Local rigidity of 3-dimensional cone-manifolds

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Zusammenfassung in deutscher Sprache

Eine 3-dimensionale Kegelmannigfaltikeit ist eine 3-Mannigfaltigkeit C, die wir als kompakt und orientiert voraussetzen, zusammen mit einer singulären geometrischen Struktur. Genauer hat man eine Längenmetrik auf C, die auf dem Komplement eines eingebetteten geodätischen Graphen Σ durch eine glatte Riemannsche Metrik konstanter Schnittkrümmung $\kappa \in \mathbb{R}$ induziert wird. Σ ist der sogenannte singuläre Ort, $M = C \setminus \Sigma$ bezeichnet man als den glatten Teil. Für Umgebungen singulärer Punkte schreibt man lokale Modelle vor, und zwar fordert man, daß eine Umgebung eines singulären Punktes modelliert sei durch den κ-Kegel über einer 2-Sphäre mit einer sphärischen Kegelmetrik, für präzise Definitionen sei der Leser auf Kapitel 2 verwiesen. Jeder Kante in Σ kann ein sogenannter Kegelwinkel zugeordnet werden, sind die Kegelwinkel höchstens π, so ist Σ trivalent.

3-dimensionale Kegelmannigfaltigkeiten treten in natürlicher Weise in der Geometrisierung 3-dimensionaler *orbifolds* auf. Der Deformationsraum dieser Strukturen spielt eine wesentliche Rolle im Beweis des Orbifold-Theorems durch M. Boileau, B. Leeb und J. Porti (vergl. [BLP1] und [BLP2]).

In dieser Arbeit untersuchen wir den lokalen Deformationsraum solcher Kegelmannigfaltigkeitsstrukturen. Das technische Hauptresultat ist ein Verschwindungssatz für L^2-Kohomologie des glatten Teils der Kegelmannigfaltigkeit.

Theorem 1 *Sei C eine 3-dimensionale Kegelmannigfaltigkeit konstanter Krümmung $\kappa \in \{-1, 0, 1\}$ mit Kegelwinkeln $\leq \pi$. Es bezeichne $(\mathcal{E}, \nabla^{\mathcal{E}})$ das Vektorbündel der infinitesimalen Isometrien auf $M = C \setminus \Sigma$ mit dem natürlich gegebenen flachen Zusammenhang. Im euklidischen Fall bezeichne $\mathcal{E}_{trans} \subset \mathcal{E}$ das parallele Unterbündel der infinitesimalen Translationen. Dann gilt im hyperbolischen und im sphärischen Fall*

$$H^1_{L^2}(M, \mathcal{E}) = 0\,,$$

im euklidischen Fall hingegen gilt

$$H^1_{L^2}(M, \mathcal{E}_{trans}) \cong \{\omega \in \Omega^1(M, \mathcal{E}_{trans}) \,|\, \nabla\omega = 0\}\,.$$

Wir verwenden hier eine Bochner-Weitzenböck Formel für den Hodge-Laplace Operator auf 1-Formen mit Werten in $\mathcal{E}$ in Kombination mit analytischen Techniken, die im wesentlichen von J. Brüning und R. Seeley stammen, vergl. [BS]. Die Hauptschwierigkeiten liegen hier in der Tatsache begründet, daß die Riemannsche Metrik auf dem glatten Teil nicht vollständig ist.

Wir bezeichnen mit $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ den Raum der Darstellungen der Fundamentalgruppe von M nach $\mathrm{SL}_2(\mathbb{C})$. $\mathrm{SL}_2(\mathbb{C})$ operiert hierauf durch Konjugation, es sei $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ der Quotient. $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ trage die kompakt-offene Topologie, $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ die Quotiententopologie. Indem man eine hyperbolische Struktur auf ihre Holonomiedarstellung hol abbildet, kann $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ lokal mit dem Raum der hyperbolischen Strukturen auf M identifiziert werden, die Kegelstrukturen bilden einen Teilraum.

Als Konsequenz aus dem L^2-Verschwindungssatz zeigen wir, daß $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ in der Nähe einer hyperbolischen Kegelstruktur glatt ist. M ist homotopieäquivalent zu einer Mannigfaltigkeit mit Rand M_ε, die man dadurch erhält, daß man eine ε-Umgebung der Singularität herausschneidet. Auf dem Rand von M_ε haben wir N Meridiankurven, wobei N die Anzahl der Kanten in Σ bezeichnet. Wir erhalten neben der Glattheit folgende lokale Parametrisierung:

Theorem 2 *Sei C eine hyperbolische Kegelmannigfaltigkeit mit Kegelwinkeln $\leq \pi$. Sei $\{\mu_1, \ldots, \mu_N\}$ die Familie der Meridiankurven, wobei $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$. Dann ist die Abbildung*

$$X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}^N, \chi \mapsto (t_{\mu_1}(\chi), \ldots, t_{\mu_N}(\chi))$$

ein lokaler Diffeomorphismus um $\chi = [\mathrm{hol}]$.

Als Konsequenz hieraus erhalten wir lokale Starrheit in folgendem starken Sinne: Der Raum der hyperbolischen Kegelmannigfaltigkeitsstrukturen wird lokal durch die Kegelwinkel parametrisiert. Eine hyperbolische Kegelmannigfaltigkeitsstruktur kann insbesondere nicht deformiert werden, ohne den Kegelwinkel zu verändern.

Im sphärischen Fall wird der Raum der Strukturen lokal mit $X(\pi_1 M, \mathrm{SU}(2)) \times X(\pi_1 M, \mathrm{SU}(2))$ identifiziert, dabei sind $R(\pi_1 M, \mathrm{SU}(2))$ und $X(\pi_1 M, \mathrm{SU}(2))$ wie im hyperbolischen Fall definiert. Die Holonomiedarstellung spaltet als ein Produkt $\mathrm{hol} = (\mathrm{hol}_1, \mathrm{hol}_2)$. Analog zum hyperbolischen Fall erhalten wir:

Theorem 3 *Sei C eine sphärische Kegelmannigfaltigkeit mit Kegelwinkeln $\leq \pi$, die nicht Seifert-gefasert ist. Sei $\{\mu_i, \ldots, \mu_N\}$ die Familie der Meridiankurven, wobei $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$. Dann ist die Abbildung*

$$X(\pi_1 M, \mathrm{SU}(2)) \to \mathbb{R}^N, \chi_i \mapsto (t_{\mu_1}(\chi_i), \ldots, t_{\mu_N}(\chi_i))$$

ein lokaler Diffeomorphismus um $\chi_i = [\mathrm{hol}_i]$ for $i \in \{1, 2\}$.

Hieraus folgt genauso wie im hyperbolischen Fall lokale Starrheit: sphärische Kegelmannigfaltigkeitsstrukturen werden lokal durch die Kegelwinkel parametrisiert. Eine sphärische Kegelmannigfaltigkeitsstruktur kann nicht deformiert werden, ohne den Kegelwinkel zu verändern.

Wir erwarten auch für das kohomologische Resultat im euklidischen Fall Anwendungen, zum jetzigen Zeitpunkt können wir allerdings kein geometrisches Resultat formulieren.

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1 Introduction

A 3-dimensional cone-manifold is a 3-manifold C equipped with a singular geometric structure. We will assume that C is compact and oriented. More precisely, one has a length-metric, which is in the complement of an embedded geodesic graph Σ induced by a smooth Riemannian metric of constant sectional curvature $\kappa \in \mathbb{R}$. $M \setminus \Sigma$ is called the smooth part of C, while Σ is called the singular locus. Neighbourhoods of singular points are modelled on κ-cones over 2-spheres with spherical cone metrics. For precise definitions we refer the reader to Chapter 2. One associates with each edge contained in Σ the so-called cone-angle, which is a positive real number. If all cone-angles are $\leq \pi$, then a connected component of Σ is either a trivalent graph or a circle.
3-dimensional cone-manifolds arise naturally in the geometrization of 3-dimensional orbifolds. The concept of cone-manifold can be viewed as a generalization of the concept of geometric orbifold, where the cone-angles are not restricted any more to the set of orbifold-angles, which are rational multiples of π.
The deformation space of cone-manifold structures (or short: cone-structures) on a cone-manifold C with fixed toplogical type (C, Σ) plays an important role in the proof of the Orbifold Theorem by M. Boileau, B. Leeb and J. Porti, cf. [BLP1] and [BLP2]. The proof of the Orbifold Theorem in the general case requires the analysis of cone-manifold structures with cone-angles $\leq \pi$, where the singularity is allowed to have trivalent vertices. The case, where the singular locus is a union of circle components, i.e. a link in C, has earlier been settled by M. Boileau and J. Porti, cf. [BP].
In this thesis we investigate local properties of the deformation space of cone-manifold structures with cone-angles $\leq \pi$. We consider the general case under this cone-angle restriction, where trivalent vertices are allowed. In particular we prove local rigidity in the spherical and in the hyperbolic case.
In the hyperbolic case there are some important results known. There is on the one hand Garland-Weil local rigidity (cf. [Gar]), which applies in any dimension ≥ 3 to the space of complete, finite-volume hyperbolic structures on a given hyperbolic manifold. On the other hand, C. Hodgson and S. Kerckhoff proved a local rigidity result for 3-dimensional hyperbolic cone-manifolds in [HK]. Their proof applies to the case, where the singular locus Σ is restricted to be a link in C, but where the cone-angles are allowed to be $\leq 2\pi$.

Our main technical result is a vanishing theorem for L^2-cohomology on the smooth part M of the cone-manifold C with coefficient in the flat vector-bundle of infinitesimal isometries. Recall that L^2-cohomology is by definition the cohomology of the subcomplex of the de Rham complex, which consists of those forms ω such that ω and $d\omega$ are L^2-bounded.

Theorem 1.1 *Let C be a 3-dimensional cone-manifold of curvature $\kappa \in \{-1, 0, 1\}$ with cone-angles $\leq \pi$. Let $(\mathcal{E}, \nabla^{\mathcal{E}})$ be the vector-bundle of infinitesimal isometries of $M = C \setminus \Sigma$ with its natural flat connection. In the Euclidean case let $\mathcal{E}_{trans} \subset \mathcal{E}$ be the parallel subbundle of infinitesimal translations. Then in the hyperbolic and the spherical case*

$$H^1_{L^2}(M, \mathcal{E}) = 0\,,$$

while in the Euclidean case

$$H^1_{L^2}(M, \mathcal{E}_{trans}) \cong \{\omega \in \Omega^1(M, \mathcal{E}_{trans}) \,|\, \nabla\omega = 0\}\,.$$

The proof of this theorem is analytic in nature. The main difficulty is caused by the non-completeness of the constant-curvature metric on M, the smooth part of the

cone-manifold. On a complete Riemannian manifold the Hodge-Laplace operator on differential forms is known to be essentially selfadjoint, cf. [BL] and the references therein. This is something we cannot expect to hold here.
On the other hand, the fact that the singularities of the metric are *conical* in a sense to be made precise, allows us to apply separation of variables techniques. This has already been explored by Cheeger, cf. [Che].
One main ingredient is a Hodge theorem for cone-manifolds, which allows to identify L^2-cohomology spaces with the kernel of a certain selfadjoint extension of the Laplacian on forms. For reasons which will become clear later we call this particular selfadjoint extension $\Delta(d_{max})$:

$$H^1_{L^2}(M,\mathcal{E}) \cong \ker \Delta^1(d_{max})$$

The second one is a Bochner-Weitzenböck formula for the Laplacian on 1-forms with values in our particular flat vector-bundle $(\mathcal{E}, \nabla^{\mathcal{E}})$, resp. the parallel subbundle $\mathcal{E}_{trans} \subset \mathcal{E}$ in the Euclidean case.
The essence of the Bochner technique is that the Weitzenböck formula may be used to bound the Laplacian on compactly supported 1-forms from below, i.e. to find $C > 0$ such that

$$\langle \Delta\omega, \omega \rangle_{L^2} \geq C \langle \omega, \omega \rangle_{L^2}$$

for all $\omega \in \Omega^1_{cp}(M,\mathcal{E})$. If one can show that this lower bound extends to hold for $\Delta^1(d_{max})$, i.e.

$$\left\langle \Delta^1(d_{max})\omega, \omega \right\rangle_{L^2} \geq C \langle \omega, \omega \rangle_{L^2}$$

for all $\omega \in \operatorname{dom} \Delta^1(d_{max})$, one obtains that $\ker \Delta^1(d_{max}) = 0$, hence via the Hodge theorem that $H^1(M,\mathcal{E}) = 0$. In the Euclidean case, where one does not get a positive lower bound, one has to vary this argument a little.
In the complete, finite-volume case this settles everything in view of the essential selfadjointness of the Hodge-Laplacian (cf. [Gar]). In our case it requires a more detailed study of the selfadjoint extensions of the Hodge-Laplacian. Here we use the techniques of Brüning and Seeley, cf. [BS], along with some basic functional analytic properties of the de-Rham complex presented in a very convenient form for us in [BL].
In the hyperbolic and in the spherical case we may conclude local rigidity from the vanishing of L^2-cohomology, let us now briefly discuss this.
If $\Sigma \subset C$ is the singular locus, for $\varepsilon > 0$ let $U_\varepsilon(\Sigma)$ be the ε-neighbourhood of Σ in C intersected with the smooth part: $U_\varepsilon(\Sigma) = B_\varepsilon(\Sigma) \cap M$. Let $M_\varepsilon = M \setminus U_\varepsilon(\Sigma)$, which is topologically a manifold with boundary. Let τ be the number of torus components in ∂M_ε, then $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$ equals the number of edges of the singular locus.
In the hyperbolic case, the holonomy representation of the smooth, but incomplete hyperbolic structure on M lifts to a representation

$$\mathrm{hol} : \pi_1 M \longrightarrow \widetilde{\mathrm{Isom}^+\mathbf{H}^3} = \mathrm{SL}_2(\mathbb{C})\,.$$

Let $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ denote the $\mathrm{SL}_2(\mathbb{C})$-representation variety of $\pi_1 M$, i.e. the set of group homomorphisms $\rho : \pi_1 M \to \mathrm{SL}_2(\mathbb{C})$ equipped with the compact-open topology. The set-theoretic quotient of $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ by the conjugation action of $\mathrm{SL}_2(\mathbb{C})$ equipped with the quotient topology is denoted by $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$.
This is not to be confused with a quotient construction by means of geometric invariant theory in the algebraic category. One feature of our presentation is rather that we can avoid these issues.
The above defined spaces may be badly behaved in general, but near the holonomy representation of a hyperbolic cone-structure we can show smoothness. This will be shown with transversality arguments as a consequence of the L^2-vanishing theorem.

Theorem 1.2 *Let C be a hyperbolic cone-manifold with cone-angles $\leq \pi$. Let $\{\mu_1, \ldots, \mu_N\}$ be the family of meridians, where $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$. Then the map*

$$X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}^N, \chi \mapsto (t_{\mu_1}(\chi), \ldots, t_{\mu_N}(\chi))$$

is a local diffeomorphism near $\chi = [\mathrm{hol}]$.

The quotient space $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ may be considered, at least locally, as the deformation space of hyperbolic structures on M. Hyperbolic cone-structures correspond to representations, where the meridians μ_i map to elliptic elements in $\mathrm{SL}_2(\mathbb{C})$. Therefore the previous theorem implies local rigidity in the following strong sense:

Corollary 1.3 (Local rigidity) *Let C be a hyperbolic cone-manifold with cone-angles $\leq \pi$. Then the set of cone-angles $\{\alpha_1, \ldots, \alpha_N\}$, where $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$, provides a local parametrization of the space of hyperbolic cone-structures near the given structure on M. In particular, there are no deformations leaving the cone-angles fixed.*

In the spherical case, the holonomy representation of the smooth, but incomplete spherical structure on M lifts to a product representation

$$\mathrm{hol} = (\mathrm{hol}_1, \mathrm{hol}_2) : \pi_1 M \longrightarrow \widetilde{\mathrm{Isom}^+ \mathbf{S}^3} = \mathrm{SU}(2) \times \mathrm{SU}(2)\,.$$

In the same way as in the hyperbolic case we may introduce $R(\pi_1 M, \mathrm{SU}(2))$ and $X(\pi_1 M, \mathrm{SU}(2))$, the $\mathrm{SU}(2)$-representation variety and its quotient by the conjugation action of $\mathrm{SU}(2)$. In the statement of the following result we have to include the additional hypothesis *C not Seifert fibered* to ensure that $\mathrm{hol}_i : \pi_1 M \to \mathrm{SU}(2)$ are non-abelian.

Theorem 1.4 *Let C be a spherical cone-manifold with cone-angles $\leq \pi$, which is not Seifert fibered. Let $\{\mu_i, \ldots, \mu_N\}$ be the family of meridians, where $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$. Then the map*

$$X(\pi_1 M, \mathrm{SU}(2)) \to \mathbb{R}^N, \chi_i \mapsto (t_{\mu_1}(\chi_i), \ldots, t_{\mu_N}(\chi_i))$$

is a local diffeomorphism near $\chi_i = [\mathrm{hol}_i]$ for $i \in \{1, 2\}$.

As in the hyperbolic case we conclude local rigidity from this.

Corollary 1.5 (Local rigidity) *Let C be a spherical cone-manifold with cone-angles $\leq \pi$, which is not Seifert fibered. Then the set of cone-angles $\{\alpha_1, \ldots, \alpha_N\}$, where $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$, provides a local parametrization of the space of spherical cone-structures near the given structure on M. In particular, there are no deformations leaving the cone-angles fixed.*

The geometric significance of the cohomological result in the Euclidean case is not yet clear to the author. This is something to be worked out in the future.

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2 Cone-manifolds

In this chapter we define the notion of cone-manifold in the 2-dimensional and the 3-dimensional case. This may easily be generalized to higher dimensions.

Recall the definition of the generalized trigonometric functions

$$\mathrm{sn}_\kappa(r) = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}r) & : \ \kappa > 0 \\ r & : \ \kappa = 0 \\ \frac{1}{\sqrt{|\kappa|}}\sinh(\sqrt{|\kappa|}r) & : \ \kappa < 0 \end{cases}$$

$$\mathrm{cs}_\kappa(r) = \begin{cases} \cos(\sqrt{\kappa}r) & : \ \kappa > 0 \\ 1 & : \ \kappa = 0 \\ \cosh(\sqrt{|\kappa|}r) & : \ \kappa < 0 \end{cases}$$

These are the unique solutions of the ODE

$$f''(r) + \kappa f(r) = 0$$

with initital conditions

$$\mathrm{sn}_\kappa(0) = 0 \quad \text{and} \quad \mathrm{sn}'_\kappa(0) = 1$$
$$\mathrm{cs}_\kappa(0) = 1 \quad \text{and} \quad \mathrm{cs}'_\kappa(0) = 0\,.$$

If (N, g^N) is a Riemannian manifold we define for $\kappa \leq 0$ the κ-**cone** over N to be the space

$$\mathrm{cone}_\kappa N = \mathbb{R} \times N$$

equipped with the Riemannian metric

$$g = dr^2 + \mathrm{sn}^2_\kappa(r) g^N\,.$$

For $\kappa > 0$ we define the κ-**suspension** to be the space

$$\mathrm{susp}_\kappa N = (0, \pi/\sqrt{\kappa})$$

equipped with the Riemannian metric

$$g = dr^2 + \mathrm{sn}^2_\kappa(r) g^N\,.$$

For all $\kappa \in \mathbb{R}$ and $\varepsilon > 0$ small enough the truncated κ-cone $\mathrm{cone}_{\kappa,(0,\varepsilon)} N$ is defined in the obvious way.

Definition 2.1 A **cone-surface** S of curvature $\kappa \in \mathbb{R}$ is a compact, oriented surface which carries a length metric with the property that there are a finite number of points $\{x_1, \ldots, x_k\} \subset S$ (the *cone-points*) and numbers $\{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{R}^k_+$ (the *cone-angles*), such that $N = S \setminus \{x_1, \ldots, x_k\}$ is a smooth Riemannian manifold of curvature κ and for each i the ε-ball around a cone-point intersected with the smooth part $U_\varepsilon(x_i) = B_\varepsilon(x_i) \cap N$ is isometric with the κ-cone over the circle of length α_i.

We will also use the notation $\mathrm{int}\, S$ for $N = S \setminus \{x_1, \ldots, x_k\}$, the smooth part of the cone-surface. For $\kappa \in \{-1, 0, 1\}$ we will call S respectively hyperbolic, Euclidean or spherical. Let us call $(S, \{x_1, \ldots, x_k\})$ the topological type of S.
The following theorem is a straightforward extension of the classical Gauß-Bonnet theorem for surfaces:

Theorem 2.2 (Gauß-Bonnet) *Let S be a cone-surface of curvature κ and $N = S \setminus \{x_1, \ldots, x_k\}$. Then*

$$2\pi\chi(S) = \int_N \kappa \, dvol + \sum_{i=1}^{k} (2\pi - \alpha_i) \, .$$

Using this theorem it is easy to classify the spherical cone-surfaces S with cone-angles are $\leq \pi$. The underlying space has to be S^2 and we obtain two types:

$$S = \begin{cases} \mathbf{S}^2(\alpha, \beta, \gamma) & \text{or} \\ \mathbf{S}^2(\alpha) & \end{cases}$$

$\mathbf{S}^2(\alpha, \beta, \gamma)$ is the double of a spherical triangle with angles $\alpha/2, \beta/2, \gamma/2$. It is often suggestively called spherical turnover. $\mathbf{S}^2(\alpha)$ is the double of a spherical bigon with angle $\alpha/2$. It is sometimes called spherical spindle.
These cone-surfaces are rigid, i.e. they are determined up to isometry by the topological type and the set of cone-angles.

Definition 2.3 A **cone-3-manifold** C of curvature $\kappa \in \mathbb{R}$ is a compact, oriented 3-manifold which carries a length metric with the property that there is a distinguished subset $\Sigma \subset C$ (the *singular locus*) such that $M = C \setminus \Sigma$ is a smooth Riemannian manifold of curvature κ and for each $x \in \Sigma$ the ε-ball around x intersected with the smooth part $U_\varepsilon(x) = B_\varepsilon(x) \cap M$ is isometric with the κ-cone over the smooth part of a spherical cone-surface S_x.

We will also use the notation $\operatorname{int} C$ for $M = C \setminus \Sigma$, the smooth part of the cone-manifold. For $\kappa \in \{-1, 0, 1\}$ we will call C respectively hyperbolic, Euclidean or spherical. Let us call (C, Σ) the topological type of C.
If $x \in \Sigma$ is a singular point and $U_\varepsilon(x) = \operatorname{cone}_{\kappa,(0,\varepsilon)} \operatorname{int} S_x$, then we call S_x the link of x in C. The hypothesis that the underlying space C is a manifold implies that the links of singular points are cone-surfaces with underlying space S^2.
If the cone-angles are $\leq \pi$ we in particular obtain that links of singular points are either $\mathbf{S}^2(\alpha, \beta, \gamma)$ or $\mathbf{S}^2(\alpha)$. This implies that the singular locus Σ is a trivalent graph embedded into C.

Cone-manifolds with cone-angles $\leq 2\pi$ may be viewed as metric length spaces with curvature bounded from below in the sense of Alexandrov. A discussion from this point of view may be found in [BLP1].

3 Analysis on cone-manifolds

By analysis on C we mean analysis on $M = C \setminus \Sigma$, the smooth part of our cone-manifold. M is a smooth Riemannian manifold, but incomplete.
In this chapter we discuss some functional analytic properties of differential operators on noncompact manifolds. In contrast to the compact situation one has to distinguish more carefully between a differential operator acting on smooth, compactly supported sections of some vector-bundle and its closed realizations as an unbounded operator on the Hilbert space of L^2-sections.

3.1 Differential operators on noncompact manifolds

Let (M, g) be a Riemannian manifold (possibly noncompact, possibly incomplete) and let $(\mathcal{E}, h^{\mathcal{E}}), (\mathcal{F}, h^{\mathcal{F}})$ be hermitian vector-bundles over M. The naturally associated L^2-spaces $L^2(\mathcal{E}), L^2(\mathcal{F})$ depend on the equivalence classes of g and $h^{\mathcal{E}}, h^{\mathcal{F}}$.

We consider a differential operator P acting on sections of $\mathcal{E}$ as an unbounded, densely defined operator with domain the compactly supported sections:

$$P : L^2(\mathcal{E}) \supset \operatorname{dom} P = C^\infty_{cp}(\mathcal{E}) \longrightarrow L^2(\mathcal{F}).$$

The **formal adjoint** of a differential operator P

$$P^t : L^2(\mathcal{E}) \supset \operatorname{dom} P^t = C^\infty_{cp}(\mathcal{F}) \longrightarrow L^2(\mathcal{E})$$

is uniquely defined by the relation $\langle Ps, t\rangle = \langle s, P^t t\rangle$ for all $s, t \in C^\infty_{cp}$. P^t is again a differential operator, hence densely defined.
P is said to be **symmetric** (or **formally selfadjoint**) if $\langle Ps, t\rangle = \langle s, Pt\rangle$ holds for all $s, t \in C^\infty_{cp}$.
The domain of the **adjoint** of P is given as follows:

$$\operatorname{dom} P^* = \{s \in L^2 | u \mapsto \langle Pu, s\rangle \text{ bounded linear functional for } u \in \operatorname{dom} P\}.$$

Since P is densely defined there is a unique $t \in L^2$ such that $\langle Pu, s\rangle = \langle u, t\rangle$ for all $u \in \operatorname{dom} P$. Then let $P^*s = t$ by definition. P^* is a closed operator. Recall that a linear operator A is called (graph-) **closed** if $\operatorname{dom} A$ equipped with the graph norm $\|x\|_A = (\|x\|^2 + \|Ax\|^2)^{\frac{1}{2}}$ is complete.
P^* obviously extends P^t ($P^t \subset P^*$) so P^* is densely defined. Note that P is symmetric if and only if $P \subset P^*$.
A natural question to ask is if P admits closed extensions, and the answer is always yes. Define

$$P_{max} = (P^t)^*$$

and

$$P_{min} = P^{**}.$$

P^{**} is well-defined since P^* is densely defined. P^{**} then equals $\overline{P}$, the (graph-) closure of P, i.e. the domain of P_{min} can be characterized as follows:

$$\operatorname{dom} P_{min} = \{s \in L^2 | \exists (s_n)_{n\in\mathbb{N}} \subset \operatorname{dom} P \text{ with } s_n \xrightarrow{L^2} s \text{ such that}$$
$$(Ps_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } L^2\}$$

and $P_{min}(s) = \lim_{n\to\infty} s_n$.
We say that $Ps = t$ in the distributional sense if $\langle s, P^t u\rangle = \langle t, u\rangle$ holds for all $u \in C^\infty_{cp}$. The domain of P_{max} may then be written as:

$$\operatorname{dom} P_{max} = \{s \in L^2 | Ps \in L^2\}$$

and $P_{max}(s) = Ps$ in the distributional sense. Clearly $P_{max} \supset P_{min}$ and both are closed extensions of P. P_{max} is maximal with respect to having C^∞_{cp} in the domain of its adjoint. This condition means that P^*_{max} still extends P^t.
If P is symmetric we ask for selfadjoint extensions. Recall that a closed symmetric operator A is called **selfadjoint** if $A = A^*$. P is called **essentially selfadjoint** if P_{min} is selfadjoint. Since for a symmetric operator $P_{max} = P^*$ this is the case if and only if $P_{min} = P_{max}$. Selfadjoint extensions need not exist in general.
On the other hand if we assume that our operator P is semibounded there is alway a distinguished selfadjoint extension which preserves the lower bound. This feature will turn out to be particularly useful.
P **semibounded** means by definition that there exists $c \in \mathbb{R}$ such that $\langle s, Ps\rangle \geq c\langle s, s\rangle$ for all $s \in \operatorname{dom} P$.
Recall that a semibounded quadratic form $q : \operatorname{dom} q \times \operatorname{dom} q \to L^2$ with lower bound c is **closed** iff $\operatorname{dom} q$ equipped with the norm $\|x\|_q = (q(x) + (1 - c)\|x\|^2)^{1/2}$ is complete.

Theorem 3.1 (the Friedrichs extension) [RS, Thm. X.23] *Let P be a semi-bounded symmetric operator and let $q(s,t) = \langle s, Pt\rangle$ for $s,t \in \operatorname{dom} P$. Then q is a closable quadratic form and the closure $\overline{q}$ is the quadratic form of a unique selfadjoint operator P_F, the so-called Friedrichs extension of P. $\operatorname{dom} P_F$ is contained in $\operatorname{dom}\overline{q}$ and P_F is the only selfadjoint extension of P with this property. Furthermore, P_F satisfies the same lower bound as P.*

Theorem 3.2 (von Neumann) [RS, Thm. X.25] *Let A be a closed densely defined operator. Then A^*A with $\operatorname{dom} A^*A = \{x \in \operatorname{dom} A | Ax \in \operatorname{dom} A^*\}$ is selfadjoint.*

For a differential operator of the form $P = D^tD$ we obtain $q(s) = \langle Ds, Ds\rangle \geq 0$ and consequently $\operatorname{dom}\overline{q} = \operatorname{dom} D_{min}$. A consequence of von Neumann's theorem is (with $A = D_{min}(D_{max})$ respectively) that $D^t_{max}D_{min}$ and $D^t_{min}D_{max}$ are selfadjoint extensions of P.
On the other hand $\operatorname{dom} D^t_{max}D_{min}$ is obviously contained in $\operatorname{dom} D_{min} = \operatorname{dom}\overline{q}$. Therefore we get as an important corollary

Corollary 3.3 *$D^t_{max}D_{min}$ is the Friedrichs extension of D^tD.*

3.2 The de-Rham complex

Let $(\mathcal{E}, \nabla^{\mathcal{E}})$ be a flat vector-bundle equipped with a hermitian metric $h^{\mathcal{E}}$. The metric will not necessarily be assumed parallel.
We denote the exterior derivative coupled with the flat connection again by d. As an operator

$$d : \Omega^{\bullet}_{cp}(M, \mathcal{E}) \to \Omega^{\bullet+1}_{cp}(M, \mathcal{E})$$

d is uniquely determined by the relation $d(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|}\alpha \otimes \nabla s$, where α is an ordinary form and s a section of $\mathcal{E}$.
Let $\omega(h^{\mathcal{E}}) = (h^{\mathcal{E}})^{-1}(\nabla^{\mathcal{E}} h^{\mathcal{E}})$. $\omega(h^{\mathcal{E}})$ measures the deviation of $h^{\mathcal{E}}$ from being parallel. Then we get for the formal adjoint of d:

$$d^t = (-1)^{np+n+1} \star (d + \omega(h^{\mathcal{E}})) \star$$

on (compactly supported) $\mathcal{E}$-valued p-forms.
Following Cheeger, a choice of a closed extension $d^i_{min} \subset \widetilde{d^i} \subset d^i_{max}$ of d^i for each i will be called an **ideal boundary condition** for the de-Rham complex if

$$\widetilde{d^i}(\operatorname{dom}\widetilde{d^i}) \subset \operatorname{dom}\widetilde{d^{i+1}}$$

and

$$\widetilde{d^{i+1}} \circ \widetilde{d^i} = 0$$

hold for all i (cf. [Che]). In particular this means that

$$\ldots \longrightarrow \operatorname{dom}\widetilde{d^i} \xrightarrow{\widetilde{d^i}} \operatorname{dom}\widetilde{d^{i+1}} \longrightarrow \ldots$$

is a complex in the sense of homological algebra. This is a particular instance of a so-called *Hilbert complex*, see [BL] for the definition and a general discussion.
Note that d^i_{min} and d^i_{max} are itself ideal boundary conditions.
Given an ideal boundary condition $\widetilde{d}$, we may form the adjoint complex

$$\ldots \longrightarrow \operatorname{dom}(\widetilde{d^i})^* \xrightarrow{(\widetilde{d^i})^*} \operatorname{dom}(\widetilde{d^{i-1}})^* \longrightarrow \ldots,$$

where clearly $(d^i)^t_{min}) \subset (\widetilde{d^i})^* \subset (d^i)^t_{max}$ for all i. Uniqueness of the ideal boundary condition also implies $d^t_{min} = d^t_{max}$.
Recall that the **Hodge-Dirac** operator $D = d + d^t$ decomposes as a direct sum $D = D^{ev} \oplus D^{odd}$, where

$$D^{ev} : \Omega^{ev}_{cp}(M,\mathcal{E}) \longrightarrow \Omega^{odd}_{cp}(M,\mathcal{E})$$

and

$$D^{odd} = (D^{ev})^t : \Omega^{odd}_{cp}(M,\mathcal{E}) \longrightarrow \Omega^{ev}_{cp}(M,\mathcal{E})\,.$$

An ideal boundary condition $\widetilde{d}$ yields closed extensions of D, D^{ev} and D^{odd}:

$$D(\widetilde{d}) = \widetilde{d} + (\widetilde{d})^*$$

and

$$D(\widetilde{d})^{ev/odd} = (\widetilde{d} + (\widetilde{d})^*)^{ev/odd}\,.$$

For the particular ideal boundary conditions d_{min} and d_{max} we obtain

$$D(d_{min}) = d_{min} + d^t_{max}$$

and

$$D(d_{max}) = d_{max} + d^t_{min}\,.$$

Note that we do *not* claim that in general any of these extensions equals the minimal or maximal extension of D itself.

Lemma 3.4 *$D(\widetilde{d})^{ev}$ is a closed extension of D^{ev}. $D(\widetilde{d})^{odd} = (D(\widetilde{d})^{ev})^*$. Furthermore $D(\widetilde{d})$ is a selfadjoint extension of D.*

Proof. Since the ranges of $\widetilde{d}$ and $(\widetilde{d})^*$ are orthogonal, the assertion follows. ◻

Recall that the **Hodge-Laplace** operator is the square of the Hodge-Dirac operator:

$$\Delta = D^2 = dd^t + d^t d$$

A selfadjoint extension $\widetilde{D}$ of D yields an extension of Δ.

$$\Delta(\widetilde{D}) = \widetilde{D}^2$$

Lemma 3.5 *$\Delta(\widetilde{D})$ is a selfadjoint extension of Δ.*

Proof. The assertion follows from von Neumann's theorem **(3.2)**. ◻

If $\widetilde{D} = D(\widetilde{d})$ then we set

$$\Delta(\widetilde{d}) = \Delta(\widetilde{D}) = \widetilde{d}(\widetilde{d})^* + (\widetilde{d})^*\widetilde{d}$$

We have that

$$\Delta(d_{min}) = d_{min}d^t_{max} + d^t_{max}d_{min}$$

and

$$\Delta(d_{max}) = d_{max}d^t_{min} + d^t_{min}d_{max}$$

Note again that these extensions need not be equal to the minimal or maximal extension of Δ itself.

Lemma 3.6 $\Delta_F = D_{max}D_{min}$

Proof. The assertion follows from Corollary **(3.3)**. ⊡

We single out the following consequence since it is the basis for our main line of argument towards the adaption of the classical Bochner technique in our singular context.

Corollary 3.7 *If D is essentially selfadjoint, then $\Delta_F = \Delta(\widetilde{d})$ for any ideal boundary condititon $\widetilde{d}$, in particular $\Delta_F = \Delta(d_{max})$.*

Proof. If D is essentially selfadjoint, then since $D(\widetilde{d})$ is a selfadjoint extension of D, we obtain $D_{min} = D(\widetilde{d}) = D_{max}$. Now the assertion follows from the previous lemma. ⊡

Once essential selfadjointness of D is established, this result allows to extend lower bounds obtained for Δ on compactly supported forms to $\Delta(d_{max})$ on its respective domain. Our interest in this particular extension will become clear from the next section.

