s)f(s,u(s),u_s)ds\\
&\quad +\int_0^t A\int_0^{t-s} F(t-s-\eta)R(\eta)f(s,u(s),u_s)d\eta ds\\
&= Au(t)+f(t,u(t),u_t)\\
&\quad +A\left[\int_0^t F(t-s)\left(R(s)\phi(0)+\int_0^s R(s-\eta)f(s,u(s),u_s)d\eta\right)ds\right]\\
&= A\left[u(t)+\int_0^t F(t-s)u(s)ds\right]+f(t,u(t),u_t).
\end{aligned}$$

This means that $u(t)$ is a classical solution of (3.1.4) on $[0,T_{\sup}(\phi,E(\cdot),f))$.

Moreover, let $u(t)$ be a classical solution of (3.1.4) on $[0, T_{\sup}(\phi, E(\cdot), f))$. Then by (1.1.8), (3.5.1) and the hypotheses, we obtain for $t \in [0, T_{\sup}(\phi, E(\cdot), f))$,

$$
\begin{aligned}
& u(t) - R(t)u(0) \\
= & \int_0^t \frac{\partial}{\partial s} R(t-s)u(s)ds \\
= & -\int_0^t \int_0^{t-s} R(t-s-\eta)AF(\eta)u(s)d\eta ds + \int_0^t R(t-s)A\int_0^s F(s-\eta)u(\eta)d\eta ds \\
& \qquad + \int_0^t R(t-s)f(s,u(s),u_s)ds \\
= & -\int_0^t \int_0^{t-s} R(t-s-\eta)AF(\eta)u(s)d\eta ds + \int_0^t \int_\eta^t R(t-s)AF(s-\eta)u(\eta)dsd\eta \\
& \qquad + \int_0^t R(t-s)f(s,u(s),u_s)ds \\
= & -\int_0^t \int_0^{t-s} R(t-s-\eta)AF(\eta)u(s)d\eta ds + \int_0^t \int_0^{t-\eta} R(t-\mu-\eta)AF(\mu)u(\eta)d\mu d\eta \\
& \qquad + \int_0^t R(t-s)f(s,u(s),u_s)ds \\
= & \int_0^t R(t-s)f(s,u(s),u_s)ds.
\end{aligned}
$$

Therefore, a classical solution of (3.1.4) is also a mild solution of (3.1.4). By Theorem 3.5.2 we get the desired conclusion.

□

Chapter 4

Regularity for abstract functional equations with infinite delay in spaces with the Radon-Nikodym property

In this chapter we investigate the regularity for abstract functional equations with infinite delay. Our attention now focus on (among others) the Cauchy problem for the functional equation (3.1.2) in a Banach space X satisfying the Radon-Nikodym property. Some regularity results are established. Theorems 4.2.6 and 4.2.7 below are entirely new, and others are generalizations of the corresponding results in our papers [57, 59].

4.1 Lipschitz continuity of solutions

This is a preliminary section and in this section X is still a general Banach space. Our purpose is to find some sufficient conditions for the "Lipschitz continuity" of solutions of the problems (4.1.2) and (4.1.7). The results given in this section will be used in the next section.

Let $0 \le \sigma \le T$, and define

$$\mathcal{Q}^{[\sigma,T]} := \Big\{\phi : R^- \to X; \ \text{ there are constants } a_\phi > T \text{ and } L_{\phi,\mathcal{P}} \text{ such that } \phi(\cdot) \text{ is Lipschitz continuous on } [-a_\phi, 0],\ \phi_{-a_\phi} \in \mathcal{P} \text{ and } \left\|\phi_{-a_\phi+\tau} - \phi_{-a_\phi}\right\|_{\mathcal{P}} \le L_{(\phi,\mathcal{P})}\tau \text{ for } \tau \in [0, T-\sigma]\Big\}. \tag{4.1.1}$$

Remark 4.1.1. Clearly, by (H1), we have $\mathcal{Q}^{[\sigma,T]} \subset \mathcal{P}$ and the set

$$\mathcal{Q}_0 = \{\phi(\theta);\ \phi : R^- \to X \text{ is Lipschitz continuous with compact support}\}$$

is a subset of $\mathcal{Q}^{[\sigma,T]}$.

For a typical case of (3.1.1)

$$\begin{cases} u(t) = g(t) + \displaystyle\int_\sigma^t E(t,s) f(s, u(s), u_s) ds \ \ (\sigma \le t \le T), \\ u_\sigma = \phi, \end{cases} \tag{4.1.2}$$

where $\{E(t,s)\}_{\sigma \le t,s \le T} \subset \mathbf{L}(X)$ is a strongly continuous family and $f : [\sigma, T] \times X \times \mathcal{P} \to X$ is a given function, we have

Theorem 4.1.2. *Let $0 \le \sigma < T$, $\mathcal{P}$ be an admissible phase space, and for every $r > 0$ there exist a constant $\widetilde{H}(r)$ such that*

$$\|f(t, x(t), x_t) - f(s, x(s), x_s)\| \le \widetilde{H}(r) \left(|t-s| + \|x(t) - x(s)\| + \|x_t - x_s\|_{\mathcal{P}}\right),$$
$$\text{for any } \ t, s \in [\sigma, T], \text{ and } \ x(t) \in \mathcal{P}^{[\sigma,T]} \text{ with } \ \max_{t\in[\sigma,T]}\{\|x(t)\|,\ \|x_t\|_{\mathcal{P}}\} \le r. \tag{4.1.3}$$

Suppose that

(1) *$\phi \in \mathcal{Q}^{[\sigma,T]}$, $g(t) : [\sigma, T] \to X$ being Lipschitz continuous with $g(\sigma) = \phi(0)$, there is a constant $L_{(E(\cdot,\cdot),f)}$ such that*

$$\int_\sigma^t \|[E(t+\eta, s+\eta) - E(t,s)] f(s, x(s), x_s)\| ds \le \eta L_{(E(\cdot,\cdot),f)}, \quad \text{for } \ t \in [\sigma, T],\ \eta \in [0, T-t],\ x(\cdot) \in \mathcal{P}^{[\sigma,T]}, \tag{4.1.4}$$

and (4.1.2) has a solution $u(t)$ on $[\sigma, T_{\sup}(\sigma, \phi, g, E(\cdot,\cdot), f))$;

or that

(2) $\phi \in \mathcal{P}$, $g(t) : [\sigma, T] \to X$ *being Lipschitz continuous with* $g(\sigma) = \phi(0)$, *there is a constant* $\widetilde{L}_{(E(\cdot,\cdot),f)}$ *such that*

$$\int_\sigma^t \|[E(t+\eta, s) - E(t,s)]f(s, x(s), x_s)\|ds \leq \eta \widetilde{L}_{(E(\cdot,\cdot),f)}, \quad \text{for} \quad t \in [\sigma, T],\ \eta \in [0, T-t],\ x(\cdot) \in \mathcal{P}^{[\sigma,T]}, \tag{4.1.5}$$

and (4.1.2) *has a solution* $u(t)$ *on* $[\sigma, T_{\sup}(\sigma, \phi, g, E(\cdot,\cdot), f))$.

Then $u(t)$ *is Lipschitz continuous on* $[\sigma, \tau_0]$ *for every* $\tau_0 \in [\sigma, T_{\sup}(\sigma, \phi, g, E(\cdot,\cdot), f))$.

Proof. The proof of case (1).

Let L_g be the Lipschitz constant for g and $\tau_0 \in [\sigma, T_{\sup}(\sigma, \phi, g, E(\cdot,\cdot), f))$. Then by (4.1.3) and (4.1.4) we deduce that for each $t \in [\sigma, \tau_0]$, $\eta \in [0, \tau_0 - t]$,

$$\begin{aligned}
&\|u(t+\eta) - u(t)\| \\
\leq\ & \|g(t+\eta) - g(t)\| + \int_\sigma^{\sigma+\eta} \|E(t+\eta, s)f(s, u(s), u_s)\|ds \\
&+ \left\| \int_{\sigma+\eta}^{t+\eta} E(t+\eta, s)f(s, u(s), u_s)ds - \int_\sigma^t E(t,s)f(s, u(s), u_s)ds \right\| \\
\leq\ & \|g(t+\eta) - g(t)\| + \int_\sigma^{\sigma+\eta} \|E(t+\eta, s)f(s, u(s), u_s)\|ds \\
&+ \int_\sigma^t \|E(t+\eta, s+\eta)\left[f\left(s+\eta, u(s+\eta), u_{s+\eta}\right) - f(s, u(s), u_s)\right]\|ds \\
&+ \int_\sigma^t \|[E(t+\eta, s+\eta) - E(t,s)]f(s, u(s), u_s)\|ds \\
\leq\ & \left[L_g + \max_{t\in[\sigma,T]} \|E(t)\| \max_{t\in[\sigma,\tau_0]} \|f(t, u(t), u_t)\| + L_{(E(\cdot,\cdot),f)}\right]\eta \\
&+ \widetilde{H}\left(\max_{[\sigma,\tau_0]} \{\|u(t)\|,\ \|u_t\|_{\mathcal{P}}\}\right) \max_{t\in[\sigma,T]} \|E(t)\| \\
&\qquad \cdot \int_\sigma^t \left[\eta + \|u(s+\eta) - u(s)\| + \|u_{s+\eta} - u_s\|_{\mathcal{P}}\right] ds \Bigg\}.
\end{aligned}$$

Noting $\phi \in \mathcal{Q}^{[\sigma,T]}$ and letting L_ϕ be the Lipschitz constant for ϕ on $[-a_\phi, 0]$, we obtain, by (4.1.2) and (H2), for every $s \in [\sigma, t]$,

$$
\begin{aligned}
& \|u_{s+\eta} - u_s\|_{\mathcal{P}} \\
\leq\ & K(s + a_\phi - \sigma) \sup_{\nu \in [-a_\phi+\sigma, s]} \|u(\eta+\nu) - u(\nu)\| \\
& \qquad + M(s + a_\phi - \sigma) \left\|\phi_{-a_\phi+\eta} - \phi_{-a_\phi}\right\|_{\mathcal{P}} \\
\leq\ & K(s + a_\phi - \sigma) \Bigg[\sup_{\nu \in [-a_\phi, -\eta]} \|\phi(\eta+\nu) - \phi(\nu)\| \\
& \qquad + \sup_{\nu \in [-\eta+\sigma, \sigma]} \|u(\nu+\eta) - \phi(\nu-\sigma)\| \\
& \qquad\quad + \sup_{\nu \in [\sigma, s]} \|u(\eta+\nu) - u(\nu)\| \Bigg] + M(s + a_\phi - \sigma) L_{(\phi,\mathcal{P})} \eta \\
\leq\ & K(s + a_\phi - \sigma) \Bigg\{ L_\phi \eta + \sup_{\nu \in [-\eta+\sigma, \sigma]} \|g(\nu+\eta) - g(\sigma)\| \\
& \qquad + \sup_{\nu \in [-\eta+\sigma, \sigma]} \|\phi(0) - \phi(\nu-\sigma)\| \\
& \qquad + \sup_{\nu \in [-\eta+\sigma, \sigma]} \left\| \int_\sigma^{\eta+\nu} E(\nu+\eta, \mu) f(\mu, u(\mu), u_\mu) d\mu \right\| \\
& \qquad + \sup_{\nu \in [\sigma, s]} \|u(\eta+\nu) - u(\nu)\| \Bigg\} + \sup_{s \in [\sigma, T]} M(s + a_\phi - \sigma) L_{(\phi,\mathcal{P})} \eta \\
\leq\ & \Bigg\{ \max\nolimits_{s \in [\sigma, T]} K(s + a_\phi - \sigma) \sup_{s \in [\sigma, T]} M(s + a_\phi - \sigma) L_{(\phi,\mathcal{P})} \\
& \qquad \left[2L_\phi + L_g + \max_{t,s \in [\sigma, T]} \|E(t,s)\| \max_{t \in [\sigma, \tau_0]} \|f(t, u(t), u_t)\| \right] \Bigg\} \eta \\
& \qquad + \max\nolimits_{s \in [\sigma, T]} K(s + a_\phi - \sigma) \sup\nolimits_{\nu \in [\sigma, s]} \|u(\eta+\nu) - u(\nu)\|.
\end{aligned}
$$

As a consequence, there are constants $\overline{H}$ and $\underline{H}$ such that

$$\sup_{\nu \in [\sigma, t]} \|u(\nu+\eta) - u(\nu)\| \leq \overline{H} \eta + \underline{H} \int_\sigma^t \sup_{\nu \in [\sigma, s]} \|u(\nu+\eta) - u(\nu)\| ds.$$

Using Gronwall-Bellman's inequality we have

$$\sup_{\nu \in [\sigma, t]} \|u(\nu+\eta) - u(\nu)\| \leq \widehat{H} \eta, \quad t \in [\sigma, \tau_0],\ \eta \in [0, \tau_0 - t],$$

for a constant $\widehat{H}$. This implies that $u(t)$ is Lipschitz continuous on $[\sigma, \tau_0]$.

The proof of case (2).

Fix $\tau_0 \in [\sigma, T_{\sup}(\sigma, \phi, g, E(\cdot,\cdot), f))$ and let L_g be the Lipschitz constant for g. Then by (4.1.5) we get for each $t \in [\sigma, \tau_0)$, $\eta \in (0, \tau_0 - t)$,

$$\begin{aligned}
& \|u(t+\eta) - u(t)\| \\
\leq \ & \|g(t+\eta) - g(t)\| + \int_t^{t+\eta} \|E(t+\eta, s) f(s, u(s), u_s)\| ds \\
& \quad + \int_\sigma^t \|[E(t+\eta, s) - E(t,s)] f\left(s, u(s), u_s\right)\| ds \\
\leq \ & \left(L_g + \max_{t,s \in [\sigma, T]} \|E(t,s)\| \max_{t \in [\sigma, \tau_0]} \|f(t, u(t), u_t)\| + \widetilde{L}_{(E(\cdot,\cdot), f)} \right) \eta,
\end{aligned}$$

i.e., the solution $u(t)$ of (4.1.2) (with respect to every $\phi \in \mathcal{P}$) is Lipschitz continuous on $[\sigma, \tau_0]$.

