Categorizing Motion: Story-Based Categorizations

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This doctoral thesis contains original research by Juan Purcalla Arrufi in collaboration with his supervisor, Dr. Alexandra Kirsch, part of which has been already published in the following papers. In brackets, we indicate in which chapters of this work such research has been primarily included.

- Purcalla Arrufi and Kirsch (2017) (Chapters 7 and 8)
- Purcalla Arrufi and Kirsch (2018a) (Chapters 7, 8 and 11)
- Purcalla Arrufi and Kirsch (2018b) (Chapter 10)
- Papenmeier, Purcalla Arrufi, and Kirsch (2023); Collaboration with Dr. Frank Papenmeier. (Chapter 9)

Whenever I use or refer to other scientists' results, I indicate it with the corresponding citation to the best of my knowledge. In case that, inadvertendly, someone might be miscited or due citation might be omited, please contact me, ORCID Nr. 0000-0002-3307-655X, for correction.

Abstract

Our main contribution is a method that generates categorizations of motion (Ch. 8)—We call such motion categorizations story-based categorizations. More specifically, a story-based categorization categorizes rigid entities that move independently of each other (Sect. 1.2); it categorizes an instantaneous description of the entities' motion, that is, a snapshot of the entities' trajectory.

Our method can generate an unlimited number of story-based motion categorizations. This is remarkable, considering the scarcity of motion categorizations in the literature (Sects. 5.1 and 5.2); in particular, we found only motion categorizations in the literature, but *no method* for systematically generating them. To demonstrate the efficacy of our method, we generated six story-based categorizations.

Another important contribution is to prove the relevant properties of the story-based motion categorizations: we investigated their properties from two complementary perspectives. From the standpoint of artificial intelligence, we proved that the story-based motion categorizations are also *qualitative calculi* (Ch. 10), which means that they have very convenient mathematical properties for solving navigation problems. From the standpoint of the psychology, we tested two story-based categorizations for cognitive plausibility, and we confirmed experimentally that they are cognitively plausible to varying degrees (Ch. 9). Our experimental tests of cognitive plausibility are groundbreaking, because we found no experimental tests on cognitive plausibility for qualitative calculi of motion in the literature, and only a reduced number for spatial qualitative calculi.

A further contribution is that we provided a solid theoretical framework for story-based categorizations: We defined mathematically the main concepts of story-based categorizations with clarity and simplicity. And we linked, or rather embedded, our framework in the vast and highly heterogeneous literature on categorization (Ch. 6). Accordingly, we related the understanding and terminology of categorization in psychology to those in artificial intelligence, which is unusual in the literature—Researchers tend to treat categorization exclusively from either a psychological or an artificial intelligence perspective.
This could not be possible without a survey of the research on categorization. We present, as a chapter (Ch. 3), such a survey, certainly incomplete, which we believe can save many hours of literature browsing—We deem our survey to be an additional contribution, because of its comprehensibility and scope.

Our achievements are far from obvious in the light of the challenges that categorization, and especially motion categorization poses (Ch. 2). We made several additional contributions which we list in Section 1.3. Last but not least, I intended that you enjoy your reading: Have a blessed journey on the quest for motion categorization!

Kurzfassung

(German abstract)

Unser Hauptbeitrag ist eine Methode zur Erstellung von Bewegungskategorisierungen (Kap. 8). Wir nennen solche Bewegungskategorisierungen story-based Kategorisierungen (story-based categorizations). Konkret kategorisiert eine story-based Kategorisierung starre Entitäten, die sich unabhängig voneinander bewegen (Abs. 1.2); sie kategorisiert eine augenblickliche Beschreibung der Bewegung der Entitäten, d.h. eine Momentaufnahme der Trajektorie der Entitäten.

Unsere Methode kann eine unbegrenzte Anzahl von story-based Bewegungskategorisierungen erstellen. Dies ist bemerkenswert, wenn man bedenkt, wie wenige Bewegungskategorisierungen in der Literatur zu finden sind (Abs. 5.1 und 5.2). Insbesondere haben wir dort nur Bewegungskategorisierungen gefunden, aber keine *Methode*, um sie systematisch zu erstellen. Um die Effektivität unserer Methode zu zeigen, haben wir sechs story-based Kategorisierungen erstellt.

Ein wichtiger weiterer Beitrag besteht darin, die relevanten Eigenschaften der story-based Bewegungskategorisierungen zu beweisen: Wir haben ihre Eigenschaften von zwei komplementären Standpunkten aus untersucht. Vom Standpunkt der künstlichen Intelligenz aus haben wir bewiesen, dass die story-based Bewegungskategorisierungen auch *'qualitative calculi'* sind (Kap. 10), was bedeutet, dass sie sehr geeignete mathematische Eigenschaften zur Lösung von Navigationsproblemen haben. Vom Standpunkt der Psychologie aus haben wir zwei story-based Kategorisierungen auf kognitive Plausibilität getestet, und wir haben experimentell bestätigt, dass sie in unterschiedlichem Maße kognitiv plausibel sind (Kap. 9). Unsere experimentellen Tests zur kognitiven Plausibilität sind bahnbrechend, da wir in der Literatur keine experimentellen Tests zur kognitiven Plausibilität für qualitative calculi für Bewegung gefunden haben, und nur eine reduzierte Anzahl für qualitative calculi für Raum.

Ein weiterer Beitrag ist, dass wir die story-based Kategorisierung mit einem soliden theoretischen Rahmen ausgestattet haben: Wir haben die Hauptkonzepte der story-based Kategorisierung mathematisch klar und einfach definiert. Und wir haben unseren Rahmen in die umfangreiche und sehr heterogene Literatur zur Kategorisierung eingebettet (Kap. 6). Dementsprechend haben wir das Verständnis und die Terminologie der Kategorisierung in der Psychologie mit denen in der künstlichen Intelligenz in Beziehung gesetzt, was in der Literatur ungewöhnlich ist – Forscher neigen dazu, sich mit Kategorisierung ausschließlich entweder aus der Perspektive der Psychologie oder der künstlichen Intelligenz zu beschäftigen. Dies wäre nicht möglich, ohne eine Übersicht über die Forschung zur Kategorisierung zu geben. Wir haben als Kapitel (Kap. 3) eine solche, sicherlich unvollständige Übersicht vorgelegt, von der wir glauben, dass sie viele Stunden der Literatursuche ersparen kann – Wir halten unsere Übersicht wegen seiner Verständlichkeit und seines Umfangs für einen Beitrag.

Angesichts der Herausforderungen, die die Kategorisierung und insbesondere die Bewegungskategorisierung mit sich bringt (Kap. 2), sind unsere Leistungen alles andere als selbstverständlich. Wir haben mehrere zusätzliche Beiträge geleistet, die wir in Abschnitt 1.3 auflisten. Zudem habe ich mich bemüht, Ihnen eine unterhaltsame Lektüre zu bereiten: Ich wünsche Ihnen eine gesegnete Reise auf der Suche nach der Bewegungskategorisierung!

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My supervisor, Dr. Alexandra Kirsch. She has mentored my research from the very beginning. In fact, she was the one who hired me for the two-years Ph.D. position which ended up in this dissertation. Here, I want to explicitly acknowledge, that she gave a full-paid position to all her Ph.D. students, which, sadly, it is not the norm in our academical landscape. I appreciate her feedback and correction; they have been instrumental for me to clearly highlight my research contributions and to achieve a mature argumentative balance. I extraordinarily appreciate her forbearance throughout these over eight years of supervision: she was always available for questions. Thank you, Alex. Bless your heart!

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My wife, Irene. She has proved to be my God-given wife throughout this last years of doctorate. Four years ago, she married a man who was “about to finish” the doctorate, but this prospect delayed beyond any imagination. Despite this initial disappointment, her love, encouragement, and patience grew day by day, steadily, to the point that she sacrificed herself for me to finish the doctorate: seeing that I was so excruciatingly slow writing my dissertation due to my part-time jobs, she compelled me to devote myself fully to the writing, and she provided with her job for the household. Thus, I could fully devote to the writing for over one year, and finish it. For that reason, she deserves to be acknowledged as co-author of this dissertation. Irene, I am humbled by your love; I hope I get to show you such a great love.

My colleagues, Alisa V., Roman R., Nataliia K., and Stefan B. They made my long office hours enjoyable, and I learned a lot in our scientific-technical discussions. A special mention for Alisa, who often shared with me literature, links, and any kind of information which could be a blessing to me—That pushed me forward. Thanks!

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My parents-in-law, Peter and Olga, which I unconditionally love.

Мои свекры, Петр и Ольга, которых я безукоризненно люблю.

Finally, to anyone that reads this work: May God abundantly bless you!

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Part I

Motivation, Problem Definition, and Novel Contributions

Chapter 1

Introduction

1.1 The Story Behind

This work originated in a research team of human-computer interaction—Jun. Prof. Alexandra Kirsch's team at the University of Tübingen—More concretely, I began this research to meet the needs in human-aware navigation. As Kruse et al. (Concl., 2013) remark in their well-known survey, all of about 120 surveyed papers dealt with "individual domains and challenges, [but] a holistic theory of human-aware navigation was not attempted yet." As in all those papers, in the very beginning of my research, I was also concerned with a very specific case of human-aware navigation, namely, I was trying to implement human-aware crossing situations between humans and robots.

Along this endeavour, I realized that a first step in reacting to a crossing situation could be the assessment, i.e., the classification or categorization, of that crossing situation. That lead me to a search for categorizations of two moving entities and I encountered the promising 'qualitative representations of motion', such as QTC (Van de Weghe 2004; e.g., Bellotto 2012; Lichtenthäler et al. 2013), but I found them wanting: In the crossing paradigm it became evident to me that motion situations with the same QTC category might evolve differently; that is, QTC did not reliably inform about the future position states of a current motion situation. Such a categorization is cognitively disputable, because information is encoded in our mind mainly in order to anticipate events, so that our behaviour can be decided (Butz and Kutter (2017, Sect. 10.2.1); Sect. 11.7.4.B); if a motion categorization loses relevant information about the future position states, great are the chances that such categorization is cognitively deficient.

Hence, I reoriented my research to develop categorizations that classify motion situations (what in this work we call *motion scenarios*) according to the expected future position states. In other words, I sought the answer to the question: How can we categorize motion situations, so that each category corresponds to a specific spatial state in the future? As a base for such categorizations I took the *qualitative spatial representations*, because they were successfully researched and applied in Artificial Intelligence (AI) for decades (Chen et al. 2015). This endeavour brought fruit forth: the contributions mentioned in Section 1.3. Amongst others, we generated two motion categorizations that are attractive both from the perspective of AI and from the perspective of human cognition; thus, they are fit for implementing human-aware navigation.

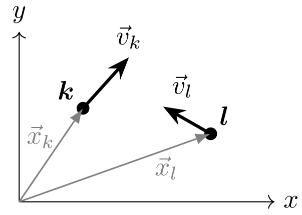


Figure 1.1: A *motion scenario* of two entities k and l in two dimensions. The scenario is characterized by their instantaneous positions, $\vec{x}_k$ and $\vec{x}_l$, and velocities, $\vec{v}_k$ and $\vec{v}_l$.

1.2 What We Categorize: *Motion Scenarios*

In this work, we categorize '*motion scenarios*'; these are like 'snapshots' of moving entities. The motion scenarios describe a single instant of moving entities using the *instantaneous values* of position and velocity (Fig. 1.1). The standard motion scenario contains two entities, k and l; it is described by four real vector values: $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$. The vectors $(\vec{x}_k, \vec{x}_l)$ describe the positions of the entities; the vectors $(\vec{v}_k, \vec{v}_l)$ describe their velocities. We often need extra positional information (e.g., orientations, sizes) that is relevant for categorization; such information is provided by additional parameters.

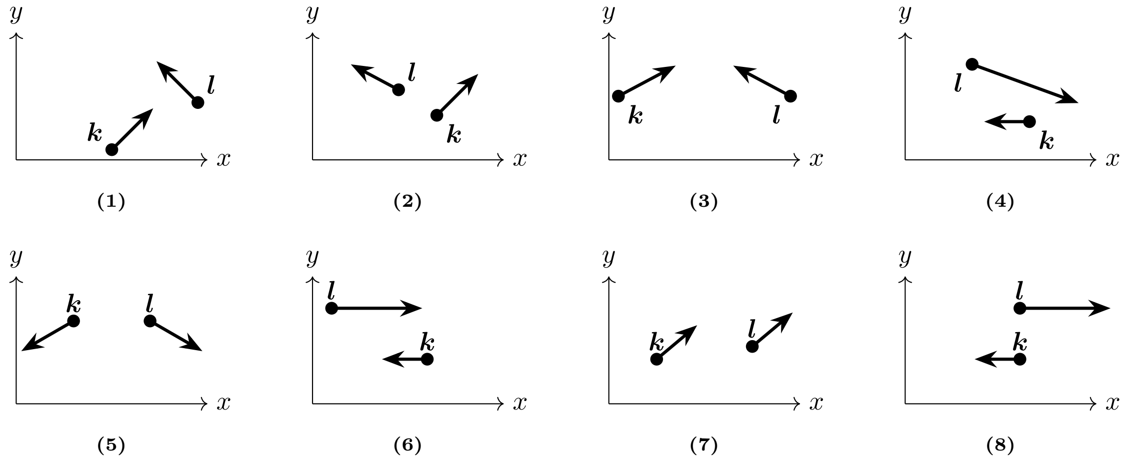


Figure 1.2: A categorization problem for motion scenarios (the discs represent the entities and the vectors represent the velocities): How should we group such motion scenarios according to their similarity?

Note that as we concentrate on the categorization of motion scenarios, we implicitly limit the influence of acceleration in our system. Truly, if the entities should experience extreme accelerations, then the changes in the scenario's configuration would be so abrupt that the categorization of scenarios would seem pointless. In fact, we posit that it would be an impossible task to obtain a motion categorization without any constraints on the acceleration.

About the number of entities and dimensions

We focus on two entities because the categorization of two entities' motion can be generalized to three or more entities by considering the pairs of entities in a scenario; we illustrate this generalization method by categorizing the motion of three entities in Section 11.6. Moreover, our categorization method is based on qualitative representations (Ch. 4) which are devised mostly for two entities—In Dylla et al. (2017, Tab. II), we find that only about 15% of the representations are intended for three entities. As a final reason, the properties of qualitative representations are studied only for two entities; the formalization of three entities relations is inexhaustive (ibid., p. 6).

Although we develop our theory and methods in a general n-dimensional kinematic space, all our examples and illustrations—including our validation experiment (Ch. 9)—are in two dimensions for the sake of simplicity: both computations and visualization are easier in two dimensions. Nonetheless, in Section 11.2, we generate a three-dimensional motion categorization with our method in order to demonstrate its viability.

About The Entities

The entities of a motion scenario need not be punctual: they can be *regions*, but they cannot alter their shape, i.e., they can only be '*rigid regions*'. To describe the shape and size of the regions, we need additional parameters. For example, if entities are discs, we need their radii—That is the case of a disc story-based categorization obtained in this work: Stories-RCC (See Section 8.2.1).

In any case, entities can only translate, i.e., they cannot spin. Otherwise, we should add the *spin angular velocity* as an extra value to the motion scenario. Summarizing, we concentrate our research on *rigid non-spinning entities*.

We justify the separation of translation and spinning (or rotation) as a common practice in classical mechanics, and, more importantly, as an observed cognitive phenomenon. Indeed, infants can independently perceive the path of an object, and the manner this object moves along the path, for example, by spinning (Pulverman et al. 2008; Pruden et al. 2013). That is, the traditional separation of translation and rotation of a movement is more than a practical mathematical device, but a cognitive feature present from infancy on.

Categorizing Trajectories from Motion Scenarios

Motion scenarios have an elementary role in motion description. The trajectories of n entities in time, $\{\vec{x}_{k_1}(t), \vec{x}_{k_2}(t), \ldots, \vec{x}_{k_n}(t)\}$, can be seen as a continuous sequence of motion scenarios $\{\vec{x}_{k_1}, \vec{v}_{k_1}; \vec{x}_{k_2}, \vec{v}_{k_2}; \ldots; \vec{x}_{k_n}, \vec{v}_{k_n}\}$ for every instant t. For that reason, we can also describe trajectories qualitatively by means of a categorization of motion scenarios, when we apply the categorization at every instant of the trajectory. We provide the method in Section 11.4.

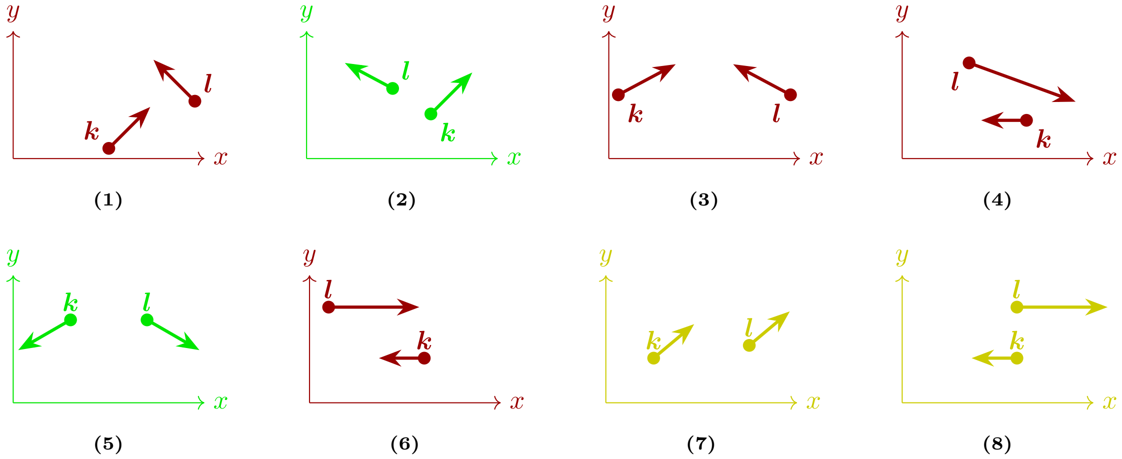
1.2.1 A Basic Example of Categorization for Motion Scenarios

In Chapter 6, we flesh out a detailed categorization model, which we also apply to motion scenarios. Here, nonetheless, we sketch motion categorization in a basic example, to introduce the reader to the subject. To categorize a *motion scenario* $K = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$, means that we label it with a category M_i. We choose the category from a pool of categories, the '*categories set*' $\mathcal{M}$; which is *finite*, and contains m categories, $\mathcal{M} = \{M_1, M_2, \ldots, M_m\}$.

As an illustration, we present here a most simple motion categorization for scenarios: Gaping. The motion categorization Gaping considers how the distance between the entities instantly changes, i.e., the sign of $\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{x}_l - \vec{x}_k\|$. It consists of three possible categories $\mathcal{M} = \{M_1, M_2, M_3\}$, also called $\{(-), (0), (+)\}$. A motion scenario belongs to a certain category according to following rules:

CATEGORY NAME	DESCRIPTION	
M_1 or $(-)$	The distance between entity k and l decreases $(\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{x}_l - \vec{x}_k\| < 0)$	(1.1a)
M_2 or (0)	The distance between entity k and l remains constant $(\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{x}_l - \vec{x}_k\| = 0)$	(1.1b)
M_3 or $(+)$	The distance between entity k and l increases $(\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{x}_l - \vec{x}_k\| > 0)$	(1.1c)

We can now categorize the scenarios in Figure 1.2 according to the Gaping motion categorization (Eq. (1.1)). The scenarios (1), (3), (4), and (6), belong to category M_1, i.e., the entities come closer; the scenarios (7) and (8) belong to category M_2, i.e., the entities keep the distance to each other; and the scenarios (2) and (5) belong to category M_3, i.e., the distance between the entities increases.



(1) (2) (3) (4)

(5) (6) (7) (8)

Figure 1.3: The motion scenarios are coloured according to the Gaping category they belong to: M_1 🟥 (approaching), M_2 🟨 (stable), M_3 🟩 (distancing).

Later, as we formalize motion categorization, we thoroughly present some well-known simple motion categorizations for motion scenarios (e.g., $\mathrm{QTC}_{\mathrm{B21}}$ and $\mathrm{QTC}_{\mathrm{B22}}$ in Sections 5.4 and 6.3).

1.3 Our Contributions

In this section, we listed our contributions in greater detail than in the Abstract in order to rend our contribution clear and distinguishable from the results of other researchers. Otherwise, our contribution might be difficult to recognize since we presented our research as a monograph integrating our results with the current state-of-the-art in related scientific disciplines—That is so because we prioritized comprehensibility and readability.

The contributions presented here all stem from the initial pursuit of a categorization of motion scenarios (as exemplified in Section 1.2.1), a categorization that we sought to be cognitively plausible.

Primary contributions (Parts III and IV)

- We provide a method for generating motion categorizations from spatial and motion representations. We call these generated motion categorizations the *story-based categorizations* (Ch. 8). This is a remarkable contribution for the following reasons:
  - In the literature, we find no method for generating motion categorizations. Note that our method is described by means of complete algorithms that generate concrete motion categorizations (e.g., Sect. 8.1.1), not just principles, guidelines, or ambiguous steps. We call these generated motion categorizations '*story-based categorizations*'.
  - We proved mathematically that, in most cases (possibly all), the story-based categorizations are also *qualitative calculi.* They have powerful operations and tools that allow reasoning and decision-making (Ch. 10), while a simple categorization (such as `fruits` or `furniture`) has only the basic set operations.
    - One of the key reasoning operations is the 'composition'. We provided a method, called '*narrative composition*', to help compute the composition in the story-based categorizations (Sect. 10.5). Even though this method does not exactly provide the composition, it considerably narrows down the choices leading to the precise result.

      Notice that not all qualitative calculi in the literature address the composition; this is also the case in qualitative motion calculi. We stress that our method helps to compute the composition of *any* story-based representation: bear in mind that story-based representations can be very different, and the method has a general validity.
  - In principle, our method generates an unlimited number of motion categorizations with a wide variety of properties (Sect. 11.1). This is an exceptional strength of our method, given the scarcity of qualitative calculi of motion. The number and variety arises from several factors:
    - Each story-based categorization is generated from a spatial qualitative calculi (i.e., a spatial categorization). In the literature, we have a large number of such spatial qualitative calculi, even families of calculi with infinite members (Sect. 11.1.1).
    - We combine the story-based categorizations by means of scalar product to create more story-based categorizations (Sect. 11.1.2).
    - Thanks to our framework for categorizations, we obtain the features of each story-based categorization and combine them to create more motion categorizations (Sect. 11.1.3).

      Furthermore, in this case, we can create *purpose-tailored* motion categorization by specifically choosing the features relevant to our purpose.
  - We generated two types of story-based categorizations. First, the *bare* story-based categorizations, which are the simplest variant, and are denoted Stories-$\mathcal{R}$ (Sect. 8.1). Second, the *beaded* story-based categorizations, denoted Motion-$\mathcal{R}$; they are a refinement of the *bare* categories in which the temporal precedence (past and future) is distinguished within a *bare* category (Sect. 8.5).
  - We did not only accurately described the method for obtaining the motion categorizations, but we obtained some concrete, ready-to-use, categorizations: We obtained both bare and beaded categorizations for the spatial representations RCC and OPRA_1, and the bare categorization for the motion representation $\mathrm{QTC}_{\mathrm{B21}}$. That is, we obtain

Stories-RCC (Sect. 8.2.1), Motion-RCC (Sect. 8.6.1), Stories-OPRA$_1$ (Sect. 8.2.2), Motion-OPRA$_1$ (Sect. 8.6.2), Stories-QTC$_{B21}$ (Sect. 8.3.1).

Thus, we proved that our generation method is effective, and we enriched the meagre panorama of qualitative calculi with these five ready-to-use qualitative motion representations.

- We provided experimental evidence that the story-based categorizations Stories-RCC and Stories-OPRA$_1$ are *cognitively plausible* (Ch. 9). Our experimental tests of cognitive plausibility are groundbreaking, because we found no experimental tests on cognitive plausibility for qualitative calculi of motion in the literature, and only a reduced number for spatial qualitative calculi.

  Concerning the results, we generally established that story-based categorizations significantly influenced the similarity choice of motion scenes. In that sense, we can affirm that the story-based categorization in our experiment, Motion-RCC and Motion-OPRA$_1$, are cognitively plausible—At least in a weak manner.

Additional contributions (Part II)

- We presented comprehensive surveys in general categorization (Ch. 3), in spatial categorization (Ch. 4), and in motion categorization (Ch. 5).
  - We related the concepts of general categorization to the particular case of spatial and motion categorization; specifically, we looked at qualitative representations from the perspective of human cognition, illustrating basic psychological and linguistic aspects.
- We worked out an elementary model of categorization that allows us to formally deal with categorizations and relate them to their description in psychology Sects. 6.1 and 6.2. The model can reproduce all the basic psychological effects of gradation by means of a featural distance upon which a similarity can be defined. Moreover, this model is designed with the qualitative representations in mind, so that we can identify their main constituents: categories, features, categorical regions; and we can obtain their important functions: feature extraction, and feature-based categorization.
  - We applied our categorization model to the spatial categorizations RCC (Sect. 8.2.1.A) and OPRA$_1$ (Sect. 8.2.2.A), to the motion categorization QTC$_B$ (Sect. 6.3), and to the story-based categorizations Stories-RCC (Sect. 8.2.1.C) and Stories-OPRA$_1$(Sect. 8.2.2.C). By doing so, we obtained their feature extraction function, along with their features, and their featural categorization function.

Clarifying our Contribution Area

In this research converge two capital topics in psychology and artificial intelligence: *categorization* and *motion*; we research the blurred boundary between engineering and science. This is an interdisciplinary endeavour with methodological tensions between disciplines (see Sect. 2.4). For this reason, we clarify our contribution: This work stands at the overlap of the two major goals of cognition: *constructive* and *explanatory* (Gärdenfors 2004, Sect. 1.1.1). First, as a *constructive* work, we offer motion categorizations that can be used in artificial intelligence, and engineering, e.g., robotics and navigation. Second, as an *explanatory* work, we investigate the cognitive plausibility of our categorizations, and test them experimentally.

1.4 Motivation: Our Topics in the Landscape of Science

It is not easy to understand the relevance of the contributions in this work without understanding the relevance of each topic—*categorization* and *motion*—on its own. Their relevance is enormous, as we justify below. Furthermore, by bringing these topics together, our research has a surplus value: indeed, *motion* is an infrequent categorization theme. In psychology, for instance, Medin and Heit (1999, Sect. I.C) observe that categorization research is predominantly concerned with natural objects and artificially created categories, as opposed to categorizing events or motion. Likewise, in AI, we found an overwhelming majority of *spatial* versus *motion* categorizations (See Section 5.1). We help correct this deficiency by providing the story-based categorizations, which are both a motion categorization with cognitive plausibility in psychology (Ch. 9), and a qualitative representation of motion with reasoning capabilities in AI (Ch. 10).

1.4.1 Relevance of Categorization

Concerning the relevance of *categorization* in cognition, Cohen and Lefebvre (2005, p. 2) assert "[Categorization] is the basis for the construction of our knowledge of the world. It is the most basic phenomenon of cognition, and consequently the most fundamental problem of cognitive science.". Harnad puts it more bluntly: "cognition is categorization: to cognize is to categorize" (Harnad 2017).

Similarly, some decades before, pioneers in categorization research, Eleanor. Rosch and Lloyd (1978, Intr., p.1), stated "categorization is important", and they specified "[It is] a basic task of all organisms (indeed, one mark of living things)". Thus, we observe an augmenting emphasis in the relevance of categorization in cognition throughout the years. Also, in artificial intelligence we find similar claims: "The organization of objects into categories is a vital part of knowledge representation" (Russell and Norvig 2014c, Sect. 2).

The reasons for that relevance are clear: categories allow an agent not to perceive every object or event as unique, but to group them according to their use or effect (e.g., Bruner et al. 1956, p. 245; Gerrig and Zimbardo 2005, pp. 229f.). For example, when we look at a bunch of grapes, we are not overwhelmed by fifty different objects dangling in it; instead, we essentially overlook the differences between each dangling object, and see fifty objects of the same kind: 'grape'—they fall into the same category. Thus, by categorizing, we save cognitive resources—'*cognitive economy*' (Eleanor. Rosch 1978, p. 28)—, we can store the bunch of fifty different objects (each with its own features) in a more parsimonious representation, i.e., fifty times an object of one kind, i.e., "fifty times grape".

Beyond *cognitive economy*, categories are essential for *reasoning* and *decision-making* (Russell and Norvig 2014c, Sect. 2 and 5; Medin and Heit 1999, Sect. II.A.1). For instance, if an agent intends to produce wine, it could apply the rule 'we can produce wine, if, and only if we have grapes' to plan the next action: 'look for grapes'. Remarkably, the execution of such action depends fully on the ability to cope with the category `grape`.

Summarizing, categorizations are an asset in artificial intelligence. Indeed, any sensor, either robotic or human, is inundated by data with no direct meaning in itself, but its numerical value. This requires both *simplification*—reducing the amount and the degree of detail of the data—, and *conceptualization*—endowing data with a more straightforward meaning. As we have seen, a meaningful categorization provides both in one stroke.

The type of categorizations we introduce in this work, the story-based, not only simplify, the raw motion data a sensor or tracking device provides—as any motion categorization does (e.g., QTC, Van De Weghe et al. 2005; QRPC, F. J. Glez-Cabrera et al. 2013), but, more importantly, they produce meaningful categories which allow for reasoning and decision-making about moving entities (Sect. 11.5). We also show how story-based categories reflect the inner structure of motions: categories correspond to relevant kinematic attributes, the *featural variables* (e.g., Sects. 8.2.1.C and 8.2.2.C). Reflecting the relevant attributes is for Eleanor. Rosch (1978, p. 28) one of the two main principles characterizing categorizations: “categories map the perceived world structure as closely as possible. [. . .] by the mapping of categories to given attribute structures”.

A Minimal History of Categorization

The contemporary history of categorization helps us appreciate the current relevance of this discipline. It was as recently as the 1970’s, when the millennia-long standing view on concepts was successfully challenged (See Medin and Heit 1999, Brief History; detailed acccount Murphy 2002, Ch. 2): the view that *definitions* are the appropriate way to characterize categories and, therefore, categories have perfectly delimited borders. This view had been implicit in all categorization work since, at least, Aristotle (See transl., Apostle 1980, pp. 6, 19–20).

It was not a single person but many scientists with a variety of contributions who made the *classical view* crumble. To name a few ones, A. J. Wilkins (1971); E. E. Smith, E. J. Shoben, and L. J. Rips (1973; 1974); and M. E. McCloskey and S. Glucksberg (1978). One of the most active and, thus, renowned contributors is Eleanor. Rosch (e.g., 1973; 1975; 1978) together with her research partner C. B. Mervis (1976) .

Wrapping up, as Murphy (2002, Ch. 1) states about the study of concepts (i.e., categorizations): “a topic that seemed straightforward in 1960 has turned out to be a much deeper and richer scientific problem than researchers expected”, so that in the current state of the art “no theory has explanation for all findings *even* within a topic”, and, what is more, neither the basic questions have been fully answered. We are confident that this work will shed more light on the topic, as well, as rise more interesting research questions.

A Note on the Classical View

We carefully avoid the classical model when we model the story-based categorizations: we make membership gradation available (Ch. 6). It might seem then contradictory, that our simplified version of the membership function (also called ‘categorization rule’, Eq. (6.1)) does not reflect a graded membership. As said, it is a *simplified* membership function; nevertheless, gradation is readily available in the ‘*featural space*’—the metric space underlying the membership function—in which we can define a similarity function based on a featural distance. Hence, when needed, we can obtain a graded membership function for story-based categorizations by means of the featural distance. In fact, we do apply the gradation properties of story-based categorization when we experimentally test the cognitive plausibility of these categorizations: we generate comparison stimuli equally similar to a reference stimulus (Sect. 9.3.3). However, we believe that implementing a graded membership function of the story-based categorizations provides no added value to our theoretical analysis; it makes mostly sense when modelling experimental results on category membership. That is why we leave it as a future practical enhancement.

1.4.2 Relevance of Motion

Concerning the relevance of *motion*, we observe, in the last decades, a marked increase in motion related research in artificial intelligence and neighbouring disciplines. Arguably, the explosion of research in autonomous vehicles has brought the major focus on motion analysis. Above all, the integration of autonomous agents in day-to-day life, e.g., robots or vehicles, heightens the need for a formalization of the entities motion (Kurata and Shi 2008a)—most notably in machine-human interaction. Not least, the notorious rise in the ability to detect and record motion has ushered a new era of motion analysis (Delafontaine et al. 2011); indeed, not only tracking devices, but the locating possibilities of smartphones and their societal pervasiveness, furnishes a massive amount of position data, and, hence, motion data that can be exploited (e.g., Roor 2018)

In this sense, a cognitively plausible motion categorization, such as, the story-based categorizations, is an asset when processing motion data: they simplify kinematic floating point data into a reduced finite set of concepts that largely reflect a human understanding of the moving system, so we can more straightforwardly implement human-like navigation rules; in addition, they lessen the overhead of floating point computations.

The understanding of motion is also a fundamental ingredient of human cognition. First of all, motion attracts infants attention earlier (from birth) and more powerfully than most other factors (Haith 1980; Fantz and Nevis 1967). At 5 months age, infants remember much better the actions in events than the objects, people, or other elements involved (Bahrick et al. 2002; Perone et al. 2008). Consequently, motion related concepts are amongst the first to be acquired (Mandler 2012); Mandler uses the early concept 'furniture' as a vivid example: "[Furniture] may mean [for the child] no more than 'things that don't move'".

Motion has even a more profound role both in artificial intelligence and cognitive sciences. On the one hand, regarding artificial intelligence, we have two different approaches: the *mental* and the *behavioural* one, according to Russell and Norvig's classification (Russell and Norvig 2014b, Sec. 1). The behavioural approach concentrates on issues such as perception, action, and manipulation in which motion is an essential component. Practical examples are, classification of video images, navigation in dynamical environments or human-aware navigation. On the other hand, regarding cognitive science, we find behavioural approaches, often termed '*embedded*'. There, the *motor* system plays a basic role in cognition through its coupling to mental and perceptual systems. In that way *motion* becomes integral part of cognitive research. Furthermore, we notice that in both artificial intelligence and cognitive science a shift from mental to behavioural paradigm; thus, motion seems to gain relevance.

Chapter 2

Challenges in Motion Categorization

Here are some of the most challenging problems that we face when categorizing motion—As you will appreciate, this is no banal research topic. Such challenges make motion categorization both more difficult and more exciting to treat than other categorization domains.

In the conclusion (Sect. 12.1), we comment on how we tackled or overcame such challenges.

2.1 High Dimensional States Space

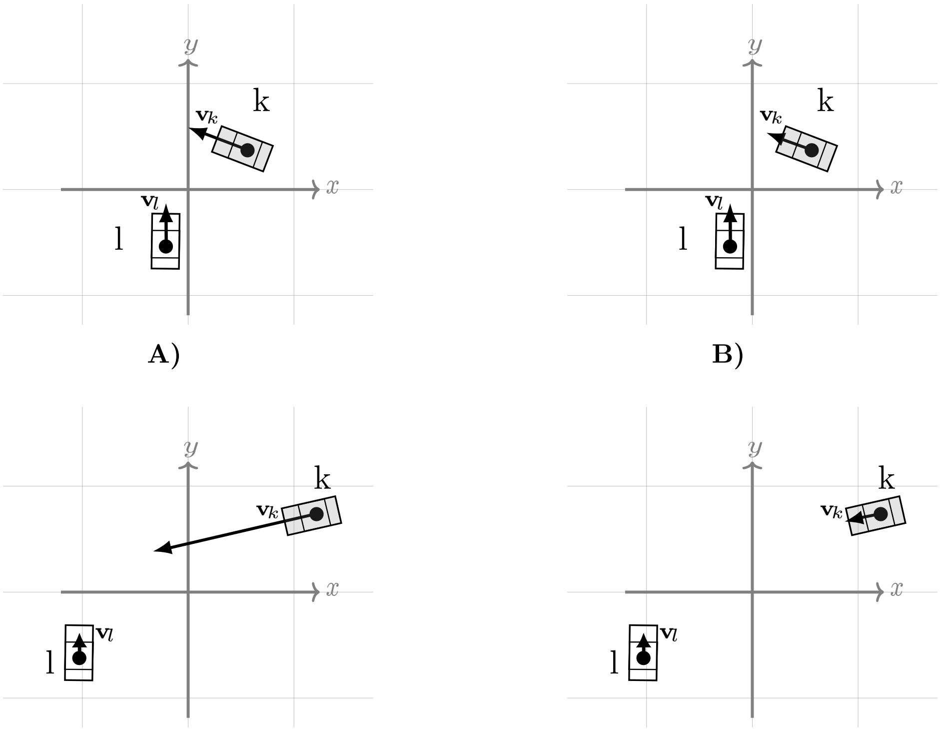
In the first place, the high number of dimensions that describe a motion scenario are a challenge for researchers in motion categorization—Two entities moving in a two-dimensional space are described as a point in $\mathbb{R}^8$, i.e., 2 entities × (2 position coordinates + 2 velocity coordinates); three entities in a three-dimensional space are described as a point in $\mathbb{R}^{18}$. For that reason, both mental and graphical representation of the space to categorize fail as a means to obtain categorizations.

We overcome this complexity by using the well-founded *spatial categorizations* (also called spatial *qualitative representations*, Hernández 1994) as the starting point of the categorization process. Out of them we obtain motion categorizations by applying a general method, the *story-based method*, that greatly reduces the mental demands of the researcher.

2.2 Variety of Categorization Criteria

A further challenge of categorizing motion is the variety of possible categorizations; we illustrate it in Figure 2.1. Which attributes should we use—and how—in order to categorize the 4 motion scenarios A, B, C, D, in this figure? For example, the pair of scenarios (A, B) and (C, D) are almost identical, and differ only in the speed of k; consequently, each pair could become a category in its own right. However—assuming velocities remain constant—in both scenarios A and C the vehicle k would cross before l without colliding, while in scenario B the vehicles collide, and in D the vehicle k crosses behind l.

Are, then, the three categories (A, C), (B), and (D) of scenarios in Figure 2.1 more meaningful than the two previous, (A, B) and (C, D)? In theory, one cannot establish an absolute scale of meaningfulness for motion categorizations. The meaningfulness of each categorization depends on how useful the categorization is for the particular task of a given agent. Thus, one ends up with a multitude of categorizations, each most meaningfully serving a specific purpose (e.g., Sect. 3.4.4). It is true that, in our categorization experiment (Ch. 9), we found that motion



A) **B)**

C) **D)**

Figure 2.1: A challenge for motion categorization: 4 scenarios $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D})$ with two moving vehicles k and l, with velocity vectors $\vec{v}_k$ and $\vec{v}_l$. Each pair (A, B) and (C, D) has identical positions, velocity angles, and fulfils $\|\vec{v}_k\| > \|\vec{v}_l\|$—each pair differs only in the speed of k, i.e., in $\|\vec{v}_k\|$.

Source: Purcalla Arrufi and Kirsch (2018a) categorization Motion-OPRA$_1$ had a much higher saliency than Motion-RCC (Sect. 9.6), but the stimuli were not general enough to make an overall statement about the meaningfulness of both categorizations.

In any case, a model of motion categorization should address the need for a variety of categorizations: it should be flexible enough to generate such wide variety of categorizations. We posit that the variety of motion categorizations originates primarily in the variety of underlying spatial categorizations. As we see in Figure 2.1, we can differently categorize A and D, due to the spatial categories `before` and `behind`, and, again, we can categorize B as a single category, due to the spatial category `overlap`, i.e., collision. This is a compelling reason to use spatial categorizations as the basis of our method to generate categorizations of motion: spatial categorizations are a source of variety for the resulting motion categorizations.

In addition to spatial categorizations, another source of variety in motion categorization are the attributes of the moving entities. For example, if the entities have volume, there are many possible types of collisions between entities, which produce many types of motion categorizations—Think of the many ways a foot can hit a ball. But, if the entities are seen as punctual, there is only one possible type of collision: to be at the same point; in that case, another aspects determine motion categorization.

Notably, spatial attributes and spatial categorizations are coupled: the considered attributes of the moving entities determine—or are determined by—the spatial categorizations that we use. If the entities have volume, we will tend to use a spatial relation that considers volume, e.g., RCC (as in Wu et al. 2014); if the entities relative orientation is relevant, e.g., ships, we will use a categorization that considers orientation, e.g., OPRA$_\mathrm{m}$ (as in Dylla et al. 2007).

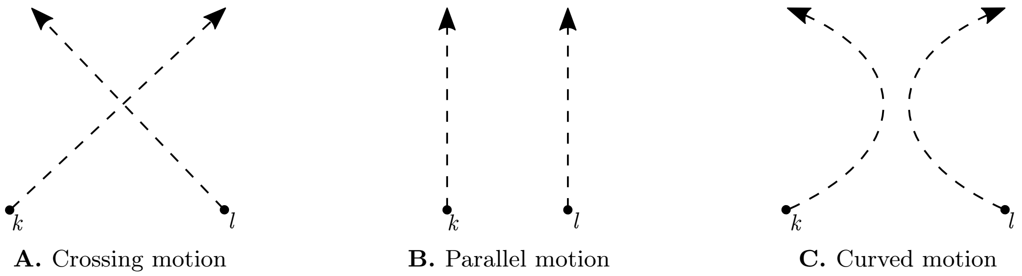
Since our method can use any spatial categorization to create motion representations, our method can create motion categorizations tailored to the attributes of the involved entities. In Section 11.1.1, we see how two story-based categorizations, Motion-RCC and Motion-OPRA$_1$, categorize the example in Figure 2.1.

2.3 Loose Cognitive Structure

Evidence has accumulated that though language influences cognitive tasks—and notably categorization—language actually reflects universals in cognitive structures (Kess 1992, Ch. 8). In Section 5.5, we infer a universal property concerning motion categorization from L. Talmy's (2000) work in cognitive linguistics, namely, that humans categorize motion quite loosely: we do not have a systematic cognitive structure for motion categorization, as we have it for spatial categorization.

The attempts to formalize motion categorization are hampered by its loose cognitive structure. For example, if we undertake to formalize motion categorization for motion scenarios, we would strive to *partition* the kinematic space of motion scenarios; in other words, we would like to assign a category—at least one—to any motion scenario. However, according to Talmy (2000a), humans do not linguistically partition the motion space, so that for most motion categorizations some regions in the motion space (i.e., some motions) remain linguistically uncategorized (See Fig. 2.2).

In contrast, spatial categorization is linguistically reflected by a partition of the space mediated by the prepositions, to each position can be assigned a category. For example, let us take a spatial categorization based on *proximity* of two entities, A and B, to an observer O. The categorization assigns three possible categories: A is *nearer* than B to the observer O, A is *farther* than B to O, A and B are *equidistant* to O. Whatever the positions of A and B relative to O are, we can always assign unambiguously a category: 'nearer', 'farther', or 'equidistant'—i.e.,



A. Crossing motion **B.** Parallel motion **C.** Curved motion

Figure 2.2: The first two motions, A and B, are linguistically categorized, respectively, as 'crossing' and 'parallel' motion. The last motion, C, is linguistically uncategorized; we use the generic term 'curved'.

the spatial categorization *partitions* the states space of the entities.

This is a first hurdle in the formalization of motion categorization: to overcome the loose structure provided by our own cognitive system in order to achieve a systematic motion categorization.

2.4 Disciplinary Fragmentation: Insularity and Tension

Topics in cognitive science, such as categorization, are prone to disciplinary fragmentation. '*Disciplinary fragmentation*' occurs when disciplines develop their own research on a topic separately or quite independently from each other, so that each discipline develops its own paradigm and methodology on that topic (See Dawson 2013, Ch. 1; See also Schwartz 1997, p. 25).

As we show below, disciplinary fragmentation is specially acute in spatio-temporal catagorization, and, consequently, in motion categorization.

Insularity The first and most dramatic effect of disciplinary fragmentation is that the other disciplines, being unaware of the results of a particular discipline, are unable to *verify*, *refine*, or *refute* such results. This is notably the case for qualitative representation and reasoning (QSR), particularly in the spatial domain.

Since the birth of qualitative representations—generally assumed to be J. F. Allen's (1983) intervals theory—AI researchers had been arguing that such representations capture the way "ordinary people conceptualize various aspects of the world around them" (Hobbs and Moore 1985, p. ix); thus, they labelled them "commonsense" or "naive" theories (ibid.). Later on, AI researches switched to the terms "cognitively plausible" (Freksa 1992a, p. 3) or "cognitively adequate" (Hernández 1994, p. 4); but they meant the same as before: that such representations are a model of, or can be found in human knowledge or reasoning (Strube 1992). Yet not until the mid of the 90's, psychologists began to proof in humans the experimental validity of such claims (Mark and Egenhofer 1994; 1995; Knauff et al. 1995; 1997).

A fragmentation had occurred between AI and psychology in the field of spatial categorization concerning cognitive plausibility of QSR. The QSR scientists had been making human-cognitive claims about their spatial representations, and had not bothered to reach out to the psychologists to verify them. Knauff et al. help us grasp such situation:

> "Whether or not a formal approach to spatial relations is a cognitively adequate [...] model of human spatial knowledge is more often based on the intuition of the researchers than on empirical data." (1997)

> "[...] attraction for qualitative spatial reasoning (QSR) rests on the claim that it is akin to human inference. Yet there is only little justification for this claim." (2004)

We faced an unpleasant situation. On the one hand, researchers in QSR emphasize how central is the link with human cognition; for example, in the words of A. G. Cohn (1995), one of the most cited QSR researchers:

> "The principal goal of Qualitative Reasoning (QR) is to represent not only our everyday commonsense knowledge about the physical world, but also the underlying abstractions used by engineers and scientists when they create quantitative models."

On the other hand, QSR researchers do not readily verify their claims by psychological experimentation—or, at least, these researchers do not call loud enough for a verification—(See notable exceptions such as Mark and Egenhofer 1994; 1995). For instance, as of 2004, one of the most basic spatial representations, RCC, created in 1992, had not yet been checked for cognitive plausibility with respect to reasoning (Knauff et al. 2004, p. 182). Still more recently, researchers criticize startled that so few studies tackle the cognitive plausibility of qualitative spatial representations, all the more because such representations are key in human-computer interaction (Yang et al. 2015, Intr.,).

Since most people involved in studies of cognitive plausibility of qualitative representations are not the ones developing them, we regrettably get the impression that the QSR community has a marginal interest on *experimentally* studying cognitive plausibility and their main concern is mathematical formalism; even though, they paradoxically continue to aim to model human cognition, or, equivalently, 'common-sense' (e.g., Wolter and Wallgrün 2012; Dylla et al. 2017).