3.3 Hodge theory

The cohomology of the compactly supported de-Rham complex $\Omega^\bullet_{cp}(M, \mathcal{E})$ is by definition compactly supported cohomology

$$H^i_{cp}(M, \mathcal{E}) = \ker d^i \cap \Omega^i_{cp}(M, \mathcal{E}) / \operatorname{im} d^{i-1} \Omega^{i-1}_{cp}(M, \mathcal{E}) \,.$$

To define L^2**-cohomology** we consider the following subcomplex of $\Omega^\bullet(M, \mathcal{E})$:

$$\Omega^i_{L^2}(M, \mathcal{E}) = \{\omega \in \Omega^i(M, \mathcal{E}) | w \in L^2 \text{and}\, dw \in L^2\} \,,$$

which we will refer to as the smooth L^2-complex. By definition L^2-cohomology is the cohomology of the smooth L^2-complex, i.e.

$$H^i_{L^2}(M, \mathcal{E}) = \ker d^i \cap \Omega^i_{L^2}(M, \mathcal{E}) / d^{i-1} \Omega^{i-1}_{L^2}(M, \mathcal{E}) \,.$$

Note that d^i considered on $\Omega^i_{cp}(M, \mathcal{E})$ or on $\Omega^i_{L^2}(M, \mathcal{E})$ will in general not be closed, i.e. does not give rise to an ideal boundary condition as defined above. Recall that an ideal boundary condition yields a complex

$$\ldots \longrightarrow \operatorname{dom} \widetilde{d}^i \xrightarrow{\widetilde{d}^i} \operatorname{dom} \widetilde{d}^{i+1} \longrightarrow \ldots$$

Let us denote the cohomology of this complex by

$$\widetilde{H}^i = \ker \widetilde{d}^i / \operatorname{im} \widetilde{d}^{i-1}.$$

We define the $\widetilde{d}$**-harmonic** i-forms to be

$$\widetilde{\mathcal{H}}^i = \ker \widetilde{d}^i \cap \ker(\widetilde{d}^{i-1})^* \,.$$

For ideal boundary conditions there is a quite general Hodge theorem, which goes back to Kodaira (cf. [Kod]) in the case of the d_{max}-complex. For a slightly more general statement in the context of Hilbert complexes we refer to [BL].

Theorem 3.8 (weak Hodge-decomposition) *Let $\widetilde{d}$ be an ideal boundary condition for the de-Rham complex. Then for each i there is an orthogonal decomposition*

$$L^2(\Lambda^i T^*M \otimes \mathcal{E}) = \widetilde{\mathcal{H}}^i \oplus \overline{\operatorname{im} \widetilde{d}^{i-1}} \oplus \overline{\operatorname{im} (\widetilde{d}^i)^*} \,,$$

and furthermore

$$\widetilde{\mathcal{H}}^i = \ker \Delta^i(\widetilde{d}) = \ker D(\widetilde{d}) \cap L^2(\Lambda^i T^*M \otimes \mathcal{E}).$$

Proof. Note that $\ker \widetilde{d}^i \subset L^2(\Lambda^i T^*M \otimes \mathcal{E})$ is a closed subspace, since $\widetilde{d}^i$ is a closed operator. Therefore we can decompose

$$\begin{aligned} L^2(\Lambda^i T^*M \otimes \mathcal{E}) =& (\ker \widetilde{d}^i)^\perp \oplus \ker \widetilde{d}^i \\ =& (\ker \widetilde{d}^i)^\perp \oplus \overline{\operatorname{im} \widetilde{d}^{i-1}} \oplus \left(\ker \widetilde{d}^i \cap (\operatorname{im} \widetilde{d}^{i-1})^\perp\right) \end{aligned}$$

Now it is a standard fact from functional analysis that for a closed operator A we have $(\ker A)^\perp = \overline{\operatorname{im} A^*}$ and $(\operatorname{im} A)^\perp = \ker A^*$. Hence we can substitute

$$\begin{aligned} (\ker \widetilde{d}^i)^\perp &= \overline{\operatorname{im} (\widetilde{d}^i)^*} \\ (\operatorname{im} \widetilde{d}^{i-1})^\perp &= \ker (\widetilde{d}^{i-1})^* \end{aligned}$$

and thus obtain the desired decomposition. The additional statement is straightforward to check. ◻

For $\widetilde{d} = d_{max}$ we obtain the d_{max}-harmonic, or L^2-**harmonic**, i-forms as

$$\mathcal{H}^i_{max} = \ker d^i_{max} \cap \ker (d^{i-1})^t_{min}$$

and the weak Hodge-decomposition has the form

$$L^2(\Lambda^i T^*M \otimes \mathcal{E}) = \mathcal{H}^i_{max} \oplus \overline{\operatorname{im} d^{i-1}_{max}} \oplus \overline{\operatorname{im} (d^i)^t_{min}}.$$

We define a map

$$\begin{aligned} \iota : \widetilde{\mathcal{H}}^i &\longrightarrow \widetilde{H}^i \\ \alpha &\longmapsto \alpha + \operatorname{im} \widetilde{d}^{i-1} \end{aligned}$$

Injectivity of ι is equivalent with $\operatorname{im} \widetilde{d}^{i-1} \cap \ker (\widetilde{d}^{i-1})^* = 0$, which is always the case, since

$$\overline{\operatorname{im} \widetilde{d}^{i-1}} = (\ker (\widetilde{d}^{i-1})^*)^\perp .$$

Surjectivity of ι is equivalent with

$$\operatorname{im} \widetilde{d}^{i-1} = \overline{\operatorname{im} \widetilde{d}^{i-1}} ,$$

therefore we obtain the following enhancement of the Hodge decomposition, which is due to Cheeger (cf. [Che]) in the case of the d_{max}-complex. Again a more general statement may be found in [BL].

Theorem 3.9 (strong Hodge-decomposition) *If $\operatorname{im} \widetilde{d}^{i-1}$ is closed for all i, then there is an orthogonal decomposition*

$$L^2(\Lambda^i T^*M \otimes \mathcal{E}) = \widetilde{\mathcal{H}}^i \oplus \operatorname{im} \widetilde{d}^{i-1} \oplus \operatorname{im} (\widetilde{d}^i)^*,$$

and furthermore $\iota : \widetilde{\mathcal{H}}^i \to \widetilde{H}^i$ is an isomorphism.

A sufficient condition for $\widetilde{d}^{i-1}$ to have closed range is finite dimensionality of $\widetilde{H}^i$ on the one hand, since $\ker \widetilde{d}^i / \operatorname{im} \widetilde{d}^{i-1}$ finite dimensional implies that $\operatorname{im} \widetilde{d}^{i-1}$ is closed in $\ker \widetilde{d}^i$, hence in $L^2(\Lambda^i T^*M \otimes \mathcal{E})$. Note that by the closed-range theorem $(\widetilde{d}^i)^*$ has closed range if and only if $\widetilde{d}^i$ has closed range.
On the other hand, if $D(\widetilde{d})^{ev}$ has closed range, then $\widetilde{d}^i$ and $(\widetilde{d}^{i+1})^*$ will have closed range for all i even. Similarly, if $D(\widetilde{d})^{odd}$ has closed range, then $\widetilde{d}^i$ and $(\widetilde{d}^{i+1})^*$ will have closed range for all i odd. Since $D(\widetilde{d})^{odd} = (D(\widetilde{d})^{ev})^*$, the closed-range theorem implies that $D(\widetilde{d})^{ev}$ has closed range if and only $D(\widetilde{d})^{odd}$ has closed range. It is easy to show that $D(\widetilde{d})^{ev}$ has closed range if $\operatorname{dom} D(\widetilde{d})^{ev}$ equipped with the graph norm embeds into $L^2(\Lambda^{ev} T^*M \otimes \mathcal{E})$ compactly.
This latter condition is related to the question of discreteness of the spectra of the operators $D(\widetilde{d})$ and $\Delta(\widetilde{d})$. Recall that an operator is said to have discrete spectrum if its spectrum consists of a discrete set of eigenvalues with finite multiplicities.

3.4 Smoothness

For any ideal boundary condition $\widetilde{d}$ we may define the following smooth subcomplex of the Hilbert-complex associated with $\widetilde{d}$:

$$\ldots \longrightarrow \operatorname{dom} \widetilde{d}^i \cap \Omega^i(M,\mathcal{E}) \xrightarrow{\widetilde{d}^i} \operatorname{dom} \widetilde{d}^{i+1} \cap \Omega^{i+1}(M,\mathcal{E}) \longrightarrow \ldots$$

Let us denote the cohomology of this complex by $\widetilde{H}^i_{C^\infty}$, i.e.

$$\widetilde{H}^i_{C^\infty} = \ker \widetilde{d}^i \cap \Omega^i(M,\mathcal{E}) / \widetilde{d}^{i-1}(\operatorname{dom} \widetilde{d}^{i-1} \cap \Omega^{i-1}(M,\mathcal{E}))\,.$$

Note that for $\widetilde{d} = d_{max}$ we obtain the smooth L^2-complex:

$$\operatorname{dom} d^i_{max} \cap \Omega^i(M,\mathcal{E}) = \Omega^i_{L^2}(M,\mathcal{E})\,.$$

The following theorem is due to Cheeger in the case of the d_{max}-complex (cf. [Che]), while [BL] consider the case of ideal boundary conditions for a general elliptic complex.

Theorem 3.10 [BL, Thm. 3.5] *Let $\widetilde{d}$ be an ideal boundary condition. Then the inclusion of the smooth subcomplex* $\operatorname{dom} \widetilde{d}^i \cap \Omega^i(M,\mathcal{E}) \to \operatorname{dom} \widetilde{d}^i$ *induces an isomorphism on the level of cohomology, i.e.* $\widetilde{H}^i_{C^\infty} \cong \widetilde{H}^i$.

Corollary 3.11 *The inclusion of the smooth L^2-complex* $\Omega^i_{L^2}(M,\mathcal{E}) \to \operatorname{dom} d^i_{max}$ *induces an isomorphism* $H^i_{L^2}(M,\mathcal{E}) \cong H^i_{max}$.

Note at this point that Hodge theory forces us to consider the particular ideal boundary condition d_{max} if we are interested in L^2-cohomology. This will be crucial for the further discussion.

4 Spectral properties of cone-manifolds

In this chapter we essentially use the techniques of Brüning and Seeley to analyze the closed extensions of the Hodge-Dirac operator on a a 3-dimensional cone-manifold. The main reference for the first order case will be [BS]. The analysis relies heavily on the fact that the spaces we consider are *locally conical*, i.e. neighbourhoods of points are isometric to (κ-)cones over spaces of lower dimension. This allows to apply separation of variables techniques.
To keep the exposition self-contained here, we describe these techniques in detail. Furthermore we adopt a more elementary viewpoint than in [BS], in particular we give a direct argument for discreteness of the relevant operators.
Let us further mention that [BS] deal with isolated conical singularities only, i.e. the links of singular points are smooth Riemannian manifolds, where in our case we have to allow the links of singular points to be again singular, namely the spherical cone-surfaces $\mathbf{S}^2(\alpha,\beta,\gamma)$ and $\mathbf{S}^2(\alpha)$.
It turns out that local analysis near singular points reduces in some respects to spectral analysis on the link of the singular point. As a consequence we will have to investigate spectral properties of cone-surfaces first, and then use this information to study 3-dimensional cone-manifolds. This will be made precise.

4.1 Separation of variables

Let us consider the following model situation: Let (N, g^N) be a Riemannian manifold of dimension n and $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} N$. We have

$$g^{U_\varepsilon}_{(r,x)} = dr^2 + \operatorname{sn}^2_\kappa(r) g^N_x\,,$$

where $r \in (0,\varepsilon)$ and $x \in N$. We may think of N as the (smooth part of the) link S_x of a singular point x in a cone-manifold, U_ε serves as a model for the (smooth part of the) ε-neighbourhood $U_\varepsilon(x)$ of a singular point x in M.
Furthermore, if a flat vector-bundle $(\mathcal{E},\nabla^{\mathcal{E}})$ is given on N, it extends in a unique way to a vector-bundle on U_ε with a flat connection, which is trivial in the radial direction. We will again denote this flat vector-bundle by $(\mathcal{E},\nabla^{\mathcal{E}})$.
We identify $\mathcal{E}$-valued p-forms on U_ε with pairs of r-dependent forms on N via

$$\begin{aligned} C^\infty_{cp}\left((0,\varepsilon),\Omega^{p-1}_{cp}(N,\mathcal{E})\oplus\Omega^p_{cp}(N,\mathcal{E})\right) &\longrightarrow \Omega^p_{cp}(U_\varepsilon,\mathcal{E}) \\ (\phi,\psi) &\longmapsto \mathrm{sn}_\kappa(r)^{(p-1)-\frac{n}{2}}\phi\wedge dr + \mathrm{sn}_\kappa(r)^{p-\frac{n}{2}}\psi. \end{aligned}$$

Since

$$\int_0^\varepsilon\int_N |\phi|^2_N\, dr dvol_N = \int_{U_\varepsilon} \mathrm{sn}_\kappa(r)^{2(p-1)-n}|\phi\wedge dr|^2_{U_\varepsilon}\, dvol_{U_\varepsilon}$$

and

$$\int_0^\varepsilon\int_N |\psi|^2_N\, dr dvol_N = \int_{U_\varepsilon} \mathrm{sn}_\kappa(r)^{2p-n}|\psi|^2_{U_\varepsilon}\, dvol_{U_\varepsilon}\ ,$$

we have a corresponding identification of L^2-spaces:

$$L^2\left((0,\varepsilon),L^2(\Lambda^{p-1}T^*N\otimes\mathcal{E})\oplus L^2(\Lambda^pT^*N\otimes\mathcal{E})\right)\cong L^2(\Lambda^pT^*U_\varepsilon\otimes\mathcal{E}).$$

Since

$$d_{U_\varepsilon}\left(\mathrm{sn}_\kappa(r)^{(p-1)-\frac{n}{2}}\phi\wedge dr\right) = \mathrm{sn}_\kappa(r)^{(p-1)-\frac{n}{2}} d_N\phi\wedge dr$$

and

$$\begin{aligned} d_{U_\varepsilon}\left(\mathrm{sn}_\kappa(r)^{p-\frac{n}{2}}\psi\right) &= (-1)^p(p-\tfrac{n}{2})\,\mathrm{sn}_\kappa(r)^{(p-1)-\frac{n}{2}}\,\mathrm{cs}_\kappa(r)\psi\wedge dr \\ &+(-1)^p\,\mathrm{sn}_\kappa(r)^{p-\frac{n}{2}}\tfrac{\partial}{\partial r}\psi\wedge dr + \mathrm{sn}_\kappa(r)^{p-\frac{n}{2}} d_N\psi \end{aligned}$$

we have that

$$d^p_{U_\varepsilon} = \begin{pmatrix} \mathrm{sn}_\kappa(r)^{-1}d^{p-1}_N & (-1)^p\left[\frac{\partial}{\partial r}+(p-\frac{n}{2})\,\mathrm{ct}_\kappa(r)\right] \\ 0 & \mathrm{sn}_\kappa(r)^{-1}d^p_N \end{pmatrix}$$

considered as an operator

$$C^\infty_{cp}\left((0,\varepsilon),\Omega^{p-1}_{cp}(N,\mathcal{E})\oplus\Omega^p_{cp}(N,\mathcal{E})\right)\longrightarrow C^\infty_{cp}\left((0,\varepsilon),\Omega^p_{cp}(N,\mathcal{E})\oplus\Omega^{p+1}_{cp}(N,\mathcal{E})\right).$$

Passing to the formal adjoints we immediately obtain

$$(d^t_{U_\varepsilon})_p = (d^{p-1}_{U_\varepsilon})^t = \begin{pmatrix} \mathrm{sn}_\kappa(r)^{-1}(d^t_N)_{p-1} & 0 \\ (-1)^p\left[\frac{\partial}{\partial r}+(\frac{n}{2}-p+1)\,\mathrm{ct}_\kappa(r)\right] & \mathrm{sn}_\kappa(r)^{-1}(d^t_N)_p \end{pmatrix}$$

considered as an operator

$$C^\infty_{cp}\left((0,\varepsilon),\Omega^{p-1}_{cp}(N,\mathcal{E})\oplus\Omega^p_{cp}(N,\mathcal{E})\right)\longrightarrow C^\infty_{cp}\left((0,\varepsilon),\Omega^{p-2}_{cp}(N,\mathcal{E})\oplus\Omega^{p-1}_{cp}(N,\mathcal{E})\right).$$

We may identify r-dependent forms on N of arbitrary degree with either even or odd forms on U_ε:

$$\begin{aligned} C^\infty_{cp}\left((0,\varepsilon),\textstyle\bigoplus_{p=0}^n\Omega^p_{cp}(N,\mathcal{E})\right) &\longrightarrow \Omega^{ev}_{cp}(U_\varepsilon,\mathcal{E}) \\ (\phi^0,\ldots,\phi^n) &\longmapsto \sum_{i=0}^{[\frac{n-1}{2}]}\mathrm{sn}_\kappa(r)^{2i+1-\frac{n}{2}}\phi^{2i+1}\wedge dr \\ &\qquad + \sum_{i=0}^{[\frac{n}{2}]}\mathrm{sn}_\kappa(r)^{2i-\frac{n}{2}}\phi^{2i} \end{aligned}$$

and

$$
\begin{aligned}
C^{\infty}_{cp}\left((0,\varepsilon), \textstyle\bigoplus_{p=0}^{n} \Omega^{p}_{cp}(N,\mathcal{E})\right) &\longrightarrow \Omega^{odd}_{cp}(U_\varepsilon, \mathcal{E}) \\
(\phi^0, \dots, \phi^n) &\longmapsto \sum_{i=0}^{[\frac{n}{2}]} \mathrm{sn}_\kappa(r)^{2i-\frac{n}{2}} \phi^{2i} \wedge dr \\
&\quad + \sum_{i=0}^{[\frac{n-1}{2}]} \mathrm{sn}_\kappa(r)^{2i+1-\frac{n}{2}} \phi^{2i+1}
\end{aligned}
$$

We obtain that the even part of the Hodge-Dirac operator

$$D^{ev}_{U_\varepsilon} : \Omega^{ev}_{cp}(U_\varepsilon, \mathcal{E}) \longrightarrow \Omega^{odd}_{cp}(U_\varepsilon, \mathcal{E})$$

may be written with respect to these decompositions as

$$D^{ev}_{U_\varepsilon} = \frac{\partial}{\partial r} + \frac{1}{\mathrm{sn}_\kappa(r)} B_\kappa(r)$$

considered as an operator

$$C^{\infty}_{cp}\left((0,\varepsilon), \textstyle\bigoplus_{p=0}^{n} \Omega^{p}_{cp}(N,\mathcal{E})\right) \longrightarrow C^{\infty}_{cp}\left((0,\varepsilon), \textstyle\bigoplus_{p=0}^{n} \Omega^{p}_{cp}(N,\mathcal{E})\right),$$

where

$$B_\kappa(r) = D_N + \begin{pmatrix} \mathrm{cs}_\kappa(r) c_0 & & \\ & \ddots & \\ & & \mathrm{cs}_\kappa(r) c_n \end{pmatrix}$$

with

$$c_p = (-1)^p (p - \tfrac{n}{2}).$$

Note that $\lim_{r\to 0} B_\kappa(r)$ is independent of $\kappa \in \mathbb{R}$, more precisely we have

$$\lim_{r \to 0} B_\kappa(r) = D_N + \begin{pmatrix} c_0 & & \\ & \ddots & \\ & & c_n \end{pmatrix}.$$

Definition 4.1 (model operator) Let $B = \lim_{r\to 0} B_\kappa(r)$ and

$$P^\kappa_B = \frac{\partial}{\partial r} + \frac{1}{\mathrm{sn}_\kappa(r)} B\,.$$

The operator P^κ_B serves as a model operator for the even part of the Hodge-Dirac operator on the cone, since it captures its essential analytic features. This is expressed by the following lemma:

Lemma 4.2 *$\mathrm{dom}(D^{ev}_{U_\varepsilon})_{max} = \mathrm{dom}(P^\kappa_B)_{max}$, moreover the graph norms $\|\cdot\|_{D^{ev}_{U_\varepsilon}}$ and $\|\cdot\|_{P^\kappa_B}$ are equivalent, in particular $\mathrm{dom}(D^{ev}_{U_\varepsilon})_{min} = \mathrm{dom}(P^\kappa_B)_{min}$.*

Proof. Since

$$\frac{B_\kappa(r) - B}{\mathrm{sn}_\kappa(r)} = \frac{\mathrm{cs}_\kappa(r) - 1}{\mathrm{sn}_\kappa(r)} \begin{pmatrix} c_0 & & \\ & \ddots & \\ & & c_n \end{pmatrix}$$

and

$$\lim_{r\to 0} \frac{\mathrm{cs}_\kappa(r) - 1}{\mathrm{sn}_\kappa(r)} = 0\,,$$

we see that $D^{ev}_{U_\varepsilon}$ differs from P^κ_B just by a bounded 0-th order term. From this we easily obtain that the domains of the maximal extensions of the operators coincide and that the graph norms are equivalent.
Since the domain of the minimal extension is just the closure of C^∞_{cp} with respect to the graph norm, we arrive at the assertion of the lemma. ◻

4.2 The radial equation

The operator B is obviously symmetric on $\Omega^\bullet_{cp}(N, \mathcal{E})$. Note also that B does not depend on the radial variable $r \in (0, \varepsilon)$ any more. If B admits a selfadjoint extension, we can use the spectral decomposition of $L^2(\Lambda^\bullet T^*N, \mathcal{E})$ to transform the model operator P^κ_B into a direct sum of operators P^κ_b on the interval $(0, \varepsilon)$ parametrized by the spectral values. This will work out particularly well, if B is essentially selfadjoint and has discrete spectrum.

For $b \in \mathbb{R}$ let

$$P^\kappa_b = \frac{\partial}{\partial r} + \frac{b}{\mathrm{sn}_\kappa(r)}\,.$$

We will consider P^κ_b acting on $C^\infty_{cp}(0,1)$. Furthermore let $P_b = P^0_b$, i.e.

$$P_b = \frac{\partial}{\partial r} + \frac{b}{r}\,.$$

It is enough to study the operator P_b in view of the following lemma.

Lemma 4.3 $\mathrm{dom}(P^\kappa_b)_{max} = \mathrm{dom}(P_b)_{max}$, *moreover the graph norms* $\|\cdot\|_{P^\kappa_b}$ *and* $\|\cdot\|_{P_b}$ *are equivalent, in particular* $\mathrm{dom}(P^\kappa_b)_{min} = \mathrm{dom}(P_b)_{min}$.

Proof. Since $P^\kappa_b - P_b = \varphi(r)b$ with

$$\varphi(r) = \frac{1}{\mathrm{sn}_\kappa(r)} - \frac{1}{r}$$

and

$$\lim_{r\to 0} \varphi(r) = 0\,,$$

we see that P^κ_b differs from P_b just by a bounded 0-th order term. In the same way as before this implies the assertion of the lemma. ◻

Note that

$$(P_b f)(r) = r^{-b}\tfrac{\partial}{\partial r}(r^b f)\,,$$

therefore $P_b f = 0$ if and only if

$$f(r) = f(1) r^{-b}\,,$$

and $P_b f = g$ if and only if

$$f(r) = f(1) r^{-b} + r^{-b}\int_1^r \varrho^b g(\varrho)d\varrho\,.$$

For any subinterval $(\delta, 1) \subset (0,1)$ the graph norm of P_b is equivalent to the ordinary H^1-norm, since $\frac{1}{r} \in L^\infty(\delta, 1)$. H^1-functions - more generally: $W^{1,1}$-functions - on $(\delta, 1)$ are absolutely continuous on $[\delta, 1]$, hence differentiable almost everywhere. For absolutely continuous functions the fundamental theorem of calculus holds, i.e. $\varphi \in AC([\delta,1])$ if and only if $\varphi(r) = \varphi(1) + \int_1^r \varphi'(\varrho)d\varrho$ for $r \in [\delta, 1]$. Therefore the above integral representation remains valid for $f \in \mathrm{dom}(P_b)_{max}$ (take $\varphi(r) = r^b f(r)$).

Let us also mention in this context that by Sobolev embedding $H^1(\delta,1) \hookrightarrow C^{0,\frac{1}{2}}([\delta,1])$ continuously.
It follows from either of these observations that $f \in \mathrm{dom}(P_b)_{max}$ is continuous on $(0,1)$ and has a continuous boundary value at $r=1$, i.e. $f \in C^0((0,1])$.

We define two integral operators acting on $L^2(0,\infty)$:

$$(T_{b,1}g)(r) = r^{-b}\int_1^r \varrho^b g(\varrho)d\varrho\,,$$

where b is arbitrary, and

$$(T_{b,0}g)(r) = r^{-b}\int_0^r \varrho^b g(\varrho)d\varrho\,,$$

for $b > -\frac{1}{2}$. Note that $b > -\frac{1}{2}$ implies that $r^b \in L^2(0,1)$ and therefore with the Cauchy-Schwarz inequality $\int_0^r \varrho^b g(\varrho)d\varrho < \infty$.

Lemma 4.4 [BS, Lemma 2.1] *For $g \in L^2(0,1)$ we have the estimates*

$$|(T_{b,0}g)(r)| \le r^{\frac{1}{2}}(2b+1)^{-\frac{1}{2}}\left(\int_0^r |g(\varrho)|^2 d\varrho\right)^{\frac{1}{2}}$$

for $b > -\frac{1}{2}$, and

$$|(T_{b,1}g)(r)| \le \begin{cases} r^{\frac{1}{2}}|2b+1|^{-\frac{1}{2}}\|g\|_{L^2(0,1)} & , \quad b < -\frac{1}{2} \\ r^{\frac{1}{2}}|\log r|^{\frac{1}{2}}\|g\|_{L^2(0,1)} & , \quad b = -\frac{1}{2} \\ r^{-b}(2b+1)^{-\frac{1}{2}}\|g\|_{L^2(0,1)} & , \quad b > -\frac{1}{2} \end{cases},$$

in particular $T_{b,1}g \in L^2(0,1)$ if $b < \frac{1}{2}$.

Proof. With the Cauchy-Schwarz inequality we have

$$\begin{aligned} |(T_{b,0}g)(r)| &\le r^{-b}\left(\int_0^r \varrho^{2b}\right)^{\frac{1}{2}}\left(\int_0^r |g(\varrho)|^2\right)^{\frac{1}{2}} \\ &= r^{-b}\left(\left[\frac{\varrho^{2b+1}}{2b+1}\right]_0^r\right)^{\frac{1}{2}}\left(\int_0^r |g(\varrho)|^2\right)^{\frac{1}{2}} \\ &= r^{\frac{1}{2}}(2b+1)^{-\frac{1}{2}}\left(\int_0^r |g(\varrho)|^2\right)^{\frac{1}{2}}, \end{aligned}$$

which proves the first estimate for $b > -\frac{1}{2}$, and for $b \neq -\frac{1}{2}$

$$\begin{aligned} |(T_{b,1}g)(r)| &\le r^{-b}\left|\left[\frac{\varrho^{2b+1}}{2b+1}\right]_1^r\right|^{\frac{1}{2}}\left|\int_1^r |g(\varrho)|^2\right|^{\frac{1}{2}} \\ &= r^{-b}\left|r^{2b+1}-1\right|^{\frac{1}{2}}|2b+1|^{-\frac{1}{2}}\left|\int_1^r |g(\varrho)|^2\right|^{\frac{1}{2}} \\ &\le r^{-b}\left|r^{2b+1}-1\right|^{\frac{1}{2}}|2b+1|^{-\frac{1}{2}}\|g\|_{L^2(0,1)}\,. \end{aligned}$$

Now for $b < -\frac{1}{2}$ and $r \in (0,1)$ we have $\left|r^{2b+1}-1\right| \le r^{2b+1}$ and therefore

$$|(T_{b,1}g)(r)| \le r^{\frac{1}{2}}\,|2b+1|^{-\frac{1}{2}}\,\|g\|_{L^2(0,1)}\,,$$

while for $b > -\frac{1}{2}$ and $r \in (0,1)$ $\left|r^{2b+1} - 1\right| \leq 1$ and therefore

$$|(T_{b,1}g)(r)| \leq r^{-b}(2b+1)^{-\frac{1}{2}} \|g\|_{L^2(0,1)} .$$

For $b = -\frac{1}{2}$ we obtain

$$\begin{aligned} |(T_{b,1}g)(r)| &\leq r^{\frac{1}{2}} |\log r|^{\frac{1}{2}} \left| \int_1^r |g(\varrho)|^2 \right|^{\frac{1}{2}} \\ &\leq r^{\frac{1}{2}} |\log r|^{\frac{1}{2}} \|g\|_{L^2(0,1)} , \end{aligned}$$

which gives the remaining estimate. ⊡

Lemma 4.5 (decay estimates) *Let $f \in \mathrm{dom}(P_b)_{max}$. Then for $r \in (0,1)$ and with $g = P_b f$ we have*

$$|f(r)| \leq \begin{cases} r^{\frac{1}{2}}(2b+1)^{-\frac{1}{2}} \left(\int_0^r |g(\varrho)|^2\right)^{\frac{1}{2}} & , \quad b \geq \frac{1}{2} \\ r^{-b}|f(1)| + r^{-b}(2b+1)^{-\frac{1}{2}}\|g\|_{L^2(0,1)} & , \quad b \in (-\frac{1}{2}, \frac{1}{2}) \\ r^{\frac{1}{2}}|f(1)| + r^{\frac{1}{2}}|\log r|^{\frac{1}{2}}\|g\|_{L^2(0,1)} & , \quad b = -\frac{1}{2} \\ r^{-b}|f(1)| + r^{\frac{1}{2}}|2b+1|^{-\frac{1}{2}}\|g\|_{L^2(0,1)} & , \quad b < -\frac{1}{2} \end{cases} .$$

Proof. The estimates for $b < \frac{1}{2}$ follow directly from the integral representation

$$f(r) = r^{-b} f(1) + (T_{b,1}g)(r)$$

and the corresponding estimates for $T_{b,1}g$ from the preceding lemma. For the case $b \geq \frac{1}{2}$ we observe that for $b \geq \frac{1}{2}$ (in fact already for $b > -\frac{1}{2}$) $r^b \in L^2(0,1)$, hence $r^b g \in L^1(0,1)$ by the Cauchy-Schwarz inequality. This implies that $r^b f$ has its distributional derivative in $L^1(0,1)$ and is therefore absolutely continuous on $[0,1]$. We obtain

$$f(r) = r^{-b} C + (T_{b,0}g)(r)$$

Now $r^{-b} \notin L^2(0,1)$ for $b \geq \frac{1}{2}$, therefore $C = 0$, so the estimate for $T_{b,0}g$ gives the result. ⊡

Corollary 4.6 *Let $f \in \mathrm{dom}(P_b)_{max}$ and $r \in (0,1)$. If $b \notin (-\frac{1}{2}, \frac{1}{2})$, then*

$$|f(r)| \leq C(b) r^{\frac{1}{2}} (1 + |\log r|^{\frac{1}{2}}) \|f\|_{P_b} ,$$

in particular $f \in C^0([0,1])$ with $f(0) = 0$, while if $b \in (-\frac{1}{2}, \frac{1}{2})$, then

$$|f(r)| \leq C(b) r^{-b} \|f\|_{P_b} .$$

Proof. The case $b \geq \frac{1}{2}$ follows directly from the above estimates. For the other cases we again refer to the integral representation

$$f(r) = r^{-b} f(1) + (T_{b,1}g)(r)$$

and observe that $r^{-b} f(1) \in L^2(0,1)$ for $b < \frac{1}{2}$. Therefore the bound on $T_{b,1}g$ translates into a bound on $|f(1)|$ in terms of $\|f\|_{L^2(0,1)}$ and $\|g\|_{L^2(0,1)}$. This plugged into the decay estimates gives the result, which clearly implies that $f(r) = o(1)$ as $r \to 0$ in the first case. ⊡

Next we prove a property analogous to the L^2-Stokes property.

Proposition 4.7 (integration by parts) *Let $\varphi \in C^\infty(0,1)$ be a cut-off function with $\varphi \equiv 1$ near 0 and $\varphi \equiv 0$ near 1. For $u \in \mathrm{dom}(P_b)_{max}$ let $f = \varphi u \in \mathrm{dom}(P_b)_{max}$, and let $g \in \mathrm{dom}(P_b^t)_{max}$. Then for $b \notin (-\frac{1}{2}, \frac{1}{2})$ the following holds:*

$$\langle (P_b)_{max} f, g\rangle_{L^2(0,1)} = \langle f, (P_b^t)_{max} g\rangle_{L^2(0,1)}$$

Proof. With $(P_b)^t = -P_{-b}$ we calculate

$$\begin{aligned}
\langle (P_b)_{max} f, g\rangle_{L^2(0,1)} &= \int_0^1 \left(\frac{\partial f}{\partial r} + \frac{rf}{b}\right) g \\
&= \lim_{\delta\to 0} \left\{ \int_\delta^1 \left(\frac{\partial f}{\partial r}\right) g + \int_\delta^1 \left(\frac{rf}{b}\right) g \right\} \\
&= \lim_{\delta\to 0} \left\{ [fg]_\delta^1 - \int_\delta^1 f\left(\frac{\partial g}{\partial r}\right) + \int_\delta^1 f\left(\frac{rg}{b}\right) \right\} \\
&= \lim_{\delta\to 0} \{f(1)g(1) - f(\delta)g(\delta)\} + \langle f, (P_b^t)_{max} g\rangle_{L^2(0,1)} \ .
\end{aligned}$$

Now $f(1) = 0$ and $\lim_{\delta\to 0} f(\delta)g(\delta) = 0$ according to the decay estimates. Therefore

$$\lim_{\delta\to 0} \{f(1)g(1) - f(\delta)g(\delta)\} = 0$$

and we obtain the result. ⊡

This statement becomes wrong, if we allow $b \in (-\frac{1}{2}, \frac{1}{2})$. To see this, let $f(r) = \varphi(r) r^{-b}$ with φ as above and $g(r) = r^b$. Note that $P_b(r \mapsto r^{-b}) = P_b^t(r \mapsto r^b) = 0$, so clearly $f \in \mathrm{dom}(P_b)_{max}$ and $g \in \mathrm{dom}(P_b^t)_{max}$. But on the other hand

$$\lim_{\delta\to 0} \{f(1)g(1) - f(\delta)g(\delta)\} = 0 - \lim_{\delta\to 0} f(\delta)g(\delta) = -1\,,$$

so we have a boundary contribution.

The preceding result allows us to conclude that we do not have to impose boundary conditions for P_b at 0, if (and only if) $b \notin (-\frac{1}{2}, \frac{1}{2})$.

Corollary 4.8 *Let $\varphi \in C^\infty(0,1)$ be a cut-off function with $\varphi \equiv 1$ near 0 and $\varphi \equiv 0$ near 1. For $u \in \mathrm{dom}(P_b)_{max}$ let $f = \varphi u \in \mathrm{dom}(P_b)_{max}$. Then $f \in \mathrm{dom}(P_b)_{min}$ for $b \notin (-\frac{1}{2}, \frac{1}{2})$.*

Put in another way, this means, that f may be approximated by compactly supported functions in the graph norm of P_b, i.e. there exists a sequence $f_n \in C^\infty_{cp}(0,1)$ with $\|f - f_n\|_{P_b} \to 0$.