□

Corollary 4.1.3. *Let $0 \leq \sigma < T$ and $\mathcal{P}$ be an admissible phase space.*

(1) *Let $f \in C([\sigma, T] \times [\sigma, T] \times X \times \mathcal{P}, X)$ satisfying that for every $r > 0$, there exist a constant $H(r)$ such that for each $s \in [\sigma, T]$,*

$$\|f(s, x, \phi) - f(s, y, \psi)\| \leq H(r) \left(\|x - y\| + \|\phi - \psi\|_{\mathcal{P}}\right),$$
$$\text{for all } \ x, y \in X, \ \phi, \psi \in \mathcal{P} \text{ with } \ \max\{\|x\|, \ \|y\|, \ \|\phi\|_{\mathcal{P}}, \ \|\psi\|_{\mathcal{P}}\} \leq r, \tag{4.1.6}$$

and (4.1.3). Then for every $\phi \in \mathcal{Q}^{[\sigma,T]}$, $g(t) : [\sigma, T] \to X$ being Lipschitz continuous with $g(\sigma) = \phi(0)$ and strongly continuous family $\{E(t)\}_{\sigma \leq t \leq T} \subset$ $\mathbf{L}(X)$, the solution of

$$\begin{cases} u(t) = g(t) + \displaystyle\int_\sigma^t E(t-s) f(s, u(s), u_s) ds, \quad t \in [\sigma, T], \\ u_\sigma = \phi, \end{cases} \tag{4.1.7}$$

on $[\sigma, T_{\sup}(\sigma, \phi, g, E(\cdot), f))$ is Lipschitz continuous on $[\sigma, \tau_0]$ for every $\tau_0 \in [\sigma,$ $T_{\sup}(\sigma, \phi, g, E(\cdot), f))$.

(2) *Let $\{E(t)\}_{\sigma \leq t \leq T}$ be a local C-regularized semigroup and $C^{-1} f \in C([\sigma, T] \times$ $[\sigma, T] \times X \times \mathcal{P}, X)$. Suppose that there is a Kamke function $\mathcal{K}(\cdot,\cdot,\cdot)$ on $[0, T] \times$*

$[0,a]\times[0,\max_{t\in[0,T]}K(t)a]$ *for some* $a>0$ *such that for every bounded set* $B\in X$ *and* $\Omega\in\mathcal{P}$,

$$\alpha(C^{-1}f(\{s\}\times B\times\Omega))\le\mathcal{K}(s,\alpha(B),\alpha(\Omega)),\quad a.e.\ s\in[0,T],$$

and that $\varpi(t)\equiv 0$ *is the unique nonnegative absolutely continuous solution to the differential equation*

$$\varpi'(t)=2\overline{\lim}_{\delta\uparrow 0}\|E(\delta)\|\sup_{t\in[0,T]}\beta(E(t))\mathcal{K}(t,\varpi(t),K(t)\varpi(t)),\quad t\in(0,T]\quad(4.1.8)$$

satisfying

$$\lim_{t\uparrow\sigma}\frac{\varpi(t)}{t-\sigma}=\varpi(\sigma)=0.$$

Then for every $\phi\in\mathcal{Q}^{[\sigma,T]}$ *with* $\phi(0)\in\mathcal{R}(C)$, *and* $g(t):[\sigma,T]\to X$ *being Lipschitz continuous with* $g(\sigma)=\phi(0)$, *the solution of* (4.1.2) *on* $[\sigma,T_{\sup}(\sigma,\phi,g,E(\cdot),f))$ *is Lipschitz continuous on* $[\sigma,\tau_0]$ *for every* $\tau_0\in[\sigma,T_{\sup}(\sigma,\phi,g,E(\cdot),f))$.

Proof. The proof of case (1).

Let

$$E(t,s)=\begin{cases}E(t-s), & t\ge s,\\ E(s-t), & t<s.\end{cases}\qquad(4.1.9)$$

Then (4.1.4) holds. This, together with Theorems 3.2.3 and 4.1.2 (1), implies (1).

The proof of case (2).

From the proof of Theorem 3.3.9, we see that (4.1.7) has a solution for every $\phi\in\mathcal{Q}^{[\sigma,T]}$ with $\phi(0)\in\mathcal{R}(C)$, and $g(t):[\sigma,T]\to X$ with $g(\sigma)=\phi(0)$. Thus by (4.1.7) and 4.1.2 (1), we get (2).

□

4.2 Regularity

We begin with the following definition of "strong solutions" of (3.1.2).

Definition 4.2.1. A function $u:(-\infty,a)\to X$ is called a *strong solution* of (3.1.2) if u is absolutely continuous on $[0,a)$ and differentiable a.e. on $[0,a)$ such that $u'(\cdot)\in L^1([0,a),X)$ satisfying (3.1.2) a.e. on $[0,a)$.

Theorem 4.2.2. *Let $T > 0$ and $\mathcal{P}$ be an admissible phase space. Let A be closed and has a local E-existence family $\{E(t)\}_{t\in[0,T]}$ satisfying that for each $z \in \mathcal{D}(A)$, $E(\cdot)z$ is an absolutely continuous X-valued function on $[0,T]$. Let* (3.3.2) *hold and $\widetilde{f} \in C([0,T]\times X \times \mathcal{P}, X)$ satisfying* (4.1.3) *for $\sigma = 0$. If X satisfies the Radon-Nikodym property, then for each $\phi \in \mathcal{Q}^{[0,T]}$ with $Ez = \phi(0)$ ($z \in \mathcal{D}(A)$), the corresponding mild solution of* (3.1.2) *(if exists) is a strong solution of* (3.1.2).

Proof. Let $\phi \in \mathcal{Q}^{[0,T]}$ with $Ez = \phi(0)$ for a $z \in \mathcal{D}(A)$, and let $u(t)$ be the corresponding mild solution of (3.1.2) on $[0, T_{\sup}(\phi))$.

By the Radon-Nikodym property of X, we get for all $z \in \mathcal{D}(A)$, $E(t)z$ is differentiable a.e. $t \in [0, T_{\sup}(\phi))$. Arguing as in the proof [30, Proposition 2.7] we deduce that $E(t)z \in \mathcal{D}(A)$ for a.e. $t \in [0, T_{\sup}(\phi))$ and

$$\int_0^t AE(s)z = E(t)z - Ez, \qquad \text{a.e.} \quad t \in [0, T_{\sup}(\phi)). \tag{4.2.1}$$

Moreover, letting $E(t,s)$ as in (4.1.9) and using Theorem 4.1.2 (1) and the Radon-Nikodym property of X, we have

$$u(t) \ \text{ is differentiable a.e. on } \ [0, T_{\sup}(\phi)). \tag{4.2.2}$$

By (3.3.1),

$$\begin{aligned} A\int_0^{t-s} E(\tau)\widetilde{f}(s,u(s),u_s)d\tau &= E(t-s)\widetilde{f}(s,u(s),u_s)ds - f(s,u(s),u_s)ds, \\ & \qquad 0 \le s \le t \le T_{\sup}(\phi). \end{aligned}$$

This, together with the closedness of A, implies that for $0 \le s \le t \le T_{\sup}(\phi)$,

$$\begin{aligned} & A\int_0^t \int_0^\tau E(\tau - s)\widetilde{f}(s,u(s),u_s)dsd\tau \\ = \ & A\int_0^t\int_0^{t-s} E(\tau)\widetilde{f}(s,u(s),u_s)d\tau ds \\ = \ & \int_0^t [E(t-s)\widetilde{f}(s,u(s),u_s)ds - f(s,u(s),u_s)]ds. \end{aligned}$$

Hence, by (4.2.1), we infer that

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s,u(s),u_s)ds + z, \quad 0 \le t \le T_{\sup}(\phi). \tag{4.2.3}$$

Using (4.2.2), (4.2.3) and the closedness of A, we obtain $u(t) \in \mathcal{D}(A)$ for a.e. $t \in [0, T_{\sup}(\phi))$ and

$$Au(t) = u'(t) - f(t, u(t), u_t), \qquad \text{a.e. } t \in [0, T_{\sup}(\phi)).$$

This means that $u(t)$ is a strong solution of (3.1.2).

□

A direct corollary of Theorem 4.2.2 and Theorem 3.3.8 is

Corollary 4.2.3. *Let the assumptions of Theorem* 4.2.2 *hold, and let* $\{E(t)\}_{t\in[0,T]}$ *and* $\widetilde{f}$ *satisfy the condition* (1) (*resp.* (2)) *of Theorem* 3.3.8. *If X satisfies the Radon-Nikodym property, then for each* $\phi \in \mathcal{Q}^{[0,T]}$ *with* $Ez = \phi(0)$ $(z \in \mathcal{D}(A))$, *there exists a real number* $T_{\sup}(\phi)$ *such that* (3.1.2) *has a* (*resp. a unique*) *strong solution* $u(t)$ *on* $[0, T_{\sup}(\phi))$.

Remark 4.2.4. Theorem 4.2.2 and Corollary 4.2.3 is new even for the corresponding case without delay (cf. [30]).

When $\{E(t)\}_{t\in[0,T]}$ is a local C-regularized semigroup, Theorem 4.2.2 can be improved as follows.

Theorem 4.2.5. *Let* $T > 0$ *and A be the generator of a local C-regularized semigroup* $\{E(t)\}_{t\in[0,T]}$. *Let $\mathcal{P}$ be an admissible phase space and* $\widetilde{f} \in C([0,T]\times X\times\mathcal{P}, X)$ *satisfying* (4.1.3) *for* $\sigma = 0$. *If X satisfies the Radon-Nikodym property, then for each* $\phi \in \mathcal{Q}^{[0,T]}$ *with* $Ez = \phi(0)$ $(z \in \mathcal{D}(A))$, *the corresponding mild solution of* (3.1.2) (*if exists*) *is a classical solution of* (3.1.2).

Proof. Let $\phi \in \mathcal{Q}^{[0,T]}$ with $Ez = \phi(0)$ for a $z \in \mathcal{D}(A)$, and let $u(t)$ be the corresponding mild solution of (3.1.2) on $[0, T_{\sup}(\phi))$.

Since $\{E(t)\}_{t\in[0,T]}$ is a local C-regularized semigroup, we have $E(t)z$ is differentiable in $[0, T_{\sup}(\phi))$ for every $z \in \mathcal{D}(A)$. On the other hand, by virtue of the Radon-Nikodym property of X, (4.1.3) and Theorem 4.1.2, we obtain $\widetilde{f}(s, u(s), u_s)ds$ is differentiable a.e. $t \in [0, T_{\sup}(\phi))$. Therefore, it can be proved that

$$t \to \int_0^t E(t-s)\widetilde{f}(s, u(s), u_s)ds$$

is differentiable in $[0, T_{\sup}(\phi))$. This implies Theorem 4.2.5 is true.

□

Moreover, we can obtain the following two results if the related operator family is supposed to have a good property.

Theorem 4.2.6. *Let $T > 0$, $\{U(t,s)\}_{0\le s\le t\le T} \subset \mathbf{L}(X)$ being a Lipschitz evolution system (cf. [65, 66]), i.e., satisfying*

$$\|U(t,s) - I\| \le (t-s)\overline{\overline{H}}e^{\omega(t-s)}, \quad 0 \le s \le t \le T, \tag{4.2.4}$$

for some constants $\overline{\overline{H}}$, $\omega \ge 0$. Let $\mathcal{P}$ be an admissible phase space and $f \in C([0,T] \times X \times \mathcal{P}, X)$ satisfying (4.1.3) (for $\sigma = 0$). Then for each $\phi \in \mathcal{P}$, the solution $u(t)$ of (3.4.1) (if exists) on $[0, T_{\sup}(\phi))$ is Lipschitz continuous on $[\sigma, \tau_0]$ for every $\tau_0 \in [\sigma, T_{\sup})$, and is differentiable a.e. $t \in [0, T_{\sup}(\phi))$ when X satisfies the Radon-Nikodym property.

Moreover, if $U(t,0)\phi(0)$ is differentiable in $t \in [0, T_{\sup}(\phi))$, then $u(t)$ is differentiable in $t \in [0, T_{\sup}(\phi))$ when X satisfies the Radon-Nikodym property.

Proof. Let $\phi \in \mathcal{P}$. By (4.2.4) we get for every $t \in [0,T]$, $\eta \in [0, \tau_0 - T]$,

$$\begin{aligned} \|U(t+\eta,0)\phi(0) - U(t,0)\phi(0)\| &\le \|U(t+\eta,t) - I\|\|U(t,0)\phi(0)\| \\ &\le \overline{\overline{H}}e^T \max_{t\in[0,T]} \|U(t,0)\|\|\phi(0)\|\eta, \end{aligned}$$

and for $t \in [\sigma, T]$, $\eta \in [0, T-t]$, and $x(\cdot) \in \mathcal{P}^{[\sigma,T]}$,

$$\begin{aligned} &\int_0^t \|[U(t+\eta,s) - U(t,s)]f(s,x(s),x_s)\|ds \\ \le\ &\int_0^t \|U(t+\eta,t) - I\|\|U(t,s)f(s,x(s),x_s)\|ds \\ \le\ &T\overline{\overline{H}}e^T \max_{t,s\in[0,T]} \|U(t,s)\| \max_{t\in[0,T]} \|f(t,x(t),x_t)\|\eta, \end{aligned}$$

i.e, (4.1.5) holds. Thus, by Theorem 4.1.2 (2), the solution $u(t)$ of (3.4.1) (if exists) on $[0, T_{\sup}(\phi))$ is Lipschitz continuous on $[\sigma, \tau_0]$ for every $\tau_0 \in [\sigma, T_{\sup})$.

The Radon-Nikodym property of X and (4.1.3) implies that $\widetilde{f}(s, u(s), u_s)ds$ is differentiable a.e. $t \in [0, T_{\sup}(\phi))$. Therefore, it can be proved that

$$t \to \int_0^t U(t,s) f(s, u(s), u_s) ds$$

is differentiable in $[0, T_{\sup}(\phi))$. This, together with (4.2.4), means that $u(t)$ is differentiable a.e. $t \in [0, T_{\sup}(\phi))$ when X satisfies the Radon-Nikodym property.

Moreover, if $U(t, 0)\phi(0)$ is differentiable in $t \in [0, T_{\sup}(\phi))$, then $u(t)$ is differentiable in $t \in [0, T_{\sup}(\phi))$ since X satisfies the Radon-Nikodym property.

□

From [70, Section 5.2], we know that under the "Parabolic Assumption", there is an operator family $\{W(t,s)\}_{0 \leq s \leq t \leq T} \subset \mathbf{L}(X)$ with the properties that it is strongly continuous for $0 \leq s \leq t \leq T$, $W'(t,s) \in \mathbf{L}(X)$ being strongly continuous on $0 \leq s < t \leq T$, and

$$\|W(t,s)\| \leq \underline{M}, \quad \|W'(t,s)\| \leq \widehat{M}(t-s)^{\alpha-1}, \quad 0 \leq s < t \leq T, \tag{4.2.5}$$

for constants $\underline{M} > 0$, $\widehat{M} > 0$ and $\alpha \in (0, 1]$, such that

$$U(t,s) = S^s(t-s) + W(t,s), \quad 0 \leq s \leq t \leq T,$$

where for every $t \in [0, T]$, $\{S^t(s)\}_{s \geq 0}$ is a strongly continuous semigroup on X generated by $A(t)$.