Anyway, we wrap up this critical view of disciplinary fragmentation between qualitative calculi and psychology with the hope-giving words of Klippel et al. (2013, Sect. 2.2):

> "While the number of behavioral [i.e., psychological] validations of spatial calculi is small compared to the number of proposed formalism, there is an active community that performs research on refining and tailoring formalisms through validating their cognitive adequacy."

We feel that we deserve to be included in this community that validates the cognitive adequacy of qualitative formalisms, because in this work we prove the experimental validity of two qualitative representations of motion. We stand against insularity in motion categorisation.

Tension A second dramatic effect of fragmentation is interdisciplinary tension. When disciplines use a different methodology or terminology, then statements in one discipline can be misunderstood or disputed by other disciplines. For example, if we use the label 'cognitive', a psychologist might understand that we talk about "human cognition", but an AI researcher might understand that we talk about "knowledge representation"—The difference is substantial.

More concretely, cognitive science has two prevailing goals: *constructive* and *explanatory* (Gärdenfors 2004, Sect. 1.1.1). On the one hand cognition in AI is mostly constructive, they create models to process information—though not necessarily mimicking human cognition—while cognition in psychology is explanatory because they describe human cognitive processes. Thus, the use of the term 'cognitive' by an AI researcher might bewilder (or even irritate) a psychologist; we see such a tension in the editorial debate of the journal *Cognitive Processing* (Ross 2019; Katsikopoulos 2019), which was sparkled by A. Kirsch's (2019) paper.

Thank God, we can defuse tension: if we are transparent in our terminology (clearly defining the terms we use), and if we found our methodology on solid principles, which need be openly and unambiguously stated. Even so, in my opinion, the key to eliminate fragmentation and the consequent tension is to strive to *integrate* our cognitive result into the other disciplines, i.e., to explicitly show the links, commonalities, and differences of our results with respect to others—This is oftentimes a hard task.

In this work, we do strive to integrate our results, which originate in the field of artificial intelligence, into other cognition fields such as psychology and linguistics: we invested effort in linking our definitions and models to well-known results and models in categorization (e.g., Sects. 5.5, 6.2 and 6.4).

Part II

Foundations of Categorization

Chapter 3

Categorizations, Categories, and Concepts in Human Cognition

> God said, "Let there be light," and there was light.
> God saw the light, and saw that it was good. God divided the light from the darkness.
> God called the light "day", and the darkness he called "night".
>
> Genesis 1:3–4 (WEB)

> God said, "Let there be light," and there was light.
> God saw that the light had a very distinctive feature than darkness: 'Goodness'.
> Based on this feature, God established two categories: `light` and `darkness`.
> God labelled the category `light` 'day', and the category `darkness` 'night'.
>
> Genesis 1:3–4 (Cognitive Paraphrase)

In this chapter, we summarize previous research on categorization. Firstly, we introduce the *basic terms* in the literature and explain how we use them in this work. We aim for clarity and consistency; we also aim to make this work understandable and integrable into any discipline that studies categories and concepts. Secondly, we mention important experimental results in human categorization, which apply to virtually *every human categorization*, regardless of the items they categorize. We consider these results in human categorization when developing our motion categorizations, because a strong motivation for categorizing motion is to apply it in human-computer interaction.

3.1 Terminology

All throughout this work, we use profusely the terms *categorization* and '*category*'. A *category* is one of the groups that result from a *categorization*, that is, one of the groups in which our motion categorizations group *motion scenarios* according to similarities, commonalities, or shared properties. Yet, depending on the context, we use alternative terms for *category*/*categorization* because they are the most common terms in another disciplines: We might use the terms 'concept/conceptualization', common in linguistics; or 'class/classification', common in machine learning; and, often, we use the terms '*qualitative representation*'/'*qualitative relation*', common in artificial intelligence, to name spatio-temporal categorizations and categories (Sect. 4.1).

Importantly, the term '*concept*' is a twin term of *category*; it has a similar meaning as *category*, but, depending on the author, it has diverse nuances. Murphy (2002, Ch. 1, p. 5) provides basic definitions that reflect the core understanding in psychology and cognitive science (e.g., Gerrig and Zimbardo 2005, p. 230; Medin and Coley 1998, Sect. II): *categories* are classes of things, and *concepts* are the mental representations of the categories—the descriptions for the categories.

'Concept' is, thus, a sophisticated term. It not only captures the commonalities and features in a category, and how they are interrelated (Sloman et al. 1998), it also works as the "categorization rule" (Bower and Clapper 1989). For that reason, we avoid the term *concept*: we advance simplicity by using more specific terms.

In our case, we are primarily interested in deciding the *membership* of items to a certain category; we intend to assign each motion scenario to a certain motion category. Thus, for us, the categorization rule is an essential component of a categorization. By the term '*categorization rule*' we refer to the function that decides the category of any item (disregarding whether the decision is deterministic or probabilistic, crisp or fuzzy). In that sense, '*category*' is the set of items grouped by the categorization rule into the same value, and a '*categorization*' consists of a categorization rule, plus all categories that such rule creates: This is the most elementary framework we can build to categorize items (More details in Section 6.1).

3.2 Classical Model and Classification

Before we delve into the nuances of how humans categorize and the categorization models that try to fit in, we must mention the oldest and simplest model: the *classical model* or *classical view*.

The '*classical model*' is equivalent to say that humans categorize items in the same way mathematicians assign set membership; in set theory, membership is only a Boolean value: '1', if the item is a member, '0', if it is not a member. In that sense, categories can be assimilated to crisp sets and the most basic operations with categories are the set operations: union $A \cup B$, intersection $A \cap B$, and set difference $A \setminus B$; we can thus apply the powerful machinery of set theory to work with categories. Furthermore, under a classical model, we can represent knowledge as a hierarchical tree of categories (i.e., a hierarchical taxonomy) which has many advantageous processing properties (Sect. 3.6.1).

Therefore, the classical model is the most popular one used in artificial intelligence: it is cheap to machinally implement and to maintain, and it has powerful reasoning capabilities relying on first order logic (Russell and Norvig 2014c). Mind you, computer scientist use unwillingly the term 'categorization' but rather 'classification'. When an AI researcher means to perform a "classification" of certain items, chances are that she refers to a categorization according to the classical model.

As we said in Sect. 1.4.1, p. 24, the classical view is outdated in human cognition as a precise account of categorization. As we will immediately see in Section 3.3, humans categorize in a graded manner; for example, `apple` is (in the Western culture) a member of the `fruit` category in a higher degree than `avocado` (McCloskey and Glucksberg 1978). Actually, gradation is one of the best established facts in human categorization. Does it mean that the classification work in AI is futile? By no means! The classical model is the most elementary approach to categorization, and, in that respect, we can always resort to it as most economical implementation which under certain conditions (e.g., artificial categories, or negligible boundary effects) can be effective enough.

3.3 Category Typicality and Membership

3.3.1 Typicality and Prototypes

Humans perceive some items as *the best examples* of a category; those items are the [highly] '*typical*' category members, for example, `apple` as fruit, or `chair` as furniture—They are often called also '*prototypical items*'. They play a pivotal role in categorization; even they underlie a traditional categorization model, the '*prototype model*' (details in Sect. 3.4.2).

Some items are less typical members, for example, `olive` as fruit, or `rug` as furniture. Others are *extremely atypical*, for example, `beet` as fruit, or `telephone` as furniture (E. Rosch and Mervis 1975; McCloskey and Glucksberg 1978). This property of how items fit a category is called '*typicality*', or '*prototypicality*'.

Typicality is a continuous value, i.e., it is graded (e.g., Barsalou 1985; E. H. Rosch 1973); subjects can numerically evaluate the typicality of an item with respect to a category. For example, subjects can assign typicality a value in the unit interval $[1.0, 0.0]$: 1.0 is extremely typical, and 0.0 is extremely atypical, i.e., unrelated to the category. In McCloskey and Glucksberg (1978), subjects evaluate the typicality of certain items as fruit; for instance, `apple` obtains an average typicality of 0.991, `olive` 0.338, `beet` 0.241; and as furniture, for instance, `chair` 0.994, `rug` 0.583, `telephones` 0.291 (we have normalized the original values to the unit interval).

Typicality is, thus, a well-established property of categorizations, and, arguably, the most influential. Eleanor. Rosch (1978, p. 38) states, "the prototypicality of items within a category can be shown to affect virtually all the major variables used as measures in psychological research." Murphy (2002, Ch. 2, p. 22), restates it, "[t]ypicality differences are probably the strongest and most reliable effects in the categorization literature." We present a non-exhaustive list of how typicality influences cognitive processes:

- Typical items are those that subjects most rapidly recognize as category members (E. H. Rosch 1973, p. 141; see Rips et al. 1973, pp. 9, 19; c.f., Wilkins 1971).

- Typical items are more frequently produced as spontaneous examples of the category (Mervis et al. 1976)—which is called '*item dominance*'; for example, if subjects are asked to give examples of fruit, they will much more probably mention `apple` than `olive`. Indeed, in the study of Battig and Montague (1969) with 442 Subjects, 97% mentioned `apple` in their fruit list, appearing in almost 60% as the first mentioned fruit; while `olive` was only mentioned by three subjects, i.e., 0.7%, never in the first place.

- Subjects learn sooner typical items than atypical ones as members of the category (Mervis et al. 1975; in natural categories, Heider 1972, p. 19), even when atypical items are shown more frequently (in artificial categories, E. Rosch, Simpson, et al. 1976, p. 498).

Even though *typicality* and, thus, the existence of real *prototypical items* in human cognition are far beyond any empirical question, the existence of *abstracted* prototypes in our mental representations of categories is heatedly debated; it is a decisive issue in the controversy of two main categorization models, the *prototype* and *exemplar* theory (Sect. 3.4.2).

A. What makes something 'typical'?

We have expounded on typicality and their effects on cognitive processes, but without addressing the question of what makes an item typical in a category.

E. Rosch, Simpson, et al. (ibid., pp. 492f.) mention essentially two different sources of typicality. First, when considering '*continuous attributes*' (e.g., lengths, weights), the most typical items (i.e., *prototypical items*) are the ones having feature values close to the average values of the category members; implying that an *abstracted prototype* is some sort of average of the category members. This occurs, for example, with sizes of animals (Rips et al. 1973, p. 19) or facial features (Reed 1972). Moreover, the most typical item in a certain category has very different feature values than the items in other categories.

Second, when considering '*binary attributes*' ('is' or 'is not'), the most typical item is the one having most common features with the category members (Note that the common features can be disjoint). Moreover, the most typical item in a certain category has the least common features with items in different categories. For example, if we consider the strings $w_1 =$ `AABDEE`, $w_2 =$ `AABKLM`, $w_3 =$ `ZXYKEE` as members of a category, then, the string w_1 would be the prototype, because it shares `AAB` with w_2, and `EE` with w_3; while w_2 and w_3 share only `K`. Coincidentally, in this case, the shared features of w_1 (`AAB` and `EE`) are disjoint, i.e., have nothing in common.

Interestingly, those two sources of typicality can be summarized in one simple principle: '*centrality*' (E. Rosch 1977, p. 36), also called '*central tendency*' (Posner and Keele 1968). If we represent the items of a category with continuous attributes in their multidimensional featural space, the prototypical element lies at the centre of the points cloud of items. Likewise, it occurs for items of a certain category with binary attributes: when we optimally represent their dissimilarities in a continuous metric space (using the MDS technique), prototypical items appear also central.

We can, thus, give an equivalent definition of typicality based on *similarity*: the most typical of a category is the item being in average *most similar* to all items in the category and *most dissimilar* to all items in the other categories.

Even though *central tendency* predicts typicality in '*common taxonomic*' categories (e.g., `fruit` and `furniture`), it fails to predict typicality in other type of categories, such as goal-derived categories (Barsalou 1985), in which typicality is rather predicted by the '*ideal*'. For example, the category `Christian` has as a typical member which is not the average features of the current Christians, but rather an *ideal* member, namely, Jesus. In other words, one considers a person to be more a member of the category `Christian` in the measure such person is similar to Christ (Oxford English Dictionary n.d.), instead of comparing him to the average Church member.

Finally, *central tendency* and *ideal* might fail to predict typicality for subjects that are unknowledgeable about a category. In that situation, subjects resort to the most primitive source of typicality: '*familiarity*' (e.g., Lynch et al. 2000), which amounts to '*frequency*' (Murphy 2002, p. 31). Accordingly, the category prototype is the item that the subject has experienced (e.g., seen, heard, or read) more frequently. We can resort to *familiarity*, even if we do not personally know the item or its features: it suffices that, in our everyday life, we hear the item's name more often than the other items in order to deem it prototypical.

3.3.2 Membership

Besides typicality—but extremely related to it—a chief term in categorization is '*membership*', i.e., whether an item belongs or not to a category. This question is answered by the *categorization rule*, either deterministically or probabilistically, either fuzzy or crisp. For example, is `apple` a `fruit`? Or, is `olive` a `fruit`?

From a practical stance, determining membership might be the most relevant question in categorization, and it is indeed the question we answer when we create our motion categorizations in Chapter 8. In fact, determining membership has so impacted research that, though categorization influences many psychological areas, in the decade of the 80's most research on concepts concentrated on the categorization rules (Medin 2011).

A right answer to the membership question is vital in daily circumstances. For instance, both humans and animals search tirelessly for items belonging to the category `edible`; also, when doing the shopping, we purchase items belonging to the categories in the grocery list (`apple`, `olive`, . . .); and, particularly in navigation, an autonomous vehicle might categorize a motion scenario as `collision` or `collision-free`, in order to start an avoidance manoeuvre.

A. Gradation and Fuzziness

According to the *classical categorization theory*, category membership is a binary value: either an item belongs or not to the category—there is no intermediate value. Even more, all items belonging to a category are equivalent members of the category: an item cannot be a preferential or a marginal member. This idea harmonizes with logical thinking, and, notably, with the classical set theory, where membership of an item to a set is a logical value, either *true* or *false*. However, category membership in real world is not a logical value, as the seminal work of McCloskey and Glucksberg (1978) clearly shows (See also, Lakoff 1973). They related *typicality* to *category membership* and gave evidence that as much as typicality is graded, so it is membership (Fig. 3.1; see also, Barsalou 1983). They showed that subjects agreed almost to 100% on the category membership of both most typical items (typicality values 1.0–0.9), as belonging to the category, and the most atypical items (typicality values 0.1–0.0), as not belonging. However, as we approached the middle typicality (0.5–0.3), on average 40% of the subjects disagreed about the membership of the items. For example, the majority of subjects accepted `olive` as member of the `fruit` category, but as much of 40% of them did not; in a similar vein, the majority rejected `rug` as `furniture`, but 48% of the subjects accepted it as member of such category. In contrast, all subjects agreed that `apple` belongs to the category `fruit`, and almost all (88%) agreed that `beet` does not.

Summarizing, the boundaries of category membership can be so *fuzzy* that virtually half of the subjects may accept item membership, while the other half may reject it. Moreover, the gradation on category membership occurs not only for members, but it is equally valid for non-members: the more typical a non-member is, the higher percent of subjects classify it as non-member, and, conversely, the more typical a member is, the lower percent of subjects classify it as non-member—As Barsalou (1985) exemplifies it, "`chair` is a better non-member of `birds` than is `butterfly`".

B. Membership and similarity: Family resemblance

Comparing membership and typicality is straightforward, because both are scalar values assigned to every single item in a category (See, for instance, Figure 3.1). In contrast, comparing membership and similarity is complex because similarity is a binary relation between items.

A fundamental result on similarity and membership is the so called '*family resemblance*': The average within-category similarity should be higher than the average between-category similarity. That is, category members are in average more similar than non category members. (ibid., p. 630; originally, E. Rosch and Mervis 1975).

3.3.3 Historical remarks

Typicality, gradation, and fuzziness in categories are far from obvious. Though now are common knowledge, they originated in the paradigm shift of categorization, which began 1970s. A notorious testimony of this paradigm shift are the words of E. H. Rosch (1973, Abstr.):

> [P]sychological and linguistic research has tended to treat categories (whether perceptual or semantic) as though they were internally unstructured—that is, as

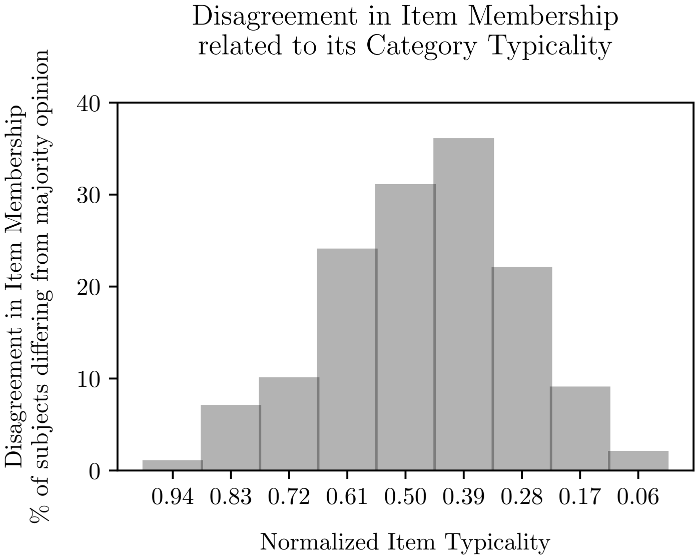


Figure 3.1: It shows, for each typicality interval, the percent of subjects that disagreed with the membership decision of the majority. For example, for the typicality interval $[0.44, 0.33]$, which has label 0.39, the average of subjects that contradicted the membership classification of the majority is 36%. That is, if the majority accepted an item as category member, on average the 36% rejected the item as member of such category, and vice versa.

Source: This figure was created with the manually taken values from Figure 2 in McCloskey and Glucksberg (1978), typicality normalized to the unit interval.

> though they were composed of undifferentiated, equivalent instances—and as though category boundaries were always "well-defined". [. . .] It is the contention of the present paper that most "real" categories are highly structured internally and do not have well-defined boundaries; thus, we may presently have a quite distorted view of how real categories are learned and how they function in cognitive processes.

Once we have settled the experimental facts in categorization: typicality, gradation, and fuzziness; we confront the awkward task to cognitively model it. To that end, two tools are frequently used: *similarity* and *fuzzy logic*. Similarity is widely used to account for category membership and comparison effects between items—we expand on it in the next section. Fuzzy logic is used to account for decision making, inference, and, more generally, operations with categories (e.g., Klir and Bělohlávek 2011, esp. Ch. 4, 5; Douven 2020; Lakoff 1973; c.f., Zadeh 1972).

3.4 [Dis-]Similarity

In the previous section, we showed that a category has internal structure: the members of a category, though having binding commonalities (i.e., *family resemblance*), are *not* perfectly equivalent; they are graded according to *typicality*. We also showed that both family resemblance and typicality can be defined in terms of *similarity*. It follows that, as a very first step in modelling a category, we can define a '*similarity function*', $\mathrm{S}(A, B)$ to compare pairs of items, (A, B), and use it to establish the category membership and the internal category structure—As is the case in most categorization models (Goldstone 1994; Kruschke 2001).

Actually, researchers often work with an alternative to similarity: the '*dissimilarity function*' $\mathrm{D}(A, B)$ (e.g., Krantz and Tversky 1975; Klippel et al. 2012, p. 244), Dissimilarity has certainly very convenient advantages over similarity. Foremost, from a practical view, dissimilarity behaves like a *distance*, which is a very intuitive concept and has a firm mathematical foundation (p. 46); moreover, distances enable the powerful technique of '*multidimensional scaling*' (MDS) in order to determine the features underlying similarity (Section 3.4.1.A).

At any rate, we can easily switch between dissimilarity and similarity because they are inversely related: the higher the dissimilarity the lower the similarity. This relation can be specified by means of a monotone decreasing function $f(x)$, so that $\mathrm{D}(A, B) = f(\mathrm{S}(A, B))$; for example, typically used are $f(x) = 1/x$ if $\mathrm{S}(A, B) > 0$, or $f(x) = -\log\left(x/\max_{A,B} \mathrm{S}(A, B)\right)$ (Mair et al. 2022). In fact, due to their direct relation, sometimes scientists use the generic term '*proximity*' to refer to any of them, similarity or dissimilarity (Borg et al. 2018, p. 2). In this work, we will explicitly mention either 'similarity' and 'dissimilarity', but always bear in mind that they are directly related. Even more, we can translate any comparison statements obtained using similarity and express them equivalently in terms of similarity, and vice versa (Tab. 3.1); such equivalence preserves the information about [dis-]similarity rankings.

SIMILARITY		DISSIMILARITY
A is more *similar* to B than to C	≡	A is less *dissimilar* to B than to C
A is less *similar* to B than to C	≡	A is more *dissimilar* to B than to C
A is the most *similar* item to B	≡	A is the least *dissimilar* item to B
A is the least *similar* item to B	≡	A is the most *dissimilar* item to B

Table 3.1: The statements in the same row are equivalent: on the left row are expressed with the term 'similarity'; on the right row are expressed with the term 'dissimilarity'

The similarity usually outputs a real value, which belongs to an '*ordinal scale*' (Stevens 1946; See, Sternberg and Pickren 2019, pp. 100–104): we can *compare* similarity values, i.e., given two pairs of items, (A, B) and (C, D), we can establish which relation is true, either $\mathrm{S}(A, B) \leq \mathrm{S}(C, D)$ or $\mathrm{S}(C, D) \leq \mathrm{S}(A, B)$. Consequently, we can *rank* items with respect to similarity: the higher the value the higher the similarity. For example, if we had $\mathrm{S}(A, B) \leq \mathrm{S}(A, D) \leq \mathrm{S}(A, C)$, we could say that C is the most similar item to A, D is less similar, and B is the least similar item.

Unfortunately, other operations, such as adding or multiplying similarities, are not necessary meaningful. For instance, in Tversky and Hutchinson (1986, Table 1) $S(\texttt{orange}, \texttt{grapefruit}) = 2.69$ and $S(\texttt{orange}, \texttt{coconut}) = 1.33$ in the 5-point scale $0 =$ unrelated, $4 =$ highly related. Based in these values, it makes, however, no sense to affirm that an `orange` is twice similar to a `grapefruit` that it is to a `coconut`.

In any case, the similarity function, or simply, the '*similarity*', is arguably the most intuitive tool to model experimental results on categorization—as Murphy and Medin (1985, p. 291) put it: "Perhaps the most powerful explanation of conceptual coherence is that objects, events, or entities form a concept because they are similar to one another". Indeed, the main types of categorization theories, i.e., classical, prototype, and exemplar model, can be modelled using a similarity function (Medin and Heit 1999, Sect. I.B). Even, not only models, but most different sorts of categories, including ad-hoc categories (Barsalou 1983), seem to possess an intern similarity function. In fact, E. J. Wisniewski (2002, pp. 467f.) finds in his survey *only* theories of concept structure and models of concept acquisition and categorization that are grounded in the similarity. About 20 years before, Murphy and Medin (1985) came to the same conclusion: all accounts on categorization "rely directly or indirectly on the notion of similarity".

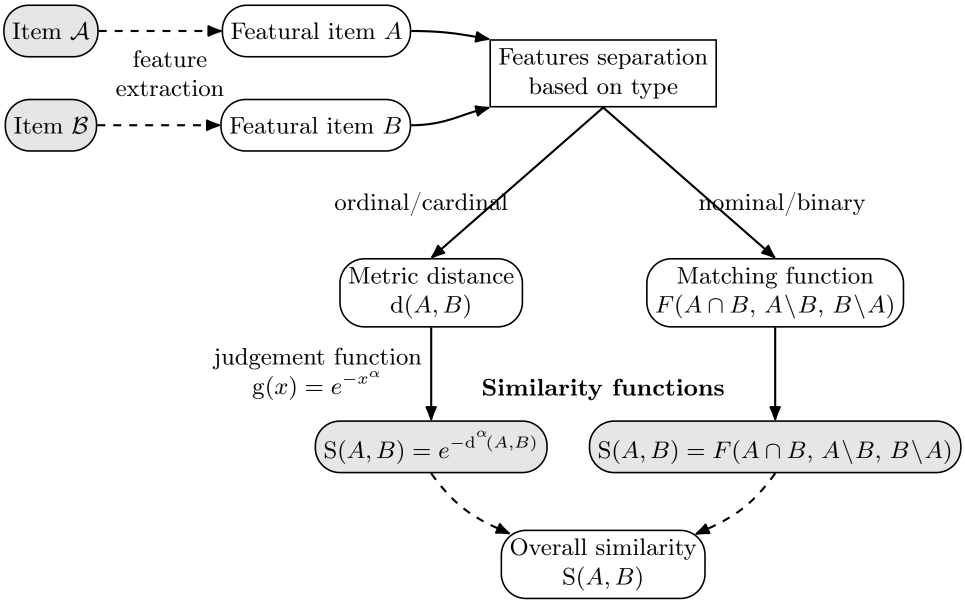


Figure 3.2: Schema of computing the similarity of two items A and B. The dashed lines indicate controverted or rather ambiguous steps (Sect. 3.4.4).

It seems, thus, that similarity as a ground of categorization is a thread throughout modern research history.

Being similarity so prominent in categorization for the last decades, we might be astounded at this remark of pioneer F. Attneave (1950) (emphasis added):

> The question 'What makes things seem alike or seem different?' is one so fundamental to psychology that *very few* psychologists have been naïve enough to ask it. We are aware of *only two papers* devoted to a consideration of the problem[.]

From which we also draw a lesson: how decisive is, in science, to concretely define, measure, or falsifiably model the key concepts. We can retrospectively affirm that, in psychology, the striving to model *similarity* has unlocked a wealth of insight.

Concerning our motion categorizations, they possess also similarity functions (Sect. 6.1.3); such functions arise effortless from the metric structure of the categorizations, as shown in Metric models. Because of the explanatory power of similarity, its pervasiveness, and success in categorization modelling (See Murphy 2002, p. 481), we consider the presence of a similarity function a strength of our model.

3.4.1 Modelling and Computing Similarity

Similarity, as an ordered relation or a graded value, has been proved to have experimental significance. Subjects understand and perform tasks that consist on assessing the similarity of items (See tasks, Dunn-Rankin et al. 2004, pp. 27–34). The most performed similarity assessment tasks are following: *a*) choose the most similar item to a reference item (e.g., Tversky and Gati 1978), *b*) '*rank*' items according to similarity to a reference item, i.e., order them without giving a numerical similarity value; *c*) numerically '*rate*' the similarity between items (e.g., Gati and Tversky 1984).

We apply such insights about similarity in this research: we experimentally check whether our story-based motion categorizations are cognitively plausible (Ch. 9). To that end, we make the subjects choose the more similar of two items compared to a reference item. Subsequently, in order to evaluate the data, we transform the pairwise similarity choices into a similarity rating (Sect. 9.2.1). Indeed, one can mathematically transform the similarity assessments between its types: pairwise comparisons, ordinal and numerical ranking.

Featural Similarity Once we have experimental similarities, one of the greatest, and most hotly debated challenges is to model the experimental similarity assessments by means of a similarity function $\mathrm{S}(A, B)$ (see more in Sect. 3.4.2). The purpose is that, given two items, A and B, we theoretically reproduce their experimental similarity values , i.e., their ordering or the rating, by means of the similarity function. The first step towards the computation of similarity is, almost inescapably, the extraction of *relevant features* from items [1]. Indeed, *feature extraction* is one of the earliest processes common to all models of categorization and concept acquisition (See, E. J. Wisniewski 2002, pp. 474f.); actually, it is a basic assumption of cognitive psychology that items in the world can be described using features (E. Wisniewski 2001, p. 5433). Surely because features seem a *transparent*, readily accessible property of items (e.g., Eleanor. Rosch 1978, but see Sect. 3.4.4). Based on the extracted features, one can compute a value for the similarity (See Fig. 3.2).

We encounter, however, two distinct types of features that we cannot process with the same methods: *nominal* and *ordinal* features:

- '*Nominal features*' (e.g., `eye colour`) cannot be assigned a meaningful value. For example, the eye colour may have following values (*brown*, *blue*, *green*); a numerical assignment of values, e.g., ($brown = 1$, $blue = 2$, $green = 3$) is as valid as any other permutation, e.g., ($brown = 3$, $blue = 1$, $green = 2$); the value works only as identifier, but has no meaning in itself. A particular case are the '*binary features*' (e.g., *canfly* vs. *cannotfly*); they have only boolean values.

- '*Ordinal features*' (e.g., `beauty`, `brightness`) can be ranked, i.e., ordered, (e.g., the *sun*, a *headlight*, a *candle*, and a *glowingcoal* can be ranked according to `brightness`). An important particular case are the '*cardinal features*', they are expressed numerically (e.g., `temperature`, `length`), so they allow arithmetical operations.

Interestingly, ordinal features can often be turned into cardinal by rating them against a maximum and minimum value. For instance, considering *sun* the maximum brightness, value 1.0, and *pitch black*, the minimum brightness, value 0.0, we can cardinally rate `brightness`: $sun = 1.0$, $headlight = 0.3$, $candle = 0.1$, $glowing\ coal = 0.01$). As a consequence, we can eventually treat ordinal features as cardinal ones (See, Bartoshuk 2019, pp. 103f.).

Summarizing, *cardinal* features (eventually also *ordinal* features) use operations that are incompatible with the *nominal* features. This has rather unpleasant consequences for the computation of similarity: We must use different mathematical operations to compute the similarity depending on the feature types. Cardinal features can be represented in a coordinate space, and, consequently, their similarities can be built on *metric distances*; categorization models based on such properties are called '*geometric*' (Tversky 1977), '*spatial*' (Hahn and Heit 2001), or '*multi-dimensional scaling*' (Kruschke 2001) models—we call them '*metric*' models. Nominal features,

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[1] Note that the 'features' are used to define the similarity by which 'categories', i.e., 'concepts', are defined. Thus, we use the *concept* 'feature' to define the concept 'concept'. We point here at the risk of *circular definition* or *infinite regress*, when trying to define 'concept' and 'feature'. As example, we may ask, "Are also the 'features' used to define the concept 'feature'?".

however, allow only *set operations*; consequently, their similarities are based on union $A \cup B$, intersection $A \cap B$, and set difference $A \setminus B$.

All considered, we might question how well-founded is comparing similarities obtained from so different models (Fig. 3.2). In practice, we find examples where both models are efficaciously combined (e.g., Volkert, Müller, and Kirsch 2018).

A. Metric models

'*Metric models*' (also called '*geometric models*' (Tversky and Gati 1982) or, as a superordinate concept, '*dimensional models*' (Tversky and Krantz 1970; E. E. Smith and Medin 1981, pp. 102ff.)) base on two assumptions. First, an item—or rather, its relevant features—can be represented as a point in a real coordinate space, commonly called '*psychological space*' (e.g., Nosofsky 2011; Tversky and Hutchinson 1986) and '*feature space*' (e.g., Rogers and McClelland 2011, pp. 112f., Tversky 1977). For example, in order to represent a beep sound, we only need two features: *pitch* and *volume*; accordingly, we can represent it in a two-dimensional coordinate space. Second, we assume that the *dissimilarity* between items is strictly increasing with respect to the *metric distance* $\mathrm{d}(A, B)$ between such featural points.

Thus, in practice, we have the following requirements for metric models: features must be *cardinal*, and we have to choose a distance between features, a '*featural distance*', widely known as '*psychological distance*'. The common distances used in categorization experiments are generalized Minkowski metrics (Eq. (3.1)) (Kruschke 2001), not only because of their computational practicability (e.g., Ashby 1992), but also for the convenient mathematical psychological properties (Beals, Krantz, and Tversky 1968; see, Glazer and Nakamoto 1991).

$$\mathrm{d}(A, B) = c \left[\sum_{i}^{N} w_i |a_i - b_i|^p \right]^{1/p} \quad p \geq 1 \tag{3.1}$$

The Minkowski distance—also known as L^p or *power* distance—measures the distance between, $\{a_i\}$, the features of item A, and $\{b_i\}$, the features of item B. The weights w_i, '*attention weights*', may refer to changes in the attention to different features, and c is a general sensitivity parameter (Nosofsky 1992a, p. 366f.).

We further require a function, $\mathrm{g}(x)$, that relates the psychological distance to the similarity:

$$\mathrm{S}(A, B) = \mathrm{g}(\mathrm{d}(A, B)) \tag{3.2}$$

Such function $\mathrm{g}(x)$ is called '*judgement function*' (e.g., Ennis 1992). Since dissimilarity is strictly increasing with the psychological distance d, the judgement function must be strictly decreasing (e.g., $\frac{1}{1+x}$, e^{-x}). Remarkably, in human and even animal psychology, the featural distance $\mathrm{d}(A, B)$ relates to the judged similarity $\mathrm{S}(A, B)$ by means of an exponential decay $\mathrm{g}(x) = e^{-x^{\alpha}}$, $\alpha > 0$: the '*universal law of generalization*' (Shepard 1987; compare with 1957; e.g., Perrin 1992, p. 124). Accordingly, we obtain the following closed formula for similarity:

$$\mathrm{S}(A, B) = e^{-\mathrm{d}^{\alpha}(A,B)} \quad \alpha > 0 \tag{3.3}$$

Metric models have successfully fitted wide range of data, moreover, they have proved to distinguish '*integral dimensions*' (types of features that can only exist simultaneously, such as the pitch and volume of a sound, or, the hue and brightness of a colour) from '*separable dimensions*' (those that can exist independently, e.g., colour brightness and sound volume) through the value of p, i.e., through the norm type (see Eq. (3.3)). Separable dimensions fit better $p = 1$, city-block distance, while integral dimensions fit better $p = 2$, i.e., Euclidean metric (Hahn and Chater 1997, pp. 55–57).

Another interesting property of the metric models, is that they reproduce a similarity rule that is frequently used in modelling: the '*multiplicative similarity*' rule (Nosofsky 1992b; see also, E. E. Smith and Medin 1981, pp. 154–156). This rule states that the overall similarity, $\mathrm{S}(A, B)$, can be computed as the product of the similarity for each feature i, $s(a_i, b_i)$ (e.g., Nosofsky 1986). This is the case, when the exponent α in Equation (3.3) equals the power p in the distance $\mathrm{d}(A, B)$ (Eq. (3.1)) then

$$\mathrm{S}(A,B) \overset{\substack{\text{Eq. (3.1)}\\ \text{Eq. (3.3)}}}{=} e^{-c^{\alpha}\left[\sum_i^N w_i|a_i-b_i|^p\right]^{\overset{\alpha=p}{\cancel{\frac{1}{p}\alpha}}}} = \prod_i^N e^{-\mathrm{d}^{\alpha}(a_i,b_i)} \overset{\text{Eq. (3.3)}}{=} \prod_i^N s(a_i,b_i)$$

Nonetheless, metric models have their weaknesses (e.g., Maddox 1992, pp. 162–165; Hahn and Chater 1997, pp. 57–60). Amongst others, the term *psychological space*, which is the ground concept of the metric models, is, at most, *vaguely* defined; and the mathematical properties of the metric distances are incompatible with numerous experimental psychological results.

Psychological Space It is an abundantly used concept with a long tradition in psychology—at least from F. Attneave (1950)—(e.g., Shepard 1957; Nosofsky 1986; Pothos and Wills 2011b). It is also called '*semantic space*' (Rips et al. 1973), or '*mental space*' (E. J. Wisniewski 2002, p. 478). Briefly, it is the space where the representation of stimuli can more naturally reproduce the experimental similarity assessments using a certain dissimilarity function called *psychological distance*—*metric models* are a kind of psychological spaces.

Unfortunately, this concept is rather indefinite in psychology, as U. Hahn and N. Chater (1997, p. 91) state:

> [T]he notion of psychological space is not particularly well defined: there are no commitments as to what exactly this space is, whether it is a long-term representation or not, nor whether it is explicitly similarity that is represented here or whether the representation of similarity it generates is merely a by-product of a general scheme for the representation of objects.

We illustrate the indefiniteness of the concept 'psychological space' by means of two examples. In his pioneer work, Attneave (1950) had subjects compare parallelograms that differed in *tilt* (the angle between the base and the other side) and in *side length* obtaining dissimilarities between the stimuli. He noticed that representing the stimuli, i.e., the parallelograms, as points in a two-dimensional space with coordinates *area* and *tilt angle* resulted in a better analogy between distance and dissimilarity than using *side length* instead of *area*. Again, he noticed that such analogy could be further improved by using the city-block metric to measure the distance between stimuli. In sum, Attneave concluded that subjects compared the parallelograms in a two-dimensional *psychological space* consisting of *area* and *tilt angle* as coordinates, in which they used the *city-block* metric, as a *psychological distance*.

In the second example, Rips, Shoben, and Smith (1973) measured the paired dissimilarities between certain mammals (e.g., bear, lion, horse, sheep), subsequently they fit the dissimilarities to the Euclidean distances of points in a two-dimensional space using *multidimensional scaling*. In the best fit, the authors argue that the horizontal axis corresponds to *size*, animals to the left (e.g., bear, deer, horse) are larger than those to the right (e.g., dog, rabbit, mouse); and the vertical axis corresponds to *predacy*, animals to the bottom are wilder (e.g., lion, cat, bear), while to the top are farm animals (e.g., goat, pig, ship) (similar results with animals, López et al.

1997; Shoben 1976). In short, Rips et al. consider that subjects mentally represent mammals in a two-dimensional space (the psychological space) with the features *size* and *predacy*, as coordinates, and Euclidean metric as distance (the psychological distance).

These examples epitomize how *psychological space* is rather a pragmatic construct than an empirical reality: it is the coordinates space in which items are featurally represented, and it is endowed with the distance that most accurately reproduces the experimental dissimilarity judgements. The coordinates used are sometimes precise measurable quantities (such as *tilt* and *area* in Attneave (1950)), but often they are also abstract values obtained by numerical analysis, e.g., multidimensional scaling, that can only be intuitively and approximatively interpreted (such as *predacy* and *size* in Rips et al. (1973))—sometimes even lacking intuitive interpretation.

Accordingly, what motivates the use of psychological spaces is not a solid cognitive theory, but rather the researchers' necessity for a featural explanation of the similarity. In the words of Rips, Shoben, and Smith (ibid., pp. 13, 14) (emphasis added):

> While we have argued that [dissimilarity] distance can be conceptualized as [a psychological] distance, this translation of constructs *has not led* to a gain in predictability. [. . .] [W]e have retained this construct because *it seemed* to capture certain conceptual relations between the noun pairs [.]

Violation of Distance Axioms One basic component of *metric spaces* is the distance d, which is directly related to the dissimilarity $\mathrm{D}(A, B)$ between items. Notably, d has sharp defined mathematical properties, i.e., the distance axioms (e.g., Carothers 2000, p. 37):

i. *self-distance constancy* $\mathrm{d}(A, A) = 0$

ii. *minimality*: $\mathrm{d}(A, B) \geq \mathrm{d}(A, A)$,

iii. *symmetry*: $\mathrm{d}(A, B) = \mathrm{d}(B, A)$

iv. *triangle inequality*: $\mathrm{d}(A, B) + \mathrm{d}(B, C) \geq \mathrm{d}(A, C)$

These distance axioms impose constraints on the similarity function, because, in metric models, similarity is computed from a distance using a strictly decreasing function, $\mathrm{g}(x)$ (e.g., Eq. (3.3)). Each distance axiom above yields its respective similarity constraint below marked with the same item label but capitalized.

I. *self-similarity constancy* $\mathrm{S}(A, A) = \mathrm{S}(B, B)\ \forall A, B$

II. *maximality* $\mathrm{S}(A, A) \geq \mathrm{S}(A, B)$

III. *symmetry* $\mathrm{S}(A, B) = \mathrm{S}(B, A)$

IV. Using the most general judgement function $\mathrm{g}(x)$, no constraint can be derived from the *triangle inequality* (item iv.). However, making assumptions on the form of $\mathrm{g}(x)$, we can obtain constraints on experimental similarity values.

   a. If $\mathrm{g}(x) = e^{-x^{\alpha}}$ with $0 \leq \alpha \leq 1$, the triangle inequality for distances becomes a multiplicative inequality for similarities: $\mathrm{S}(A, C) \geq \mathrm{S}(A, B)\ \mathrm{S}(B, C)$ (Prop. A.5.1)

   b. If the distance d is a *Minkowski metric* (Eq. (3.1)), then a tighter constraint for the distances, the '*corner inequality*', can be derived from the triangle inequality (Tversky and Gati 1982). The *corner inequality* has a powerful practical purpose: we can use it to rebut the triangle inequality for the psychological distance using experimental values of the psychological distance without the need to specify or know the function $\mathrm{g}(x)$.

It is experimentally verified that similarity properties I. and III. do not always hold (Tversky 1977; Tversky and Gati 1978), the triangle inequality has also been rebutted experimentally by Tversky and Gati (1982) under the conditions in IV.b.. Briefly, it seems that the *metric assumption*, i.e., that dissimilarity is strictly increasing related to a distance, is only an acceptable approximation, but not a faithful psychological model. For that reasons, the metric models have been modified to accommodate the discrepancies, mainly extending the distance with asymmetrical terms (e.g., Krumhansl 1978; Appelman and Mayzner 1982), but often opening more questions than solving them (see critique, Corter 1987; 1988; Krumhansl 1988).

Finally, Nosofsky (1991a) sets a milestone in the attempts to create valid modified distances, as he notes that various modifications to the metric models (e.g., distance-density model, Krumhansl 1978; hybrid tree-euclidean Carroll 1976)—and even some nominal models (e.g., the *additive* contrast model, Tversky 1977)—are particular cases of a general model due to E. W. Holman (1979), where

$$\mathrm{S}(A, B) = F\left[s(A, B) + r(A) + c(B)\right]$$

F is a strictly increasing function, $s(A, B)$ is a *symmetric* similarity function, $r(A)$ and $c(B)$ are bias of the particular items. The key point is that the additive extra terms, r and c, account for the *asymmetry* and violations of *self-similarity*.

Summarizing, a distance-based similarity is not an ideal psychological model, though in most cases a very good approximation, as Tversky and Gati (1978) explain:

> Although the violations of [similarity properties that base on distance] are statistically significant and experimentally reliable [...], the effects are relatively small. Consequently, [such properties] may provide good first approximations to similarity data. [Multidimensional] [s]caling models that are based on these assumptions, therefore, should not be rejected off-hand. A Euclidean map may provide a very useful and parsimonious description of complex data, even though its underlying assumptions (e.g., symmetry, or the triangle inequality) may be incorrect. At the same time, one should not treat such a representation, useful as it might be, as an adequate psychological theory of similarity.

B. Nominal models

When features are nominal or binary values, the items are described as a *set of features*. As example, we show how Keren and Baggen (ibid.) described *seven-segmented* digits (the numbers appearing in digital displays) using their segments as features. They labelled each segment with an integer value, as shown in Figure 3.3. Thus, each digit was a set of such labels. For instance, the digit `1` $= \{\mathit{3}, \mathit{6}\}$ and the digit `6` $= \{\mathit{1}, \mathit{2}, \mathit{5}, \mathit{7}, \mathit{6}, \mathit{4}\}$.

Since, in nominal models, items are described as sets of features, we can only apply *set operations* for whatsoever computation between items, particularly, when computing similarity. That is, we are restricted to union $X \cup Y$, intersection $X \cap Y$, and set difference $X \setminus Y$. On this fact bases the most influential model to compute similarity between items with nominal

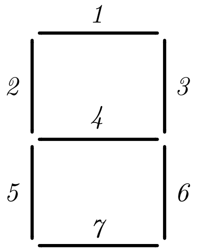

Figure 3.3: Features (*1*–*7*) of a seven-segmented numeral according to Keren and Baggen (ibid.)

or binary features: A. Tversky's (1977) '*contrast model*'. He
postulated certain cogent assumptions for the similarity function, $\mathrm{S}(A,B)$, between two items A and B: *matching*, *monotonicity*, *independence*, *invariance*, and *solvability*. Because of relevance and simplicity, we present only the first two ones.

The *matching* assumption holds that the similarity of nominal values is only a function of following set operations, namely $A \cap B$, $A \backslash B$, and $B \backslash A$. That is,

$$\mathrm{S}(A,B) = F(A \cap B,\, A \backslash B,\, B \backslash A) \tag{3.4}$$

It seems perfectly reasonable that the '*common features*' of both items A and B, i.e., $A \cap B$, should mainly determine how similar two items are—things are similar inasmuch as they have commonalities. It seems also reasonable that the features belonging exclusively to an item, i.e., the '*distinctive features*', $A \backslash B$ and $B \backslash A$, also influence similarity—for example `horse` and `zebra` have countless common features; however, the very distinctive feature of *being striped* differentiate unequivocally both concepts. Finally, Tversky excluded the whole features set, i.e., $A \cup B$, from the computation of similarity—this decision is disputable, we cannot find convincing arguments for this exclusion, neither Tversky does in his paper (See Sect. Contrast Model's Criticism).

The monotonicity assumption holds that the similarity function F in Equation (3.4) is monotone with respect to inclusion in its variables. That is

$$\mathrm{S}(A,B) \geq \mathrm{S}(A,C) \iff \begin{cases} A \cap B \supseteq A \cap C & (3.5a) \\ A \setminus B \subseteq A \setminus C & (3.5b) \\ B \setminus A \subseteq C \setminus A & (3.5c) \end{cases}$$

The inequality is strict whenever any inclusion is proper.

Monotonicity fits also common sense: the more common features, $A \cap B$, and the less distinctive features, $A \backslash B$, $B \backslash A$, the more similar the items A and B are.

In conclusion, based on the above mentioned assumptions (*matching*, *monotonicity*, *independence*, *invariance*, and *solvability*), Tversky proved the main result of the contrast model: it exists a similarity function, $\mathrm{S}(A,B)$, that monotonically matches the results of the experimental similarity, $s(A,B)$. Furthermore, the similarity function can be expressed as a linear combination of a non-negative function $f(X)$ on the features sets, which is a measure of the *salience* of each features set .

$$\mathrm{S}(A,B) = \theta f(A \cap B) - \alpha f(A \backslash B) - \beta f(B \backslash A) \qquad \theta, \alpha, \beta \geq 0 \tag{3.6}$$

Using this similarity function, without specifying f, Tversky could explain several unintuitive experimental results of similarity—mainly those challenging the metric models: the violation of *self-similarity constancy*, *maximality*, and *symmetry* (See Items I. to III.). For example, from Equation (3.6), we have $\mathrm{S}(A,A) = \theta f(A)$, hence, if $\theta \neq 0$, the self-similarity constancy, $\mathrm{S}(A,A) = \mathrm{S}(B,B)$, can be easily violated when $f(A) \neq f(B)$, that is, when the features of each item have different salience. Likewise, from the equation, we have $\mathrm{S}(A,B) - \mathrm{S}(B,A) = (\alpha - \beta)\,[f(B \backslash A) - f(A \backslash B)]$, that is, the degree of deviation from symmetry is settled by $\alpha - \beta$; only for $\alpha = \beta$ is symmetry guaranteed. (See Items i. to ii., i. and ii.)