Proof. For all $g \in \mathrm{dom}(P_b^t)_{max}$ we have

$$\langle (P_b)_{max} f, g\rangle_{L^2(0,1)} = \langle f, (P_b^t)_{max} g\rangle_{L^2(0,1)}$$

This means that $f \in \mathrm{dom}(P_b^t)^*_{max} = \mathrm{dom}(P_b)_{min}$. ⊡

Let $P_B^\kappa = \frac{\partial}{\partial r} + \mathrm{sn}_\kappa(r)^{-1} B$ acting on $C^\infty_{cp}((0,1)\times N)$. We will assume that B is essentially selfadjoint on $C^\infty_{cp}(N)$, i.e. in equivalent terms $B_{max} = B_{min}$, since B is symmetric,. We will furthermore assume that B has discrete spectrum. Let $\{\Psi_b\}_{b\in\mathrm{spec}\,B}$ be the collection of normed eigensections repeated according to multiplicities such that $B\Psi_b = b\Psi_b$. By interior elliptic regularity, the Ψ_b are smooth. We have an orthogonal decomposition of L^2-spaces

$$L^2(N) = \overline{\bigoplus_{b\in\mathrm{spec}\,B} \mathbb{R}\otimes\Psi_b}$$

and

$$L^2\left((0,1)\times N\right)=\overline{\bigoplus_{b\in\operatorname{spec}B} L^2(0,1)\otimes\Psi_b}\,.$$

The closure is to be taken with respect to the L^2-norm. For $f\in L^2\left((0,1)\times N\right)$ we have an L^2-convergent expansion

$$f=\sum_{b\in\operatorname{spec}B} f_b\otimes\Psi_b\,,$$

where

$$f_b(r)=\int_N \left(f(r,x),\Psi_b(x)\right)dx.$$

Obviously we have

$$\|f\|^2_{L^2((0,1)\times N)}=\sum_{b\in\operatorname{spec}B}\|f_b\|^2_{L^2(0,1)}\,.$$

Lemma 4.9 *Let $f\in L^2\left((0,1)\times N\right)$. Then $P^\kappa_B f=g$ with $g\in L^2\left((0,1)\times N\right)$ if and only if $P^\kappa_b f_b=g_b$ for all $b\in\operatorname{spec}B$. In particular $f\in\operatorname{dom}(P^\kappa_B)_{max}$ if and only if $f_b\in\operatorname{dom}(P^\kappa_b)_{max}$ for all $b\in\operatorname{spec}B$.*

Proof. Let us assume first that $P^\kappa_B f=g$ holds with $f,g\in L^2\left((0,1)\times N\right)$. By definition this means that $\langle f,P^{\kappa,t}_B\phi\rangle_{L^2}=\langle g,\phi\rangle_{L^2}$ for all $\phi\in C^\infty_{cp}\left((0,1)\times N\right)$.
If $\varphi\in C^\infty_{cp}(0,1)$ is an arbitrary cut-off function, we claim that this relation extends to hold for $\phi=\varphi\Psi_b$ and $b\in\operatorname{spec}B$. Since the Ψ_b won't in general be compactly supported, we choose sequences $\Psi_{b,n}\in C^\infty_{cp}(N)$, which approximate Ψ_b with respect to $\|\cdot\|_B$. This can be done since by assumption $B_{max}=B_{min}$. We obtain that

$$P^{\kappa,t}_B\left(\varphi(\Psi_b-\Psi_{b,n})\right)=\left(-\frac{\partial}{\partial r}\varphi\right)(\Psi_b-\Psi_{b,n})+\varphi\frac{1}{\operatorname{sn}_\kappa(r)}B(\Psi_b-\Psi_{b,n})\,.$$

Then it follows that $\varphi\Psi_{b,n}$ converges to $\varphi\Psi_b$ with respect to $\|\cdot\|_{P^{\kappa,t}_B}$ as $n\to\infty$. Since $\varphi\Psi_{b,n}\in C^\infty_{cp}\left((0,1)\times N\right)$ we have

$$\langle f,P^{\kappa,t}_B(\varphi\Psi_{b,n})\rangle_{L^2}=\langle g,\varphi\Psi_{b,n}\rangle_{L^2}$$

for all n. By continuity we obtain

$$\langle f,P^{\kappa,t}_B(\varphi\Psi_b)\rangle_{L^2}=\langle g,\varphi\Psi_b\rangle_{L^2}.$$

Now the left-hand side of this equation equals

$$\begin{aligned}\int_0^1\int_N\left(f,P^{\kappa,t}_B(\varphi\Psi_b)\right)&=\int_0^1\int_N\left(f,\left(-\frac{\partial}{\partial r}\varphi\right)\Psi_b+\varphi\frac{1}{\operatorname{sn}_\kappa(r)}B\Psi_b\right)\\&=\int_0^1 P^{\kappa,t}_b\varphi\int_N(f,\Psi_b)=\int_0^1 f_bP^{\kappa,t}_b\varphi,\end{aligned}$$

whereas the right-hand side is given by

$$\int_0^1\int_N(g,\varphi\Psi_b)=\int_0^1\varphi\int_N(g,\Psi_b)=\int_0^1 g_b\varphi\,.$$

Since φ was arbitrary, this means that $P^\kappa_b f_b=g_b$ for all $b\in\operatorname{spec}B$.
Conversely, if $P^\kappa_b f_b=g_b$ holds for all $b\in\operatorname{spec}B$, we have to show that

$$\langle f,P^{\kappa,t}_B\phi\rangle_{L^2}=\langle g,\phi\rangle_{L^2}$$

is true for all $\phi \in C^\infty_{cp}((0,1)\times N)$. Now

$$\langle f, P_B^{\kappa,t}\phi\rangle_{L^2} = \sum_{b\in \operatorname{spec} B} \langle f_b, (P_B^{\kappa,t}\phi)_b\rangle_{L^2(0,1)}$$

and

$$\langle g, \phi\rangle_{L^2} = \sum_{b\in \operatorname{spec} B} \langle g_b, \phi_b\rangle,$$

so we clearly get the result, if $(P_B^{\kappa,t}\phi)_b = P_b^{\kappa,t}\phi_b$. This is verified by the following computation:

$$\begin{aligned}\left(P_B^{\kappa,t}\phi\right)_b &= \int_N \left(P_B^{\kappa,t}\phi, \Psi_b\right) = \int_N \left(-\frac{\partial}{\partial r}\phi + \frac{1}{\operatorname{sn}_\kappa(r)}B\phi, \Psi_b\right) \\ &= \left(-\frac{\partial}{\partial r} + \frac{b}{\operatorname{sn}_\kappa(r)}\right)\int_N (\phi, \Psi_b) = P_b^{\kappa,t}\phi_b\,,\end{aligned}$$

which in turn finishes the proof. ▫

Corollary 4.10 *We have an orthogonal decomposition*

$$\operatorname{dom}(P_B^\kappa)_{max} = \overline{\bigoplus_{b\in \operatorname{spec} B} \operatorname{dom}(P_b^\kappa)_{max} \otimes \Psi_b}\,,$$

where $\operatorname{dom}(P_B^\kappa)_{max}$ *is equipped with the graph inner product* $\langle\cdot\,,\cdot\rangle_{P_B^\kappa}$ *and the closure is taken with respect to the corresponding graph norm* $\|\cdot\|_{P_B^\kappa}$.

Lemma 4.11 *Let* $f \in \operatorname{dom}(P_B^\kappa)_{max}$. *Then* $f \in \operatorname{dom}(P_B^\kappa)_{min}$ *if and only if* $f_b \in \operatorname{dom}(P_b^\kappa)_{min}$ *for all* $b \in \operatorname{spec} B$.

Proof. The proof essentially uses the observation that $f \in \operatorname{dom}(P_B^\kappa)_{min}$ if and only if $\langle P_B^\kappa f, g\rangle_{L^2} = \langle f, P_B^{\kappa,t} g\rangle_{L^2}$ for all $g \in \operatorname{dom}(P_B^{\kappa,t})_{max}$. Now the left-hand side of the equation in question equals

$$\sum_{b\in \operatorname{spec} B} \langle (P_B^\kappa f)_b, g_b\rangle_{L^2(0,1)} = \sum_{b\in \operatorname{spec} B} \langle P_b^\kappa f_b, g_b\rangle_{L^2(0,1)}\ ,$$

since $f_b \in \operatorname{dom}(P_b^\kappa)_{max}$ and $g_b \in \operatorname{dom}(P_b^{\kappa,t})_{max}$, while the right-hand side is given by

$$\sum_{b\in \operatorname{spec} B} \left\langle f_b, (P_B^{\kappa,t} g)_b\right\rangle_{L^2(0,1)} = \sum_{b\in \operatorname{spec} B} \left\langle f_b, P_b^{\kappa,t} g_b\right\rangle_{L^2(0,1)}\ .$$

We obtain that $f \in \operatorname{dom}(P_B^\kappa)_{min}$ if and only if $\langle P_b^\kappa f_b, g_g\rangle_{L^2(0,1)} = \langle f_b, P_B^{\kappa,t} g_b\rangle_{L^2(0,1)}$ for all $g_b \in \operatorname{dom}(P_b^{\kappa,t})_{max}$, i.e. that $f_b \in \operatorname{dom}(P_b^\kappa)_{min}$ for all $b \in \operatorname{spec} B$. ▫

Corollary 4.12 *We have an orthogonal decomposition*

$$\operatorname{dom}(P_B^\kappa)_{min} = \overline{\bigoplus_{b\in \operatorname{spec} B} \operatorname{dom}(P_b^\kappa)_{min} \otimes \Psi_b}\,,$$

where $\operatorname{dom}(P_B^\kappa)_{min}$ *is equipped with the graph inner product* $\langle\cdot\,,\cdot\rangle_{P_B^\kappa}$ *and the closure is taken with respect to the corresponding graph norm* $\|\cdot\|_{P_B^\kappa}$.

The following lemma will turn out to be decisive in the question of essential selfadjointness of D on cone-manifolds.

Lemma 4.13 *Let $\varphi \in C^\infty(0,1)$ be a cut-off function with $\varphi \equiv 1$ near 0 and $\varphi \equiv 0$ near 1. For $u \in \operatorname{dom}(P_B^\kappa)_{max}$ let $f = \varphi u \in \operatorname{dom}(P_B^\kappa)_{max}$. Then $f \in \operatorname{dom}(P_B^\kappa)_{min}$ if* $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$.

Proof. This follows from the above discussion together with Corollary **(4.8)** and Lemma **(4.3)**. ◽

In the following we derive certain compactness properties which will be relevant for the question of discreteness of $D(d_{max})$ and $\Delta(d_{max})$ on cone-manifolds.

Lemma 4.14 *The embedding* $\operatorname{dom}(P_b)_{max} \hookrightarrow L^2(0,1)$ *is compact for all* $b \in \mathbb{R}$.

Proof. Given a sequence $f_n \in \operatorname{dom}(P_b)_{max}$ with a bound $\|f_n\|_{P_b} \le C$ independent of n, we have to extract a subsequence convergent in $L^2(0,1)$. On any subinterval $(\delta, 1) \subset (0,1)$ the graph norm of P_b is equivalent to the ordinary H^1-norm, since $\frac{1}{r} \in L^\infty(\delta, 1)$. Recall that the embedding $H^1(\delta,1) \hookrightarrow C^0([\delta,1])$ is compact by Rellich's theorem. Therefore we obtain a locally uniformly convergent subsequence, which we again denote by f_n.
As a consequence of the decay estimates (cf. Corollary **(4.6)**) we have

$$|f_n(r)| \le C(b) r^{\frac{1}{2}}(1 + |\log r|^{\frac{1}{2}}) \|f_n\|_{P_b} \le C'(b) r^{\frac{1}{2}}(1 + |\log r|^{\frac{1}{2}})$$

if $b \notin (-\frac{1}{2}, \frac{1}{2})$, and

$$|f_n(r)| \le C(b) r^{-b} \|f_n\|_{P_b} \le C'(b) r^{-b}$$

if $b \in (-\frac{1}{2}, \frac{1}{2})$. The functions $r^{\frac{1}{2}}(1 + |\log r|^{\frac{1}{2}})$ and r^{-b} with $b < \frac{1}{2}$ are certainly in $L^2(0,1)$. In any case we conclude with Lebesgue's dominated convergence theorem, that f_n is convergent in $L^2(0,1)$. ◽

Corollary 4.15 *The embedding* $\operatorname{dom}(P_b^\kappa)_{max} \hookrightarrow L^2(0,1)$ *is compact for all* $b \in \mathbb{R}$.

Proof. This follows from the previous lemma in view of Lemma **(4.3)**. ◽

For $b \in \mathbb{R}$ we define

$$\widetilde{P}_b^\kappa = \begin{cases} (P_b^\kappa)_{max} & , \quad b \in (-\frac{1}{2}, \frac{1}{2}) \\ (P_b^\kappa)_{min} & , \quad b \notin (-\frac{1}{2}, \frac{1}{2}) \end{cases}$$

This determines a closed extension $\widetilde{P}_B^\kappa$ of P_B^κ such that

$$\operatorname{dom} \widetilde{P}_B^\kappa = \overline{\bigoplus_{b \in \operatorname{spec} B} \operatorname{dom} \widetilde{P}_b^\kappa \otimes \Psi_b}\,,$$

where the closure is taken with respect to the graph norm $\|\cdot\|_{P_B^\kappa}$. Note in particular that $\widetilde{P}_B^\kappa = (P_B^\kappa)_{min}$ if $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$.

Lemma 4.16 *The embedding* $\operatorname{dom} \widetilde{P}_B^\kappa \hookrightarrow L^2((0,1) \times N)$ *is compact.*

Proof. The previous lemma implies that $(L_b^\kappa)_{max} : \operatorname{dom}(P_b^\kappa)_{max} \hookrightarrow L^2(0,1)$ is a compact embedding for all $b \in \operatorname{spec} B$. We derive an upper bound for the operator norm of $(L_b^\kappa)_{min} : \operatorname{dom}(P_b^\kappa)_{min} \hookrightarrow L^2(0,1)$, where $\operatorname{dom}(P_b^\kappa)_{min}$ is equipped with the graph norm $\|\cdot\|_{P_b^\kappa}$. For $f \in C_{cp}^\infty(0,1)$ we have

$$P_b^{\kappa,t} P_b^\kappa f = -\frac{\partial^2 f}{\partial r^2} + \frac{b(b + \mathrm{cs}_\kappa(r)) f}{\mathrm{sn}_\kappa^2(r)}\,,$$

and therefore integration by parts applied twice yields

$$\begin{aligned}\|P_b^\kappa f\|^2_{L^2(0,1)} &= \langle P_b^{\kappa,t} P_b^\kappa f, f\rangle_{L^2(0,1)} \\ &= \int_0^1 \left|\frac{\partial f}{\partial r}\right|^2 + \int_0^1 \frac{b(b+\mathrm{cs}_\kappa(r))f^2}{\mathrm{sn}_\kappa^2(r)} \\ &\geq C_\kappa(b)\, \|f\|^2_{L^2(0,1)}\ ,\end{aligned}$$

where $C_\kappa(b) \nearrow \infty$ as $|b| \to \infty$. Since $C^\infty_{cp}(0,1)$ is dense in $\mathrm{dom}(P_b^\kappa)_{min}$ we obtain

$$\begin{aligned}\|(L_b^\kappa)_{min}\|^2 &= \sup_{f\in C^\infty_{cp}(0,1)\setminus\{0\}} \frac{\|f\|^2}{\|f\|^2 + \|P_b^\kappa f\|^2} \\ &\leq \frac{1}{1+C_\kappa(b)}\ ,\end{aligned}$$

i.e. for large eigenvalues of B the operator norm of $(L_b^\kappa)_{min}$ is uniformly small.
Let L denote the embedding $\mathrm{dom}\,\widetilde{P}_B^\kappa \hookrightarrow L^2\left((0,1)\times N\right)$, furthermore for $a>0$ let $\pi^{<a}$ denote the projection onto the eigenspaces corresponding to eigenvalues b with $|b|<a$. Since there are only finitely many such eigenvalues,

$$L^{<a} = \pi^{<a} \circ L$$

is a compact operator and by the above estimates

$$\|L - L^{<a}\|^2 = \sup_{|b|\geq a} \|(L_b^\kappa)_{min}\|^2 \leq \frac{1}{1+C_\kappa(a)}\ ,$$

for a large enough. In particular, for $a\to\infty$ we obtain that L is a limit of compact operators with respect to the operator norm and is therefore itself compact. ⊡

4.3 Spectral properties of cone-surfaces

Let $\mathrm{cone}_{\kappa,(0,\varepsilon)}\, S^1_\alpha$ be the truncated cone over S^1_α of constant curvature κ, i.e.

$$\mathrm{cone}_{\kappa,(0,\varepsilon)}\, S^1_\alpha = (0,\varepsilon)\times S^1_\alpha$$

with metric

$$dr^2 + \mathrm{sn}_\kappa^2(r)d\theta^2$$

where $r\in(0,\varepsilon)$ and $\theta\in\mathbb{R}/\alpha\mathbb{Z}$. Performing the change of variables $\theta = \frac{\alpha}{2\pi}t$ we may also write

$$\mathrm{cone}_{\kappa,(0,\varepsilon)}\, S^1_\alpha = (0,\varepsilon)\times S^1$$

with metric

$$g_{\kappa,\alpha} = dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 \mathrm{sn}_\kappa^2(r)dt^2$$

where $r\in(0,\varepsilon)$ and $\theta\in\mathbb{R}/2\pi\mathbb{Z}$. In particular we see that all the metrics $g_{\kappa,\alpha}$ are Bi-Lipschitz equivalent on $(0,\varepsilon)\times S^1$.
Nevertheless, it will turn out that certain analytic properties of the Hodge-Dirac, resp. the Hodge-Laplace operator do depend on the cone-angle in a strong way. If we consider the de-Rham complex with coefficients in a flat vector-bundle, then also the holonomy around the cone-points affects the analysis.

4.3.1 Discreteness

Here we investigate the discreteness of the operators $D(d_{max})$ and $\Delta(d_{max})$ on a compact, oriented cone-surface S. Particular attention will be paid to the spherical cone-surfaces $\mathbf{S}^2(\alpha,\beta,\gamma)$ and $\mathbf{S}^2(\alpha)$, which appear as the links of points in the singular locus $\Sigma \subset C$, where C is a 3-dimensional cone-manifold.
A selfadjoint operator A is called discrete if its spectrum is discrete, i.e. if $\operatorname{spec} A$ consists of a discrete set of eigenvalues with finite multiplicities. A necessary and sufficient condition for A to be discrete is the compactness of the embedding $\operatorname{dom} A \hookrightarrow L^2$, where $\operatorname{dom} A$ is equipped with the graph norm $\|\cdot\|_A$.
The results concerning discreteness will be valid without further restricting the cone-angles or the holonomy of the flat vector-bundle $(\mathcal{F}, \nabla^{\mathcal{F}})$ around the cone-points $\{x_1, \ldots, x_k\} \subset S$.

Proposition 4.17 *The embedding* $\operatorname{dom} D^{ev}_{max} \hookrightarrow L^2(\Lambda^{odd} T^*N \otimes \mathcal{F})$ *is compact on* $N = \operatorname{int} S$, *where* S *is a cone-surface and* $(\mathcal{F}, \nabla^{\mathcal{F}})$ *a flat vector-bundle over* N.

Proof. We construct a partition of unity on C in the following way: Let $\{x_1, \ldots, x_k\}$ be the set of cone-points, we choose $\varepsilon > 0$ such that the $U_\varepsilon(x_i)$ are disjoint. We choose cut-off functions φ_i supported inside $U_\varepsilon(x_i)$ with $\varphi_i = \varphi_i(r)$ and $\varphi_i \equiv 1$ near $r = 0$. Then we define $\varphi_{int} = 1 - \sum_{i=1}^k \varphi_i$.
Now let $u_n \in \operatorname{dom} D^{ev}_{max}$ be a sequence with $\|u_n\|_{D^{ev}} \le C$. We have to extract a subsequence convergent in L^2.
Clearly $\varphi_{int} u_n$ has a subsequence convergent in L^2: Let $\Omega \subset N$ be a relatively compact domain with smooth boundary, such that $\operatorname{supp} \varphi_{int} \subset \Omega$. Then by the usual elliptic regularity results, $\varphi_{int} u_n \in H^1_0(\Omega)$. Furthermore by the standard elliptic estimate we have control over the H^1-norm:

$$\|\varphi u_n\|^2_{H^1(\Omega)} \le C \left(\|\varphi u_n\|^2_{L^2(\Omega)} + \|D^{ev} \varphi u_n\|^2_{L^2(\Omega)} \right) = C \, \|\varphi u_n\|^2_{D^{ev}_\Omega}.$$

Now by Rellich's theorem $H^1_0(\Omega)$ embeds into $L^2(\Omega)$ compactly, which proves the subclaim.
Thus we are reduced to a situation on the cone $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} S^1_\alpha$, i.e. given a sequence $f_n = \varphi u_n$ with $\|f_n\|_{P^\kappa_B} \le C$, we have to extract a subsequence convergent in $L^2((0,1) \times S^1_\alpha)$. The operator B is essentially selfadjoint and discrete, since the cross-section of the cone is nonsingular in this case. Therefore the discussion from the last section applies. It is a consequence of Corollary **(4.8)** that $\varphi u_n \in \operatorname{dom} \widetilde{P}^\kappa_B$, therefore Lemma **(4.16)** yields the result. ◻

As a consequence we obtain that strong Hodge-decomposition holds for any ideal boundary condition for the de-Rham complex on a cone-surface with coefficients in a flat vector-bundle $(\mathcal{F}, \nabla^{\mathcal{F}})$, in particular for the d_{max}-complex. Here we refer the reader again to Theorem **(3.9)** and the remark thereafter.
We summarize the results concerning Hodge-decomposition on cone-surfaces relevant to L^2-cohomology in the following statement:

Theorem 4.18 (strong Hodge theorem for cone-surfaces) *Let* S *be a cone-surface and* $(\mathcal{F}, \nabla^{\mathcal{F}})$ *a flat vector-bundle over* $N = \operatorname{int} S$ *together with a fixed metric* $h^{\mathcal{F}}$. *Then there is an orthogonal decomposition*

$$L^2(\Lambda^i T^*N \otimes \mathcal{F}) = \mathcal{H}^i_{max} \oplus \operatorname{im} d^{i-1}_{max} \oplus \operatorname{im}(d^i)^t_{min},$$

and furthermore $\iota : \mathcal{H}^i_{max} \to H^i_{max}$ *is an isomorphism. The inclusion of the smooth* L^2*-complex* $\Omega^i_{L^2}(N, \mathcal{F}) \to \operatorname{dom} d^i_{max}$ *induces an isomorphism* $H^i_{L^2}(N, \mathcal{F}) \cong H^i_{max}$.

Since $D^{odd} = (D^{ev})^t$, the same arguments yield that $\operatorname{dom} D^{odd}_{max} \hookrightarrow L^2(\Lambda^{ev} T^* N \otimes \mathcal{F})$ and in particular $\operatorname{dom} D_{max} \hookrightarrow L^2(\Lambda^\bullet T^* N \otimes \mathcal{F})$ are again compact embeddings.

Proposition 4.19 *The operators $D(d_{max})$ and $\Delta(d_{max})$ are discrete on a cone-surface S for $(\mathcal{F}, \nabla^{\mathcal{F}})$ a flat vector-bundle over $N = \operatorname{int} S$.*

Proof. This follows from the compactness of $\operatorname{dom} D_{max} \hookrightarrow L^2(\Lambda^\bullet T^* N \otimes \mathcal{F})$ since $\operatorname{dom} D(d_{max})$ and $\operatorname{dom} \Delta(d_{max})$ are continuously contained in $\operatorname{dom} D_{max}$. ◻

4.3.2 Selfadjointness

In this section we will address the question of essential selfadjointness of the Hodge-Dirac operator D on a cone-surface with coefficients in a flat vector-bundle $(\mathcal{F}, \nabla^{\mathcal{F}})$. The answer to this question will essentially depend on the cone-angles and the holonomy of the flat vector-bundle $(\mathcal{F}, \nabla^{\mathcal{F}})$ around the cone-points, since these data determine the spectrum of the operator B on the cross-section of the cone.
If x is a cone-point, the smooth part of the ε-ball around x will be isometric with the κ-cone over the circle of length α, i.e. $U_\varepsilon(x) = \operatorname{cone}_{\kappa,(0,\varepsilon)} S^1_\alpha$. In this situation the model operator for the even part of the Hodge-Dirac operator on the cone is given by

$$P^\kappa_B = \frac{\partial}{\partial r} + \frac{1}{\operatorname{sn}_\kappa(r)} B$$

with

$$B = D_{S^1_\alpha} + \begin{pmatrix} -\frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & d^t_{S^1_\alpha} \\ d_{S^1_\alpha} & -\frac{1}{2} \end{pmatrix}.$$

We determine the spectrum of the operator B, let us discuss the case with trivial coefficient bundle first. If we identify functions and 1-forms on S^1_α via

$$\begin{aligned} C^\infty(S^1_\alpha) &\longrightarrow \Omega^1(S^1_\alpha) \\ g &\longmapsto g \cdot d\theta, \end{aligned}$$

we may write

$$D_{S^1_\alpha} = \begin{pmatrix} 0 & -\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\theta} & 0 \end{pmatrix}.$$

It is easily verified that

$$\operatorname{spec} D_{S^1_\alpha} = \left\{ \frac{2\pi n}{\alpha}, n \in \mathbb{Z} \right\},$$

and therefore we obtain

$$\operatorname{spec} B = \left\{ -\frac{1}{2} + \frac{2\pi n}{\alpha}, n \in \mathbb{Z} \right\}.$$

We see that $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ if $\alpha \le 2\pi$ in the case of trivial coefficients.
Let us now add a flat bundle to the situation. Let $\mathbb{C}(a)$ be the flat $U(1)$-bundle on S^1_α with holonomy $e^{ia}, a \in \mathbb{R}$. Without loss of generality we may assume that $a \in [0, 2\pi)$. Note that the bundles $\mathbb{C}(a)$ are topologically trivial. Any unitarily flat bundle on S^1_α decomposes as a direct sum of these. A flat connection is given by

$$\nabla^{\mathbb{C}(a)} = d - i\frac{a}{\alpha} d\theta.$$

The associated Hodge-Dirac operator may be written as

$$D_{S^1_\alpha, \mathbb{C}(a)} = \begin{pmatrix} 0 & -\frac{\partial}{\partial\theta} + i\frac{a}{\alpha} \\ \frac{\partial}{\partial\theta} - i\frac{a}{\alpha} & 0 \end{pmatrix}$$

We obtain

$$\operatorname{spec} D_{S^1_\alpha,\mathbb{C}(a)} = \left\{ \pm \left| \frac{2\pi n - a}{\alpha} \right|, n \in \mathbb{Z} \right\},$$

and therefore

$$\operatorname{spec} B = \left\{ -\frac{1}{2} \pm \left| \frac{2\pi n - a}{\alpha} \right|, n \in \mathbb{Z} \right\}.$$

We see that $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ if either $a = 0$ and $\alpha \leq 2\pi$ or $\alpha \leq a \leq 2\pi - \alpha$. In the latter case we must in particular have that $\alpha \leq \pi$.

Definition 4.20 Let S be a cone-surface and $(\mathcal{F}, \nabla^{\mathcal{F}})$ a flat vector-bundle over $N = \operatorname{int} S$. If $\{x_i\}$ are the cone-points and $P^\kappa_{B_i}$ is the model operator for D^{ev} on $U_\varepsilon(x_i)$, then we call $(\mathcal{F}, \nabla^{\mathcal{F}})$ **cone-admissible** if $\operatorname{spec} B_i \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ for all i.

Note that this definition contains in an implicit way restrictions on the cone-angles of S and the holonomy of the flat bundle $(\mathcal{F}, \nabla^{\mathcal{F}})$.

Remark 4.21 The previous discussion shows that if S has cone-angles $\leq \pi$ and $\mathcal{F}$ decomposes locally around the cone-points as a direct sum of trivial bundles $\mathbb{R}$ and bundles of type $\mathbb{C}(a)$ with $\alpha \leq a \leq 2\pi - \alpha$, then $\mathcal{F}$ will be cone-admissible in the sense of Definition **(4.20)**.

Proposition 4.22 *$D^{ev}_{max} = D^{ev}_{min}$ on N if $(\mathcal{F}, \nabla^{\mathcal{F}})$ is cone-admissible.*

Proof. Given $u \in \operatorname{dom} D^{ev}_{max}$ we have to show that already $u \in \operatorname{dom} D^{ev}_{min}$. We choose a partition of unity on S as in the proof of Proposition **(4.17)**.
Clearly $\varphi_{int} u \in \operatorname{dom} D^{ev}_{min}$: As we have already observed in the proof of Proposition **(4.17)**, if $\Omega \subset N$ is a relatively compact domain with smooth boundary such that $\operatorname{supp} \varphi_{int} \subset \Omega$, then $\varphi_{int} u \in H^1_0(\Omega)$. Now $C^\infty_{cp}(\Omega)$ is dense in $H^1_0(\Omega)$, therefore we find a sequence $f_n \in C^\infty_{cp}(\Omega)$ such that f_n approximates $f := \varphi_{int} u$ with respect to the H^1-norm. But since D^{ev} maps $H^1(\Omega)$ continuously to $L^2(\Omega)$, f_n approximates f also with respect to the graph norm of D^{ev} on Ω, which proves the claim.
It remains to prove that $\varphi_i u \in \operatorname{dom} D^{ev}_{min}$ for $i \in \{1, \ldots, k\}$. But here we are again in a situation on the cone $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} S^1_\alpha$. It is therefore sufficient to show that $f = \varphi u \in \operatorname{dom}(P^\kappa_B)_{min}$ for $u \in \operatorname{dom}(P^\kappa_B)_{max}$ and φ a cut-off function of the above type. Now since $(\mathcal{F}, \nabla^{\mathcal{F}})$ is cone-admissible, $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ will be satisfied. Then Lemma **(4.13)** implies that $f \in \operatorname{dom}(P^\kappa_B)_{min}$, hence in $\operatorname{dom} D^{ev}_{min}$. ◻

Corollary 4.23 *D is essentially selfadjoint on N if $(\mathcal{F}, \nabla^{\mathcal{F}})$ is cone-admissible.*

Proof. We have

$$D = \begin{pmatrix} 0 & (D^{ev})^t \\ D^{ev} & 0 \end{pmatrix}$$

considered as an operator

$$\Omega^{ev}_{cp}(N, \mathcal{F}) \oplus \Omega^{odd}_{cp}(N, \mathcal{F}) \longrightarrow \Omega^{ev}_{cp}(N, \mathcal{F}) \oplus \Omega^{odd}_{cp}(N, \mathcal{F})$$

and therefore

$$D_{min} = \begin{pmatrix} 0 & (D^{ev})^t_{min} \\ D^{ev}_{min} & 0 \end{pmatrix}$$

and

$$D_{max} = \begin{pmatrix} 0 & (D^{ev})^t_{max} \\ D^{ev}_{max} & 0 \end{pmatrix}$$

This shows that $D_{max} = D_{min}$, i.e. that D is essentially selfadjoint. ◻

Corollary 4.24 *$\Delta_F = \Delta(d_{max})$ on N if $(\mathcal{F}, \nabla^{\mathcal{F}})$ is cone-admissible.*

Proof. This follows from the essential selfadjointness of D with Corollary **(3.7)**. ◻

4.3.3 The first eigenvalue

Let λ_1 be the smallest positive eigenvalue of $\Delta^0(d_{max})$ on the smooth part of $\mathbf{S}^2(\alpha,\beta,\gamma)$ (resp. $\mathbf{S}^2(\alpha)$) with coefficients in a flat bundle $(\mathcal{F},\nabla^{\mathcal{F}})$. Here we will give a lower bound on λ_1, which will be sufficient for later purposes. Comparison with the smooth case suggests that this bound might not be optimal.
The positivity of the curvature on the smooth part together with restrictions on the cone-angles and on the holonomy of the flat bundle as in the previous section will be the important assumptions here.

Proposition 4.25 *Let S be either $\mathbf{S}^2(\alpha,\beta,\gamma)$ or $\mathbf{S}^2(\alpha)$ and let $(\mathcal{F},\nabla^{\mathcal{F}})$ be a flat vector-bundle over $N = \operatorname{int} S$. If $(\mathcal{F},\nabla^{\mathcal{F}})$ is orthogonally flat and cone-admissible, then $\mathcal{H}^1_{max} = 0$. Moreover, under the same hypothesis, if λ_1 denotes the smallest positive eigenvalue of $\Delta^0(d_{max})$, then $\lambda_1 \geq 1$.*

Proof. Since $(\mathcal{F},\nabla^{\mathcal{F}})$ is orthogonally flat, we may apply the standard Weitzenböck formula on $\mathcal{F}$-valued 1-forms

$$\Delta\omega = \nabla^t\nabla\omega + (\mathrm{Ric}\otimes \mathrm{id})\omega\,,$$

where the action of the Ricci tensor on a scalar-valued 1-form α is determined by the relation

$$g(\mathrm{Ric}(\alpha),\beta) = \mathrm{Ric}(\alpha,\beta)$$

for all $\beta \in \Omega^1(N,\mathbb{R})$. In two dimensions the Ricci tensor of a spherical metric (i.e. of constant curvature $\kappa = 1$) is given by

$$\mathrm{Ric}(\cdot\,,\cdot) = g(\cdot\,,\cdot),$$

so we end up with

$$\Delta\omega = \nabla^t\nabla\omega + \omega.$$

For $\omega \in \Omega^1_{cp}(N,\mathcal{F})$ integration by parts yields

$$\begin{aligned}\int_N (\Delta\omega,\omega) &= \int_N (\nabla^t\nabla\omega,\omega) + \int_N |\omega|^2 \\ &= \int_N |\nabla\omega|^2 + \int_N |\omega|^2 \geq \int_N |\omega|^2\,.\end{aligned}$$

This means we have a lower bound for Δ on $\Omega^1_{cp}(N,\mathcal{F})$:

$$\langle\Delta\omega,\omega\rangle_{L^2} \geq \|\omega\|^2_{L^2}\,.$$

Since we know that $\Delta(d_{max}) = \Delta_F$ if $(\mathcal{F},\nabla^{\mathcal{F}})$ is cone-admissible and the Friedrichs extension preserves lower bounds, we obtain

$$\langle\Delta(d_{max})\omega,\omega\rangle_{L^2} \geq \|\omega\|^2_{L^2}$$

for all $\omega \in \operatorname{dom}\Delta^1(d_{max})$. This proves the first part of the assertion. Now for $f \in E_{\lambda_1}$, the λ_1-eigenspace of $\Delta^0(d_{max})$, $f \neq 0$, let $\omega := d_{max}f$. Then $w \neq 0$ and $\Delta^1(d_{max})\omega = d_{max}d^t_{min}d_{max}f = \lambda_1\omega$. This yields the estimate $\lambda_1 \geq 1$. ◻

4.4 Spectral properties of cone-3-manifolds

For the local analysis around the singularity, we consider two cases:

1. x is a vertex

2. x lies on a singular edge

In the first case, a neighbourhood of x in M is isometric with

$$U_\varepsilon(x) \cong \mathrm{cone}_{\kappa,(0,\varepsilon)}\, \mathrm{int}\, \mathbf{S}^2(\alpha,\beta,\gamma)$$

and in the second

$$U_\varepsilon(x) \cong \mathrm{cone}_{\kappa,(0,\varepsilon)}\, \mathrm{int}\, \mathbf{S}^2(\alpha)$$

The two cases can be treated simultaneously, let N denote either $\mathrm{int}\, \mathbf{S}^2(\alpha,\beta,\gamma)$ or $\mathrm{int}\, \mathbf{S}^2(\alpha)$ in the following.