Theorem 4.2.7. *Let $\{W(t,s)\}_{0 \leq s \leq t \leq T}$ be the evolution system as above, and $f \in C([0,T] \times X \times \mathcal{P}, X)$ satisfying (4.1.3) for $\sigma = 0$. Then for each $\phi \in \mathcal{P}$ with $\phi(0) \in D$, the solution $u(t)$ of the Cauchy problem*

$$u(t) = \begin{cases} W(t,0)\phi(0) + \displaystyle\int_0^t W(t,s) f(s, u(s), u_s) ds, & t \in [0, T], \\ \phi(t), & t \in (-\infty, 0] \end{cases} \tag{4.2.6}$$

(if exists) on $[0, T_{\sup}(\phi))$ is Lipschitz continuous on $[\sigma, \tau_0]$ for every $\tau_0 \in [\sigma, T_{\sup})$, and is differentiable in $t \in [0, T_{\sup}(\phi))$ when X satisfies the Radon-Nikodym property.

Proof. Let $\phi \in \mathcal{P}$ and $\tau_0 < T_{\sup}(\phi)$. For $t \in [\sigma, T]$, $\eta \in [0, T-t]$, and $x(\cdot) \in \mathcal{P}^{[\sigma,T]}$, we obtain, from (4.2.5) and (4.2.6),

$$
\begin{aligned}
& \int_0^t \|[W(t+\eta, s) - W(t,s)]f(s, u(s), u_s)\| ds \\
\leq\ & \int_0^t \int_0^\eta \left\| \frac{\partial W}{\partial \mu}(t+\mu, s) f(s, u(s), u_s) \right\| d\mu ds \\
\leq\ & \max_{t \in [0, \tau_0]} \|f(t, u(t), u_t)\| \int_0^\eta \int_0^t \widehat{M}(t+\mu-s)^{\alpha-1} ds d\mu \\
\leq\ & \underline{M}\eta,
\end{aligned}
$$

for a constant $\underline{M}$. This means (4.1.5) holds for $\{W(t,s)\}_{0 \leq s \leq t \leq T}$ and f.

On the other hand, by (3.4.3), we have for $t \in [0, T]$, $\eta \in [0, T-t]$,

$$
\begin{aligned}
& \|W(t+\eta, 0)\phi(0) - W(t, 0)\phi(0)\| \\
\leq\ & [\|A(t_0)U(t_0, 0)A(0)^{-1}A(0)\phi(0)\| + \max_{t \in [0,T]} \|S^0(t)\| \|A(t_0)A(0)^{-1}A(0)\phi(0)\|]\eta \\
\leq\ & \underline{\underline{M}}\eta,
\end{aligned}
$$

where $t_0 \in [t, t+\eta]$ and $\underline{\underline{M}}$ is a constant.

Therefore, by Theorem 4.1.2 (2) we get the desired Lipschitz continuity of the solution $u(t)$ of (4.2.6).

Similar reasoning as in the proof of Theorem 4.2.6 gives that $u(t)$ is differentiable in $t \in [0, T_{\sup}(\phi))$ when X satisfies the Radon-Nikodym property.

□

Chapter 5

Wellposedness of the Cauchy problem for abstract functional equations with infinite delay

In the previous chapter, we investigated the regularity for abstract functional equations with infinite delay in spaces with the Radon-Nikodym property. We are now interested in the wellposedness of (3.1.2) and (3.1.3) in the general setting of Banach spaces. In Chapter 3, we gave a few wellposedness theorems (Theorems 3.3.11, 3.4.5 and 3.4.6) under an assumption (among others) on the range of nonlinear term f. Our objective here is to establish wellposedness theorems for (3.1.2) and (3.1.3) when f is Fréchet differentiable. In Section 1, we introduce a new concept for a continuously differentiable function $\phi \in \mathcal{P}$, called *one-point-property*. In terms of it, we set up a wellposedness result for (3.1.2), which generalizes the corresponding results in [3, 8, 13, 22, 23, 35, 36, 45, 46, 48, 51, 58, 59, 71, 77, 78, 84, 86, 87]). Section 2 is devoted to the nonautonomous problem (3.1.3). The wellposedness result given there is new even for the finite delay case.

5.1 Wellposedness of (3.1.2)

Definition 5.1.1. A continuously differentiable function $\phi \in \mathcal{P}$ is said to have *one-point-property* if there exists a point $a = a(\phi) > 0$ such that $\phi'_{-a} \in \mathcal{P}$ and the derivative of ϕ_t ($\in \mathcal{P}$) at point $t = -a$ in $\mathcal{P}$ is ϕ'_{-a}.

Remark 5.1.2. (1) If $\phi : (-\infty, 0] \to X$ is continuously differentiable with compact support, then ϕ has one-point-property.

(2) Suppose that $\|\phi\|_{\mathcal{P}} \leq \text{const}\|\psi\|_{\mathcal{P}}$ for every ϕ, $\psi \in \mathcal{P}$ with $\|\phi(\theta)\| \leq \|\psi(\theta)\|$ a.e. $\theta \in (-\infty, 0]$. Then ϕ has one-point-property if ϕ ($\in \mathcal{P}$) is continuously differentiable in $(-\infty, 0]$ and there exists $a = a(\phi) > 0$ such that ϕ_{-a} and $\phi'_{-a} \in \mathcal{P}$.

Theorem 5.1.3. *Let A have a local E-existence family $\{E(t)\}_{t\in[0,T]}$. Let $\mathcal{P}$ be an admissible phase space and $\widetilde{f}$ be continuously differentiable from $[0,T] \times X \times \mathcal{P}$ into X. Then* (3.1.2) *has a unique classical solution for any $\phi \in \mathcal{Q}_0 = \{\phi;\quad \phi : (-\infty, 0] \to X$ with one-point-property, $\phi'(0) = A\phi(0) + f(0, \phi(0), \phi)$, and there is a $z \in \mathcal{D}(A)$ such that $Ez = \phi(0)\}$. Moreover, if $u(t)$ and $\widehat{u}(t)$ are classical solutions of* (3.1.2) *on $[0,T]$ with respect to $\phi \in \mathcal{P}$ and to $\widehat{\phi} \in \mathcal{P}$ respectively, then there is a constant $\overline{\overline{L}}(u, \widehat{u})$ such that*

$$\|u(t) - \widehat{u}(t)\|_{\mathcal{P}[0,T]} \leq \overline{\overline{L}}(u, \widehat{u}) \left(\|\phi(0) - \widehat{\phi}(0)\|_{[\mathcal{R}(E)]} + \|\phi - \widehat{\phi}\|_{\mathcal{P}}\right).$$

Proof. It is clear that $\widetilde{f}$ satisfies

$$\begin{aligned} \|\widetilde{f}(t, x, \phi) - \widetilde{f}(t, y, \psi)\| \quad &\leq \quad L_{\widetilde{f}}\left(\|x - y\| + \|\phi - \psi\|_{\mathcal{P}}\right), \\ &\text{for } t \in [0,T]\ x, y \in X,\ \phi, \psi \in \mathcal{P}, \end{aligned} \tag{5.1.1}$$

for a constant $L_{\widetilde{f}} > 0$. Thus, according to Theorem (3.3.8) (2), we infer that for any $\phi \in \mathcal{Q}_0$, there exists $u(t) \in \mathcal{P}_{\phi}^{[0,T]}$ such that

$$\begin{aligned} u(t) \quad &= \quad \begin{cases} E(t)z + \displaystyle\int_0^t E(t-s)\widetilde{f}(s, u(s), u_s)ds, \quad t \in [0,T], \\ \phi(t), \quad t \in (-\infty, 0], \end{cases} \\ &= \quad \begin{cases} E(t)z + \displaystyle\int_0^t E(t)\widetilde{f}(t-s, u(t-s), u_{t-s})ds, \quad t \in [0,T], \\ \phi(t), \quad t \in (-\infty, 0]. \end{cases} \end{aligned} \tag{5.1.2}$$

We claim that it is sufficient to prove both $u(t)$ (from $[0,T]$ into X) and u_t (from $[0,T]$ into $\mathcal{P}$) are continuously differentiable if we want to show that $u(t)$ is also the classical solution of (3.1.2). In fact, if we know that $u(t)$ and u_t are continuously differentiable, then by [89, Corollary 3.2] we have

$$\begin{aligned} Au(t) &= AE(t)z + E(t)\widetilde{f}(0,\phi(0),\phi) - E\widetilde{f}(t,u(t),u_t) \\ &\quad + \int_0^t E(t-s)\frac{d}{ds}\widetilde{f}(s,u(s),u_s)dr, \end{aligned}$$

$$u'(t) = AE(t)z + E(t)\widetilde{f}(0,\phi(0),\phi) + \int_0^t E(t-s)\frac{d}{ds}\widetilde{f}(s,u(s),u_s)dr.$$

This yields

$$u'(t) = Au(t) + f(t,u(t),u_t).$$

This means $u(t)$ is a classical solution of (1.1).

Write

$$\begin{cases} \widetilde{f}'_{(1)}(s,x,\phi) = \dfrac{\partial}{\partial s}\widetilde{f}(s,x,\phi), \\ \widetilde{f}'_{(2)}(s,x,\phi) = \dfrac{\partial}{\partial x}\widetilde{f}(s,x,\phi), \\ \widetilde{f}'_{(3)}(s,x,\phi) = \dfrac{\partial}{\partial \phi}\widetilde{f}(s,x,\phi). \end{cases}$$

Then

$$\sup_{s\in[0,T]}\left\|\widetilde{f}'_{(2)}(s,u(s),u_s)\right\|, \quad \sup_{s\in[0,T]}\left\|\widetilde{f}'_{(3)}(s,u(s),u_s)\right\| \leq \text{ const.} \tag{5.1.3}$$

We set for each $\tau > 0$ and $\phi \in \mathcal{P}$,

$$\mathcal{P}_\phi^{[0,\tau]} = \left\{ u : (-\infty,\tau] \to X; \quad u\Big|_{[0,\tau]} \in C([0,\tau],X) \text{ and } u_0 = \phi \right\}.$$

Then $\mathcal{P}_\phi^{[0,\tau]}$ is a Banach space under the norm

$$\|u\|_{\mathcal{P}^{[0,\tau]}} := \max_{t\in[0,\tau]}\|u(t)\| + \|\phi\|_{\mathcal{P}}.$$

From $\phi \in \mathcal{Q}_0$ and by (H1) it follows that $\phi' \in \mathcal{P}$. For any $\zeta(t) \in \mathcal{P}_{\phi'}^{[0,T]}$, define

$$(\mathbf{F}(\zeta))(t) = \begin{cases} A(t)E(t)\phi(0) + E(t)\widetilde{f}(0,\phi(0),\phi) + \displaystyle\int_0^t E(t-s)\widetilde{f}'_{(1)}(s,u(s),u_s)ds \\ \qquad + \displaystyle\int_0^t E(t-s)\widetilde{f}'_{(2)}(s,u(s),u_s)\zeta(s)ds \\ \qquad + \displaystyle\int_0^t E(t-s)\widetilde{f}'_{(3)}(s,u(s),u_s)\zeta_s ds, \quad 0 \leq t \leq T, \\ \phi'(t), \quad t \in (-\infty,0]. \end{cases}$$

Then $\mathbf{F}\zeta \in \mathcal{P}_{\phi'}^{[0,T]}$ and $\mathbf{F}$ has a unique fixed point $\zeta(t)$ in $\mathcal{P}_{\phi'}^{[0,T]}$ by standard arguments. Let

$$\zeta_s^\delta = \frac{1}{\delta}(u_{s+\delta} - u_s) - \zeta_s.$$

Now we show that

$$\lim_{\delta \to 0^+} \zeta_s^\delta = 0. \tag{5.1.4}$$

Clearly, $\phi \in \mathcal{Q}_0$ implies that $\phi_{-a+\delta}$ $(\delta > 0)$, ϕ_{-a}, $\phi'_{-a} \in \mathcal{P}$. So we get by (H2), for any $s \in [0, T)$, $\delta \in (0, T - s)$,

$$\begin{aligned}
&\left\|\zeta_s^\delta\right\|_{\mathcal{P}} \\
\leq\ & \max_{t\in[0,T]} K(t+a) \max\left\{ \max_{\eta\in[0,s]} \left\|\frac{1}{\delta}(u(\eta+\delta) - u(\eta)) - \zeta(\eta)\right\|, \right. \\
& \qquad \max_{\eta\in[-\delta,0]} \left\|\frac{1}{\delta}(u(\eta+\delta) - \phi(\eta)) - \phi'(\eta)\right\|, \\
& \qquad \left. \max_{\eta\in[-a,-\delta]} \left\|\frac{1}{\delta}(\phi(\eta+\delta) - \phi(\eta)) - \phi'(\eta)\right\| \right\} \\
& + \sup_{t\in[0,T]} M(t+a) \left\|\frac{1}{\delta}\left(\phi_{-a+\delta} - \phi_{-a}\right) - \phi'_{-a}\right\|_{\mathcal{P}}.
\end{aligned} \tag{5.1.5}$$

Our next task is to estimate every item on the right of (5.1.5).