Despite the appealing simplicity and explanation power of Equation (3.6), note that function $f(X)$ is quite general: it is a function from the power set of all features, $\mathcal{P}(A \cup B \cup C \cup \ldots)$, into $\mathbb{R}^+$; for instance, for three items A, B, and C, the domain of $f(X)$ is $\mathcal{P}(A \cup B \cup C)$, which can be a large set. It has only one restriction (due to Equation (3.5)), it has to be a *strictly monotonic set function*, i.e., $f(X) < f(Y) \iff X \subsetneq Y$.

We find interesting particular cases of $f(X)$ that fulfil the above restriction. For example, the set of *additive functions*, i.e., $f(X \cup Y) = f(X) + f(Y)$ whenever $X \cap Y = \emptyset$; which is the case in most experimental research (e.g. Tversky 1977; Keren and Baggen 1981; but see, Gati and Tversky 1984, supporting subadditivity)—in that way, f is fully defined by its value on each single feature, i.e., $f(\{x\}) \quad x \in X$. Another example is any monotonically increasing function, $\mathrm{g}(x)$, of the *cardinality* $|X|$, i.e., $f(X) = \mathrm{g}(|X|)$ (see, Coletti and Bouchon-Meunier 2019). The simplest particular case of both mentioned examples is the *set cardinality* $f(X) = |X|$, which is equivalent to consider f additive, and assign value 1 to each feature, i.e., $f(\{x\}) = 1 \; \forall x \in X$.

C. Contrast Model's Criticism

As we mentioned above the exclusion of the whole features set, i.e., $A \cup B$, from the computation of $\mathrm{S}(A, B)$ is questionable. Even more, when we find featural similarity measures in the literature where $A \cup B$ is used; for example in the similarity measure $\mathrm{S}(A, B) = \frac{f(A \cap B)}{f(A \cup B)}$ (Gregson 1975, p. 47), already mentioned in 1901 by Jaccard with $f(X) = |X|$ (formally defined, Jaccard 1902).

Interestingly, for the usual case of $f(X)$ being *additive*, we have that $f(X \cup Y) = f(X \cap Y) + f(X \backslash Y) + f(Y \backslash X)$, which means that, Eq. (3.6) can be rewritten as

$$\mathrm{S}(A, B) = \theta f(A \cup B) + \alpha' f(X \backslash Y) + \beta' f(Y \backslash X) \qquad \theta \geq 0$$

And, thus, in the additive case, $A \cup B$ is actually present in the computation of similarity.

A more elaborated criticism of the contrast model is that it makes a shallow distinction of the *distinctive features*. In Equation (3.6) only two groups of distinctive features are considered: those features belonging exclusively to item A, i.e., $A \backslash B$, and those belonging exclusively to item B, i.e., $B \backslash A$. This distinction fails to capture the contribution of the *alignable* differences to the similarity.

'*Alignable differences*' are distinctive features that take the same place in a certain structure, while '*non-alignable*' do not. Importantly, the *aligned* differences have a higher impact on similarity than *non-alignable* do (E. J. Wisniewski 2002, p. 499; see also, Hahn and Heit 2001, p. 13879).

As example, we compare event A. to events B. and C.:

A. `a man` *`driving`* `to work` *`on a sunny day`*

B. `a family` *`flying`* `on holiday with their pet`

C. `a family` *`buying`* `food` *`on a sunny day`*

The words `driving` and `flying` are *different* features, but they are *aligned*, because each expresses how the subjects in each event move. On the other hand, the words `driving` and `buying`, though being both verbs, are *not aligned* compared to `driving` and `flying`, since they express very different types of actions. Because of the aligned different actions, subjects may find sentence A. more similar to B. than to C., even though A. has exactly the feature `on a sunny day` with C. in common. Such alignment effects would contradict the contrast model.

3.4.2 Category Membership: Prototype, Exemplar, and Clustering

One of the principal applications of similarity is to decide *category membership*. But even assuming we agree in the form of the similarity function we use, $\mathrm{S}(A, B)$, we still can work out a myriad of methods to decide category membership based on that $\mathrm{S}(A, B)$. Anyway, two methods (or models) have achieved overriding prominence: the *prototype* and the *exemplar* model.

Prototype Model The '*prototype model*', also called '*probabilistic model*' (E. E. Smith and Medin 1981), assumes that a category M_i has a *most typical* representative item: the '*prototype*', we call it P_i—caveat, usually the prototype does not exist as a real-world item, but it is computed as the modal or average value of the items' features in a category (See lucid intro., Minda and J. D. Smith 2011). With this assumption, we define a *prototype* P_i for each category M_i, and we obtain the category membership of an item A using the similarity of A to each prototype P_i. For example, a possible method bases on a basic requirement of the theory: amongst all prototypes, an item A has the highest similarity to the prototype P_i of its category $M(A)$ (ibid., p. 59; Murphy 2002, p. 96):

$$M(A) = M_i \text{ such that } \mathrm{S}(A, P_i) = \max_{j=1\dots n} \{\mathrm{S}(A, P_j)\} \tag{3.7}$$

Although Equation (3.7) can coarsely model a categorization rule, it has two drawbacks. First, even when the similarities of one item A to two prototypes are comparable or equal, we may only decide for one of both categories: it seems unfit to affirm that such a boundary item, A, belongs only to one of the two categories. Second, it misses the gradation effect of categorization, namely, that humans rarely choose category membership with an absolute certainty: they assign membership according to a certain probability. For that reason, the *prototype theory* resorts to probabilistic formulae; these are based in Shepard's (1957) and Luce's (1963) stimulus-response choice model (Minda and J. D. Smith 2011). In such formulae (Eq. (3.8)), similarity remains a central constituent: it is computed between an item A and each prototype.

$$P(M_i|A) = \frac{\mathrm{S}(A, P_i)}{\sum_{j=1\dots n} \mathrm{S}(A, P_j)} \tag{3.8}$$

Exemplar Model The '*exemplar model, of categoriz.*'—originally known as '*context model*' (Medin and Schaffer 1978), and tweaked as '*generalized context model*' (Nosofsky 1986)—computes the probability of categorization using the same pattern of the stimulus-response choice model. But, in this case, instead of the similarity to the prototypes P_i, the model uses the similarity to each known '*exemplar*' I of a category M_i, i.e., $\{I \mid I \in M_i\}$ (2011).

$$P(M_i|A) = \frac{\sum_{I \in M_i} \mathrm{S}(A, I)}{\sum_{j=1\dots n} \sum_{I \in M_j} \mathrm{S}(A, I)} \tag{3.9}$$

Note that similarity, in its own, does not determine category membership in any categorization model, but helps us to define it. On the one hand, doubtless, similarity is the most fundamental parameter to decide category membership. On the other hand, we have mostly to resort to intuition both to define similarity and a categorization model, since they are so flexible that otherwise they are overly undefined.

A further limitation for the prototype and exemplar models—as seen in Equations (3.8) and (3.9)—is that they do not deal with the '*complementary category*': the item A can only belong to the presented n possible categories $\{M_1, \dots, M_n\}$. But, how can we compute the probability that the item A does *not* belong to any of those categories? For example, if we consider only the categories `cat`, `dog`, and `horse`, we can never classify an item as `fox`.

The problem of the complementary category boils down to the disjunctive question 'whether an item A belongs to a certain category M or not'. To answer such question, we are left with one single option: to intuitively choose a threshold value (fuzzy or crisp) on the similarity between the item A and the category prototype or the category exemplars: above this threshold, we tend to accept category membership of the item, below it, we tend to reject it (See, E. J. Wisniewski 2002, p. 470; see also, Hampton 1995).

Finally, a more elaborated similarity-based categorization method is '*clustering*' (McDonnell and Gureckis 2011). A cluster model, like prototype and exemplar models, is founded on similarity to assign objects to clusters (i.e., categorize) or create new clusters (create new categories). The great assets in clustering are the flexibility in categorization—which allows combination of prototype and exemplar effects—and the possibility that the model proactively creates new categories (which avoids the problem of the complementary category). Even so, the process of category creation is tuned by, at least, one parameter. Thus, once again, our intuition guides the process of computational category formation.

A. The controversy: prototype vs. exemplar model

One of the most lively and long-standing controversies in categorization, or psychology of concepts is which model of categorization fits better human cognition: *prototype* or *exemplar* model—This debate began at the end of the 70's and is still active. Due to its relevance, I summarize key facts that lead to an answer (See, Murphy 2002, pp. 95–114; Minda and J. D. Smith 2011, Sect. Motivation).

At first glance, in the 80's and 90's, numerous experiments showed a higher validity of the exemplar than of the prototype model (See inexhaustive list, Nosofsky 1992b, p. 149)—Experiments that, admittedly, were mostly conducted by D. L. Medin (the founder of the exemplar theory) and his colleagues, including R. M. Nosofsky (See acknowledgments, 1984), the founder of an extended version of the example theory, the '*generalized context model*' (2011). Yet, such experiments relied on categories with a remarkable unnatural structure (precisely opposite to the natural categories): lower within-category and higher between-category similarity, minuscule number of features (usually four), binary features (i.e., present or absent), categories of reduced size. Most remarkably, these characteristics, as a whole, blatantly favour the memorization of the single exemplars by the subjects (Blair and Homa 2003), as in the exemplar model, instead of synthesizing a summary representation category, as in the prototype model.

In the above mentioned experiments, a final blow deceptively enhanced the bias towards the exemplar model: the *data aggregation*. J. D. Smith et al. (1997, pp. 669f.) proved that fitting the *group* data instead of fitting *each subject* individually favours the exemplar model. They simulated groups with half of subjects categorizing according to prototypes and the other half according to exemplars, and the aggregated data showed an overall best fit of the exemplar model.

When we correct for the above mentioned experimental bias, we find compelling evidence that prototypical categorization is far more prevailing than exemplar in humans—also in animals (2008). For example, the *default strategy* for categorization in humans seems to be prototypical, that is, the subjects begin categorization experiments using prototypes and change strategy in case the categorization feedback shows unsatisfactory results; but this change occurs only after numerous trials (J. D. Smith and Minda 1998, p. 1419f.), that is, subjects retain a deficient prototypical categorization even when they repeatedly received negative feed-back on exception items (J. D. Smith et al. 1997, p. 666).

Another issue that severely contradicts exemplar model and rather underpins the prototype theory is *learning performance*. Exemplar models do not significantly predict differences in learning performance for categories with different geometry; that is, in exemplar theory, although a category had a convoluted domain (e.g., with disconnected or highly non-convex regions) the learning performance should be similar to a category with a simple domain (e.g., with a single straight boundary). However, it is well established that simple domains, e.g., '*linearly separable*', (See def. Russell and Norvig 2014d), are much easier to learn than more complicated ones, e.g., '*non-linearly separable*' (J. D. Smith et al. 1997, p. 679; Ashby and Maddox 2005, p. 159). In that sense, the prototype theory shows a greater experimental validity: it reproduces phenomena that subjects exhibit in the category learning process. Certainly, if we allow only *one* prototype per category, prototype theory can only fully learn categories that are linearly separable: the exceptions remain unlearned. But such exceptions can be added by direct memorization, if we augment the prototype model into the '*mixture model*' (J. D. Smith and Minda 2000, p. 12)—whose performance is better than exemplar model (ibid., Tab. 2)

In conclusion, we deem prototypical categorization one of the most ready categorization methods in human cognition, which is obviously tailored to process natural basic categories (those with many members, features, and usually sharp differentiation). The extensive memorization of exemplars—thus, eventually, the exemplar model—occurs only in the very odd cases when the abstraction of a prototype is burdensome, which is not the case in the vast majority of natural environments (ibid.; 1998; but see reply, Nosofsky 2000).

3.4.3 Backwards: From Dissimilarity to Features

Given a certain distribution of dissimilarity values between items, we can try to elucidate the '*diagnostic features*': those features who decisively determine the categorization rule (Goldstone 1996, p. 611). To that end, we search for a metric space in which each item must be represented as a point, and the distances between them must fit the given dissimilarities—such technique is called '*multidimensional scaling*' (Dunn-Rankin et al. 2004), abbreviated as '*MDS*'.

The coordinates of the points (i.e., the items) in such metric space can be assimilated to *continuous features*. That is, the features arise from the 'dimensionalization' of scalar dissimilarity results. For example, Rips et al. (1973) obtain the MDS of experimental dissimilarities between animals as a two-dimensional space with the Euclidean distance; the two coordinates are interpreted as *size* and *predacity*.

MDS has, regrettably, two main downsides. First, we cannot unequivocally determine the dimensionality of our solution space. There is no optimal number of dimensions: we can always reduce the fitting error by increasing the dimensionality—It is obvious. Since any dimensional space can be perfectly embedded in a space of higher dimensionality, the fitting error must be lower or, at most, equal for higher dimensions. Although, in concrete cases, some methods provide a criterion to choose the dimensionality (e.g., M. D. Lee 2001), truth is that, ultimately, lower dimensions are chosen (2 or 3) because the data can be better visualized and the dimensions more easily interpreted. For example, Rips et al. (1973, p. 11) acknowledge that they cannot interpret as features the coordinates of higher dimensions, even though for higher dimensions they obtain a better data fitting:

> [T]he correlations in question can eventually be made significant by further increases in dimensionality, but this seemed a pointless endeavor since solutions for the [...] mammals spaces in 4 and 5 dimensions already resulted in relatively uninterpretable dimensions.

Additionally, and connected to the first, the interpretation of the dimensions, as valid intelligible or perceivable features, is not always straightforward nor clear even in the lowest dimensionalities (See Sect. 3.4.1.A).

Altogether, MDS solutions are highly dependent in the judgement of the researcher. Nevertheless, we concede it is a powerful tool for data reduction: we represent $\frac{n(n-1)}{2}$ scalar distances as $n \cdot d$ scalar components, where n are the number of items and d the number of dimension. Thus, as long as $d < \frac{n-1}{2}$, we effectively reduce data size. Moreover, if the solution has 2 or

3 dimensions, it can be humanly interpreted—as we said above. In other words, we obtain a more parsimonious model than the dissimilarity matrix (See also, Irwing et al. 2018), even if the cognitive validity of the transformation might be disputable.

3.4.4 Objections to Similarity

Despite all their successes, similarity models have their downside: they have many *parameters* that must be determined, and those parameters vary depending on external variables, such as the stimulus context, or the categorization task (See, Tversky 1977). Actually, we could argue that a similarity function has *too many* parameters, and, even worse, we cannot determine all of them systematically. Therefore, researchers restrict manually the parameters considered for similarity, and, furthermore, they fix the external variables, e.g., the categorization context.

Even in the most simplified similarity models, we must keep following elementary parameters: the items *features*, and, also, their associated *weights*, which materialize as the *attention weights* w_i, in the metric models (Nosofsky 1992a, p. 367), and as the *salience function* f, in the nominal models—Such parameters are ubiquitous. Even if we restrict ourselves only to such parameters alone, to determine the similarity between objects seems a daunting task, if not a lost battle.

For example, if we want to determine the similarity between `apple`, `olive`, and `soccer ball`, we first should determine how many features are available. Unfortunately, there are, virtually, infinity features available. Let us see: we could consider *size*, *colour*, *form*. . . ; and also consider *whether it rots*, *whether it has seeds*, *who/what produces it*. . . ; and even more features: *whether it is edible*, *how good we can play soccer with it*, *whether it floats on water*. . . A next step on similarity-based categorization is to *choose* the features, or rather its relevance, i.e., the weight of each feature (the irrelevant features have zero weight). Obviously, for practical reasons, researchers *choose* a finite number of features.

Now we can show, how the weight, or, equivalently, the choice of features can yield complete different similarity results. If we consider following features: *who/what produces it*, *whether it has seeds*, and *whether it is edible*; `apple` is much similar to `olive` than it is to `soccer ball`. But, if we consider following features: *form* (`apple` is more spherical than `olive`), *whether it floats on water* (`apple` floats `olive` does not), and *how good we can play soccer with it* (`apple` can be better kicked than `olive`); then `apple` is more similar to `soccer ball` than it is to `olive`.

Hence, the necessity of manually choosing from a set of infinity features and determining its weights makes similarity flexible enough to produce any possible result and wipes away any trace of falsifiability (Popper 1935).

The weak, if any, explanatory power of *similarity* was at fiercest criticized by Goodman (1972a, p. 437):

> Similarity, I submit, is insidious [. . .], ever ready to solve philosophical problems and overcome obstacles, is a pretender, an impostor, a quack. It has, indeed, its place and its uses, but is more often found where it does not belong, professing powers it does not possess. [. . .] [O]nly recently have I come to realize how often I have encountered this false friend and had to undo his work.

In sum, though, in the literature, we find similarity functions everywhere yielding good categorization results (e.g., Nosofsky 1991b; Volkert et al. 2018; 2019), similarity alone cannot account for categorization—researchers *have to choose* the computation model (e.g., metric spaces, set-theoretic functions, ultrametric trees), its appropriate similarity or distance function (e.g., euclidean or city-block metric), and the relevant features (e.g., size, colour, predacy). Thus, similarity has a limited explanation power, since the researchers' choices play a dominant role.

In other words, similarity help us to express categorization in terms of models, parameters, and features, but sadly it cannot explain why certain choices are made to the detriment of others.

3.5 Boundary models

‘*Boundary models*’ are, in a sense, a case of *metric models*. Items are likewise represented as points in a featural metric space, the *psychological space*. However, they differ from the traditional metric models, i.e., *prototype* and *exemplar* models, in how categories are represented: *boundary models* represent a category by means of its borders rather than its members, as we see in Figure 3.4. Note that since boundary models are metric spaces, they can be endowed with a similarity function—this enriches the model with the expected boundary *fuzziness* and membership *gradation* of natural categories.

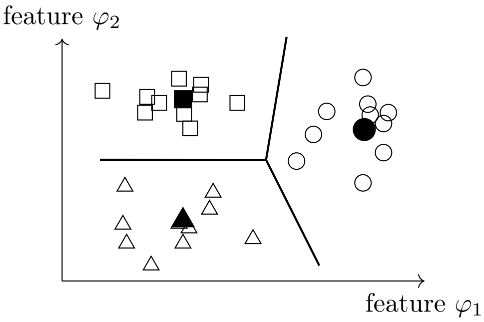

Figure 3.4: Three categories: `square`, `triangle`, and `circle`. The filled figures mark the *prototypes*, the unfilled figures mark the *exemplars*, and the lines mark the *boundaries* between categories.

To a great extent, the three category models are formally equivalent (boundary vs. prototype, Goldstone et al. 2012, p. 616; boundary vs. exemplar, Shanks 2001, p. 2492) and each has its representational advantages. Nonetheless, we underscore the virtues of the boundary models, since the story-based categorizations are constructed as such.

First of all, the boundary models allow a more immediate *decision making*: they define the border between categories, and, consequently, the computation of *category membership* is straightforward. In fact, boundary models can be also seen as a particular case of *classical* or *rule-based* models (Shanks 2001, p. 2492): a sharp border works as a *logical rule* to decide for category membership.

Most importantly, we argue that boundary models are not only a formal device to define categories using their borders—as opposed to using, for example, their prototypes or exemplars—but boundary models are the most natural approach for dealing with certain types of categories, notably, those generated by ‘*conceptually meaningful borders*’; that is, when the borders between categories are also categories. This is the case of geometric concepts, such as parallelism, perpendicularity, etc.

In this work, we deal with motion categorizations that originate from *spatial* categorizations; for that reason, geometric concepts are pervasive. Accordingly, our motion categories are naturally defined as boundary models, and, moreover, the borders between motion categories constitute also new categories. For that reason we further illustrate boundary models and borders below.

An application of boundary model

We take the concept `skew` into consideration. Is this concept more naturally defined through a prototype or through a border? According to the Oxford English Dictionary, `skew` means “having an oblique direction or position, turning to one side, slanting, squint”. This definition uses *borders*, i.e., paraphrasing, `skew` means “having angle between 0° and 90°”. If we attempt to define `skew` by means of a *prototype*, then we come short, as we next argue.

We assume that the prototype of `skew` is the middle point, i.e., 45°, of the categorical region, the interval (0° 90°). This assumption relies on the key property of *central tendency*: the most typical items are those most similar to the average values of the category members (Sect. 3.3.1).

Thus, a *prototypical definition* of `skew` could be verbalized as "having a direction similar to an angle of 45°". It seems, though, that such a prototypical definition is equivocal, because it contradicts the *family resemblance* principles. For example, the prototype of `parallel` is, doubtless, 0°, and, given that 10° is more similar to 0° than to 45° (building the similarity on the difference of degrees), we conclude that 10° should be rather categorized as `parallel` than `skew`—That seems not to be the case. Alternatively, we check, whether 45° fulfils the typicality properties of a prototype, i.e., whether it is classified as more "*typically skew*" than any other angle: again, the result is unclear. Is it a 45° *skewer* than, for example, a 10° angle? Geometrically, 45°, though undoubtedly `oblique`, has a sense of geometric stability that we cannot so readily reconcile with the "slanting" or "squint" belonging to `skew`. This is not the case of a 10° angle: we can more directly label it `oblique`, `slant`, and `squint` than 45°.

Summarizing, the `skew` seems to be most naturally modelled by means of boundaries. Additionally—as we claimed for the boundary models—the category borders are conceptually meaningful; for example, the borders of `skew` are 0°, the concept `parallel`, and 90°, the concept `perpendicular`.

3.5.1 Meaningful borders

We are persuaded that '*meaningful borders*' is the distinctive mark of the boundary models, as we here argue. In any psychological space, categories have boundaries which can be both mathematically sharpened or blurred at will—This is also the case for the prototypical or exemplar models. However, for both the prototypical and exemplar models, the boundaries are cognitively irrelevant regions of the psychological space.

To illustrate, we consider the categorical space of `fruit`, which fits in the prototype model. The border between `orange` and `grapefruit` does not correspond to any meaningful category: We can neither easily imagine how the fruit at the border between `orange` and `grapefruit` look like nor call such border with a specific term. In contrast, we can easily visualize the borders between the concepts `right` and `left`; and we have terms for it: these are the concepts '`front`' (the border ahead) and '`back`' (the border behind) when we consider an oriented entity, such as a human.

Note that the borders of categories have a lower dimension than the categories themselves. Thus, even when such borders are meaningful and form new categories, they are distinguished from the original categories by their dimensionality. For example, on the plane, `right` and `left` are half-planes (i.e., two-dimensional), and the borders `front` and `back` are a lines (i.e., one-dimensional).

We can recursively find the borders of border categories, and they are again meaningful. For example, the border categories `front` and `back` (both one-dimensional) have a border, which is the `origin` or `centre` (zero-dimensional), that is, the very location of an entity.

Apparently, when we claim that, in boundary models, the borders constitute categories related to meaningful concepts, we contradict the conclusive evidence that category boundaries are fuzzy transition regions (See Section 3.3). But the solution to this apparent contradiction lies on considering the transition between the categories of different dimensionality. We saw above the two-dimensional categories `right` and `left` separated by the one-dimensional category `front`. In that case, there is indeed a fuzzy border between the categories—as we should expect—though not between `left` and `right`, but rather between `left` and `front`, or between `right` and `front`. Certain positions cannot be exclusively categorized as `front` or `right`, but 'rather front' or 'extremely right', which confirms both *fuzziness* and *gradation*. The key is that fuzziness and gradation appear not directly between the two-dimensional categories (i.e., `right` and `left`), but between the two-dimensional categories and its one-dimensional border (i.e., `front`).

3.6 Between-Category Structure: Hierarchical Taxonomy

Similarity, as we have seen, provides essentially knowledge about the within-category (i.e., inner) structure of categories—it relates pairs of category items. In this section, we want to take a look at the between-category (i.e., intercategory) structure.

It is widely accepted that people organize categories as a *hierarchical taxonomy* (Murphy and Lassaline 1997, p. 96; E. J. Wisniewski 2002, pp. 506f.). A '*hierarchical taxonomy*' is a structure whose elements (e.g., the categories) are related by *inclusion*—We will use the simplified term '*taxonomy*' as a synonym, because we only deal with hierarchical ones. We can refer to the inclusion relation in a taxonomy as the '*is-a*' relation (Collins and Quillian 1969), e.g., '`apple` *is a* `fruit`' means that each object belonging to the category `apple` belongs also to the category `fruit`. An example of everyday hierarchical taxonomy is displayed as a tree in Figure 3.5: `apple`, `pear`, and `banana` are `fruit`; `bean`, `lentil`, and `pea` are `legume`; both `fruit` and `legume` are `vegetable`.

Noteworthy, in a hierarchical taxonomy of categories, we exploit the set properties of categories (i.e., union, inclusion, and intersection). Since, in the taxonomy, each category is included in a '*supercategory*', we have that each category can be extensively defined as the *union* of its '*subcategories*'; for example, `fruit` is the union of `apple`, `pear`, `banana`, and so forth.

Additionally, we might freely use the union of categories. For example, we might create a new category $\texttt{favouriteFruit} = \texttt{apple} \cup \texttt{lychee}$, which captures someone's fruit preferences, and, therefore might be relevant on a personally relational level. Also, beyond this personal categories, we might observe the union operation in ordinary situations: We can generate a new category $\texttt{citrus} = \texttt{orange} \cup \texttt{lemon} \cup \cdots \cup \texttt{pomelo}$ which shows coarser information. We might use such a category the first time we see a `lime`: if we do not know its specific name, we might resort to call it or identify it as a `citrus`. Even to verbalize the undefined taste of a soda, we might say the soda has a `citrus` taste. It seems, thus, that union of categories to define more general categories is a cognitively plausible process.

Featural Approach Furthermore, if we use a *featural approach*, by which we define a category as the set of elements that have certain features, then, each category must share all its features with each subcategory. For example, `apple` shares its features (juicy, fist-sized, having seeds, and so on) with the subcategories `granny smith`, `red delicious`, `royal gala`, ... And this recursively applies to each subcategory, e.g., `fruit` shares its features with `apple` and `granny smith`. From a set-theoretical perspective, the features of subsequent subcategories in a taxonomy yield a sequence of inclusions; for instance,

$$\cdots \subset \text{features}(\texttt{fruit}) \subset \text{features}(\texttt{apple}) \subset \text{features}(\texttt{granny smith}) \subset \cdots$$

Remarkably, in the featural approach, we can create hierarchical taxonomies with maximum flexibility: The uppermost category contains no feature; we define the next subcategories according to the values of a certain feature, e.g., φ_1; we subdivide the subcategories according to another feature, e.g., φ_2, and so on; so that each category is defined as the series of values that the features have on the levels above such category. For example, beginning on the uppermost category `vegetable` we can take the feature 'sweetness' and create subcategories according to the sweetness values; subsequently we can take the feature 'juiciness' to further subcategorize

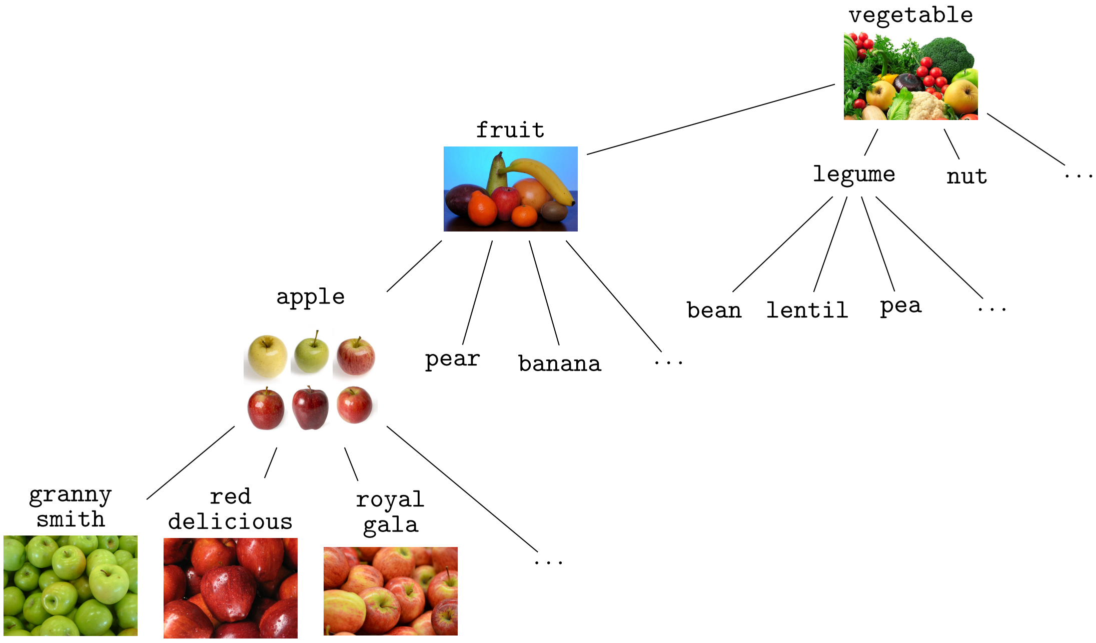

Figure 3.5: Hierarchical taxonomy of `vegetable` subcategories represented as a tree according to inclusion. The features of a category (e.g., `fruit`) are included in its children (e.g., `apple`, `pear`, `banana`, . . .). Children of a category are called *subcategories* and parents are called *super-categories*. The *basic level* of this taxonomy is arguably `apple`, `pear`, `banana`, `bean`, . . .

Photos sources: `vegetable`, by Olearys; `fruit`, by Charli Lopez; `apple`, derived work from fotos by Apple and Pear Australia Ltd; `granny smith`, by Deborah Fitchet; `red delicious`, by Zajac; `royal gala` by Apple and Pear Australia Ltd. All used under CC BY 2.0.

the subcategories, and so on. Note that, by modifying the order in which the features are chosen, we obtain different taxonomies. Later, when we categorize motion, we can see this effect: two different choices of features yield two different hierarchical taxonomies of the categorization Stories-OPRA$_1$ (Figs. 8.5 and 11.7)

We have seen that each category has more features than its containing supercategory. This fact is expressed using the concept '*informativeness*' (Murphy and Lassaline 1997, p. 106), also called '*specificity*' (Murphy and Brownell 1985), which is the amount of information that a category provides. Obviously, the greater the amount of features a category has, the greater its informativeness; and the lower a category lies in the taxonomy the more informative such category is.

Note that *informativeness*, plays a primary role in deciding what category one should use to refer to an object. One can decide to provide more information about an object by moving from a supercategory (e.g., `fruit`) into a subcategory (e.g., `apple`). We call this process of being more specific '*specialization*'. However, sometimes one can decide to remain general, and opt for a supercategory—for example, a physician may recommend, for health reasons, eating more fruit, not exclusively apples[2]. We call this process '*generalization*'. In that way, we see the taxonomy as consisting in '*levels of abstraction*', each level associated to a certain informativeness. The higher a category is in the hierarchy (e.g., `vegetable`) the higher level of abstraction: categories are more general and less informative. The lower the category is in the hierarchy (e.g., `granny smith`) the lower level of abstraction: categories are more specific and informative.

3.6.1 Category taxonomies in artificial intelligence

Category taxonomies have useful properties for knowledge representation and reasoning, so that they are broadly adopted in artificial intelligence; in fact, they constitute the most basic implementation of an '*ontology*' (Russell and Norvig 2014c). The key property, is the possibility to quickly draw '*inferences*'. For example, if we know that `vegetable` is edible, and `legume` is a vegetable, we can *infer* that *legume* is edible, and—iterating the process—that beans, lentils, and peas are edible. In other words, we can say that the features of categories are preserved by *specialization*.

As advantageous side effect, inference allows us store information more efficiently: The features of a certain category need not be stored in all their subcategories, thus, saving storage space. For example, the feature of 'being edible' need only be stored in the category *vegetable* and not in any subcategory (i.e., fruit, legumes, apples, etc.), since we know, due to inference, that a category shares its features with all its subcategories.

Importantly, both the ability to draw inferences and store information efficiently lie wholly in a featural approach of the categories in which *all* members of a category share certain features, or, equivalently, the features of a category are the intersection of the features of all its members (or subcategories). That is, if there is no common feature to all members of a category, then, inference is either impossible, or restricted to probabilistic values, or forced to deal with exceptions—The storage of information is similarly affected. A paradigmatic example is the item `penguin` which belongs to the category `bird`: `penguin` lacks the feature `can fly`; therefore, drawing inferences and storing information is hampered by the item `penguin`—but also by `ostrich`—in the category `bird`.

All things considered, hierarchical taxonomies are a significant structure regarding the applications of categories. For that reason, it is worthwhile to build—at least, to explore the possibility of building—hierarchical taxonomies for any given categorization. In this work, we undertake

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[2] Here, we are not questioning the validity of the folk wisdom: "An apple a day, keeps the doctor away".

this task for our motion categorizations (Sect. 11.7.3): we briefly show the possibilities of the featural approach to build taxonomies in the story-based categories.

3.6.2 The Basic Level

In view of the hierarchical structure of categories, a question arises: which is the most suitable hierarchical level to refer to items? For example, if we are shown a *granny smith apple*, as in Figure 3.6, how should we call it? We can call it just `granny smith`. But we may also name it using a *supercategory*: `apple`, `fruit`, or `vegetable`, as shown in the taxonomy of Figure 3.5.

It is cognitively well-established—both psychologically (E. Rosch, Mervis, et al. 1976) and anthropologically (Berlin 1978; 2014, pp. 31–35)—that humans have a preferred level to name or refer to things: the '*basic level*'. For example, the basic level for the item in Figure 3.6 is `apple`; thus, under ordinary circumstances, we would refer to Figure 3.6 as `apple` and not as `vegetable` or `granny smith`.

Figure 3.6: Granny Smith apple.

Source: by Apple and Pear Australia Ltd used under CC BY 2.0.

The choice of a basic level is far from obvious. Indeed, all four mentioned categories (`vegetable`, `fruit`, `apple`, and `granny smith`) are true of the item in Figure 3.6, and each category offers advantages. On the one hand, using general categories (i.e., `vegetable`) we save processing resources—we can identify an item as member of a category considering less features than for specific categories. And, using general categories also increase the chances of accurate identification—we are more certain of an item belonging to a general category (e.g., `fruit`) than a specific one (e.g., `granny smith`). On the other hand, specific categories provide a much richer featural information, i.e., they have greater informativeness. For example, knowing that an object is `granny smith` provides additional information of colour (green), and taste (sour), than `apple` alone cannot provide.

The most common explanation for the existence of the basic level is the '*differentiation*' of a hierarchical level (Murphy and Lassaline 1997, pp. 106–107; originally, Mervis and Crisafi 1982; also, Murphy 2002, pp. 217–223), i.e., at the basic level the concepts are most *differentiated*. The degree of differentiation of the hierarchy levels increases with the increase of two factors: *informativeness* and *distinctiveness*.

The *informativeness* is related to the within-category similarity. The more similar the items are within a category, the greater amount of features they share, that is, the greater the informativeness is. The '*distinctiveness*' is related to the between-category dissimilarity at the same hierarchical level. The more different a category is, compared to the other categories at the same level, the more distinctive such category is.

Notice that, as we move from the higher to the lower hierarchy levels, the informativeness, the within-category similarity, increases; e.g., the average similarity between items of the category `apple` is higher than between items of `fruit`, and those items of `fruit` are more similar between them that the items of `vegetable`. But, as we move to lower hierarchical levels, distinctiveness, between-category similarity, decreases; e.g., the dissimilarity between the subcategories of `apple` is lower than the similarity between the subcategories of `fruit`. Consequently, the most differentiated hierarchy level is the the level with the *higher combined value* of informativeness and distinctiveness. Admittedly, how the "higher combined value" is computed remains open—we can use endless mathematical methods. For example, Mervis and Crisafi (1982) computed the differentiation as *within-category similarity* − *between-category similarity*, which is equivalent to *differentiation* = *informativeness* + *distinctiveness*, where *informativeness* = *within-category similarity* and *distinctiveness* = −*between-category similarity*.

All in all, the differentiation theory provides a plausible principle to explain and compute the basic level, which has been successfully validated experimentally.

In this work, we did not try to determine the *basic level* in motion categorizations, though it manifests in a variety of cognitive areas: language, mental representation, identification (E. Rosch, Mervis, et al. 1976). For that reason, because it connects experimentally different cognitive areas, we regard the basic level as a powerful measure of cognitive plausibility of a categorization. We expect to determine it as future work.

3.7 Categorization in Philosophy

In this section, we establish a link between the scientific results presented above and the work in philosophy. We chiefly endeavour to ease the reading of philosophical works by relating the specific terms about categorization between science and philosophy. Further, we deem it necessary to mention the philosophers that have influenced or, rather, inspired the categorization research.

A fundamental area of philosophy dealing with categorization is '*ontology*'. It is solely concerned with the study of reality at the most basic level, e.g., matter, persons, events, concepts, ideas; that is—as ambitious as it may sound—a *general theory of being*. *Ontology* is often seen as a part of '*metaphysics*', and sometimes even as a synonym for it. It all depends on how we define each of both concepts—an issue upon which philosophers disagree (See, Macdonald 2005, pp. 3–8; Bunnin and Yu 2004, pp. 429, 491).

Interestingly, based on the definition of *ontology*, we expect no imperative relation to *categorization*, but truth is that ontology seeks primarily to answer *which kinds* of beings exist and how these kinds relate to each other. In fact, Macdonald (2005) gives a more colloquial definition of ontology: "it is the study of what kinds or sorts of things there are in the world" (e.g., also, Chisholm 1992; Cumpa and Tegtmeier 2011). In order to show the tight link between ontology and categorization, we reformulate the target questions of ontology as "Which are the most basic category levels ever?" and "What is the relation among such categories?".

Concluding, *ontology* contains the philosophical view of what, in science, we call *categorization*. In that sense, philosophers have inspired work in categorization, chiefly W. V. O. Quine (1969), advocating for modelling, for using theories to determine *category membership* (Murphy and Medin 1985), and also N. Goodman (1972b) with his devastating critique on *similarity* (Sect. 3.4.4).

Chapter 4

Spatial Categorizations

Spatial Representations

4.1 Introduction and Related Work

'*Spatial categorizations*'—a very particular sort of categorizations—are at the foundation of our story-based categorizations of motion. Spatial categorizations further permeate many areas of science, which study them under diverse approaches. In the following, we explain what makes spatial categorizations so peculiarly relevant, how some science areas approach them, and, foremost, which of such approaches on spatial categorizations we have embraced to develop our motion categorizations. Thus, here, we pave the way to understanding both the generation method of our motion categorizations and their properties, which largely base on the spatial categorizations called *qualitative spatial representations*.

4.1.1 Spatial categories as relations between entities

Spatial categorizations describe spatial configurations of entities. As an illustration, we express some spatial categorizations for the entities in Figure 4.1; we use an everyday spatial categorization: the allocentric reference system `front`, `behind`, `left`, `right`. According to this system we can say following:

Example 4.1 The *bottle* is in `front` of the *plate*

Note that unlike most categories, the spatial ones cannot refer to a single entity, but they relate at least *two* entities—In the previous example, we relate *bottle* and *plate*. This is the first particular property of spatial categories: they are '*relations*', that is, they link several entities, mostly two entities. In the following example, we relate three entities.

Example 4.2 The *cup* is `between` the *bottle* and the *plate*

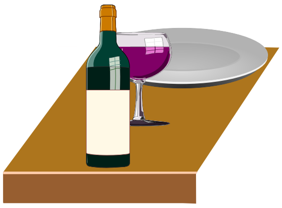

Figure 4.1: Spatial configuration of a *bottle*, *cup*, and *plate* used in Examples 4.1 to 4.4

Source: Derivative from pictures in publicdomainvectors.org used under Creative Commons Deed CC0. Licensed by J. Purcalla A. under CC BY 4.0.

Because they *relate* entities, '*spatial categories*' are largely known as '*spatial relations*', and so we will predominantly use this term, although we will occasionally opt for the term *spatial category*.

4.1.2 Cognitive and linguistic aspects of spatial categorization

Another peculiarity: spatial relations have a fundamental role in cognitive development and linguistics. Concerning cognitive development, it is staggering how early children learn spatial relations; for instance, the relation `contained`, i.e., an entity fully inside another, begins to be learned at only 2½ months, when children already know that the container must have an opening, and is fully learned at 7½ months, when children know that an object cannot be totally inside a shorter container (Hespos and Baillargeon 2001a; b; See summary, Mandler 2004, p. 111–115); similarly, the relations `above` and `below` are grasped already by 3-month-old children (Quinn 1994; Quinn et al. 1996).

Concerning linguistics, the spatial relations play a central role, because they are encoded in a cornerstone of our language: the '*prepositions*', e.g., *in front of*, *above*, *left*. Certainly, the basic elements to verbalize an object's location in standard English are three (See Example 4.1): the '*located object*'[1] (e.g., bottle), a '*reference object*'[2] (e.g., plate), and their *relationship* (e.g., in front of); and, importantly, "the relationship is encoded as a spatial preposition" (Landau and Jackendoff 1993, Sec. 2.1). Moreover, as Landau and Jackendoff remark, the number of prepositions is extremely small compared to other lexical units, such as nouns or verbs—English has about 90 prepositions, and only about 10 of them are exclusively time prepositions. Hence, *spatial relations* strongly identify with *prepositions*, a very salient lexical unit.

4.1.3 Formalization of spatial relations

Finally, and most relevant to this work, spatial relations are peculiar in that they have a *precise* and *powerful* mathematical formalization.

Why can spatial relations be precisely formalized? Because spatial, or rather geometrical, features are at the very core of mathematics; in contrast, other features can only be, at most, vaguely defined. For example, as we saw in Section 3.4, Rips et al. (1973) analysed the category `mammal` and, through dimensional analysis, suggested two main features that determine the similarity in such category: *size* and *predacy*. The concept *size* is quite ambiguous: it can be its length, its height, its volume, or combinations of any; and, in any case, can only be statistically defined over a certain population. Even worse, the concept *predacy* is difficult to define at all. Thus, it follows that animal categories, though intuitive, are difficult to define precisely in mathematics. On the other hand, consider our egocentric reference system: `front`, `behind`, `left`, `right`. We can define `front` as being at 0° and `behind` as 180°, `left` as the interval $(0°, 180°)$, and `right` as the interval $(-180°, 0°)$—mathematically precise and simple. Another issue, that we do not treat here, is the '*cognitive plausibility*' of such a definition, i.e., to which extent humans mentally use such a spatial formalization.

Why are the formalized spatial relations powerful? Because we can mathematically operate with spatial relations to obtain new relations. For example, spatial relations usually have an '*converse*', also called '*inverse*'. We obtain it by swapping the spatially related entities from *main* object to *reference* object. As illustration, consider Example 4.1 "The *bottle* is in `front` of the *plate*". In order to swap *bottle* and *plate*, we must use the new relation `behind`, as in the following Example.

Example 4.3 The *plate* is `behind` the *bottle*.

Hence, `behind` is the *converse* (or *inverse*) of `front`; analogously, `front` is the *converse* of `behind`.

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[1] The *located* object is also called '*primary object*' (Hernández 1994, p. 28, Sect. 4.1.1) or '*figure*' (Talmy 1983, p. 232, Sect. 3.2)

[2] The *reference* object is also called '*ground*' (ibid., p. 232, Sect. 3.2)

Another example of operation is the '*composition*', which allows us to derive new relations by combining known ones. For instance, combining both relations in Example 4.1, "The *bottle* is in `front` of the *plate*", and in Example 4.2, "The *cup* is `between` the *bottle* and the *plate*", we can derive a new relation:

Example 4.4 The *cup* is in `front` of the *plate*

Thus, spatial relations have both a precise mathematical definition and a rich variety of operations, which gave rise, in the '90s, to a research area spanning the fields of computer science and artificial intelligence (Pioneer work, Egenhofer 1991; Informative summary, Hernández 1994)—This area is commonly called *qualitative spatial* (or *spatio-temporal*) *reasoning* (QSTR), because of how central *reasoning* is there. Truth is that not all spatial relations have a reasoning apparatus; that is why we find this name inaccurate, and we will seldom refer to it. We will mostly use the terms '*spatial relations*' and '*spatial categorizations*', even though AI researchers refer to them respectively as '*qualitative spatial relations*' and '*qualitative spatial representations*' (e.g., Chen et al. 2015)—sometimes they call the qualitative spatial representations also '*qualitative spatial calculi*' (e.g., Dylla et al. 2017), when they emphasize the ability to operate with them. In Table 4.1, we relate the equivalent terminology used in different science areas.

COGNITION / PSYCHOLOGY	AI / COMPUTER SCIENCE	LINGUISTICS
spatial category	[qualitative] spatial *relation*	spatial preposition
spatial categorization	[qualitative] spatial *representation*	—

Table 4.1: Terminology of spatial categorizations in different science areas. The word 'qualitative' often drops, as indicated by the brackets.

4.1.4 Reasoning with qualitative spatial representations

As said before, when we examine the applications of qualitative spatial representations (e.g., Cohn and Hazarika 2001a; Renz and Nebel 2007; Dylla and Wallgrün 2007), we recognize that one of their main purposes is '*reasoning*'—in the broad sense of the term. More concretely, reasoning with spatial representations helps us to derive new spatial information (new relations) from certain known spatial information (known relations), or to check whether the known spatial information is '*consistent*', i.e., contradiction-free; it also helps us to use the available spatial information to make decisions, to plan, trajectories or movements.

Spatial representations provide two main instruments for reasoning: first, the '*conceptual neighbourhood diagrams*' (Freksa 1992a), which enable decision-making (e.g., Dylla and Moratz 2005; Dylla et al. 2007)); second, they provide operations between qualitative relations, the '*converse*' (also called '*inverse*'; Ligozat (2012, p. xviii)), and the '*composition*', which are the base for the methods that find new relations and check consistency—mostly through '*constraint satisfaction techniques*' (e.g., Cohn and Renz 2008; Ligozat 2012, p. xii; A pedagogic intro, Russell and Norvig 2014a).

It is much easier to show that a qualitative representation is suitable as a spatial categorization (i.e., suitable to qualitatively represent spatial knowledge), than to present its conceptual neighbourhood diagrams, (e.g., Van de Weghe and De Maeyer 2005), or to show its suitability for reasoning through constraint satisfaction techniques, (e.g., Van De Weghe et al. 2005). Indeed, the most important reasoning tool, *composition*, is not even handled in more than 10% of the qualitative representations surveyed by Dylla et al. (2017), while about 30% only describe how to compute it without computing the composition tables.