Recall that the model operator for the even part of the Hodge-Dirac operator on the κ-cone with two-dimensional cross-section N is given by

$$P_B^\kappa = \frac{\partial}{\partial r} + \frac{1}{\mathrm{sn}_\kappa(r)} B$$

with

$$B = D_N + \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} -1 & d_N^t & \\ d_N & 0 & d_N^t \\ & d_N & 1 \end{pmatrix}.$$

Let us assume that $(\mathcal{F}, \nabla^{\mathcal{F}})$ restricted to N is cone-admissible. Then D_N and in particular the operator B will be essentially selfadjoint. Let us further assume that $(\mathcal{F}, \nabla^{\mathcal{F}})$ restricted to N is orthogonally flat. Then the Hodge-$\star$-operator defines a linear isometry

$$\star : L^2(\Lambda^p T^* N \otimes \mathcal{F}) \longrightarrow L^2(\Lambda^{n-p} T^* N \otimes \mathcal{F}),$$

where in this case $n = 2$. Note furthermore that these two conditions together imply that $\mathcal{H}^1_{max} = 0$ via Proposition **(4.25)**. We determine $\mathrm{spec}\, B$ in the following.
For $\lambda \geq 0$ let E_λ be the λ-eigenspace of

$$\Delta(d_{max}) = \Delta^0(d_{max}) \oplus \Delta^1(d_{max}) \oplus \Delta^2(d_{max}).$$

Let $\lambda > 0$ be an eigenvalue and f_λ a corresponding eigensection of $\Delta^0(d_{max})$ with $\|f_\lambda\|_{L^2}$=1. Then

$$\left\{ f_\lambda, \frac{1}{\sqrt{\lambda}} df_\lambda, \frac{1}{\sqrt{\lambda}} \star df_\lambda, \star f_\lambda \right\}$$

form an orthonormal basis of a D_N-invariant subspace $E_{f_\lambda} \subset E_\lambda$. It is a consequence of Theorem **(4.18)** that the E_{f_λ} provide an orthogonal decomposition of E_λ for f_λ pairwise orthogonal. With respect to the given basis of E_{f_λ} we have

$$D_N|_{E_{f_\lambda}} = \begin{pmatrix} 0 & \sqrt{\lambda} & & \\ \sqrt{\lambda} & 0 & & \\ & & 0 & -\sqrt{\lambda} \\ & & -\sqrt{\lambda} & 0 \end{pmatrix}$$

and correspondingly

$$B|_{E_{f_\lambda}} = \begin{pmatrix} -1 & \sqrt{\lambda} & & \\ \sqrt{\lambda} & 0 & & \\ & & 0 & -\sqrt{\lambda} \\ & & -\sqrt{\lambda} & 1 \end{pmatrix}.$$

For $\lambda = 0$ we observe that if there is $f_0 \in \mathcal{H}^0_{max}$ with $\|f_0\|_{L^2} = 1$, then $\{f_0, f_0 \otimes dvol\}$ form an orthonormal basis of $E_{f_0} \subset E_0 = \mathcal{H}^0_{max} \oplus \mathcal{H}^2_{max}$ and we obtain

$$B|_{E_{f_0}} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}.$$

Note that E_0 may well be 0. Therefore we obtain for the spectrum of B

$$\operatorname{spec} B \subset \{-1, 1\} \cup \left\{ \pm\frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda} \,\middle|\, \lambda \in \operatorname{spec} \Delta^0(d_{max}), \lambda > 0 \right\}.$$

We see that $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ if $\lambda_1 \geq \frac{3}{4}$, which we can guarantee under the given conditions by means of Proposition **(4.25)**.

Definition 4.26 Let C be a 3-dimensional cone-manifold and $(\mathcal{E}, \nabla^{\mathcal{E}})$ a flat vector-bundle over $M = \operatorname{int} C$. If x is a singular point and $P^{\kappa}_{B_x}$ is the model operator for D^{ev} on $U_\varepsilon(x)$, then we call $(\mathcal{E}, \nabla^{\mathcal{E}})$ **cone-admissible** if B_x is essentially selfadjoint and $\operatorname{spec} B_x \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ for all $x \in \Sigma$.

If we compare this definition with the cone-surface case, we note that a new issue arises, namely that we have to include essential selfadjointness of the operator B on the cross-section of the model cone into the definititon. This issue was not present in the cone-surface case, since there the cross-section of the model cone was compact.

Remark 4.27 As a consequence of the previous discussion we observe that a sufficient condition for $(\mathcal{E}, \nabla^{\mathcal{E}})$ to be cone-admissible in the sense of Definititon **(4.26)** is that the restriction of $(\mathcal{E}, \nabla^{\mathcal{E}})$ to the link S_x of a singular point x is orthogonally flat and cone-admissible in the sense of Definition **(4.20)** for all $x \in \Sigma$.

4.4.1 Discreteness

Here we investigate the discreteness of the operators $D(d_{max})$ and $\Delta(d_{max})$ on a 3-dimensional cone-manifold. In contrast to the 2-dimensional case we have to include essential selfadjointness of the operator B on the links of the singular points into the hypothesis to make the separation of variables approach work.
For simplicity we state the results concerning discreteness under the stronger hypothesis that $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible, though we do not need the assumption on the spectrum of B as far as discreteness is concerned.

Proposition 4.28 *The embedding* $\operatorname{dom} D^{ev}_{max} \hookrightarrow L^2(\Lambda^{odd} T^*M \otimes \mathcal{E})$ *is compact if* $(\mathcal{E}, \nabla^{\mathcal{E}})$ *cone-admissible.*

Proof. Since Σ is compact we find finitely many $x_i \in \Sigma$ such that the $B_\varepsilon(x_i)$ cover Σ. Then $\{M, B_\varepsilon(x_i)\}$ is a finite open cover of C. We fix a partition of unity $\{\varphi_{int}, \varphi_i\}$ subordinate to this cover. Let $U_\varepsilon(x_i) = B_\varepsilon(x_i) \cap M$.
Now let $u_n \in \operatorname{dom} D^{ev}_{max}$ be a sequence with $\|u_n\|_{D^{ev}} \leq C$. We have to extract a convergent subsequence in L^2.
Since $\varphi_{int} \in C^\infty_{cp}(M)$, clearly $\varphi_{int} u_n$ has a convergent subsequence in L^2: This follows in exactly the same fashion as in the cone-surface case (cf. Proposititon **(4.17)** and its proof).
On the other hand $U_\varepsilon(x)$ will be isometric with $\operatorname{cone}_{\kappa,(0,\varepsilon)} \operatorname{int} \mathbf{S}^2(\alpha, \beta, \gamma)$ if x is a vertex or $\operatorname{cone}_{\kappa,(0,\varepsilon)} \operatorname{int} \mathbf{S}^2(\alpha)$ if x is an edge point. Thus we are reduced to a situation on the cone $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} N$. Without loss of generality we may assume that $\varphi = \varphi(r)$ if r is the radial variable and $\varphi(r) = 1$ for r small. If this is not the case we just replace φ by a second cut-off function $\widetilde{\varphi} \in C^\infty_{cp}(U_\varepsilon(x))$ which satisfies these

assumptions and in addition $\widetilde{\varphi} = 1$ near $\operatorname{supp}\varphi$, and we replace u_n by $\widetilde{u}_n = \varphi u_n$. Since $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible, the operator B will be essentially selfadjoint. B will have discrete spectrum as a consequence of Proposition **(4.17)**. As in the cone-surface case we obtain that $\varphi u_n \in \operatorname{dom} \widetilde{P}^{\kappa}_B$. We may now use Lemma **(4.16)** to conclude the result. ▫

As a consequence we obtain that strong Hodge-decomposition holds for any ideal boundary condition for the de-Rham complex on a 3-dimensional cone-manifolds if $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible, in particular for the d_{max}-complex. Here we refer the reader again to Theorem **(3.9)** and the remark thereafter.
We summarize the results concerning Hodge-decomposition on 3-dimensional cone-manifolds relevant to L^2-cohomology in the following statement:

Theorem 4.29 (strong Hodge theorem for cone-manifolds) *Let C be a 3-dimensional cone-manifold and $(\mathcal{E}, \nabla^{\mathcal{E}})$ a flat vector-bundle over $M = \operatorname{int} C$ together with a fixed metric $h^{\mathcal{E}}$. If $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible, there is an orthogonal decomposition*

$$L^2(\Lambda^i T^*M \otimes \mathcal{E}) = \mathcal{H}^i_{max} \oplus \operatorname{im} d^{i-1}_{max} \oplus \operatorname{im}(d^i)^t_{min},$$

and furthermore $\iota : \mathcal{H}^i_{max} \to H^i_{max}$ is an isomorphism. The inclusion of the smooth L^2-complex $\Omega^i_{L^2}(M, \mathcal{E}) \to \operatorname{dom} d^i_{max}$ induces an isomorphism $H^i_{L^2}(M, \mathcal{E}) \cong H^i_{max}$.

Since $D^{odd} = (D^{ev})^t$, the same arguments yield that $\operatorname{dom} D^{odd}_{max} \hookrightarrow L^2(\Lambda^{ev} T^*M \otimes \mathcal{E})$ and in particular $\operatorname{dom} D_{max} \hookrightarrow L^2(\Lambda^\bullet T^*M \otimes \mathcal{E})$ are again compact embeddings.

Proposition 4.30 *The operators $D(d_{max})$ and $\Delta(d_{max})$ are discrete on $M = \operatorname{int} C$ if $(\mathcal{E}, \nabla^{\mathcal{E}})$ cone-admissible.*

Proof. This follows from the compactness of $\operatorname{dom} D_{max} \hookrightarrow L^2(\Lambda^\bullet T^*M \otimes \mathcal{E})$ since $\operatorname{dom} D(d_{max})$ and $\operatorname{dom} \Delta(d_{max})$ are continuously contained in $\operatorname{dom} D_{max}$. ▫

4.4.2 Selfadjointness

In this section we establish essential selfadjointness of the Hodge-Dirac operator D on the smooth part of a 3-dimensional cone-manifold $M = \operatorname{int} C$, if the coefficient bundle $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible over M. Here the condition on the spectrum of the operator B on the links of the singular points is essential.

Proposition 4.31 $D^{ev}_{max} = D^{ev}_{min}$ *on M if $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible.*

Proof. Given $u \in \operatorname{dom} D^{ev}_{max}$ we have to show that already $u \in \operatorname{dom} D^{ev}_{min}$. We choose a partition of unity on C as in the proof of Proposition **(4.28)**.
Since $\varphi_{int} \in C^\infty_{cp}(M)$ clearly $\varphi_{int} u \in \operatorname{dom} D^{ev}_{min}$: This again follows in the same way as in the surface-case (cf. the proof of Proposition **(4.22)**).
It remains to prove that $\varphi_i u \in \operatorname{dom} D^{ev}_{min}$. Again this brings us back to a situation on the cone $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} N$, where $N = \operatorname{int} \mathbf{S}^2(\alpha, \beta, \gamma)$ or $N = \operatorname{int} \mathbf{S}^2(\alpha)$. It is therefore sufficient to show that $f := \varphi u \in \operatorname{dom}(P^{\kappa}_B)_{min}$ for $u \in \operatorname{dom}(P^{\kappa}_B)_{max}$ and φ a cut-off function of the above type. Since $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible, B is essentially selfadjoint and has discrete spectrum. Moreover, the condition $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ will be satisfied. Then Lemma **(4.13)** implies that $f \in \operatorname{dom}(P^{\kappa}_B)_{min}$, hence in $\operatorname{dom} D^{ev}_{min}$. ▫

Corollary 4.32 *D is essentially selfadjoint on M if $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible.*

Proof. This follows as in the two-dimensional case from $D^{ev}_{max} = D^{ev}_{min}$. ▫

Corollary 4.33 $\Delta_F = \Delta(d_{max})$ *on* M *if* $(\mathcal{E}, \nabla^{\mathcal{E}})$ *is cone-admissible.*

Proof. This follows from the essential selfadjointness of D with Corollary **(3.7)**. ◽

5 The Bochner technique

5.1 Infinitesimal isometries

For simplicity consider $\mathbf{M}^3_\kappa$ for $\kappa \in \{-1, 0, 1\}$. Let $G = \mathrm{Isom}^+ \mathbf{M}^3_\kappa$ and $\mathfrak{g}$ its Lie-algebra. $\mathfrak{g}$ may be identified with the Lie-algebra of Killing vectorfields. Note however, that the Lie-bracket in $\mathfrak{g}$ corresponds to the negative of the vectorfield commutator under this identification:

$$ad_{\mathfrak{g}}(X)Y = [X, Y]_{\mathfrak{g}} = -[X, Y] = -\mathcal{L}_X Y.$$

Fix a point $p \in \mathbf{M}^3_\kappa$ and let $K = \mathrm{Stab}_G(p)$. Note that $K \cong \mathrm{SO}(T_p\mathbf{M}^3_\kappa)$, since G acts simply transitively on frames in constant curvature. Then we get the usual decomposition (depending on p):

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie-algebra of K. Recall that

$$\mathfrak{k} = \{X \in \Gamma(T\mathbf{M}^3_\kappa) \mid X(p) = 0\}$$

and

$$\mathfrak{p} = \{X \in \Gamma(T\mathbf{M}^3_\kappa) \mid (\nabla X)(p) = 0\}.$$

We have isomorphisms

$$\begin{aligned} \mathfrak{p} &\xrightarrow{\cong} T_p\mathbf{M}^3_\kappa \\ X &\longmapsto X(p) \end{aligned}$$

and (in our constant-curvature situation)

$$\begin{aligned} \mathfrak{k} &\xrightarrow{\cong} \mathfrak{so}(T_p\mathbf{M}^3_\kappa) \\ X &\longmapsto A_X(p) := (\nabla X)(p). \end{aligned}$$

We know that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, since $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the $+1$ (-1) eigenspace of the Cartan-involution on $\mathfrak{g}$ induced by the geodesic involution on $\mathbf{M}^3_\kappa$ about p.

Lemma 5.1 *Under the identification* $\mathfrak{g} \cong \mathfrak{so}(T_p\mathbf{M}^3_\kappa) \oplus T_p\mathbf{M}^3_\kappa$ *the Lie-bracket corresponds to*

$$[(A, X), (B, Y)] = ([A, B] - R(X, Y), AY - BX),$$

where $[A, B]$ *is the commutator in* $\mathfrak{so}(T_p\mathbf{M}^3_\kappa)$ *and* R *the Riemannian curvature tensor.*

Proof. Let $X, Y \in \mathfrak{k}$, $Z \in \mathfrak{p}$.

$$\begin{aligned} A_{[X,Y]_{\mathfrak{g}}} Z(p) &= -\nabla_Z [X, Y](p) = -\nabla_{[X,Y]} Z(p) - [Z, [X, Y]](p) \\ &= [X, [Y, Z]](p) + [Y, [Z, X]](p) \\ &= [X, \nabla_Y Z - \nabla_Z Y](p) + [Y, \nabla_Z X - \nabla_X Z](p) \\ &= [X, \nabla_Z, Y](p) + [Y, \nabla_Z X](p) \\ &= -\left(\nabla_X \nabla_Z Y - \nabla_{\nabla_Z Y} X\right)(p) - \left(\nabla_Y \nabla_Z X - \nabla_{\nabla_Z X} Y\right)(p) \end{aligned}$$

$$= \nabla_{\nabla_Z Y} X(p) - \nabla_{\nabla_Z X} Y(p)$$
$$= [A_X, A_Y] Z(p)$$

Let $X \in \mathfrak{k}, Y \in \mathfrak{p}$.

$$\begin{aligned}[X,Y]_{\mathfrak{g}}(p) &= -\left(\nabla_X Y - \nabla_Y X\right)(p) \\ &= \nabla_Y X(p) \\ &= A_X Y(p)\end{aligned}$$

Let X be a Killing vectorfield. Let γ be a geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = Y(p)$. Then X will be a Jacobi vectorfield along γ. We obtain

$$\begin{aligned}0 &= \nabla_{\dot\gamma}\nabla_{\dot\gamma} X + R(X,\dot\gamma)\dot\gamma \\ &= \nabla_{\dot\gamma}\{A_X\dot\gamma\} + R(X,\dot\gamma)\dot\gamma \\ &= (\nabla_{\dot\gamma} A_X)\dot\gamma + R(X,\dot\gamma)\dot\gamma\end{aligned}$$

Therefore we have

$$(\nabla_Y A_X)Y + R(X,Y)Y = 0$$

We claim that the expression $(\nabla_Y A_X)Z + R(X,Y)Z$ is symmetric in Y and Z. Without loss of generality, we may assume $[X,Y] = 0$:

$$\begin{aligned}&(\nabla_Y A_X)Z + R(X,Y)Z \\ =&\nabla_Y\nabla_Z X - \nabla_{\nabla_Y Z} X + R(X,Y)Z \\ =&\nabla_Y\nabla_Z X + R(Y,Z)X - \nabla_{\nabla_Z Y} X - R(Y,Z)X - R(Z,X)Y \\ =&(\nabla_Z A_X)Y + R(X,Z)Y\end{aligned}$$

Therefore we obtain by polarization

$$(\nabla_Y A_X)Z + R(X,Y)Z = 0 \ , \ X \text{ Killing vectorfield} \qquad (*)$$

Let $X, Y \in \mathfrak{p}, Z \in \mathfrak{p}$.

$$\begin{aligned}A_{[X,Y]_{\mathfrak{g}}} Z(p) &= -\nabla_Z [X,Y](p) = -\{\nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X\}(p) \\ &= \{-(\nabla_Z A_Y)X + (\nabla_Z A_X)Y\}(p) \\ &= -\{R(Y,Z)X + R(Z,X)Y\} \\ &= -R(X,Y)Z\end{aligned}$$

This is sufficient, since we know a priori that $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. ◻

Note that the usual formula for the curvature tensor of a symmetric space

$$R(X,Y)Z(p) = -[[X,Y],Z](p), \ X, Y, Z \in \mathfrak{p}$$

is contained in the statement.

Corollary 5.2 *$Ad_G(g)(A,X) = (Ad_K(g)A, gX)$ for $g \in K \cong \mathrm{SO}(T_p\mathbf{M}^3_\kappa)$.*

Let $\mathcal{E} = \mathfrak{so}(T\mathbf{M}^3_\kappa) \oplus T\mathbf{M}^3_\kappa$. It is a vector-bundle of Lie-algebras. Extension of Killing vectorfields induces a flat structure on $\mathcal{E}$, such that a section $\sigma = (A, X)$ is parallel if and only if X is a Killing vectorfield and $A = A_X$.

Lemma 5.3 *The flat connection on $\mathcal{E}$ is given by*

$$\nabla^{\mathcal{E}}_Y (A, X) = (\nabla_Y A - R(Y,X), \nabla_Y X - AY),$$

where ∇ is the Levi-Civita connection on $T\mathbf{M}^3_\kappa$ and $\mathfrak{so}(T\mathbf{M}^3_\kappa)$ respectively.

Proof. If ∇^0 and ∇^1 are two connections on a vector-bundle $\mathcal{E}$, then the difference $\nabla^0 - \nabla^1 =: \alpha$ is a 1-form with values in $\mathrm{End}(\mathcal{E})$. If $\nabla^0\sigma = 0$, then $-\nabla^1_Y\sigma = \alpha(Y)\sigma$ for all $Y \in T\mathbf{M}^3_\kappa$.
Let $\nabla^0 = \nabla^{\mathcal{E}}$ and $\nabla^1 = \nabla$, the Levi-Civita connection on $\mathcal{E}$. Let X be a Killing vectorfield. X determines a parallel section $\sigma_X = (A_X, X) \in \Gamma(\mathcal{E})$. From equation $(*)$ we have

$$(\nabla_Y A_X)Z = -R(X,Y)Z = R(Y,X)Z,$$

and from the very definition

$$\nabla_Y X = A_X Y,$$

hence

$$\alpha(Y)(A,X) = (-R(Y,X), -AY).$$

This proves the claim, taking into account that $\nabla^{\mathcal{E}} = \nabla + \alpha$. ▫

In fact $\mathcal{E} = \mathbf{M}^3_\kappa \times \mathfrak{g}$ and $\nabla^{\mathcal{E}}$ is just the trivial connection d written in terms of the subbundles $T\mathbf{M}^3_\kappa$ and $\mathfrak{so}(T\mathbf{M}^3_\kappa)$.

Corollary 5.4 *$\nabla^{\mathcal{E}}_Y\sigma = \nabla_Y\sigma + ad(Y)\sigma$ for $\sigma \in \Gamma(\mathcal{E}), Y \in T\mathbf{M}^3_\kappa$.*

Proof. Lemma **(5.1)** implies that $\alpha(Y)\sigma = ad(Y)\sigma$. ▫

We have a natural metric on $\mathcal{E}$, namely

$$h^{\mathcal{E}} = (\,\cdot\,,\cdot)_{\mathfrak{so}(T\mathbf{M}^3_\kappa)} \oplus (\,\cdot\,,\cdot)_{T\mathbf{M}^3_\kappa},$$

where

$$(A,B)_{\mathfrak{so}(T\mathbf{M}^3_\kappa)} = -\frac{1}{2}\,\mathrm{tr}(AB).$$

Recall the definition of the Killing form

$$B_{\mathfrak{g}}(a,b) = \mathrm{tr}(ad_{\mathfrak{g}}(a)ad_{\mathfrak{g}}(b))$$

for $a, b \in \mathfrak{g}$. $B_{\mathfrak{g}}$ is a symmetric bilinear form, which is $Ad_G(g)$-invariant for all $g \in G$. This implies in particular that $ad_{\mathfrak{g}}(a)$ is antisymmetric with respect to $B_{\mathfrak{g}}$ for all $a \in \mathfrak{g}$.
Let us compute $B_{\mathfrak{g}}$ in terms of the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. First of course the relations $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ imply that $\mathfrak{k}$ and $\mathfrak{p}$ are $B_{\mathfrak{g}}$-orthogonal.

Lemma 5.5 *The restrictions of $B_{\mathfrak{g}}$ to $\mathfrak{k} \cong \mathfrak{so}(T_p\mathbf{M}^3_\kappa)$ and $\mathfrak{p} \cong T_p\mathbf{M}^3_\kappa$ are given as follows:*

$$B_{\mathfrak{g}}|_{\mathfrak{k}}\,(\,\cdot\,,\cdot) = -4(\,\cdot\,,\,\cdot)_{\mathfrak{so}(T_p\mathbf{M}^3_\kappa)}$$
$$B_{\mathfrak{g}}|_{\mathfrak{p}}\,(\,\cdot\,,\cdot) = -4\kappa(\,\cdot\,,\,\cdot)_{T_p\mathbf{M}^3_\kappa}$$

Proof. Let us first calculate the restriction of $B_{\mathfrak{g}}$ to $\mathfrak{k}$. Note that $ad_{\mathfrak{g}}(A)$ preserves $\mathfrak{k}$ and $\mathfrak{p}$ for $A \in \mathfrak{k}$. It is easily verified that

$$\mathrm{tr}_{\mathfrak{k}}\,(ad_{\mathfrak{g}}(A)ad_{\mathfrak{g}}(B)) = \mathrm{tr}\,(ad_{\mathfrak{k}}(A)ad_{\mathfrak{k}}(B)) = \mathrm{tr}(AB)$$

for $A, B \in \mathfrak{k}$. Obviously we have

$$\mathrm{tr}_{\mathfrak{p}}\,(ad_{\mathfrak{g}}(A)ad_{\mathfrak{g}}(B)) = \mathrm{tr}(AB)$$

On the other hand

$$(A,B)_{\mathfrak{so}(T_p\mathbf{M}^3_\kappa)} = -\frac{1}{2}\,\mathrm{tr}(AB).$$

This implies

$$B_{\mathfrak{g}}(A,B) = -4(A,B)_{\mathfrak{so}(T_p\mathbf{M}^3_\kappa)}$$

for $A, B \in \mathfrak{k}$.

Now consider the restriction of $B_{\mathfrak{g}}$ to $\mathfrak{p}$. $ad_{\mathfrak{g}}(X)$ switches $\mathfrak{k}$ and $\mathfrak{p}$. For $A \in \mathfrak{k}$ we have

$$ad_{\mathfrak{g}}(X)ad_{\mathfrak{g}}(Y)A = R(X, AY),$$

and therefore with $\{e_{ij} = e^i \otimes e_j - e^j \otimes e_i\}_{i<j}$ the standard orthonormal basis of $\mathfrak{so}(T_p\mathbf{M}^3_\kappa)$ associated with a frame $\{e_i\}_i$ and the dual coframe $\{e^j\}_j$:

$$\begin{aligned}
\operatorname{tr}_k\left(ad_{\mathfrak{g}}(X)ad_{\mathfrak{g}}(Y)\right) &= \sum_{i<j}\left(e_{ij}, R(X, e_{ij}Y)\right)_{\mathfrak{so}(T_p\mathbf{M}^3_\kappa)}\\
&= \sum_{i<j}(e_i,Y)\left(e_{ij}, R(X,e_j)\right) - (e_j,Y)\left(e_{ij}, R(X,e_i)\right)\\
&= \sum_{i<j}(e_i,Y)\left(R(X,e_j)e_i, e_j\right) - (e_j,Y)\left(R(X,e_i)e_i, e_j\right)\\
&= -\sum_i (e_i,Y)\operatorname{Ric}(X,e_i) = -2\kappa(X,Y)_{T_p\mathbf{M}^3_\kappa}
\end{aligned}$$

For $Z \in \mathfrak{p}$ we have

$$ad_{\mathfrak{g}}(X)ad_{\mathfrak{g}}(y)Z = R(Y,Z)X$$

and therefore

$$\begin{aligned}
\operatorname{tr}_{\mathfrak{p}}\left(ad_{\mathfrak{g}}(X)ad_{\mathfrak{g}}(Y)\right) &= \sum_i \left(e_i, R(Y,e_i)X\right)_{T_p\mathbf{M}^3_\kappa}\\
&= -\operatorname{Ric}(X,Y) = -2\kappa(X,Y)_{T_p\mathbf{M}^3_\kappa}
\end{aligned}$$

This proves the claim. ⧈

Corollary 5.6 ($\kappa = 1$) *$ad(Y)$ is antisymmetric with respect to $h^{\mathcal{E}}$ for all $Y \in T\mathbf{M}^3_\kappa$, in particular $\nabla^{\mathcal{E}}$ is compatible with $h^{\mathcal{E}}$, i.e. $\nabla^{\mathcal{E}}h^{\mathcal{E}} = 0$.*

Corollary 5.7 ($\kappa = -1$) *$ad(Y)$ is symmetric with respect to $h^{\mathcal{E}}$ for all $Y \in T\mathbf{M}^3_\kappa$.*

Let us now consider M, the nonsingular part of our cone-manifold X. The condition that M is locally modelled on $\mathbf{M}^3_\kappa$ is usually expressed in terms of the *developing map*

$$\operatorname{dev} : (\widetilde{M}, p_0) \longrightarrow (\mathbf{M}^3_\kappa, p)$$

and the *holonomy representation*

$$\operatorname{hol} : \pi_1(M, x_0) \longrightarrow G = \operatorname{Isom}^+\mathbf{M}^3_\kappa\,,$$

where dev is a local isometry and $\pi_1(M)$-equivariant with respect to the deck-action on $\widetilde{M}$ and the action via hol on $\mathbf{M}^3_\kappa$.
We again denote by $\mathcal{E}$ the bundle $\mathfrak{so}(TM) \oplus TM$. Since being a Killing vectorfield is a purely local condition, we again have a flat connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ with the property that parallel sections correspond to Killing vectorfields. The formula for $\nabla^{\mathcal{E}}$ given in Lemma **(5.3)** applies as well.
In contrast to the model-space situation, $\mathcal{E}$ will now have holonomy. It is easy to see that the holonomy of $\mathcal{E}$ along a loop $\gamma \in \pi_1(M, x_0)$ is given by $Ad \circ \operatorname{hol}(\gamma)$ if we identify $\mathcal{E}_{x_0}$ with $\mathfrak{g}$. Therefore we obtain an alternative description of $\mathcal{E}$:

$$\mathcal{E} = \widetilde{M} \times_{Ad \circ \operatorname{hol}} \mathfrak{g}$$

The by this representation obvious Lie-algebra structure on $\mathcal{E}$ is consistent with the one given in Lemma **(5.1)**.

The same considerations apply to the two-dimensional situation as well if we replace $\mathbf{M}^3_\kappa$ and its isometry group with the corresponding two-dimensional objects. Here we restrict our attention to the spherical case. Let

$$S = \begin{cases} \mathbf{S}^2(\alpha,\beta,\gamma) & \text{or} \\ \mathbf{S}^2(\alpha) & \end{cases}$$

in the following. Since S is a spherical cone-surface and $\mathrm{Isom}^+ \mathbf{S}^2 = \mathrm{SO}(3)$ we obtain a holonomy representation

$$\mathrm{hol} : \pi_1(\mathrm{int}\, S) \longrightarrow \mathrm{Isom}^+ \mathbf{S}^2 = \mathrm{SO}(3)$$

and developing map

$$\mathrm{dev} : \widetilde{\mathrm{int}\, S} \longrightarrow \mathbf{S}^2 \,.$$

Let us denote the vector-bundle of infinitesimal isometries with its natural flat connection in this situation by $(\mathcal{F}, \nabla^{\mathcal{F}})$. We have

$$\mathcal{F} = \widetilde{\mathrm{int}\, S} \times_{Ad \circ \mathrm{hol}} \mathfrak{so}(3) \,.$$

Since the adjoint representation of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)$ is isomorphic with the standard representation of $\mathrm{SO}(3)$ on $\mathbb{R}^3$, we have alternatively

$$\mathcal{F} = \widetilde{\mathrm{int}\, S} \times_{\mathrm{hol}} \mathbb{R}^3 \,.$$

Now if $x_i \in S$ is a cone-point with cone-angle α_i and $\gamma_i \in \pi_1(\mathrm{int}\, S)$ a loop around x_i, then $\mathrm{hol}(\gamma_i)$ is just rotation about the cone-angle α_i around some fixed axis in $\mathbb{R}^3$. Note that the axis of $\mathrm{hol}(\gamma_i)$ and the axis of $\mathrm{hol}(\gamma_j)$ need not coincide for $x_i \neq x_j$.
This gives us a quite explicit description of $\mathcal{F}$. In particular we see that locally around the cone-points we have the following splitting

$$\mathcal{F}|_{S^1_{\alpha_i}} = \mathbb{C}(\alpha_i) \oplus \mathbb{R},$$

where $\mathbb{C}(\alpha_i)$ denotes the flat $U(1)$-bundle over $S^1_{\alpha_i}$ with holonomy $e^{i\alpha_i}$.

Next we describe the restriction of $\mathcal{E}$ to the links of singular points. Recall that if $x \in \Sigma$ is a singular point and S_x is its link, then

$$S_x \cong \mathbf{S}^2(\alpha,\beta,\gamma)$$

if x is a vertex, and

$$S_x \cong \mathbf{S}^2(\alpha)$$

if x is an edge point.

Lemma 5.8 *Let S_x be the link of a singular point $x \in \Sigma$. Then the restriction of $\mathcal{E}$ to $\mathrm{int}\, S_x$ is given by:*

$$\mathcal{E}|_{\mathrm{int}\, S_x} = \mathcal{F} \oplus \mathcal{F},$$

where $\mathcal{F}$ is the flat vector-bundle of infinitesimal isometries on S_x.

Proof. The holonomy of $\pi_1(\mathrm{int}\, S_x)$ fixes a point $p \in \mathbf{M}^3_\kappa$ and is therefore contained in $K = \mathrm{Stab}_G(p) \cong \mathrm{SO}(T_p\mathbf{M}^3_\kappa)$. We have seen in Corollary **(5.2)** that $Ad_G(g) = (Ad_K(g), g)$ for $g \in K$ with respect to the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Again, since the adjoint representation and the standard representation of $\mathrm{SO}(3)$ are isomorphic, we obtain two copies of $\mathcal{F}$. ⊡

Proposition 5.9 *Let C be a cone-manifold with cone-angles $\leq \pi$. Then $(\mathcal{E}, \nabla^{\mathcal{E}})$, the vector-bundle of infinitesimal isometries of $M = \operatorname{int} C$ with its natural flat connection, is cone-admissible.*

Proof. If $x \in \Sigma$ is a singular point and $P^{\kappa}_{B_x}$ is the model operator for D^{ev} on $U_\varepsilon(x)$, according to Definition **(4.26)** we have to check that B_x is essentially selfadjoint and $\operatorname{spec} B_x \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$. If S_x is the link of x, via the previous results we have

$$\mathcal{E}|_{\operatorname{int} S_x} = \mathcal{F} \oplus \mathcal{F}\,.$$

Clearly $(\mathcal{F}, \nabla^{\mathcal{F}})$ is orthogonally flat and, via Remark **(4.21)**, cone-admissible over $\operatorname{int} S_x$ if the cone-angles are $\leq \pi$. Then we may apply Remark **(4.27)** to conclude that $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible over M. ◻

In the Euclidean case for fixed $p \in \mathbf{E}^3$ we have a group homomorphism

$$\begin{aligned} \operatorname{rot} : \operatorname{Isom}^+ \mathbf{E}^3 &\longrightarrow \operatorname{Stab}_G(p) \cong \operatorname{SO}(T_p\mathbf{E}^3) \\ g &\longmapsto g + (p - g(p)) \end{aligned}$$

We may form the rotational part of the holonomy

$$\operatorname{rot} \circ \operatorname{hol} : \pi_1(M) \longrightarrow \operatorname{Stab}_G(p) \cong \operatorname{SO}(T_p\mathbf{E}^3)\,.$$

On the other hand

$$\mathcal{E}_{trans} := TM \subset \mathcal{E} = \mathfrak{so}(TM) \oplus TM$$

is via the explicit formula for $\nabla^{\mathcal{E}}$ in Lemma **(5.3)** easily seen to be a parallel subbundle of $\mathcal{E}$. Note that in contrast

$$\mathcal{E}_{rot} := \mathfrak{so}(TM) \subset \mathcal{E} = \mathfrak{so}(TM) \oplus TM$$

is *not* parallel. We rather obtain an exact sequence of flat vector-bundles

$$0 \longrightarrow \mathcal{E}_{trans} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_{trans} \longrightarrow 0\,,$$

which does not split in general.
Since the rotational part of the holonomy is nothing but the holonomy of the flat tangent bundle, we obtain

$$\mathcal{E}_{trans} = \widetilde{M} \times_{\operatorname{rot} \circ \operatorname{hol}} \mathbb{R}^3\,.$$

In the same way as before one shows:

Lemma 5.10 *Let C be a Euclidean cone-manifold. The restriction of $\mathcal{E}_{trans}$ to the link S_x of a singular point $x \in \Sigma$ is given as*

$$\mathcal{E}_{trans}|_{\operatorname{int} S_x} = \mathcal{F},$$

where $\mathcal{F}$ is the flat vector-bundle of infinitesimal isometries on S_x. Furthermore $\mathcal{E}_{trans}$ is cone-admissible if the cone-angles are $\leq \pi$.

5.2 Weitzenböck formulas

5.2.1 The spherical and the Euclidean case

Let $(\mathcal{E}, \nabla^{\mathcal{E}})$ be an orthogonally flat vector-bundle. Recall the standard Weitzenböck formula on $\mathcal{E}$-valued 1-forms (for this formula to hold we really need that the metric on $\mathcal{E}$ is parallel!):

$$\Delta\omega = \nabla^t\nabla\omega + (\operatorname{Ric} \otimes \operatorname{id})\omega$$

The action of the Ricci tensor on a scalar-valued 1-form α is determined by the relation

$$g(\mathrm{Ric}(\alpha), \beta) = \mathrm{Ric}(\alpha, \beta)$$

for all $\beta \in \Omega^1(M, \mathbb{R})$. In three dimensions the Ricci tensor of a metric with constant curvature κ is given by

$$\mathrm{Ric}(\cdot\,, \cdot) = 2\kappa \cdot g(\cdot\,, \cdot),$$

so we end up with

$(\kappa > 0)$

$$\Delta\omega = \nabla^t \nabla\omega + 2\kappa \cdot \omega$$

$(\kappa = 0)$

$$\Delta\omega = \nabla^t \nabla\omega$$

in the non-negatively curved cases.

5.2.2 The hyperbolic case

Here we use a different kind of Weitzenböck formula, essentially due to [MM]. We use the notation of [HK]. Let $\mathcal{E}$ be the vector-bundle of infinitesimal isometries and $\nabla^{\mathcal{E}}$ its natural flat connection. By abuse of notation we continue to denote by $\nabla^{\mathcal{E}}$ the tensor-product connection on $\Lambda^\bullet T^*M \otimes \mathcal{E}$ induced by the Levi-Civita connection and $\nabla^{\mathcal{E}}$, in a similar twofold fashion we use the symbol ∇.
Recall the relation $\nabla^{\mathcal{E}}_Y = \nabla_Y + ad(Y)$ for $Y \in TM$, where the endomorphism $ad(Y)$ is symmetric with respect to $h^{\mathcal{E}}$. Let in the following

$$\varepsilon : T^*M \otimes \Lambda^\bullet T^*M \to \Lambda^{\bullet+1} T^*M$$

denote exterior multiplication, and

$$\iota : TM \otimes \Lambda^\bullet T^*M \to \Lambda^{\bullet-1} T^*M$$

denote interior multiplication. Then we have

$$d = \sum_i \varepsilon(e^i) \nabla^{\mathcal{E}}_{e_i} = \sum_i \varepsilon(e^i) \left(\nabla_{e_i} + ad(e_i)\right).$$

This implies

$$d^t = -\sum_i \iota(e_i) \left(\nabla_{e_i} - ad(e_i)\right).$$

Define

$$D := \sum_i \varepsilon(e^i) \nabla_{e_i} \text{ and } T := \sum_i \varepsilon(e^i) ad(e_i),$$

this implies

$$D^t = -\sum_i \iota(e^i) \nabla_{e_i} \text{ and } T^t = \sum_i \iota(e_i) ad(e_i).$$

We obviously have $d = D + T$ and $d^t = D^t + T^t$, let $\Delta_D := DD^t + D^t D$ and $H := TT^t + T^t T$. H is symmetric and non-negative.