First, by (5.1.1), (5.1.2) and (3.3.1), we get for each $\eta \in [0, s]$,

$$\begin{aligned}
& u(\eta+\delta) - u(\eta) \\
=\ & E(\eta+\delta)\phi(0) - E(\eta)\phi(0) + \int_0^\delta E(\eta+\delta-\tau)\widetilde{f}(\tau, u(\tau), u_\tau)d\tau \\
& + \int_\delta^{\eta+\delta} E(\eta+\delta-\tau)\widetilde{f}(\tau, u(\tau), u_\tau)d\tau - \int_0^\eta E(\eta-\tau)\widetilde{f}(\tau, u(\tau), u_\tau)d\tau \\
=\ & \int_\eta^{\eta+\delta} A(\tau)E(\tau, 0)\phi(0)d\tau + \int_0^\delta E(\eta+\delta-\tau)\widetilde{f}(\tau, u(\tau), u_\tau)d\tau \\
& + \int_0^\eta E(\eta-\tau)\left\{ \left[\widetilde{f}\left(\tau+\delta, u(\tau+\delta), u_{\tau+\delta}\right) - \widetilde{f}\left(\tau, u(\tau+\delta), u_{\tau+\delta}\right)\right] \right. \\
& + \left[\widetilde{f}\left(\tau, u(\tau+\delta), u_{\tau+\delta}\right) - \widetilde{f}\left(\tau, u(\tau), u_{\tau+\delta}\right)\right] \\
& \left. + \left[\widetilde{f}\left(\tau, u(\tau), u_{\tau+\delta}\right) - \widetilde{f}\left(\tau, u(\tau), u_\tau\right)\right] \right\} d\tau.
\end{aligned}$$

So, for any $\eta \in [0, s]$,

$$
\begin{aligned}
& \left\| \frac{1}{\delta}(u(\eta+\delta) - u(\eta)) - \zeta(\eta) \right\| \\
\leq \; & \left\| \frac{1}{\delta} \int_{\eta}^{\eta+\delta} A(\tau)E(\tau)\phi(0)d\tau - A(\eta)E(\eta)\phi(0) \right\| \\
& + \left\| \frac{1}{\delta} \int_{0}^{\delta} E(\eta+\delta-\tau)\widetilde{f}(\tau, u(\tau), u_\tau)d\tau - E(\eta)\widetilde{f}(0, \phi(0), \phi) \right\| \\
& + \left\| \int_0^\eta E(\eta-\tau) \left[\widetilde{f}'_{(1)}(\tau, u(\tau+\delta), u_{\tau+\delta}) - \widetilde{f}'_{(1)}(\tau, u(\tau), u_\tau) + \omega_1(\tau, \delta) \right] d\tau \right\| \\
& + \left\| \int_0^\eta E(\eta-\tau) \left\{ \left[\widetilde{f}'_{(2)}(\tau, u(\tau), u_{\tau+\delta}) - \widetilde{f}'_{(2)}(\tau, u(\tau), u_\tau) \right] \right. \right. \\
& \qquad \times \left(\frac{1}{\delta}(u(\tau+\delta) - u(\tau)) - \zeta(\tau) \right) \\
& + \left[\widetilde{f}'_{(2)}(\tau, u(\tau), u_{\tau+\delta}) - \widetilde{f}'_{(2)}(\tau, u(\tau), u_\tau) \right] \zeta(\tau) \\
& \left. \left. + \widetilde{f}'_{(2)}(\tau, u(\tau), u_\tau) \left(\frac{1}{\delta}(u(\tau+\delta) - u(\tau)) - \zeta(\tau) \right) + \omega_2(\tau, \delta) \right\} d\tau \right\| \\
& + \left\| \int_0^\eta E(\eta-\tau) \left[\widetilde{f}'_{(3)}(\tau, u(\tau), u_\tau) \left(\frac{1}{\delta}(u_{\tau+\delta} - u_\tau) - \zeta_\tau \right) + \omega_3(\tau, \delta) \right] d\tau \right\|,
\end{aligned}
$$

where $\lim_{\delta \to 0^+} \|\omega_i(\tau, \delta)\| = 0$ $(i = 1, 2, 3)$ which is implied by the continuous differentiability of $\widetilde{f}$. Thus, noting that

$$
\sup_{s \in [0,T]} \left\| \widetilde{f}(s, u(s), u_s) \right\| \leq \text{ const}, \tag{5.1.6}
$$

and using (5.1.1), (5.1.3) and Gronwall-Bellman's inequality, we have

$$
\sup_{\eta \in [0,s]} \left\| \frac{1}{\delta}(u(\eta+\delta) - u(\eta)) - \zeta(\eta) \right\| \leq N_1(\delta) + \text{ const } \int_0^s \sup_{\tau \in [0,s]} \left\| \zeta_\tau^\delta \right\|_{\mathcal{P}} d\tau, \tag{5.1.7}
$$

where $\lim_{\delta \to 0^+} N_1(\delta) = 0$.

Second, observing

$$\begin{aligned}
&\sup_{\eta\in[-\delta,0]}\left\|\frac{1}{\delta}(u(\eta+\delta)-\phi(\eta))-\phi'(\eta)\right\| \\
\le\ &\sup_{\eta\in[-\delta,0]}\left\|\frac{1}{\delta}[u(\eta+\delta)-\phi(0)]-\frac{\eta+\delta}{\delta}\phi'(0)\right. \\
&\qquad\qquad\left.+\frac{1}{\delta}[\phi(0)-\phi(\eta)]+\frac{\eta}{\delta}\phi'(0)+\phi'(0)-\phi'(\eta)\right\| \\
\le\ &\max\left\{\left\|\frac{1}{\delta}[\phi(0)-\phi(-\delta)]-\phi'(-\delta)\right\|,\right. \\
&\qquad\sup_{\eta\in(-\delta,0]}\left\|\frac{\eta+\delta}{\delta}\left[\frac{u(\eta+\delta)-\phi(0)}{\eta+\delta}-\phi'(0)\right]\right\| \\
&\qquad\left.+\sup_{\eta\in(-\delta,0]}\left\|\frac{1}{\delta}[\phi(0)-\phi(\eta)]+\frac{\eta}{\delta}\phi'(0)\right\|+\sup_{\eta\in(-\delta,0]}\|\phi'(0)-\phi'(\eta)\|\right\},
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{\eta\in(-\delta,0]}\left\|\left[\frac{u(\eta+\delta)-\phi(0)}{\eta+\delta}-\phi'(0)\right]\right\| \\
\le\ &\sup_{\eta\in(-\delta,0]}\left\|\frac{1}{\eta+\delta}[E(\eta+\delta)\phi(0)-\phi(0)]-A(0)\phi(0)\right\| \\
&+\sup_{\eta\in(-\delta,0]}\left\|\frac{1}{\eta+\delta}\int_0^{\eta+\delta}E(\eta+\delta-\tau)\widetilde{f}(\tau,u(\tau),u_s)d\tau-\widetilde{f}(0,\phi(0),\phi)\right\|,
\end{aligned}$$

we obtain

$$\lim_{\delta\to0^+}\sup_{\eta\in[-\delta,0]}\left\|\frac{1}{\delta}(u(\eta+\delta)-\phi(\eta))-\phi'(\eta)\right\|=0. \tag{5.1.8}$$

It is easy to see that

$$\lim_{\delta\to0^+}\sup_{\eta\in[-a,-\delta]}\left\|\frac{1}{\delta}(\phi(\eta+\delta)-\phi(\eta))-\phi'(\eta)\right\|=0. \tag{5.1.9}$$

Finally, the one-point-property of ϕ means that

$$\lim_{\delta\to0^+}\left\|\frac{1}{\delta}\left(\phi_{-a+\delta}-\phi_{-a}\right)-\phi'_{-a}\right\|_{\mathcal{P}}=0. \tag{5.1.10}$$

Combining (5.1.7) – (5.1.10) together, we get that for any $s\in[0,T)$ and $\delta\in(0,T-s)$,

$$\sup_{\eta\in[0,t]}\left\|\zeta_\eta^\delta\right\|_{\mathcal{P}}\le N_2(\delta)+\text{const}\int_0^t\sup_{\eta\in[0,s]}\left\|\zeta_\eta^\delta\right\|_{\mathcal{P}}ds,$$

where $\lim_{\delta\to 0^+} N_2(\delta) = 0$. According to Gronwall-Bellman's inequality, (5.1.4) holds. Clearly, (5.1.4) and (5.1.7) imply that

$$\lim_{\delta\to 0^+} \sup_{\eta\in[0,s]} \left\| \frac{1}{\delta}(u(\eta+\delta) - u(\eta)) - \zeta(\eta) \right\| = 0.$$

Hence, $u(t)$ is right continuously differentiable on $[0, T)$.

On the other hand, for any $s \in (0, T]$, $\delta \in (-s, 0)$,

$$\begin{aligned} \left\|\zeta_s^\delta\right\|_{\mathcal{P}} \leq \; & \sup_{s\in[0,T]} K(s+a) \Bigg\{ \sup_{\eta\in[-\delta,s]} \left\| \frac{1}{\delta}(u(\eta+\delta) - u(\eta)) - \zeta(\eta) \right\| \\ & + \sup_{\eta\in[0,-\delta]} \left\| \frac{1}{\delta}(\phi(\eta+\delta) - u(\eta)) - \zeta(\eta) \right\| \\ & + \sup_{\eta\in[-a,0]} \left\| \frac{1}{\delta}(\phi(\eta+\delta) - \phi(\eta)) - \phi'(\eta) \right\| \Bigg\} \\ & + \sup_{s\in[0,T]} M(s+a) \left\| \frac{1}{\delta}\left(\phi_{-a+\delta} - \phi_{-a}\right) - \phi'_{-a} \right\|_{\mathcal{P}}. \end{aligned}$$

Noting

$$\begin{aligned} & \sup_{\eta\in[0,-\delta]} \left\| \frac{1}{\delta}(\phi(\eta+\delta) - u(\eta)) - \zeta(\eta) \right\| \\ \leq \; & \sup_{\eta\in[0,-\delta]} \Bigg\| \frac{1}{\delta}[\phi(\eta+\delta) - \phi(0)] - \frac{\eta+\delta}{\delta}\phi'(0) \\ & \qquad + \phi'(0) - \zeta(\eta) + \frac{1}{\delta}[\phi(0) - u(\eta)] + \frac{\eta}{\delta}\phi'(0) \Bigg\| \\ \leq \; & \max \Bigg\{ \sup_{\eta\in[0,-\delta)} \left\| \frac{\eta+\delta}{\delta} \left[\frac{\phi(\eta+\delta) - \phi(0)}{\eta+\delta} - \phi'(0) \right] \right\| + \sup_{\eta\in[0,-\delta)} \|\zeta(\eta) - \phi'(0)\| \\ & \qquad + \sup_{\eta\in[0,-\delta)} \left\| \frac{1}{\delta}[\phi(0) - u(\eta)] + \frac{\eta}{\delta}\phi'(0) \right\|, \\ & \qquad \left\| \frac{1}{\delta}[\phi(0) - u(-\delta)] - \zeta(0) \right\| + \|\zeta(0) - \zeta(-\delta)\| \Bigg\}, \end{aligned}$$

we conclude by $\phi \in \mathcal{Q}_0$, $\zeta(t) \in \mathcal{P}_{\phi'}^{[0,T]}$, and the right continuous differentiability of $u(t)$ that

$$\lim_{\delta\to 0^-} \sup_{\eta\in[0,-\delta]} \left\| \frac{1}{\delta}(\phi(\eta+\delta) - u(\eta)) - \zeta(\eta) \right\| = 0. \tag{5.1.11}$$

Using the fact that for any $s \in (0,T]$, $\delta \in (-s,0)$, $\eta \in [-\delta, s]$,

$$\begin{aligned}
& u(\eta+\delta) - u(\eta) \\
= \;& E(\eta+\delta)\phi(0) - E(\eta)\phi(0) + \int_{-\delta}^{\eta} E(\eta-\tau)\widetilde{f}(\tau+\delta, u(\tau+\delta), u_{\tau+\delta})d\tau \\
& \quad - \int_0^{\eta} E(\eta-\tau)\widetilde{f}(\tau, u(\tau), u_s)d\tau \\
= \;& -\int_{\eta+\delta}^{\eta} A(\tau)E(\tau)\phi(0)d\tau - \int_0^{-\delta} E(\eta-\tau)\widetilde{f}(\tau, u(\tau), u_\tau)d\tau \\
& + \int_{-\delta}^{\eta} E(\eta-\tau)\left[\widetilde{f}(\tau+\delta, u(\tau+\delta), u_{\tau+\delta}) - \widetilde{f}(\tau, u(\tau), u_\tau)\right] d\tau,
\end{aligned} \tag{5.1.12}$$

we get by similar arguments as in getting (5.1.7) that

$$\sup_{\eta\in[-\delta,s]} \left\| \frac{1}{\delta}(u(\eta+\delta) - u(\eta)) - \zeta(\eta) \right\| \leq N_3(\delta) + \text{const} \int_0^s \sup_{\tau\in[-\delta,s]} \left\|\zeta_\tau^\delta\right\|_{\mathcal{P}} d\tau, \tag{5.1.13}$$

where $\lim_{\delta\to 0^-} N_3(\delta) = 0$. Moreover, by the one-point-property of ϕ, we know that

$$\lim_{\delta\to 0^-} \left\| \frac{1}{\delta}\left(\phi_{-a+\delta} - \phi_{-a}\right) - \phi'_{-a} \right\|_{\mathcal{P}} = 0.$$

Combining (5.1.11) – (5.1.12) and the obvious fact

$$\lim_{\delta\to 0^-} \sup_{\eta\in[-a,0]} \left\| \frac{1}{\delta}(\phi(\eta+\delta) - \phi(\eta)) - \phi'(\eta) \right\| = 0,$$

with the Gronwall-Bellman inequality, we obtain

$$\lim_{\delta\to 0^-} \zeta_s^\delta = 0. \tag{5.1.14}$$

This implies that u_t is left continuously differentiable on $(0,T]$. (5.1.13) shows that $u(t)$ is left continuously differentiable on $(0,T]$.

□

5.2 Wellposedness of (3.1.3)

Theorem 5.2.1. *Let $T > 0$, $\{A(t)\}_{t\in[0,T]}$ be an operator family such that*

(i) *there are constants* M *and* ω *such that*

$$(\omega, \infty) \in \rho(A(t)) \quad \text{for } t \in [0, T],$$

and

$$\left\| \prod_{j=1}^{k} (\lambda - A(t_j))^{-1} \right\| \leq \frac{M}{(\lambda - \omega)^k} \quad \text{for } \lambda > \omega$$

and every finite sequence $\{t_j\}_1^k \subset [0, T]$ $(k \in N)$,

(ii) $\mathcal{D}(A(t)) = D$ *is independent of* $t \in [0, T]$,

(iii) *for every* $x \in D$*, the function* $A(t)x$ *is continuously differentiable in* $[0, T]$.

Let $\mathcal{P}$ *be an admissible phase space and* f *be continuously differentiable from* $[0, T] \times X \times \mathcal{P}$ *into* X. *Then for any* $\phi \in \mathcal{Q}_0 = \{\phi; \quad \phi : (-\infty, 0] \to X$ *with one-point-property,* $\phi(0) \in D$ *and* $\phi'(0) = A(0)\phi(0) + f(0, \phi(0), \phi)\}$, (3.1.3) *has a unique classical solution.*

Moreover, let $u(t)$ *and* $\widehat{u}(t)$ *be classical solutions of* (3.1.3) *for* $\phi \in \mathcal{P}$ *and for* $\widehat{\phi} \in \mathcal{P}$ *respectively. Then there is a constant* $\underline{M}$ *such that*

$$\|u(t) - \widehat{u}(t)\|_{\mathcal{P}[0,\tau_0]} \leq \underline{M} \left(\|\phi(0) - \widehat{\phi}(0)\| + \|\phi - \widehat{\phi}\|_{\mathcal{P}} \right). \tag{5.2.1}$$

Proof. Endowing D with the graph norm of $A(0)$:

$$\|x\|_D := \|x\| + \|A(0)x\|, \quad x \in D,$$

we get a Banach space $(D, \|\cdot\|_D)$. By the hypotheses, we know that there is a $\lambda \in R$ large enough such that

$$J(t) = \lambda I - A(t), \quad t \in [0, T]$$

is an isomorphism of $(D, \|\cdot\|_D)$ onto X for every $t \in [0, T]$, and $J(t)y$ is continuously differentiable in X for any $y \in (D, \|\cdot\|_D)$ and $t \in [0, T]$. So $J^{-1}(t)$ is strongly continuously differentiable in $t \in [0, T]$ and

$$\frac{d}{dt} J^{-1}(t)x = J^{-1}(t)J'(t)J^{-1}(t)x, \quad x \in X, \ t \in [0, T],$$

where $J'(t)$ denotes the strong derivative of $J(t)$.