A main cause is that both *converse* and *composition* can only be computed by using the semantics of the relations (Renz and Nebel 2007). Consequently, the composition often requires a burdensome manual case analysis, (e.g., Cohn et al. 1997, p. 292; Van de Weghe et al. 2005; Mossakowski and Moratz 2010), which, once computed, it is kept in tabular form as a '*composition table*' (e.g., Randell, Cohn, et al. 1992).

The story-based motion categorizations are generated from qualitative spatial representations, and, thus, they naturally inherit their properties: formalization, operations, and reasoning methods. This offers many advantages: we can easily implement the story-based categorizations in artificial intelligence and related disciplines, such as robotics; we can apply methods for trajectory control (Sect. 11.5); and we can develop reasoning methods with the converse and composition (Sects. 10.3 to 10.5)

For all aforementioned reasons, we readily use the formalism of the *qualitative spatial relations*—notation and terminology—when dealing with the story-based motion categorizations. Notwithstanding, we keep the link of story-based categorizations to cognition and psychology, mostly using the related vocabulary. Accordingly, throughout this work, we use three rather equivalent terminologies as shown in Table 4.1

4.2 Spatial Representations: Notorious Examples

Here, we present two qualitative spatial representations, i.e., spatial categorizations, RCC and OPRA_1 that we use to obtain story-based categorizations (Ch. 8). They are well-known and most cited spatial representations: the foundational paper on RCC of Randell, Cui, and Cohn (1992b) surpasses 2300 citations, and the OPRA paper of Moratz (2006) has over 100, according to Semantic Scholar.

From RCC we generate the motion story-based categorizations Stories-RCC and Motion-RCC, from OPRA_1 we generate Stories-OPRA_1 and Motion-OPRA_1 (Ch. 8). These motion categorizations exemplify, throughout this work, most properties and applications of the story-based representations. For that reason, we encourage the reader to acquaint herself with RCC and OPRA_1.

4.2.1 RCC: A Topological Spatial Representation

The '*Region Connection Calculus*', broadly known as '*RCC*', relates two finite regions in a topological space according to their connectedness (Randell, Cui, et al. 1992b). We apply RCC concretely to the two-dimensional euclidian space and convex regions. In this case, RCC simply categorizes the overlapping between regions, resulting in 8 possible relations (Fig. 4.2): DC, regions do not overlap; EC, regions are connected but non-overlapping; PO, regions overlap in the interior, but none is contained in the other; TPP, region k is contained in l and is tangent to the border; TPPI, region l is contained in k and is tangent to the border; EQ, both regions overlap completely; NTPP, k is contained in l and does not overlap the border of l; NTPPI, l is contained in k and does not overlap the border of k.

Note that the relative size of the regions, k and l, constrains the relations EQ, TPP$*$, and NTPP$*$—namely, EQ can only occur when both regions have the same size, TPP and NTPP when k is smaller than l, TPPI and NTPPI when k is larger than l. For the sake of simplicity, we will assume in our examples that both regions are discs, and k is smaller than l (e.g., Fig. 8.2 and Sects. 11.4 and 11.5). That is, only in our examples, we omit the relations TPPI, NTPPI, and EQ, but we include them in our theoretical results, chiefly when we develop our motion categorizations with RCC (e.g., Sect. 8.2.1).

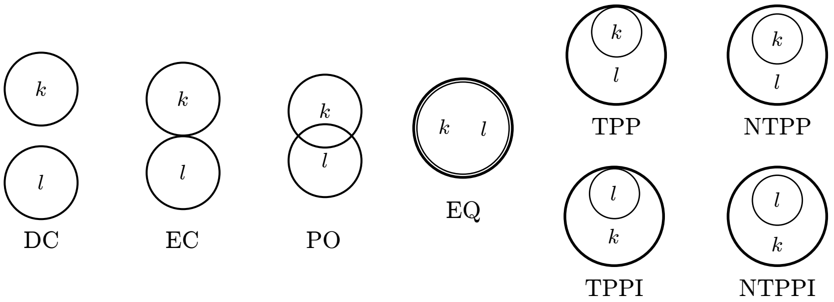


Figure 4.2: The 8 qualitative relations of RCC; namely, DC, EC, PO, TPP, NTPP, EQ, TPPI, and NTPPI; they depend on how two entities, k and l, overlap. Note that the relative size of the entities constraints the possible relations: EQ can only occur when both regions have the same size, TPP and NTPP when k is smaller than l, TPPI and NTPPI when k is larger than l.

The RCC representation presented in this Section is more accurately known as '*RCC8*', because it has 8 base relations. In fact, we find in the literature a family of RCC representations: each family member is called '*RCCn*' where n is the number of base relations. Important variants are RCC5 (Cohn et al. 1997, Sect. 8), which disregards the tangent relations (EC, TPP, TPPI); RCC15, which extends RCC5 for concave entities; and RCC23, which analogously extends RCC8 (ibid., Sect. 5). Even more detailed extension for concave entities are possible, such as RCC62 (OuYang et al. 2007). The most popular of all is RCC8, and, thus, simply known as RCC.

4.2.2 OPRA$_1$: A Directional Spatial Categorization

OPRA$_1$ (Moratz 2006) describes two punctual oriented entities according to their relative orientation, it regards also the case whether the entities are at the same point or not. A single punctual oriented entity partitions the space into four regions (Fig. 4.3) that are numbered as following: `0` is the half line beginning at the entity and extending forwards in the entity's orientation sense, `1` is the half plane at the left of the entity, `2` is the half line beginning at the entity and extending backwards opposite to the orientation sense, `3` is the half plane at the right of the entity.

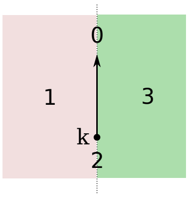


Figure 4.3: The regions that a punctual oriented particle defines under OPRA$_1$: `0` the frontal half line, `1` left half plane, `2` the back half line, `3` the right half plane. Note that the regions have different dimensionality: `0` and `2` are half lines, while `1` and `3` are half planes.

Source: Purcalla Arrufi and Kirsch (2018a)

The relation between two non-overlapping entities (Fig. 4.4) is expressed as the region that each entity occupies with respect to the other entity by using the symbol $\angle$ with the following

(a) Spatial relation $\angle^{0}_{3}$ **(b)** Spatial relation $\angle^{1}_{3}$

Figure 4.4: Examples of OPRA$_1$ spatial relations, $\angle^{b}_{a}$, between two entities k and l that are at different points. The syntax is $\angle^{\text{region of } l \text{ where } k \text{ is}}_{\text{region of } k \text{ where } l \text{ is}}$.

Source: Purcalla Arrufi and Kirsch (2018a)

(a) Spatial relation $\angle_{1}$ **(b)** Spatial relation $\angle_{0}$

Figure 4.5: Examples of OPRA$_1$ spatial relations, $\angle_{a}$, between two entities k and l that are at the same point. The syntax is $\angle_{\text{region of } k \text{ at which } l \text{ points}}$.

Source: Purcalla Arrufi and Kirsch (2018a)

syntax:

$$\angle^{\text{region of } l \text{ where } k \text{ is}}_{\text{region of } k \text{ where } l \text{ is}}$$

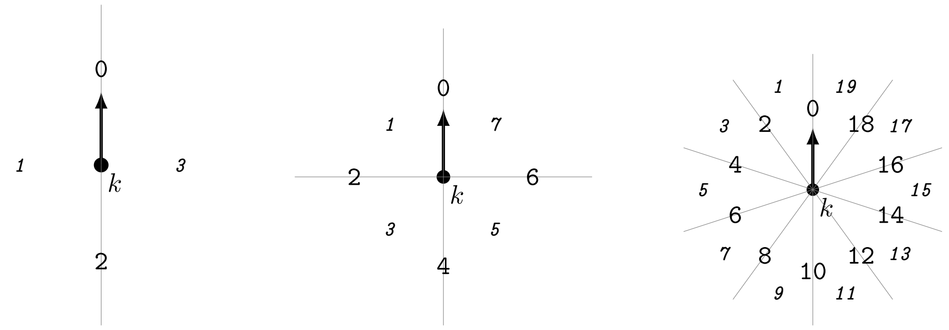
For example, in Figure 4.4a, the second entity l is on region `3` of the first entity k, and k is on region `0` of l. Accordingly, the relation between both entities is expressed as $\angle^{0}_{3}$.

There is, though, the singular case in OPRA$_1$, when the entities overlap, i.e., they are at the same point (see Figure 4.5). In that case, we obtain new categories defined as

$$\angle_{\text{region of } k \text{ at which } l \text{ points}}$$

Similarly as RCC, OPRA is a family of qualitative representations represented as OPRA$_\text{n}$, in which the n parameter sets the '*granularity*' (Moratz 2006, Sect. 2.2). The granularity n is related to the number of regions of the representation: $n = (\textit{number of regions})/4$. For example OPRA$_1$ has 4 regions, and OPRA$_5$ has 20 regions (Figs. 4.6a and 4.6c). Note that the granularity, n, indicates how accurate the OPRAn representation is—namely, the higher granularity, the higher the accuracy. For example, the region `1` of OPRA$_1$ (Fig. 4.6a) can be more accurately subdivided in OPRA$_2$ through the regions `1`, `2`, and `3` (Fig. 4.6b).

Of all OPRA$_\text{n}$, we only use in this work OPRA$_1$, the one with the lowest granularity; and we obtain a quite elaborate and expressive motion categorization: Motion-OPRA$_1$ (Sect. 8.6.2). We anticipate, thus, that future work with higher granularity representations, such as OPRA$_2$, will produce much more expressive motion categorizations, e.g., Motion-OPRA$_2$.



(a) OPRA$_1$: 4 regions **(b)** OPRA$_2$: 8 regions **(c)** OPRA$_5$: 20 regions

Figure 4.6: An entity k and its regions under different OPRA$_\text{n}$ representations with *granularity* $n = 1, 2, 5$. For every representation, the even numbered regions are one-dimensional (half lines) and the odd numbered regions are two-dimensional (infinite plane sectors).

4.3 Properties of Qualitative Spatial Representations

Qualitative spatial representations are spatial categorizations with extended mathematical properties. The building blocks of such representations are the '*base relations*', which are a *finite* set of relations that determine a spatial representation—not only a finite set, but very reduced: about 70% of spatial representations have less the 100 base relations (Dylla et al. 2017, Table II). For example, the RCC representation is determined by the 8 *base relations*, $\{\text{DC}, \text{EC}, \text{PO}, \text{EQ}, \text{TPP}, \text{TPPI}, \text{NTPP}, \text{NTPPI}\}$ (Fig. 4.2). OPRA$_1$ has 20 base relations, the 16 $\angle_a^b$ and 4 $\angle^a$ base relations, where $a, b \in \{0, 1, 2, 3\}$ (Figs. 4.4 and 4.5).

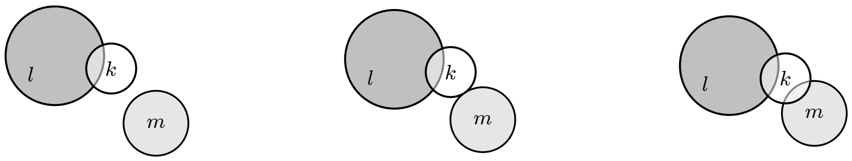
Importantly, the *base relations* are Jointly Exhaustive and Pairwise Disjoint (JEPD), which practically means following: the entities in a certain spatial configuration must fulfil *one and only one* of the base relations—This is a crucial property when operating or reasoning with spatial relations. We can check this property in RCC (Fig. 4.2): two solid figures in the plane must necessarily overlap in one of the 8 basic relations, and they cannot simultaneously fulfil two of such basic relations.

Equivalently, the JEPD property means that a qualitative spatial representation is a '*partition*' of the space that it categorizes. For instance, OPRA$_5$ categorizes relative directions, and in Figure 4.6c we realize how all possible directions are divided, i.e, partitioned, into the 20 regions. A categorization partitions a continuum into categories, and analogously do spatial representations, they partition the space into qualitative relations.

In spatial representations the proper name for a *relation* between entities is '*qualitative spatial relation*', though we shorten it into '*spatial relation*' or, even, '*relation*' when the meaning is clear through the context.

4.3.1 Operations with Qualitative Spatial Relations

We can both perform operations and create new relations with the *base relations*. The most basic operation is the '*disjunction*', i.e., the '*or*' operation. For example, we can define a new relation NTO (Not Totally Overlapped) as $\text{NTO} = \{\text{DC} \vee \text{EC} \vee \text{PO}\}$, which means that if two entities fulfil the relation NTO they must fulfil "DC *or* EC *or* PO", that is, they do not totally overlap. We call '*composite relations*' those relations created by disjunction, i.e., those that are not base relations. Note that, in RCC, with the $|\text{RCC}| = 8$ base relations, we can generate by



(a) $k\{\text{DC}\}m$ **(b)** $k\{\text{EC}\}m$ **(c)** $k\{\text{PO}\}m$

Figure 4.8: Graphical representation of the result of the composition PO $\circ$ DC, which is the composite relation $\{\text{DC} \vee \text{EC} \vee \text{PO}\}$. The relation PO, between k and l, is composed with the relation DC, between l and m, yielding three possible relations between k and m, namely, (a) DC, (b) EC, and (c) PO.

disjunction $2^{|\text{RCC}|} - |\text{RCC}| = 198$ composite relations.

We have two important operations besides disjunction: /converse, of a qualit. relation]converse and *composition*. The '*converse, of a spatial relation*' operation—also called '*inverse*'—consists in finding the new relation that originates when we swap the order of the entities (as already shown in Example 4.3). For example, in Figure 4.7, k has relation NTPP with respect to l; we write it $k\{\text{NTPP}\}l$. The converse to such relation is asking "Which relation has l with respect to k?"; the answer is NTPPI, i.e., l has relation NTPPI with respect to k; we write it $l\{\text{NTPPI}\}k$. Thus, the converse of NTPP is NTPPI, which we express by means of the operator $()^{\smile}$, $\text{NTPPI} = \text{NTPP}^{\smile}$.

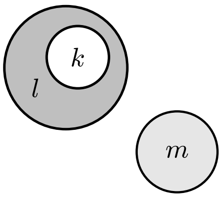


Figure 4.7: Three entities, k, l, and m, displaying different RCC relations: $k\{\text{NTPP}\}l$, $l\{\text{DC}\}m$, $k\{\text{DC}\}m$.

The '*composition*' answers the transitivity question in qualitative relations (Randell, Cohn, et al. 1992). That is, if we have 3 entities, k, l, and m; a relation R_{kl} between the pair, (k, l); and a relation R_{lm} between the pair (l, m), what is the relation R_{km} between the pair (k, m)? As an illustration take Figure 4.7, we have $k\{\text{NTPP}\}l$ and $l\{\text{DC}\}m$, and we ask, then, what is the relation for the pair k, m? The only possible relation is DC, as we see in the figure; thus, we should write $\text{DC} = \text{NTPP} \circ \text{DC}$ with the '*composition operator*' '$\circ$', meaning that DC is the result of the composition of DC and NTPP.

Although, in the previous example, the composition of two base relations ($\text{NTPP} \circ \text{DC}$) yields also a base relation (DC), in most cases, the composition yields a *composite relation*. As example, we show in Figure 4.8 how the composition PO $\circ$ DC yields three possible base relations: DC, EC, and PO (respectively in subfig. (a, b, c)); in other words, it yields the composite relation $\{\text{DC} \vee \text{EC} \vee \text{PO}\}$.

Remember that the aforementioned operations (disjunction, converse, and composition) are the basic tools for solving *reasoning* problems in the paradigm of qualitative representations (Ch. 10).

4.4 Overview on Spatial Representations

We importantly claim in this work that we can generate story-based motion representations from any qualitative spatial representation and substantiate our claim by generating two story-based representations (Motion-RCC and Motion-OPRA$_1$) from the corresponding spatial representations, RCC and OPRA$_1$. Indeed, as these two spatial representations have distinctive features (Tab. 4.2), they compellingly uphold our claim.

Now, the question is "how vast is the domain of applicability of the story-based method?" Or equivalently, "how large is the field of qualitative spatial representations?" As an answer, we provide an overview on qualitative spatial representations (Tab. 4.2) from a survey by Dylla et al. (2017): they list as many as 33 main types of spatial representations, of which some unfold in an endless number of spatial representations by modifying granularity or spatial dimensions. In view of such large number of spatial representations, the possibilities of the story-based generating method are huge.

Qualitative spatial representations in the overview are classified according to two features: the *dimensionality* of the entities involved and the *spatial aspects* described. Regarding the entities' dimensionality, some representations describe points (dimension $= 0$); others lines, curves, segments, intervals (dimension $= 1$); and also surfaces or volumes (dimension > 2). Regarding the spatial aspects, spatial representations describe absolute direction (direction relative to an absolute reference system), relative direction (direction relative to an entity's reference system), distance, topology (mostly overlap), and shape. In sum, there are 15 (3×5) possible classes of representations with only 4 classes without representation—namely, regarding zero dimensional entities, topology and shape are missing, and regarding one dimensional entities, distance and shape are missing.

Table 4.2: Classification of qualitative *spatial* representations according to the dimensionality of their entities and the spatial aspect they describe. Coloured cells indicate representations that span more than one aspect or dimensionality, e.g., Elevated OPRA$_n$ describes *relative direction* and *distance*. The representations used in this work, RCC and OPRA1, are framed in a rectangle for visibility.

Source: Adapted from Dylla et al. (2017, Fig. 4, 5)

<table>
<tr><th>ENTITIES'S DIMENSIONS
SPATIAL ASPECT DESCRIBED</th><th>zero-dimensional (point)</th><th>one-dimensional</th><th>two-dimensional or higher</th></tr>
<tr><th rowspan="2">topology</th><td></td><td>Dipole Connectivity[1]</td><td>RCCn[2]
Nine-Intersection Model[3]</td></tr>
<tr><td colspan="3">Calculus Based Method[4]
Closed Disk Algebra[5]
9^+-Intersection Calculi[6]</td></tr>
<tr><th>absolute direction</th><td>Star Calculi[8]
Cardinal Direction Calc.[9]
Point Calculus[10]</td><td>Alg. of Cyclic Intervals[7]</td><td>Cardinal Direction Relat.[11]
Rect. Card. Dir. Calc.[12]
Block Algebra[13]</td></tr>
<tr><th rowspan="2">relative direction</th><td>LR Calculus[14]
OPRA$_n$[15]
Ternary Point Config. Calc.[16]
Ternary Projective Relat.[17]
StarVars[18]
Single/Double Cross Calc.[19]
3-D Orientation Model[20]</td><td>Dipole Calculus[21]
Alg. of Bipartite Arrang.[22]
Cyclic Ordering[23]</td><td>Visibility Relations[24]
Ternary Projective Relat.[25]</td></tr>
<tr><td>Elevated OPRA$_n$[26]</td><td colspan="2">Region-in-the-frame-of-Directed-Line[27]</td></tr>
<tr><th>distance</th><td>Elevated Point Rel. Alg.[28]</td><td></td><td>Lines of Sight[29]
Region Occlusion Calc.[30]
Occlusion Calculus[31]
(V)RCC-3D(+)[32]</td></tr>
<tr><th>shape</th><td></td><td></td><td>MC-4[33]</td></tr>
</table>

[1] (Wallgrün et al. 2010) [2] (Randell, Cui, and Cohn 1992a) [3] (Egenhofer 1991) [4] (Clementini et al. 1993) [5] (Egenhofer and Sharma 1993) [6] (Kurata 2010) [7] (Balbiani and Osmani 2000) [8] (Renz and Mitra 2004) [9] (Frank 1991) [10] (Vilain and Kautz 1986) [11] (Skiadopoulos and Koubarakis 2004) [12] (Navarrete et al. 2013) [13] (Balbiani et al. 1998) [14] (Scivos and Nebel 2001) [15] (Moratz 2006) [16] (Moratz and Ragni 2008) [17] (Clementini and Billen 2006) [18] (J. H. Lee et al. 2013) [19] (Freksa and Zimmermann 1992) [20] (Pacheco et al. 2001) [21] (Moratz et al. 2000) [22] (Gottfried 2004) [23] (Isli and Cohn 2000) [24] (Tarquini et al. 2007) [25] (Clementini and Billen 2006) [26] (Moratz and Wallgrün 2012) [27] (Kurata and Shi 2008a) [28] (Moratz and Wallgrün 2012) [29] (A. P. Galton 1994) [30] (Randell et al. 2001) [31] (Köhler 2002) [32] (Sabharwal and Leopold 2014) [33] (Cristani 1999)

Chapter 5

Motion Categorizations

Qualitative Representations of Motion

Motion categorization is increasingly attracting the interest of researchers. One obvious reason is that motion categorization is a relatively untouched area compared to the extensively researched spatial categorization; another more pungent reason is the massive rise of mobile positioning (Chen et al. 2015, Sect. 3.4).

Yet “motion categorization” is a daunting endeavour due to the overwhelming variety of motions, unless we restrict the kind of motions we categorize. In Section 1.2, we specified the type of motion that we categorize in this work, i.e., *motion scenarios*—which are two or more entities moving at a certain time instant. We now additionally establish which formalism we use to handle the motion categorizations: we formalize them as *qualitative representations*. Thus, we deal with ‘*qualitative motion representations*’, motion categorizations that possess analogous properties as the qualitative spatial representations (See Sect. 4.3).

In this chapter, we firstly survey qualitative motion representations and classify them in a taxonomy. Next, we describe a simple mathematical technique, concatenation of representations, that is frequently used in the literature to create motion categorizations—We make extensive use of this technique, notably, when building the story-based representations, and we apply it also in Section 11.1.3. Finally, we compare motion and spatial categorization through cognitive linguistics; our aim is to explain a striking result of the survey: in the literature, we find much fewer motion representations than spatial representations.

5.1 Survey of Qualitative Motion Representations

Dylla et al. (2017) surveyed a total of 40 qualitative representations from which they only classify *three* ($< 8\%$) as representations of motion (“relative motion”): QRPC (F. J. Glez-Cabrera et al. 2013), RfDL-3-12 (Kurata and Shi 2008a), and, the most used, QTC (Van de Weghe 2004).

QTC refers to a varied family of representations characterized by suffixes (see concise and complete summary, Delafontaine et al. 2011, Sect. 2): $\mathrm{QTC_{Bxy}}$, $\mathrm{QTC_{C2y}}$. The subindex ‘B’ specifies that only relative approaching and speeds difference between entities are considered, while the subindex ‘C’ specifies that, additionally, relative movement to the right or left is considered; subindex ‘x’ $= \{1, 2\}$ is the dimensionality of the space where the entities move (i.e., line or plane), and subindex ‘y’ $= \{1, 2\}$ relates to the number of kinematic features considered. For example, e.g., $\mathrm{QTC_{B21}}$ (See full description in Sect. 6.3) describes the relative approaching of entities (subindex ‘B’) in 2 dimensions (subindex ‘2’), but it disregards speeds difference (subindex ‘1’). We find also a three-dimensional of QTC, namely, QTC_{3D} (Mavridis et al. 2015), however it is ill-defined for uniform motions and motionless particles.

The categories in a QTC representation are written as a *n-tuple*, each element in the tuple describes a kinematic feature always by means of three symbols $\{-, 0, +\}$. For example, consider the kinematic feature ‘relative approaching of entity k to entity l’. The ‘$-$’ symbol means that k moves towards l, the ‘$+$’ symbol means that k moves away from l, and ‘0’ means that k remains stationary with respect to l.

The RfDL-3-12 considers the straight path described by a point with respect to a two-dimensional static region. Thus, we classify it as a ‘*path representation*’—a subclass of the motion representations. A similar path representation is Double Cross (Freksa 1992b; Zimmermann and Freksa 1996), which is defined like the RfDL-3-12, but instead of using a region as reference, it uses a static point to describe the straight path of a displaced punctual entity. In these path representations much kinematic information is lost: only the entity’s displacement is measured—time is irrelevant—therefore, speed and acceleration are disregarded. In fact, the path considered in these representations need not be an entity moving, it suffices that it is the path between two different landmarks. For that reason, we see these path representations rather marginally as motion representations.

More elaborated path representations are those describing polygonal trajectories qualitatively, as the QMV (Qualitative Motion Vector) sequences (Musto et al. 1998, 1999). The QMV sequences include time since they are obtained by scanning at a fixed rate the positions of an entity’s trajectory. QMV sequences include, amongst others, qualitative information about the speed of the entity (`slow`, `medium-vel`, `fast`, . . .). Nonetheless, this qualitative description addresses *only one* entity’s motion—we do not have a relation between two entities—thus, it falls without the scope of this work.

In another survey of qualitative representations, Chen et al. (2015, Sect. 3.4) find *three* motion representations: QTC (already mentioned), and two additional representations, Dipole Calculus (Moratz et al. 2000) and DIA (Directed Intervals Algebra) (Renz 2001).

Interestingly, Chen et al. and Dylla et al. only coincide with classifying one representation as motion, namely, QTC, while they disagree in classifying DIA and Dipole Calculus as motion representations. We explain such disagreement by remarking a fine distinction: representations such as Dipole Calculus and DIA (also OPRA (Moratz 2006)) are primarily *relative direction* or *orientation* representations rather than *motion* representations—they ignore speed. However, they can be used to represent moving entities by equating *orientation* with *velocity direction* (e.g., Dylla et al. 2007). For that reason, they are sometimes classified as motion representations.

Besides the mentioned motion representations, we also found one developed by Wu et al. (2014). That is the only one, to our knowledge, that deals explicitly with regions. It combines the spatial representation RCC with the distance between regions, hence, we call it RCC-d (see details, Sect. 5.4.2.A).

Concluding, all the aforementioned representations, excepting the *path representations*, can be used to categorize *motion scenarios*—our research endeavour (Sect. 1.2). Notwithstanding, from the representations that categorize motion scenarios, few can be unambiguously classified as motion categorization, namely, QTC, QRPC, and RCC-d. These are ‘*genuine*’ qualitative motion representations, the other ones are primarily directional representations.

5.2 Identified Shortcomings and our Story-Based Solutions

Observing our survey (Sect. 5.1), we identify relevant shortcomings in the current landscape of qualitative representations of motion.

Sparse work on motion representations As we can observe from our survey, the work in qualitative representations of *motion* is unusually sparse when compared to the vast research in *spatial* representations (Cohn and Hazarika 2001a, Sect. 5.1.2, p. 16; Delafontaine et al. 2011, p. 5187).

Mostly applicable in low dimensional domains Our surveyed motion representations are quite limited in their application domain: they are mainly restricted to point-like entities moving in one or two dimensions (e.g., $\mathrm{QTC_C}$, $\mathrm{QTC_B}$, QRPC), while spatial representations deal also with region-like entities located in three-dimensional spaces (e.g., Egenhofer 1991; Albath et al. 2010) (See overview in Sect. 4.4).

Ill-defined categorization of motionless entities Some motion categorizations (e.g., $\mathrm{QTC_{3D}}$) are ill-defined when at least one entity is motionless; they rely on the *motion direction* of the entities to determine the qualitative values, and a motionless entity ($\vec{v} = \vec{0}$) has an undefined motion direction.

The situation is not much better with the directional categorizations, e.g., OPRA and QRPC. There, the objects have an intrinsic orientation $\vec{o}$; hence such categorizations are well-defined even when entities are motionless. However, we cannot solely rely on the velocity vector $\vec{v}$, for motionless entities we must resort to the intrinsic orientation in Section 8.2.2.A, when we obtain Stories-$\mathrm{OPRA_1}$.

Neglected composition Another problem in motion representations—which also occurs in spatial representations (See Section 4.1.4)—is that, to some extent, researchers neglect the *composition*: neither compute their composition tables nor describe how to do it. If we consider the *genuine* motion representations, only QTC (Van de Weghe et al. 2005) and QRPC[1] (Alvarez Bravo and F. Glez-Cabrera 2022) compute the composition tables; while RCC-d remains, so far, without such a reasoning apparatus.

Our story-based motion representations overcome these limitations by presenting a method to create qualitative representations of motion that work in any number of dimensions, with any kind of entities, i.e., point-like or regions, and, that can categorize scenarios with motionless entities as long as the generating representation is well-defined in ($\vec{v} = \vec{0}$). Likewise, we oppose the trend of neglecting the composition: In Chapter 10, we provide a method to facilitate the computation of the composition for story-based motion representations.

5.3 A Taxonomy of Qualitative Relations of Motion

We present the above surveyed motion representations as a taxonomy according to two criteria. First, on a higher level, we classify them according to the temporal duration of the categorized motions. That is, some representations categorize *instantaneous* motion (a punctual instant of time), while others categorize motion in a time interval, i.e., along a trajectory. In between are

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[1] We mentioned in our paper of 2018 that, since 2010, QRPC lacked composition. The composition was shown first in 2022.

categorizations that, in addition to instantaneous motion, consider also the *expected* trajectory. Second, on a lower level, we classify them according to the kinematic features captured; for example, some representations consider only the direction of motion, while others consider the entities' speeds and distances.

A. *Instantaneous motion*

- Orientation of velocity vectors:
  The family of representations 'Oriented Point Relation Algebra' (OPRA) (Mossakowski and Moratz 2010)
- Orientation of velocity vectors + compared speeds and angles:
  The family of representations 'Qualitative Trajectory Calculus' (QTC) relations (Van de Weghe 2004; Delafontaine et al. 2011)
- Overlapping (RCC representation) + relative region approach:
  RCC-d (Wu et al. 2014)

B. *Instantaneous motion + expected trajectory*

- Configuration of the velocity vectors + proximity and relative position to expected crossing point: QRPC (F. J. Glez-Cabrera et al. 2013)
- Kinematic aspects are inherited from the generating (either spatial or motion) representation:
  - Story-based representations from spatial representations, e.g., Motion-RCC and Motion-OPRA$_1$ (Purcalla Arrufi and Kirsch 2018a)
  - Story-based representations from qualitative representations of motion, e.g., Story-QTC$_\mathrm{B}$ (Section 8.3).

C. *Trajectory segments*

- Landmark based and path-centred frames:
  Double Cross (Zimmermann and Freksa 1996; Freksa 1992b) and RfDL-3-12 (Kurata and Shi 2008a; b)
- Oriented trajectory segments:
  Dipole Calculus (Moratz et al. 2000)

D. *Arbitrary trajectories*
Described as temporal sequences of motion relations that belong to representations describing instantaneous motion (item A.):

- Permutable 4th order sequences of QTC$_\mathrm{C}$ relations (Delafontaine et al. 2011)
- Sequences of QTC$_\mathrm{C}$ relations with same start and end time (Hanheide et al. 2012)

5.4 Expanding Qualitative Representations of Motion

In this section, we present a standard method for creating or expanding—making finer— representations of motion. The method relies on Cartesian product, that is, on the concatenation of representations: We can concatenate arbitrary motion representations to form new qualitative representations. For example, if we have motion representations $\mathcal{M}_A$ and $\mathcal{M}_B$, we can create a new representation by concatenating $\mathcal{M}_B$ to $\mathcal{M}_A$; we obtain then the '*product representation*'

$\mathcal{M}_C = \mathcal{M}_A \times \mathcal{M}_B$, in which each motion relation is expressed as a tuple $R_C = (R_A, R_B)$ where $R_A \in \mathcal{M}_A$ and $R_B \in \mathcal{M}_B$. Nonetheless, it is up to interpretation, whether we consider $\mathcal{M}_C$ a *brand new* qualitative motion representation, or just an expansion of the representation $\mathcal{M}_A$—or, alternatively, an expansion of $\mathcal{M}_B$.

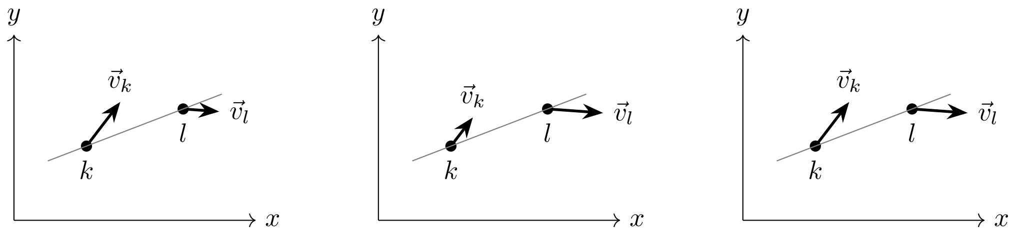
Now, we illustrate the method with a concrete example: we use N. Van de Weghe's (2004) QTC family of categorizations (Fig. 5.1). Initially, we take two qualitative motion representations, namely $\mathrm{QTC}_{\mathrm{B21}}$ (fully described in Sect. 6.3), and a very simple representation that we will call Speed.

A motion relation in $\mathrm{QTC}_{\mathrm{B21}}$ describes two properties: $\mathcal{P}_k$, 'relative approaching of entity k with respect to entity l', and, conversely, $\mathcal{P}_l$, 'the relative approaching of entity l to entity k'; each property, $\mathcal{P}_*$, takes three possible symbols $\{-, 0, +\}$. The symbol '$-$' means 'moving towards', '$+$' means 'moving away', and '0' means 'remaining stationary'. Accordingly, a motion category is represented by the 2-tuple in which each element is a property. For instance, a motion scenario where k chases l—that is, k moves towards l, and l moves away from k—is described as $(-, +)$ (See Figures 5.1a and 5.1b)

A motion category in the Speed representation simply describes which of the entities moves faster. It also takes three possible symbols $\{-, 0, +\}$: the symbol '$-$' means 'k moves faster than l', '$+$' means 'k moves slower than l', and '0' means 'k and l move equally fast' (See, respectively, Figures 5.1a to 5.1c).

Both $\mathrm{QTC}_{\mathrm{B21}}$ and Speed are qualitative motion representations in their own right—they fulfil the properties of qualitative representations (Sect. 4.3). Thus, following the indications above, we can create a new motion representation $\mathcal{M}_C = \mathrm{QTC}_{\mathrm{B21}} \times \mathsf{Speed}$. In $\mathcal{M}_C$, a motion relation is represented through the tuple $(R_{\mathrm{QTC}_{\mathrm{B21}}}, R_{\mathsf{Speed}})$; for example, the tuple $((-, +), +)$, which describes following: k moves towards l, l moves away from k, and k moves slower than l (See Fig. 5.1a). We can clearly appreciate the advantages of concatenating the extra representation: in case k is chasing l, relation $(-, +)$ in representation $\mathrm{QTC}_{\mathrm{B21}}$, the extra relation in representation Speed tells us how successful the chase is—if the relation is $((-, +), -)$, k may reach l, otherwise, with Speed relations '$+$' and '0', k cannot reach l.

This "new" motion categorization $\mathrm{QTC}_{\mathrm{B21}} \times \mathsf{Speed}$ is already known as $\mathrm{QTC}_{\mathrm{B22}}$ (e.g., Van de Weghe et al. 2007, Sect. 3). For the sake of simplicity, in the QTC family, the relations are written as a simple tuple, i.e., without nested tuples; for example $((-, +), -)$ is written $(-, +, -)$.



(a)

$\mathrm{QTC}_{\mathrm{B21}} = (-, +)$
$\mathsf{Speed} = (-)$
$\mathrm{QTC}_{\mathrm{B22}} = (-, +, -)$

(b)

$\mathrm{QTC}_{\mathrm{B21}} = (-, +)$
$\mathsf{Speed} = (+)$
$\mathrm{QTC}_{\mathrm{B22}} = (-, +, +)$

(c)

$\mathrm{QTC}_{\mathrm{B21}} = (-, +)$
$\mathsf{Speed} = (0)$
$\mathrm{QTC}_{\mathrm{B22}} = (-, +, 0)$

Figure 5.1: Motion scenarios classified according to $\mathrm{QTC}_{\mathrm{B21}}$, Speed, and their *product representation* $\mathrm{QTC}_{\mathrm{B22}} = \mathrm{QTC}_{\mathrm{B21}} \times \mathsf{Speed}$. All these scenarios share the same $\mathrm{QTC}_{\mathrm{B21}}$ relation, but differ in Speed; consequently, they have different $\mathrm{QTC}_{\mathrm{B22}}$ relations.

Theoretically, we can concatenate as many representations as we wish in order to obtain new product representations, i.e., $\mathcal{M}_C = \mathcal{M}_{A_1} \times \mathcal{M}_{A_2} \times \cdots \times \mathcal{M}_{A_i} \times \cdots$. Every product representation is at least as fine as each of its components, that is, $\mathcal{M}_C$ has at least the same motion relations that each component representations $\mathcal{M}_{A_i}$ has. For example, in any QTC_{B22} motion relation, we can find the corresponding QTC_{B21} and Speed relations: these are, respectively, the first two coefficients and the last coefficient of a QTC_{B22} tuple (e.g., Fig. 5.1).

In fact, the product representation $\mathcal{M}_C$ is usually *finer* than any of the component representations. In other words, a category in any of the component representations $\mathcal{M}_{A_i}$ is further *subcategorized* into the categories of another component representation $\mathcal{M}_{A_j}$. For example, the QTC_{B21} relation $(-,+)$ is subcategorized in three QTC_{B22} relations, namely, $(-,+,-)$, $(-,+,+)$, and $(-,+,0)$ (respectively, Figs. 5.1a to 5.1c).

Even though we might concatenate an *unlimited* number of motion representations $\{\mathcal{M}_{A_1}, \mathcal{M}_{A_2}, \ldots \mathcal{M}_{A_i}, \ldots\}$ to generate the corresponding product representation, we are ultimately constrained: only a product representation with a few of those $\mathcal{M}_{A_i}$ can be handily implemented and used. Hence, the question arises: which criteria could help us choose the more convenient $\mathcal{M}_{A_i}$ representations to concatenate?

Such a desired criterion is furnished in the next section, based on a key concept: the *multiplicativity* of concatenated representations.

5.4.1 Multiplicativity of Concatenated Representations

Ideally, we might expect that, in a product representation, $\mathcal{M}_C = \mathcal{M}_A \times \mathcal{M}_B$, every concatenated representation, $\mathcal{M}_B$, subdivides (i.e., subcategorizes) each motion relation in $\mathcal{M}_A$ according to its number of elements, $|\mathcal{M}_B|$. In short, we might expect following equality, $|\mathcal{M}_A \times \mathcal{M}_B| = |\mathcal{M}_A| \cdot |\mathcal{M}_B|$, between the total number of relations in the product representation, $|\mathcal{M}_A \times \mathcal{M}_B|$, and the number of relations in the component representations, namely, $|\mathcal{M}_A|$ and $|\mathcal{M}_B|$.

For instance, in the product motion relation $\text{QTC}_{\text{B22}} = \text{QTC}_{\text{B21}} \times$ Speed (Fig. 5.1), we should ideally expect that each of the 3^2 QTC_{B21} motion relations is subcategorized by the 3 Speed relations. If so, QTC_{B22} should have a total of $3^2 \cdot 3 = 27$ motions relations—which is precisely the case.

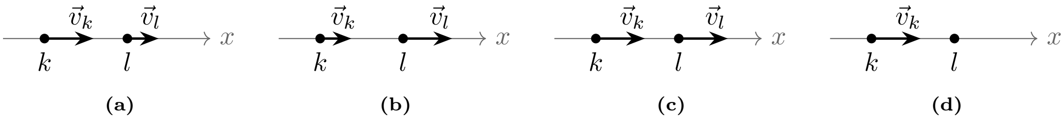
Unfortunately, the ideal case observed in QTC_{B22}, $|\text{QTC}_{\text{B22}}| = |\text{QTC}_{\text{B21}}| \cdot |\text{Speed}|$, i.e., $|\mathcal{M}_A \times \mathcal{M}_B| = |\mathcal{M}_A| \cdot |\mathcal{M}_B|$, does not apply to every concatenated representation. Most generally, we can only confirm the inequality in Equation (5.1), which sets the upper bound for the number of relations in the product representation. In the optimal case, where the equality holds, we say that the representations $\mathcal{M}_A$ and $\mathcal{M}_B$ are '*multiplicative*'.

$$|\mathcal{M}_A \times \mathcal{M}_B| \leq |\mathcal{M}_A| \cdot |\mathcal{M}_B| \tag{5.1}$$

In fact, component representations are often *non*-multiplicative, that is, usually only the strict inequality holds in Equation (5.1). Non-multiplicativity arises when some combinations of relations are physically unrealizable—This occurs, for example, with the one-dimensional motion representation QTC_{B12}.

QTC_{B12} is a motion representation for entities moving in one dimension. It is constructed as the product representation of QTC_{B11} (exactly defined as QTC_{B21} but restricted to entities moving in one dimension), and the representation Speed (as defined above but likewise restricted to entities in one dimension). If QTC_{B11} and Speed were multiplicative, then $|\text{QTC}_{\text{B12}}| = |\text{QTC}_{\text{B11}}| \cdot |\text{Speed}| = 3^2 \cdot 3 = 27$. But QTC_{B12} has only 17 relations, because 10 combinations of QTC_{B11} and Speed relations are physically unrealizable. In Figure 5.2, we exemplify this behaviour. The QTC_{B11} relation $(-,+)$ combines smoothly with any of the Speed relations yielding three different QTC_{B12} relations: $(-,+,-)$, $(-,+,+)$, and $(-,+,0)$ (Figs. 5.2a to 5.2c).

Yet the $\mathrm{QTC_{B11}}$ relation $(-,0)$ is only compatible with the Speed relation '$-$', yielding $(-,0,-)$. Truly, the component '0' of $(-,0)$ necessarily means—in one dimension—that l is motionless, which entails that, if k moves towards l, the Speed relation can only be '$-$'. Consequently, the relations $(-,0,0)$ and $(-,0,+)$ are physically unrealizable in one dimension.



(a) (b) (c) (d)

$\mathrm{QTC_{B11}} = (-,+)$	$\mathrm{QTC_{B11}} = (-,+)$	$\mathrm{QTC_{B11}} = (-,+)$	$\mathrm{QTC_{B11}} = (-,0)$
Speed $= (-)$	Speed $= (+)$	Speed $= (0)$	Speed $= (-)$
$\mathrm{QTC_{B12}} = (-,+,-)$	$\mathrm{QTC_{B12}} = (-,+,+)$	$\mathrm{QTC_{B12}} = (-,+,0)$	$\mathrm{QTC_{B12}} = (-,0,-)$

Figure 5.2: Motion scenarios classified according to $\mathrm{QTC_{B11}}$, Speed, and their *product representation* $\mathrm{QTC_{B12}} = \mathrm{QTC_{B11}} \times$ Speed. Scenarios (a), (b), and (c) display all three possible Speed variants of the $\mathrm{QTC_{B11}}$ relation $(-,+)$. Scenario (d) displays the *only* possible Speed variant, $(-)$, of the $\mathrm{QTC_{B11}}$ relation $(-,0)$; the Speed relations $(+)$ and (0) are physically unrealizable.

We resume our determination of the bounds for $|\mathcal{M}_A \times \mathcal{M}_B|$. The lower bound for occurs when one of the representations contains the other, that is either $\mathcal{M}_A \subset \mathcal{M}_B$ or $\mathcal{M}_A \supset \mathcal{M}_B$. In that case, the corresponding subrepresentation does not provide any extra categorization information at all, and, therefore, the product representation has the same number of relations as the finest of both representations $\mathcal{M}_A$ and $\mathcal{M}_B$. We express mathematically this fact in Equation (5.2)

$$\max(|\mathcal{M}_A|, |\mathcal{M}_B|) \leq |\mathcal{M}_A \times \mathcal{M}_B| \tag{5.2}$$

Summarizing, we have the following bounds for $|\mathcal{M}_A \times \mathcal{M}_B|$:

$$\max(|\mathcal{M}_A|, |\mathcal{M}_B|) \leq |\mathcal{M}_A \times \mathcal{M}_B| \leq |\mathcal{M}_A| \cdot |\mathcal{M}_B| \tag{5.3}$$

Equation (5.3) hints about the criterion for choosing the representation $\mathcal{M}_B$ that we should more conveniently concatenate to $\mathcal{M}_A$. On the one hand, if $\mathcal{M}_A$ and $\mathcal{M}_B$ are *multiplicative* the product representation, $\mathcal{M}_A \times \mathcal{M}_B$ has the greatest possible number of categories (related to the size of $\mathcal{M}_B$)—the product representation has the greatest granularity, namely, $|\mathcal{M}_A| \cdot |\mathcal{M}_B|$. On the other hand, if we concatenate a subrepresentation, i.e., $\mathcal{M}_B \subset \mathcal{M}_A$, the cardinality of the product representation, $|\mathcal{M}_A \times \mathcal{M}_B|$, is the same as the cardinality of $\mathcal{M}_A$. As a result, we should avoid the concatenation of subrepresentations and strive to concatenate multiplicative representations.

However, not all pairs of motion representations are either multiplicative or are subrepresentations of one another—we have cases in between. For that reason, instead of just defining *multiplicativity* as a boolean feature—whether a motion representation is multiplicative or not—we define multiplicativity as a graded value. The bounds of $|\mathcal{M}_A \times \mathcal{M}_B|$ in Equation (5.3) help us to define a '*measure of multiplicativity*', $\mu(\mathcal{M}_A, \mathcal{M}_B)$, with values in $[0,1]$ (Eq. (5.4); see proof, Prop. A.1.2). The maximum value, 1, is attained if the representations are multiplicative; the minumum, 0, is attained if one categorization is subcategory of the other—The higher the value the more independent are the concatenated representations, that is, they generate more effectively product relations by yielding finer subcategorizations.

$$\mu\left(\mathcal{M}_A, \mathcal{M}_B\right) = \frac{\frac{|\mathcal{M}_A \times \mathcal{M}_B|}{\max(|\mathcal{M}_A|,|\mathcal{M}_B|)} - 1}{\min(|\mathcal{M}_A|,|\mathcal{M}_B|) - 1} \tag{5.4}$$

$$0 \leq \mu\left(\mathcal{M}_A, \mathcal{M}_B\right) \leq 1$$

In conclusion, the *multiplicativity*, $\mu\left(\mathcal{M}_A, \mathcal{M}_B\right)$, provides us with a criterion to choose amongst representations that we want to concatenate. If we have some representation candidates $\{\mathcal{M}_{B_1}, \ldots, \mathcal{M}_{B_n}\}$ to concatenate with a motion representation $\mathcal{M}_A$, we can choose the one with the highest multiplicativity, i.e., $\mathcal{M}_B = \max_{i=1\ldots n}\{\mu\left(\mathcal{M}_A, \mathcal{M}_{B_i}\right)\}$. In that way, we ensure that the concatenated representation optimizes the subcategorization of the motion categories of $\mathcal{M}_A$ in the new product representation $\mathcal{M}_A \times \mathcal{M}_B$.