Lemma 5.11 $\Delta = \Delta_D + H$

Proof. From the definitions we have

$$\begin{aligned}\Delta &= dd^t + d^t d \\ &= \Delta_D + H + DT^t + TD^t + D^tT + T^tD\end{aligned}$$

Therefore we have to show that $DT^t + TD^t + D^tT + T^tD = 0$.
Note that the Lie-bracket is parallel with respect to the flat connection $\nabla^{\mathcal{E}}$, i.e.

$$\nabla^{\mathcal{E}}_X([\sigma,\tau]) = [\nabla^{\mathcal{E}}_X\sigma,\tau] + [\sigma,\nabla^{\mathcal{E}}_X\tau].$$

This means that ad is $\nabla^{\mathcal{E}}$-parallel:

$$[\nabla^{\mathcal{E}}_X, ad(\sigma)] = ad(\nabla^{\mathcal{E}}_X\sigma)$$

With $\nabla_X = \nabla^{\mathcal{E}}_X + ad(X)$ we obtain

$$\begin{aligned}[\nabla_X, ad(\sigma)] &= \left[\nabla^{\mathcal{E}}_X - ad(X), ad(\sigma)\right] \\ &= ad(\nabla^{\mathcal{E}}_X\sigma) - [ad(X), ad(\sigma)] \\ &= ad\left(\nabla_X\sigma + ad(X)\sigma\right) - ad\left([X,\sigma]\right) \\ &= ad(\nabla_X\sigma),\end{aligned}$$

such that ad is also ∇-parallel. ε and ι are certainly ∇-parallel. In normal coordinates we may therefore assume that $[\nabla_{e_i}, ad(e_j)] = [\nabla_{e_i}, \varepsilon(e^j)] = [\nabla_{e_i}, \iota(e_j)] = 0$.
Using the relation $\varepsilon(e^i)\iota(e_j) + \iota(e_j)\varepsilon(e^i) = \delta_{ij}$ we obtain

$$\begin{aligned}DT^t + T^tD &= \sum_{i,j}\left(\varepsilon(e^i)\iota(e_j) + \iota(e_j)\varepsilon(e^i)\right)\nabla_{e_i}ad(e_j) \\ &= \sum_i \nabla_{e_i}ad(e_i).\end{aligned}$$

Similarly, we have

$$\begin{aligned}TD^t + D^tT &= -\sum_{i,j}\left(\varepsilon(e^j)\iota(e_i) + \iota(e_i)\varepsilon(e^j)\right)\nabla_{e_i}ad(e_j) \\ &= -\sum_i \nabla_{e_i}ad(e_i).\end{aligned}$$

This proves the claim. ◻

Lemma 5.12 $H = \sum_i ad(e_i)^2 + \sum_{i,j}\varepsilon(e^i)\iota(e_j)ad\left([e_i,e_j]\right)$

Proof. We have, again using $\varepsilon(e^k)\iota(e_l) + \iota(e_l)\varepsilon(e^k) = \delta_{kl}$, that

$$\begin{aligned}TT^t + T^tT &= \sum_{i,j}\left(\varepsilon(e^i)\iota(e_j) + \iota(e_i)\varepsilon(e^j)\right)ad(e_i)ad(e_j) \\ &= \sum_i ad(e_i)^2 + \sum_{i\neq j}\left(\varepsilon(e^i)\iota(e_j) - \varepsilon(e^j)\iota(e_i)\right)ad(e_i)ad(e_j) \\ &= \sum_i ad(e_i)^2 + \sum_{i<j}\left(\varepsilon(e^i)\iota(e_j) - \varepsilon(e^j)\iota(e_i)\right)ad\left([e_i,e_j]\right) \\ &= \sum_i ad(e_i)^2 + \sum_{i,j}\varepsilon(e^i)\iota(e_j)ad\left([e_i,e_j]\right)\end{aligned}$$

This proves the claim. ◻

Proposition 5.13 [Wei] *There is a constant $C > 0$ such that*

$$(H\omega, \omega)_x \geq C(\omega, \omega)_x$$

for all $\omega \in \Omega^1(M, \mathcal{E})$ and $x \in M$.

Proof. We first observe that $H : \Lambda^1 T^*M \otimes \mathcal{E} \longrightarrow \Lambda^1 T^*M \otimes \mathcal{E}$ preserves the decomposition $\mathcal{E} = \mathfrak{so}(TM) \oplus TM$. Since H is symmetric with respect to $h^{\mathcal{E}}$, we may choose an orthonormal basis of eigenvectors $\{e^k \otimes (e_k \oplus e_{\alpha\beta})\}_{k,l,\alpha<\beta}$. We have

$$H(e^k \otimes e_l) = \sum_i e^k \otimes ad(e_i)^2 e_l + \sum_i e^i \otimes ad([e_i, e_k])\, e_l$$

Now $ad(e_i)^2 e_l = R(e_i, e_l)e_i$ and $ad([e_i, e_k])\, e_l = -R(e_i, e_k)e_l$, therefore

$$H(e^k \otimes e_l) = \sum_i e^k \otimes R(e_i, e_l)e_i - \sum_i e^i \otimes R(e_i, e_k)e_l$$

and

$$\begin{aligned}\big(H(e^k \otimes e_l), e^k \otimes e_l\big) &= -\sum_i (R(e_l, e_i)e_i, e_l) - \sum_i \delta_{ik}\,(R(e_i, e_k)e_l, e_l) \\ &= -\operatorname{Ric}(e_l, e_l) - 0 = -2\kappa = 2\end{aligned}$$

With $e_{\alpha\beta} = e^\alpha \otimes e_\beta - e^\beta \otimes e_\alpha$ we have

$$H(e^k \otimes e_{\alpha\beta}) = \sum_i e^k \otimes ad(e_i)^2 e_{\alpha\beta} + \sum_i e^i \otimes ad([e_i, e_k])\, e_{\alpha\beta}$$

Now $ad(e_i)^2 e_{\alpha\beta} = \delta_{\alpha i} R(e_i, e_\beta) - \delta_{\beta i} R(e_i, e_\alpha)$ and $ad([e_i, e_k])\, e_{\alpha\beta} = -\,[R(e_i, e_k), e_{\alpha\beta}]$, therefore

$$H(e^k \otimes e_{\alpha\beta}) = 2e^k \otimes R(e_\alpha, e_\beta) - \sum_i e^i \otimes [R(e_i, e_k), e_{\alpha\beta}]$$

and

$$\begin{aligned}\big(H(e^k \otimes e_{\alpha\beta}), e^k \otimes e_{\alpha\beta}\big) &= 2\,(R(e_\alpha, e_\beta), e_{\alpha\beta}) - \sum_i \delta_{ik}\,([R(e_i, e_k), e_{\alpha\beta}]\,, e_{\alpha\beta}) \\ &= 2\,(R(e_\alpha, e_\beta), e_{\alpha\beta}) - 0 \\ &= -2\,(R(e_\alpha, e_\beta)e_\beta, e_\alpha) \\ &= -2\kappa = 2\end{aligned}$$

Therefore we have shown that $(H\omega, \omega)_x = 2(\omega, \omega)_x$ for all $\omega \in \Omega^1(M, \mathcal{E})$ and $x \in M$. In particular with $C = 2$ we obtain the proposition. ◻

5.3 A vanishing theorem

In this section we prove the main result about L^2-cohomology spaces of cone-manifolds with values in the flat vector-bundle of infinitesimal isometries. This completes the analytic part of our argument.

Theorem 5.14 *Let C be a 3-dimensional cone-manifold of curvature $\kappa \in \{-1, 0, 1\}$ with cone-angles $\leq \pi$. Let $(\mathcal{E}, \nabla^{\mathcal{E}})$ be the vector-bundle of infinitesimal isometries of $M = C \setminus \Sigma$ with its natural flat connection. In the Euclidean case let $\mathcal{E}_{trans} \subset \mathcal{E}$ be the parallel subbundle of infinitesimal translations. Then in the hyperbolic and the spherical case*

$$H^1_{L^2}(M, \mathcal{E}) = 0\,,$$

while in the Euclidean case

$$H^1_{L^2}(M, \mathcal{E}_{trans}) \cong \{\omega \in \Omega^1(M, \mathcal{E}_{trans}) \,|\, \nabla\omega = 0\}\,.$$

For convenience we give a proof for each constant-curvature geometry separately.

5.3.1 The spherical case

Theorem 5.15 *Let C be a spherical cone-manifold with cone-angles $\leq \pi$. Let $M = C \setminus \Sigma$ and $(\mathcal{E}, \nabla^{\mathcal{E}})$ be the vector-bundle of infinitesimal isometries of M with its natural flat connection. Then*

$$H^1_{L^2}(M, \mathcal{E}) = 0 \, .$$

Proof. We recall the Weitzenböck formula for the Hodge-Laplace operator on $\mathcal{E}$-valued 1-forms, which in the spherical case (i.e. $\kappa = 1$) amounts to

$$\Delta\omega = \nabla^t \nabla \omega + 2\omega$$

for $\omega \in \Omega^1(M, \mathcal{E})$. For $\omega \in \Omega^1_{cp}(M, \mathcal{E})$ integration by parts yields

$$\begin{aligned}
\int_M (\Delta\omega, \omega) &= \int_M (\nabla^t \nabla \omega, \omega) + 2 \int_M |\omega|^2 \\
&= \int_M |\nabla\omega|^2 + 2 \int_M |\omega|^2 \\
&\geq 2 \int_M |\omega|^2
\end{aligned}$$

This means we have a positive lower bound for Δ on $\Omega^1_{cp}(M, \mathcal{E})$:

$$\langle \Delta\omega, \omega \rangle_{L^2} \geq C \, \langle \omega, \omega \rangle_{L^2}$$

with $C = 2$. Since $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible according to Proposition **(5.9)**, we obtain

$$\Delta_F = \Delta(d_{max})$$

via Corollary **(4.33)**. Since the Friedrichs extension preserves lower bounds, we conclude

$$\mathcal{H}^1_{max} = \ker \Delta^1(d_{max}) = 0 \, .$$

Finally the strong Hodge theorem for cone-manifolds, Theorem **(4.29)**, identifies L^2-cohomology with the d_{max}-harmonic forms. This implies $H^1_{L^2}(M, \mathcal{E}) = 0$ and therefore proves the theorem. ◻

5.3.2 The Euclidean case

Theorem 5.16 *Let C be a Euclidean cone-manifold with cone-angles $\leq \pi$. Let $\mathcal{E}_{trans} \subset \mathcal{E}$ be the parallel subbundle of infinitesimal translations of $M = C \setminus \Sigma$. Then*

$$H^1_{L^2}(M, \mathcal{E}_{trans}) \cong \{\omega \in \Omega^1(M, \mathcal{E}_{trans}) \, | \, \nabla\omega = 0\} \, .$$

Proof. The Weitzenböck formula for the Hodge-Laplace operator on $\mathcal{E}_{trans}$-valued 1-forms in the Euclidean case (i.e. $\kappa = 0$) amounts to

$$\Delta\omega = \nabla^t \nabla \omega$$

for $\omega \in \Omega^1(M, \mathcal{E}_{trans})$. This implies with Corollary **(3.3)** that

$$\Delta_F = \nabla^t_{max} \nabla_{min} \, .$$

Since $\mathcal{E}_{trans} \subset \mathcal{E}$ is cone-admissible according to Lemma **(5.10)**, we obtain

$$\Delta_F = \Delta(d_{max}) \, .$$

via Corollary **(4.33)**. This implies that

$$\Delta(d_{max}) = \nabla^t_{max}\nabla_{min}\,.$$

For $\omega \in \ker \Delta^1(d_{max})$ we have

$$0 = \langle \Delta(d_{max})\omega, \omega\rangle_{L^2} = \langle \nabla^t_{max}\nabla_{min}\omega, \omega\rangle_{L^2} = \|\nabla_{min}\omega\|^2_{L^2}$$

We conclude that $\omega \in \ker \nabla_{min}$. On the other hand if $\omega \in L^2(\Lambda^1 T^*M \otimes \mathcal{E}_{trans})$ satisfies $\nabla\omega = 0$ in the distributional sense, then

$$d\omega = (\varepsilon \circ \nabla)\omega = 0$$

and

$$d^t\omega = -(\iota \circ \nabla)\omega = 0$$

will be satisfied in the distributional sense, in particular $\omega \in \ker D_{max}$. Since D is essentially selfadjoint according to Corollary **(4.32)**, we obtain $\omega \in \ker D(d_{max})$. Thus $\omega \in \ker \Delta^1(d_{max})$. We obtain

$$\ker \nabla_{max} \subset \mathcal{H}^1_{max} = \ker \Delta^1(d_{max}) \subset \ker \nabla_{min}\,,$$

in particular

$$\mathcal{H}^1_{max} = \ker \nabla_{max} = \{\omega \in L^2(\Lambda^1 T^*M \otimes \mathcal{E}_{trans}) \,|\, \nabla\omega = 0\}\,.$$

This implies via Theorem **(4.29)**, the strong Hodge theorem for cone-manifolds, that

$$H^1_{L^2}(M, \mathcal{E}_{trans}) \cong \{\omega \in \Omega^1(M, \mathcal{E}_{trans}) \,|\, \nabla\omega = 0\}\,,$$

since $\ker \Delta(d_{max})$ consists of smooth sections. Note also that a parallel form ω will automatically be L^2-bounded, since ∇ is compatible with metric on $\mathcal{E}_{trans}$. This proves the theorem. ⊡

5.3.3 The hyperbolic case

Theorem 5.17 *Let C be a hyperbolic cone-manifold with cone-angles $\leq \pi$. Let $M = C \setminus \Sigma$ and $(\mathcal{E}, \nabla^{\mathcal{E}})$ be the vector-bundle of infinitesimal isometries of M with its natural flat connection. Then*

$$H^1_{L^2}(M, \mathcal{E}) = 0\,.$$

Proof. The proof follows the same scheme as in the spherical case. For convenience of the reader we also give full details in this case.
We recall that in the hyperbolic case we have a Weitzenböck formula for the Hodge-Laplace operator for $\mathcal{E}$-valued 1-forms of the type

$$\Delta\omega = D^t D\omega + DD^t\omega + H\omega\,,$$

where

$$\langle H\omega, \omega\rangle_{L^2} \geq C\,\langle \omega, \omega\rangle_{L^2}$$

for $C > 0$ independent of $\omega \in \Omega^1(M, \mathcal{E})$. For $\omega \in \Omega^1_{cp}(M, \mathcal{E})$ integration by parts yields

$$\begin{aligned}\int_M (\Delta\omega, \omega) &= \int_M (D^t D\omega, \omega) + \int_M (DD^t\omega, \omega) + \int_M (H\omega, \omega)\\ &= \int_M |D\omega|^2 + \int_M |D^t\omega|^2 + \int_M (H\omega, \omega)\end{aligned}$$

$$\geq c \int_M |\omega|^2$$

This means we have a positive lower bound for Δ on $\Omega^1_{cp}(M,\mathcal{E})$:

$$\langle \Delta\omega, \omega\rangle_{L^2} \geq C \langle \omega, \omega\rangle_{L^2}$$

for $C > 0$. Since $(\mathcal{E}, \nabla^{\mathcal{E}})$ is cone-admissible according to Proposition **(5.9)**, we obtain

$$\Delta_F = \Delta(d_{max})$$

via Corollary **(4.33)**. Since the Friedrichs extension preserves lower bounds, we conclude

$$\mathcal{H}^1_{max} = \ker \Delta^1(d_{max}) = 0\,.$$

Finally the strong Hodge theorem for cone-manifolds, Theorem **(4.29)**, identifies L^2-cohomology with the d_{max}-harmonic forms. This implies $H^1_{L^2}(M,\mathcal{E}) = 0$ and therefore proves the theorem. ⧈

6 Deformation theory

In this chapter we study the deformation space of cone-manifold structures on a 3-dimensional cone-manifold of given topological type (C,Σ). It is convenient to use the more general framework of (X,G)-structures and deformations thereof, in particular since there is a quite general theorem of [Gol], which relates the local structure of the deformation space of (X,G)-structures to the local structure of $X(\pi_1 M, G)$. By $X(\pi_1 M, G)$ we denote the quotient of $R(\pi_1 M, G)$, the space of representations $\pi_1 M$ in G, by the conjugation action of G.
In our case the relevant (X,G)-structure will be $X = \mathbf{M}^3_\kappa$ and $G = \mathrm{Isom}^+ \mathbf{M}^3_\kappa$, in fact by a theorem of [Cul], the holonomy representation of a 3-dimensional cone-manifold structure may always be lifted to the universal covering group of $\mathrm{Isom}^+ \mathbf{M}^3_\kappa$, which in the hyperbolic case is $\mathrm{SL}_2(\mathbb{C})$ and in the spherical case is $\mathrm{SU}(2) \times \mathrm{SU}(2)$. In the Euclidean case the rotational part of the holonomy lifts to $\mathrm{SU}(2)$.
We will use the L^2-vanishing theorem to analyze local properties of $\mathrm{SL}_2(\mathbb{C})$-, and $\mathrm{SU}(2)$-representation spaces. From this we will be able to conclude local rigidity in the hyperbolic and in the spherical case.

6.1 (X,G)-structures

Let (X, g^X) be a Riemannian manifold upon which a Lie group G acts transitively by isometries. Let M be manifold of the same dimension as X. Then we say that M carries an (X,G)-structure if M is locally modelled on X, i.e. there is a covering of M by charts $\{\varphi_i : U_i \to X\}_{i\in I}$ such that for each connected component of C of $U_i \cap U_j$ there exists $g_{C,i,j} \in G$ such that $g_{C,i,j} \circ \varphi_i = \varphi_j$ on C. The collection of charts $\{\varphi_i : U_i \to X\}_{i\in I}$ is called an (X,G)-atlas and an (X,G)-structure on M is a maximal (X,G)-atlas. A detailed discussion of this kind of structure may be found in [Gol], which we will mainly refer to in this section.
The most important examples are for our purposes of course 3-dimensional Riemannian manifolds of constant sectional curvature κ, which can be understood as (X,G)-manifolds with $X = \mathbf{M}^3_\kappa$ and $G = \mathrm{Isom}^+ \mathbf{M}^3_\kappa$.
Let us fix a basepoints $x_0 \in M$ and $p_0 \in \pi^{-1}(x_0)$, where $\pi : \widetilde{M} \to M$ is the universal covering of M. Then an (X,G)-structure on M together with the germ of an (X,G)-chart $\varphi : U \to X$ around x_0 determines by analytic continuation of φ a local diffeomorphism

$$\mathrm{dev} : \widetilde{M} \longrightarrow X\,,$$

the developing map, and a representation

$$\text{hol} : \pi_1(M, x_0) \longrightarrow G\,,$$

the holonomy representation, such that dev is equivariant with respect to hol, i.e.

$$\text{dev} \circ \gamma = \text{hol}(\gamma) \circ \text{dev}$$

for all $\gamma \in \pi_1(M, x_0)$. Conversely, a local diffeomorphism dev : $\widetilde{M} \longrightarrow X$, which is equivariant with respect to a representation hol : $\pi_1(M, x_0) \longrightarrow G$ as above, defines an (X, G)-structure on M together with the germ of an (X, G)-chart at x_0. Note that hol is uniquely determined by dev and the equivariance condition.

Let $\mathcal{D}'_{(X,G)}(M)$ be the space of developing maps with the topology of C^∞-convergence on compact sets. As usual we equip $R(\pi_1(M, x_0), G)$, the set of representations of $\pi_1(M, x_0)$ in G, with the compact-open topology. Associating its holonomy representation with a developing map yields a continuous map

$$\begin{aligned} \mathcal{D}'_{(X,G)}(M) &\longrightarrow R(\pi_1(M, x_0), G) \\ \text{dev} &\longmapsto \text{hol}\ . \end{aligned}$$

For simplicity we assume that M is diffeomorphic to the interior of a compact manifold with boundary $M \cup \partial M$, which is certainly the case for the object of our main concern, namely the smooth part of a 3-dimensional cone-manifold.

Following [CHK] we introduce the equivalence relation $\sim$ on the space of developing maps, which is generated by *isotopy* and *thickening*. Clearly $\text{Diff}_0(M)$, the group of diffeomorphisms of M isotopic to the identity, acts on the space of developing maps, two structures equivalent under this action will be called *isotopic*. On the other hand, if an (X, G) structure on M extends to $M \cup \partial M \times [0, \varepsilon)$ for some $\varepsilon > 0$, this gives rise to an (X, G)-structure on M, which we will call a *thickening* of the original structure. Let

$$\mathcal{D}_{(X,G)}(M) = \mathcal{D}'_{(X,G)}(M)/\sim\ .$$

We obtain a G-equivariant map

$$\begin{aligned} \mathcal{D}_{(X,G)}(M) &\longrightarrow R(\pi_1(M, x_0), G) \\ [\text{dev}] &\longmapsto \text{hol}\ . \end{aligned}$$

We define the *deformation space* of (X, G)-structures to be the quotient

$$\mathcal{T}_{(X,G)}(M) := \mathcal{D}_{(X,G)}(M)/G\,.$$

Let $X(\pi_1(M, x_0), G)$ denote the G-quotient of $R(\pi_1(M, x_0), G)$ by conjugation, properties of this quotient in our particular context will be discussed in greater detail in subsequent sections.

Let us assume that the action of G on $R(\pi_1(M, x_0), G)$ by conjugation is proper, this implies in particular by the G-equivariance of the above map, that the action of G on $\mathcal{D}_{(X,G)}(M)$ is also proper. In this situation the arguments of [Gol] (cf. also the discussion in [CHK]) yield the following theorem about the local structure of the deformation space of (X, G)-structures:

Theorem 6.1 (Deformation theorem) [Gol] *If the conjugation action of G on $R(\pi_1(M, x_0), G)$ is proper, then the map*

$$\begin{aligned} \mathcal{T}_{(X,G)}(M) &\longrightarrow X(\pi_1(M, x_0), G) \\ [\text{dev}] &\longmapsto [\text{hol}] \end{aligned}$$

is a local homeomorphism.

This theorem explains the meaning of representation spaces in the study of deformations of (X,G)-structures: Local properties of $X(\pi_1(M,x_0),G)$ translate into local properties of the deformation space of (X,G)-structures on M.

By a theorem of M. Culler (cf. [Cul]) the holonomy representation of a cone-manifold may be lifted to the universal covering group of $\mathrm{Isom}^+\,\mathbf{M}^3_\kappa$:

$$\widetilde{\mathrm{hol}} : \pi_1 M \longrightarrow \widetilde{\mathrm{Isom}^+\mathbf{M}^3_\kappa}$$

In the hyperbolic case $\widetilde{\mathrm{Isom}^+\mathbf{H}^3} = \mathrm{SL}_2(\mathbb{C})$. We obtain that the flat vector-bundle of infinitesimal isometries may be written as

$$\mathcal{E} = \widetilde{M} \times_{Ad\circ\widetilde{\mathrm{hol}}} \mathfrak{sl}_2(\mathbb{C})\,.$$

As a consequence $\mathcal{E}$ has a parallel complex structure, such that in particular all the cohomology spaces $H^i(M,\mathcal{E})$ are complex vector spaces.
In the spherical case $\widetilde{\mathrm{Isom}^+\mathbf{S}^3} = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Therefore the lift of the holonomy splits as a product representation

$$\widetilde{\mathrm{hol}} = (\mathrm{hol}_1, \mathrm{hol}_2) : \pi_1 M \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2)\,,$$

in particular the flat vector-bundle of infinitesimal isometries splits as a direct sum of parallel subbundles:

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2\,,$$

where

$$\mathcal{E}_i = \widetilde{M} \times_{Ad\circ\mathrm{hol}_i} \mathfrak{su}(2)\,.$$

Consequently $H^i(M,\mathcal{E}) = H^i(M,\mathcal{E}_1) \oplus H^i(M,\mathcal{E}_2)$ for all i.
In the Euclidean case $\widetilde{\mathrm{Isom}^+\mathbf{E}^3} = \mathrm{SU}(2) \ltimes \mathbb{R}^3$. We may lift the rotational part $\mathrm{rot} : \mathrm{Isom}^+\,\mathbf{E}^3 \to \mathrm{SO}(3)$ to the universal covering groups:

$$\widetilde{\mathrm{rot}} : \mathrm{SU}(2) \ltimes \mathbb{R}^3 \longrightarrow \mathrm{SU}(2)\,.$$

We have an exact sequence

$$0 \to \mathbb{R}^3 \to \widetilde{\mathrm{Isom}^+\mathbf{E}^3} \xrightarrow{\widetilde{\mathrm{rot}}} \mathrm{SU}(2) \to 1\,.$$

The lift of the translational part $\mathrm{trans} : \mathrm{Isom}^+\,\mathbf{E}^3 \to \mathbb{R}^3$

$$\widetilde{\mathrm{trans}} : \mathrm{SU}(2) \ltimes \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

is not a group homomorphism, but rather a group 1-cocycle of with coefficients in the representation $\widetilde{\mathrm{rot}} : \mathrm{SU}(2) \ltimes \mathbb{R}^3 \longrightarrow \mathrm{SU}(2)$, cf. the next section for definitions. We obtain for the flat vector-bundle of infinitesimal translations

$$\begin{aligned}\mathcal{E}_{trans} &= \widetilde{M} \times_{\widetilde{\mathrm{rot}}\circ\widetilde{\mathrm{hol}}} \mathbb{R}^3 \\ &= \widetilde{M} \times_{Ad\circ\widetilde{\mathrm{rot}}\circ\widetilde{\mathrm{hol}}} \mathfrak{su}(2)\,,\end{aligned}$$

where we identify $\mathfrak{su}(2)$ and $\mathbb{R}^3$ as $\mathrm{SU}(2)$-modules.

For notational convenience we will drop the distinction between hol and $\widetilde{\mathrm{hol}}$ (resp. between rot, $\widetilde{\mathrm{rot}}$ and trans, $\widetilde{\mathrm{trans}}$) from here.

6.2 The representation variety

Let Γ be a finitely generated discrete group. Once and for all we fix a presentation $\langle \gamma_1, \ldots, \gamma_n | (r_i)_{i \in I} \rangle$ of Γ. The cardinality of the indexset I may a priori be infinite, however most of the groups we will deal with will turn out to be finitely presented. Let $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SU}(2)$. The **representation variety** $R(\Gamma, G)$ is defined to be the set of group homomorphisms $\rho : \Gamma \to G$. $R(\Gamma, G)$ endowed with the compact-open topology will be a Hausdorff space, compact in the case of $\mathrm{SU}(2)$.
The relations r_i define functions $f_i : G^n \to G$ such that $R(\Gamma, G)$ may be identified with the set $\{(A_1, \ldots, A_n) \in G^n | f_i(A_1, \ldots, A_n) = 1 \forall i \in I\}$. Since $\mathrm{SL}_2(\mathbb{C})$ is a $\mathbb{C}$-algebraic (resp. $\mathrm{SU}(2)$ a $\mathbb{R}$-algebraic) group and the f_i are polynomial maps, $R(\Gamma, G)$ acquires the structure of a $\mathbb{C}$-algebraic (resp. $\mathbb{R}$-algebraic) set. Note that $R(\Gamma, G)$ won't be a smooth space in general.
The action of G on G^n by simultaneous conjugation leaves the set $R(\Gamma, G) \subset G^n$ invariant. Therefore the set-theoretic quotient $X(\Gamma, G) = R(\Gamma, G)/G$ is well defined. We endow $X(\Gamma, G)$ with the quotient topology. $X(\Gamma, G)$ will in general be neither smooth nor even Hausdorff. $X(\Gamma, G)$ as we have defined it is not to be confused with a quotient constructed in the algebraic category. This usually requires arguments from geometric invariant theory, which we can avoid to use here.
A smooth family of representations $\rho_t : \Gamma \to G$ with $\rho_0 = \rho$ defines a group 1-cocycle $z : \Gamma \to \mathfrak{g}$, where

$$z(\gamma) = \frac{d}{dt}\Big|_{t=0} \rho_t(\gamma)\rho(\gamma)^{-1}$$

for $\gamma \in \Gamma$. Recall that $Z^1(\Gamma, \mathfrak{g})$, the space of 1-cocycles of Γ with coefficients in the representation $Ad \circ \rho : \Gamma \to \mathrm{GL}(\mathfrak{g})$, is the the space of maps $z : \Gamma \to \mathfrak{g}$ such that

$$z(ab) = z(a) + Ad \circ \rho(a) z(b)$$

for all $a, b \in \Gamma$. A cocycle z is a coboundary if there exists some $v \in \mathfrak{g}$ such that

$$z(a) = v - Ad \circ \rho(a) v$$

for all $a \in \Gamma$. Let $B^1(\Gamma, \mathfrak{g})$ be the space of 1-coboundaries. Now by definition

$$H^1(\Gamma, \mathfrak{g}) = Z^1(\Gamma, \mathfrak{g}) / B^1(\Gamma, \mathfrak{g})$$

is the first **group cohomology** group of Γ with coefficients in the representation $Ad \circ \rho : \Gamma \to \mathrm{GL}(\mathfrak{g})$. $H^1(\Gamma, \mathfrak{g})$ is a real vector space.
We refer to $Z^1(\Gamma, \mathfrak{g})$ as the space of infinitesimal deformations of the representation ρ. We call a 1-cocycle z integrable, if there exists a (local) deformation ρ_t, which is tangent to z in the above sense.
It is easy to see that $z \in B^1(\Gamma, \mathfrak{g})$ if and only if z is tangent to the orbit of G through ρ, i.e. there exists a smooth curve g_t in G with $g_0 = 1$ such that

$$z(\gamma) = \frac{d}{dt}\Big|_{t=0} g_t \rho(\gamma) g_t^{-1} \rho(\gamma)^{-1}$$

for $\gamma \in \Gamma$. A deformation $\rho_t(\gamma) = g_t \rho(\gamma) g_t^{-1}$ will be considered trivial.
We use the following observation due to A. Weil (cf. [Wei]): A map $z : \Gamma \to \mathfrak{g}$ defines a group 1-cocycle if and only if the map

$$\begin{aligned}(Ad \circ \rho, z) : \Gamma &\longrightarrow \mathrm{GL}(\mathfrak{g}) \ltimes \mathfrak{g} \\ \gamma &\longmapsto (Ad \circ \rho(\gamma), z(\gamma))\end{aligned}$$

is a group homomorphism. $\mathrm{GL}(\mathfrak{g}) \ltimes \mathfrak{g}$ is the affine group of the vector-space $\mathfrak{g}$. Using the fixed presentation of Γ, this identifies $Z^1(\Gamma, \mathfrak{g})$ with a linear subspace of

$\mathfrak{g}^n$. More precisely, the relations r_i determine linear functions $g_i : \mathfrak{g}^n \to \mathrm{GL}(\mathfrak{g}) \ltimes \mathfrak{g}$, such that

$$Z^1(\Gamma, \mathfrak{g}) = \{(a_1, \ldots, a_m) \in \mathfrak{g}^n | g_i(a_1, \ldots, a_n) = 0 \, \forall i \in I\} .$$

On the other hand, $\ker df_i$ may be identified with a subspace of $\mathfrak{g}^n$ via

$$(\dot{A}_1, \ldots, \dot{A}_n) \mapsto (\dot{A}_1 A_1^{-1}, \ldots, \dot{A}_n A_n^{-1}) .$$

With these identifications we have the following lemma:

Lemma 6.2 [Wei] $Z^1(\Gamma, \mathfrak{g}) = \cap_{i \in I} \ker df_i$.

Proof. An easy calculation shows that $df_i(\dot{A}_1, \ldots, \dot{A}_n) = 0$ for $\dot{A}_i \in T_{A_i}G$ if and only if $g_i(a_1, \ldots, a_n) = 0$, where $a_i = \dot{A}_i A_i^{-1}$. ◽

If the equations $(f_i)_{i \in I}$ cut out $R(\Gamma, G)$ transversely near ρ, then the previous lemma identifies $Z^1(\Gamma, \mathfrak{g})$ with the tangent space of $R(\Gamma, G)$ at the point ρ. In particular ρ will be a smooth point. If furthermore the G-action on $R(\Gamma, G)$ by conjugation is free and proper, then $X(\Gamma, G)$ will be smooth near $\chi = [\rho]$ and the tangent space at χ may be identified with $H^1(\Gamma, \mathfrak{g})$.

6.3 Integration and group cohomology

We wish to represent group cocycles of $\pi_1 M$ with coefficients in the representation $Ad \circ \mathrm{hol} : \pi_1 M \to \mathfrak{g} = \mathfrak{isom}^+ \mathbf{M}^3_\kappa$ by differential forms on M with values in $\mathcal{E}$. This will be achieved by means of integration.

Let x_0 be a base point in M, then for $\gamma \in \pi_1(M, x_0)$ and $\omega \in \Omega^1(M, \mathcal{E})$ closed we define

$$\int_\gamma \omega = \int_0^1 \tau^{-1}_{\gamma(t)} \omega(\dot{\gamma}(t)) dt \in \mathcal{E}_{x_0},$$

where $\tau_{\gamma(t)}$ denotes the parallel transport along γ from $x_0 = \gamma(0)$ to $\gamma(t)$. Since ω is closed, the integral depends only on the homotopy class of γ. If we identify $\mathcal{E}_{x_0}$ with $\mathfrak{g}$, then we may set

$$z_\omega(\gamma) = \int_\gamma \omega \in \mathfrak{g} .$$

Alternatively, we may proceed as follows: The flat bundle $\mathcal{E}$ may be described as an associated bundle $\mathcal{E} = \widetilde{M} \times_{Ad \circ \mathrm{hol}} \mathfrak{g}$. A function $f : \widetilde{M} \to \mathfrak{g}$ descends to a section $\sigma \in \Gamma(M, \mathcal{E})$ if and only if

$$f(\gamma p) = Ad \circ \mathrm{hol}(\gamma) \cdot f(p)$$

for all $p \in \widetilde{M}, \gamma \in \pi_1(M, x_0)$. Similarly, a 1-form $\widetilde{\omega} \in \Omega^1(\widetilde{M}, \mathfrak{g})$ descends to a form $\omega \in \Omega^1(M, \mathcal{E})$ if and only if

$$\gamma^* \widetilde{\omega} = Ad \circ \mathrm{hol}(\gamma) \cdot \widetilde{\omega}$$

for all $\gamma \in \pi_1(M, x_0)$. For $\omega \in \Omega^1(M, \mathcal{E})$ closed consider $\pi^* \omega \in \Omega^1(\widetilde{M}, \pi^* \mathcal{E})$. Let $p_0 \in \pi^{-1}(x_0)$ be a base point in $\widetilde{M}$. After identifying $\mathcal{E}_{x_0}$ with $\mathfrak{g}$ this yields a form $\widetilde{\omega} \in \Omega^1(\widetilde{M}, \mathfrak{g})$, which satisfies the above equivariance condition. Now since $\widetilde{M}$ is simply connected, there exists a primitive $F \in C^\infty(M, \mathfrak{g})$ such that $dF = \widetilde{\omega}$. For $\gamma \in \pi_1(M, x_0)$ we define

$$z_\omega(\gamma) = \int_\gamma \omega = F(\gamma p_0) - F(p_0) \in \mathfrak{g} .$$

Since F is determined up to an additive constant, this is well defined. Both definitions of the map $z_\omega : \pi_1 M \to \mathfrak{g}$ associated with the closed form $\omega \in \Omega^1(M, \mathcal{E})$ clearly agree.