By virtue of [81, Theorem 7.4], there exists a unique evolution system $\{U(t,s)\}_{0\le s\le t\le T}$ satisfying

$$U(t,s)D \subset D, \quad \text{for } 0 \le s \le t \le T,$$

$$\frac{\partial}{\partial t}U(t,s)y = -A(t)U(t,s)y, \quad \text{for } y \in (D, \|\cdot\|_D), 0 \le s \le t \le T,$$

and

$$\frac{\partial}{\partial s}U(t,s)y = U(t,s)A(s)y, \quad \text{for } y \in (D, \|\cdot\|_D), 0 \le s \le t \le T.$$

The uniform boundedness principle implies that

$$\|J'(t)J^{-1}(t)\| \le \text{const}, \quad \text{for } t \in [0,T],$$

$$\|J(t)U(t,s)J^{-1}(t)\| \le \text{const}, \quad \text{for } 0 \le s \le t \le T.$$

Since f is continuously differentiable from $[0,T] \times X \times \mathcal{P}$ into X, we know that f is uniformly Lipschitz continuous. This implies that for any $\phi \in \mathcal{P}$, there exists $x(t) \in \mathcal{P}_\phi^{[0,T]}$ such that

$$x(t) = \begin{cases} U(t,0)\phi(0) + \displaystyle\int_0^t U(t,s)f(s,x(s),x_s)ds, & t \in [0,T], \\ \phi(t), & t \in (-\infty, 0]. \end{cases} \tag{5.2.2}$$

Now we prove that $x(t)$ is a classical solution of (3.1.3).

Define

$$\widetilde{\mathcal{P}}^{[0,T]} = \left\{ u : (-\infty, T] \to X; \;\; u\Big|_{[0,T]} \in C^1([0,T], X) \;\text{ and }\; u_0 \in \mathcal{P} \right\},$$

endowed with the norm

$$\|u\|_{\widetilde{\mathcal{P}}^{[0,T]}} := \max_{t\in[0,T]} \|u(t)\| + \max_{t\in[0,T]} \|u'(t)\| + \|u_0\|_{\mathcal{P}}.$$

Then $\widetilde{\mathcal{P}}^{[0,T]}$ is a Banach space.

Let $\phi \in \mathcal{P}$ being continuously differentiable in $(-\infty, 0]$, and set

$$\widetilde{\mathcal{P}}_\phi^{[0,T]} := \left\{ u \in \widetilde{\mathcal{P}}^{[0,T]}; \;\; u_0 = \phi, \; u_0' = \phi' \right\}.$$

Then $\widetilde{\mathcal{P}}_\phi^{[0,T]}$ is a nonempty closed convex subset of $\widetilde{\mathcal{P}}^{[0,T]}$.

For every $\phi \in \mathcal{Q}_0$ and $u \in \widetilde{\mathcal{P}}_\phi^{[0,T]}$, define

$$\mu_t(\theta) := \begin{cases} u'(t+\theta), & 0 \le t+\theta \le T, \\ \phi'(t+\theta), & t+\theta < 0. \end{cases}$$

Clearly, $\mu_t \in \mathcal{P}$. Moreover, for each $s \in [0,T)$,

$$\begin{aligned} &\left\| \frac{u_{s+\delta} - u_s}{\delta} - \mu_s \right\|_{\mathcal{P}} \\ \le \;& \max_{t\in[0,T]} K(t+a) \max \left\{ \max_{\eta\in[0,s]} \left\| \frac{1}{\delta}(u(\eta+\delta) - u(\eta)) - u'(\eta) \right\|, \right. \\ &\qquad \max_{\eta\in[-\delta,0]} \left\| \frac{1}{\delta}(u(\eta+\delta) - \phi(\eta)) - \phi'(\eta) \right\|, \\ &\qquad\qquad \left. \max_{\eta\in[-a,-\delta]} \left\| \frac{1}{\delta}(\phi(\eta+\delta) - \phi(\eta)) - \phi'(\eta) \right\| \right\} \\ &+ \sup_{t\in[0,T]} M(t+a) \left\| \frac{1}{\delta}\left(\phi_{-a+\delta} - \phi_{-a}\right) - \phi'_{-a} \right\|_{\mathcal{P}}, \quad \text{for } 0 < \delta < T-s, \end{aligned}$$

and for each $s \in (0,T]$,

$$\begin{aligned} &\left\| \frac{u_{s+\delta} - u_s}{\delta} - \mu_s \right\|_{\mathcal{P}} \\ \le \;& \sup_{t\in[0,T]} K(t+a) \left\{ \sup_{\eta\in[-\delta,s]} \left\| \frac{1}{\delta}(u(\eta+\delta) - u(\eta)) - u'(\eta) \right\| \right. \\ &\qquad + \sup_{\eta\in[0,-\delta]} \left\| \frac{1}{\delta}(\phi(\eta+\delta) - u(\eta)) - u'(\eta) \right\| \\ &\qquad\qquad \left. + \sup_{\eta\in[-a,0]} \left\| \frac{1}{\delta}(\phi(\eta+\delta) - \phi(\eta)) - \phi'(\eta) \right\| \right\} \\ &+ \sup_{t\in[0,T]} M(t+a) \left\| \frac{1}{\delta}\left(\phi_{-a+\delta} - \phi_{-a}\right) - \phi'_{-a} \right\|_{\mathcal{P}}, \quad \text{for } -s < \delta < 0. \end{aligned}$$

Observing that

$$
\begin{aligned}
&\sup_{\eta\in[-\delta,0]}\left\|\frac{1}{\delta}(u(\eta+\delta)-\phi(\eta))-\phi'(\eta)\right\| \\
\leq\ & \max\left\{\left\|\frac{1}{\delta}[\phi(0)-\phi(-\delta)]-\phi'(-\delta)\right\|,\right. \\
&\qquad\sup_{\eta\in(-\delta,0]}\left\|\frac{\eta+\delta}{\delta}\left[\frac{u(\eta+\delta)-\phi(0)}{\eta+\delta}-\phi'(0)\right]\right\| \\
&\qquad\left.+\sup_{\eta\in(-\delta,0]}\left\|\frac{1}{\delta}[\phi(0)-\phi(\eta)]+\frac{\eta}{\delta}\phi'(0)\right\|+\sup_{\eta\in(-\delta,0]}\|\phi'(0)-\phi'(\eta)\|\right\}, \\
\leq\ & \max\left\{\left\|\frac{1}{\delta}[\phi(0)-\phi(-\delta)]-\phi'(0)\| + \|\phi'(-\delta)-\phi'(0)\right\|\right. \\
&\qquad\sup_{\eta\in(-\delta,0]}\left\|\frac{\eta+\delta}{\delta}\left[\frac{u(\eta+\delta)-u(0)}{\eta+\delta}-u'(0)\right]\right\| \\
&\qquad\left.+\sup_{\eta\in(-\delta,0]}\left\|\frac{1}{\delta}[\phi(0)-\phi(\eta)]+\frac{\eta}{\delta}\phi'(0)\right\|+\sup_{\eta\in(-\delta,0]}\|\phi'(0)-\phi'(\eta)\|\right\}, \\
&\qquad\qquad\text{for } s\in[0,T),\ 0<\delta<T-s,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{\eta\in[0,-\delta]}\left\|\frac{1}{\delta}(\phi(\eta+\delta)-u(\eta))-u'(\eta)\right\| \\
\leq\ & \max\left\{\sup_{\eta\in[0,-\delta)}\left\|\frac{\eta+\delta}{\delta}\left[\frac{\phi(\eta+\delta)-\phi(0)}{\eta+\delta}-\phi'(0)\right]\right\|+\sup_{\eta\in[0,-\delta)}\|u'(\eta)-u'(0)\|\right. \\
&\qquad+\sup_{\eta\in[0,-\delta)}\left\|\frac{1}{\delta}[u(0)-u(\eta)]+\frac{\eta}{\delta}\phi'(0)\right\|, \\
&\qquad\left.\left\|\frac{1}{\delta}[u(0)-u(-\delta)]-u'(0)\right\|+\|\phi'(0)-\phi'(-\delta)\|\right\}, \\
&\qquad\qquad\text{for } s\in(0,T],\ -s<\delta<0,
\end{aligned}
$$

we obtain for $s\in[0,T]$,

$$\left\|\frac{u_{s+\delta}-u_\delta}{\delta}-\mu_s\right\|_{\mathcal{P}}\to 0,\qquad \text{as } \delta\to 0,$$

that is, u_t is continuously differentiable in $[0,T]$. Therefore, $f(t,u(t),u_t)$ is continuously differentiable in $[0,T]$. If we set

$$\begin{cases} f'_{(1)}(t,x,\phi) := \dfrac{\partial}{\partial t} f(t,x,\phi), \quad t \in [0,T],\ x \in X,\ \phi \in \mathcal{P} \\ f'_{(2)}(t,x,\phi) := \dfrac{\partial}{\partial x} f(t,x,\phi), \quad t \in [0,T],\ x \in X,\ \phi \in \mathcal{P} \\ f'_{(3)}(t,x,\phi) := \dfrac{\partial}{\partial \phi} f(t,x,\phi), \quad t \in [0,T],\ x \in X,\ \phi \in \mathcal{P}, \end{cases}$$

then for $t \in [0,T],\ x \in X,\ \phi \in \mathcal{P}$,

$$\frac{d}{dt} f(t,u(t),u_t) = f'_{(1)}(t,u(t),u_t) + f'_{(2)}(t,u(t),u_t)u'(t) + f'_{(3)}(t,u(t),u_t)(u_t)'.$$

Take $\phi \in \mathcal{Q}_0$ and for every $u \in \widetilde{\mathcal{P}}_\phi^{[0,T]}$, write

$$(\mathbf{F}u)(t) := \begin{cases} U(t,0)\phi(0) + \displaystyle\int_0^t U(t,s) f(s,u(s),u_s)ds, \quad t \in [0,T], \\ \phi(t), \quad t \in (-\infty, 0]. \end{cases}$$

Observing that for $t \in [0,T]$,

$$\begin{aligned} & \int_0^t U(t,s) f(s,u(s),u_s)ds \\ = \ & -\int_0^t \frac{\partial U(t,s)}{\partial s} J^{-1}(s) f(s,u(s),u_s)ds + \lambda \int_0^t U(t,s) J^{-1}(s) f(s,u(s),u_s)ds \\ = \ & -J^{-1}(t) f(t,u(t),u_t) + U(t,0) J^{-1}(0) f(0,\phi(0),\phi) \\ & + \int_0^t U(t,s) \frac{d}{ds} J^{-1}(s) f(s,u(s),u_s))ds + \lambda \int_0^t U(t,s) J^{-1}(s) f(s,u(s),u_s)ds \\ = \ & -J^{-1}(t) f(t,u(t),u_t) + U(t,0) J^{-1}(0) f(0,\phi(0),\phi) \\ & \qquad + \int_0^t U(t,s) J^{-1}(s) J'(s) J^{-1}(s) f(s,u(s),u_s)ds \\ & + \int_0^t U(t,s) J^{-1}(s) \frac{d}{ds}(f(s,u(s),u_s))ds + \lambda \int_0^t U(t,s) J^{-1}(s) f(s,u(s),u_s)ds, \end{aligned}$$

and that

$$
\begin{aligned}
& \left\| \frac{1}{\delta}[(\mathbf{F})u(\delta) - \phi(0)] - \phi'(0) \right\| \\
= & \left\| \frac{1}{\delta}\left[U(\delta,0)\phi(0) + \int_0^\delta U(\delta,s) f(s,u(s),u_s)ds - \phi(0) \right] - \phi'(0) \right\| \\
= & \left\| \frac{U(\delta,0)\phi(0) - U(0,0)\phi(0)}{\delta} + \frac{1}{\delta}\int_0^\delta U(\delta,s) f(s,u(s),u_s)ds - \phi'(0) \right\| \\
\to & \ \|A(0)\phi(0) + f(0,\phi(0),\phi) - \phi'(0)\| = 0, \quad \text{as } \delta \to 0^+,
\end{aligned}
$$

we have

$$
\begin{aligned}
(\mathbf{F}u)'(t) &= \begin{cases} -J^{-1}(t)J'(t)J^{-1}(t)f(t,u(t),u_t) - J^{-1}(t)\dfrac{d}{dt}(f(t,u(t),u_t)) \\ \qquad +A(t)U(t,0)\phi(0) + A(t)U(t,0)J^{-1}(0)f(0,\phi(0),\phi) \\ \qquad -\displaystyle\int_0^t A(t)U(t,s)J^{-1}(s)J'(s)J^{-1}(s)f(s,u(s),u_s)ds \\ \qquad +J^{-1}(t)J'(t)J^{-1}(t)f(t,u(t),u_t) + J^{-1}(t)\dfrac{d}{dt}(f(t,u(t),u_t)) \\ \qquad -\displaystyle\int_0^t A(t)U(t,s)J^{-1}(s)\frac{d}{ds}(f(s,u(s),u_s))ds \\ \qquad -\lambda\displaystyle\int_0^t A(t)U(t,s)J^{-1}(s)f(s,u(s),u_s)ds \\ \qquad\qquad +\lambda J^{-1}(t)f(t,u(t),u_t), \quad t \in [0,T], \\ \phi'(t), \quad t \in (-\infty,0], \end{cases} \\
&= \begin{cases} A(t)U(t,0)[\phi(0) + J^{-1}(0)f(0,\phi(0),\phi)] + \lambda J^{-1}(t)f(t,u(t),u_t) \\ \qquad -\displaystyle\int_0^t A(t)U(t,s)\left[\lambda + J^{-1}(s)J'(s)\right]J^{-1}(s)f(s,u(s),u_s)ds \\ \qquad -\displaystyle\int_0^t A(t)U(t,s)J^{-1}(s)[f'_{(1)}(s,u(s),u_s) \\ \qquad +f'_{(2)}(s,u(s),u_s)u'(s) + f'_{(3)}(s,u(s),u_s)(u_s)']ds, \quad t \in [0,T], \\ \phi'(t), \quad t \in (-\infty,0]. \end{cases}
\end{aligned}
$$