A. An application of multiplicativity

We illustrate here the application of multiplicativity. Consider the motion representation $\mathcal{M}_A = \mathrm{QTC}_{\mathrm{B11}}$ and two candidates for expanding it, namely, $\mathcal{M}_{B_1} = \mathsf{Speed}$ and $\mathcal{M}_{B_2} = \mathsf{Gaping}$. The motion representation Gaping was wholly presented in Section 1.2.1. In brief, it takes, like Speed, three possible values $\{-, 0, +\}$: the symbol '$-$' means that the distance between the entities decreases, the symbol '$+$' means that the distance increases, and the symbol '0' means that the distance between the entities remains constant.

In the case we want to expand $\mathcal{M}_A$ by concatenating one representation, we must choose between $\mathcal{M}_{B_1}$ or $\mathcal{M}_{B_2}$. The *multiplicativity measure* provides the criterion for the choice. In order to obtain the multiplicativity (Eq. (5.4)), we must compute following values: $|\mathcal{M}_A| = 9$, $|\mathcal{M}_{B_1}| = 3$, $|\mathcal{M}_{B_2}| = 3$, $|\mathcal{M}_A \times \mathcal{M}_{B_1}| = 17$, $|\mathcal{M}_A \times \mathcal{M}_{B_2}| = 13$. We obtain $\mu\left(\mathcal{M}_A, \mathcal{M}_{B_1}\right) = 0.44$ and $\mu\left(\mathcal{M}_A, \mathcal{M}_{B_2}\right) = 0.22$. This motivates the choice for $\mathcal{M}_{B_1}$, because it subcategorizes $\mathcal{M}_A$ more effectively.

Admittedly, in this case, it suffices to choose the representation whose product representation has more elements (i.e., $|\mathcal{M}_A \times \mathcal{M}_{B_1}| > |\mathcal{M}_A \times \mathcal{M}_{B_2}|$), because both representations $\mathcal{M}_{B_1}$ and $\mathcal{M}_{B_2}$ have the same number of relations, i.e., three relations. Notwithstanding, the multiplicativity measure has the advantage that it compares also representations with different number of relations (i.e, when $|\mathcal{M}_{B_1}| \neq |\mathcal{M}_{B_2}|$).

5.4.2 Further Types of Concatenation: Hybrids

So far, we have expanded motion representations by concatenating additional *motion* representations. However, we can also expand a motion representation $\mathcal{M}_A$ by concatenating a *spatial* representation $\mathcal{D}_B$, so that the product representation, $\mathcal{M}_C = \mathcal{M}_A \times \mathcal{D}_B$, contains both spatial and motion relations. Such a representation—which we call '*hybrid motion representation*'—retains all the properties of a genuine motion representation, for instance, the computation of the composition table or the multiplicativity.

Most importantly, a main result of this work—the *beaded* story-based representations—are mostly *hybrid motion representation*, as we comment at the end of this section. But first, we detail an example of *hybrid representation* in the literature: the already mentioned RCC-d.

A. A hybrid motion representation: RCC-d

The RCC-d (Wu et al. 2014) relates two moving regions A and B (See Figure 5.3)—In the following, we restrict to the case where region A is smaller than B. RCC-d is the Cartesian product of the *spatial* representation RCC, and the *motion* representation "variation of distance

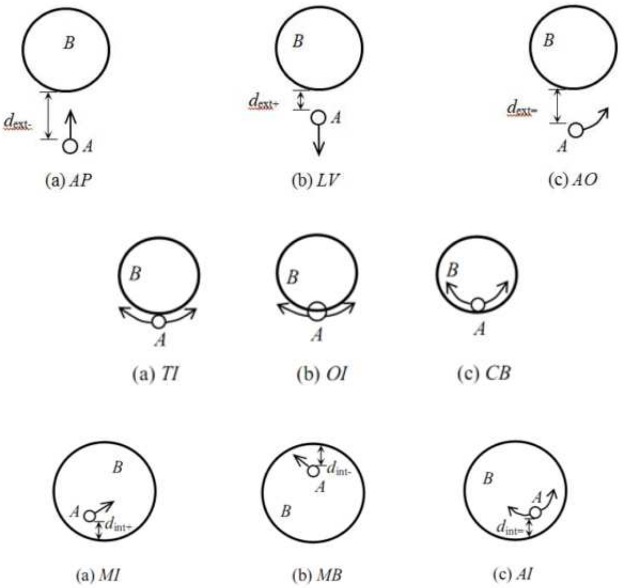


Figure 5.3: Simplified iconic representation of the RCC-d relations—These cases are restricted to region A being smaller than B. (Source: Wu et al. (2014), composed of Figures 1, 2, and 3. © Springer Nature Customer Service Centre GmbH. Used with permission 5443550499269)

between regions"—we call it Gaping2, equivalent to the Gaping representation but for regions. Consequently, a motion relation in RCC-d can be formalized as (R_1, M_1).

- R_1 is the RCC value of the motion scenario (explained in Sect. 4.2.1), which, for entity A smaller than entity B, can take following values: DC, EC, PO, TPP, NTPP.
- M_1 is the distance variation between regions. Although Wu et al. (ibid.) omit any definition of the distance between regions, from *their* fig. 1 and 2 (our Fig. 5.3) one can infer that the "distance" they meant is the minimum distance between the *border points* of the regions—notice, however, that this is not a 'distance' in the mathematical sense.
  M_1 can take following values: '+', '0', '−'; they respectively indicate if the *nearest* points of the borders move away, remain at a constant distance, or approach[2].

Accordingly, the RCC-d relations are AP = (DC, −), LP = (DC, +), AO = (DC, 0), TI = (EC, 0), OI = (PO, 0), CB = (TPP, 0), MI = (NTPP, +), MB = (NTPP, −), AI = (NTPP, 0).

As to the rest, RCC-d functions as any motion representation. For example, we can calculate the multiplicativity of RCC-d to see how effectively the spatial representation subcategorizes the motion representation: $|\mathcal{M}_{\text{RCC}}| = 5$, $|\mathcal{M}_{\mathsf{Gaping2}}| = 3$, $|\mathcal{M}_{\text{RCC}} \times \mathcal{M}_{\mathsf{Gaping2}}| = 9$, thus, $\mu\,(\mathcal{M}_{\text{RCC}}, \mathcal{M}_{\mathsf{Gaping2}}) = 0.4$.

B. Hybrid story-based representations

In this work, we present two main types of story-based representations: the *bare* and the *beaded* ones (Ch. 8). Even though the beaded story-based representations are thoroughly expounded in

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[2] Wu et al. (2014) use the symbols $d_{\text{ext}-}$, $d_{\text{ext}=}$, $d_{\text{ext}+}$, $d_{\text{int}-}$, $d_{\text{int}=}$, $d_{\text{int}+}$ instead of $\{-, 0, +\}$. Note, however, that the additional identifiers 'ext' and 'int' are redundant when combined with the RCC relations; indeed, 'ext' associates to DC, and 'int' to NTPP.

Section 8.5, we briefly mention them here because, in most cases, they are *hybrid representations*. A beaded story-based representation is a product representation $\mathcal{M}_A \times \mathcal{R}_B$, where $\mathcal{R}_B$ is a spatial or motion representation, and $\mathcal{M}_A$ is a very specific motion representation, namely, $\Sigma_{\mathcal{R}_B}$, which is obtained straightforwardly—but not effortlessly—from the original spatial (or motion) representation $\mathcal{R}_B$ (Sect. 8.1). We call the motion relations in $\Sigma_{\mathcal{R}_B}$ '*stories*', and hence the name '*story-based representations*'. Note that when the original representation $\mathcal{R}_B$ is a *spatial* representation, i.e., $\mathcal{R}_B = \mathcal{D}_B$, then the product representation $\mathcal{M}_A \times \mathcal{R}_B$ is also hybrid representation, $\mathcal{M}_A \times \mathcal{D}_B$.

Relations in a beaded story-based representation are expressed as $S_i(R_j)$ where S_i is a story, that is, a *motion relation* in $\Sigma_{\mathcal{R}_B}$; and R_j is a *spatial relation* in $\mathcal{R}_B$. The notation $S_i(R_j)$ is equivalent to the traditional tuple notation (S_i, R_j)—with our notation, we just highlight that $R_j \in S_i$.

5.5 Motion and spatial categorization: linguistic comparison

As we mentioned in our survey (Sect. 5.2), it is odd to find a much greater number of spatial representations than motion representations. All the more, because both categorizations are directly related—motion is *spatial* change in time—and have some fundamental cognitive commonalities.

In principle, we might expect that the abundant work in *spatial* representations should prompt a similar number of results in *motion* representations; but this is not the case: scientists cannot lightly transfer the formalism used in qualitative spatial representations to motion representations. We hypothesize that such transfer difficulties arise from the different way humans cognitively process spatial and motion relations.

We can best appreciate through our language how we cognitively process motion and spatial categorization. Indeed, the differences—but also the commonalities—between these two categorizations show up when we verbalize them. For that reason, in this section, we resort to '*cognitive linguistics*' to contrast space and motion relations: "[cognitive linguistics deals with] the linguistic representation of conceptual structure" (Talmy 2000b, p. 1, Introduction). We are inspired in our cognitive linguistic analysis by the work of L. Talmy, collected in his opus magnum *Toward a Cognitive Semantics*.

The main insights we draw from cognitive linguistics are following. First, the powerful mathematical formalism of *qualitative spatial representations*, used in spatial categorization (Sect. 4.1.3), essentially mirrors the properties of spatial relations in language. Second, motion relations are expressed through different linguistic devices than spatial relations, and, this is a compelling reason why scientists cannot straightforwardly transfer the formalism of qualitative representations from spatial into motion categorization. Admittedly, we exemplify such insights in English language, but they are applicable to other European languages, and possibly to general human linguistics.

5.5.1 Commonalities

Before we tackle the differences, we remark the most fundamental similarity between spatial and motion categorization: they *often* relate at least two entities that have differentiated roles. The first entity, a *main object* (called '*figure*'), is related to a second entity, a *reference object* (called '*ground*') (ibid., Ch. 5). In Example 5.1, we can see *figure* and *ground* for each categorization: spatial (item a.) and motion (item b.).

Example 5.1 *Figure* and *ground* in spatial and motion relations.

a. The bottle is `in front of` the plate. (spatial relation)
figure *preposition* ground

b. The alpinist `climbed` the mountain. (motion relation)
figure *verb* ground

5.5.2 Differences

In this section, we present the differences between the linguistic description of *spatial* and *motion* relations. We use as a paradigm the differences between Example 5.1.a. and b..

A. Prepositions for spatial relations, verbs for motion relations

First of all, we argue that, in most cases, two entities are *spatially* related mostly by means of a *preposition*. Indeed, in English—as in many other languages—places or position of objects are predominantly represented by prepositions (Landau and Jackendoff 1993, p. 218). In Example 5.1.a., the preposition `in front of` relates the two entities spatially. Secondly, we argue that, in most cases, two *moving* entities are related by means of a *verb* which is eventually assisted by and *adverb*. In Example 5.1.b. the two moving entities are related by means of a *verb*, such as `climbed`.

One may object, though, that the spatial relation in Example 5.1.a. is also a verb, namely, *`is`* `in front of`. However, if we turn both examples into *adjective phrases* (Ex. 5.2), then we see which the decisive elements are. Indeed, in Example 5.2.a. the verb ‘to be’ drops from the sentence gracefully—it is superfluous. Consequently, the preposition `in front of` is the key element. Whereas in Example 5.2.b. the verb ‘to climb’, in the present participle form `climbing`, cannot be stripped from the setence: it is the necessary key element to refer to the *alpinist’s* motion.

Example 5.2 Adjective phrases obtained from Example 5.1.

a. I see the *bottle* `in front of` the *plate.* (spatial relation)
preposition

b. I see the *alpinist* `climbing` the *mountain.* (motion relation)
verb

One might further object that the preposition in Example 5.1.a. becomes the relevant element in the spatial relation because the verb ‘to be’ is a dull verb. But notice that, whatever verb of position we use (e.g., ‘to stand’, ‘to lie’, ‘to place’), the preposition remains most relevant. Indeed, we can proceed as following: first, we exchange the verb ‘to be’ in Ex. 5.1.a. for ‘to lie’ and we get ‘the bottle lay `in front of` the plate’; next, we turn the sentence into an adjective phrase (Ex. 5.3.a.), and test which element is most relevant by stripping it from the sentence (Ex. 5.3.b. and c.). Then, we observe that only when we strip the prepositional information (Item c.), any spatial relation between *bottle* and *plate* is lost; whereas when we strip the verbal information (Item b.), the spatial relation remains.

Example 5.3

a. I see the bottle lying `in front of` the plate.

b. I see the bottle ~~lying~~ `in front of` the plate.

c. I see the bottle lying ~~`in front of`~~ the plate.*

Some prepositions pose a challenge because they are closely related to motion, amongst others the preposition conveying direction; for example, `towards` or `through`. However, such prepositions still depend on the verb to express a motion or spatial relation (Ex. 5.4). At the very most, just a couple of prepositions express only motion, i.e., they are not used to express position; for instance, `into`.

Example 5.4 The prepositions below, `toward` and `through`, express a spatial or motion relation depending on the verb.

	`toward`	`through`
Spatial relation	The child *is pointing* `toward` his mom.	The broken bone *sticks out* `through` the skin.
Motion relation	The athlete *runs* `toward` the goal.	The birds *fly* `through` the trees.

In sum, it is virtually impossible to describe the *spatial relation* between two objects without a *preposition* (as explained in Sect. 4.1.2). Likewise, it is very difficult to describe the *motion* between two objects without a *verb*. This linguistic difference has profound cognitive implications.

B. Cognitive differences between prepositions and verbs

According to L. Talmy (2000b, Intr. pp. 22f.) prepositions and verbs are cognitively very different elements. Prepositions are '*grammatical terms*', whereas verbs are '*lexical terms*'. In another words, prepositions form a '*closed class*' (like determiners or conjunctions do), which means that the set of prepositions are relatively small compared to '*open classes*', such as verbs or adjectives. Moreover, the closed classes have a very stable membership—a language rarely adopts new prepositions. However, open classes, such as verbs, are in steady change growing with new members; for instance, verbs can be readily created from nouns and adjectives, e.g., the *-ize* verbs.

More relevantly, the difference between *closed-class* and *open-class* elements manifests in different mathematical properties. Closed-class elements describe "largely a relativistic, topological, qualitative, or approximative rather than a absolute, Euclidean, quantitative, or precisional", as open-class elements do (ibid., p. 28). Specifically, clossed-class elements comprise notions such as *partition*, region, locatedness.

Finally, in spatial relations, the roles of *figure* and *ground* are exchangeable in a *more symmetrical* way than in motion relations. We illustrate this phenomenon in Example 5.5: when we exchange the roles in the *spatial* relation, we obtain a sentence grammatically equivalent to the first with a different preposition (`behind`); however, when we exchange roles in the *motion* relation, we are forced to use the *passive* construction (i.e., was climbed by). As a consequence, the motion relation with figure-ground exchanged is not grammatically equivalent to the original one, but it is a different grammatical structure: a passive sentence. Certainly, though *mountain* is the subject, it is not the one who performs the action, in fact, we have no verb which expresses the mountain as a performer of the action on a climbing alpinist.

Example 5.5 We exchange ground and figure in Example 5.1.

	Original sentence	Figure-Ground exchanged
Spatial relation	The *bottle* is `in front of` the *plate*	The *plate* is `behind` the *bottle*
Motion relation	The *alpinist* `climbed` the *mountain.*	The *mountain* `was climbed` by the *alpinist.*

Admittedly, as noted by Talmy (1983, Sect. 3.1), spatial relations are not fully symmetric concerning meaning. For instance, when we exchange figure and ground as in Example 5.6, Item a. sounds more familiar than Item b., and, therefore, we may argue that it has different meaning nuances. Nevertheless, grammatically both expressions are equivalent. Consequently, we confirm our claim: spatial relations are grammatically equivalent when we exchange figure and ground.

Example 5.6

a. The *bicycle* is `near` the *house.*

b. The *house* is `near` the *bicycle.*

By the way, motion relations are strongly asymmetrical, because usually one of the entities performs a certain motion that the other does not. For example, an entity moves, while the other remains motionless—as in Example 5.1.b.—or when an entity, e.g., a motorcycle, overtakes another, e.g., a car. Spatial relations, however, remain to a great extent symmetrical because both objects are motionless. We have, though, in spatial relations softer asymmetry sources (For a complete list, see ibid., table (3) pp. 230f.); one of them relates to the item *mobility.* For instance, in Example 5.6, 'bicycle' sounds more common as *figure* (item a.) than as *ground* (item b.), because 'bicycle' is more movable than 'house'.

5.5.3 Implications for Qualitative Representations

Section 5.5.2 considered, it is noteworthy how good the properties of a linguistic *closed-class*, i.e., a *grammatical term*, fit the properties of qualitative relations (Tab. 5.1). One of such closed-class elements, the *prepositions*, is the standard way in language to express spatial relations. Consequently, we argue that when researchers mathematically formalize spatial relations, e.g., in a *qualitative spatial representation*, they intuitively reflect the linguistic properties of prepositions in their formalization. It might be the other way round: *prepositions* and *spatial representations* show similar properties, because they are, respectively, linguistical and mathematical manifestations of the same cognitive reality: the way our understanding processes space.

Conversely, it is also remarkable how different are the properties of linguistic *open-classes* from the properties of qualitative representations. Thus, since motion is largely expressed through *verbs* (open-class elements), it becomes awkward to formalize motion in a qualitative representation—Notwithstanding, researchers have constructed a handful of such qualitative motion representation.

To wrap up, language seems to show that we cognitively process space in a very different manner than we process motion. In that sense, the mathematical formalization of spatial relations as qualitative representations is for humans more straightforward than, regrettably, the formalization of motion relations as qualitative representations. Howbeit, in this work, we provide a cognitively plausible method to generate qualitative motion representations.

Table 5.1: Comparison of properties shown by three domains: qualitative representations, open-class elements, closed-class elements.

DOMAIN / PROPERTIES		Qualitative representns. (e.g., RCC, OPRA$_1$)	Closed-class elemts. (e.g., prepositions)	Open-class elemts. (e.g., verbs)
Type of measure		qualitative	qualitative	quantitative
Number of elements		fixed, finite, low number	fixed, finite, low number	variable, growing
Topological Structure		disjoint partition	partitions, regions (amongst others)	—
Operations	Converse	Yes	Yes (different preposition used)	Arguably, No (passive form used)
	Composition	Yes	Yes	No

Part III

Developing Story-Based Categorizations

Chapter 6

Describing [Motion] Categorizations: A Framework

> *In some respects, it seems as if the study of concepts is the study of theories that do not work for one reason or another.*
>
> N. Braisby (2012)
> "Concepts", p. 162

In the literature, we find a great variety of categorization models, e.g., probabilistic models based on prototypes, based on exemplars, categorization through clustering, and so on (Ch. 3). Unfortunately, each model is built on a particular formalization, i.e., each model is described using its own mathematical tools (See comprehensive survey, Pothos and Wills 2011a). This makes it a toilsome task trying to compare, or computationally implement, categorization models; even more, because the formalization of many categorizations models is rather vague, notably, the formalization of the prototype model (Murphy 2002, pp. 41–45). We avoid these drawbacks by mathematically specifying our categorization model and relating it to the psychological terminology.

The precise definition of our story-based motion categorizations takes place in Chapter 8, but, in this chapter, we provide the mathematical building blocks that, then, enable a solid definition of the story-based motion categorizations. Here, we define a mathematical framework to formalize categorizations, concretely, motion categorizations. In plain words, here we describe the categorization process mathematically. Admittedly, our framework suits more naturally a boundary categorization model, but it allows for modifications to accommodate other types, such as prototype or exemplar model.

We proceed as follows. At first, in Sections 6.1 and 6.2, we define our categorization framework: we express the categorization process mathematically, in particular, we link the mathematical expressions to fundamental psychological categorization terms, such as *category*, *attribute*, and *feature*. Subsequently, in Section 6.3, we illustrate our framework by describing a simple motion categorization, $\mathrm{QTC}_{\mathrm{B12}}$, which is also a qualitative motion representation (Van De Weghe et al. 2006). Finally, in Section 6.4, we go beyond pure descriptive tasks: we use our framework to test which similarity function and which categorization model (prototype or boundary) most suitably describes a motion categorization; we take again as example the $\mathrm{QTC}_{\mathrm{B12}}$ categorization.

We undertake to formalize categorization, because of its advantages (See, Murphy 2011): we expect to be detailed enough, here, in our categorization framework, and, later, in our story-based categorizations, that there be nothing left to intuition, but anyone may implement our methods in the same way we do; so that we ensure that she obtains the same results we do (*reproducibility*), and make the same predictions we would (*effectiveness*) (Anderson 1976, p. 17–18). We understand our categorization framework as an unambiguous formalization of the rather vague concept of *psychological space* (Sect. 3.4.1.A, p. 45), and has, thus, many commonalities with it. All throughout our formalization and subsequent modelling, we keep Braisby's admonishing words in mind: we know that for some reason our theory will be found wanting (See quotation above).

That said, we acknowledge that our categorization framework is comparable to many implementations of a psychological space—for example, the 'conceptual spaces' of Gärdenfors (2004, 2014)—but with some customizations. The main virtues of our framework are following: the terms are clearly, unambiguously, and mathematically defined; moreover, the terminology is explained so that both researchers in psychology and AI should sympathize with the framework.

6.1 The Categorization Process: Formalization

In the following, we describe the categorization process using terms related to motion categorization; more accurately, we use the terms concerning *scenarios categorization* because this is the focus of this work. Thus, it might seem that we limit our description to the scenarios categorization—It is not so. We could describe the categorization process in this chapter using the more general terms shown in Table 6.1. For instance, we could substitute everywhere 'motion scenario' by the term '*motion state*' which is a more general description of the motion of two entities, not limited to the current positions and velocities. Even more, we could substitute 'motion scenario' and 'kinematic space' by the most general terms 'item' and 'items space'; and our description would still be consistent.

Scenarios categorization	⊂	Motion categorization	⊂	General categorization
Kinematic space	=	Kinematic space	⊂	Items space
Motion scenario	⊂	Motion state	⊂	Item

Table 6.1: Hierarchy of terms according to the degree of specificity. From most specific ('scenarios categorization') to most general ('general categorization').

6.1.1 Basic Elements: Kinematic space & categorization rule

First off, to categorize a *motion scenario* $K = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ means that we label it with a category M_i. Now, in our formalization, this categorization process is defined by the basic elements that we list below:

i. The '*categories set*' $\mathcal{M}$ contains the categories $\mathcal{M} = \{M_1, M_2, \dots, M_m\}$ in which a motion scenario can be categorized. We assume the categories are a finite number m (e.g., the nine QTC$_{B12}$ categories in Eq. (6.9))—Both the requirement of *cognitive economy* and the finite number of human concepts seem to enforce this assumption.

ii. The '*kinematic space*', $\mathcal{K}$ is the set of all possible motion scenarios K; they are represented as a vector of positions, $\vec{x}$, and velocities, $\vec{v}$, for each entity. We call '*kinematic coordinates*', K, these parameters (positions and velocities) that uniquely characterize a motion scenario—Depending on the context, we refer to them also as '*kinematic variables*'. In that sense, $\mathcal{K}$ is a coordinate space in which each motion scenario K is described as a point.

iii. The '*categorization rule*' f_μ labels each motion scenario with a category. In other words, the categorization rule is a function that maps each *motion scenario*, K, into the corresponding *motion category*, M_i (e.g., Eq. (6.10)). Alternatively, we can say that the categorization rule defines the *category membership* of the motion scenarios.

$$\begin{aligned} \mathrm{f}_\mu : \mathcal{K} &\longrightarrow \mathcal{M} \\ K &\longmapsto M_i \end{aligned} \tag{6.1}$$

Summarizing, a '*motion categorization*', $\mathbb{K}$, is most basically defined as (*kinematic space* $\mathcal{K}$, *categorization rule* f_μ, *motion categories* $\mathcal{M}$), which is actually a formalization of the classical model:

$$\mathbb{K} = (\mathcal{K}, \mathrm{f}_\mu, \mathcal{M}) \tag{6.2}$$

Equivalently, we can define the motion categorization providing the *categorical regions* instead of providing the *categorization rule.* A '*categorical region*', $\boldsymbol{K}_i$, is the set of scenarios that belong to a certain category M_i. In this case, a motion categorization $\mathbb{K}$ is defined as (*kinematic space* $\mathcal{K}$, *categorical regions* $\boldsymbol{K}_i$, *motion categories* $\mathcal{M}$):

$$\mathbb{K} = (\mathcal{K}, \{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_m\}, \mathcal{M}) \tag{6.3}$$
$$\text{where } \boldsymbol{K}_i = {\mathrm{f}_\mu}^{-1}(M_i) \subset \mathcal{K} \quad M_i \in \mathcal{M}$$

In our most simple understanding of categorization, the categorical regions, $\boldsymbol{K}_i$, should verify two properties: a motion scenario K cannot belong to more than one category, i.e., categorical regions should be disjoint (Eq. (6.4a)); and every motion scenario should be categorized, i.e., the categorical regions cover the whole kinematic space (Eq. (6.4b)). Hence, the categorical regions, $\boldsymbol{K}_i$, are a *partition* of the kinematic space $\mathcal{K}$.

$$\forall i, j \quad \boldsymbol{K}_i \cap \boldsymbol{K}_j = \emptyset \tag{6.4a}$$

$$\mathcal{K} = \bigcup_{i=1}^{m} \boldsymbol{K}_i \tag{6.4b}$$

Both equivalent definitions of motion categorization, $(\mathrm{f}_\mu, \mathcal{M})$ (Eq. (6.2)) and $(\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_n\}, \mathcal{M})$ (Eq. (6.3)), correspond respectively to the '*intensional*', by rules, and '*extensional*', by members, definitions of a category or concept (e.g., Houdé 2004, pp. 78f.). Note that the intensional definition—in our case by means of the *categorization rule* f_μ—is cognitively more useful than just specifying all category members, because it allows us to more easily grasp a categorization, that is, to understand what makes a category different from other, or which are the key features upon which a category is built. As Hampton (2011) acknowledges "Intensions are of more interest to psychologists, since they reflect the way in which we represent the concept[, i.e., the category,] internally".

6.1.2 Additional Elements: Featural space & Feature extraction

The elements described in Section 6.1.1 (kinematic space, categorization rule, and categories set) are indispensable to define a motion categorization. Nevertheless, there is no trace of two capital concepts in categorization according to psychology: *features* and *dissimilarity* (Sect. 3.4). Here, we work out these additional elements (features and dissimilarity) from the basic ones (kinematic space, categorization rule, and categories set).

Please note that we work inversely as in traditional categorization: Instead of defining features and a feature-based dissimilarity in order to create a categorization rule based on them (Bottom-up method in Fig. 6.1), we extract the features and a feature-based categorization rule from the general *categorization rule*, and, finally, define a dissimilarity in the obtained *featural space* (Top-down method in Fig. 6.1). At the end, the result is equivalent as we had first defined the features, and, of course, we can apply our framework also in a *bottom-up* context, when the features and the dissimilarity are first given.

This *top-down* method is the only effective one when we deal with the story-based categories, because our method for generating story-based categorizations provides, in first place, their categorization rule (which we call *story map*, Sect. 8.1)—This is also the case for qualitative representations (Sects. 4.4 and 5.1), where the categorization rule is defined without singling out the features in advance.

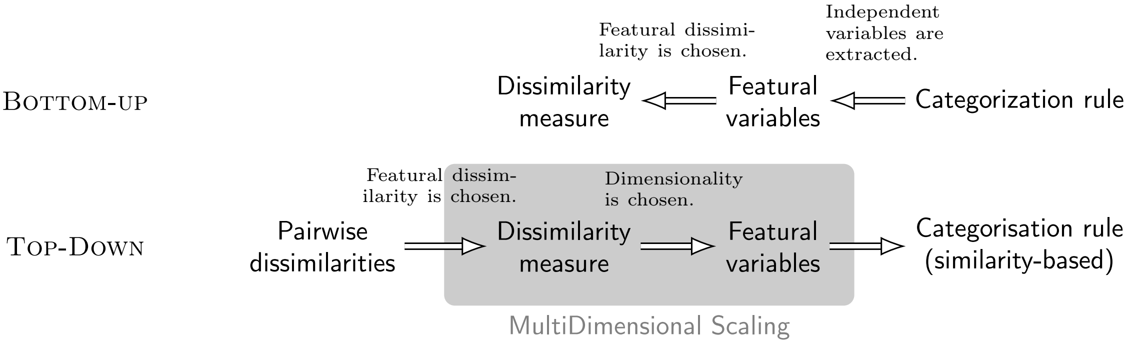


Figure 6.1: Two methods of describing a categorization: '*bottom-up*' and '*top-down*'. We denote *bottom-up* when we start with the categorization rule, and, from it, we derive the remaining categorization properties (first, features and, then, dissimilarity)—This is the case of story-based representations. We denote *top-down* when we begin with the pairwise dissimilarities and, from them, we obtain the best fit for dissimilarity measure and features (e.g., using MDS techniques).

In Equation (6.5), we apply the top-down analysis to our categorization method; we express the categorization rule f_μ in Equation (6.1) as a two-step process. First, the *featural variables* $F = (\varphi_1, \ldots, \varphi_n)$ are obtained from the *kinematic variables* K through function Φ—comparable to a *feature extraction*. Second, based on the featural values F, the *motion category* M_i is obtained by means of f_Φ—comparable to a *feature based categorization*.

$$\begin{tikzcd}
\mathcal{K} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{M} \\
K = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \arrow[r, mapsto, "\Phi"] \arrow[rr, bend right, "\mathrm{f}_\mu \quad (\text{Eq. } (6.1))"'] & F = (\varphi_1, \ldots, \varphi_n) \arrow[r, mapsto, "\mathrm{f}_\Phi"] & M_i
\end{tikzcd} \tag{6.5}$$

Summing up, the decomposition of the categorization rule, f_μ, as shown in Equation (6.5), originates new elements: the *features extraction function* $\varPhi$, the *featural variables* F, the *featural space* $\mathcal{F}$, and the *featural categorization rule* $\mathrm{f}_\varPhi$. Below, we list these additional elements, we define them explaining how we obtain them from the categorization rule, and briefly illustrate them with the motion categorizations $\mathrm{QTC}_{\mathrm{B21}}$ and Stories-RCC—one of the motion categorizations derived in this work (Sect. 8.2.1). Later in Section 6.3 we provide a full detailed example in which we apply our categorization framework—and concretely this decomposition of the categorization rule—to the $\mathrm{QTC}_{\mathrm{B21}}$ motion categorization.

i. The '*featural variables*' (or, simply, '*features*') are the *smallest* set of independent variables, $F = (\varphi_1, \ldots, \varphi_n)$, that determine the category for any given scenario, and fulfil the requirements in Section 6.1.3. The featural variables must be obtained as function of the kinematic variables, i.e., $\forall i\ \varphi_i = \varphi_i(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$; we obtain them by inspection of f_μ.

   Example 6.1.i When we describe the motion categorization $\mathrm{QTC}_{\mathrm{B21}}$ according to our framework (Sect. 6.3), we obtain two featural variables, $\varphi_k(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \cos(\angle(\vec{x}_l - \vec{x}_k, \vec{v}_k))$ and $\varphi_l(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \cos(\angle(\vec{x}_k - \vec{x}_l, \vec{v}_l))$, which are the cosine of the relative motion angles of each entity, $\gamma_k = \angle(\vec{x}_l - \vec{x}_k, \vec{v}_k)$ and $\gamma_l = \angle(\vec{x}_k - \vec{x}_l, \vec{v}_l)$ (Fig. 6.2, also see Eqs. 6.12). In short, the featural variables are $F = (\varphi_k, \varphi_l)$.

   Example 6.2.i The categorization Stories-RCC is determined by two featural variables, $d_{\min}$ and $\mathrm{dif}_{\mathrm{V}}$, i.e., $F = (\varphi_1, \varphi_2)$ where $\varphi_1 = d_{\min}$ and $\varphi_2 = \mathrm{dif}_{\mathrm{V}}$. The first variable, $\mathrm{d}_{\min}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \|\vec{x}_l - \vec{x}_k\| |\det(\vec{x}_l - \vec{x}_k, \vec{v}_l - \vec{v}_k)|$, is the minimum distance between entities along a uniform trajectory; the second one, $\mathrm{dif}_{\mathrm{V}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \frac{\|\vec{v}_l - \vec{v}_k\|}{\|\vec{v}_k\| + \|\vec{v}_l\|}$, is real value in $[0, 1]$ telling how different are the velocities ('0' if they are equal, $\vec{v}_k = \vec{v}_l$, and '1' if they are opposite $\vec{v}_k = -|\alpha|\vec{v}_l$) (Eq. (8.3)).

ii. The '*featural space*' $\mathcal{F}$, is the real coordinate space generated by the *featural variables* F.

   Example 6.1.ii The featural variables of $\mathrm{QTC}_{\mathrm{B21}}$ (Ex. 6.1.i), generate the featural space $\mathcal{F} = [-1, 1] \times [-1, 1]$ (See Fig. 6.4), because they are cosine values.

   Example 6.2.ii The featural variables of Stories-RCC (Ex. 6.2.i), generate following featural space $\mathcal{F} = [0, +\infty) \times [0, 1]$, because the first variable is a distance, and the second a speeds ratio.

iii. The '*features extraction function*' $\varPhi$, yields the values of the features $F = (\varphi_1, \ldots, \varphi_n)$ for each motion scenario $K = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$.

$$\begin{aligned} \varPhi : \mathcal{K} &\longrightarrow \mathcal{F} \\ K &\longmapsto F = (\varphi_1, \ldots, \varphi_n) \end{aligned} \tag{6.6}$$

   Example 6.1.iii The featural variables of $\mathrm{QTC}_{\mathrm{B21}}$, φ_k and φ_l, along with their formulae define the features extraction function of $\mathrm{QTC}_{\mathrm{B21}}$ as following (Eqs. 6.12):

$$(\varphi_k, \varphi_l) = \varPhi(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \big(\cos(\angle(\vec{x}_l - \vec{x}_k, \vec{v}_k)),\ \cos(\angle(\vec{x}_k - \vec{x}_l, \vec{v}_l))\big)$$

   Example 6.2.iii The featural variables of Stories-RCC, $\mathrm{d}_{\min}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ and $\mathrm{dif}_{\mathrm{V}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ (Ex. 6.2.i), along with their formulae define the features extraction function of Stories-RCC as following (See Eq. (8.4)):

$$(\varphi_1, \varphi_2) = \varPhi(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = (\mathrm{d}_{\min}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l),\ \mathrm{dif}_{\mathrm{V}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l))$$

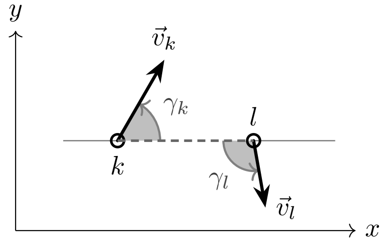


Figure 6.2: In this motion scenario, we show the angles γ_k and γ_l, which fully determine the scenario categorization in $\mathrm{QTC_{B21}}$ through their cosine, $\varphi_k = \cos(\gamma_k)$ and $\varphi_l = \cos(\gamma_l)$ (Ex. 6.1.i)

iv. The '*featural categorization rule*' f_Φ maps the feature values $F = (\varphi_1, \ldots, \varphi_n)$ into the corresponding category M_i.

$$\begin{aligned} \mathrm{f}_\Phi : \mathcal{F} &\longrightarrow \mathcal{M} \\ F = (\varphi_1, \ldots, \varphi_n) &\longmapsto M_i \end{aligned} \tag{6.7}$$

Example 6.1.iv Here we show the featural categorization rule of $\mathrm{QTC_{B12}}$. The categorization rule partitions the featural space in the categorical regions, as we will see in Figure 6.4.

$$\mathrm{f}_\Phi(\varphi_k, \varphi_l) := \begin{cases} M_1 & \text{if} \quad \varphi_k > 0,\ \varphi_l > 0 \\ M_2 & \text{if} \quad \varphi_k > 0,\ \varphi_l = 0 \\ M_3 & \text{if} \quad \varphi_k > 0,\ \varphi_l < 0 \\ M_4 & \text{if} \quad \varphi_k = 0,\ \varphi_l > 0 \\ M_5 & \text{if} \quad \varphi_k = 0,\ \varphi_l = 0 \\ M_6 & \text{if} \quad \varphi_k = 0,\ \varphi_l < 0 \\ M_7 & \text{if} \quad \varphi_k < 0,\ \varphi_l > 0 \\ M_8 & \text{if} \quad \varphi_k < 0,\ \varphi_l = 0 \\ M_9 & \text{if} \quad \varphi_k < 0,\ \varphi_l < 0 \end{cases}$$

Example 6.2.iv In Section 8.2.1, we obtain the featural categorization rule, f_Φ, of the Stories-RCC categorization (which we later call σ_Φ). Recognizably, the domain variables of f_Φ are the featural variables, $F = (\varphi_1, \varphi_2) = (d_{\mathrm{min}}, \mathrm{dif_V})$. The categories are here noted as S_{ij} instead of M_i. Here, we show only a part of the categories set: S_{01}, S_{02}, S_{03}, S_{11}, S_{12}, and S_{13}.

$$\mathrm{f}_\Phi(\mathrm{d_{min}}, \mathrm{dif_V}) := \begin{cases} S_{01} & \text{if} \quad \mathrm{d_{min}} > d_2,\ \mathrm{dif_V} = 0 \\ S_{02} & \text{if} \quad \mathrm{d_{min}} = d_2,\ \mathrm{dif_V} = 0 \\ S_{03} & \text{if} \quad d_2 > \mathrm{d_{min}} > d_4,\ \mathrm{dif_V} = 0 \\ \vdots & \qquad\qquad \vdots \\ S_{11} & \text{if} \quad \mathrm{d_{min}} > d_2,\ \mathrm{dif_V} > 0 \\ S_{12} & \text{if} \quad \mathrm{d_{min}} = d_2,\ \mathrm{dif_V} > 0 \\ S_{13} & \text{if} \quad d_2 > \mathrm{d_{min}} > d_4,\ \mathrm{dif_V} > 0 \\ \vdots & \qquad\qquad \vdots \end{cases}$$

where $d_2 = |r_k + r_l|$ and $d_4 = |r_k - r_l|$, being r_k and r_l the radii of the entities.

6.1.3 Properties of Features and Definition of Dissimilarity

Here we define decisive properties of the features in our framework: *lower dimensionality* and *self-density*. We devote a section to it because features are central to categorization (Sect. 3.4.1). Features allow us to comprehend, to conceptualize a categorization: Roughly, they are the concepts on which a categorization bases (See Sect. 11.7.2). Furthermore, features are fundamental to compute similarity, or, equivalently, dissimilarity (Sect. 3.4.1).

Additionally, as a by-effect, the property of *self-density* helps us to extract the features, i.e., the featural variables. Certainly, the process of extracting the featural variables from the categorization rule (item i., Sect. 6.1.2) is somewhat ambiguous: we can choose diverse sets, or combinations, of featural variables which fulfil the requirement of *minimality*; by enforcing *self-density*, we reduce the ambiguity in the choice of featural variables.

Lower Dimensionality The *featural space* $\mathcal{F}$ has, by definition (item i., Sect. 6.1.2), lower or equal dimensionality than the *kinematic space* $\mathcal{K}$, i.e., $\dim(\mathcal{F}) \leq \dim(\mathcal{K})$. In fact, the equality hardly ever holds, because the featural space is always a simplification of the kinematic space—the featural space focuses only in certain aspects of the motion—that is, generally, $\dim(\mathcal{F}) < \dim(\mathcal{K})$.

As an illustration, consider the featural space Stories-RCC, which consists only of two real parameters (Ex. 6.2.i); it has much lower dimensionality than the kinematic space, $\mathcal{K}$, which consists of 8 real parameters for the simple case of two entities moving in the plane. Therefore, in most cases, the featural space has not only lower but *much lower* dimensionality than the kinematic space, i.e., $\dim(\mathcal{F}) \ll \dim(\mathcal{K})$.

Because of its much lower dimensionality, the *featural space* $\mathcal{F}$ allows a more direct understanding of the categorization than the *kinematic space* $\mathcal{K}$ does; the numerous dimensions of $\mathcal{K}$ (normally $\dim(\mathcal{K}) \geq 8$ (Sect. 2.1)) make this space hard—de facto impossible—to visualize and, thus, to treat intuitively. Furthermore, in most cases, the coordinates in $\mathcal{F}$, the '*featural coordinates*', correspond to *natural kinematic concepts*, such as distances or angles.

Self-Density The featural space $\mathcal{F}$ should have two additional characteristics that we summarize in the property '*self-density*': it should be a metric space, and it should be *dense-in-itself*.

First, the very purpose of requiring the featural space to be a metric space is to endow it with a *dissimilarity function* $\mathrm{D}(F_1, F_2)$; so that we can obtain a dissimilarity function in the *kinematic space*, $\mathrm{D}(K_1, K_2)$—which is equivalent to obtaining a similarity function (See begin Sect. 3.4).

In more detail, if the featural space has a metric $d_{\mathcal{F}}$, then we can measure the distances between the featural points F_1 and F_2, i.e., $\mathrm{d}_{\mathcal{F}}(F_1, F_2)$, as we do in Section 6.3—Note that once we have a distance, we automatically have a *dissimilarity* (Sect. 3.4). The featural distance, $d_{\mathcal{F}}$, leads to a distance between scenarios, $d_{\mathcal{K}}$, by mapping the scenarios into the featural space using the features extraction function $\varPhi$. In other words, we extract the features of each scenario, $F_i = \varPhi(K_i)$, and, then, we measure the featural distance $d_{\mathcal{F}}$ between the extracted features, F_1 and F_2, to obtain the dissimilarity between scenarios K_1 and K_2.

$$\mathrm{d}_{\mathcal{K}}(K_1, K_2) = \mathrm{d}_{\mathcal{F}}(\varPhi(K_1), \varPhi(K_2)) = \mathrm{d}_{\mathcal{F}}(F_1, F_2) \tag{6.8}$$

Second, a featural space with a metric is, additionally, *dense-in-itself* when the featural variables are *continuum-like* values—that is, intervals of $\mathbb{R}$ or $\mathbb{Q}$ values—instead of, for instance, boolean or discrete values. As an illustration, consider a featural space with two featural variables: φ_x, which takes values in $[-3, 3]$ and φ_y, which takes values in $[0, 2]$; both take values in an interval, in consequence, such featural space is dense. In contrast, a featural space with the variable φ_x which takes discrete values $\{-3, 3\}$ and φ_y, which takes discrete values $\{0, 2\}$ is non-dense because we have discrete featural values.

A crucial implication is following: a dense-in-itself space is more appropriate to describe human categorization because the distances and the corresponding dissimilarities in such space have an ideal gradation effect (See Sect. 3.3.2). In contrast, in a non-dense space, the distances and the corresponding dissimilarities are discrete, which can lead to a coarse gradation or even no gradation at all. Therefore, its featural space is less appropriate for describing human categorization.

In conclusion, since we have a certain flexibility in choosing the featural variables, we should strive to define the featural variables as continuum values. This is the striving in our story-based categorizations (e.g., Sect. 8.2). For example, for Stories-RCC we have chosen the featural variable $\mathrm{dif}_{\mathrm{V}} = \frac{\|\vec{v}_l - \vec{v}_k\|}{\|\vec{v}_k\| + \|\vec{v}_l\|}$ with values in the interval $[0, 1]$, although we could have chosen the discrete variable $\mathrm{dif}_{\mathrm{V}} = \{0 \text{ if } \vec{v}_k = \vec{v}_l; 1 \text{ if } \vec{v}_k \neq \vec{v}_l\}$. In the latter case, the featural variable would not be *dense*, because it takes few discrete values, and it would separate scenarios into just two groups according to the 0 or 1 value: within-category gradation would be lost for such a variable.

6.2 Link to Psychological and Philosophical Terms

Our framework reproduces basic categorization principles in human cognition. Indeed, as mentioned in Section 3.4.1, p. 43, all models of categorization and concept acquisition lie on the *feature extraction*; our framework contains also a feature extraction by means of the function Φ (item iii., Sect. 6.1.2). In addition, as mentioned in Section 3.4, p. 41, many models base on similarity; in our framework, we can create a similarity measure based on the featural distance $d_{\mathcal{F}}$ (Eq. (6.8)).

Consequently, we can link the elements in our framework to terms and properties in human categorization, as we do in the following.

Features We identify the *featural variables* in $\mathcal{F}$ with the '*features*', as defined in the *APA Dictionary of Psychology*, p. 370 (2007) "[A]ttribute of an object or event that plays an important role in distinguishing it from other objects or events and in the formation of category judgements." Whereas the *kinematic variables* and all other non-featural variables are simply '*attributes*', the *featural variables* are a very particular type of *attributes*: they univocally determine the category. Still, alternative terminology is used in the literature, which we mention for the sake of completeness: often researchers use the term 'feature' to generally refer to 'attribute' (e.g., Colman 2015, pp. 812f.); because of this, they distinguish the relevant features by adding an adjective such as '*characteristic*', '*defining*' (Hampton 1979), or '*diagnostic*' (Goldstone 1996), e.g., 'diagnostic features'.

Our framework stresses the feature extraction, Φ, as an intermediate step in categorization. Even though we do not lie on the features from start—They are subsequently extracted—we can rightly call our framework a '*featural framework*'. This might be seen as a limitation, or even as opposed to '*neural networks*' methods, but truth is that modern neural networks, e.g., deep learning, strive to automate the finding of *features* (Skansi 2018, p. 55), and, thus, the quest after features seems a constant in categorization research (e.g., Sect. 3.4.1 and Fig. 3.2).

Interestingly, we give clear mathematical requirements in order to define the features, which is rare in psychology.

Similarity function and Psychological Space In our framework, we can implement a similarity function between motion scenarios, $\mathrm{S}(K_1, K_2)$, because the featural space is a metric space: we can directly transform the distance function in the featural space into a dissimilarity, or into a similarity. In addition, such dissimilarity has the property of being graded due to the self-density of the featural space; this increases its cognitive plausibility.