Lemma 6.3 *If $\omega \in \Omega^1(M, \mathcal{E})$ is closed, then z_ω defines a group 1-cocycle, i.e. $z_\omega \in Z^1(\pi_1 M, \mathfrak{g})$. ω is exact if and only if z_ω is a coboundary, i.e. $z_\omega \in B^1(\pi_1 M, \mathfrak{g})$.*

Proof. If ω is closed, we have to check the cocycle condition

$$z_\omega(ab) = z_\omega(a) + Ad \circ \mathrm{hol}(a) \cdot z_\omega(b)$$

for $a, b \in \pi_1(M)$. Let F be a primitive for $\widetilde{\omega} = \pi^*\omega$. By definition of z_ω we have

$$\begin{aligned} z_\omega(ab) &= F(abp_0) - F(p_0) \\ &= F(abp_0) - F(ap_0) + F(ap_0) - F(p_0)\,. \end{aligned}$$

Since $\widetilde{\omega}$ is equivariant, for each $\gamma \in \pi_1 M$ we obtain

$$d\,(F \circ \gamma - Ad \circ \mathrm{hol}(\gamma) F) = 0\,,$$

i.e. F satisfies the equations

$$F \circ \gamma - Ad \circ \mathrm{hol}(\gamma) F = C_\gamma$$

for constants $C_\gamma \in \mathfrak{g}$. In particular

$$F(abp_0) - F(ap_0) = Ad \circ \mathrm{hol}(a)\,(F(bp_0) - F(p_0))\;,$$

which proves the first claim.
For the second claim we have to check that ω is exact if and only if there exist $v \in \mathfrak{g}$ such that

$$z_\omega(\gamma) = Ad \circ \mathrm{hol}(\gamma) v - v \qquad (*)$$

for all $\gamma \in \pi_1 M$. We first observe that $\omega \in \Omega^1(M, \mathcal{E})$ is exact if and only if there exists $C \in \mathfrak{g}$ such that $f = F + C$ is equivariant. Now f equivariant implies that

$$\int_\gamma \omega = Ad \circ \mathrm{hol}(\gamma) f(p_0) - f(p_0)$$

for all $\gamma \in \pi_1 M$, such that with $v = f(p_0)$ we arrive at relation $(*)$. Conversely, given $(*)$ for some $v \in \mathfrak{g}$, by choosing $C \in \mathfrak{g}$ appropriately we may achieve that $f = F + C$ satisfies $f(p_0) = v$. Then again by the equivariance of $\widetilde{\omega}$ we obtain

$$\int_\gamma \omega = Ad \circ \mathrm{hol}(\gamma) v + C_\gamma - v$$

for some $C_\gamma \in \mathfrak{g}$. Comparing this with $(*)$ now implies that $C_\gamma = 0$ for all $\gamma \in \pi_1 M$, which means that f is equivariant. ⊡

As a consequence of the preceding lemma, we obtain that the period map

$$\begin{aligned} P : H^1(M, \mathcal{E}) &\longrightarrow H^1(\pi_1 M, \mathfrak{g}) \\ [\omega] &\longmapsto [\gamma \mapsto \textstyle\int_\gamma \omega] \end{aligned}$$

is well defined and injective. Since we know from general considerations (cf. [Bro, Thm. 5.2, see also p. 59]), that $H^1(M, \mathcal{E}) \cong H^1(\pi_1 M, \mathfrak{g})$, we find that the period map provides an explicit isomorphism between $H^1(M, \mathcal{E})$ and $H^1(\pi_1 M, \mathfrak{g})$.

6.4 Isometries

6.4.1 Isometries of $\mathbf{H}^3$

The action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbf{H}^3$ by Poincaré extension identifies $\mathrm{SL}_2(\mathbb{C})$ with the universal covering group of $\mathrm{Isom}^+ \mathbf{H}^3 = \mathrm{PSL}_2(\mathbb{C})$. Here we use the upper half space model. Let $\phi : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Isom}^+ \mathbf{H}^3$ be the covering projection.
Semisimple elements in $\mathrm{SL}_2(\mathbb{C})$ project to semisimple isometries. A semisimple isometry ϕ has an invariant axis, this is the geodesic where δ_ϕ, the displacement of ϕ, assumes its minimum. If this minimum is positive, we call ϕ hyperbolic, otherwise elliptic. Parabolic elements in $\mathrm{SL}_2(\mathbb{C})$ project to parabolic isometries. Parabolic isometries have a unique fixed point at infinity.

Lemma 6.4 *$A, B \in \mathrm{SL}_2(\mathbb{C})$ commute if and only if $\phi(A), \phi(B)$ are either semisimple isometries and preserve the same axis γ or $\phi(A), \phi(B)$ are parabolic isometries with the same fixed point at infinity.*

The stabilizer of an oriented geodesic γ is isomorphic to $\mathbb{C}^*$, more precisely, if we work in the upper half space model $\mathbf{H}^3 = \mathbb{C} \times \mathbb{R}_+$, then for $\gamma = \{0\} \times \mathbb{R}_+$ we obtain

$$\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{C})}(\gamma) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}^* \right\} .$$

$S^1 \subset \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{C})}(\gamma)$ corresponds to pure rotations around γ, while $\mathbb{R} \subset \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{C})}(\gamma)$ corresponds to pure translations along γ. In particular, if we choose cylindrical coordinates (r, θ, z) around γ, we see that

$$\sigma_{\partial/\partial\theta} = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

and

$$\sigma_{\partial/\partial z} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) .$$

The factor $1/2$ comes from the fact that $\mathrm{SL}_2(\mathbb{C})$ is a twofold cover of $\mathrm{Isom}^+ \mathbf{H}^3$.

Let $\phi = \phi(A) \in \mathrm{Isom}^+ \mathbf{H}^3$ be semisimple. Then A is conjugate to $\mathrm{diag}(\lambda, \lambda^{-1})$ in $\mathrm{SL}_2(\mathbb{C})$ for $\lambda \in \mathbb{C}^*$. Let $z \in \mathbb{C}/2\pi i\mathbb{Z}$ such that $\lambda = \exp(z)$. We define

$$\mathcal{L} = 2z \in \mathbb{C}/2\pi i\mathbb{Z} .$$

Then $\mathcal{L}$ is determined by A (resp. by the set of eigenvalues $\{\lambda, \lambda^{-1}\}$) up to sign. $\mathcal{L}$ is called the *complex length* of A. If we orient γ, the axis of ϕ, the sign ambiguity of $\mathcal{L}$ can be removed consistently for all elements in a neighbourhood of A. The real part of $\mathcal{L}$ equals the (signed) translation length of ϕ along γ, while the imaginary part equals the angle of rotation around γ. We obtain

$$\mathrm{tr}\, A = \exp(z) + \exp(-z) = \cosh(z) .$$

Since the map $z \mapsto \mathcal{L}$ is a twofold covering, we obtain the following lemma:

Lemma 6.5 *The implicitly defined function* $\mathrm{tr} \mapsto \mathcal{L}$ *is locally biholomorphic if* $\mathrm{tr} \neq \pm 2$.

Let us denote by $Z(\rho(\Gamma))$ the centralizer of the representation ρ. Recall that a representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is called *irreducible* if there is no invariant line in $\mathbb{C}^2$ for the action of Γ via ρ.

Lemma 6.6 *Let Γ be a discrete group. For a representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ the following statements are equivalent:*

1. *ρ is irreducible.*
2. *$\rho(\Gamma)$ has no fixed point at infinity.*

If ρ is irreducible, then $Z(\rho(\Gamma)) = \{\pm 1\}$.

Proof. ρ reducible means by definition that $\rho(\Gamma)$ is conjugate to a group of upper triangular matrices. This is equivalent with $\rho(\Gamma)$ fixing $\infty \in \mathbb{C} \cup \{\infty\}$ if we look at the upper half-space model of hyperbolic space.
Let us now assume that there is a non-trivial element $A \in Z(\rho(\Gamma))$. If A is semisimple, then the whole group $\rho(\Gamma)$ is conjugate to a group of diagonal matrices. Similarly, if A is parabolic, $\rho(\Gamma)$ is conjugate to a group of upper triangular matrices. In both cases ρ is reducible. ◻

6.4.2 Isometries of $\mathbf{S}^3$

We identify $\mathbf{S}^3$ with the unit quaternions, i.e. $\mathbf{S}^3 = \{x \in \mathbb{H} : |x| = 1\}$. If we view the quaternions as a subalgebra of $\mathbb{C}^{2\times 2}$ via

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \, i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \, j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \, k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$\mathbf{S}^3$ gets identified with the group $\mathrm{SU}(2)$ via

$$\begin{aligned} \mathbf{S}^3 &\longrightarrow \mathrm{SU}(2) \\ a + bj &\longmapsto \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, \end{aligned}$$

where $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. The map

$$\begin{aligned} \phi : \mathrm{SU}(2) \times \mathrm{SU}(2) &\longrightarrow \mathrm{SO}(4) \\ (A, B) &\longmapsto (x \mapsto AxB^{-1}) \end{aligned}$$

exhibits $\mathrm{SU}(2) \times \mathrm{SU}(2)$ as the universal covering group of $\mathrm{Isom}^+ \mathbf{S}^3 = \mathrm{SO}(4)$.
Note that the diagonal matrices

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} : \lambda \in S^1 \right\} \subset \mathrm{SU}(2)$$

correspond to the geodesic $\gamma = \mathbb{C} \cap \mathbf{S}^3$. Here $\mathbb{C}$ is identified with $\mathbb{R} \oplus \mathbb{R}i \subset \mathbb{H}$. For any geodesic $\gamma \subset \mathbf{S}^3$ let us denote by $\gamma^\perp$ the geodesic which lies in the plane orthogonal to γ. In the above case $\gamma^\perp = \mathbb{C}j \cap \mathbf{S}^3 = (\mathbb{R}j \oplus \mathbb{R}k) \cap \mathbf{S}^3$, which corresponds to the set of matrices

$$\left\{ \begin{pmatrix} 0 & \lambda \\ -\overline{\lambda} & 0 \end{pmatrix} : \lambda \in S^1 \right\} \subset \mathrm{SU}(2).$$

A spherical isometry may be put in standard form, namely if $\phi = \phi(A, B)$ with $A, B \in \mathrm{SU}(2)$, then by conjugation we may achieve that $A = \mathrm{diag}(\lambda, \overline{\lambda})$ and $B = \mathrm{diag}(\mu, \overline{\mu})$ with $\lambda, \mu \in S^1$. The matrix A corresponds to $\lambda \in \mathbb{C} \cap \mathbf{S}^3$ and B to $\mu \in \mathbb{C} \cap \mathbf{S}^3$ if we identify $\mathrm{SU}(2)$ with $\mathbf{S}^3$ as above. Then for $x \in \mathbf{S}^3$ we have

$$\phi(x) = \lambda x \overline{\mu},$$

such that ϕ preserves the Hopf-fibrations, which are associated with the complex structures $x \mapsto ix$ and $x \mapsto xi$ on $\mathbb{H}$. In particular, ϕ preserves $\gamma = \mathbb{C} \cap \mathbf{S}^3$ and $\gamma^\perp = \mathbb{C}j \cap \mathbf{S}^3$, more precisely we have

$$\phi(\eta) = \lambda\overline{\mu}\eta$$

for $\eta \in S^1 = \mathbb{C} \cap \mathbf{S}^3$, and

$$\phi(\eta j) = (\lambda\mu\eta)j$$

for $\eta j \in \mathbb{C}j \cap \mathbf{S}^3$. Note that γ and $\gamma^\perp$ are the common leaves of the two foliations, which are transverse everywhere else.
If $\mu = 1$, then ϕ translates along the fibres of the Hopf-fibration obtained by left-multiplication with S^1, in particular the displacement of ϕ is constant on $\mathbf{S}^3$. Similarly, if $\lambda = 1$, then ϕ translates along the fibres of the Hopf-fibration by right-multiplication with S^1. Again the displacement of ϕ will be constant on $\mathbf{S}^3$.
If $\lambda = \mu$, then ϕ is a pure rotation around γ, or equivalently, a pure translation along $\gamma^\perp$. Similarly, if $\lambda = \overline{\mu}$, then ϕ is a pure rotation around $\gamma^\perp$, or equivalently, a pure translation along γ.
In particular, if we choose cylindrical coordinates (r, θ, z) around γ, we see that

$$\sigma_{\partial/\partial\theta} = \left(\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

and

$$\sigma_{\partial/\partial z} = \left(\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)\,.$$

The factors $1/2$ arise from the fact that $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is a twofold cover of $\mathrm{Isom}^+ \mathbf{S}^3$.

Lemma 6.7 *$\phi_1, \phi_2 \in \mathrm{Isom}^+ \mathbf{S}^3$ commute if and only they preserve the same pair of orthogonal axes $\{\gamma, \gamma^\perp\}$.*

By the term *axis* we mean the same thing as *oriented geodesic*.

Proof. Let $\phi_1 = \phi(A_1, B_1)$ and $\phi_2 = \phi(A_2, B_2)$ with $A_i, B_i \in \mathrm{SU}(2)$. If ϕ_1 and ϕ_2 commute, we may assume that A_1, B_1, A_2 and B_2 are simultaneously diagonal. Then it is easy to check that ϕ_1 and ϕ_2 preserve the axes $\gamma = \mathbb{C} \cap \mathbf{S}^3$ and $\gamma^\perp = \mathbb{C}j \cap \mathbf{S}^3$.
On the other hand, if ϕ_1 and ϕ_2 preserve a pair of orthogonal axes, we may assume that $\gamma = \mathbb{C} \cap \mathbf{S}^3$ and $\gamma^\perp = \mathbb{C}j \cap \mathbf{S}^3$. Again it is easy to check that then A_1, B_1, A_2 and B_2 have to be diagonal, hence ϕ_1 and ϕ_2 commute. ▫

Lemma 6.8 *A spherical isometry $\phi = \phi(A, B)$ with $A, B \in \mathrm{SU}(2)$ has a fixed point if and only if A is conjugate to B within $\mathrm{SU}(2)$.*

Proof. Without loss of generality we may assume that 1 is a fixed point of ϕ. This implies immediately that $A = B$. ▫

We want to define a complex length in the spherical case, too. If $\phi = \phi(A, B)$ with A conjugate to $\mathrm{diag}(\lambda, \overline{\lambda})$ and B conjugate to $\mathrm{diag}(\mu, \overline{\mu})$, then let $x \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\lambda = \exp(ix)$$

and $y \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\mu = \exp(iy)\,.$$

We define $\mathcal{L}_1 = x - y$ and $\mathcal{L}_2 = x + y$. Then $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) \in \mathbb{R}^2/2\pi\mathbb{Z}^2$ is determined by A and B up to sign and up to switching components. If ϕ preserves a pair of orthogonal axes $\{\gamma, \gamma^\perp\}$, these ambiguities can be removed in a neighbourhood of (A, B) by orienting γ and $\gamma^\perp$. Let us again call $\mathcal{L}$ the "complex" length of $(A, B) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$. $\mathcal{L}_1$ equals the (signed) translation length along γ, while $\mathcal{L}_2$ equals the (signed) translation length along $\gamma^\perp$. We obtain

$$\operatorname{tr} A = \exp(ix) + \exp(-ix) = 2\cos x$$

and

$$\operatorname{tr} B = \exp(iy) + \exp(-iy) = 2\cos y\,.$$

We set $\mathrm{Tr}_1(A, B) = \operatorname{tr} A$ and $\mathrm{Tr}_2(A, B) = \operatorname{tr} B$. Since the map $(x, y) \mapsto \mathcal{L}$ is a twofold covering, we obtain the following lemma:

Lemma 6.9 *The implicitly defined function* $\mathrm{Tr} = (\mathrm{Tr}_1, \mathrm{Tr}_2) \mapsto \mathcal{L}$ *is a local diffeomorphism as long as* $\mathrm{Tr}_1 \neq \pm 2$ *and* $\mathrm{Tr}_2 \neq \pm 2$.

We finish this section with the following lemma about SU(2)-representations:

Lemma 6.10 *Let* Γ *be a discrete group. For a representation* $\rho : \Gamma \to \mathrm{SU}(2)$ *the following statements are equivalent:*

1. ρ *is irreducible.*
2. $\rho(\Gamma)$ *is non-abelian.*
3. $Z(\rho(\Gamma)) = \{\pm 1\}$.

Proof. ρ reducible implies, since SU(2) is a compact group, that $\rho(\Gamma)$ is conjugate to a group of diagonal matrices, hence that $\rho(\Gamma)$ is abelian. Conversely, an abelian subgroup of SU(2) is clearly conjugate to a group of diagonal matrices.
In particular, the centralizer of ρ will be non-trivial for a reducible representation. Conversely, if $A \in Z(\rho(\Gamma))$ is non-trivial, $\rho(\Gamma)$ will be conjugate to a group of diagonal matrices, hence ρ will be reducible. ◻

6.4.3 Isometries of $\mathbf{E}^3$

Here we concentrate on the discussion of the rotational parts of isometries of $\mathbf{E}^3$. The map

$$\begin{aligned} \mathrm{SU}(2) \times \operatorname{Im}\mathbb{H} &\longrightarrow \operatorname{Im}\mathbb{H} \\ (A, x) &\longrightarrow AxA^{-1} \end{aligned}$$

exhibits SU(2) as the universal covering group of SO(3) if we identify $\operatorname{Im}\mathbb{H}$ with $\mathbb{R}^3$. It is at the same time the adjoint representation of SU(2) if we identify $\operatorname{Im}\mathbb{H}$ with $\mathfrak{su}(2)$. Let $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ be the covering projection.

Lemma 6.11 $A, B \in \mathrm{SU}(2)$ *commute if and only if* $\phi(A), \phi(B) \in \mathrm{SO}(3)$ *have the same rotation axis.*

Proof. Let $x \in \operatorname{Im}\mathbb{H} \cap \mathbf{S}^3$. Then $\phi(A)x = x$ if and only if $Ax = xA$. From this the statement is clear. ◻

Let $\gamma = \mathbb{R}i \subset \operatorname{Im}\mathbb{H}^3$. Then

$$\mathrm{Stab}_{\mathrm{SU}(2)}(\gamma) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} : \lambda \in S^1 \right\}.$$

In particular, if we choose cylindrical coordinates (r,θ,z) around γ, we see that

$$\sigma_{\partial/\partial z} = \sigma_{\partial/\partial\theta} = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2)\,.$$

Here we identify $\mathbb{R}^3$ and $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ as SU(2)-modules.

6.5 Cohomology computations

Let C be cone-manifold with singular locus Σ. A connected component of Σ is either a circle or a trivalent graph. Let $M_\varepsilon = M \setminus B_\varepsilon(\Sigma)$, where $B_\varepsilon(\Sigma)$ is the open ε-tube around Σ. Let $U_\varepsilon(\Sigma) = B_\varepsilon(\Sigma)\setminus\Sigma$. Then M_ε is topologically a manifold with boundary, which is a deformation retract of M. ∂M_ε consists of tori and surfaces of higher genus. $\partial M_\varepsilon = \partial U_\varepsilon(\Sigma)$ is a deformation retract of $U_\varepsilon(\Sigma)$.
Without loss of generality we may assume in the following that Σ is connected.

6.5.1 The torus case

Let $\Sigma = S^1$. Then $U_\varepsilon(\Sigma)$ is given as $(0,\varepsilon)\times T^2$, where $T^2 = \mathbb{R}/\Lambda$ and Λ is the lattice generated by $(\theta,z)\mapsto(\theta+\alpha,z)$ and $(\theta,z)\mapsto(\theta-t,z+l)$. The metric is given as $g = dr^2 + \mathrm{sn}_\kappa^2(r)d\theta^2 + \mathrm{cs}_\kappa^2(r)dz^2$. Here α, t and l are the parameters, which determine the geometry of $U_\varepsilon(\Sigma)$, namely the cone-angle, the twist and the length of the singular tube. Note that a function f in the coordinates (r,θ,z) descends to a function on $U_\varepsilon(\Sigma)$ if and only if $f(r,\theta,z) = f(r,\theta+\alpha,z)$ and $f(r,\theta,z+l) = f(r,\theta+t,z)$. Note also that $H^i(U_\varepsilon(\Sigma),\cdot) = H^i(T^2,\cdot)$ for any local coefficient system.
The forms $d\theta$ and dz are invariant under Λ and descend to forms on T^2, which generate the de-Rham cohomology of the torus, i.e. $H^1(T^2,\mathbb{R}) = \mathbb{R}\cdot[d\theta]\oplus\mathbb{R}\cdot[dz]$. Similarly, $\partial/\partial\theta$ and $\partial/\partial z$ descend to Killing-vectorfields on $U_\varepsilon(\Sigma)$. To be more specific, $\partial/\partial\theta$ is an infinitesimal rotation around the singular axis and $\partial/\partial z$ an infinitesimal translation along the same axis. Consequently, $\sigma_{\partial/\partial\theta}$ and $\sigma_{\partial/\partial z}$ make up parallel sections of the bundle $\mathcal{E}$, i.e. $\sigma_{\partial/\partial z}, \sigma_{\partial/\partial z} \in H^0(T^2,\mathcal{E})$. In the Euclidean case, $\sigma_{\partial/\partial z}$ is a parallel section of $\mathcal{E}_{trans}$, i.e. $\sigma_{\partial/\partial z} \in H^0(T^2,\mathcal{E}_{trans})$. Recall that in the Euclidean case the infinitesimal translations form a parallel subbundle $\mathcal{E}_{trans}\subset\mathcal{E}$, which is isomorphic to the flat tangent-bundle.

Lemma 6.12 *If the cone-angle is not an integer multiple of 2π, then in all cases we have*

$$H^0(T^2,\mathcal{E}) = \mathbb{R}\cdot\sigma_{\partial/\partial\theta}\oplus\mathbb{R}\cdot\sigma_{\partial/\partial z},$$

and in the Euclidean case

$$H^0(T^2,\mathcal{E}_{trans}) = \mathbb{R}\cdot\sigma_{\partial/\partial z}.$$

Proof. Let λ be the longitudinal and μ be the meridian loop. Then $\pi_1 T^2 = \mathbb{Z}\lambda\oplus\mathbb{Z}\mu$. Clearly $H^0(T^2,\mathcal{E}) \cong Z^0(\pi_1 T^2;\mathfrak{g}) = \{v\in\mathfrak{g}\,|\,Ad\circ\mathrm{hol}(\gamma)v = v\ \forall\gamma\in\pi_1 T^2\}$, which we view as the infinitesimal centralizer of the representation $\mathrm{hol} : \pi_1 T^2 \to \widetilde{G}$, where $\widetilde{G}$ is the universal covering group of $G = \mathrm{Isom}^+\mathbf{M}^3_\kappa$. We compute the centralizer $Z(\mathrm{hol}(\pi_1 T^2))\subset\widetilde{G}$ in each case.
In the hyperbolic case, let $A = \mathrm{hol}(\lambda)\in\mathrm{SL}_2(\mathbb{C})$ and $B = \mathrm{hol}(\mu)\in\mathrm{SL}_2(\mathbb{C})$. Since hol is the holonomy of a hyperbolic cone-structure with cone-angle not a multiple of 2π, we may assume that $A = \mathrm{diag}(\eta,\eta^{-1})$ and $B = \mathrm{diag}(\xi,\xi^{-1})$ with $\eta,\xi\neq\pm1$. Then it is easy to see that $Z(\mathrm{hol}(\pi_1 T^2)) = \{\mathrm{diag}(\zeta,\zeta^{-1}),\zeta\in\mathbb{C}^*\}$. This clearly implies that $Z^0(\pi_1 T^2,\mathfrak{sl}_2(\mathbb{C}))\cong\mathbb{R}^2$. Since $\sigma_{\partial/\partial\theta}$ and $\sigma_{\partial/\partial z}$ are closed and linearly independent, the result follows.
In the spherical case, $\mathrm{hol} : \pi_1 T^2 \to \mathrm{SU}(2)\times\mathrm{SU}(2)$ splits as a product representation $\mathrm{hol} = (\mathrm{hol}_1,\mathrm{hol}_2)$ with $\mathrm{hol}_i : \pi_1 T^2 \to \mathrm{SU}(2)$ for $i\in\{1,2\}$. We then clearly have

$Z(\mathrm{hol}(\pi_1 T^2)) = Z(\mathrm{hol}_1(\pi_1 T^2)) \times Z(\mathrm{hol}_2(\pi_1 T^2))$. Let $A_i = \mathrm{hol}_i(\lambda) \in \mathrm{SU}(2)$ and $B_i = \mathrm{hol}_i(\mu) \in \mathrm{SU}(2)$. Without loss of generality we assume that $A_i = \mathrm{diag}(\eta_i, \overline{\eta}_i)$ and $B_i = \mathrm{diag}(\xi_i, \overline{\xi}_i)$ with $\eta_i, \xi_i \in S^1$. Since hol is the holonomy of a spherical cone structure with cone-angle not a multiple of 2π, $\mathrm{hol}(\mu)$ must be a nontrivial rotation. This implies that $\{\xi_1, \overline{\xi}_1\} = \{\xi_2, \overline{\xi}_2\} \neq \{\pm 1\}$. Then it follows that $Z(\mathrm{hol}_i(\pi_1 T^2)) = \{\mathrm{diag}(\zeta, \overline{\zeta}), \zeta \in S^1\}$ and $Z^0(\pi_1 T^2, \mathfrak{su}(2)) \cong \mathbb{R}$. As above $\sigma_{\partial/\partial\theta}$ and $\sigma_{\partial/\partial z}$ provide a basis for $H^0(T^2, \mathcal{E})$.
In the Euclidean case, we consider hol : $\pi_1 T^2 \to \mathrm{SU}(2) \ltimes \mathbb{R}^3$ and the rotational part rot $\circ$ hol : $\pi_1 T^2 \to \mathrm{SU}(2)$, where we continue to denote by rot the lift of the rotational part rot : $\mathrm{Isom}^+ \mathbf{E}^3 \to \mathrm{SO}(3)$ to the universal covering groups. Let $(A, a) = \mathrm{hol}(\lambda)$ and $(B, b) = \mathrm{hol}(\mu)$. We may assume without loss of generality that $(A, a) = (\mathrm{diag}(\eta, \overline{\eta}), a)$ and $(B, b) = (\mathrm{diag}(\xi, \overline{\xi}), 0)$. Since hol is the holonomy of a Euclidean cone-structure with cone-angle not a multiple of 2π, $\mathrm{hol}(\mu)$ must be a nontrivial rotation. Therefore we obtain that $Z(\mathrm{hol}(\pi_1 T^2)) = \{(\mathrm{diag}(\zeta, \overline{\zeta}), c), \zeta \in S^1, c \text{ parallel to the axis of } \phi(B)\}$. Here ϕ denotes the projection $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Similarly $Z(\mathrm{rot} \circ \mathrm{hol}(\pi_1 T^2)) = \{(\mathrm{diag}(\zeta, \overline{\zeta}), c), \zeta \in S^1\}$. Then $Z^0(\pi_1 T^2, \mathfrak{isom}^+ \mathbf{E}^3) \cong \mathbb{R}^2$ and $Z^0(\pi_1 T^2, \mathfrak{su}(2)) \cong \mathbb{R}$. Note finally that $\sigma_{\partial/\partial z}$ is an infinitesimal translation in the Euclidean case. ⊡

We define forms

$$\begin{aligned} \omega_{ang} &= d\theta \otimes \sigma_{\partial/\partial\theta} \\ \omega_{shr} &= d\theta \otimes \sigma_{\partial/\partial z} \\ \omega_{tws} &= dz \otimes \sigma_{\partial/\partial\theta} \\ \omega_{len} &= dz \otimes \sigma_{\partial/\partial z}. \end{aligned}$$

Since $\sigma_{\partial/\partial\theta}$ and $\sigma_{\partial/\partial z}$ are parallel, these forms are closed. These forms will be tangent to the corresponding geometric deformations of the singular tube, i.e. ω_{ang} is supposed to change the cone-angle α, similarly for t and l. ω_{shr} will be tangent to a deformation, which leads out of the class of cone-metrics (which may be called a shearing-deformation). This will be made precise.

Lemma 6.13 *The forms ω_{ang} and ω_{shr} are not L^2 on $U_\varepsilon(\Sigma)$, whereas the forms ω_{tws} and ω_{len} are bounded on $U_\varepsilon(\Sigma)$ and hence L^2.*

Proof. We recall that the metric on $U_\varepsilon(\Sigma)$ is given by $g = dr^2 + \mathrm{sn}_\kappa^2(r) d\theta^2 + \mathrm{cs}_\kappa^2(r) dz^2$. Hence $dvol = \mathrm{sn}_\kappa(r) cs(r) dr \wedge d\theta \wedge dz$. For $\omega = \alpha \otimes \sigma_X$ with $\alpha \in \Omega^1(U_\varepsilon(\Sigma))$ and $X \in \Gamma(TU_\varepsilon(\Sigma))$ we have $|\omega|^2 = |\alpha|^2 \left(|\nabla X|^2 + |X|^2\right)$.
Clearly

$$|d\theta|^2 = \frac{1}{\mathrm{sn}_\kappa^2(r)},\ |dz|^2 = \frac{1}{\mathrm{cs}_\kappa^2(r)},\ \left|\tfrac{\partial}{\partial\theta}\right|^2 = \mathrm{sn}_\kappa^2(r),\ \left|\tfrac{\partial}{\partial z}\right|^2 = \mathrm{cs}_\kappa^2(r).$$

Let

$$\left\{e_1 = \tfrac{\partial}{\partial r},\ e_2 = \mathrm{sn}_\kappa(r)^{-1} \tfrac{\partial}{\partial\theta},\ e_3 = \mathrm{cs}_\kappa(r)^{-1} \tfrac{\partial}{\partial z}\right\}$$

be an orthonormal frame for $TU_\varepsilon(\Sigma)$. A straightforward calculation shows that with respect to this frame

$$\nabla \tfrac{\partial}{\partial\theta} = \begin{pmatrix} 0 & -\mathrm{cs}_\kappa(r) & 0 \\ \mathrm{cs}_\kappa(r) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \Gamma(\mathfrak{so}(TU_\varepsilon(\Sigma))$$

and

$$\nabla \tfrac{\partial}{\partial z} = \begin{pmatrix} 0 & 0 & \kappa\,\mathrm{sn}_\kappa(r) \\ 0 & 0 & 0 \\ -\kappa\,\mathrm{sn}_\kappa(r) & 0 & 0 \end{pmatrix} \in \Gamma(\mathfrak{so}(TU_\varepsilon(\Sigma)),$$

such that

$$\left|\nabla \frac{\partial}{\partial \theta}\right|^2 = \mathrm{cs}_\kappa^2(r),\ \left|\nabla \frac{\partial}{\partial z}\right|^2 = \kappa^2 \,\mathrm{sn}_\kappa^2(r).$$

We obtain

$$|\omega_{ang}|^2 = \frac{\mathrm{sn}_\kappa^2(r) + cs^2(r)}{\mathrm{sn}_\kappa^2(r)},\ |\omega_{shr}|^2 = \frac{\mathrm{cs}_\kappa^2(r) + \kappa^2 \,\mathrm{sn}_\kappa^2(r)}{\mathrm{sn}_\kappa^2(r)}$$

and

$$|\omega_{tws}|^2 = \frac{\mathrm{sn}_\kappa^2(r) + cs^2(r)}{\mathrm{cs}_\kappa^2(r)},\ |\omega_{len}|^2 = \frac{\mathrm{cs}_\kappa^2(r) + \kappa^2 \,\mathrm{sn}_\kappa^2(r)}{\mathrm{cs}_\kappa^2(r)}.$$

In the first case we observe that $|\omega_{ang}|^2 dvol \sim |\omega_{shr}|^2 dvol \sim \mathrm{sn}_\kappa(r)^{-1}$, which is not integrable for $r \in (0, \varepsilon)$. In the second case we find ω_{tws} and ω_{len} bounded and therefore L^2-integrable. ⊡

Lemma 6.14 *If the cone-angle not an integer multiple of 2π, then we have in the hyperbolic and the spherical case*

$$H^1(T^2, \mathcal{E}) = \mathbb{R} \cdot [\omega_{ang}] \oplus \mathbb{R} \cdot [\omega_{shr}] \oplus \mathbb{R} \cdot [\omega_{tws}] \oplus \mathbb{R} \cdot [\omega_{len}],$$

while in the Euclidean case

$$H^1(T^2, \mathcal{E}_{trans}) = \mathbb{R} \cdot [\omega_{shr}] \oplus \mathbb{R} \cdot [\omega_{len}].$$

Proof. Since $H^0(T^2, \mathcal{E}) = \mathbb{R} \cdot \sigma_{\partial/\partial\theta} \oplus \mathbb{R} \cdot \sigma_{\partial/\partial z}$, we obtain a short exact sequence of flat bundles

$$0 \to \underline{\mathbb{R}}^2 \to \mathcal{E} \to \mathcal{E}/\underline{\mathbb{R}}^2 \to 0,$$

and in the Euclidean case, since $H^0(T^2, \mathcal{E}_{trans}) = \mathbb{R} \cdot \sigma_{\partial/\partial z}$,

$$0 \to \underline{\mathbb{R}} \to \mathcal{E}_{trans} \to \mathcal{E}_{trans}/\underline{\mathbb{R}} \to 0.$$

Here we denote by $\underline{\mathbb{R}}^2$ (resp. $\underline{\mathbb{R}}$) the trivial flat bundle of $\mathbb{R}$-rank 2 (resp. 1). We claim that the natural maps

$$H^1(T^2, \mathbb{R}^2) \to H^1(T^2, \mathcal{E})$$

in the hyperbolic and the spherical case, and

$$H^1(T^2, \mathbb{R}) \to H^1(T^2, \mathcal{E}_{trans})$$

in the Euclidean case, are isomorphisms. In the spherical and the Euclidean case we use the parallel metric on $\mathcal{E}$ (resp. $\mathcal{E}_{trans}$) to embed the quotient bundle into $\mathcal{E}$ (resp. $\mathcal{E}_{trans}$) transversally to $\underline{\mathbb{R}}^2$ (resp. $\underline{\mathbb{R}}$). Then clearly $H^0(\mathcal{E}/\underline{\mathbb{R}}^2) = H^0(\mathcal{E}_{trans}/\underline{\mathbb{R}}) = 0$ and we may use Poincaré duality to conclude that $\mathcal{E}/\underline{\mathbb{R}}^2$ (resp. $\mathcal{E}_{trans}/\underline{\mathbb{R}}$) is acyclic. Then the result follows from the long exact cohomology sequence associated with the respective coefficient sequence.
In the hyperbolic case we use the nondegeneracy of the parallel Killing form B to embed $\mathcal{E}/\underline{\mathbb{R}}^2$ into $\mathcal{E}$. Since B is indefinite, to ensure transversality we have to check that B restricted to $\underline{\mathbb{R}}^2$ is nondegenerate. We use the local formula for the Killing form in Lemma **(5.5)** with $\kappa = -1$:

$$B(\sigma_X, \sigma_Y) = -4(\nabla X, \nabla Y) + 4(X, Y).$$

From the calculations in the previous lemma we obtain

$$B(\sigma_{\partial/\partial\theta}, \sigma_{\partial/\partial\theta}) = -4\cosh^2(r) + 4\sinh^2(r) = -4$$
$$B(\sigma_{\partial/\partial z}, \sigma_{\partial/\partial z}) = -4\sinh^2(r) + 4\cosh^2(r) = 4$$

$$B(\sigma_{\partial/\partial\theta}, \sigma_{\partial/\partial z}) = 0,$$

which shows that $B|_{\mathbb{R}^2}$ is nondegenerate. Then the result follows as above. ◻

We calculate the periods of the differential forms $\omega_{ang}, \omega_{shr}, \omega_{tws}$ and ω_{len}. Let $x_0 = (0,0)$ be the basepoint of T^2. For $\gamma \in \pi_1 T^2$ and $\omega \in \Omega^1(T^2, \mathcal{E})$ closed, we have a well-defined integral

$$\int_\gamma \omega = \int_0^1 \tau_{\gamma(t)}^{-1} \omega(\dot\gamma(t))dt,$$

where $\tau_{\gamma(t)}$ denotes the parallel transport along γ from $x_0 = \gamma(0)$ to $\gamma(t)$. Recall that the map $\gamma \mapsto z_\omega(\gamma) := \int_\gamma \omega$ defines a group 1-cocycle, i.e. $z_\omega \in Z^1(\pi_1 T^2, \mathfrak{g})$, if we identify $\mathcal{E}_{x_0}$ with $\mathfrak{g}$. The period map

$$\begin{aligned} P : H^1(T^2, \mathcal{E}) &\longrightarrow H^1(\pi_1 T^2, \mathfrak{g}) \\ [\omega] &\longmapsto [z_\omega] \end{aligned}$$

is an isomorphism. Note that if ω is of the form $\omega = \alpha \otimes \sigma$ with $\nabla\sigma = 0$, then $\int_\gamma \omega$ is very easy to compute:

$$\int_\gamma \omega = \int_\gamma \alpha \cdot \sigma_{x_0}.$$

This remark applies in particular to $\omega_{ang}, \omega_{shr}, \omega_{tws}$ and ω_{len}. Let $\mu \in \pi_1 T^2$ be the meridian and $\lambda \in \pi_1 T^2$ the longitude, $\mu(0) = \lambda(0) = x_0$. Then

$$\begin{aligned}
z_{ang}(\mu) &= \int_\mu \omega_{ang} = \alpha \cdot (\sigma_{\partial/\partial\theta})_{x_0}, & z_{ang}(\lambda) &= \int_\lambda \omega_{ang} = -t \cdot (\sigma_{\partial/\partial\theta})_{x_0} \\
z_{shr}(\mu) &= \int_\mu \omega_{shr} = \alpha \cdot (\sigma_{\partial/\partial z})_{x_0}, & z_{shr}(\lambda) &= \int_\lambda \omega_{shr} = -t \cdot (\sigma_{\partial/\partial z})_{x_0} \\
z_{tws}(\mu) &= \int_\mu \omega_{tws} = 0, & z_{tws}(\lambda) &= \int_\lambda \omega_{tws} = l \cdot (\sigma_{\partial/\partial\theta})_{x_0} \\
z_{len}(\mu) &= \int_\mu \omega_{len} = 0, & z_{len}(\lambda) &= \int_\lambda \omega_{len} = l \cdot (\sigma_{\partial/\partial z})_{x_0}.
\end{aligned}$$

6.5.2 The higher genus case

Let Σ be a connected graph with trivalent vertices. Then $F_g = \partial U_\varepsilon(\Sigma)$ is a surface of genus $g = (N+3)/3$, where N is the number of edges of Σ. The smooth part of the ε-ball $U_\varepsilon(v)$ of a vertex $v \in \Sigma$ is homotopy equivalent to a pair of pants P.