For each $\phi \in \mathcal{P}$ and $r > 0$, write

$$\overline{B}_r(\phi) := \{\psi \in \mathcal{P};\ \ \|\psi - \phi\|_{\mathcal{P}} \le r\},$$

$$\overline{B}_r(\phi(0)) := \{x \in X;\ \ \|x - \phi(0)\| \le r\}.$$

Obviously, there is a number $r > 0$ such that

$$\begin{aligned} M &:= \max\Big\{f(t,x,\psi),\ f'_{(1)}(t,x,\psi),\ f'_{(2)}(t,x,\psi),\ f'_{(3)}(t,x,\psi); \\ &\qquad\qquad t \in [0,T],\ x \in \overline{B}_r(\phi(0)),\ \psi \in \overline{B}_r(\phi)\Big\} \\ &< \infty. \end{aligned}$$

For every $t \in [0,T]$, let

$$\begin{aligned} \bar{f}(t,x,\psi) &:= f(t,x,\psi)\Big|_{x\in\overline{B}_r(\phi(0)),\ \psi\in\overline{B}_r(\phi)}, \\ \bar{f}_{(i)}(t,x,\psi) &:= f'_{(i)}(t,x,\psi)\Big|_{x\in\overline{B}_r(\phi(0)),\ \psi\in\overline{B}_r(\phi)}, \quad i = 1,2,3. \end{aligned}$$

and let F, $F_{(1)}$, $F_{(2)}$ and $F_{(3)}$ be the extensions of $\bar{f}$, $\bar{f}_{(1)}$, $\bar{f}_{(2)}$ and $\bar{f}_{(3)}$ respectively to the whole space $R \times X \times \mathcal{P}$ such that

$$\max_{t\in R, x\in X, \psi\in\mathcal{P}} \left\{F(t,x,\psi), F_{(1)}(t,x,\psi), F_{(2)}(t,x,\psi), F_{(3)}(t,x,\psi)\right\} \le M.$$

By virtue of [26, Lemma 1.1], we get sequences $\{F^n\}_{n\in N}$, $\{F^n_{(1)}\}_{n\in N}$, $\{F^n_{(2)}\}_{n\in N}$, and $\{F^n_{(3)}\}_{n\in N}$ of locally Lipschitz functions, satisfying

$$\|F^n(t,x,\psi) - F(t,x,\psi)\| \le \frac{1}{n}, \quad n \in N,$$

$$\|F^n_{(i)}(t,x,\psi) - F_{(i)}(t,x,\psi)\| \le \frac{1}{n}, \quad n \in N,\ i = 1,2,3,$$

and

$$\max_{t\in R, x\in X, \psi\in\mathcal{P}} \left\{F^n(t,x,\psi), F^n_{(1)}(t,x,\psi), F^n_{(2)}(t,x,\psi), F^n_{(3)}(t,x,\psi)\right\} \le M + 1.$$

For each $n \in N$, $\phi \in \mathcal{Q}_0$ and $u \in \widetilde{\mathcal{P}}_\phi^{[0,T]}$, define

$$(\mathbf{G}^n u)(t) \quad := \quad \begin{cases} [U(t,0)-I][\phi(0)+J^{-1}(0)f(0,\phi(0),\phi)]+\phi(0) \\ \qquad +\lambda \int_0^t J^{-1}(s)F^n(s,u(s),u_s)ds - \int_0^t \int_0^\tau A(\tau)U(\tau,s) \\ \qquad \times \left[\lambda + J^{-1}(s)J'(s)\right] J^{-1}(s)F^n(s,u(s),u_s)dsd\tau \\ \qquad - \int_0^t \int_0^\tau A(\tau)U(\tau,s)J^{-1}(s)[F^n_{(1)}(s,u(s),u_s) \\ \qquad +F^n_{(2)}(s,u(s),u_s)u'(s)+F^n_{(3)}(s,u(s),u_s)(u_s)']dsd\tau, \\ \qquad\qquad\qquad\qquad t\in[0,T], \\ \phi(t), \quad t \in (-\infty, 0]. \end{cases}$$

Take $r_n \leq r$ such that

$$\|F^n(t,x,\psi)-F^n(t,y,\zeta)\| \leq L(n,r_n)(\|x-y\|+\|\psi-\zeta\|),$$

$$t\in[0,T],\ x,y\in \overline{B}_{r_n}(\phi(0)),\ \psi,\zeta\in \overline{B}_{r_n}(\phi),$$

$$\|F^n_{(i)}(t,x,\psi)-F^n_{(i)}(t,y,\zeta)\| \leq L(n,r_n)(\|x-y\|+\|\psi-\zeta\|),$$

$$t\in[0,T],\ x,y\in \overline{B}_{r_n}(\phi(0)),\ \psi,\zeta\in \overline{B}_{r_n}(\phi),\ i=1,2,3,$$

and $\bar{r}_n \leq r_n \leq r$ such that

$$\bar{r}_n \left(\max_{t\in[0,T+a]} K(t) + \sup_{t\in[0,T+a]} M(t) \right) \leq r_n.$$

Moreover, for each $b>0$, $r>0$, define

$$\widetilde{\mathcal{P}}_\phi^{[0,b]}(r) \quad = \quad \Big\{ u\in \widetilde{\mathcal{P}}_\phi^{[0,b]}; \quad \text{for every } t\in[0,b],\ u(t)\in \overline{B}_r(\phi(0)) \\ \qquad \text{and} \quad u'(t)\in \overline{B}_r(A(0)[\phi(0)+J^{-1}(0)f(0,\phi(0),\phi)]) \Big\},$$

then $\widetilde{\mathcal{P}}_\phi^{[0,b]}(r)$ is nonempty, convex and closed. Based on our analysis above, we obtain for fixed $\phi \in \mathcal{Q}_0$ and $n \in N$, there is a real number $b_n > 0$ such that for every

$u \in \widetilde{\mathcal{P}}_\phi^{[0,b_n]}(\bar{r}_n)$,

$$\|(\mathbf{G}^n u)(t) - \phi(0)\| \leq \bar{r}_n, \quad t \in [0, b_n],$$

$$\|[(\mathbf{G}^n u)(t)]' - A(0)[\phi(0) + J^{-1}(0) f(0, \phi(0), \phi)]\| \leq \bar{r}_n, \quad t \in [0, b_n],$$

by noting that when b_n is sufficiently small we have

$$\begin{aligned} \max_{s\in[0,b_n]} \|(u_s)'\|_{\mathcal{P}} &\leq \max_{s\in[0,b_n]} \|(u_s)' - \mu_s\|_{\mathcal{P}} + \max_{s\in[0,b_n]} \|\mu_s\|_{\mathcal{P}} \\ &\leq 1 + \max_{t\in[0,T]} K(t) \max_{s\in[0,b_n]} \|u'(s)\| + \sup_{t\in[0,T]} M(t) \|\phi'\|_{\mathcal{P}}. \end{aligned}$$

This means that

$$\mathbf{G}^n \left\{ \widetilde{\mathcal{P}}_\phi^{[0,b_n]}(\bar{r}_n) \right\} \subset \widetilde{\mathcal{P}}_\phi^{[0,b_n]}(\bar{r}_n).$$

In addition, for every $u, v \in \widetilde{\mathcal{P}}_\phi^{[0,b_n]}(\bar{r}_n)$, $t \in [0, b_n]$,

$$\begin{aligned} &\|(\mathbf{G}^n u)(t) - (\mathbf{G}^n v)(t)\| \\ \leq\ & \overline{V} \Bigg[\max_{s\in[0,b_n]} \|F^n(s, u(s), u_s) - F^n(s, v(s), v_s)\| + \|F^n_{(1)}(s, u(s), u_s) - F^n_{(1)}(s, v(s), v_s)\| \\ &\qquad + \|F^n_{(2)}(s, u(s), u_s) u'(s) - F^n_{(2)}(s, v(s), v_s) v'(s)\| \\ &\qquad + \left\| F^n_{(3)}(s, u(s), u_s)(u_s)' - F^n_{(3)}(s, v(s), v_s)(v_s)' \right\| \Bigg] \\ \leq\ & \left[2 + \max_{s\in[0,b_n]} \|v'(s)\| + \max_{s\in[0,b_n]} \|(v_s)'\| \right] \overline{V} L(n, r_n) \max_{s\in[0,b_n]} [\|u(s) - v(s)\| + \|u_s - v_s\|_{\mathcal{P}}] \\ &\qquad + \overline{V}(M+1) \left[\max_{s\in[0,b_n]} \|u'(s) - v'(s)\| + \max_{s\in[0,b_n]} \|(u_s)' - (v_s)'\| \right] \\ \leq\ & \left[2 + \max_{s\in[0,b_n]} \|v'(s)\| + \max_{s\in[0,b_n]} \|(v_s)'\| \right] \overline{V} L(n, r_n) \\ &\qquad \times \left(1 + \max_{s\in[0,T]} K(s) \right) \max_{s\in[0,b_n]} \|u(s) - v(s)\| \\ &\qquad + \overline{V}(M+1) \left(1 + \max_{s\in[0,T]} K(s) \right) \max_{s\in[0,b_n]} \|u'(s) - v'(s)\|, \end{aligned}$$

and similarly, for every $u, v \in \widetilde{\mathcal{P}}_{\phi}^{[0,b_n]}(\bar{r}_n)$,

$$
\begin{aligned}
& \|(\mathbf{G}^n u)'(t) - (\mathbf{G}^n v)'(t)\| \\
\leq \;& \underline{V}\Big[\max_{s\in[0,b_n]} \|u(s) - v(s)\| + \max_{s\in[0,b_n]} \|u'(s) - v'(s)\|\Big], \quad t \in [0, b_n],
\end{aligned}
$$

where $\overline{V}$, $\underline{V}$ are constants. Therefore, $\mathbf{G}^n$ is uniformly Lipschitz continuous in $\widetilde{\mathcal{P}}_{\phi}^{[0,b_n]}(\bar{r}_n)$. Consequently, we know that there is $u^n(\cdot) \in \widetilde{\mathcal{P}}_{\phi}^{[0,b_n]}(\bar{r}_n)$ such that

$$
(\mathbf{G}^n u^n)(t) = u^n(t), \quad t \in (-\infty, b_n], \tag{5.2.3}
$$

and satisfying

$$
[u^n(t)]' = \begin{cases}
A(t)U(t,0)[\phi(0) + J^{-1}(0) f(0, \phi(0), \phi)] + \lambda J^{-1}(t) F^n(t, u^n(t), u^n_t) \\
\quad + \displaystyle\int_0^t A(t)U(t,s)\left[\lambda + J^{-1}(s)J'(s)\right] J^{-1}(s) F^n(s, u^n(s), u^n_s) ds \\
\quad - \displaystyle\int_0^t A(t)U(t,s)J^{-1}(s)[F^n_{(1)}(s, u^n(s), u^n_s) \\
\quad + F^n_{(2)}(s, u^n(s), u^n_s)(u^n)'(s) + F^n_{(3)}(s, u^n(s), u^n_s)(u^n_s)']ds, \\
\qquad\qquad t \in [0, T], \\
\phi'(t), \quad t \in (-\infty, 0].
\end{cases} \tag{5.2.4}
$$

Let $(-\infty, \bar{b}_n)$ be the maximal interval with respect to the existence of the solution of (5.2.3) satisfying (5.2.4). Then there is a constant $\widetilde{V}$ which is independent of n and δ such that for every $n \in N$ and $0 < \delta < \bar{b}_n$,

$$
\begin{aligned}
& \|(u^n(t))'\| \\
\leq \;& \widetilde{V}\left(1 + \int_0^t \|(u^n(s))'\| ds + \int_0^t \|(u^n_s)'\|_{\mathcal{P}} ds\right) \\
\leq \;& \widetilde{V}T\left(1 + \sup_{t\in[0,T]} M(t)\|\phi'\|_{\mathcal{P}}\right) \\
& \quad + \widetilde{V}\left(1 + \max_{t\in[0,T]} K(t)\right) \int_0^t \max_{\tau\in[0,s]} \|(u^n(\tau))'\| ds, \quad t \in [0, \bar{b}_n - \delta],
\end{aligned}
$$

that is

$$\|u^n(t)\|,\ \|(u^n(t))'\| \leq \widehat{M}, \quad t \in [0, \bar{b}_n),\ n \in N, \tag{5.2.5}$$

where $\widehat{M}$ is a constant being independent of n and δ.