Furthermore, we can readily identify the *featural space* with the *psychological space* because it is in the featural space where the dissimilarities are computed (Sect. 3.4.1.A, p. 45). We see this clearly in Equation (6.8): the distance between scenarios, $\mathrm{d}_{\mathcal{K}}(K_1, K_2)$, is effectively computed in the featural space, $\mathrm{d}_{\mathcal{F}}(\varphi_1, \varphi_2)$, after the features are extracted $\varphi_1 = \Phi(K_1)$ and $\varphi_2 = \Phi(K_2)$. In other words, scenarios with exactly the same featural values are categorized as “identical”.

All considered, we advocate for the use of the term ‘featural space’ over the term ‘psychological space’ to refer to a metric space in which similarities are measured (Sect. 3.4.1.A). The adjective ‘featural’ is more objective: it refers to the fact that such metric space consists of coordinates which are called, by definition, ‘features’; and there is no psychological implication about such space: it is only a practical modelling device. On the contrary, the adjective ‘psychological’ implies that such space is part of the mind or, at least, that is cognitively plausible.

Standard Categorization Models The categorization rule f_μ—or rather its featural version f_Φ—tells us which standard categorization model is at work for a given categorization. By inspecting the categorization rule according to our model, we can see whether a given categorization implements a rule-based model (i.e., a boundary model, Sect. 3.5) or a similarity-based model (i.e., a metric model, Sect. 3.4.2); moreover, we can see whether the similarity model requires exemplars or category prototypes, or whether clustering is performed.

Regarding philosophical terms, we already mentioned above how the *intensional* and *extensional* types of concept definitions are expressed respectively by each definition of categorization: given the categorization rule $(\mathrm{f}_\mu, \mathcal{M})$ (Eq. (6.2)) and given the categorical regions $(\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_n\}, \mathcal{M})$ (Eq. (6.3)).

6.3 Example: Scenario Categorization by QTC$_{B21}$

We illustrate the definitions and claims of the previous section through the well-known and simple motion categorization QTC$_{B21}$ (Van De Weghe et al. 2006). Although it is already briefly defined in Sect. 5.4, we offer below a more extensive definition.

In QTC$_{B21}$, each motion category is a 2-tuple $(\mathcal{P}_k, \mathcal{P}_l)$, where each coefficient, $\mathcal{P}_*$, can take one of three symbols, $\{+, 0, -\}$. Accordingly, we have a total of 9 categories that we can alternatively name using the coefficients $\mathcal{P}_*$ or the standard terms for motion categories M_i (item i., Sect. 6.1.1), e.g., $(-,-)$ or M_1.

$$\begin{aligned} \mathcal{M}_{\mathrm{QTC_{B21}}} &= \{(\mathcal{P}_k, \mathcal{P}_l)\} \\ &= \{\overset{M_1}{(-,-)}, \overset{M_2}{(-,0)}, \overset{M_3}{(-,+)}, \overset{M_4}{(0,-)}, \overset{M_5}{(0,0)}, \overset{M_6}{(0,+)}, \overset{M_7}{(+,-)}, \overset{M_8}{(+,0)}, \overset{M_9}{(+,+)}\} \end{aligned} \tag{6.9}$$

The coefficients $\mathcal{P}_k$ and $\mathcal{P}_l$ are determined according to following rules:

- $\mathcal{P}_k$ is the relative motion of the entity k regarding l still.
  - − if k moves towards l
  - 0 if k keeps distance to l
  - \+ if k moves away from l
- $\mathcal{P}_l$ is the relative motion of the entity l regarding k still:
  - − if l moves towards k

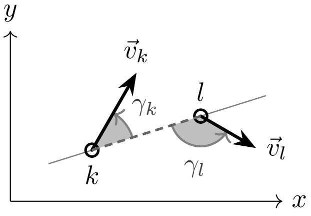


Figure 6.3: In this motion scenario, we show the angles γ_k and γ_l, which fully determine the scenario's categorization in $\mathrm{QTC}_{\mathrm{B21}}$ through their cosine, $\cos(\gamma_k)$ and $\cos(\gamma_l)$ (Eqs. 6.12 and Eq. (6.10)). This motion scenario is categorized as $(-,+)$, i.e., M_3.

0	if l keeps distance to k
+	if l moves away from k

Categorization rule f_μ

Based on the above definitions, we can explicitly define the categorization rule f_μ.

$$\begin{aligned} \mathrm{f}_\mu : \mathcal{K} &\longrightarrow \mathcal{M}_{\mathrm{QTC}_{\mathrm{B21}}} \\ (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) &\longmapsto M_i = (\underbrace{\mathtt{Sign}(-\frac{\vec{v}_k \cdot (\vec{x}_l - \vec{x}_k)}{\|\vec{v}_k\|\|\vec{x}_l - \vec{x}_k\|})}_{\mathcal{P}_k}, \underbrace{\mathtt{Sign}(-\frac{\vec{v}_l \cdot (\vec{x}_k - \vec{x}_l)}{\|\vec{v}_l\|\|\vec{x}_k - \vec{x}_l\|})}_{\mathcal{P}_l}) \end{aligned} \tag{6.10}$$

where $\mathtt{Sign}(x)$ is the '*sign symbol function*'; it outputs a character.

$$\mathtt{Sign}(x) = \begin{cases} \text{'}-\text{'} & \text{if } x < 0 \\ \text{'0'} & \text{if } x = 0 \\ \text{'+'} & \text{if } x > 0 \end{cases} \tag{6.11}$$

Feature extraction $\mathit{\Phi}$ and feature categorization $\mathrm{f}_{\mathit{\Phi}}$

The categorization rule f_μ in Equation (6.10) becomes more graspable, if we decompose it into the feature extraction $\mathit{\Phi}$ and the feature categorization $\mathrm{f}_{\mathit{\Phi}}$, i.e., $\mathrm{f}_\mu = \mathrm{f}_{\mathit{\Phi}} \circ \mathit{\Phi}$, according to Equation (6.5).

By inspecting f_μ we discern the possible featural variables φ_k and φ_l (Eqs. 6.12), which correspond to cosine values of the relative motion angles (γ_k and γ_l) of the entities k and l.

$$\varphi_k(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) := \frac{\vec{v}_k \cdot (\vec{x}_l - \vec{x}_k)}{\|\vec{v}_k\|\|\vec{x}_l - \vec{x}_k\|} = \cos(\angle(\vec{x}_l - \vec{x}_k, \vec{v}_k)) = \cos(\gamma_k) \tag{6.12a}$$

$$\varphi_l(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) := \frac{\vec{v}_l \cdot (\vec{x}_k - \vec{x}_l)}{\|\vec{v}_l\|\|\vec{x}_k - \vec{x}_l\|} = \cos(\angle(\vec{x}_k - \vec{x}_l, \vec{v}_l)) = \cos(\gamma_l) \tag{6.12b}$$

These features, φ_k and φ_l, fulfil the aforementioned main conditions for featural values (item i., Sect. 6.1.2 and Sect. 6.1.3): they are the smallest set of independent variables that determine the categorization; the dimensionality of the features space, $\dim(\mathcal{F}) = 2$, is much lower than the dimensionality of the kinematic space, $\dim(\mathcal{K}) = 8$; they correspond to natural kinematic concepts, and they are self-dense (they are real values).

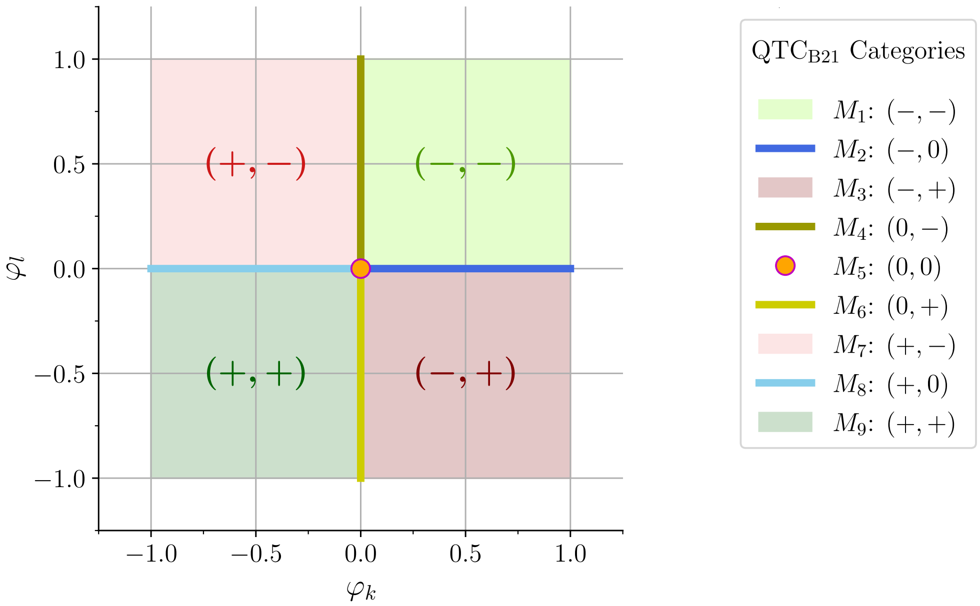


Figure 6.4: Featural space of QTC$_{B21}$, $\mathcal{F}_{\text{QTC}_{\text{B21}}}$, which is constituted by two variables $\varphi_k = \cos(\gamma_k)$ and $\varphi_l = \cos(\gamma_l)$ (Eq. (6.14)). The different coloured regions correspond to the QTC$_{B21}$ categories; i.e., they are *categorical regions*. They have different dimensionality; for example, M_5 is 0-dimensional, M_6 is 1-dimensional, and M_7 is 2-dimensional.

Now, we define the *feature extraction function*, Φ, which maps every motion scenario into the QTC$_{B21}$ featural space, $\mathcal{F}_{\text{QTC}_{\text{B21}}}$, as follows.

$$\begin{aligned} \Phi : \mathcal{K} &\longrightarrow \mathcal{F}_{\text{QTC}_{\text{B21}}} = [-1,1] \times [-1,1] \\ (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) &\longmapsto (\varphi_k, \varphi_l) \end{aligned} \tag{6.13}$$

with φ_k and φ_l defined, respectively, in Equations (6.12a) and (6.12b)

As last step, we obtain the *feature categorization function*, f_Φ, which yields the categories based on the features φ_k and φ_l. We deduce the following f_Φ function by filling the gap in Equation (6.13) that allow us to obtain Equation (6.10). We can visualize this function, f_Φ, in Figure 6.4.

$$\begin{aligned} \mathrm{f}_\Phi : \mathcal{F}_{\text{QTC}_{\text{B21}}} &\longrightarrow \mathcal{M}_{\text{QTC}_{\text{B21}}} = \{-,0,+\}^2 \\ (\varphi_k, \varphi_l) &\longmapsto M_i = (\mathcal{P}_k, \mathcal{P}_l) = (\texttt{Sign}(-\varphi_k), \texttt{Sign}(-\varphi_l)) \end{aligned} \tag{6.14}$$

with `Sign` defined in Equation (6.11)

At last, we have the main ingredients of a categorization: the *categorization rule*, f_μ, and its resolution into *feature extraction*, Φ, and *feature categorization*, f_Φ. These three functions are

$$\begin{tikzcd}
\mathcal{K} & & \\
(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \arrow[d, mapsto, "\mathit{\Phi}"] \arrow[drr, mapsto, "\mathrm{f}_\mu = \mathrm{f}_{\mathit{\Phi}} \circ \mathit{\Phi}"] & & \\
(\varphi_k, \varphi_l) \arrow[rr, mapsto, "\mathrm{f}_{\mathit{\Phi}}"] & & M_i = (\mathcal{P}_k, \mathcal{P}_l) \\
\mathcal{F}_{\mathrm{QTC}_{\mathrm{B21}}} = [-1, 1]^2 & & \mathcal{M}_{\mathrm{QTC}_{\mathrm{B21}}} = \{-, 0, +\}^2
\end{tikzcd}$$

Figure 6.5: This diagram is a particular case of Equation (6.5) applied to $\mathrm{QTC}_{\mathrm{B12}}$. The categorization rule f_μ categorizes a motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ into a $\mathrm{QTC}_{\mathrm{B12}}$ category $(\mathcal{P}_k, \mathcal{P}_l)$ (Eq. (6.10)). The categorization process is decomposed in, first, the extraction of features (φ_k, φ_l), by $\mathit{\Phi}$ (Eq. (6.13)), and the subsequent feature-based classification, by $\mathrm{f}_{\mathit{\Phi}}$ (Eq. (6.14)).

related by the Equation (6.5), which we adapt in Figure 6.5 to this specific case of the $\mathrm{QTC}_{\mathrm{B21}}$ representation.

Notice how in describing $\mathrm{QTC}_{\mathrm{B21}}$ according to our framework, we have proceeded in a *top-down* way (Fig. 6.1). As in any other qualitative representation of motion, $\mathrm{QTC}_{\mathrm{B21}}$ was defined without explicit mention of features or featural space; nevertheless, these were implicit in the definition, and our framework brings them to light.

6.4 Testing Dissimilarities and Traditional Models

In this section, we show further capabilities of our categorization framework. We apply it to test the dissimilarity measure in three different spaces and two traditional categorization models for motion scenarios—We use as example the motion categorization $\mathrm{QTC}_{\mathrm{B12}}$, which we described in Section 6.3. Concerning the dissimilarity measures, we test in which representation space the dissimilarity best captures the properties of a categorization, either in the *kinematic space* ($\mathcal{K}$), or in the *featural space* ($\mathcal{F}$), or in the *categorical space* ($\mathcal{M}$). Further, using the fittest dissimilarity, we test which traditional model, either *prototype* or *boundary* (Sects. 3.4.2 and 3.5), is more suitable to describe such motion categorization. Keep in mind that the results, in terms of *dissimilarities*, can be directly translated in terms of *similarities* (as shown in Tab. 3.1).

Our prime aim is not to perform an in-depth analysis of all possible representation spaces that are apt for computing dissimilarity measures, but to outline how dissimilarity measures behave regarding the space in which they are computed. For that reason, we drastically restrict the number of analyzed scenarios and representation spaces: 3 reprentation spaces and 9 motion scenarios. The chosen representation spaces and scenarios are representative enough to show common issues, which can be generalized to categorizations other than categorization of motion scenarios.

We have chosen 9 scenarios, K_*, arrayed in three groups according to their $\mathrm{QTC}_{\mathrm{B21}}$ categories: the group K_a belongs to category $(-, -)$, K_b belongs to $(0, 0)$, and K_c to $(+, +)$ (Fig. 6.6 and Tab. 6.2). We compare all scenarios pairwise using three different dissimilarity measures: $d_{\mathcal{K}}$, $d_{\mathcal{F}}$, and $d_{\mathcal{M}}$ (Eqs. 6.15). For each measure, we use the Euclidean norm, i.e., $\|\vec{x}\|_2 = \left(\sum_{i=1}^{n} {x_i}^2\right)^{1/2}$ to compute the dissimilarities. Another possibility would be to use $\|\cdot\|_2$ only for $d_{\mathcal{K}}$, because many features are *integral* (they cannot exist independently of one another); while we could use

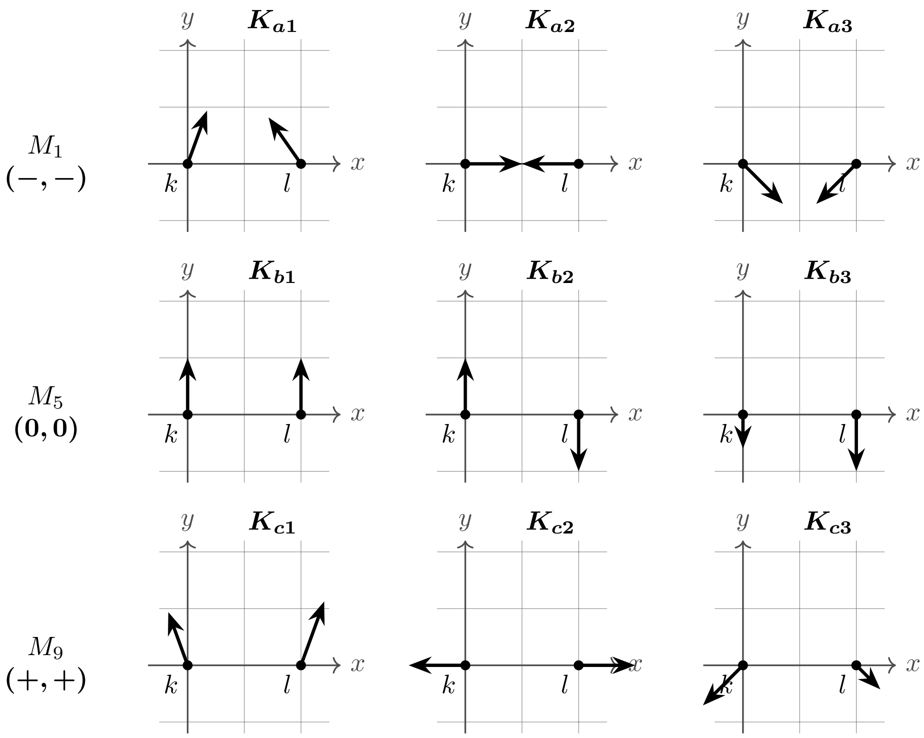


Figure 6.6: Motion scenarios that we use to analyze the three different dissimilarity measures in Table 6.3. The scenarios are grouped by $\mathrm{QTC_{B12}}$ categorization: scenarios K_{a*} belong to category M_1, i.e., $(-,-)$, K_{b*} to category M_5, i.e., $(0,0)$, and K_{c*} to M_9, i.e., $(+,+)$. The complete kinematic, featural, and categorical data of the scenarios is listed in Table 6.2

	$\mathcal{K}$ kinematic space	$\mathcal{F}_{\mathrm{QTC_{B21}}}$ featural space	$\mathcal{M}_{\mathrm{QTC_{B21}}}$
	$(\vec{x}_k\ ,\ \vec{v}_k\ ;\ \vec{x}_l\ ,\ \vec{v}_l)$	$(\varphi_k, \varphi_l) = (\cos(\gamma_k), \cos(\gamma_l))$	$(\mathcal{P}_k, \mathcal{P}_l)$
K_{a1}	$((0,0),\ (1.0, 70°)\ ;\ (2{,}0),\ (1.0, 125°))$	$(0.34, 0.57) = (\cos(70°), \cos(-55°))$	$M_1 = (-,-)$
K_{a2}	$((0,0),\ (1.0, 0°)\ ;\ (2{,}0),\ (1.0, 180°))$	$(1.00, 1.00) = (\cos(0°), \cos(0°))$	$M_1 = (-,-)$
K_{a3}	$((0,0),\ (1.0, -45°);\ (2{,}0),\ (1.0, 225°))$	$(0.71, 0.71) = (\cos(-45°), \cos(45°))$	$M_1 = (-,-)$
K_{b1}	$((0,0),\ (1.0, 90°)\ ;\ (2{,}0),\ (1.0, 90°))$	$(0.00, 0.00) = (\cos(90°), \cos(-90°))$	$M_5 = (0,0)$
K_{b2}	$((0,0),\ (1.0, 90°)\ ;\ (2{,}0),\ (1.0, -90°))$	$(0.00, 0.00) = (\cos(90°), \cos(90°))$	$M_5 = (0,0)$
K_{b3}	$((0,0),\ (0.6, -90°);\ (2{,}0),\ (1.0, -90°))$	$(0.00, 0.00) = (\cos(-90°), \cos(90°))$	$M_5 = (0,0)$
K_{c1}	$((0,0),\ (1.0, 110°);\ (2{,}0),\ (1.2, 70°))$	$(-0.34, -0.34) = (\cos(110°), \cos(-110°))$	$M_9 = (+,+)$
K_{c2}	$((0,0),\ (1.0, 180°);\ (2{,}0),\ (1.0, 0°))$	$(-1.0, -1.0) = (\cos(180°), \cos(180°))$	$M_9 = (+,+)$
K_{c3}	$((0,0),\ (1.0, 225°);\ (2{,}0),\ (0.6, -45°))$	$(-0.71, -0.71) = (\cos(-135°), \cos(135°))$	$M_9 = (+,+)$

Table 6.2: Complete kinematic, featural, and categorical data of each scenario K_* in Figure 6.6 according to the $\mathrm{QTC_{B21}}$ categorization (Eq. (6.9)). The first column, '$\mathcal{K}$ kinematic space', displays the kinematic data (positions $\vec{x}_*$ and velocities $\vec{v}_*$); the length units are not specified, one unit corresponds to the separation between grid lines at Fig. 6.6; the velocities are expressed in polar coordinates (*speed*, *angle*). The featural coordinates φ_* are those obtained through Eqs. 6.12. The categories are determined through the categorization rule in Equation (6.10)

$\|\cdot\|_1$ for $d_{\mathcal{F}}$ and $d_{\mathcal{M}}$, because the features are *separable* (See Sect. 3.4.1.A). However, in our example, the results are qualitatively the same independently of the norm used. We choose the same norm for all measures, Euclidean, so that the difference lies only in the chosen representation space. In that way, we can better appreciate how important is the choice of the representation space (i.e., kinematic, featural, or categorical) to appropriately model the categorization.

The dissimilarity measures are defined as follows:

$$\mathrm{d}_{\mathcal{K}}(K_A, K_B) = \left[(\vec{x}_{kB} - \vec{x}_{kA})^2 + (\vec{v}_{kB} - \vec{v}_{kA})^2 + (\vec{x}_{lB} - \vec{x}_{lA})^2 + (\vec{v}_{lB} - \vec{v}_{lA})^2\right]^{1/2} \tag{6.15a}$$

$$\mathrm{d}_{\mathcal{F}}(K_A, K_B) = \left[(\varphi_{kB} - \varphi_{kA})^2 + (\varphi_{lB} - \varphi_{lA})^2\right]^{1/2} \tag{6.15b}$$

$$\mathrm{d}_{\mathcal{M}}(K_A, K_B) = \left[(p_{kB} - p_{kA})^2 + (p_{lB} - p_{lA})^2\right]^{1/2} \tag{6.15c}$$

The parameters used in Equations 6.15 mean following:

(a) $(\vec{x}_{kA}, \vec{v}_{kA}, \vec{x}_{lA}, \vec{v}_{lA})$ and $(\vec{x}_{kB}, \vec{v}_{kB}, \vec{x}_{lB}, \vec{v}_{lB})$ are, respectively, the *kinematic coordinates* of scenario K_A and K_B, that is, their coordinates in the kinematic space $\mathcal{K}$.

(b) $(\varphi_{kA}, \varphi_{lA})$ and $(\varphi_{kB}, \varphi_{lB})$ are, respectively, the *featural coordinates* of scenario K_A and K_B, that is, their coordinates in the featural space $\mathcal{F}$. They are computed from the kinematic coordinates via Equations 6.12.

(c) (p_{kA}, p_{lA}) and (p_{kB}, p_{lB}) are the *categorical coordinates* of the scenarios K_A and K_B, that is, the values $\{-1, 0, +1\}$ that univocally determine the $\mathrm{QTC_{B21}}$ category of the scenarios, $(\mathcal{P}_k, \mathcal{P}_l)$ (Eq. (6.9)). That is, they are their coordinates in the categorical space $\mathcal{M}$. For example, if scenario K_A belongs to the category $(-, +)$, then $(p_{kA}, p_{lA}) = (-1, 1)$; and if scenario K_B belongs to the category $(+, 0)$, then $(p_{kB}, p_{lB}) = (+1, 0)$.

6.4.1 Dissimilarity results: analysis

In order to analyze the dissimilarity values (Tab. 6.3) of the test scenarios (Tab. 6.2 and Fig. 6.6), we regard as benchmark some desirable and plausible properties for dissimilarities, which we derive from our exposition in Section 3.3—More accurately, there, we mentioned the properties of similarity in human categorization, which, here, we translate into properties for dissimilarities (items I. to III.). A dissimilarity having such properties should suitably reflect both the inner and the outer category structure in human categorization.

I. (*minimality*) The lowest dissimilarity of an item should be the self-dissimilarity (Compare with item ii. in distance axioms).

II. (‘*family resemblance*’) The average within-category dissimilarity should be lower than the average between-category dissimilarity (Sect. 3.3.2.B).

III. (*prototypes adequacy*) If we assume that the categories are represented by prototypes following properties should apply:

   A. The protypes are the central (sort of average) items of a category (Sect. 3.3.1).

   B. A prototype is, in average, least dissimilar to the items within the category, and overall most dissimilar to the items in other categories—This is a kind of *family resemblance* property for prototypes.

   C. An item’s category is determined by the least dissimilar prototype to such item (Sect. 3.4.2). Equivalently, the category members should be less dissimilar to the prototypes in their category than to those in other categories.

$\mathrm{d}_{\mathcal{K}}(K_A, K_B)$	K_{a1}	K_{a2}	K_{a3}	K_{b1}	K_{b2}	K_{b3}	K_{c1}	K_{c2}	K_{c3}
K_{a1}	0.00	1.75	2.81	0.70	1.94	2.91	1.24	2.59	3.01
K_{a2}		0.00	1.29	2.24	2.24	1.93	2.61	2.83	2.47
K_{a3}			0.00	3.12	2.63	1.05	3.34	2.71	1.83
K_{b1}				0.00	2.00	3.02	0.56	2.24	2.92
K_{b2}					0.00	2.26	2.20	2.24	2.62
K_{b3}						0.00	3.09	1.93	1.02
K_{c1}							0.00	1.95	2.82
K_{c2}								0.00	1.26
K_{c3}									0.00

(a) Dissimilarity matrix between scenarios obtained from $\mathrm{d}_{\mathcal{K}}(K_A, K_B)$, which is the Euclidean norm applied to the *kinematic* coordinates of the scenarios K_A and K_B (Eq. (6.15a)).

$\mathrm{d}_{\mathcal{F}}(K_A, K_B)$	K_{a1}	K_{a2}	K_{a3}	K_{b1}	K_{b2}	K_{b3}	K_{c1}	K_{c2}	K_{c3}
K_{a1}	0.00	0.78	0.39	0.67	0.67	0.67	1.14	2.07	1.66
K_{a2}		0.00	0.41	1.41	1.41	1.41	1.90	2.83	2.41
K_{a3}			0.00	1.00	1.00	1.00	1.48	2.41	2.00
K_{b1}				0.00	0.00	0.00	0.48	1.41	1.00
K_{b2}					0.00	0.00	0.48	1.41	1.00
K_{b3}						0.00	0.48	1.41	1.00
K_{c1}							0.00	0.93	0.52
K_{c2}								0.00	0.41
K_{c3}									0.00

(b) Dissimilarity matrix between scenarios obtained from $\mathrm{d}_{\mathcal{F}}(K_A, K_B)$, which is the Euclidean norm applied to the *featural* coordinates of the scenarios K_A and K_B (Eq. (6.15b)).

$\mathrm{d}_{\mathcal{M}}(K_A, K_B)$	K_{a1}	K_{a2}	K_{a3}	K_{b1}	K_{b2}	K_{b3}	K_{c1}	K_{c2}	K_{c3}
K_{a1}	0.00	0.00	0.00	1.41	1.41	1.41	2.83	2.83	2.83
K_{a2}		0.00	0.00	1.41	1.41	1.41	2.83	2.83	2.83
K_{a3}			0.00	1.41	1.41	1.41	2.83	2.83	2.83
K_{b1}				0.00	0.00	0.00	1.41	1.41	1.41
K_{b2}					0.00	0.00	1.41	1.41	1.41
K_{b3}						0.00	1.41	1.41	1.41
K_{c1}							0.00	0.00	0.00
K_{c2}								0.00	0.00
K_{c3}									0.00

(c) Dissimilarity matrix between scenarios obtained from $\mathrm{d}_{\mathcal{M}}(K_A, K_B)$, the Euclidean norm applied to the categorical values (p_k, p_l), of the scenarios K_A and K_B (Eq. (6.15c)).

Table 6.3: Dissimilarity matrices that pairwise relate the 9 scenarios in Items I. to III.. In each matrix a different dissimilarity measure is used: all dissimilarities are Euclidean, but each is computed in a different space. The level of dissimilarity is proportional to the blue saturation of each cell: lowest dissimilarity (0.00) corresponds to white cells and highest (depending on the measure) corresponds to blue saturated cells. The brown coloured rectangles are *within-category* dissimilarities; the rest are *between-category* dissimilarities.

Minimality In the Tables 6.3a to 6.3c, we observe how all dissimilarities fulfil the first property, *minimality*. By definition all self-dissimilarities are zero, which is the minimum possible value.

Family resemblance We test the second property—*family resemblance*—aided by the brown rectangles at the tables. The rectangles contain the within-category dissimilarities, i.e., dissimilarities between scenarios belonging to the same category. For that reason, the dissimilarities inside the brown rectangles should be lower than those outside them.

In the dissimilarity $d_{\mathcal{K}}$ (Tab. 6.3a), the values inside the rectangles are not particularly low in whatever measure we take (mean or median). Actually, by visual inspection, the kinematic dissimilarities do not seem to follow any particular pattern related to category membership, which makes $d_{\mathcal{K}}$ an entirely useless dissimilarity with regard to the $\mathrm{QTC}_{\mathrm{B12}}$ categorization. This becomes more evident, if we notice that in our examples (Fig. 6.6), the only contributions to $d_{\mathcal{K}}$ arise from differences in the velocity vectors—since the positions of k and l are the same in each scenario—and, hence, by modifying the entities' positions alone, we could obtain any arbitrarily high dissimilarity values between scenarios.

On the other hand, Tables 6.3b and 6.3c show family resemblance: we observe a clearly low within-category and high between-category dissimilarity, though with some objections.

First, we realize that the categorical dissimilarity, $d_{\mathcal{M}}$, discerns no *intra-categorical* structure; the differences of $d_{\mathcal{M}}$ are only due to the different categorical membership, while all items of the same category have the same dissimilarity. For example, all scenarios within the same category have $d_{\mathcal{M}} = 0.0$, and, between all scenarios of category $(-,-)$ and category $(+,+)$ there is the same dissimilarity, $d_{\mathcal{M}} = 2.83$. Thus, in practice, $d_{\mathcal{M}}$ is a dissimilarity between categories not between items; it is only a little finer than the '*discrete dissimilarity*' between categories which assigns 1.0 to items of different categories and 0.0 to items of the same category. The dissimilarity $d_{\mathcal{M}}$ takes advantage of the fact that $\mathrm{QTC}_{\mathrm{B12}}$ is a concatenated representation, i.e., $\mathrm{QTC}_{\mathrm{B12}} = \mathcal{P}_k \times \mathcal{P}_l$, so that $d_{\mathcal{M}}$ reflects the between-category structure provided by the Cartesian product, but it obliterates all trace of within-category structure.

Second, the dissimilarity $d_{\mathcal{F}}$ presents a marked effect of family resemblance. In fact, if we take $\tau_{\mathcal{F}} = 1.00$ as threshold, the rule $\mathrm{d}_{\mathcal{F}}(K_A, K_B) < \tau_{\mathcal{F}}$ correctly classifies most pair of scenarios, (K_A, K_B), as belonging to the same category. Even more, any scenario but scenario K_{c1} belongs to the same category of the scenario with which it has lowest $d_{\mathcal{F}}$; in other words,

$$\text{If } \mathrm{d}_{\mathcal{F}}(K_A, K_B) < \mathrm{d}_{\mathcal{F}}(K_A, K_C)\ \forall K_C \neq K_B \quad \Rightarrow \quad K_A \text{ and } K_B \text{ belong to the same category}$$

In this sense, $d_{\mathcal{F}}$ would be cognitively the most adequate of all similarities but with a caveat: its family resemblance is imperfect. For example, family resemblance is flagrantly violated by the scenario K_{c1}. This scenario—which according to the categorization rule (Eq. (6.10)) belongs to the category M_9, i.e., $(+,+)$—has lower dissimilarity, $d_{\mathcal{F}} = 0.48$, to all scenarios in category M_5, i.e., $(0,0)$, than to those in its own category, $\mathrm{d}_{\mathcal{F}}(K_{c1}, K_{c3}) = 0.52$ and $\mathrm{d}_{\mathcal{F}}(K_{c1}, K_{c2}) = 0.93$.

Family resemblance is deficient in $\mathrm{QTC}_{\mathrm{B12}}$, because this categorization is defined as a boundary model in which the borders are categories: category $(0,0)$ is the border between categories $(-,-)$ and $(+,+)$. For that reason, the scenarios of category $(+,+)$ that are closer to the *boundary category* $(0,0)$—that is, those scenarios of $(+,+)$ with $(\varphi_k > -0.5,\ \varphi_l > -0.5)$, i.e., $(-120^\circ < \gamma_k < 120^\circ,\ -120^\circ < \gamma_l < 120^\circ)$—are in average more similar to the scenarios in $(0,0)$ than to those in its own category, $(+,+)$. An analogous effect occurs with category $(-,-)$.

Prototypes adequacy If $\mathrm{QTC}_{\mathrm{B12}}$ motion categorization were suitably modelled by prototypes, the properties recounted in Item III. should apply. However, a prototype model cannot fully account for its category structure: The deficiencies that arose when we tested family resemblance reverberate throughout the testing of the *prototypes adequacy.*

For a start, the categorical distance, $d_{\mathcal{M}}$ (Tab. 6.3c), cannot define prototypes because it displays no inner structure in the categories; only the featural distance $d_{\mathcal{F}}$ remains as possible distance for defining the prototypes. Accordingly, the prototypes in the featural space $\mathcal{F}$ correspond to the K_{*3} scenarios (i.e., K_{a3}, K_{b3}, and K_{c3}); certainly, we observe that those scenarios have the lowest dissimilarity $d_{\mathcal{F}}$ to the other scenarios in their own category (Tab. 6.3b)—As side note, the scenarios of the category $(0, 0)$ are indistinguishable in the featural space: any of them have the same featural distance $d_{\mathcal{F}}$ to any other scenario; we have chosen K_{b3} for ease of exposition.

Therefore, by definition, the prototypical scenarios, K_{*3}, fulfil the centrality property (item III.A.), and, thus, have the lowest within-category dissimilarity; this is the first condition of Item III.B.. However, they fail to fulfil the second condition of Item III.B., namely, that the prototypes have the highest between-category dissimilarity. Indeed, the highest between-category dissimilarity is given by the K_{*2} scenarios: K_{a2} in category $(-, -)$, and by K_{c2} in category $(+, +)$—As said before, in the border category $(0, 0)$ all scenarios have the same between-category dissimilarity, so we can choose K_{b2}. Sadly, we find no scenarios that completely fulfil Item III.B.; we are forced into a trade-off: if we increase the between-category dissimilarity, the within-category dissimilarity will also increase, so we cannot increase one, while decreasing the other to find an optimum.

On top of that, the prototypes fail to fully account for category membership (item III.C.), which is a primary goal of the prototype theory. For example, the item K_{c1} has a lower dissimilarity $d_{\mathcal{F}}$ to the prototype in the category $(0, 0)$—what is more, to any member of such category—than to any possible prototype in its own category $(+, +)$, i.e., to K_{c2} or K_{c3}. Actually, all members of category $(+, +)$ with featural coordinates $(\varphi_k > -0.34,\ \varphi_l > -0.34)$, i.e., $(-70^\circ > \gamma_k > 70^\circ,\ -70^\circ > \gamma_l > 70^\circ)$, have a higher dissimilarity to any of their possible prototypes than to *any* member of the category $(0, 0)$.

Much as we explained above for the *family resemblance*, the prototype model fails to precisely describe $\mathrm{QTC}_{\mathrm{B12}}$ because this motion categorization is defined as a boundary model. Even though we can define prototypes in $\mathrm{QTC}_{\mathrm{B12}}$ that retain some of their basic properties (for instance, lowest within-category or highest between-category dissimilarity), we cannot find a prototype that fulfils all properties. Moreover, prototypes can only approximatively account for category membership; they fail when category members are near to category borders (as it is the case of K_{c1}).

Although we did not prove it, it is evident that neither the exemplar model can provide a detailed categorization account for similar reasons as the prototype model: one fails to reproduce the effect of a border category by means of a *finite* number of exemplars. Technically, it is not possible because, in a real continuum, a border has always at least one topologically open side. In sum, only the boundary model can precisely describe the categorization originated by $\mathrm{QTC}_{\mathrm{B12}}$.

6.4.2 Conclusion

By means of the simple example, $\mathrm{QTC}_{\mathrm{B12}}$, we have seen how the motion categorizations behave that are created as qualitative representations. The most meaningful dissimilarity function works in the featural space, i.e., it is based on the featural distance $d_{\mathcal{F}}$. It yields acceptable results regarding the expected properties of a dissimilarity (items I. to III.): In psychological jargon, we would say that the featural space $(\mathcal{F}, d_{\mathcal{F}})$ is the most faithful representation of the *psychological space* for the exemplified categorization.

Prototypes can be used as a rough description of the categorization; they fulfil to a certain degree their expected properties (item III.A. to item III.C.). Nevertheless, prototypes fail to provide a precise categorization account because a motion categorization—at least in the form of a qualitative representation—has *meaningful borders*, i.e., borders that are also categories; this is the case of the category $(0, 0)$, the border between categories $(-, -)$ and $(+, +)$. Moreover, as we mentioned, the border categories make also the exemplar model inadequate.

All counted, we have established that only the *boundary model* captures the categorical essence of a qualitative representation of motion, and, furthermore, we have also validated our framework as a tool capable of describing and analyzing motion categorizations.

Chapter 7

Stories and Temporal Sequences of Relations

> *The qualitative is supervenient on the quantitative. There can be no qualitative distinctions without underlying quantitative ones. This does not in itself necessarily imply that qualitative distinctions arise from quantitative ones, [however,] that conclusion is all but irresistible in very many cases.*
>
> A. Galton (2000, p. 341)

So far, we have presented a broad and multidisciplinary understanding of categorization (Ch. 3), and have explained how qualitative representations, specifically those concerning motion, are—or, rather, can be—integrated in such categorization domain (Chs. 4 to 6). In this chapter, we begin the genesis of the story-based representations: motion categorizations derived from spatial and motion representations. The keystone of story-based representations is the concept of 'story'; the most part of this chapter is devoted to defining it.

7.1 Motivation and Background for Temporal Sequences

Fundamental onset conditions We base the categorization of the movement on the following conditions: (See Sects. 1.1 and 1.2):

i. We have '*motion scenarios*' (Sect. 1.2), which are described by instantaneous position-velocity pairs of vectors in a certain instant $t = t_0$ (a vector pair for each entity). In the most common case of two entities, k and l, a motion scenario is described as $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$.

ii. We have a '*qualitative spatial representation*' ($\mathcal{D}$, $\delta_{\mathcal{X}}$) (Ch. 4). It categorizes *position scenarios* $(\vec{x}_k, \vec{x}_l)$ of two entities, into n possible categories, the qualitative spatial relations $\{R_1, R_2, \ldots, R_n\}$. The set of all possible position scenarios is the '*position space*' $\mathcal{X} \times \mathcal{X}$. Note that a '*position scenario*' $(\vec{x}_k, \vec{x}_l)$ describes not only the "positions" of two entities, but *any positional information* (e.g., positions, orientations, sizes) that is relevant for the spatial categorization $\mathcal{D}$. Here, we briefly show how we integrate the qualitative spatial relations, as defined in the literature, into our kinematic spaces—These are formal details. As a result, we define the '*spatial map*', δ, in Equation (7.1).

The spatial categorization $\mathcal{D}$ is usually defined *extensively*. For example, a '*qualitative spatial relation*' is a region $\boldsymbol{P}_i$ of the '*position space*' $\mathcal{X}\times\mathcal{X}$, i.e., $R_i = \{(\vec{x}_k, \vec{x}_l) \in \boldsymbol{P}_i \subset \mathcal{X}\times\mathcal{X}\}$ (e.g., Ligozat and Renz 2004; Dylla et al. 2017) (See gen. def. Sect. 10.1.1). In our work, it is more convenient to define $\mathcal{D}$ *intensionally* (Eq. (7.2)); we call $\delta_{\mathcal{X}}$ this categorization rule that assigns the corresponding spatial relation R_i to each *position scenario*, i.e., $R_i = \delta_{\mathcal{X}}(\vec{x}_k, \vec{x}_l)$.

As we want to obtain the spatial relations R_i for motion scenarios, we extend the spatial categorization rule $\delta_{\mathcal{X}}$ to motion scenarios and call it '*spatial map*', δ. To obtain δ is a trivial process (Eq. (7.1): first, we extract the positional information, $(\vec{x}_k, \vec{x}_l)$, from the motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$—we use a projection, $\pi_{\mathcal{X}}$—then, we apply the categorization rule $\delta_{\mathcal{X}}$ to the extracted position scenario $(\vec{x}_k, \vec{x}_l)$.

$$\begin{tikzcd} \mathcal{K} \arrow[r] & \mathcal{X}\times\mathcal{X} \arrow[r] & \mathcal{D} \\ (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \arrow[r, mapsto, "\pi_{\mathcal{X}}"] \arrow[rr, mapsto, bend right, "\delta"'] & (\vec{x}_k, \vec{x}_l) \arrow[r, mapsto, "\delta_{\mathcal{X}}"] & R_i \end{tikzcd} \tag{7.1}$$

In sum, δ categorizes motion scenarios *spatially* by means of a spatial representation, $\delta_{\mathcal{X}}$; δ assigns a spatial relation R_i to each motion scenario.

$$\begin{tikzcd} \mathcal{X}\times\mathcal{X} \arrow[r] & \mathcal{D} \\ (\vec{x}_k, \vec{x}_l) \arrow[r, mapsto, "\delta_{\mathcal{X}}"] & R_i \end{tikzcd} \tag{7.2}$$

Trajectories and sequences of relations We endeavour to categorize motion by taking advantage of the spatial representations; for that reason, the next step is to devise a method by which spatial relations may describe motion. A most straightforward method is to describe the entities' trajectories as a *temporal sequence of spatial relations*. Based on such sequences, researchers implemented successful methods for motion analysis: Delafontaine et al. (2011) analyzed complex trajectories in a squash game by looking at its 4-relations subsequences, Hanheide et al. (2012) compared navigation trajectories in human-robot interaction by means of the Levenshtein distance between sequences, and Chavoshi et al. (2015) compared dance motions through sequence alignment methods (SAM).

Therefore, we embrace temporal sequences of spatial relations as a decisive step towards motion categorization. But, is this a sensible decision? How can then a motion scenario, which occurs in a punctual time instant, be expressed as a sequence of relations, which extends throughout a time interval? The solution is to embed each scenario in a certain motion *trajectory*, and, then obtain the temporal sequence of such embedding trajectory.

$$\begin{tikzcd}[column sep=huge] \textsc{Scenario} \arrow[r, Rightarrow, "\substack{\text{scenario is embedded}\\ \text{in a trajectory}}"'] & \textsc{Trajectory} \arrow[r, Rightarrow, "\substack{\text{trajectory's temporal}\\ \text{sequence is extracted}}"'] & \textsc{Story} \end{tikzcd} \tag{7.3}$$

The summarized final outcome is that we map each scenario onto a particular temporal sequence which we call 'story', and this 'story' serves as a category for the scenario. Indeed, most spatial representations yield a finite number of stories, and, therefore, mapping a scenario K onto a story S is equivalent to label a scenario K with the category S.

Interestingly, we obtain stories not only from *spatial* relations, but we also obtain stories from certain representations of *motion*, concretely, from those motion representations that categorize instantaneous motion (those motion representations listed as Items A. and B. in Section 5.3). For that reason, we generalize Equation (7.1), in which scenarios are *spatially* categorized, into Equation (7.4), in which scenarios are *generally* categorized: 'ρ' represents the '*qualitative map*' of any kind of qualitative representation $\mathcal{R}$ that categorizes scenarios, notably, *spatial* (δ, $\mathcal{D}$) or *motion* (μ, $\mathcal{M}$) representations. Concluding, we have broadened the condition in Item *ii.* to include also qualitative motion representations with which create story-based motion representations.

$$\begin{array}{rcl} \mathcal{K} & \longrightarrow & \mathcal{R} \\ (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) & \overset{\rho}{\longmapsto} & R_i \end{array} \tag{7.4}$$

In the following, we reveal how to map scenarios onto stories. Before defining the 'stories', we define temporal sequences of relations *generally*, i.e., for *any* kind of representation (Sects. 7.2 to 7.4). Moreover, we delve into the properties of general temporal sequences: we relate them to fundamental concepts in the field of qualitative representations, and touch on properties that broaden our understanding of such sequences. Subsequently, we define *stories* as a particular case of the temporal sequences of relations (Sects. 7.5 and 7.6). All throughout this chapter, we exemplify the concepts with the spatial relation RCC (Sect. 4.2.1). Later, in Chapter 8, we build the full-fledged stories for RCC and OPRA$_1$ spatial representations, as well as stories of *motion* representations such as QTC$_{\mathrm{B21}}$.

7.2 Temporal Sequences: A Qualitative Trajectory Description

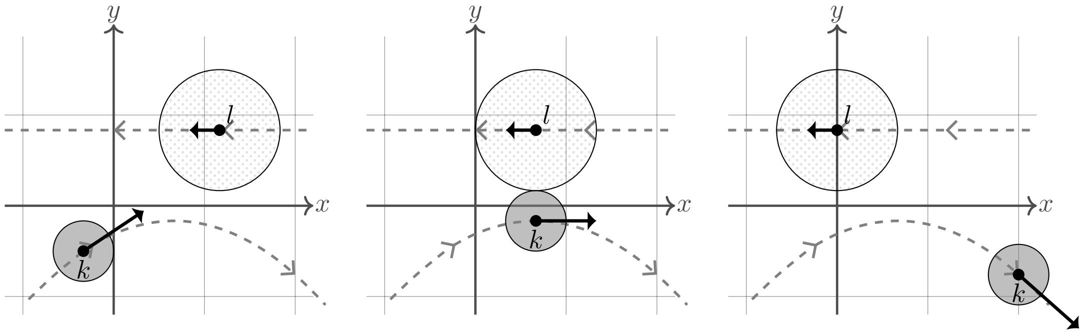
In this section, we deal with *trajectories of entities* $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$ and how to describe them using sequences of qualitative representations. As seen in the previous section, qualitative representations categorize *motion scenarios*, that is, they categorize only each instantaneous point of a trajectory, e.g., $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t_0)$. As a result, we might describe a trajectory by assigning to every time instant t_0 in the trajectory the corresponding qualitative relation R_i: This lead us directly to the concept of temporal sequence (Def. 7.2.1).

We remark that a trajectory is a particular case of a more general concept, namely, *continuous transformations*. Thus, for the sake of generality, we often refer in the following sections to 'continuous transformations'; but, keep in mind that our focus are trajectories.