Lemma 6.15 *If the cone-angles are $\leq \pi$, then we have $H^0(F_g, \mathcal{E}) = 0$ in all cases, in particular in the Euclidean case $H^0(F_g, \mathcal{E}_{trans}) = 0$.*

Proof. If we restrict the holonomy of M to the smooth part of the ε-ball $U_\varepsilon(v)$ around a vertex $v \in \Sigma$, then $\mathrm{hol}(\pi_1(U_\varepsilon(v))$ fixes a point $p \in \mathbf{M}^3_\kappa$. $U_\varepsilon(v)$ deformation-retracts to $P \subset \partial U_\varepsilon(\Sigma) = F_g$. Using the presentation $\pi_1(P) = \langle \mu_1, \mu_2, \mu_3 | \mu_1\mu_2\mu_3 = 1\rangle$, we obtain that $A_i = \mathrm{hol}(\mu_i) \in \mathrm{SU}(2)$ project to rotations with mutually distinct axes. This implies that $Z(\mathrm{hol}(\pi_1 F_g)) = Z(\mathrm{hol}(\pi_1 P)) = \{\pm 1\}$, and therefore in particular $Z^0(\pi_1 F_g, \mathfrak{g}) = H^0(F_g, \mathcal{E}) = 0$. ◻

Corollary 6.16 *If the cone-angles are $\leq \pi$, then in the hyperbolic case we have $H^1(F_g, \mathcal{E}) \cong \mathbb{C}^{6g-6}$, in the spherical case $H^1(F_g, \mathcal{E}_i) \cong \mathbb{R}^{6g-6}$ for $i \in \{1,2\}$, and in the Euclidean case $H^1(F_g, \mathcal{E}_{trans}) \cong \mathbb{R}^{6g-6}$.*

Proof. Using the parallel Killing form B on $\mathcal{E}$ in the hyperbolic case, resp. the parallel metric on $\mathcal{E}_i$ in the spherical case and on $\mathcal{E}_{trans}$ in the Euclidean case we conclude with Poincaré duality that $H^2(F_g,\mathcal{E}) = H^2(F_g,\mathcal{E}_i) = H^2(F_g,\mathcal{E}_{trans}) = 0$. Now for any flat bundle $\mathcal{F}$ one has $\chi(F_g,\mathcal{F}) = \dim\mathcal{F}\cdot\chi(F_g) = \dim\mathcal{F}\cdot(2-2g)$. Therefore $\dim H^1(F_g,\mathcal{F}) = -\dim\mathcal{F}\cdot(2-2g)$ if $H^0(F_g,\mathcal{F}) = H^2(F_g,\mathcal{F}) = 0$, which yields the result. ▫

Away from the vertices, the singular locus $U_\varepsilon(\Sigma)$ can be given coordinates (r,θ_i,z_i) with $r\in(0,\varepsilon)$, $\theta_i\in\mathbb{R}/\alpha_i\mathbb{Z}$ and $z_i\in(0,l_i)$. Here α_i is the cone-angle around the i-th edge and l_i its length. Then the metric is given by $g = dr^2 + \mathrm{sn}_\kappa^2(r)d\theta_i + \mathrm{cs}_\kappa^2(r)dz_i$. We choose a function $\varphi_i = \varphi_i(z_i)$ such that $\varphi_i(0) = 0$, $\varphi_i(l_i) = l_i$ and $d\varphi_i|_{(0,\delta)} = d\varphi_i|_{(l_i-\delta,l_i)} = 0$ for $\delta > 0$. Then $d\varphi_i\in\Omega^1(U_\varepsilon(\Sigma))$ is well-defined and so are

$$\omega^i_{tws} = d\varphi_i\otimes\sigma_{\partial/\partial\theta_i}$$
$$\omega^i_{len} = d\varphi_i\otimes\sigma_{\partial/\partial z_i}\,.$$

Note that these forms are supported away from the vertices of the singularity.

Lemma 6.17 *The forms ω^i_{tws} and ω^i_{len} are bounded on $U_\varepsilon(\Sigma)$ and hence in particular L^2.*

Proof. This is essentially the same computation as in the torus case. ▫

Lemma 6.18 *The de-Rham cohomology classes of the differential forms*

$$\{\omega^1_{tws},\omega^1_{len},\ldots,\omega^N_{tws},\omega^N_{len}\}$$

are linearly independent in $H^1(F_g,\mathcal{E})$.

Proof. Suppose we have a nontrivial linear relation between the above classes in $H^1(F_g,\mathcal{E})$, say

$$t_1\omega^1_{tws} + l_1\omega^1_{len} + \ldots + t_N\omega^N_{tws} + l_N\omega^N_{len} = d\sigma$$

for some $\sigma\in\Gamma(F_g,\mathcal{E})$. Since the forms ω^i_{tws} and ω^i_{len} are supported away from the vertices, we obtain $d\sigma = 0$ in a neigbourhood of each vertex v_i. A neighbourhood $U_\varepsilon(v_i)$ of a vertex is homotopy equivalent to the thrice-punctured sphere P. Since $H^0(P,\mathcal{E}) = 0$, we have $\sigma|_{U_\varepsilon(v_i)} = 0$ for each vertex.
We obtain nontrivial linear relations on the tori $T^2_i = \mathbb{R}^2/\alpha_i\mathbb{Z} + l_i\mathbb{Z}$, where σ_i denotes the restriction of σ to a neighbourhood of the i-th edge:

$$t_1\omega^1_{tws} + l_1\omega^1_{len} = d\sigma_1$$
$$\vdots$$
$$t_N\omega^N_{tws} + l_N\omega^N_{len} = d\sigma_N\,,$$

which is a contradiction in view of Lemma **(6.14)**, since $d\varphi_i$ is cohomologous to dz_i on T^2_i. ▫

6.6 Local structure of the representation variety

6.6.1 The torus case

Let $\iota : T^2\to M$ be the inclusion of a torus boundary component. ι induces a group homomorphism $\iota_* : \pi_1T^2\to\pi_1M$ and a map $\iota^* : R(\pi_1M,G)\to R(\pi_1T^2,G)$ for $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SU}(2)$ respectively.

Lemma 6.19 *The restriction of the holonomy of a hyperbolic cone-structure to a torus boundary component, $\rho = \iota^*_{T^2}\,\mathrm{hol} : \pi_1 T^2 \to \mathrm{SL}_2(\mathbb{C})$, is a smooth point of $R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$. The $\mathbb{C}$-dimension of $R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ around ρ equals 4. $T_\rho R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ may be identified with $Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$.*

Proof. We identify $R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ with the (affine algebraic) set $\{(A,B) \in \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \,|\, [A,B] = 1\}$. The kernel of the differential of the commutator map $\ker d_{(A,B)}[\,\cdot\,,\cdot\,]$ may be identified with the space of 1-cocycles $Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$. We know that $\dim_{\mathbb{C}} Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C})) = 4$ from the cohomology computations. Note that this implies that $d_{(A,B)}[\,\cdot\,,\cdot\,]$ is *not* surjective at $(A,B) = \rho$. Without loss of generality we may assume that $\rho = (\mathrm{diag}(\lambda, \lambda^{-1}), \mathrm{diag}(\mu, \mu^{-1}))$ with $\lambda, \mu \in \mathbb{C}^*$. We define a map

$$\begin{aligned} F : \mathbb{C}^* \times \mathbb{C}^* \times \mathrm{SL}_2(\mathbb{C}) &\longrightarrow \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \\ (\lambda, \mu, A) &\longmapsto \big(A\,\mathrm{diag}(\lambda, \lambda^{-1})A^{-1}, A\,\mathrm{diag}(\mu, \mu^{-1})A^{-1}\big) \end{aligned}$$

We claim that $\mathrm{rank}_{\mathbb{C}} F = 4$ at $(\lambda, \mu, 1)$. The image of F is certainly contained in $R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$, such that an easy application of the implicit function theorem (cf. [Wei], [Rag, Lemma 6.8]) yields the result. Consider the standard $\mathbb{C}$-basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$\left\{ x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Clearly $\mathbb{C} \cdot h$ exponentiates to $Z(\rho(\pi_1 T^2)) = \{\mathrm{diag}(\eta, \eta^{-1}) | \eta \in \mathbb{C}^*\}$, the stabilizer of ρ under the conjugation action of $\mathrm{SL}_2(\mathbb{C})$. Now it is easily verified that

$$\{dF(1,0,0), dF(0,1,0), dF(0,0,x), dF(0,0,y)\}_{(\lambda,\mu,1)}$$

are linearly independent if $\lambda \neq \pm 1$ or $\mu \neq \pm 1$. This implies that $\mathrm{rank}_{\mathbb{C}} F$ at $(\lambda, \mu, 1)$ is at least 4, but since $\mathrm{im}\, d_{(\lambda,\mu,1)} F \subset Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$, it has to equal 4. ◻

Corollary 6.20 $\chi = [\iota^*_{T^2}\,\mathrm{hol}]$ *is a smooth point of $X(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$. The $\mathbb{C}$-dimension of $X(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ around χ equals 2. $T_\chi X(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ may be identified with $H^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C})))$.*

Proof. The restriction of F to $\mathbb{C}^* \times \mathbb{C}^* \times \{1\}$ provides a local slice to the action through ρ, upon which the stabilizer of ρ acts trivially. The tangent space to the orbit through ρ may be identified with $B^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$. From the cohomology computations we have $\dim_{\mathbb{C}} H^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C})) = 2$. ◻

For $\gamma \in \Gamma$ we define a function $t_\gamma : R(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}$ by $t_\gamma(\rho) = \mathrm{tr}\,\rho(\gamma)$. If ρ is a smooth point of $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, then t_γ is smooth near ρ. Since tr is invariant under conjugation, t_γ descends to a map on the quotient $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, which we again refer to as t_γ. If $\chi = [\rho]$ is a smooth point of $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, then t_γ is smooth near χ.

Let $\rho = \iota^*_{T^2}\,\mathrm{hol}$ and let $z \in Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$ be given. If we have a deformation of ρ, i.e. a family of representations $\rho_t : \pi_1 T^2 \to \mathrm{SL}_2(\mathbb{C})$ with $\rho_0 = \rho$, which is tangent to z, i.e. $z(\gamma) = \frac{d}{dt}\big|_{t=0} \rho_t(\gamma)\rho(\gamma)^{-1}$ for all $\gamma \in \pi_1 T^2$, we have that the infinitesimal change of the trace of $\rho(\gamma)$ is given as

$$dt_\gamma(z) = \frac{d}{dt}\Big|_{t=0} \mathrm{tr}\,\rho_t(\gamma) = \mathrm{tr}\,(z(\gamma)\rho(\gamma)).$$

We wish to apply this to $z_{ang}, z_{shr}, z_{tws}$ and z_{len}. Let $\mu \in \pi_1 T^2$ be the meridian and $\lambda \in \pi_1 T^2$ the longitude. We assume that

$$\rho(\lambda) = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

and

$$\rho(\mu) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

with $\eta, \xi \neq \pm 1$. Then ρ preserves the axis $\gamma = \{0\} \times \mathbb{R}_+ \subset \mathbf{H}^3$, if we work in the upper half-space model $\mathbf{H}^3 = \mathbb{C} \times \mathbb{R}_+$. If we use cylindrical coordinates (r, θ, z) around γ, then we have already observed that

$$\sigma_{\partial/\partial\theta} = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

and

$$\sigma_{\partial/\partial z} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) .$$

Let us concentrate on the value of the cocycles $z_{ang}, z_{shr}, z_{tws}$ and z_{len} on the meridian $\mu \in \pi_1 T^2$. We obtain

$$z_{ang}(\mu) = \frac{\alpha}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$
$$z_{shr}(\mu) = \frac{\alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) ,$$

while

$$z_{tws}(\mu) = z_{len}(\mu) = 0 .$$

As a consequence we obtain for the infinitesimal change of trace

$$dt_\mu(z_{ang}) = (i\alpha/2)(\xi - \xi^{-1}) \in \mathbb{C}$$
$$dt_\mu(z_{shr}) = (\alpha/2)(\xi - \xi^{-1}) \in \mathbb{C},$$

while

$$dt_\mu(z_{tws}) = dt_\mu(z_{len}) = 0 .$$

Note that $\xi - \xi^{-1} \neq 0$ since $\xi \neq \pm 1$. Since the cohomology classes of the cocycles $\{z_{ang}, z_{shr}, z_{tws}, z_{len}\}$ provide a $\mathbb{R}$-basis of $H^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$, we obtain as a consequence of the above calculations:

Lemma 6.21 *The function t_μ has $\mathbb{C}$-rank 1 in a neighbourhood of $\chi = [\iota_{T^2}^* \mathrm{hol}]$. In particular, the level-set $V = \{t_\mu \equiv t_\mu(\chi)\}$ is locally around χ a smooth, half-dimensional submanifold of $X(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$. Furthermore, the cohomology class of the cocycle z_{len} provides a $\mathbb{C}$-basis for $T_\chi V$. The cohomology classes of the cocycles $\{z_{tws}, z_{len}\}$ provide a $\mathbb{R}$-basis of $T_\chi V$.*

We now turn to the spherical case.

Lemma 6.22 *Let $\rho_i = \iota_{T^2}^* \mathrm{hol}_i : \pi_1 T^2 \to \mathrm{SU}(2), i \in \{1, 2\}$, be the restriction of a component of the holonomy of a spherical cone-structure to a torus boundary component. Then ρ_i is a smooth point of $R(\pi_1 T^2, \mathrm{SU}(2))$. The $\mathbb{R}$-dimension of $R(\pi_1 T^2, \mathrm{SU}(2))$ around ρ_i equals 4. $T_{\rho_i} R(\pi_1 T^2, \mathrm{SU}(2))$ may be identified with $Z^1(\pi_1 T^2, \mathfrak{su}(2))$.*

Proof. As above we define a map

$$F : S^1 \times S^1 \times \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2)$$
$$(\lambda, \mu, A) \longmapsto \left(A \,\mathrm{diag}(\lambda, \lambda^{-1}) A^{-1}, A \,\mathrm{diag}(\mu, \mu^{-1}) A^{-1}\right)$$

We consider the standard $\mathbb{R}$-basis of $\mathfrak{su}(2)$:

$$\left\{ i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}.$$

Now $\mathbb{R} \cdot i$ exponentiates to $Z(\rho(\pi_1 T^2)) = \{\operatorname{diag}(\eta, \eta^{-1}) | \eta \in S^1\}$, the stabilizer of ρ under the conjugation action of $\mathrm{SU}(2)$. It is easily verified that

$$\{dF(1,0,0), dF(0,1,0), dF(0,0,j), dF(0,0,k)\}_{(\lambda,\mu,1)}$$

are linearly independent if $\lambda \neq \pm 1$ or $\mu \neq \pm 1$. The result follows as above. ⊡

Corollary 6.23 $\chi_i = [\iota_{T^2}^* \operatorname{hol}_i]$, $i \in \{1,2\}$, *is a smooth point of* $X(\pi_1 T^2, \mathrm{SU}(2))$. *The* $\mathbb{R}$*-dimension of* $X(\pi_1 T^2, \mathrm{SU}(2))$ *around* χ_i *equals* 2. $T_{\chi_i} X(\pi_1 T^2, \mathrm{SU}(2))$ *may be identified with* $H^1(\pi_1 T^2, \mathfrak{su}(2))$.

Proof. The restriction of F to $S^1 \times S^1 \times \{1\}$ provides a local slice to the action through ρ_i, upon which the stabilizer of ρ acts trivially. The tangent space to the orbit through ρ_i may be identified with $B^1(\pi_1 T^2, \mathfrak{su}(2))$. From the cohomology computations we have $\dim_{\mathbb{R}} H^1(\pi_1 T^2, \mathfrak{su}(2)) = 2$. ⊡

For $\gamma \in \Gamma$ we define a function $t_\gamma : R(\Gamma, \mathrm{SU}(2)) \to \mathbb{R}$ by $t_\gamma(\rho) = \operatorname{tr} \rho(\gamma)$. If ρ is a smooth point of $R(\Gamma, \mathrm{SU}(2))$, then t_γ is smooth near ρ. Since tr is invariant under conjugation, t_γ descends to a map on the quotient $X(\Gamma, \mathrm{SU}(2))$, which we again refer to as t_γ. If $\chi = [\rho]$ is a smooth point of $X(\Gamma, \mathrm{SU}(2))$, then t_γ is smooth in a neighbourhood of χ.
For a representation $\rho = (\rho_1, \rho_2) : \Gamma \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\gamma \in \Gamma$ let $T_\gamma^i(\rho) = t_\gamma(\rho_i)$. This defines a function $T_\gamma = (T_\gamma^1, T_\gamma^2) : R(\Gamma, \mathrm{SU}(2) \times \mathrm{SU}(2)) \to \mathbb{R}^2$, which we view as a "complex" trace function.

Let $\rho = \iota_{T^2}^* \operatorname{hol}$ and let $z = (z_1, z_2) \in Z^1(\pi_1 T^2, \mathfrak{su}(2) \oplus \mathfrak{su}(2))$ be given. The infinitesimal change of the trace of $\rho(\gamma)$ is given as

$$dT_\gamma(z) = (dt_\gamma(z_1), dt_\gamma(z_2))$$

We wish to apply this to $z_{ang}, z_{shr}, z_{tws}$ and z_{len}. Let $\lambda \in \pi_1 T^2$ be the meridian and $\mu \in \pi_1 T^2$ the longitude. We assume that

$$\rho(\lambda) = \left(\begin{pmatrix} \eta_1 & 0 \\ 0 & \overline{\eta}_1 \end{pmatrix}, \begin{pmatrix} \eta_2 & 0 \\ 0 & \overline{\eta}_2 \end{pmatrix} \right) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$$

and

$$\rho(\mu) = \left(\begin{pmatrix} \xi_1 & 0 \\ 0 & \overline{\xi}_1 \end{pmatrix}, \begin{pmatrix} \xi_2 & 0 \\ 0 & \overline{\xi}_2 \end{pmatrix} \right) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$$

with $\xi_1 = \xi_2 =: \xi$ and $\xi \neq \pm 1$, since $\rho(\mu)$ is a nontrivial rotation. Then ρ preserves the pair of axes $\{\gamma, \gamma^\perp\}$, where $\gamma = \mathbb{C} \cap \mathbf{S}^3$ and $\gamma^\perp = \mathbb{C}j \cap \mathbf{S}^3$. If we use cylindrical coordinates (r, θ, z) around γ, then we have already observed that

$$\sigma_{\partial/\partial\theta} = \left(\frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

and

$$\sigma_{\partial/\partial z} = \left(\frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2).$$

In particular, this implies that $\sigma_{\partial/\partial\theta} + \sigma_{\partial/\partial z} \in \Gamma(U_\varepsilon(\Sigma), \mathcal{E}_1)$, and on the other hand $\sigma_{\partial/\partial\theta} - \sigma_{\partial/\partial z} \in \Gamma(U_\varepsilon(\Sigma), \mathcal{E}_2)$. Therefore we have

$$\omega_{tws} + \omega_{len} \in \Omega^1(U_\varepsilon(\Sigma), \mathcal{E}_1)$$

and

$$\omega_{tws} - \omega_{len} \in \Omega^1(U_\varepsilon(\Sigma), \mathcal{E}_2)\,.$$

Again, we concentrate on the value of the cocycles $z_{ang}, z_{shr}, z_{tws}$ and z_{len} on the meridian $\mu \in \pi_1 T^2$. We obtain

$$\begin{aligned} z_{ang}(\mu) &= \left(\frac{\alpha}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{\alpha}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2) \\ z_{shr}(\mu) &= \left(\frac{\alpha}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{\alpha}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)\,, \end{aligned}$$

while

$$z_{tws}(\mu) = z_{len}(\mu) = 0\,.$$

As a consequence we obtain for the infinitesimal change of trace

$$\begin{aligned} dT_\mu(z_{ang}) &= \alpha(-\operatorname{Im}\xi, -\operatorname{Im}\xi) \in \mathbb{R}^2 \\ dT_\mu(z_{shr}) &= \alpha(-\operatorname{Im}\xi, +\operatorname{Im}\xi) \in \mathbb{R}^2\,, \end{aligned}$$

while

$$dT_\mu(z_{tws}) = dT_\mu(z_{len}) = 0\,.$$

Note that $\operatorname{Im}\xi = \frac{1}{2i}(\xi - \overline{\xi}) \neq 0$ since $\xi \neq \pm 1$. Since the cohomology classes of the cocycles $\{z_{ang}, z_{shr}, z_{tws}, z_{len}\}$ provide a $\mathbb{R}$-basis of $H^1(\pi_1 T^2, \mathfrak{su}(2) \oplus \mathfrak{su}(2))$, we obtain as a consequence of the above calculations:

Lemma 6.24 *The function t_μ has $\mathbb{R}$-rank 1 in a neighbourhood of $\chi_i = [\iota_{T^2}^* \operatorname{hol}_i]$. In particular, the level-set $V_i = \{t_\mu \equiv t_\mu(\chi_i)\}$ is locally around χ_i a smooth, half-dimensional submanifold of $X(\pi_1 T^2, \mathrm{SU}(2))$. Furthermore the cohomology class of the cocycle $z_{tws} + z_{len}$ provides a $\mathbb{R}$-basis of $T_{\chi_1} V_1$, the cohomology class of the cocycle $z_{tws} - z_{len}$ provides a $\mathbb{R}$-basis of $T_{\chi_2} V_2$.*

In the Euclidean case, let $\rho = \iota_{T^2}^*(\mathrm{rot} \circ \mathrm{hol})$. Then the arguments given in the spherical case apply directly to yield

Lemma 6.25 *Let $\rho = \iota_{T^2}^*(\mathrm{rot} \circ \mathrm{hol}) : \pi_1 T^2 \to \mathrm{SU}(2)$ be the restriction of the rotational part of the holonomy of a Euclidean cone-structure to a torus boundary component. Then ρ is a smooth point of $R(\pi_1 T^2, \mathrm{SU}(2))$. The $\mathbb{R}$-dimension of $R(\pi_1 T^2, \mathrm{SU}(2))$ around ρ_i equals 4. $T_\rho R(\pi_1 T^2, \mathrm{SU}(2))$ may be identified with $Z^1(\pi_1 T^2, \mathfrak{su}(2))$.*

and as a consequence

Corollary 6.26 *$\chi = [\iota_{T^2}^*(\mathrm{rot} \circ \mathrm{hol})]$ is a smooth point of $X(\pi_1 T^2, \mathrm{SU}(2))$. The $\mathbb{R}$-dimension of $X(\pi_1 T^2, \mathrm{SU}(2))$ around χ equals 2. $T_\chi X(\pi_1 T^2, \mathrm{SU}(2))$ may be identified with $H^1(\pi_1 T^2, \mathfrak{su}(2))$.*

We assume that

$$\rho(\lambda) = \begin{pmatrix} \eta & 0 \\ 0 & \overline{\eta} \end{pmatrix} \in \mathrm{SU}(2)$$

and

$$\rho(\mu) = \begin{pmatrix} \xi & 0 \\ 0 & \overline{\xi} \end{pmatrix} \in \mathrm{SU}(2)$$

with $\xi \neq \pm 1$, since $\rho(\mu)$ is a nontrivial rotation around the axis $\gamma = \mathbb{R}i \subset \operatorname{Im}\mathbb{H}$. After choosing cylindrical coordinates (r, θ, z) around γ we have

$$\sigma_{\partial/\partial z} = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2)\,,$$

and thus we obtain

$$z_{shr}(\mu) = \frac{\alpha}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2)$$

and

$$z_{len}(\mu) = 0\,.$$

From this we conclude $dt_\mu(z_{shr}) = -\alpha \operatorname{Im}\xi$ and $dt_\mu(z_{len}) = 0$.

Lemma 6.27 *The function t_μ has $\mathbb{R}$-rank 1 in a neighbourhood of $\chi = [\iota_{T^2}^*(\mathrm{rot}\circ\mathrm{hol})]$. In particular, the level-set $V = \{t_\mu \equiv t_\mu(\chi)\}$ is locally around χ a smooth, half-dimensional submanifold of $X(\pi_1 T^2, \mathrm{SU}(2))$. Furthermore the cohomology class of the cocycle z_{len} provides a $\mathbb{R}$-basis of $T_\chi V$.*

6.6.2 The higher genus case

Let $\iota : F_g \to M$ be the inclusion of a boundary component of higher genus $g \geq 2$. ι induces a group homomorphism $\iota_* : \pi_1 F_g \to \pi_1 M$ and a map $\iota^* : R(\pi_1 M, G) \to R(\pi_1 F_g, G)$ for $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SU}(2)$ respectively.

Lemma 6.28 *Let $\rho : \pi_1 F_g \to \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation. Then ρ is a smooth point of $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$. The $\mathbb{C}$-dimension of $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ around ρ equals $6g-3$. $T_\rho R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ may be identified with $Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C}))$.*

Proof. We identify $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ with the (affine algebraic) set

$$\left\{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{SL}_2(\mathbb{C})^{2g} \,|\, f(A_1, B_1, \ldots, A_g, B_g) = 1\right\},$$

where $f(A_1, B_1, \ldots, A_g, B_g) = [A_1, B_1]\cdot\ldots\cdot[A_g, B_g]$. $\ker d_\rho f$ may be identified with the space of 1-cocycles $Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C}))$. We know that $\dim_{\mathbb{C}} Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C})) = 6g-3$ from the cohomology computations. Hence $\operatorname{rank}_{\mathbb{C}} d_\rho f = 3$, i.e. $d_\rho f$ is surjective. Now the implicit function theorem implies that $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ is smooth at ρ with $T_\rho R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C})) = Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C}))$. ◻

Corollary 6.29 *The restriction of the holonomy of a hyperbolic cone-structure to a boundary component of higher genus, $\rho = \iota_{F_g}^* \mathrm{hol} : \pi_1 F_g \to \mathrm{SL}_2(\mathbb{C})$, is a smooth point of $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$. The $\mathbb{C}$-dimension of $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ around ρ equals $6g-3$. $T_\rho R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ may be identified with $Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C}))$.*

Proof. Clearly ρ is irreducible: If $v \in \Sigma$ is a singular vertex and we restrict hol further to $U_\varepsilon(v)$, which deformation-retracts to a pair of pants $P \subset F_g$, then $\iota_P^* \mathrm{hol}$ preserves a point $p \in \mathbf{H}^3$. Now if ρ was reducible, then $\iota_P^* \mathrm{hol}$ would preserve a geodesic, which is a contradiction. ◻

Lemma 6.30 *The action of $\mathrm{SL}_2(\mathbb{C})$ on $R_{irr}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is proper for Γ a finitely generated group.*

Proof. Recall that by definition a group action $G \times X \to X$ is proper if the map

$$G \times X \longrightarrow X \times X\,, (g, x) \longmapsto (gx, x)$$

is proper. If we have a G-equivariant map from X to a proper G-space Y, then X itself will be a proper G-space. We construct a continuous, $\mathrm{SL}_2(\mathbb{C})$-equivariant map

$$\begin{aligned} R_{irr}(\Gamma, \mathrm{SL}_2(\mathbb{C})) &\longrightarrow \mathbf{H}^3 \\ \rho &\longmapsto \mathrm{center}(\rho)\,, \end{aligned}$$

where the "center" of a representation will be the point in $\mathbf{H}^3$, which is displaced the least in average by the generators of the group. This is a variant of a construction by M. Bestvina (cf. [Bes]). More precisely, let us fix a presentation $\langle \gamma_1, \dots, \gamma_n | (r_i)_{i \in I} \rangle$ of Γ. We call a function $f : \mathbf{H}^3 \to \mathbb{R}$ (strictly) convex if it is (strictly) convex along any geodesic. Note that the (modified) displacement function of $A \in \mathrm{SL}_2(\mathbb{C})$

$$\begin{aligned} \delta_A : \mathbf{H}^3 &\longrightarrow \mathbb{R} \\ x &\longmapsto \cosh d(x, Ax) - 1 \end{aligned}$$

is a convex function in general. It is strictly convex if A is parabolic. If A is semisimple, it is strictly convex along any geodesic different from the axis of A. We define

$$f_\rho(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{\rho(\gamma_i)}\,.$$

If we have a sequence $x_n \in \mathbf{H}^3$ which converges to $x_\infty \in \partial_\infty \mathbf{H}^3$, then since ρ is irreducible, there has to be at least one $\rho(\gamma_i)$ that does not fix x_∞. Then it follows that $\delta_{\rho(\gamma_i)}(x_n) \to \infty$. Therefore f_ρ is proper. If we take any geodesic γ, again since ρ is irreducible, there has to be at least one $\rho(\gamma_i)$ such that $\delta_{\rho(\gamma_i)}$ is strictly convex. Therefore f_ρ is strictly convex.
As a proper and strictly convex function, f_ρ assumes its minimum at a unique point in $\mathbf{H}^3$, which we define to be the center of ρ.
If we have a sequence of representations ρ_n converging to ρ with respect to the compact-open topology on $R_{irr}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, then f_{ρ_n} converges to f_ρ uniformly on compact sets. Therefore the map $\rho \mapsto \mathrm{center}(\rho)$ is continuous.
Since $\delta_{BAB^{-1}}(x) = \delta_A(B^{-1}x)$ we have that $\mathrm{center}(B\rho B^{-1}) = B\,\mathrm{center}(\rho)$, i.e. that $\rho \mapsto \mathrm{center}(\rho)$ is $\mathrm{SL}_2(\mathbb{C})$-equivariant.
This together with the fact that the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbf{H}^3$ is proper proves the lemma. ⊡

Corollary 6.31 *$\chi = [\iota_{F_g}^* \mathrm{hol}]$ is a smooth point of $X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$. The $\mathbb{C}$-dimension of $X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ around χ equals $6g - 6$. $T_\chi X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ may be identified with $H^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C})))$.*

Proof. Since the action of $\mathrm{SL}_2(\mathbb{C})$ is proper, we have a local slice to the action. We recall that the stabilizer of ρ, $Z(\rho(\pi_1 F_g))$, equals $\{\pm 1\}$. Therefore $X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ is locally around χ the quotient of a free $\mathrm{PSL}_2(\mathbb{C})$ action and therefore smooth. ⊡

The meridian curves around the singularity give rise to a pair-of-pants decomposition of F_g. Let $\{\mu_1, \dots, \mu_N\}$ be the family of meridians, where $N = 3g - 3$. This may be used to give an alternative construction of $R(\pi_1(F_g), \mathrm{SL}_2(\mathbb{C}))$, which is better suited for our purposes.
Let P denote the thrice-punctured sphere, i.e. a pair of pants. The fundamental group of P is the free group on 2 generators. We will use the following slightly redundant presentation:

$$\pi_1 P = \langle \mu_1, \mu_2, \mu_3 | \mu_1 \mu_2 \mu_2 = 1 \rangle\,.$$

It follows that

$$R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) = \{(A_1, A_2, A_3) \in \mathrm{SL}_2(\mathbb{C})^3 | A_1 A_2 A_3 = 1\}.$$

Clearly the map $f : \mathrm{SL}_2(\mathbb{C})^3 \to \mathrm{SL}_2(\mathbb{C}), (A_1, A_2, A_3) \mapsto A_1A_2A_3$ is a submersion, such that $R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) = f^{-1}(1)$ is a smooth submanifold of $\mathbb{C}$-dimension 6. Let $\iota_i : S^1 \to P$ be the inclusion of the i-th boundary circle. Then the induced map $\iota_i^* : R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S^1, \mathrm{SL}_2(\mathbb{C}))$ corresponds to the projection $pr_i : R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) \to \mathrm{SL}_2(\mathbb{C}), (A_1, A_2, A_3) \mapsto A_i$, which is also a submersion.

Lemma 6.32 *Let $\rho = \iota_P^* \mathrm{hol}$ be the restriction of the holonomy of a hyperbolic cone-structure to a pair of pants P. Then the differentials $\{dt_{\mu_1}, dt_{\mu_2}, dt_{\mu_3}\}$ are linearly independent over $\mathbb{C}$ in $T_\rho^* R(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$.*

Proof. It is enough to show that there is a deformation $t \mapsto (A_1(t), A_2(t), A_3(t)) \in R(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$ with $\mathrm{tr}[\dot{A}_1] = \mathrm{tr}[\dot{A}_2] = 0$, but $\mathrm{tr}[\dot{A}_3] \neq 0$. Without loss of generality we may assume that $A_1 = \mathrm{diag}(\lambda, \lambda^{-1})$ with $\lambda \neq 0$. Now since

$$\tfrac{d}{dt}\Big|_{t=0} \mathrm{tr}\,[A_1(t)A_2(t)] = \mathrm{tr}[\dot{A}_1 A_2] + \mathrm{tr}[A_1 \dot{A}_2]\,,$$

we may choose $A_1(t) = A_1$ and $A_2(t) = B(t)A_2B(t)^{-1}$ for $t \mapsto B(t) \in \mathrm{SL}_2(\mathbb{C})$ with $B(0) = 1$. Then $\dot{A}_1 = 0$ and $\dot{A}_2 = \dot{B}A_2 - A_2\dot{B}$, so we obtain $\mathrm{tr}[\dot{A}_1] = \mathrm{tr}[\dot{A}_2] = 0$ and

$$\tfrac{d}{dt}\Big|_{t=0} \mathrm{tr}\,[A_1(t)A_2(t)] = \mathrm{tr}[A_1(\dot{B}A_2 - A_2\dot{B})]\,.$$

Note that A_2 is not a diagonal matrix since ρ is irreducible. Therefore we can find $\dot{B} \in \mathfrak{sl}_2(\mathbb{C})$ such that $\dot{B}A_2 - A_2\dot{B} \in \mathfrak{sl}_2(\mathbb{C})$ has non-vanishing diagonal. This implies that $\mathrm{tr}[A_1(\dot{B}A_2 - A_2\dot{B})] \neq 0$. Since $\mathrm{tr}[A] = \mathrm{tr}[A^{-1}]$ in $\mathrm{SL}_2(\mathbb{C})$, we obtain that $\mathrm{tr}[\dot{A}_3] \neq 0$. ◻

Since ρ is irreducible, we can use Lemma **(6.30)** to conclude that $\chi = [\rho]$ is a smooth point in $X(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$. The local $\mathbb{C}$-dimension of $X(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$ around χ is 3. The functions $\{t_{\mu_1}, t_{\mu_2}, t_{\mu_3}\}$ are local holomorphic coordinates on $X(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$ near χ.