Choose $\bar{b} \in [0, T]$ such that for all $t \in [0, \bar{b}]$,

$$\|u_t^1 - \phi\|_{\mathcal{P}} \leq \frac{r}{2} \tag{5.2.6}$$

and

$$\begin{aligned} & \max\left\{1, 2\max_{t\in[0,T]} K(t)\right\}\Bigg\{[U(t,0)-I][\phi(0)+J^{-1}(0)f(0,\phi(0),\phi)] \\ & \quad +t|\lambda|(M+1)\sup_{t\in[0,T]}\|J^{-1}(t)\| + t^2(M+1)\Bigg[\Bigg(|\lambda|+1+\widehat{M} \\ & \quad +\widehat{M}\max_{t\in[0,T]} K(t) + \sup_{t\in[0,T]} M(t)\|\phi\|_{\mathcal{P}}\Bigg)\sup_{0\leq s\leq t\leq T}\|A(t)U(t,s)J^{-1}(s)\| \\ & \quad +\sup\nolimits_{t\in[0,T]}\|J^{-1}(t)J'(t)J^{-1}(t)\|\Bigg]\Bigg\} \\ \leq\ & \frac{r}{2}. \end{aligned} \tag{5.2.7}$$

Next, we prove that $\bar{b}_n \geq \bar{b}$ for all $n \in N$, If this is false, then there is an $\bar{n} \in N$ such that $\bar{b}_{\bar{n}} < \bar{b}$. Because $u^{\bar{n}}(\cdot)$ satisfies (5.2.3) and (5.2.4) for $t \in [0, \bar{b}_{\bar{n}})$, we deduce that for every $\bar{t}$, $t \in [0, \bar{b}_{\bar{n}})$ with $\bar{t} < t$,

$$\begin{aligned} & \|u^{\bar{n}}(t) - u^{\bar{n}}(\bar{t})\| \\ \leq\ & \|[U(t,0)-U(\bar{t},0)][\phi(0)+J^{-1}(0)f(0,\phi(0),\phi)]\| \\ & \quad +(t-\bar{t})\Bigg\{|\lambda|(M+1)\sup_{t\in[0,T]}\|J^{-1}(t)\| + T(M+1)\Bigg[\Bigg(|\lambda|+1+\widehat{M} \\ & \quad +\widehat{M}\max_{t\in[0,T]} K(t) + \sup_{t\in[0,T]} M(t)\|\phi\|_{\mathcal{P}}\Bigg)\sup_{0\leq s\leq t\leq T}\|A(t)U(t,s)J^{-1}(s)\| \\ & \quad +\sup\nolimits_{t\in[0,T]}\|J^{-1}(t)J'(t)J^{-1}(t)\|\Bigg]\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} & \|u_t^{\bar{n}} - u_{\bar{t}}^{\bar{n}}\|_{\mathcal{P}} \\ \leq\ & \max_{s\in[0,T]} K(s)\max_{s\in[0,t-\bar{t}]}\|u^{\bar{n}}(s)-u^{\bar{n}}(0)\| + \sup_{s\in[0,T]} M(s)\left\|u_{t-\bar{t}}^{\bar{n}} - \phi\right\|_{\mathcal{P}}. \end{aligned}$$

Therefore, there exist $v \in X$ and $\xi \in \mathcal{P}$ such that

$$\lim_{t \to \bar{b}_{\bar{n}}} u^{\bar{n}}(t) = v, \quad \lim_{t \to \bar{b}_{\bar{n}}} u_t^{\bar{n}} = \xi. \tag{5.2.8}$$

In view of the local Lipschitz continuity of $F^{\bar{n}}$, we know that there is a number $\hat{r} > 0$ such that

$$\|F^{\bar{n}}(t,x,\psi) - F^{\bar{n}}(t,y,\zeta)\| \leq L(\hat{r})(\|x-y\| + \|\psi - \zeta\|),$$
$$t \in [0,T], \ x, y \in \overline{B}_{\hat{r}}(v), \ \psi, \zeta \in \overline{B}_{\hat{r}}(\xi).$$

Choosing $\hat{t} \in [0, \bar{b}_{\bar{n}})$ such that

$$u^{\bar{n}}(t) \in \overline{B}_{\hat{r}}(v), \quad u_t^{\bar{n}} \in \overline{B}_{\hat{r}}(\xi), \quad t \in [\hat{t}, \bar{b}_{\bar{n}}),$$

we have for every $\bar{t}$, $t \in [\hat{t}, \bar{b}_{\bar{n}})$ with $\bar{t} < t$,

$$\begin{aligned}
& \|F^{\bar{n}}(t, u^{\bar{n}}(t), u_t^{\bar{n}}) - F^{\bar{n}}(\bar{t}, u^{\bar{n}}(\bar{t}), u_{\bar{t}}^{\bar{n}})\| \\
\leq \ & \|F^{\bar{n}}(t, u^{\bar{n}}(t), u_t^{\bar{n}}) - F^{\bar{n}}(t, u^{\bar{n}}(\bar{t}), u_{\bar{t}}^{\bar{n}})\| + \|F^{\bar{n}}(t, u^{\bar{n}}(\bar{t}), u_{\bar{t}}^{\bar{n}}) - F^{\bar{n}}(t, v, \xi)\| \\
& + \|F^{\bar{n}}(t, v, \xi) - F^{\bar{n}}(\bar{t}, v, \xi)\| + \|F^{\bar{n}}(\bar{t}, v, \xi) - F^{\bar{n}}(\bar{t}, u^{\bar{n}}(\bar{t}), u_{\bar{t}}^{\bar{n}})\| \\
\leq \ & L(\hat{r})(\|u^{\bar{n}}(t) - u^{\bar{n}}(\bar{t})\| + \|u_t^{\bar{n}} - u_{\bar{t}}^{\bar{n}}\| + 2\|u^{\bar{n}}(\bar{t}) - v\| + 2\|u_{\bar{t}}^{\bar{n}} - \xi\|) \\
& \qquad + \|F^{\bar{n}}(t, v, \xi) - F^{\bar{n}}(\bar{t}, v, \xi)\|.
\end{aligned} \tag{5.2.9}$$

Since for any $\bar{t}$, $t \in [\hat{t}, \bar{b}_{\bar{n}})$ with $\bar{t} < t$,

$$\begin{aligned}
& \|(u^{\bar{n}})'(t) - (u^{\bar{n}})'(\bar{t})\| \\
\leq \ & \|[A(t)U(t,0) - A(\bar{t})U(\bar{t},0)][\phi(0) + J^{-1}(0)f(0,\phi(0),\phi)]\| \\
& + |\lambda|(M+1)\|J^{-1}(t) - J^{-1}(\bar{t})\| + |\lambda| \sup_{t \in [0,T]} \|J^{-1}(t)\| \\
& \times \|F^{\bar{n}}(t, u(t), u_t) - F^{\bar{n}}(\bar{t}, u(\bar{t}), u_{\bar{t}})\|
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\bar{t}} \Big\| [A(t)U(t,s) - A(\bar{t})U(\bar{t},s)] \Big[(\lambda + J^{-1}(s)J'(s)) \\
&\qquad \times J^{-1}(s)F^{\bar{n}}(s, u^{\bar{n}}(s), u_s^{\bar{n}}) + J^{-1}(s)\Big(F_{(1)}^{\bar{n}}(s, u^{\bar{n}}(s), u_s^{\bar{n}}) \\
&\qquad\qquad + F_{(2)}^{\bar{n}}(s, u^{\bar{n}}(s), u_s^{\bar{n}})(u^{\bar{n}})'(s) + F_{(3)}^{\bar{n}}(s, u^{\bar{n}}(s), u_s^{\bar{n}})(u_s^{\bar{n}})' \Big) \Big] \Big\| ds \\
&+(t-\bar{t})|\lambda|(M+1) \sup_{0 \le s \le t \le T} \|A(t)U(t,s)J^{-1}(s)\| \\
&\qquad \times \left\{ \left[|\lambda| + \sup_{s\in[0,T]} \|J'(s)J^{-1}(s)\| \right] \right. \\
&\qquad\qquad \left. \times \left[1 + \widehat{M}\left(1 + \max_{t\in[0,T]} K(t) \right) + \sup_{t\in[0,T]} M(t)\|\phi'\| \right] \right\},
\end{aligned}
$$

we infer, by (5.2.5), (5.2.9) and noting that $F^{\bar{n}}(t, v, \xi)$, $J^{-1}(t)$, $A(t)U(t,s)y$ ($y \in (D, \|\cdot\|_D)$, $s \le t$) are uniformly continuous on $[0, T]$, that

$$
\lim_{t\to \bar{b}_{\bar{n}}} (u^{\bar{n}})'(t) \text{ exists in } X, \text{ and } \lim_{t\to \bar{b}_{\bar{n}}} (u_t^{\bar{n}})' \text{ exists in } \mathcal{P}. \tag{5.2.10}
$$

By similar arguments as in the proof of Theorem 3.2.4 and the proof of existence of solution of (5.2.3) satisfying (5.2.4), we see that $u^{\bar{n}}(\cdot)$ can be extended beyond $\bar{b}_{\bar{n}}$ contradicting the definition of $\bar{b}_{\bar{n}}$. Hence, for all $n \in N$, (5.2.3) has a solution $u^n(\cdot)$ on $[0, \bar{b}]$ satisfying (5.2.4). Moreover, by (5.2.7) and (5.2.3), we have

$$
u^n(t) \in \overline{B}_r(\phi(0)), \quad t \in [0, \bar{b}],\ n \in N. \tag{5.2.11}
$$

From (5.2.6) and (5.2.7), it follows that

$$
\begin{aligned}
& \|u_t^n - \phi\|_{\mathcal{P}} \\
\le\ & \|u_t^n - u_t^1\|_{\mathcal{P}} + \|u_t^1 - \phi\|_{\mathcal{P}} \\
\le\ & \max_{t\in[0,T]} K(t) \max_{t\in[0,\bar{b}]} \|u^n(t) - u^1(t)\| + \max_{t\in[0,\bar{b}]} \|u_t^1 - \phi\|_{\mathcal{P}} \\
\le\ & \max_{t\in[0,T]} K(t) \left(\max_{t\in[0,\bar{b}]} \|u^n(t) - \phi(0)\| + \max_{t\in[0,\bar{b}]} \|u^1(t) - \phi(0)\| \right) + \max_{t\in[0,\bar{b}]} \|u_t^1 - \phi\|_{\mathcal{P}} \\
\le\ & r, \quad t \in [0, \bar{b}],\ n \in N,
\end{aligned}
$$

that is,

$$
u_t^n \in \overline{B}_r(\phi), \quad t \in [0, \bar{b}],\ n \in N. \tag{5.2.12}
$$

This, together with (5.2.11), implies that if we put

$$\begin{aligned} G^n(s) &:= F^n(s,u^n(s),u^n_s) - f(s,u^n(s),u^n_s), \quad s\in[0,\bar b],\ n\in N,\\ G^n_{(0)}(s) &:= F^n_{(1)}(s,u^n(s),u^n_s) - f'_{(1)}(s,u^n(s),u^n_s)\\ &\quad +F^n_{(2)}(s,u^n(s),u^n_s)(u^n(s))' - f'_{(2)}(s,u^n(s),u^n_s)(u^n(s))'\\ &\quad +F^n_{(3)}(s,u^n(s),u^n_s)(u^n_s)' - f'_{(3)}(s,u^n(s),u^n_s)(u^n_s)', \quad s\in[0,\bar b],\ n\in N, \end{aligned}$$

then

$$\lim_{n\to\infty}\|G^n(s)\|=0,\quad \lim_{n\to\infty}\|G^n_{(0)}(s)\|=0 \quad \text{uniformly for all } s\in[0,\bar b],$$

and

$$u^n(t) = \begin{cases} U(t,0)\phi(0) + \int_0^t U(t,s)f(s,u^n(s),u^n_s)\\ \quad + \int_0^t \left\{U(t,s)[\lambda + J^{-1}(s)J'(s)]J^{-1}(s) - J^{-1}(s)J'(s)J^{-1}(s)\right\} G^n(s)ds\\ \qquad - \int_0^t [U(t,s)-I]J^{-1}(s)G^n_{(0)}(s)ds, \quad t\in[0,T],\\ \phi'(t), \quad t\in(-\infty,0], \end{cases} \tag{5.2.13}$$

$$(u^n(t))' = \begin{cases} A(t)U(t,0)[\phi(0)+J^{-1}(0)f(0,\phi(0),\phi)] + \lambda J^{-1}(t)f(t,u^n(t),u^n_t)\\ \quad - \int_0^t A(t)U(t,s)\left[\lambda + J^{-1}(s)J'(s)\right]J^{-1}(s)f(s,u^n(s),u^n_s)ds\\ \quad - \int_0^t A(t)U(t,s)J^{-1}(s)[f'_{(1)}(s,u^n(s),u^n_s)\\ \quad + f'_{(2)}(s,u^n(s),u^n_s)(u^n(s))' + f'_{(3)}(s,u^n(s),u^n_s)(u^n_s)']ds,\\ \quad + \lambda J^{-1}(t)G^n(t) - \int_0^t A(t)U(t,s)\left[\lambda + J^{-1}(s)J'(s)\right]J^{-1}(s)G^n(s)ds\\ \quad - \int_0^t A(t)U(t,s)J^{-1}(s)G^n_{(0)}(s)ds, \quad t\in[0,T],\\ \phi'(t), \quad t\in(-\infty,0]. \end{cases} \tag{5.2.14}$$

Therefore, using the Lipschitz continuity of f, we get for any n, $m \in N$,

$$\|u^n(t) - u^m(t)\| \leq \overline{\overline{V}} \left(\max_{t\in[0,\bar{b}]} \|G^n(t)\| + \max_{t\in[0,\bar{b}]} \|G^n_{(0)}(t)\| \right)$$

$$\underline{\underline{V}} \int_0^t \max_{\tau\in[0,s]} \|u^n(\tau) - u^m(\tau)\| d\tau, \quad t \in [0, \bar{b}],$$

where $\overline{\overline{V}}$, $\underline{\underline{V}}$ are constants being independent of $t \in [0, \bar{b}]$, n and m. This means that

$$\lim_{n\to\infty} u^n(t) = v(t) \quad \text{uniformly on } [0, \bar{b}],$$

and

$$\lim_{n\to\infty} u^n_t = \chi_t \in \mathcal{P} \quad \text{uniformly on } [0, \bar{b}],$$

where

$$\chi_t(\theta) := \begin{cases} v(t+\theta), & 0 \leq t+\theta \leq \bar{b}, \\ \phi(t+\theta), & t+\theta < 0. \end{cases}$$

Hence, for any $\varepsilon > 0$, we have if n is large enough then

$$\max_{s\in[0,\bar{b}]} \|f(s, u^n(s), u^n_s) - f(s, v(s), \chi_s)\| \leq \varepsilon,$$

$$\max_{s\in[0,\bar{b}]} \|f'_{(i)}(s, u^n(s), u^n_s) - f(s, v(s), \chi_s)\| \leq \varepsilon, \quad i = 1, 2, 3.$$

Thus, (5.2.14) implies the uniform convergence of $(u^n(\cdot))'$ as $n \to \infty$, and

$$\lim_{n\to\infty} (u^n(t))' = v'(t) \quad \text{uniformly on } [0, \bar{b}],$$

$$\lim_{n\to\infty} (u^n_t)' = (\chi_t)' \in \mathcal{P} \quad \text{uniformly on } [0, \bar{b}].$$

Consequently, the function

$$w(t) := \begin{cases} v(t), & t \in [0, \bar{b}], \\ \phi(t), & t < 0 \end{cases}$$

is a fixed point of $\mathbf{F}$ and satisfies

$$w'(t) = (\mathbf{F}w)'(t), \quad t \in (\infty, \bar{b}].$$

So $w(t) = x(t)$ on $[0, \bar{b}]$.

It remains to prove the $w(t)$ can be extended to $(-\infty, T]$ such that the extension is still a fixed point of $\mathbf{F}$ and continuously differentiable in $[0, T]$.