Definition 7.2.1 ***Temporal sequence of qualitative relations*** We borrow this term from Hanheide et al. (2012), and we define it as follows: A '*temporal sequence of qualitative relations*' is a chronologically ordered sequence of qualitative relations of any kind, e.g., space or motion, generated by a continuous transformation of two entities in a time interval $[t_a, t_b]$. In each temporal sequence, $(R_1, R_2, \ldots, , R_i, \ldots)$, we avoid *consecutive repeated* relations—in case they appear we merge them—so that, the resultant temporal sequence fulfils $\forall i \; R_i \neq R_{i+1}$.

We remark following important details in the definition:

- A temporal sequence of relations is as generally defined as possible. Firstly, the sequence describes '*continuous transformations*', which includes *trajectories* (as a continuous change of position in time); but also *deformations* (as continuous change of form in time), in the case that entities are regions; and even changes on non-spatial domains. Admittedly, later, we will concentrate on *trajectories* and disregard any other kind of continuous transformations, especially deformations, because, in this work, we restrict ourselves to rigid entities (Sect. 1.2).



$t_1 = -1.5$, DC $t_2 = 0.0$, EC $t_3 = 2.0$, DC

Figure 7.1: A trajectory, $K_{\mathrm{A}}(t)$, of two entities, k and l. We display three time instants (t_1, t_2, and t_3) of the trajectory along with their corresponding RCC relations. The entities are circular with radii $r_k = 1$ and $r_l = 2$; $K_{\mathrm{A}}(t) = \big((2t+2, -\frac{4}{9}t^2 - 0.5), (-t+2, -\frac{8}{9}t); (2-t, 2.5), (-1, 0)\big)$.

Secondly, we do not restrict to sequences of *spatial* relations, temporal sequences can contain any kind of qualitative relations—as long, as we do not mix kinds in the same sequence. Although we predominantly use sequences of spatial relations, we also use sequences of *motion* relations; for example, in Section 8.3.

- The time interval $[t_a, t_b]$ that determines the temporal sequnce can be freely chosen. For example, we may consider a *punctual* interval by setting $t_a = t_b$, we may consider the limits $\lim_{t_a \to -\infty}$ or $\lim_{t_b \to +\infty}$, and, thus, have a half-bounded, e.g., $[t_a, \infty)$, or totally unbounded interval $(-\infty, \infty)$. In fact, *stories* are defined over a totally unbounded interval $(-\infty, \infty)$.

- We avoid repetitions of consecutive relations by assigning only one symbol R_i to each relation that occurs in a subinterval of $[t_a, t_b]$. That means that the sequence captures every transition of relations (from one relation R_i to the next different relation R_{i+1}) within the continuous transformation (e.g., within the trajectories).

- The definition of *temporal sequence of relations* assumes implicitly the existence of a map called $\mathrm{f}_{\rho_{[\mathrm{t_a},\mathrm{t_b}]}}$ that, given a continuous transformation $K(t) = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$ in a certain interval $[t_a, t_b]$ "generates" the corresponding temporal sequence of relations. Such map exists and we describe it as follows.

$$\begin{tikzcd}
\mathcal{K}(t) \arrow[r] & \mathcal{R}(t) \arrow[r] & \Sigma^* \\
K(t) = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t) \arrow[r, mapsto, "\rho"] \arrow[rr, bend right, mapsto, "\mathrm{f}_{\rho_{[\mathrm{t_a},\mathrm{t_b}]}}"'] & R(t) \arrow[r, mapsto, "\mathrm{TS}_{[\mathrm{t_a},\mathrm{t_b}]}"] & s = (R_1, R_2, \ldots, , R_i, \ldots)
\end{tikzcd} \tag{7.5}$$

In Equation (7.5), $R(t) = \rho\left((\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)\right)$ is the function that provides the qualitative relation at any time instant t of the continuous transformation $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$. We can call it '*qualitative transformation*' or '*qualitative trajectory*'. Subsequently, $\mathrm{TS}_{[\mathrm{t_a},\mathrm{t_b}]}$ extracts from $R(t)$ the temporal sequence of relations s, i.e., the relations in the temporal order as they occur in the interval $[t_a, t_b]$.

Example 7.1 We apply Equation (7.5) to the trajectory $K_{\mathrm{A}}(t)$ in Figure 7.1.

1. The trajectory in the figure is $K_{\mathrm{A}}(t) = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t) = \big((2t{+}2, -\frac{4}{9}t^2{-}0.5), (-t{+}2, -\frac{8}{9}t); (2-t, 2.5), (-1, 0)\big)$

2. If we apply the spatial representation RCC, i.e., $\rho = \delta_{\mathrm{RCC}}$, to the trajectory $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$, we obtain the *qualitative trajectory* $R(t)$:

$$R(t) = \begin{cases} \mathrm{DC} & t < 0 \\ \mathrm{EC} & t = 0 \\ \mathrm{DC} & t > 0 \end{cases}$$

   $R(t)$ can be easily obtained graphically from the trajectory snapshots displayed in Figure 7.1.

3. Finally, we apply the function $\mathrm{TS}_{[\mathrm{t_a},\mathrm{t_b}]}$ to $R(t)$ and obtain a qualitative temporal sequence of relations in the interval $[t_a, t_b]$. For instance, in Figure 7.1 the considered interval is $[-1.5, 2.0]$; so we have $\mathrm{TS}_{[\text{-}1.5,2.0]}$ which applied to $R(t)$ produces the three-elements sequence $(\mathrm{DC}, \mathrm{EC}, \mathrm{DC})$.

   To further illustrate how $\mathrm{TS}_{[\mathrm{t_a},\mathrm{t_b}]}$ works, we apply the function $\mathrm{TS}_{[\mathrm{t_a},\mathrm{t_b}]}$ to the same trajectory in Figure 7.1 considering other intervals.

   (a) If we consider the interval $[-10.0, -1.0]$, we have $\mathrm{TS}_{[\text{-}10.0,\text{-}1.0]}$ which applied to $R(t)$ produces the one-element sequence (DC). This is the same sequence produced by any interval $[t_a, t_b]$ in which $t_b < 0$.

   (b) If we consider the interval $[-5.0, 0.0]$, we have $\mathrm{TS}_{[\text{-}5.0,0.0]}$ which applied to $R(t)$ produces the two-elements sequence $(\mathrm{DC}, \mathrm{EC})$. This is the same sequence produced by any interval $[t_a, t_b]$ in which $t_b = 0.0$ and $t_a \neq 0.0$, e.g., the interval $[-0.4, 0.0]$.

   (c) If we consider the interval $[0.0, 3.0]$, we have $\mathrm{TS}_{[0.0,3.0]}$ which applied to $R(t)$ produces the two-elements sequence $(\mathrm{EC}, \mathrm{DC})$. This is the same sequence produced by any interval $[t_a, t_b]$ in which $t_a = 0$ and $t_b \neq 0$, e.g., the interval $[0.0, 0.01]$.

   (d) If we consider the interval $[-1.7, 20.4]$, we have $\mathrm{TS}_{[\text{-}1.7,20.4]}$ which applied to $R(t)$ produces the three-elements sequence $(\mathrm{DC}, \mathrm{EC}, \mathrm{DC})$. In fact, any interval $[t_a, t_b]$ containing the instant $t = 0$ not in the boundaries produces the sequence $(\mathrm{DC}, \mathrm{EC}, \mathrm{DC})$.

The whole process described in Items 1 to 3 is condensed in the mapping $\mathrm{f}_{\mathrm{RCC}_{[\mathrm{t_a},\mathrm{t_b}]}}$. For example, for the time interval in Figure 7.1, $[-1.5, 2.0]$, we have

$$(\mathrm{DC}, \mathrm{EC}, \mathrm{DC}) = \mathrm{f}_{\mathrm{RCC}_{[\text{-}1.5,2.0]}}\big(K_{\mathrm{A}}(t)\big)$$

The mapping $\mathrm{f}_{\mathrm{RCC}_{[\mathrm{t_a},\mathrm{t_b}]}}$ for Items 3a to 3d can be represented as following:

(a) $(\mathrm{DC}) = \mathrm{f}_{\mathrm{RCC}_{[\text{-}10.0,\text{-}1.0]}}\big(K_{\mathrm{A}}(t)\big)$

(b) $(\mathrm{DC}, \mathrm{EC}) = \mathrm{f}_{\mathrm{RCC}_{[5.0,0.0]}}\big(K_{\mathrm{A}}(t)\big)$

(c) $(\mathrm{EC}, \mathrm{DC}) = \mathrm{f}_{\mathrm{RCC}_{[0.0,3.0]}}\big(K_{\mathrm{A}}(t)\big)$

(d) $(\mathrm{DC}, \mathrm{EC}, \mathrm{DC}) = \mathrm{f}_{\mathrm{RCC}_{[\text{-}1.7,20.4]}}\big(K_{\mathrm{A}}(t)\big)$

A 'temporal sequence of relations' is a rather general term. In the following, we specify some types of temporal sequences of relations which are specially interesting.

Definition 7.2.2 ***Finite and infinite sequences*** '*finite sequences*' are those containing a finite number of relations, and '*infinite sequences*' are those containing an infinite number of relations.

For instance, all sequences obtained in Example 7.1, (DC), (DC, EC), (EC, DC), (DC, EC, DC) are obviously finite.

Note that these two concepts, *finite* and *infinite sequences*, express the *number of relations* that a sequence contains; what is usually called '*length*' of a sequence. The length of the temporal sequences influences heavily their practical application. We can deal much easier with *finite sequences* than with *infinite* ones. For that reason, the most important sequences defined in this work, the 'stories', are finite (item i., Sect. 7.6).

Definition 7.2.3 ***Complete temporal sequence of relations*** A complete temporal sequence of relations is a temporal sequence in the whole unbounded time interval $(-\infty, \infty)$. All possible complete temporal sequences define the set Σ^*.

We can define the complete temporal sequence s of a certain trajectory $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$ by means of function $\mathrm{f}_{\rho_{[\mathrm{t_a},\mathrm{t_b}]}}$ in Equation (7.5) and seeking its limits $\lim_{t_a \to -\infty}$ and $\lim_{t_b \to +\infty}$: $s = \mathrm{f}_{\rho_{(\text{-}\infty,\infty)}}\big((\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)\big)$. A compelling property of complete temporal sequence is that they are invariant under monotone non-decreasing unbounded time transformations $\mathrm{g}(t)$:

$$s = \mathrm{f}_{\rho_{(\text{-}\infty,\infty)}}\big((\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)\big) = \mathrm{f}_{\rho_{(\text{-}\infty,\infty)}}\big((\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(\mathrm{g}(t))\big) \tag{7.6}$$

For example, a complete story is invariant under time translations and dilations, namely, under $\mathrm{g}(t) = at + b$ where $a > 0$.

As example of *complete sequence*, let us consider the trajectory $K_{\mathrm{A}}(t) = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$ in Example 7.1, which was illustrated in Figure 7.1. In Item 3d, we said that, for such trajectory, any interval $[t_a, t_b]$ containing the instant $t = 0$ not in the boundaries, i.e., $t = 0 \in [t_a, t_b]$ and $t_a, t_b \neq 0$, produces the temporal sequence of relations (DC, EC, DC). And, we can easily see that the limit $\lim_{\substack{t_a \to -\infty \\ t_b \to +\infty}} \mathrm{f}_{\mathrm{RCC}_{[\mathrm{t_a},\mathrm{t_b}]}}\big(K_{\mathrm{A}}(t)\big)$ is the sequence (DC, EC, DC) too. Hence, we can compute the *complete* temporal sequence of the trajectory $K_{\mathrm{A}}(t)$:

$$(\mathrm{DC}, \mathrm{EC}, \mathrm{DC}) = \mathrm{f}_{\mathrm{RCC}_{(\text{-}\infty,\infty)}}\big((\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)\big)$$

Certainly, we obtain the same sequence (DC, EC, DC) as complete sequence, i.e., in the interval $(-\infty, \infty)$, and as non-complete sequence in the interval $I_1 = [-1.7, 20.4]$. What makes then a complete sequence particular compared to a non-complete sequence?

In first place, a complete sequence is invariant under certain time transformations. For instance, the sequence (DC, EC, DC) obtained from the interval I_1 changes into the sequence (EC, DC) if we shift the time $t' = t + 1.7$, since the interval I_1 becomes $[0, 22.1]$. In second place, if we restrict our analysis to the certain particular trajectories or motions the resultant number of complete sequences is very limited. For instance, restricted to uniform linear motions the sequence (DC, EC) can never be a complete sequence because it ends at EC but a uniform linear motion in the interval $(-\infty, \infty)$ which begins in the disconnected relation DC cannot get stuck at the tangent relation EC for the rest of the time.

We remark that Definitions 7.2.2 and 7.2.3 are independent: we can have both finite and infinite temporal sequences of relations that are complete sequences or not. In fact, the temporal sequences defined in this work, the *stories*, are *complete* sequences of relations, i.e., defined in the time interval $(-\infty, \infty)$, and happen also to be *finite* sequences (Sect. 7.5). The most rare case of temporal sequences, however, are the infinite temporal sequences that are not complete, i.e., which occur in a finite time interval; for that to happen, we need either entities or motions that are extremely convoluted, actually, those only found in theoretical settings.

To conclude, a *temporal sequence of relations* is a parsimonious description of the entities' continuous transformations (e.g., motion trajectories) in the space of qualitative relations $\mathcal{R}$, as compared to the most detailed description, $R(t)$, the *qualitative trajectory*, which provides the qualitative relation at any time instant t. In a temporal sequence of relations the precise instant at which every qualitative representation occurs is lost, but a coarse temporal information remains: the temporal order in which the qualitative relations occur (analogously to the events models of Warglien et al. (2012)). The temporal sequence of relations is equivalent to an ordered listing of the *transitions* between relations.

7.3 Transitions between relations: Conceptual Neighbourhood Diagram

As seen above, temporal sequences of relations are mainly about *transitions* between relations. For example, in the temporal sequence of spatial relations $(\mathrm{DC}, \mathrm{EC}, \mathrm{DC})$ (See Fig. 7.1), we recognize two transitions: $\mathrm{DC} \rightsquigarrow \mathrm{EC}$ and $\mathrm{EC} \rightsquigarrow \mathrm{DC}$. Actually, because the temporal sequences are defined for continuous transformations, temporal sequences show more than just transitions: they show *direct transitions* (as defined below).

Direct and indirect transitions A '*direct transition*', $R_a \rightsquigarrow R_b$ is a transition between two relations, R_a and R_b, that occurs when two entities go through a *continuous* transformation, and there is *no intermediate relation* R_c involved. Now, we take as example the RCC relations and their transitions as visualized in Figure 7.2. For instance, the transition $\mathrm{DC} \rightsquigarrow \mathrm{EC}$ is a direct transition: two entities that are disconnected, DC, move *continuously* towards each other until they become connected through their border, EC—There is no other intermediate relation involved. The transition $\mathrm{DC} \rightsquigarrow \mathrm{NTPP}$, however, cannot be direct—it is '*indirect*'—because there is no continuous trajectory that brings two disconnected entities, DC, into being one contained in the other, NTPP, without going through an intermediate relation, such as partial overlap PO.

Transitions are not 'direct' in absolute terms. A transition $R_a \rightsquigarrow R_b$ might be *direct* in a certain type of continuous transformation and *indirect* in another type. For example, the transition $\mathrm{DC} \rightsquigarrow \mathrm{EC}$ is direct both for translations or deformations; however, the transition $\mathrm{TPP} \rightsquigarrow \mathrm{EQ}$ (from being contained and tangent to overlap perfectly) is a direct transition for deformations but not for translations (Fig. 7.2).

Note, however, that if we had allowed *discontinuous* transformations in the definition of direct transitions, then every transition between any relations would be a direct transition, and, hence, the concept of 'direct transition' would be useless: we could instantly leap from one motion state to any other motion state in order to generate whatsoever transition of relations. Therefore, *continuity* is a fundamental assumption when considering transitions of qualitative relations.

By definition (Def. 7.2.1), a temporal sequence of relations shows *direct transitions* of two entities occurring in a certain time interval. Restricting ourselves to continuous trajectories, if a temporal sequence happens to include an *indirect transition*—e.g., the transition $\mathrm{DC} \rightsquigarrow \mathrm{NTPP}$, as in the sequence $(\mathrm{PO}, \mathrm{EC}, \mathbf{DC}, \mathbf{NTPP})$—then such a temporal sequence cannot be obtained from continuous trajectories of entities. In other words, such sequence is *unrealizable* under common physical conditions.

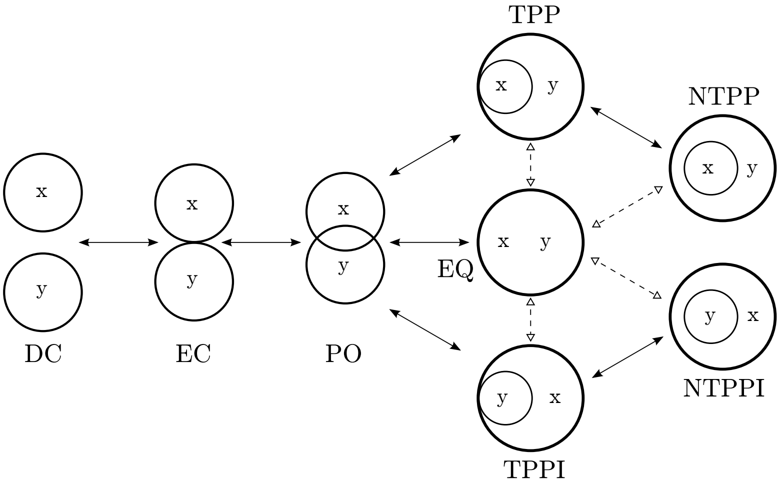


Figure 7.2: Conceptual Neighbourhood Diagram of RCC The RCC qualitative relations depend on how two entities overlap. This figure depicts the 8 RCC relations: DC, EC, PO, TPP, NTPP, EQ, TPPI, and NTPPI as a *conceptual neighbourhood diagram.* The arrows correspond to *direct transitions*, i.e., relations that are directly connected by any continuous transformation: dashed arrows represent only deformations, solid arrows represent translations and deformations.

Realizable and unrealizable temporal sequences A temporal sequence of relations is '*unrealizable*', when it does not exist any continuous transformation (e.g., trajectories) of entities that produces such sequence. Conversely, any temporal sequence of relations that can be produced from continuous transformations is '*realizable*'. Equivalently, expressed in terms of transitions, a temporal sequence of relations is *realizable* if and only if each transition is a *direct transition.*

In conclusion, when we use temporal sequences of relations to describe trajectories, it is essential to know which transitions in a qualitative representation, under which type of continuous transformation, are direct. This is when 'conceptual neighbourhood diagrams' come into play.

Conceptual Neighbourhood Diagram The '*conceptual neighbourhood diagram*' (CND) is simply a tool to display the *direct transitions* of a qualitative representation. It is a graph linking the pairs of qualitative relations that have a direct transition, i.e., pairs of relations directly connected by *continuous transformation.* Again, we remark a CND is not restricted to spatial representations, but it can apply to any kind of representations; for example, motion representations, and, more concretely, story-based representations.

Note that, as the CND displays the direct transitions, if we lift the continuity requirement from the definition of direct transition, any possible transition becomes realizable, and, thus, the CND becomes a complete graph. In that case, the CND would be pointless: it provides no extra information about transitions.

As a CND example, we show the diagram for the spatial representation RCC in Figure 7.2. We can apply this CND to analyze temporal sequences. For example, the temporal sequence $(\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{TPP})$ is realizable as all transitions are represented as edges in the diagram. The temporal sequence $(\mathrm{PO}, \mathrm{TPP}, \mathrm{EQ}, \mathrm{PO})$ is only realizable if the entities are deformed, because the transition TPP ⇝ EQ is represented with a dashed arrow in the diagram, and, thus, is only possible by deformation. In any case, sequences such as (EC, PO, NTPP) or (DC, PO, TPP) are not realizable because, in the CND, it exists no direct transition (i.e., no edge) PO ⇝ NTPP, neither DC ⇝ PO. Notwithstanding, such sequences can become realizable by adding intermediate relations (bold faced) (EC, PO, **TPP**, NTPP) and (DC, **EC**, PO, TPP).

We can formulate the connection between *realizable temporal sequences* and the CND in an elegant way: Any *realizable* temporal sequence of relations in a certain qualitative representation is a *path* in the CND of such qualitative representation—The CND let us visualize realizable sequences as paths. Moreover, in the CND, we can visualize at once different types of direct transitions, if we format the arrows connecting the relations according to the type of continuous transition. For example, in Figure 7.2, the direct transitions obtained only through deformation are denoted by dashed arrows, while those possible both by deformation and translation are denoted by solid arrows.

In this work, we certainly benefit from the CND when we generate the story-based categorizations (i.e., story-based qualitative representations of motion), because a necessary step in such generation is finding temporal sequences of spatial relations, the *stories*; these are particular paths in the CND of the spatial representation. Interestingly, the story-based categorizations are also qualitative representations of motion, and, thus, we could also obtain their CND in order to analyse direct transitions between their relations: we refer to such analysis in the control of trajectories (Sect. 11.5). Anyway, we leave for future work building whole CNDs of the story-based categorizations.

CND in the literature

The conceptual neighbourhood diagram (CND) is widely acknowledged as one of the most elementary concepts concerning qualitative representations. It was introduced by C. Freksa (1992a), though another names used are '*transition graph*' (Cohn et al. 1994) or '*continuity network*' (1997, p. 295)—compare also with the similar concept '*closest-topological-relationship-graph*' (Egenhofer and Al-Taha 1992).

When dealing with motion, the conceptual neighbourhood diagram becomes a ubiquitous concept, because motion is a *continuous transformation* of the position in time (See thorough analysis, Hazarika and Cohn 2001; Cohn and Hazarika 2001b). Therefore, when we represent motion qualitatively, we are using (at least implicitly) the CND.

Furthermore, the conceptual neighbourhood diagram is a basic tool for implementation of decision-making, planing, and control algorithms (e.g., Dylla and Wallgrün 2007; Dylla 2008; Dylla et al. 2012)—Qualitatively steering a moving entity consists on making the entity transition through the desired relations in the conceptual neighbourhood diagram (Sect. 11.5). As mentioned above, CND are not only created for spatial relations; we can also display motion relations (e.g., Van de Weghe and De Maeyer 2005).

7.4 Dominance Theory

The '*dominance theory*'—introduced by A. Galton (1995)—seeks a deeper mathematical insight in the continuous transformations between qualitative relations than the conceptual neighbourhood diagrams (CND) (See didactical introduction, 2001). One result of the dominance theory are the '*dominance diagrams*', which are conceptual neighbourhood diagrams with additional information about continuity at the transition of qualitative relations. In fact, the dominance diagrams define a topology in the qualitative representations, the *dominance topology*, which

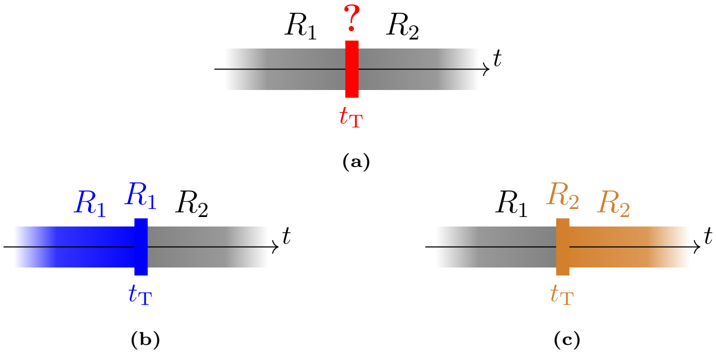


Figure 7.3: A visual representation of the *dominance theory*. When entities transition from a relation, R_1, into another relation, R_2, in a certain instant t_T (Fig. (a)), the dominance theory determines which relation occurs always at the transition instant t_T (the boundary instant): In Figure (b), the relation R_1 *dominates* R_2, i.e., R_1 occurs always at the transition instant t_T; conversely, in Figure (c), the relation R_2 *dominates* R_1, i.e., R_2 occurs always at the transition instant t_T.

characterizes the continuous transformations of entities. On this account, the dominance theory underlie any analysis of continuous transformations of entities, particularly, continuous trajectories.

This theory has very practical applications: it provides tools to build conceptual neighbourhood diagrams (CND), and determines properties of the time intervals in which the relations occur. For example, the dominance theory provides a method for obtaining the CND of a qualitative representation formed by Cartesian product of properties or representations whose CNDs are known. Such method is used by Van de Weghe 2004 to obtain the CNDs for variants of the QTC motion representation, since such variants are all constructed through Cartesian product of simpler properties or representations (Sect. 5.4). Many story-based categorizations are also built as product representations (as explained in Sect. 5.4.2.B), and, consequently, we could apply the dominance theory to construct their CNDs.

In this work we resort twice to the dominance theory. In Section 7.6.2 we use the *dominance relation* to differentiate two types of story-based relations, the *rigid* and *singleton* stories. Later, in Section 10.5.2, the definition of *position state* will simplify the computation of the composition of stories.

We stress that the dominance theory characterizes not only continuous map onto qualitative representations, but more generally, continuous maps into *discrete spaces*. Examples of discrete spaces are the *discrete properties* $\mathcal{P}_*$ in Section 5.4, which are used to build qualitative relations. For that reason, and in view of its limited success, we believe that the dominance theory is currently undervalued, and still has much to say in qualitative description.

Dominance relation A. Galton (Def. 2.1, 2001) defines the dominance relation as follows: a state R_1 *dominates* another state R_2, when R_1 can occur at the boundary of an open interval (t_a, t_b) in which R_2 occurs, i.e., R_1 can occur at t_a or t_b.

Another way to define the *dominance relation* is as in Figure 7.3: the relation that dominates over the other is the one that must occur at the transition instant t_T between both relations. For instance, if R_1 dominates R_2, then whenever we transition from R_1 to R_2—or vice versa—R_1 will be the relation occurring at the boundary, i.e., at the transition point t_T (e.g., Fig. 7.3b). If on the contrary, R_2 dominates R_1, then it is R_2 that will be occurring at the transition point t_{T} (e.g., Fig. 7.3c).

Example 7.2 In RCC, if we allow deformations, the following temporal sequence of relations is realizable, $(\mathrm{EQ}, \mathrm{TPP}, \mathrm{PO})$. If we inspect the transition $\mathrm{EQ} \rightsquigarrow \mathrm{TPP}$, we realize that it is the state EQ that must occur at the boundary instant t_{T} of the transition; for instance, EQ occurs in $(t_a, t_{\mathrm{T}}]$ and TPP in (t_{T}, t_b). However, a transition of the form (t_a, t_{T}) for EQ and $[t_{\mathrm{T}}, t_b)$ for TPP would be impossible: In the dominance theory, one says that EQ dominates TPP. Analogously, TPP dominates PO, because TPP must occur at the transition boundary, t_{T}, of the interval $(t_{\mathrm{T}}, \dots)$ in which PO occurs.

Position and motion states Note that, in RCC, the states EQ and EC always dominate over their neighbour states (no matter whether deformations or translations are allowed). Accordingly, the dominance relation defines two new types of states: the *position* and the *motion* states.

A '*position state*' is such that always dominates over its neighbour states. Thus, it can only occur in a *closed* interval $[t_a, t_b]$. Particularly, a position state can occur in a single time instant $t_0 = [t_0, t_0]$. For example, the relation EC is a position state.

A '*motion state*' is such that always is dominated by its neighbour states. Thus, it can only occur in an *open* interval (t_a, t_b). Particularly, a motion state can never occur in a single time instant t_0. For example, the relation DC is a motion state—Do not confuse the 'motion states' according to the dominance relation with the 'motion states' $K = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ which are kinematic descriptions of the entities, as sketched in Tab. 7.1.

Not all relations in a qualitative representation need to be either *position* or *motion states*: they can be neither. In that case, we have a '*non-motion non-position state*', which can occur in a *half-open* interval, i.e., $(t_a, t_b]$ or $[t_a, t_b)$. If, in a qualitative representation, each relation is either a position or a motion state then such representation constitutes a '*regular space*' according to the dominance relation. Remarkably, both RCC (without deformations) and $\mathrm{OPRA}_{\mathrm{n}}$ are *regular*.

Example 7.3 If we allow only translations and the entities have different size (Fig. 7.2), the RCC relations EC, TPP, and TPPI are *position states*, while DC, PO, NTPP, and NTPPI are *motion states*. It is evident, if we think of continuous translations, that EC can occur in a single time instant (as in Fig. 7.1), and that PO cannot; the latter can only occur in open intervals (See realization in Fig. 7.8).

Concluding, the dominance theory classifies the qualitative relations between entities according to how they can occur at the boundaries of open time intervals. This is essential for analyzing or building temporal sequences of relations, because the occurrence of relations at the boundaries of time intervals determines how relations can be strung together to form temporal sequences.

Yet, behind this simple "property of time intervals", lies a bigger picture: the '*dominance topology*' (ibid., p. 60), which is the finest topology that makes a qualitative representation continuous—specifically, it makes continuous its categorization rule. It is beyond the scope of this work to delve into the topological properties of the story-based representations, though it is a relevant topic for a deeper mathematical understanding of such representations.

7.5 Stories: From Scenarios to Temporal Sequences

Here, we aim to map each motion scenario into a particular temporal sequence of qualitative relations: the 'story'. To that end, we use the concepts in Section 7.2 but specialized to motion trajectories instead of general continuous transformations.

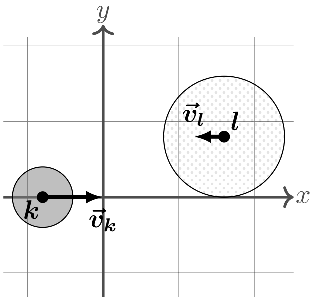


Figure 7.4: A motion scenario K_{B} of two entities k and l with kinematic coordinates $K_{\mathrm{B}} = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \big((-2,0),(2,0);(4,2),(-1,0)\big)$.
.

Which type of qualitative representations can we use to create a story? Whatever. We can choose the type of qualitative representation that is appropriate for our current purpose. For this reason, we strive to keep as *general* as possible the qualitative representations in this section. Although the examples and illustrations here have only spatial relations—mostly RCC—for the sake of simplicity, in Section 8.3, we specifically deal with stories of qualitative relations of motion. Throughout the section, we will use the scenario K_{B} (Fig. 7.4) to illustrate the new defined concepts.

7.5.1 Embedding a scenario in a uniform motion

In our aim to map scenarios into temporal sequences, the critical step is to embed a motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ (e.g., K_{B} in Fig. 7.4) into a trajectory $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$; because, once we map the scenario into a trajectory, we can obtain a temporal sequence rather straightforwardly. We decide to embed a motion scenario into a *uniform motion* trajectory, i.e., a trajectory without accelerations, because of the many persuading arguments:

- First of all, a motion scenario contains no information about the acceleration, only velocities. Therefore, as a best guess we can choose a central value, i.e., zero acceleration, for the trajectory containing the scenario.

- At the very moment, when we consider it worth to categorize motion scenarios, we are assuming a limited effect of acceleration on the entities—The extreme case is constant zero acceleration. Conversely, if scenarios should belong to trajectories with high and greatly variable acceleration, then the changes in the scenario's configuration would be so abrupt that the categorization of scenarios would be meaningless.

- Uniform motion is the state of steady motion of any entity (a person, an animal, a car); it is a typical state, and, thus, has highly ecological validity.

Finally, we have relevant computational and mathematical reasons:

- In uniform motion, we can most easily compute the temporal sequence of relations for a trajectory.

- And, perhaps, most decisive, we have only *finite* cardinalities (App. A.2.1): any temporal sequence in uniform motion has a finite number of relations (i.e., it is *finite*), and, also, the total number of temporal sequences that originate from a qualitative representation is finite.

The latter is a necessary condition for stories to be a *qualitative calculus* (Sect. 10.1.1), and also a desirable property for stories to act as categories (item i., Sect. 6.1.1)

After all, by embedding scenarios into uniform motion trajectories, we are *not* limiting our categorization of scenarios to uniform motion in any case: the categorized scenario can belong to any kind of trajectories (See examples in Sect. 11.4). The use of uniform motion embedding is only a device that helps us categorize scenarios, and, as an advantageous effect, it enables us to qualitatively detect accelerations when we categorize trajectories (See Sect. 11.5).

Thus, to resume, we embed the motion scenario $K = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ into its uniform motion trajectory $K(t) = (\vec{x}_k + \vec{v}_k t, \vec{v}_k; \vec{x}_l + \vec{v}_l t, \vec{v}_l)$, as given by the equation $\vec{x}(t) = \vec{x}_0 + \vec{v}(t - t_0)$ that describes the position of an entity in uniform motion. This embedding is univocal, and, hence, equivalent to the following map.

$$\begin{array}{rcl} \mathrm{U_t} : \mathcal{K} & \longrightarrow & \mathcal{K}(t) \\ K = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) & \longmapsto & (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t) = (\vec{x}_k + \vec{v}_k t, \vec{v}_k;\ \vec{x}_l + \vec{v}_l t, \vec{v}_l) \end{array} \tag{7.7}$$

Example 7.4 We embed the scenario K_B in Figure 7.4 into a uniform motion and obtain the trajectory $K_\mathrm{B}(t)$.

$$\begin{array}{rcl} \mathrm{U_t} : \mathcal{K} & \longrightarrow & \mathcal{K}(t) \\ K_\mathrm{B} = \big((-2,0),(2,0);(4,2),(-1,0)\big) & \longmapsto & K_\mathrm{B}(t) = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t) \\ & & = ((-2,0) + (2,0)t, (2,0);\ (4,2) + (-1,0)t, (-1,0)) \end{array} \tag{7.8}$$

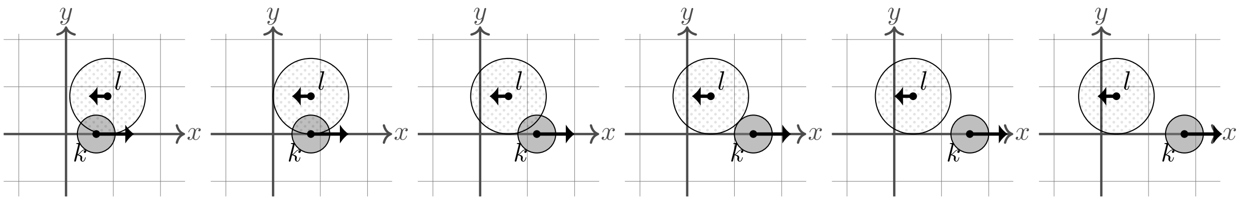
We display the trajectory $K_\mathrm{B}(t)$ both in Figure 7.5 at interval $[1.8, 3.2]$, and in Figure 7.6 at interval $[-1.0, 4.0]$.

Note that the mapping $\mathrm{U_t}$ creates a uniform trajectory $K(t) = \mathrm{U_t}(K)$, that contains the embedded scenario at $t = 0$, i.e., $K = K(0)$. Alternatively, we could have defined the mapping $\mathrm{U_{t\text{-}t_0}}$, which is the same as $\mathrm{U_t}$ but with time shift $t \to t - t_0$. In that case the uniform trajectory $K(t)$ reproduces the embedded scenario at $t = t_0$, i.e., $K = K(t_0)$. Taking $\mathrm{U_t}$ or $\mathrm{U_{t\text{-}t_0}}$, i.e., centering the embedded scenario at $t = 0$ or $t = t_0$, will make no difference for our purpose at hand: the generation of stories.

7.5.2 Mapping uniform motion into sequences

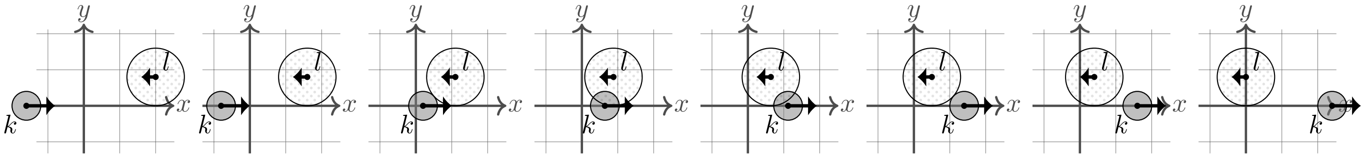
The next step in the *stories* generation is to obtain a qualitative temporal sequence from the uniform trajectory $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$ defined by the embedding $\mathrm{U_t}$ in Equation (7.7). To that end, we apply onto the trajectory the map $\mathrm{f}_{\rho_{[\mathrm{t_a},\mathrm{t_b}]}}$ that we defined in Equation (7.5). Remember that, for the map to work, we must specify the qualitative representation ρ (e.g., RCC) and the time interval $[t_a, t_b]$ in which the sequence is computed. In summary, starting with a given scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$, we apply upon it the composition of both the map $\mathrm{U_t}$, which embeds the scenario in a uniform trajectory, and, subsequently, we apply the map $\mathrm{f}_{\rho_{[\mathrm{t_a},\mathrm{t_b}]}}$, which generates the temporal sequence in the interval $[t_a, t_b]$ (Eq. (7.9)). The end result is the mapping $\sigma_{[\mathrm{t_a},\mathrm{t_b}]}$, which assigns to a scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ a temporal sequence $s = (R_{i_1}, R_{i_2}, \ldots, R_{i_m})$ in the interval $[t_a, t_b]$.

$$\begin{tikzcd} \mathcal{K} \arrow[r] & \mathcal{K}(t) \arrow[r] & \Sigma_{\mathrm{U_t}} \subset \Sigma^* \\ (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \arrow[r, mapsto, "\mathrm{U_t}"] \arrow[rr, bend right, "\sigma_{[\mathrm{t_a},\mathrm{t_b}]}"'] & (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t) \arrow[r, mapsto, "\mathrm{f}_{\rho_{[\mathrm{t_a},\mathrm{t_b}]}}"] & s = (R_{i_1}, R_{i_2}, \ldots, R_{i_m}) \end{tikzcd} \tag{7.9}$$



$t = 1.8$, PO $t = 2.0$, PO $t = 2.5$, PO $t = 2.75$, EC $t = 3.0$, DC $t = 3.2$, DC

Figure 7.5: Each picture is a snapshot of the trajectory obtained by embedding the scenario K_B (Fig. 7.4) into uniform motion, i.e., by applying $\mathrm{U_t}$ to K_B, as in Eq. (7.8). Below each picture, we display the time t of the trajectory and its corresponding RCC qualitative relation. The scenario K_B, which occurs at $t = 0$, is not visible in the figure, because we consider the time interval $[1.8, 3.2]$.



$t = -1.0$, DC $t = 0.0$, DC $t = 1.25$, EC $t = 1.8$, PO $t = 2.4$, PO $t = 2.75$, EC $t = 3.0$, DC $t = 4.0$, DC

Figure 7.6: Each picture is a snapshot of the uniform motion trajectory obtained by applying $\mathrm{U_t}$ to the scenario K_B, $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \big((-2,0),(2,0);(4,2),(-1,0)\big)$ of Figure 7.4; we consider the time interval $[-1.0, 4.0]$. Below each picture, we display the time t of the trajectory and its corresponding RCC qualitative relation. The scenario K_B occurs at $t = 0$.

Example 7.5 We use Figure 7.5 to illustrate the mapping $\sigma_{[\mathrm{t_a},\mathrm{t_b}]}$ of Equation (7.9). The trajectory in Figure 7.5 is obtained by applying $\mathrm{U_t}$ to the scenario K_B, $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \big((-2,0),(2,0);(4,2),(-1,0)\big)$ of Figure 7.4 with time restricted to the interval $[1.8, 3.2]$. Its temporal sequence of RCC relations is (PO, EC, DC). We can summarize all these operations by means of $\sigma_{[\mathrm{t_a},\mathrm{t_b}]}$ as follows:

$$(\mathrm{PO}, \mathrm{EC}, \mathrm{DC}) = \sigma_{[1.8,3.2]}\big((-2,0),(2,0);(4,2),(-1,0)\big) = \sigma_{[1.8,3.2]}(K_\mathrm{B})$$

Meaningful intervals $[t_a, t_b]$ for categorization It is a non-trivial issue to decide the specific values of the time interval $[t_a, t_b]$ that map the scenario into a temporal sequence of relations. As an illustration, consider two scenarios at the instant t_a: $K_1 = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ and scenario $K_2 = (\vec{x}_k, \alpha\vec{v}_k; \vec{x}_l, \alpha\vec{v}_l)$; K_2 has the same positions as K_1 and proportional velocities by a factor α. If we take the time interval $[t_a, t_b]$ in which $\sigma_{[\mathrm{t_a},\mathrm{t_b}]}(K_1)$ yields a certain temporal sequence, e.g., (R_1, R_2, R_3), then, by slowing K_2 enough, i.e., making α arbitrarily near to zero, $\sigma_{[\mathrm{t_a},\mathrm{t_b}]}(K_2)$ would yield the one-element sequence (R_1).[*] The consequence would be that if we categorize scenarios according to temporal sequences in *finite* intervals $[t_a, t_b]$, scenarios with proportional velocities might belong to different categories. We deem such behaviour undesirable, as we see both scenarios (K_1 and K_2) being fully analogous, since they differ only in how fast they temporally evolve. Now, if we take as time interval the *unbounded* interval $(-\infty, \infty)$, the scenarios with proportional velocities, such as K_1 and K_2, yield the same temporal sequences, and, hence,

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[*] We have tacitly assumed that R_1 occurs in an interval $[t_a, t_c]$ with $t_a < t_c < t_b$ and not only at the instant t_a.

belong to the same category. Following equations define the mapping $\sigma_{[\mathrm{t_a},\mathrm{t_b}]}$ for the unbounded interval $(-\infty, \infty)$.

$$\sigma_{(\text{-}\infty,\infty)}(K_1) = \sigma_{(\text{-}\infty,\infty)}(K_2) \quad \forall K_1 = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l), K_2 = (\vec{x}_k, \alpha\vec{v}_k; \vec{x}_l, \alpha\vec{v}_l) \tag{7.10}$$

$$\text{where } \sigma_{(\text{-}\infty,\infty)} = \lim_{\substack{t_a \to -\infty \\ t_b \to +\infty}} \sigma_{[\mathrm{t_a},\mathrm{t_b}]} \tag{7.11}$$

Therefore, we choose the unbounded interval $(-\infty, \infty)$ to generate temporal sequences of trajectories in uniform motion by applying the mapping $\sigma_{(\text{-}\infty,\infty)}$. In other words, we choose to work with *complete sequences of relations* because of the time invariance properties that they possess (See Definition 7.2.3).

Example 7.6 We use Figure 7.6 to illustrate computation and meaning of the mapping $\sigma_{(\text{-}\infty,\infty)}$ of Equation (7.11). The uniform trajectory in Figure 7.6 is the uniform motion trajectory obtained from scenario K_B, $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \big((-2,0),(2,0);(4,2),(-1,0)\big)$ of Figure 7.4 with time restricted to the interval $[-1.0, 4.0]$. Therefore, its temporal sequence of relations is obtained as $(\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC}) = \sigma_{[\text{-}1.0,4.0]}\big((-2,0),(2,0);(4,2),(-1,0)\big) = \sigma_{[\text{-}1.0,4.0]}(K_\mathrm{B})$.

From the pictures in Figure 7.6, it is clear that for every $t \leq -1.0$ or $t \geq 4.0$ the qualitative relation RCC remains DC, so we conclude that

$$\begin{aligned}(\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC}) = \sigma_{(\text{-}\infty,\infty)}(K_\mathrm{B}) &= \\ \lim_{\substack{t_a \to -\infty \\ t_b \to +\infty}} \sigma_{[\mathrm{t_a},\mathrm{t_b}]} \big((-2,0),(2,0);(4,2),(-1,0)\big) &= \sigma_{[\text{-}1.0,4.0]}(K_\mathrm{B})\end{aligned} \tag{7.12}$$

7.5.3 The story map σ, stories S_i, and stories set Σ

Recapitulating, in last section, we obtained the mapping $\sigma_{[\mathrm{t_a},\mathrm{t_b}]}$ which maps a scenario into a temporal sequence s by evolving the scenario into a uniform motion trajectory in the interval $[t_a, t_b]$. Next, we saw that the unbounded interval $(-\infty, \infty)$ was the most meaningful interval to categorize scenarios using the $\sigma_{[\mathrm{t_a},\mathrm{t_b}]}$ mapping. Accordingly, the application $\sigma_{(\text{-}\infty,\infty)}$ is the most appropriate as categorization rule for scenarios. Using these results, we establish the *fundamental definitions* of this work.

Story map σ and Stories S_i The mapping $\sigma_{(\text{-}\infty,\infty)}$ possesses appropriate categorizations properties. We display $\sigma_{(\text{-}\infty,\infty)}$ in Equation (7.13); it is obtained by mapping composition in two steps:

1. $\mathrm{U_t}$ embeds the scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ into a uniform motion trajectory $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$.
2. $\mathrm{f}_{\rho_{(\text{-}\infty,\infty)}}$ yields the complete sequence $S_i = (R_{i_1}, R_{i_2}, \ldots, R_{i_m})$ for the trajectory $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$.

$$\begin{tikzcd}
\mathcal{K} \arrow[r] & \mathcal{K}(t) \arrow[r] & \Sigma \subset \Sigma_{\mathrm{U_t}} \subset \Sigma^* \\
(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \arrow[r, mapsto, "\mathrm{U_t}"] \arrow[rr, bend right, "\sigma := \sigma_{(\text{-}\infty,\infty)}"'] & (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t) \arrow[r, mapsto, "\mathrm{f}_{\rho_{(\text{-}\infty,\infty)}}"] & S_i = (R_{i_1}, R_{i_2}, \ldots, R_{i_m})
\end{tikzcd} \tag{7.13}$$

For the sake of simplicity, we denote as σ the mapping $\sigma_{(\text{-}\infty,\infty)}$. We call the mapping σ '*story map*' because we call 'story' each sequence of relations S_i that the mapping σ generates. A ‘*story*’ is, thus, the complete temporal sequence of qualitative relations generated by a uniform motion of entities.

In Equation (7.14), we represent the *story map* σ compactly by hiding the intermediate steps of Equation (7.13):

$$\begin{array}{rcl} \sigma : \mathcal{K} & \longrightarrow & \Sigma \\ (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) & \longmapsto & S_i = (R_{i_1}, R_{i_2}, \ldots, R_{i_m}) \end{array} \tag{7.14}$$

Equation (7.14) is a fundamental formula in this work: it shows how a motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ is mapped into a story, S_i. This is the cornerstone of the categorization of motion scenarios, as we see below. Remember that we set modest prerequisites for obtaining the story map (See items *i.* and *ii.*, Sect. 7.1); we only need the description of entities as motion scenarios, and a *qualitative representation* $\mathcal{R}$, which provides the function ρ.