We build up $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ from $R(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$ using two basic operations:

1. glue a pair of pants P to a connected surface with boundary S along a boundary circle, call the resulting connected surface S'
2. glue a connected surface S along two different boundary circles, call the resulting connected surface S'

In the first case $\pi_1 S' = \pi_1 S \amalg_{\pi_1 S^1} \pi_1 P$ by van Kampen's theorem and we have

$$R(\pi_1 S', \mathrm{SL}_2(\mathbb{C})) = R(\pi_1 S, \mathrm{SL}_2(\mathbb{C})) \times_{R(\pi_1 S^1, \mathrm{SL}_2(\mathbb{C}))} R(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$$

via the maps

$$\iota_{S^1 \hookrightarrow S}^* : R(\pi_1 S, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S^1, \mathrm{SL}_2(\mathbb{C}))$$

and

$$\iota_{S^1 \hookrightarrow P}^* : R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S^1, \mathrm{SL}_2(\mathbb{C}))\,,$$

which will be transversal since the latter one is a submersion. Therefore $\rho = \iota_{S'}^* \mathrm{hol}$ is a smooth point in $R(\pi_1 S', \mathrm{SL}_2(\mathbb{C}))$ since $\rho_S = \iota_S^* \mathrm{hol}$ is a smooth point in $R(\pi_1 S, \mathrm{SL}_2(\mathbb{C}))$ and $\rho_P = \iota_P^* \mathrm{hol}$ is a smooth point in $R(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$.

In the second case $\pi_1 S'$ splits as an HNN-extension of $\pi_1 S$. More precisely, if $\mu_1, \mu_2 \in \pi_1 S$ are the loops around the boundary circles, which will be identified, then $\pi_1 S' = \langle \pi_1 S, \lambda | \lambda \mu_1 \lambda^{-1} = \mu_2 \rangle$. In this case we have

$$R(\pi_1 S', \mathrm{SL}_2(\mathbb{C})) = \{(\rho_S, B) \in R(\pi_1 S, \mathrm{SL}_2(\mathbb{C})) \times \mathrm{SL}_2(\mathbb{C}) | B\rho_S(\mu_1)B^{-1} = \rho_S(\mu_2)\}$$

as a consequence. We show that the map

$$\begin{aligned} f: R(\pi_1 S, \mathrm{SL}_2(\mathbb{C})) \times \mathrm{SL}_2(\mathbb{C}) &\longrightarrow \mathrm{SL}_2(\mathbb{C}) \\ (\rho_S, B) &\longmapsto B\rho_S(\mu_1)B^{-1}\rho_S(\mu_2)^{-1} \end{aligned}$$

is a submersion near $\rho = \iota_{S'}^* \,\mathrm{hol}$. This implies that $\rho = (\rho_S, B)$ is a smooth point in $R(\pi_1 S', \mathrm{SL}_2(\mathbb{C}))$.
Surjectivity of df at ρ can be established as follows: Let $A_1 = \rho_S(\mu_1)$ and $A_2 = \rho_S(\mu_2)$. Clearly the map $B \mapsto BA_1B^{-1}A_2^{-1}$ has $\mathbb{C}$-rank 2. Since $\{dt_{\mu_1}, dt_{\mu_2}\}$ are linearly independent, we can construct a deformation $t \mapsto (\rho_S)_t$ with $(\rho_S)_t(\mu_2) = A_2$ and $dt_{\mu_1}(\dot\rho_S) \neq 0$. This deformation will be transverse to $\mathrm{im}(B \mapsto BA_1B^{-1}A_2^{-1})$.

Lemma 6.33 *The differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ with $N = 3g - 3$ are linearly independent over $\mathbb{C}$ in $T_\rho^* R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ for $\rho = \iota_{F_g}^* \,\mathrm{hol}$.*

Proof. This follows from the construction above by an inductive argument. We assume for $\{\mu_i\}$ the family of meridians for S that the differentials $\{dt_{\mu_i}\}$ are linearly independent.
In the first case we use Lemma **(6.32)** to conclude that for $\{\mu_i'\}$ the family of meridians for S' the differentials $\{dt_{\mu_i'}\}$ are linearly independent. In fact, any deformation of $(\rho_S, \rho_{S'})$ gives rise to a deformation of $\rho_{S'}$ as long as $\iota_{S^1}^*\rho_S$ and $\iota_{S^1}^*\rho_P$ remain the same.
In the second case we observe that any deformation of ρ_S gives rise to a deformation of $\rho_{S'}$ as long as $\rho_S(\mu_1)$ and $\rho_S(\mu_2)$ remain conjugate, i.e. t_{μ_1} and t_{μ_2} remain the same. Therefore if $\{\mu_i'\}$ is family of meridians for S' the differentials $\{dt_{\mu_i'}\}$ will be linearly independent. ◻

Clearly

$$z_{tws}^i(\mu_j) = \int_{\mu_j} \omega_{tws}^i = 0$$

and

$$z_{len}^i(\mu_j) = \int_{\mu_j} \omega_{len}^i = 0\,.$$

Therefore

$$dt_{\mu_j}(z_{tws}^i) = 0$$

and

$$dt_{\mu_j}(z_{len}^i) = 0\,.$$

As a consequence of the above discussion we obtain

Lemma 6.34 *The level-set $V = \{t_{\mu_1} \equiv t_{\mu_1}(\chi), \ldots, t_{\mu_N} \equiv t_{\mu_N}(\chi)\}$ is locally around $\chi = [\iota_{F_g}^* \,\mathrm{hol}]$ a smooth, half-dimensional submanifold of $X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$. Furthermore, the cohomology classes of the cocycles $\{z_{len}^1, \ldots, z_{len}^N\}$ provide a $\mathbb{C}$-basis of $T_\chi V$. The cohomology classes of the cocycles $\{z_{tws}^1, z_{len}^1, \ldots, z_{tws}^N, z_{len}^N\}$ provide a $\mathbb{R}$-basis for $T_\chi V$.*

We now turn to the spherical case.

Lemma 6.35 *Let $\rho : \pi_1 F_g \to \mathrm{SU}(2)$ be an irreducible representation. Then ρ is a smooth point of $R(\pi_1 F_g, \mathrm{SU}(2))$. The $\mathbb{R}$-dimension of $R(\pi_1 F_g, \mathrm{SU}(2))$ around ρ equals $6g - 3$. $T_\rho R(\pi_1 F_g, \mathrm{SU}(2))$ may be identified with $Z^1(\pi_1 F_g, \mathfrak{su}(2))$.*

Proof. This follows as in the case of $\mathrm{SL}_2(\mathbb{C})$ from the cohomology computations and the implicit function theorem. ◻

Corollary 6.36 *Let $\rho_i = \iota_{F_g}^* \operatorname{hol}_i : \pi_1 F_g \to \mathrm{SU}(2), i \in \{1,2\}$, be the restriction of a component of the holonomy of a spherical cone-structure to a boundary component of higher genus. Then ρ_i is a smooth point of $R(\pi_1 F_g, \mathrm{SU}(2))$. The $\mathbb{R}$-dimension of $R(\pi_1 F_g, \mathrm{SU}(2))$ around ρ_i equals $6g-3$. $T_{\rho_i} R(\pi_1 F_g, \mathrm{SU}(2))$ may be identified with $Z^1(\pi_1 F_g, \mathfrak{su}(2))$.*

Proof. Clearly the ρ_i are both irreducible: If $v \in \Sigma$ is a singular vertex and we restrict $\operatorname{hol} = (\operatorname{hol}_1, \operatorname{hol}_2)$ further to $U_\varepsilon(v)$, which deformation-retracts to a pair of pants $P \subset F_g$, then $\iota_P^* \operatorname{hol}$ preserves a point $p \in \mathbf{S}^3$. Without loss of generality we may assume that $p = 1 \in \mathbf{S}^3 \subset \mathbb{H}$. Then since

$$\mathrm{Stab}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}(1) = \{(A,A) : A \in \mathrm{SU}(2)\},$$

we obtain that $\iota_P^* \operatorname{hol}_1 = \iota_P^* \operatorname{hol}_2$. Now if ρ_1 or ρ_2 was reducible, then $\iota_P^* \operatorname{hol}$ would preserve a geodesic, which is a contradiction. ⊡

Corollary 6.37 *$\chi_i = [\iota_{F_g}^* \operatorname{hol}_i]$, $i \in \{1,2\}$, is a smooth point of $X(\pi_1 F_g, \mathrm{SU}(2))$. The $\mathbb{R}$-dimension of $X(\pi_1 F_g, \mathrm{SU}(2))$ around χ_i equals $6g-6$. $T_{\chi_i} X(\pi_1 F_g, \mathrm{SU}(2))$ may be identified with $H^1(\pi_1 F_g, \mathfrak{su}(2))$.*

Proof. Since the group $\mathrm{SU}(2)$ is compact, the properness of the action is granted. We recall that the stabilizer of ρ_i, $Z(\rho_i(\pi_1 F_g))$, equals $\{\pm 1\}$. Therefore $X(\pi_1 F_g, \mathrm{SU}(2))$ is near χ_i a quotient of a free $\mathrm{PSU}(2)$ action and therefore smooth. ⊡

Lemma 6.38 *The differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ with $N = 3g-3$ are linearly independent in $T_{\rho_i}^* R(\pi_1 F_g, \mathrm{SU}(2))$ for $\rho_i = \iota_{F_g}^* \operatorname{hol}_i$.*

Proof. The arguments in the hyperbolic case apply without essential change. ⊡

We obtain finally

Lemma 6.39 *The level-set $V_i = \{t_{\mu_1} \equiv t_{\mu_1}(\chi_i), \ldots, t_{\mu_N} \equiv t_{\mu_N}(\chi_i)\}$ is locally around $\chi_i = [\iota_{F_g}^* \operatorname{hol}_i]$ a smooth, half-dimensional submanifold of $X(\pi_1 F_g, \mathrm{SU}(2))$. The cohomology classes of the cocycles $\{z_{tws}^1 + z_{len}^1, \ldots, z_{tws}^N + z_{len}^N\}$ provide a basis for $T_{\chi_1} V_1$, while the cohomology classes of the cocycles $\{z_{tws}^1 - z_{len}^1, \ldots, z_{tws}^N - z_{len}^N\}$ provide a basis for $T_{\chi_2} V_2$.*

In the Euclidean case we obtain along the same lines

Corollary 6.40 *Let $\rho = \iota_{F_g}^*(\mathrm{rot} \circ \operatorname{hol}) : \pi_1 F_g \to \mathrm{SU}(2)$ be the restriction of the rotational part of the holonomy of a Euclidean cone-structure to a boundary component of higher genus. Then ρ is a smooth point of $R(\pi_1 F_g, \mathrm{SU}(2))$. The $\mathbb{R}$-dimension of $R(\pi_1 F_g, \mathrm{SU}(2))$ around ρ equals $6g-3$. $T_\rho R(\pi_1 F_g, \mathrm{SU}(2))$ may be identified with $Z^1(\pi_1 F_g, \mathfrak{su}(2))$.*

Proof. Clearly ρ is irreducible, otherwise it would preserve a line in $\mathbb{R}^3$, which is absurd in the presence of vertices. ⊡

Corollary 6.41 *$\chi = [\iota_{F_g}^*(\mathrm{rot} \circ \operatorname{hol})]$ is a smooth point of $X(\pi_1 F_g, \mathrm{SU}(2))$. The $\mathbb{R}$-dimension of $X(\pi_1 F_g, \mathrm{SU}(2))$ around χ equals $6g-6$. $T_\chi X(\pi_1 F_g, \mathrm{SU}(2))$ may be identified with $H^1(\pi_1 F_g, \mathfrak{su}(2))$.*

We obtain as in the spherical case

Lemma 6.42 *The differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ with $N = 3g-3$ are linearly independent in $T_\rho^* R(\pi_1 F_g, \mathrm{SU}(2))$ for $\rho = \iota_{F_g}^*(\mathrm{rot} \circ \operatorname{hol})$.*

and finally

Lemma 6.43 *The level-set* $V = \{t_{\mu_1} \equiv t_{\mu_1}(\chi), \dots, t_{\mu_N} \equiv t_{\mu_N}(\chi)\}$ *is locally around* $\chi = [\iota^*_{F_g}(\mathrm{rot} \circ \mathrm{hol})]$ *a smooth, half-dimensional submanifold of* $X(\pi_1 F_g, \mathrm{SU}(2))$*. The cohomology classes of the cocycles* $\{z^1_{len}, \dots, z^N_{len}\}$ *provide a basis for* $T_\chi V$*.*

6.7 Local rigidity

Lemma 6.44 *Let* C *be a hyperbolic or a spherical cone-manifold with cone-angles* $\leq \pi$*. Then:*

1. *The natural map* $H^1(M, \mathcal{E}) \to H^1(\partial M_\varepsilon, \mathcal{E})$ *is injective.*
2. $\dim H^1(M, \mathcal{E}) = \frac{1}{2} \dim H^1(\partial M_\varepsilon, \mathcal{E})$*.*

In the spherical case, the assertions hold for the parallel subbundles $\mathcal{E}_i \subset \mathcal{E}$ *separately,* $i \in \{1, 2\}$*.*

Proof. Let us look at a part of the long exact cohomology sequence of the pair $(M_\varepsilon, \partial M_\varepsilon)$ with coefficients in $\mathcal{E}$. The natural map $q : H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) \to H^1(M_\varepsilon, \mathcal{E})$ factors through L^2-cohomology, since $H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) = H^1_{cp}(M, \mathcal{E})$:

$$\begin{array}{ccccccc}
\longrightarrow & H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) & \xrightarrow{q} & H^1(M_\varepsilon, \mathcal{E}) & \xrightarrow{r} & H^1(\partial M_\varepsilon, \mathcal{E}) & \longrightarrow \\
& \| & & \uparrow & & & \\
& H^1_{cp}(M, \mathcal{E}) & \longrightarrow & H^1_{L^2}(M, \mathcal{E}) & & &
\end{array}$$

Since by our vanishing theorem $H^1_{L^2}(M, \mathcal{E}) = 0$, we have that q is the zero map and $r : H^1(M_\varepsilon, \mathcal{E}) \to H^1(\partial M_\varepsilon, \mathcal{E})$ is injective.
Since the Killing form B on $\mathcal{E}$ (resp. the parallel metric $h^{\mathcal{E}}$ in the spherical case) provides a non-degenerate coefficient pairing, we can apply Poincaré duality to conclude that $H^2(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) \cong H^1(M_\varepsilon, \mathcal{E})^*$ and $H^2(M_\varepsilon, \mathcal{E}) \cong H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E})^*$. The Poincaré duality isomorphisms are natural, such that we obtain the following commutative diagram:

$$\begin{array}{ccccc}
& H^1(M_\varepsilon, \mathcal{E})^* & \xrightarrow{q^*} & H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E})^* & \\
& \cong \uparrow \text{P.D.} & & \cong \uparrow \text{P.D.} & \\
\longrightarrow & H^2(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) & \longrightarrow & H^2(M_\varepsilon, \mathcal{E}) & \longrightarrow
\end{array}$$

Since $q^* = 0$, we obtain the following short exact sequence:

$$\begin{array}{ccccccccc}
& & & & & & H^1(M_\varepsilon, \mathcal{E})^* & & \\
& & & & & & \cong \uparrow \text{P.D.} & & \\
0 & \longrightarrow & H^1(M_\varepsilon, \mathcal{E}) & \longrightarrow & H^1(\partial M_\varepsilon, \mathcal{E}) & \longrightarrow & H^2(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) & \longrightarrow & 0
\end{array}$$

This implies that $\dim H^1(M_\varepsilon, \mathcal{E}) = \frac{1}{2} \dim H^1(\partial M_\varepsilon, \mathcal{E})$. In the spherical case these arguments apply to the parallel subbundles $\mathcal{E}_i \subset \mathcal{E}$ separately, $i \in \{1, 2\}$. ⊡

6.7.1 The hyperbolic case

The following is a well-known fact about the holonomy representation of a hyperbolic cone-structure.

Lemma 6.45 *The holonomy of a hyperbolic cone-structure* $\mathrm{hol}: \pi_1 M \to \mathrm{SL}_2(\mathbb{C})$ *is irreducible.*

Proof. For convenience of the reader we give a sketch of proof. Let us assume that the holonomy representation is reducible. Then there is a point $x_\infty \in \partial_\infty \mathbf{H}^3$ fixed by the holonomy. This means that the horospherical fibration centered at x_∞ is also preserved by the holonomy. The volume decreasing flow, which moves each point x with unit speed towards x_∞ along the unique geodesic connecting x and x_∞, may then be pulled back via the developing map to a volume decreasing flow on M. This is absurd since M has finite volume. ◻

Lemma 6.46 *Let* $\mathrm{hol}: \pi_1 M \to \mathrm{SL}_2(\mathbb{C})$ *be the holonomy of a hyperbolic cone-structure with cone-angles* $\leq \pi$. *Then* hol *is a smooth point of* $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$. *The* $\mathbb{C}$*-dimension of* $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ *around* hol *equals* $\tau + 3 - \frac{3}{2}\chi(\partial M_\varepsilon)$, *where* τ *is the number of torus components in* ∂M_ε. $T_{\mathrm{hol}} R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ *may be identified with* $Z^1(\pi_1 M, \mathfrak{sl}_2(\mathbb{C}))$.

Proof. We follow the discussion in M. Kapovich's book (cf. [Kap]), which essentially amounts to a transversality argument. The key to the proof is the following splitting of M_ε, which may be viewed as a generalization of the Heegard-splitting to 3-manifolds with boundary.

Lemma 6.47 *There exists a system of disjoint* 1*-handles* $\{H_1, \ldots, H_t\}$ *in* M_ε *attached to* ∂M_ε *such that* $M_1 := M_\varepsilon \setminus \mathrm{int}(\cup_i H_i)$ *is a handlebody.*

Proof. For a proof we refer to [Kap, Lemma 8.46]. ◻

The H_i are called *tunnels*, the minimal number t of tunnels in such a splitting is called *tunnel number* of M_ε. Without loss of generality (though this is not essential to the argument) we assume minimality of the splitting in this sense.

As a consequence M_ε may be written as a union

$$M_\varepsilon = M_1 \cup_S M_2 \,,$$

where S is a surface of genus $g = 1 + t - \chi(\partial M_\varepsilon)/2$. M_2 is homotopy equivalent to the wedge product of the components of ∂M_ε and $t - b + 1$ circles, where b is the number of components of ∂M_ε. Therefore we obtain by van Kampen's theorem

$$\pi_1 M_\varepsilon = \pi_1 M_1 \amalg_{\pi_1 S} \pi_1 M_2 \,,$$

where $\pi_1 M_1$ is the free group on g generators, and $\pi_1 M_2$ splits as a free product of the fundamental groups of the components of ∂M_ε and $t - b + 1$ $\mathbb{Z}$-factors. Consequently we obtain for the representation varieties

$$R(\pi_1 M_\varepsilon, \mathrm{SL}_2(\mathbb{C})) = R(\pi_1 M_1, \mathrm{SL}_2(\mathbb{C})) \times_{R(\pi_1 S, \mathrm{SL}_2(\mathbb{C}))} R(\pi_1 M_2, \mathrm{SL}_2(\mathbb{C}))\,,$$

via the maps

$$\mathrm{res}_1 = \iota^*_{S \hookrightarrow M_1} : R(\pi_1 M_1, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S, \mathrm{SL}_2(\mathbb{C}))$$

and

$$\mathrm{res}_2 = \iota^*_{S \hookrightarrow M_2} : R(\pi_1 M_2, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S, \mathrm{SL}_2(\mathbb{C}))\,.$$

$R(\pi_1 M_1, \mathrm{SL}_2(\mathbb{C}))$ and $R(\pi_1 M_2, \mathrm{SL}_2(\mathbb{C}))$ are smooth near the restriction of the holonomy of a hyperbolic cone-structure. Note that $\pi_1 S$ surjects onto $\pi_1 M_\varepsilon$. Since hol is irreducible, this will also be the case for $\iota_S^*\,\mathrm{hol}$, which is therefore seen to be a smooth point of $R(\pi_1 S, \mathrm{SL}_2(\mathbb{C}))$.
Therefore it is sufficient to show that res_1 and res_2 meet transversally at $\iota_S^*\,\mathrm{hol}$. This will follow from the equation

$$\begin{aligned} &\dim_{\mathbb{C}} Z_1(\pi_1 M_1, \mathfrak{sl}_2(\mathbb{C})) + \dim_{\mathbb{C}} Z_1(\pi_1 M_2, \mathfrak{sl}_2(\mathbb{C})) \\ =\, &\dim_{\mathbb{C}} Z_1(\pi_1 S, \mathfrak{sl}_2(\mathbb{C}) + \dim_{\mathbb{C}} Z^1(\pi_1 M_\varepsilon, \mathfrak{sl}_2(\mathbb{C}))\,, \end{aligned}$$

if we identify $Z^1(\pi_1 M_\varepsilon, \mathfrak{sl}_2(\mathbb{C}))$ with

$$\{(z_1, z_2) \in Z^1(\pi_1 M_1, \mathfrak{sl}_2(\mathbb{C})) \oplus Z^1(\pi_1 M_2, \mathfrak{sl}_2(\mathbb{C})) \,|\, d\,\mathrm{res}_1(z_1) = d\,\mathrm{res}_2(z_2)\}\,.$$

To obtain the desired equation, we have to calculate the dimensions of the cocycle spaces. Note that $Z^1(\Gamma \amalg \Gamma', \mathfrak{g}) = Z^1(\Gamma, \mathfrak{g}) \oplus Z^1(\Gamma', \mathfrak{g})$.

- $\dim_{\mathbb{C}} Z^1(\pi_1 M_1, \mathfrak{sl}_2(\mathbb{C})) = 3 + 3t - \frac{3}{2}\chi(\partial M_\varepsilon)$, since $\pi_1 M_1$ is the free group on $g = 1 + t - \chi(\partial M_\varepsilon)/2$ generators.
- $\dim_{\mathbb{C}} Z^1(\pi_1 M_2, \mathfrak{sl}_2(\mathbb{C})) = \tau - 3\chi(\partial M_\varepsilon) + 3t + 3$, since $\dim_{\mathbb{C}} Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C})) = 4$ at $\iota_{T^2}^*\,\mathrm{hol}$, $\dim_{\mathbb{C}} Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C})) = -3\chi(F_g) + 3$ at $\iota_{F_g}^*\,\mathrm{hol}$ and the fundamental group of a wedge of $t - b + 1$ circles is the free group on that number of generators.
- $\dim_{\mathbb{C}} Z^1(\pi_1 S, \mathfrak{sl}_2(\mathbb{C})) = 6t - 3\chi(\partial M_\varepsilon) + 3$, since $\iota_S^*\,\mathrm{hol}$ is irreducible. This implies in particular that $Z^0(\pi_1 S, \mathfrak{sl}_2(\mathbb{C})) = 0$.
- $\dim_{\mathbb{C}} Z^1(\pi_1 M_\varepsilon, \mathfrak{sl}_2(\mathbb{C})) = \tau - \frac{3}{2}\chi(\partial M_\varepsilon) + 3$, since we have by the cohomology computations that $\dim_{\mathbb{C}} H^1(M_\varepsilon, \mathcal{E}) = \frac{1}{2}\dim_{\mathbb{C}} H^1(\partial M_\varepsilon, \mathcal{E})$, furthermore hol is irreducible, therefore $Z^0(\pi_1 M_\varepsilon, \mathfrak{sl}_2(\mathbb{C})) = 0$.

This finishes the proof. ◻

Corollary 6.48 $\chi = [\mathrm{hol}]$ *is a smooth point of* $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})$. *The* $\mathbb{C}$*-dimension of* $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})$ *around* χ *equals* $\tau - \frac{3}{2}\chi(\partial M_\varepsilon)$, *where* τ *is the number of torus components in* ∂M_ε. $T_\chi X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ *may be identified with* $H^1(\pi_1 M, \mathfrak{sl}_2(\mathbb{C}))$.

Proof. $Z(\mathrm{hol}(\pi_1 M)) = \{\pm 1\}$ since hol is irreducible. Using Lemma **(6.30)** we proceed in the same way as in the surface case. ◻

We are now ready to state and prove the main result in the hyperbolic case.

Theorem 6.49 *Let* C *be a hyperbolic cone-manifold with cone-angles* $\leq \pi$. *Let* $\{\mu_1, \ldots, \mu_N\}$ *be the family of meridians, where* $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$. *Then the map*

$$X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}^N, \chi \mapsto (t_{\mu_1}(\chi), \ldots, t_{\mu_N}(\chi))$$

is a local diffeomorphism near $\chi = [\mathrm{hol}]$.

Proof. Without loss of generality we may assume that Σ is connected. Then we have to consider two cases:

1. Σ is a circle, i.e. $\partial M_\varepsilon = T^2$
2. Σ is a connected, trivalent graph, i.e. $\partial M_\varepsilon = F_g$

Let us recall what we already know. The level-set of the trace functions

$$V = \{t_{\mu_i} \equiv t_{\mu_i}(\chi), \ldots, t_{\mu_N} \equiv t_{\mu_N}(\chi)\}$$

is a smooth, half-dimensional submanifold of $X(\pi_1\partial M_\varepsilon, \mathrm{SL}_2(\mathbb{C}))$ in each case, since the differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ are $\mathbb{C}$-linearly independent in $H^1(\pi_1\partial M_\varepsilon, \mathrm{SL}_2(\mathbb{C}))^*$ at χ. If we work in the de-Rham realization of $H^1(\pi_1\partial M_\varepsilon, \mathrm{SL}_2(\mathbb{C}))$, the classes of the differential forms

$$\{\omega^1_{tws}, \omega^1_{len}, \ldots, \omega^N_{tws}, \omega^N_{len}\}$$

provide a basis of $T_\chi V$. Furthermore, these forms are L^2-bounded on $U_\varepsilon(\Sigma)$.
On the other hand we know that the restriction map $H^1(M, \mathcal{E}) \to H^1(\partial M_\varepsilon, \mathcal{E})$ is injective with half-dimensional image. This means that $X(\pi_1 M, SL_2(\mathbb{C}))$ is immersed into $X(\pi_1\partial M_\varepsilon, SL_2(\mathbb{C}))$ as a half-dimensional submanifold.
We claim that V and $X(\pi_1 M, SL_2(\mathbb{C}))$ are transversal in $X(\pi_1\partial M_\varepsilon, SL_2(\mathbb{C}))$ at χ. It is sufficient to show that $T_\chi V$ and $\mathrm{im}(H^1(M,\mathcal{E}) \to H^1(\partial M_\varepsilon, \mathcal{E}))$ intersect trivially in $H^1(\partial M_\varepsilon, \mathcal{E})$.
Let $\omega \in \Omega^1(M, \mathcal{E})$ be a closed form such that $[\omega]|_{\partial M_\varepsilon} \in T_\chi V$. In particular, since ω^i_{tws} and ω^i_{len} are L^2-bounded on $U_\varepsilon(\Sigma)$, $\omega + d\sigma$ will be L^2-bounded on $U_\varepsilon(\Sigma)$ for some $\sigma \in \Gamma(U_\varepsilon(\Sigma), \mathcal{E})$. We choose a cut-off function φ, which is 1 in a neighbourhood of Σ and which is supported in $U_\varepsilon(\Sigma)$. Then $\varphi\sigma$ extends to a section on M, such that $\omega + d(\varphi\sigma)$ is L^2-bounded on M. Since $H^1_{L^2}(M, \mathcal{E}) = 0$, this implies that $[\omega] = 0$ in $H^1(M, \mathcal{E})$ and therefore $[\omega]|_{\partial M_\varepsilon} = 0$.
It follows that the differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ are $\mathbb{C}$-linearly independent already in $H^1(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))^*$. ◻

The complex length $\mathcal{L}_i$ of the i-th meridian is related to its trace via

$$t_{\mu_i}(\rho) = \pm 2\cosh(\mathcal{L}_i/2)\,.$$

Locally the set of representations $\rho : \pi_1 M \to \mathrm{SL}_2(\mathbb{C}))$ such that $\mathcal{L}_i$ is purely imaginary for all $i \in \{1, \ldots, N\}$ corresponds to hyperbolic cone-structures on M. The cone-angle α_i is just the imaginary part of $\mathcal{L}_i$.

Corollary 6.50 (Local rigidity) *Let C be a hyperbolic cone-manifold with cone-angles $\leq \pi$. Then the set of cone-angles $\{\alpha_1, \ldots, \alpha_N\}$, where $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$, provides a local parametrization of the space of hyperbolic cone-structures near the given structure on M. In particular, there are no deformations leaving the cone-angles fixed.*

6.7.2 The spherical case

Lemma 6.51 [BLP2], [Por, Lemma 9.1] *Let* $\mathrm{hol} : \pi_1 M \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ *be the holonomy of a spherical cone-structure. Then* hol_1 *and* hol_2 *are both non-abelian, unless Σ is a link and M is Seifert fibered.*

Proof. For convenience of the reader we give a sketch of proof. Let us assume that hol_1 is abelian. Then we may assume that the holonomy is contained in $S^1 \times \mathrm{SU}(2)$. This means that the Hopf-fibration on $\mathbf{S}^3 \subset \mathbb{H}$ obtained by left-multiplication with $S^1 \subset \mathbb{H}$ is preserved by the holonomy and may be pulled back via the developing map to a Seifert fibration on M. If hol_2 is abelian, then the Hopf-fibration obtained by right-multiplication with $S^1 \subset \mathbb{H}$ will be invariant under the holonomy, and the same argument applies. In both cases the singular locus Σ has to be a link, since in the presence of vertices hol_1 and hol_2 are clearly irreducible. ◻

Lemma 6.52 *Let* $\mathrm{hol}_i : \pi_1 M \to \mathrm{SU}(2)$ *be a component of the holonomy of a spherical cone-structure with cone-angles* $\leq \pi$. *If* M *is not Seifert fibered, then* hol_i *is a smooth point of* $R(\pi_1 M, \mathrm{SU}(2))$. *The* $\mathbb{R}$*-dimension of* $R(\pi_1 M, \mathrm{SU}(2))$ *around* hol_i *equals* $\tau + 3 - \frac{3}{2}\chi(\partial M_\varepsilon)$, *where* τ *is the number of torus components in* ∂M_ε. $T_{\mathrm{hol}_i} R(\pi_1 M, \mathrm{SU}(2))$ *may be identified with* $Z^1(\pi_1 M, \mathfrak{su}(2))$.

Proof. The arguments in the hyperbolic case apply directly, the $\mathbb{R}$-dimensions of the $\mathfrak{su}(2)$-cocycle spaces are equal to the $\mathbb{C}$-dimensions of the corresponding $\mathfrak{sl}_2(\mathbb{C})$-cocycle spaces. ◻

Corollary 6.53 $\chi_i = [\mathrm{hol}_i]$ *is a smooth point of* $X(\pi_1 M, \mathrm{SU}(2))$. *The* $\mathbb{R}$*-dimension of* $X(\pi_1 M, \mathrm{SU}(2))$ *around* χ_i *equals* $\tau - \frac{3}{2}\chi(\partial M_\varepsilon)$, *where* τ *is the number of torus components in* ∂M_ε. $T_{\chi_i} X(\pi_1 M, \mathrm{SU}(2))$ *may be identified with* $H^1(\pi_1 M, \mathfrak{su}(2))$.

Proof. The action of $\mathrm{SU}(2)$ on $R(\pi_1 M, \mathrm{SU}(2))$ is proper since $SU(2)$ is a compact group. Since hol_i is non-abelian by Lemma **(6.51)**, we have that $Z(\mathrm{hol}_i(\pi_1 M)) = \{\pm 1\}$. Now the result follows as in the surface case. ◻

The main result in the spherical case is the following theorem:

Theorem 6.54 *Let* C *be a spherical cone-manifold with cone-angles* $\leq \pi$, *which is not Seifert fibered. Let* $\{\mu_i, \dots, \mu_N\}$ *be the family of meridians, where* $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$. *Then the map*

$$X(\pi_1 M, \mathrm{SU}(2)) \to \mathbb{R}^N, \chi_i \mapsto (t_{\mu_1}(\chi_i), \dots, t_{\mu_N}(\chi_i))$$

is a local diffeomorphism near $\chi_i = [\mathrm{hol}_i]$ *for* $i \in \{1, 2\}$.

Proof. The proof proceeds exactly along the same lines as in the hyperbolic case. The level-sets of the trace-functions

$$V_i = \{t_{\mu_1} \equiv t_{\mu_1}(\chi_i), \dots, t_{\mu_N} \equiv t_{\mu_N}(\chi_i)\}$$

are smooth, half-dimensional submanifolds of $X(\pi_1 M_\varepsilon, \mathrm{SU}(2))$ near χ_i for $i \in \{1, 2\}$. The classes of the differential forms

$$\{\omega^1_{tws} + \omega^1_{len}, \dots, \omega^N_{tws} + \omega^N_{len}\}$$

provide a basis for $T_{\chi_1} V_1$, while the classes of the forms

$$\{\omega^1_{tws} - \omega^1_{len}, \dots, \omega^N_{tws} - \omega^N_{len}\}$$

provide a basis for $T_{\chi_1} V_1$. These forms are L^2-bounded on $U_\varepsilon(\Sigma)$. The same argument as in the hyperbolic case shows, that $T_{\chi_i} V_i$ and $\mathrm{im}(H^1(M, \mathcal{E}_i) \to H^1(\partial M_\varepsilon, \mathcal{E}_i))$ are transversal for $i \in \{1, 2\}$. It follows that the differentials $\{dt_{\mu_1}, \dots, dt_{\mu_N}\}$ are $\mathbb{R}$-linearly independent already in $H^1(\pi_1 M, \mathrm{SU}(2))^*$ at χ_i for $i \in \{1, 2\}$. ◻

Locally around hol the set of representations $\rho = (\rho_1, \rho_2)$ such that $t_{\mu_i}(\rho_1) = t_{\mu_i}(\rho_2)$ for all $i \in \{1, \dots, N\}$ corresponds to spherical cone-structures on M. The cone-angle α_i is related to the trace of the meridian via

$$t_{\mu_i}(\rho_1) = t_{\mu_i}(\rho_2) = \pm 2\cos(\alpha_i/2)\,.$$

Corollary 6.55 (Local rigidity) *Let* C *be a spherical cone-manifold with cone-angles* $\leq \pi$, *which is not Seifert fibered. Then the set of cone-angles* $\{\alpha_1, \dots, \alpha_N\}$, *where* $N = \tau - \frac{3}{2}\chi(\partial M_\varepsilon)$, *provides a local parametrization of the space of spherical cone-structures near the given structure on* M. *In particular, there are no deformations leaving the cone-angles fixed.*

6.7.3 The Euclidean case

The arguments given in the hyperbolic and the spherical case do not apply directly in the Euclidean case. Note that $H^1_{L^2}(M, \mathcal{E}_{trans})$ is always at least one-dimensional, since the identity transformation viewed as an element

$$\mathrm{id} \in \Gamma(M, T^*M \otimes TM) = \Omega^1(M, \mathcal{E}_{trans})$$

is certainly parallel and therefore contributes nontrivially to $H^1_{L^2}(M, \mathcal{E}_{trans})$. It is therefore subject to further investigation if smoothness of $R(\pi_1 M, \mathrm{SU}(2))$ near $\mathrm{rot} \circ \mathrm{hol}$ can be established and if a geometric rigidity result can be deduced from this.
Furthermore, it would be desirable to relate this to work of J. Porti (cf. [Por]) concerning the question of regeneration of Euclidean into hyperbolic or spherical structures.

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