Let $(-\infty, b)$ with $b < T$ be the maximal interval of existence of $w(t)$. Then $w(t) = x(t)$ on $[0, T]$. This implies that

$$\lim_{t \in b^-} w(t) \text{ exists in } X, \text{ and } \lim_{t \in b^-} w_t \text{ exists in } \mathcal{P}.$$

Hence, there is a constant $\widetilde{\widetilde{\widetilde{M}}}$ such that

$$\|f(t, w(t), w_t)\| \leq \widetilde{\widetilde{\widetilde{M}}}, \quad t \in [0, b),$$

$$\|f'_{(i)}(t, w(t), w_t)\| \leq \widetilde{\widetilde{\widetilde{M}}}, \quad t \in [0, b), \quad i = 1, 2, 3.$$

Thus, by

$$w'(t) = (\mathbf{F}w)'(t), \quad t \in (\infty, b),$$

we obtain

$$\lim_{t \in b^-} w'(t) \text{ exists in } X, \text{ and } \lim_{t \in b^-} (w_t)' \text{ exists in } \mathcal{P}.$$

By similar arguments as above, we know that $w(t)$ can be extended to an interval $[0, b+\delta]$ $(\delta > 0)$ such that the extension is still a fixed point of $\mathbf{F}$ and $w'(t) = (\mathbf{F}w)'(t)$ in $[0, b + \delta]$. The choice of b makes this no sense. So $b = T$. The analysis above implies that the maximal interval of existence of $w(t)$ should be $(-\infty, T]$. This means that $x(t)$ is a classical solution of (3.1.3).

The uniqueness and (5.2.1) is easy to see. This ends the proof.

□

Bibliography

[1] P. Acquistapace, Abstract linear nonautonomous parabolic equations: a survey, Differential equations in Banach spaces (G. Dore, A. Favini, E. Obrecht, A. Venni (Eds), Proceedings, Bologna 1991), 1–19, Lecture Notes in Pure and Appl. Math. 148, Dekker, New York, 1993.

[2] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[3] M. Adimy, H. Bouzahir, and K. Ezzinbi, Existence for a class of partial functional differential equations with infinite delay, Nonlinear Anal. Ser. A: Theory Methods 46 (2001), 91–112.

[4] A. V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them, Pacific J. Math. 10 (1960), 419-437.

[5] K. Balachandran and M. Chandrasekaran, The non-local Cauchy problem for semilinear integrodifferential equations with deviating argument, Proc. Edinburgh Math. Soc. 44 (2001), 63-70.

[6] J. Banas and K. Goebel, Measures of Noncompactness, Lecture Notes in Pure Appl. Math. 60, Dekker, New York, 1980

[7] H. T. Banks and A. Burns, Hereditary control problems: Numerical methods based on averaging approximations, SIAM J. Control Optimization 16 (1978), 169–208.

[8] A. Bàtkai and S. Piazzera, Damped wave equations with delay, Topics in functional differential and difference equations (Lisbon, 1999), 51–61, Fields Inst. Commun. 29, Amer. Math. Soc., Providence, RI, 2001.

[9] A. Boucherif, First-order differential inclusions with nonlocal initial conditions, Appl. Math. Lett. 15 (2002), 409-414.

[10] J. A. Burns and T. L. Herdman, Adjoint semigroup theory for a class of functional differential equations, SIAM J. Math. Anal. 5 (1976), 729–745.

[11] J. A. Burns, T. L. Herdman and H. W. Stech, Linear functional differential equations as semigroups on product spaces, SIAM J. Math. Anal. 14 (1983), 98-116.

[12] J. A. Burns, T. L. Herdman and J. Turi, Neutral functionals integro-differential equations with weakly singular kernels, J. Math. Anal. Appl. 145 (1990), 371-401.

[13] T. A. Burton, Stability and periodic solutions of ordinary and functional-differential equations, Academic Press, Inc., Orlando, FL, 1985.

[14] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl. 162 (1991), 494-505.

[15] L. Byszewski, Uniqueness of solutions of parabolic semilinear nonlocal-boundary problems, J. Math. Anal. Appl. 165 (1992), 472-478.

[16] L. Byszewski, Application of montone iterative method to a system of parabolic semilinear functional-differential problems with nonlocal conditions, Nonlinear Anal. TMA 28 (1997), 1347-1357.

[17] L. Byszewski, Application of properties of the right-hand sides of evolution equations to an investigation of nonlocal evolution problems, Nonlinear Anal. TMA 33 (1998), 413–426.

[18] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, Applicable Anal. 40 (1991), 11-19.

[19] L. Byszewski and V. Lakshmikantham, Montone iterative technique for nonlocal hyperbolic differential problem, J. Math. Phys. Sci. 26 (4) (1992), 345-359.

[20] G. Chen, Control and stabilization for the wave equation in a bounded domain, SIAM J. Control 17 (1979), 66-81.

[21] R. M. Christensen, The Theory of Viscoelasticity: An Introduction, 2nd ed., Academic Press, New York, 1982.

[22] C. Corduneanu and V. Lakshmikantham, Equations with unbounded delay: a survey, Nonlinear Anal. TMA 4 (1979), 831-877.

[23] J. M. Cushing, Integrodifferential Equations and Delay Models in Population Dynamics, Lect. Notes in Biomath. 20, Springer, Berlin, 1977.

[24] E. B Davies, One Parameter Semigroups, Academic Press, London, 1980.

[25] E. B. Davies and M. M. Pang, The Cauchy problem and a generalization of the Hille-Yosida theorem, Proc. London Math. Soc. 55 (1987) 181-208.

[26] K. Deimling, Ordinary Differential Equations in Banach spaces, Lect. Notes in Math. 596, Springer-Verlag, Berlin, New York, 1977.

[27] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, New York, 1985.

[28] K. Deimling, Multivalued Differential Equations, Walter de Gruyter, Berlin, New York, 1992.

[29] R. deLaubenfels, Existence and uniqueness families for the abstract Cauchy problems, J. London Math. Soc. 44 (1991) 310-338.

[30] R. deLaubenfels, Existence Families, Functional Calculi and Evolution Equations, Lect. Notes in Math. 1570, Springer-Verlag, Berlin, New York, 1994.

[31] R. deLaubenfels, F. Yau, S. Wang, Fractional powers of operators of regularized type, J. Math. Anal. Appl. 199 (1996), 910-933.

[32] M. C. Delfour and S. K. Mitter, Hereditary differential systems with constant delays. I. General case, J. Differential Equations 12 (1972), 213–235.

[33] W. Desch, R. Grimmer and W. Schappacher, Some considerations for linear integrodifferential equations, J. Math. Anal. Appl. 104 (1984), 219-234.

[34] W. Desch, R. Grimmer and W. Schappacher, Well-posedness and wave propagation for a class of integrodifferential equations in Banach space, J. Differential Equations 74 (1988), 391–411.

[35] O. Diekmann, S. van Gils, S. M. Verduyn Lunel and H. O. Walther, Delay Equations. Functional-, Complex-, and Nonlinear Analysis, Appl. Math. Sci. 110, Springer-Verlag, Berlin, New York, 1995.

[36] K. -J. Engel and R. Nagel, One-parameter Semigroups for Linear Evolution Equations, GTM 194, Springer, Berlin, New York, 2000.

[37] H. O. Fattorini, The Cauchy Problem, Encyclopedia of Math. and Appl. 18, Addison-Wesley Publishing Co., Reading, Mass., 1983.

[38] J. A. Goldstein, Abstract evolution equations, Trans. Amer. Math. Soc. 141 (1969), 159-185.

[39] J. A. Goldstein, Semigroups of Linear Operators and Applications, Oxford Univ. Press, New York, 1985.

[40] R. Grimmer, Resolvent operators for integral equations in Banach space, Trans. Amer. Math. Soc. 273 (1982), 333-349.

[41] R. Grimmer and Liu, Integrated semigroups and integrodifferential equations, Semigroup Forum 48 (1994), 79-95.

[42] R. Grimmer and W. Schappacher, Weak solutions of integro-differential equations and resolvent operators, J. Integral Equations 6 (1984), 205–229.

[43] G. Gripenberg, S. -O. Londen and O. Staffans, Volterra Integral and Functional Equations, Cambridge Uni. Press, Cambridge, 1990.

[44] G. Gühring, F. Räbiger and R. Schunaubelt, A characteristic equation for non-autonomous partial functional differential equations, J. Differential Equations 181 (2002), 439–462.

[45] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978), 11-41.

[46] J. K. Hale and S. M. Verduyn Lunel, Introduction to Functional-Differential Equations, Springer-Verlag, New York, 1993.

[47] H. R. Henriquez, Periodic solutions of quasi-linear partial differential equations with unbounded delay, Funkcial. Ekvac. 37 (1994), 329-344.

[48] H. R. Henriquez, Regularity of solutions of abstract retarded functional-differential equations with unbounded delay, Nonlinear Anal. TMA 28 (1997), 513-531.

[49] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lect. Notes in Math. 840, Springer-Verlag, Berlin, New York, 1981.

[50] H. P. Heinz, On the behavior of measures of noncompactness with respect to differential and integration of vector-valued function, Nonlinear Anal. TMA, 7 (1983), 1351-1371.

[51] Y. Hino, S. Murakami and T. Naito, Functional-Differential Equations with Infinite Delay. Lect. Notes in Math. 1473, Springer-Verlag, Berlin, 1991.

[52] D. Jackson, Existence and uniqueness of solutions of a semilinear nonlocal parabolioc equations, J. Math. Anal. Appl. 172 (1993), 256-265.

[53] F. Kappel and W. Schappacher, Some considerations to the fundamental theory of infinite delay equations, J. Differential Equations 37 (1980), 141–183.

[54] A. Kartsatos and M. E. Parrott, The weak solution of a functional-differential equation in a general Banach space, J. Differential Equations 75 (1988), 290–302.

[55] W. Kerscher and R. Nagel, Positivity and stability for Cauchy problems with delay. In: Partial Differential Equations, Lect. Notes in Math. 1324, Springer, Berlin, New York, 1987.

[56] J. Liang and T. J. Xiao, Functional differential equations with infinite delay in Fréchet spaces, Sichuan Daxue Xuebao 26 (1989), 382-390.

[57] J. Liang and T. J. Xiao, Functional differential equations with infinite delay in Banach spaces, Inter. J. Math. Math. Sci. 14 (1991), 497-508.

[58] J. Liang and T. J. Xiao, Solutions of abstract functional differential equations with infinite delay, Acta Math. Sinica 34 (1991), 631-644.

[59] J. Liang and T. J. Xiao, The Cauchy problem for nonlinear abstract functional differential equations with infinite delay, Computers Math. Appl. 40 (2000), 693-703.

[60] J. Liang, T. J. Xiao and J. van Casteren, A note on semilinear abstract functional differential and integrodifferential equations with infinite delay, Appl. Math. Letter, to appear

[61] Y. Lin, J. H. Liu, Semilinear integrodifferential equations with nonlocal Cauchy problem, Nonlinear Anal. TMA 26 (1996), 1023-1033.

[62] J. H. Liu, Resolvent operators and weak solutions of integrodifferential equations. Differential Integral Equations 7 (1994), 523–534.

[63] J. H. Liu, Periodic solutions of infinite delay evolution equations, J. Math. Anal. Appl. 247 (2000), 627-644.

[64] J. L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol. I, Springer-Verlag, New York, 1972.

[65] R. Nagel and G. Nickel, Wellposedness for nonautonomous abstract Cauchy problems, Progr. Nonlinear Differential Equations Appl. 50, 279-293, Birkhöuser, Basel, 2002.

[66] R. Nagel and A. Rhandi, A characterization of Lipschitz continuous evolution families on Banach spaces, Operator Theory in Function Spaces and Banach Lattices, 275–288, Oper. Theory Adv. Appl. 75, Birkhäuser, Basel, 1995.

[67] R. Nagel and E. Sinestrari, Nonlinear hyperbolic Volterra integrodifferential equations, Nonlinear Anal. TMA 27 (1996), 167–186.

[68] S. K. Ntouyas and P. Ch. Tsamatos, Global existence for semilinear evolution equations with nonlocal conditions, J. Math. Anal. Appl. 210 (1997), 679-687.

[69] R. D. Nussbaum, A generalization of the Ascoli theorem and an application to functional differential equations, J. Math. Anal. Appl. 35 (1971), 600-610.

[70] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin, 1983.

[71] M. E. Parrott, Linearized stability and irreducibility for a functional-differential equation, SIAM J. Math. Anal. 23 (1992), 649–661.

[72] J. Prüss, Evolutionary Integral Equations and Applications, Monographs Math. 87, Birkhäuser Verlag, 1993.

[73] M. Reed and B. Simon, Methods of Modern Mathematical Physics, I-IV, Academic Press, New York, 1980.

[74] W. M. Ruess, The evolution operator approach to functional differential equations with delay, Proc. Amer. Math. Soc. 119 (1993), 783-791.

[75] W. M. Ruess and W. H. Summers, Operator semigroups for functional differential equations with delay, Trans. Amer. Math. Soc. 341 (1994), 695-719.

[76] W. M. Ruess and W. H. Summers, Almost periodicity and stability for solutions to functional-differential equations with infinite delay, Differential Integral Equations 9 (1996), 1225–1252.

[77] K. Schumacher, Existence and continuous dependence for functional differential equations with unbounded delay, Arch. Rat. Mech. Ana. 67 315-335 (1978).

[78] J. S. Shin, Existence of solutions and Kamke's theorem for functional equations in Banach spaces, J. Differential Equations 81 (1989), 294-312.

[79] J. S. Shin and T. Naito, Existence and continuous dependence of mild solutions to semilnear functional differential equations in Banach spaces, Tohoku Math. J. 51 (1999), 555-583.

[80] B. Straub, Fractional powers of operators with polynomially bounde6 resolvent and the semigroups generated by them, Hiroshima Math. J. 24 (1994), 529-548.

[81] H. Tanabe, Equations of Evolution, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.

[82] N. Tanaka and N. Okazawa, Local C-semigroups and local integrated semigroups, Proc. London Math. Soc. 61(1990), 63-90.

[83] G. Tadmor and J. Turi, Neutral equations and associated semigroups, J. Differential Equations 116 (1995), 59–87.

[84] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, Trans. Amer. Math. Soc. 200 (1974), 395-418.

[85] J. van Casteren, Generators of Strongly Continuous Semigroups, Pitman, London, 1985.

[86] G. F. Webb, Theory of Nonlinear Age-dependent Population Dynamics, Monographs and Textbooks in Pure and Applied Mathematics 89. Marcel Dekker, Inc., New York, 1985.

[87] J. Wu, Theory and Applications of Partial Functional-differential Equations. Applied Mathematical Sciences 119, Springer-Verlag, New York, 1996.

[88] T. J. Xiao and J. Liang, The Cauchy Problem for Higher Order Abstract Differential Equations, Lect. Notes in Math. 1701, Springer, Berlin, New York, 1998.

[89] T. J. Xiao and J. Liang, Higher order abstract Cauchy problems and their existence, uniqueness families, J. London Math. Soc., to appear.

[90] A. Yagi, On the abstract evolution equations of parabolic type, Osaka J. Math. 14 (1977), 557-568.

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