A story originates from a motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ by observing the time evolution of the qualitative relations, considering that the velocities remain unchanged. That is, we list which past relations of motion could have occurred (*past inference*), the current relation, and which future relations of motion could follow (*future inference*).

Example 7.7 We already saw in the last section (Sect. 7.5.2) a single example of *story*, an RCC story: The story obtained from the motion scenario K_B, i.e., $(\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC}) = \sigma_{(\text{-}\infty,\infty)}(K_\mathrm{B})$ (Eq. (7.12)). Now, in Figure 7.7, we present some additional examples of RCC stories: the stories S_{C_i} which we obtain from the scenarios K_{C_i}; these stories are namely, (DC), (DC, EC, DC), (DC, EC, PO, EC, DC), (DC, EC, PO, TPP, PO, EC, DC), (DC, EC, PO, TPP, NTPP, TPP, PO, EC, DC).

Example 7.8 In Figure 7.8, we depict the uniform trajectory $K_\mathrm{D}(t)$, which is generated by the motion scenario K_D (sketched at $t=2$). This trajectory yields the RCC story $S_{K_\mathrm{D}} = (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{TPP}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$. This is the same as the story $S_{K_{\mathrm{C}_5}}$ generated by the uniform trajectory $K_{\mathrm{C}_5}(t)$ in Figure 7.7.

We explicitly show here a trajectory in which the velocities are not antiparallel, as in the other examples in this section, so that one can better grasp how the story map works in general. Additionally, the uniform trajectory $K_\mathrm{D}(t)$ contains the embedded scenario at $t=2$, $K_\mathrm{D} = K_\mathrm{D}(2)$, which means that it was embedded by $\mathrm{U}_{\text{t-2}}$. It is trivial to see that whether we embed the scenario at $t=0$, U_t, or $t=2$, $\mathrm{U}_{\text{t-2}}$, the generated story remains the same. In that way, we exemplify our previous statements: first, that we can choose any $\mathrm{U}_{\text{t-t}_0}$ to embed the scenario which generates the story, and, second, equivalent to the first, that the stories are invariant to time translations.

Stories set The set that contains all possible stories is called ‘*stories set*’, and we notate it with Σ; it is a subset of the set of complete temporal sequences Σ^*. The stories set, as any set of categories, can be endowed with a hierarchical structure, forming a tree of subsets (i.e., subcategories). We call such subsets Σ_*, where ‘$*$’ is the name of the subset. For instance, in the stories set of the spatial representation OPRA_1, we have, amongst others, the subsets Σ_C (crossing trajectories, both entities moving) and Σ_P (parallel non-superposed trajectories) (Fig. 11.7).

Example 7.9 In Example 7.7, we have seen six RCC stories. Since the stories set of RCC, Σ_RCC, contains all RCC stories, these six stories offer us a first glance into Σ_RCC.

$$\begin{aligned} &\big\{(\mathrm{DC}), (\mathrm{DC}, \mathrm{EC}, \mathrm{DC}), (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC}), \\ &\quad (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC}), (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{TPP}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})\big\} \subset \Sigma_\mathrm{RCC} \end{aligned}$$

We emphasize that we can use relations *of any kind* to create a story. Once we choose a specific kind of relations, e.g., spatial relations, we obtain the corresponding kind of sequence. In this work, we study the stories of *spatial* relations (Section 8.2) and stories of *motion* relations (Section 8.3). There might be more types of relations that depend on the positions and velocities, if so, the definition of stories extends to that types of relations as well.

We might add a subindex $\mathcal{R}$ to the story map, i.e., $\sigma_{\mathcal{R}}$, that indicates the qualitative representation participating in the story generation. As an illustration, the story map σ_{RCC} application uses the qualitative spatial representation RCC to generate stories; $\sigma_{\text{QTC}_\text{B}}$ uses the qualitative motion representation QTC_B. But, often, the $\mathcal{R}$ drops, and we have σ, because the qualitative representation $\mathcal{R}$ is clear from the context, or because we mean a general qualitative representation. For instance, in Figure 7.7, we wrote $S_{\text{C}_i} = \sigma(K_{\text{C}_i})$ to express that the RCC story S_{C_i} was obtained from scenario K_{C_i}. We could have written $S_{\text{C}_i} = \sigma_{\text{RCC}}(K_{\text{C}_i})$, but it was clear by the context and the name of the qualitative relations that we were dealing with RCC.

7.6 Stories become Categories: Story-based categorizations

Stories have two crucial properties that made them apt to form both a *motion categorization* and, specifically, a *qualitative representation of motion*:

i. Stories are *finite* sequence of relations, i.e., stories have a finite number of relations (Prop. A.2.1). In that way, computations are facilitated or, rather, made possible: we can obtain the full temporal sequence of each story, and we can operate with stories as with qualitative representation, for instance, to compute inverse and composition (Ch. 10).

ii. The set of all possible stories in uniform motion is *finite* (Prop. A.2.1). Only then makes it sense to use stories as categories or qualitative relations—It would defeat *cognitive economy* to categorize with a categorization that has an infinite number of categories.

Consequently, we have a *motion categorization* as we formalized it in Chapter 6, namely (σ, Σ); we illustrate this in Table 7.1. The story map σ is the *categorization rule*, because it maps the categorized objects (i.e., motion scenarios) into the categories (i.e., stories S_i), which form the set of stories Σ. In brief, the motion categorization induced by the stories is fully defined as the categorization rule σ and the set of categories Σ.

We call '*story-based*' any categorization that uses stories as a categorization cue.

Moreover, stories are not only categories, but they are also qualitative relations of motion—we show it rigorously in Chapter 10. Thus, the stories set Σ is not only a categorization, but also a qualitative representation of motion.

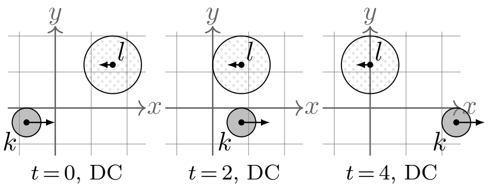
7.6.1 Types of Story-Based Categorization: *Bare* and *Beaded*

In this work, we treat two types of story-based categorizations: 'bare' and 'beaded'. We expound them at length in Chapter 8. Here, we mention them in an example (Ex. 7.10) without delving into details; we want to illustrate that not only the simple stories, S_i, are story-based categorizations, but, using the stories, we can create further types of story-based categorizations.

Example 7.10 (Bare and beaded story-based categorizations) In Figure 7.7c, we have five scenarios, the five snapshots of the trajectory $K_{\text{C}_3}(t)$, which can be categorized according to a 'bare' or a 'beaded' categorization, as shown in Table 7.2, both based in the spatial representation RCC.

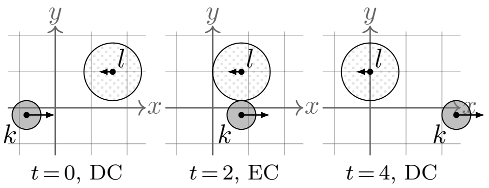
The *bare* categorization uses only the story, S_i, to which the scenario belongs, as category. For example, in Figure 7.7c, each scenario belongs to the same category, namely, S_{C_3}.

$$K_{\mathrm{C}_1} = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(0) = \big((-2,-1),\ (2,0);\ (4,3),\ (-1,0)\big)$$



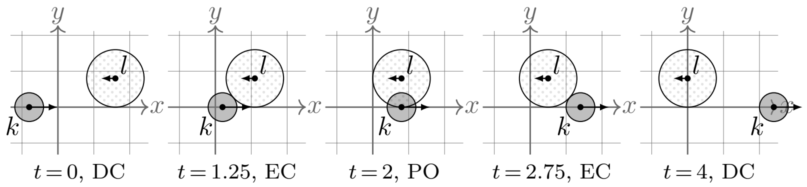
(a) Story $S_{\mathrm{C}_1} = (\mathrm{DC})$

$$K_{\mathrm{C}_2} = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(0) = \big((-2,-0.5),\ (2,0);\ (4,2.5),\ (-1,0)\big)$$



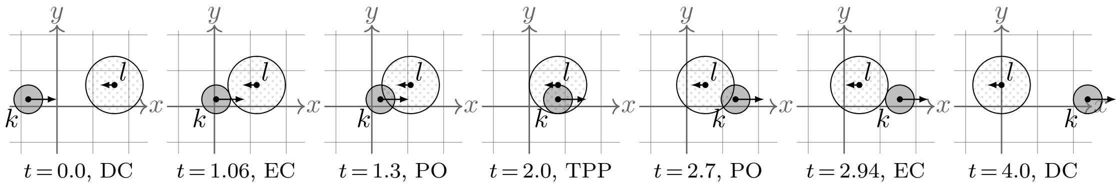
(b) Story $S_{\mathrm{C}_2} = (\mathrm{DC}, \mathrm{EC}, \mathrm{DC})$

$$K_{\mathrm{C}_3} = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(0) = \big((-2,0),\ (2,0);\ (4,2),\ (-1,0)\big)$$



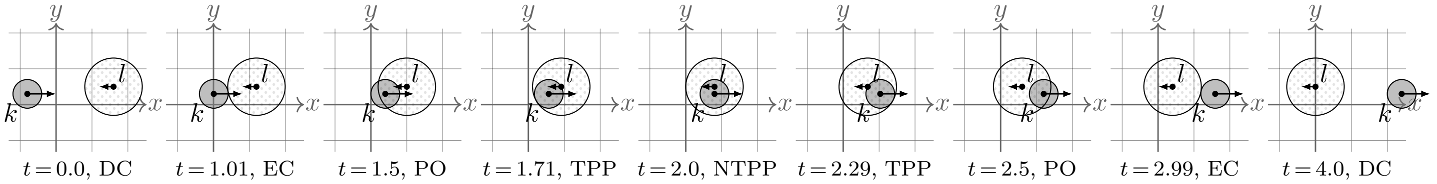
(c) Story $S_{\mathrm{C}_3} = (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$

$$K_{\mathrm{C}_4} = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(0) = \big((-2,0.5),\ (2,0);\ (4,1.5),\ (-1,0)\big)$$



(d) Story $S_{\mathrm{C}_4} = (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$

$$K_{\mathrm{C}_5} = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(0) = \big((-2,0.75),\ (2,0);\ (4,1.25),\ (-1,0)\big)$$



(e) Story $S_{\mathrm{C}_5} = (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{TPP}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$

Figure 7.7: A sample of uniform trajectories, $K_{\mathrm{C}_i}(t) = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(t)$, of two circular entities k and l with radii $r_k = 1$ and $r_l = 2$. Each trajectory displays a story. Although we show only the time interval in $[0,4]$, the trajectory can be effortlessly extrapolated to the interval $(-\infty, \infty)$; and, since each trajectory $K_{\mathrm{C}_i}(t)$ is the uniform embedding, $\mathrm{U_t}$, of the motion scenario K_{C_i} (sketched at $t = 0$), each obtained qualitative temporal sequence S_{C_i} is the *story* for each scenario K_{C_i}. In other words, these qualitative temporal sequences are obtained by story map σ as $S_{\mathrm{C}_i} = \sigma(K_{\mathrm{C}_i})$.

$$K_{\mathrm{D}} = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)(2) = \big((2, -0.7), (1, -1); (2, 0), (-1, 0)\big)$$

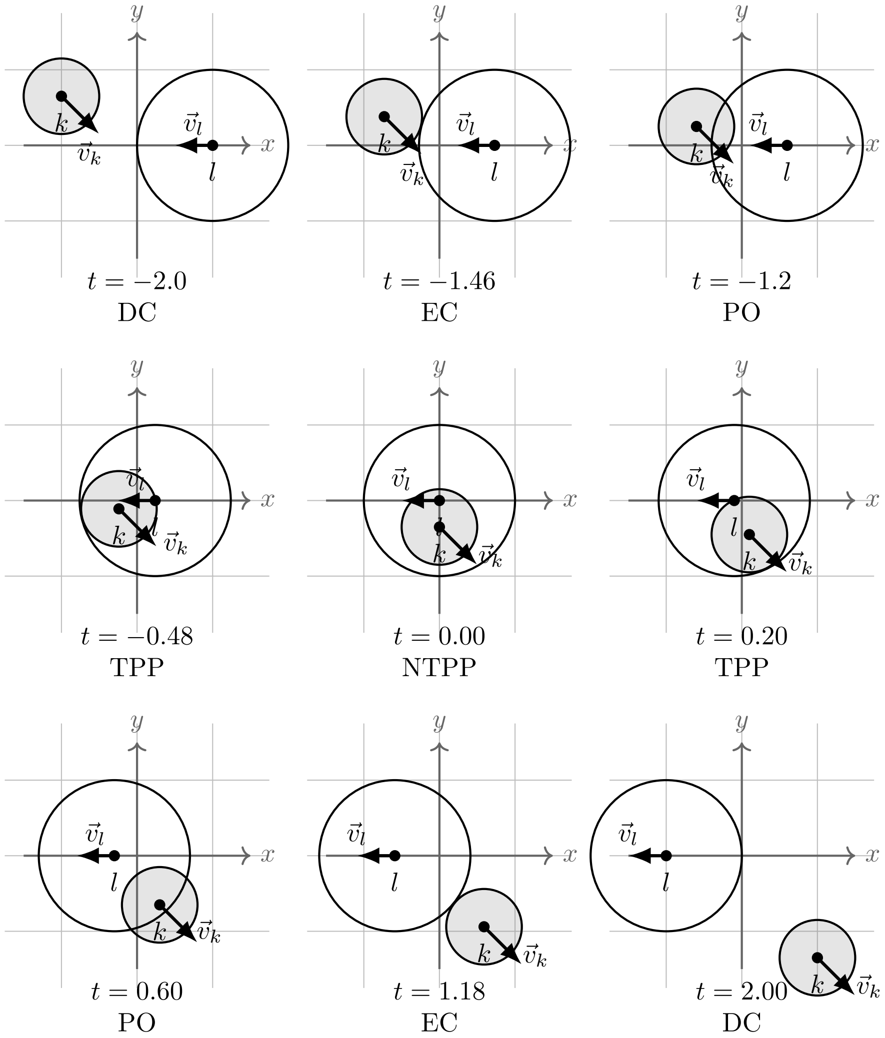


Figure 7.8: A RCC story. The snapshots (each in a specific time t) depict the temporal sequence of RCC relations $S_{K_{\mathrm{D}}} = (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}, \mathrm{TPP}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$ of the uniform trajectory created by the scenario K_{D} $(t=2)$. The entities k and l are discs with $r_k = 1$ and $r_l = 2$. We display only the sequence for the interval $[-2.0, 2.0]$, but, obviously, it is the same sequence as for $(-\infty, \infty)$. Therefore, this sequence is a story, $S_{K_{\mathrm{D}}} = \sigma(K_{\mathrm{D}})$.

Source: Purcalla Arrufi and Kirsch (2017)

General	term	Motion state Kinematic space	Categorization rule	Category Category set	Categorization (extensional def.)
	symbol	$K \in \mathcal{K}$	f_μ	$M_i \in \mathcal{M}$	$(\mathrm{f}_\mu, \mathcal{M})$
Bare Story-based	term	Motion scenario Kinematic space	Story map	Story Stories set	Story categorization (extensional def.)
	symbol	$(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \in \mathcal{K}$	σ	$S_i \in \Sigma$	(σ, Σ)

Table 7.1: Terms and their symbols for the 'general' motion categorization model (Ch. 6) compared with analogous terms and symbols of motion categorization through stories, i.e., 'bare story-based' categorization.

Scenario Name	Scenario as Trajectory instant $K_{\mathrm{C}_3}(t)$	Bare Category	Beaded Category
$K_{\mathrm{C}_{31}}$	$K_{\mathrm{C}_3}(0.00)$	S_{C_3}	$S_{\mathrm{C}_3}(\mathrm{DC}_-)$
$K_{\mathrm{C}_{32}}$	$K_{\mathrm{C}_3}(1.25)$	S_{C_3}	$S_{\mathrm{C}_3}(\mathrm{EC}_-)$
$K_{\mathrm{C}_{33}}$	$K_{\mathrm{C}_3}(2.00)$	S_{C_3}	$S_{\mathrm{C}_3}(\mathrm{PO})$
$K_{\mathrm{C}_{34}}$	$K_{\mathrm{C}_3}(2.75)$	S_{C_3}	$S_{\mathrm{C}_3}(\mathrm{EC}_+)$
$K_{\mathrm{C}_{35}}$	$K_{\mathrm{C}_3}(4.00)$	S_{C_3}	$S_{\mathrm{C}_3}(\mathrm{DC}_+)$

Table 7.2: The scenarios in Figure 7.7c categorized according to its story S_{C_3}, i.e., $(\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$, (*bare category*), and according to its story along with the corresponding spatial representation $S_i(R_j)$ (*beaded category*), e.g., $S_{\mathrm{C}_3}(\mathrm{PO})$.

The *beaded* categorization uses both the story, S_i, and the qualitative relation, R_j, to which the scenario belongs. For example, in Figure 7.7c, the scenario at $t = 2$ belongs to the story S_{C_3} and to the relation PO, thus, its beaded category is $S_{\mathrm{C}_3}(\mathrm{PO})$; in the same way, the scenario at $t = 4.0$ belongs to the beaded category $S_{\mathrm{C}_3}(\mathrm{DC}_+)$, because it belongs to the story S_{C_3} and to the relation DC.

The RCC stories have the particularity that some relations appear twice in the temporal sequence; so we have added the signs '+' and '−' to discern which of both relations in the sequence is meant, the one occurring first or second. For example the relations DC and EC in the story sequence of S_{C_3} would be noted as $(\mathrm{DC}_-, \mathrm{EC}_-, \mathrm{PO}, \mathrm{EC}_+, \mathrm{DC}_+)$.

As a third example, the scenario at $t = 0.0$ in Figure 7.7c belongs to the beaded category $S_{\mathrm{C}_3}(\mathrm{DC}_-)$, because the scenario belongs to the story S_{C_3} and to the first appearance of relation DC in the story sequence.

7.6.2 Rigid and Singleton Stories

We subdivide the stories that have only *one* relation in either 'rigid' or 'singleton' stories. An example of rigid story is $S_{01} = (\mathrm{DC})$, and of singleton story is $S_{11} = (\mathrm{DC})$ (Fig. 7.9). It might seem bewildering, that we classify S_{01} and S_{11} as different stories, while they have the same one element 'DC'. Admittedly, we are slightly refining our categorization rule, σ. We differentiate stories that according to the categorization rule should be the same story, when certain properties hold. In the following, we explain the properties that make *rigid* stories, such as $S_{01} = (\mathrm{DC})$, different from *singleton* stories, such as $S_{11} = (\mathrm{DC})$.

Figure 7.9: Two motion scenarios that seemingly have the same story (DC), but are differently classified according to a variety of criteria, such as Gestalt psychology, dynamic operations, or dominance theory. The scenario left, in which both velocity vectors are equal, i.e., $\vec{v}_k = \vec{v}_l$, can occur only in closed time intervals; such scenario is a rigid story $S_{01} = (\text{DC})$. The scenario right can only occur in opened time intervals, velocity vectors are different, i.e., $\vec{v}_k \neq \vec{v}_l$; and it is, thus, a singleton story $S_{11} = (\text{DC})$.

Source: Purcalla Arrufi and Kirsch (2018a)

Rigid stories A '*rigid story*' is the story of two entities that move with the same velocity, i.e., $\vec{v}_k = \vec{v}_l$, including the case where both objects are motionless $\vec{v}_k = \vec{v}_l = \vec{0}$ (Thereby, we state that spatial categorization can be seen as a particular case of motion categorization).

Singleton Stories A '*singleton story*' is a story consisting of a single relation, where the velocities of the entities are different, i.e., $\vec{v}_k \neq \vec{v}_l$.

According to the definitions above. We will consider as two different stories, if the same story that consist on only one relation can be generated both by scenarios with the same velocity, i.e., $\vec{v}_k = \vec{v}_l$, or different velocity, i.e.,$\vec{v}_k \neq \vec{v}_l$. As in the case of story $S = (\text{DC})$, that results in *rigid story*, such as $S_{01} = (\text{DC})$, and *singleton story*, such as $S_{11} = (\text{DC})$.

We can immediately see that the story map, σ, yields rigid stories for every spatial representation. That is, every story-based motion representation contains rigid stories. However, not every representation yields *singleton* stories; for instance, OPRA_2 yields no singleton stories, because any trajectory in uniform motion goes at least through two different relations. In any case, the two spatial representations used in this work, RCC and OPRA_1, yield singleton stories, and, therefore, we devote next section to show the differences between singleton and rigid stories.

Differences between rigid and singleton stories

We have several ways to differentiate between rigid and singleton stories. The most direct way is possibly the *principle of the common fate* which belongs to the *Gestalt psychology*—It has a long tradition and profound influence in psychology. We can also distinguish rigid and singleton stories through dynamic operations (rigid stories react differently to accelerations than singleton stories) or through continuous transformations (in that case we resort to the commutative diagrams and dominance theory).

Gestalt psychology *Rigid* stories are integral part of '*Gestalt psychology*'. Indeed, they are central in the '*law of common fate*' (in German, *gemeinsames Schicksal*). This law is a particular case of the '*good continuation*' grouping law, which is one of the four original '*grouping laws*' or '*gestalt principles of organization*' defined by M. Wertheimer (1923). The law of common fate states that entities "moving in time in the same direction and at the same speed" appear to belong together, to form a single unit (Colman 2015, p. 148; compare, VandenBos 2007, p. 198).

Actually, as we have seen, Gestalt psychology does not expressly distinguish between both types of stories: it singles out the *rigid stories* as being extremely salient, and leaves *singleton stories* unmentioned, most assuredly because they lack such saliency.

Dynamic operations Murphy and Medin (1985, p. 295f.) explain that further differences between categories arise "from our knowledge about transformations and operations associated with [them]"; they compare it with finding "higher order features" in the categories. Concerning the motion scenarios, which are such "operations" associated with them? We argue that they are the *dynamic* operations, i.e., *accelerations*: stories (i.e., the motion categories) react differently depending on the acceleration applied on the entities.

Applying a tangential acceleration, i.e., changing speed, reveals a difference between singleton and rigid stories. A *singleton story* remains unchanged for low enough speed changes, while most *rigid stories* can switch into another story when either entity changes speed, no matter how low the change might be—plainly expressed, rigid stories are extremely sensitive to tangential accelerations. This is related to the fact that the velocities in a rigid story have a tighter, a lower dimension constraint, i.e., $\vec{v}_k = \vec{v}_l$, than the velocities in a singleton story, e.g., $\vec{v}_k \parallel \vec{v}_l$. This is a practical distinction, since it repercusses on navigation control (Sect. 11.5).

Continuous operations Rigid and singleton stories behave differently when we apply continuous operations on the entities. We note that *dynamic operations* are a case of continuous operations limited to exerting forces on the particles. When we deal with continuous operations, we can apply the insights of Sections 7.3 and 7.4: direct transitions and dominance theory. We consider two continuous operations: translation, and speed variation (which is also a dynamic operation, the tangential acceleration).

Most remarkably, the rigid stories are closed under translation; the non-rigid stories (e.g., the singleton stories) are also closed under translation. Thus, there is no direct transition between a rigid and a non-rigid story, if we allow only translations. However, the rigid↔non-rigid transitions are possible, if we allow speed change.

The *dominance theory* also differentiates between rigid and singleton stories. The rigid stories dominate over the non-rigid ones, and, particularly, over the singleton stories, because the rigid stories correspond to the single value $\|\vec{v}_k - \vec{v}_l\| = 0$ of the parameter $\|\vec{v}_k - \vec{v}_l\|$. Simply put, if we change only speed, the rigid stories can occur in closed intervals while the non-rigid stories cannot.

All the above arguments distinguishing rigid and singleton stories are presumably related. Through dynamic and continuous operations, we show a particular behaviour of the rigid stories that might explain the saliency captured in the law of common fate. Remember that the *Gestalt principles of organization* are not explained from higher principles, but postulated from experimental results on human perception (see, Ullman 1979, Sect. 4.3, p. 144).

Chapter 8

Creating Story-Based Categorizations: Bare and Beaded

In Chapter 7, we have shown that *stories* are fit to categorize motion scenarios, and we have defined stories for any type of qualitative representations. Particularly, in Section 7.6, we established that any categorization using stories as a categorization cue is a *story-based categorization*. Further, in Section 7.6.1, we mentioned two main types of story-based categorizations: 'bare' (using stories, S_i, as categories) and 'beaded' (using pairs stories-relations, $S_i(R_j)$, as categories). Expanding on the definition in Chapter 7:

- The *bare* story-based representations are motion categorizations in which the categories are simply the stories, S_i. We call any of such representations '*Stories-$\mathcal{R}$*', where $\mathcal{R}$ is the qualitative representation that generates it.
  For example, we have the representations Stories-RCC or Stories-QTC$_{\mathrm{B21}}$. Some categories of Stories-RCC are S_{11} or S_{13}, some categories of Stories-QTC$_{\mathrm{B21}}$ are S_{00} or $S_{1\text{-}1}$.

- The *beaded* story-based representation are motion categorizations in which each category is a pair formed by a story, S_i, and some of its constituting relations, R_j, i.e., $S_i(R_j)$ where $R_j \in S_i$. We call any of such representations '*Motion-$\mathcal{R}$*', where $\mathcal{R}$ is the qualitative representation that generates it.
  For example, we have the representations Motion-RCC, or Motion-OPRA$_1$. Some categories of Motion-RCC are $S_{11}(\mathrm{DC}_{-})$ or $S_{13}(\mathrm{PO})$, some categories of Stories-OPRA$_1$ are $S_{\mathrm{C21}}(\angle_1^3)$ or $S_{\mathrm{T\text{-}1}}(\angle 2)$.

In this chapter, we define more precisely both types of categorizations—bare and beaded—and create full-fledged instances. Concretely, we create the *bare* story-based categorizations Stories-RCC, Stories-OPRA$_1$ (Sect. 8.2), and Stories-QTC$_{\mathrm{B21}}$ (Sect. 8.3). We also create the *beaded* story-based categorizations Motion-RCC, Motion-OPRA$_1$ (Sect. 8.6), and Motion-QTC$_{\mathrm{B21}}$ (Sect. 8.7).

Whatever the type of categorization (bare or beaded), the very first step in a story-based categorization is to find the *stories set* Σ—We provide an algorithm to find it in Section 8.1. When finding the stories set, we obtain, as a by-product, the *story map* σ compactly defined. Once we have the stories set and the story map, we already have the bare story-based categorization.

A *beaded* story-based categorization is obtained by adding the corresponding relation R_j to the story S_i given by the bare story-based categorization (Sect. 8.5). Equivalently, in the beaded categorization, the categorization rule is the Cartesian product of the story map, σ, and the qualitative representation used to create the story map, ρ, i.e., $\mathrm{f}_\mu = \sigma_{\mathcal{R}} \times \rho$ (Sect. 8.5.1) .

8.1 Finding the Stories Set Σ

To obtain the story of a single scenario might be laborious but straightforward: we manually apply the story map σ to the given scenario $K = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$. However, to obtain the stories set Σ seems an overwhelming process. Theoretically, we should apply the storymap σ to every scenario K in the kinematic space $\mathcal{K}$—there are infinitely uncountable scenarios—to obtain the stories set Σ. This is why we present a practical method to obtain Σ.

Conveniently, as we find the stories set, we find also a compact form of the story map. Indeed, when we find the stories, we find simultaneously their *categorical regions* $\{\boldsymbol{K}_1, \dots, \boldsymbol{K}_n\}$, i.e., the region $\boldsymbol{K}_i$ for each story S_i (Sect. 6.1, p. 89). Since the categorical regions are finite, the story map is defined as a *piecewise constant* function in the kinematic space. Therefore, to obtain the categorical regions is equivalent to obtain the story map.

Summing up, the method for generating the stories set below, determines at once the stories set Σ with their categorical regions $\{\boldsymbol{K}_1, \dots, \boldsymbol{K}_n\}$, and the story map σ based on such categorical regions. The method fully determines the *bare story-based categorization* (σ, Σ)—actually, $(\sigma_{\mathcal{R}}, \Sigma_{\mathcal{R}})$, when we spell out the used representation, $\mathcal{R}$—and we call such categorization '*Stories-$\mathcal{R}$*'.

8.1.1 Algorithm for stories set generation

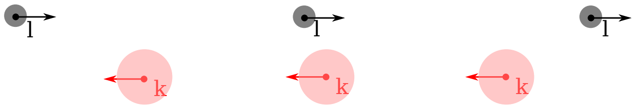
At the beginning we have the empty stories set, $\Sigma_{\mathcal{R}} = \{\}$, the empty regions set, $\mathbf{K} = \{\}$, and the story map σ defined by a qualitative representation $\mathcal{R}$ (Eqs. (7.13) and (7.14))

1. We pick a random motion scenario $K_a = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \in \mathcal{K}$ that does not belong to any categorical region $\boldsymbol{K}_i$ in the regions set $\mathbf{K}$. That is, we pick $K_a \in \{\mathcal{K} \setminus \bigcup_{\boldsymbol{K}_i \in \mathbf{K}} \boldsymbol{K}_i\}$.
2. We obtain the scenario's story with the story map, $S_a = \sigma_{\mathcal{R}}(K_a)$.
3. Add the story to the stories set, $\Sigma_{\mathcal{R}} = \Sigma_{\mathcal{R}} \cup \{S_a\}$
4. We find the categorical region of the story S_a, i.e., all scenarios in the kinematic space that belong to such story, i.e., $\boldsymbol{K}_a = {\sigma_{\mathcal{R}}}^{-1}(S_a)$.
5. We add the categorical region to the regions set $\mathbf{K} = \mathbf{K} \cup \{\boldsymbol{K}_a\}$
6. Repeat steps 1 to 5 until the whole kinematic space is partitioned in categorical regions, each with its corresponding story.

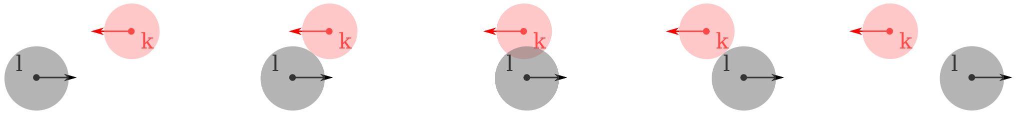
At the end we have the stories set $\Sigma_{\mathcal{R}} = \{S_1, \dots, S_n\}$, the categorical regions $\mathbf{K} = \{\boldsymbol{K}_1, \dots, \boldsymbol{K}_n\}$, and the story map $\sigma_{\mathcal{R}}$ defined according to the categorical regions. Importantly, Proposition A.2.1 guarantees that this algorithm ends after a finite number of steps, since in almost all representations the stories set is finite.

8.2 Stories-$\mathcal{R}$ of Spatial Representations

We show now concrete examples of the stories set Σ for spatial representations. We choose two spatial representations which are simple but non-trivial: RCC and OPRA$_1$. Since we use the method in Section 8.1 to generate Σ, we will obtain for each spatial representation a *bare story-based categorization*, namely, '*Stories-RCC*' and '*Stories-OPRA$_1$*'.



(a) A story S_{11} = (DC)



(b) A story S_{13} = (DC, EC, PO, EC, DC)

Figure 8.1: Examples of stories in RCC. (Source: Purcalla Arrufi and Kirsch (2018a))

8.2.1 Stories-RCC

We generate the categorization Stories-RCC by applying the method for stories set generation (Sect. 8.1) with the spatial representation $\mathcal{R}$ = RCC. Our start point is the spatial map δ provided by RCC, i.e., δ_{RCC}; this map relates each motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ with a spatial relation RCC.

The spatial map of RCC, δ_{RCC}, and, consequently, the ensuing story map σ_{RCC} depend heavily on the geometry of the entities. In this work, we assume that the entities (k and l) are discs with radii r_k and r_l; because, then, we have the simplest formulation of δ_{RCC} and σ_{RCC}.

Regarding notation, we tend to drop the suffix RCC and write simply δ and σ, whenever it is clear that we deal with the representation RCC.

A. Spatial Map δ_{RCC}

For disc entities, k and l, the spatial relation of a motion scenario depends exclusively on their radii and on the distance between their centres: r_k, r_l, $\mathrm{d}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \|\vec{x}_l - \vec{x}_k\|$.

$$\delta_{\text{RCC}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) := \begin{cases} \text{DC} & \text{if } \ \mathrm{d} > d_2 \\ \text{EC} & \text{if } \ \mathrm{d} = d_2 \\ \text{PO} & \text{if } \ d_2 > \mathrm{d} > d_4 \\ \left.\begin{array}{ll} \text{TPP} & \text{if } \ \mathrm{d} = d_4 \\ \text{NTPP} & \text{if } \ d_4 > \mathrm{d} \end{array}\right\} & \text{if } r_k < r_l \\ \left.\begin{array}{ll} \text{TPPI} & \text{if } \ \mathrm{d} = d_4 \\ \text{NTPPI} & \text{if } \ d_4 > \mathrm{d} \end{array}\right\} & \text{if } r_k > r_l \\ \left.\begin{array}{ll} \text{EQ} & \text{if } \ \mathrm{d} = d_4 \ (= 0) \end{array}\right\} & \text{if } r_k = r_l \end{cases} \tag{8.1}$$

$$d_2 = |r_k + r_l|, \text{distance at spatial relation EC}$$
$$d_4 = |r_k - r_l|, \text{distance at spatial relation TPP}$$

B. Stories Set Σ_{RCC}

With the spatial map δ_{RCC}, we need only apply the definition of story map (Eq. (7.13)) to generate the stories—we show some examples of stories in Fig. 8.1. By applying our method, we find the whole stories set (Table 8.1). We subdivide the set into two subsets: $\Sigma_0 = \{S_{01}, S_{02}, \ldots, S_{08}\}$, the rigid stories, and $\Sigma_1 = \{S_{11}, S_{12}, \ldots, S_{18}\}$, the non-rigid stories. Some stories are

	Story Sequence	Story Name	Verbal Description
	Rigid stories $\vec{v}_k = \vec{v}_l$		**Flock motion** (entities move together)
Σ_0	(DC)	S_{01}	Entities move keeping distance
	(EC)	S_{02}	Entities move connected
	(PO)	S_{03}	Entities move overlapping
	$r_k < r_l$		*Entity k is carried inside l*
	(TPP)	S_{04}	... tangent to the inner border
	(NTPP)	S_{05}	... no tangent
	$r_k > r_l$		*Entity l is carried inside k*
	(TPPI)	S_{06}	... tangent to the inner border
	(NTPPI)	S_{07}	... no tangent
	$r_k = r_l$		Entities move perfectly stapled
	(EQ)	S_{08}	
	Non-rigid stories $\vec{v}_k \neq \vec{v}_l$		**Independent motion**
Σ_1	(**DC**)	S_{11}	Entities pass each other by
	(DC, **EC**, DC)	S_{12}	Entities brush past each other
	(DC, EC, **PO**, EC, DC)	S_{13}	Entities collision partially
	$r_k < r_l$		*Entity k goes fully through l*
	(DC, EC, PO, **TPP**, PO, EC, DC)	S_{14}	... moving tangent to the inner border
	(DC, EC, PO, TPP, **NTPP**, TPP, PO, EC, DC)	S_{15}	... no tangent to the inner border
	$r_k > r_l$		*Entity l goes fully through k*
	(DC, EC, PO, **TPPI**, PO, EC, DC)	S_{16}	... moving tangent to the inner border
	(DC, EC, PO, TPPI, **NTPPI**, TPPI, PO, EC, DC)	S_{17}	... no tangent to the inner border
	$r_k = r_l$		Entities coincide exactly at the crossing
	(DC, EC, PO, **EQ**, PO, EC, DC)	S_{18}	

Table 8.1: Stories set, Σ, of RCC consisting of 16 stories. It is subdivided in rigid stories, Σ_0 (8 stories), and non-rigid stories, Σ_1 (8 stories). For each story, we provide a description in natural language.
Some stories depend on which of both entities is larger, as indicated through the radii relations (r_k and r_l). The non-rigid stories are symmetrical, so we have marked bold-faced the middle relation for clarity.

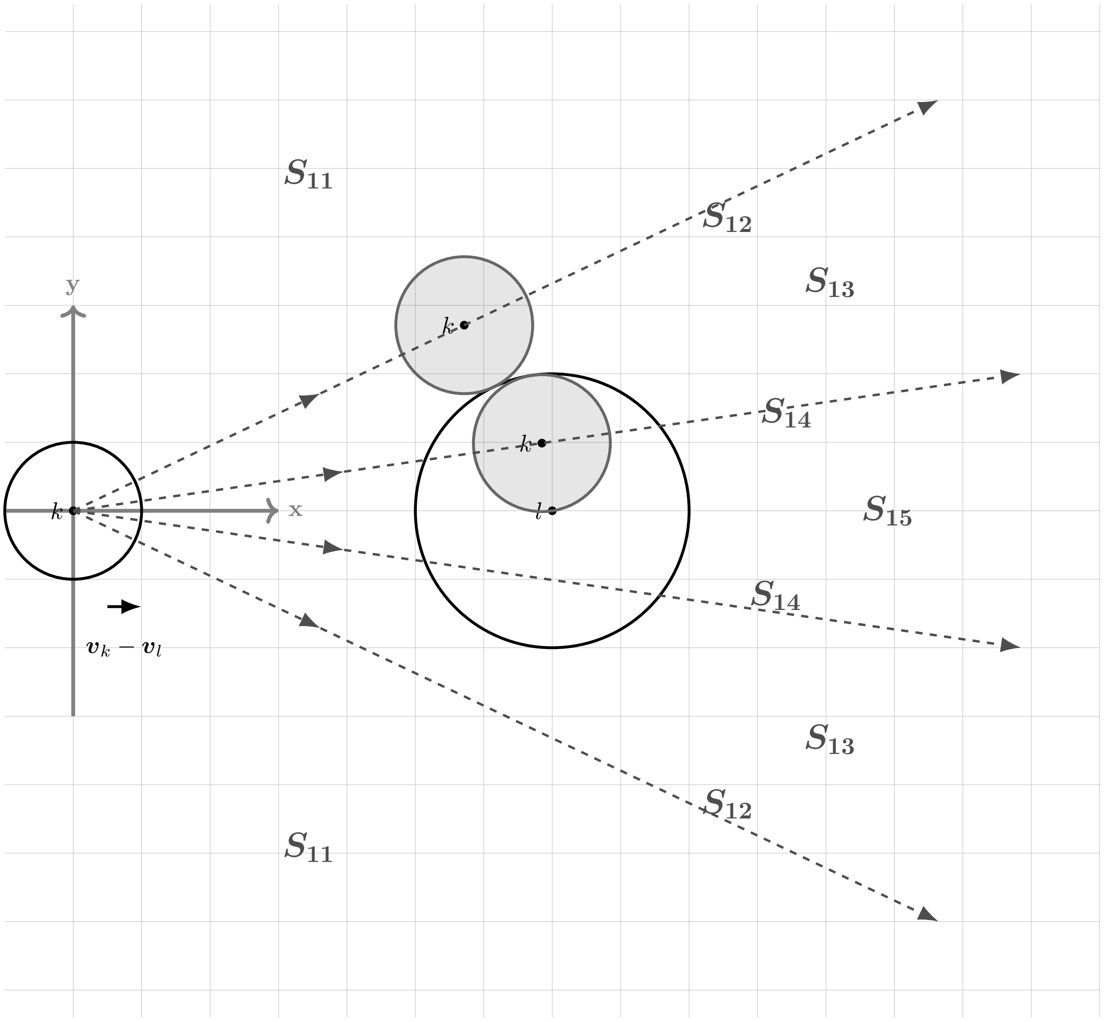


Figure 8.2: Depiction of $\{S_{11}, S_{12}, S_{13}, S_{14}, S_{15}\}$, the non-rigid stories of Σ_{RCC} for two entities k and l with $r_k < r_l$ and $\vec{v}_k \neq \vec{v}_l$ (Tab. 8.1). Two stories correspond to one-dimensional regions: $S_{12} = (\text{DC}, \text{EC}, \text{DC})$, and $S_{14} = (\text{DC}, \text{EC}, \text{PO}, \text{TPP}, \text{PO}, \text{EC}, \text{DC})$. The remaining three stories correspond to two-dimensional regions: $S_{11} = (\text{DC})$, $S_{13} = (\text{DC}, \text{EC}, \text{PO}, \text{EC}, \text{DC})$, $S_{15} = (\text{DC}, \text{EC}, \text{PO}, \text{TPP}, \text{NTPP}, \text{TPP}, \text{PO}, \text{EC}, \text{DC})$.
We depict l as being motionless and k moving with the difference of velocities $\vec{v}_{kl} = \vec{v}_k - \vec{v}_l$, which is an equivalent depiction as l moving with $\vec{v}_l$ and k moving with $\vec{v}_k$. The stories depend on the direction of $\vec{v}_{kl}$.

Source: Purcalla Arrufi and Kirsch (2017) restricted to certain ratios of the entities' radii: $r_k < r_l$, $r_k > r_l$, or $r_k = r_l$. The ratios remain constant, because deformations are not allowed in our work (Sect. 1.2). In Figure 8.2, we depict how the non-rigid stories originate in two entities with different size, $r_k < r_l$.

C. Story Map σ_{RCC} and Featural Variables

Here, we define the core of the Stories-RCC categorization: the story map σ_{RCC}, which is the categorization rule; and we relate it to our categorization framework in Section 6.1.

As we generate the RCC stories, we observe that the categorical regions are fully determined by two real variables, and, consequently, they are the featural variables for this categorization: the minimum distance between entities in the story, $\mathrm{d_{min}}$ (Eq. (8.3a)), and the dissimilarity between velocity vectors, $\mathrm{dif_V}$ (Eq. (8.3b))—additional parameters are the radii, r_k and r_l, but they are constant for all stories of the same entities; for this reason, we do not consider them featural variables. It is, thus, more natural to define the story map as the composition of two functions: $\sigma_{\varPhi}(\mathrm{d_{min}}, \mathrm{dif_V})$ and $\varPhi(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ (Eq. (8.2)). We call the first function, $\varPhi$ (Eq. (8.4)), *feature extraction function*, because it maps the kinematic variables $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ into $(\mathrm{d_{min}}, \mathrm{dif_V})$, which act as *featural variables*, or simply, as *features*. We call the second function, $\sigma_{\varPhi}$ (Eq. (8.5)), '*featural story map*', because it maps the featural variables into stories.

Story map

$$\sigma_{\mathrm{RCC}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \sigma_{\varPhi}(\mathrm{d_{min}}, \mathrm{dif_V}) \circ \varPhi(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \tag{8.2}$$

Featural variables and feature extraction function

$$\mathrm{d_{min}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \|\vec{x}_l - \vec{x}_k\| |\det(\vec{x}_l - \vec{x}_k, \vec{v}_l - \vec{v}_k)| \tag{8.3a}$$

$$\mathrm{dif_V}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \begin{cases} \dfrac{\|\vec{v}_l - \vec{v}_k\|}{\|\vec{v}_k\| + \|\vec{v}_l\|} & \|\vec{v}_k\| \neq 0, \|\vec{v}_l\| \neq 0 \\ 0 & \|\vec{v}_k\| = \|\vec{v}_l\| = 0 \end{cases} \tag{8.3b}$$

$$\varPhi(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = (\mathrm{d_{min}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l); \mathrm{dif_V}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)) \tag{8.4}$$

$$\begin{tikzcd}
\mathcal{K} \arrow[ddd, bend right] \arrow[dddr, bend left] & \\
(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \arrow[d, mapsto, "\Phi"] \arrow[dr, mapsto, "\sigma_{\mathrm{RCC}}=\sigma_{\Phi} \circ \Phi"] & \\
(\mathrm{d_{min}}, \mathrm{dif_V}) \arrow[r, mapsto, "\sigma_{\Phi}"] & S_i \\
\mathcal{F}_{\mathrm{RCC}} \arrow[r, bend right] & \Sigma_{\mathrm{RCC}}
\end{tikzcd}$$

Figure 8.3: Categorization of Stories-RCC reflecting the framework in Section 6.1 (Eq. (6.5)). The story map of RCC, σ_{RCC}, assigns a story S_i—a motion category—to every scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ in the kinematic space $\mathcal{K}$. The story map σ_{RCC} can be expressed as a two-step process: first, the featural variables, $\mathrm{d_{min}}$ and $\mathrm{dif_V}$, are extracted from the kinematic variables $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ through function Φ; second, based on the features, the featural story map σ_{Φ} assigns a story to the scenario features, i.e., it performs a feature based categorization.

Featural story map

$$\sigma_{\Phi}(\mathrm{d_{min}}, \mathrm{dif_V}) := \begin{cases}
\left.\begin{array}{l}
S_{01} \quad \text{if} \quad \mathrm{d_{min}} > d_2 \\
S_{02} \quad \text{if} \quad \mathrm{d_{min}} = d_2 \\
S_{03} \quad \text{if} \quad d_2 > \mathrm{d_{min}} > d_4 \\
\left.\begin{array}{l} S_{04} \quad \text{if} \quad \mathrm{d_{min}} = d_4 \\ S_{05} \quad \text{if} \quad d_4 > \mathrm{d_{min}} \end{array}\right\} \quad \text{if} \quad r_k < r_l \\
\left.\begin{array}{l} S_{06} \quad \text{if} \quad \mathrm{d_{min}} = d_4 \\ S_{07} \quad \text{if} \quad d_4 > \mathrm{d_{min}} \end{array}\right\} \quad \text{if} \quad r_k > r_l \\
\left.\begin{array}{l} S_{08} \quad \text{if} \quad \mathrm{d_{min}} = d_4 \end{array}\right\} \quad \text{if} \quad r_k = r_l
\end{array}\right\} & \text{if} \quad \mathrm{dif_V} = 0 \\
\left.\begin{array}{l}
S_{11} \quad \text{if} \quad \mathrm{d_{min}} > d_2 \\
S_{12} \quad \text{if} \quad \mathrm{d_{min}} = d_2 \\
S_{13} \quad \text{if} \quad d_2 > \mathrm{d_{min}} > d_4 \\
\left.\begin{array}{l} S_{14} \quad \text{if} \quad \mathrm{d_{min}} = d_4 \\ S_{15} \quad \text{if} \quad d_4 > \mathrm{d_{min}} \end{array}\right\} \quad \text{if} \quad r_k < r_l \\
\left.\begin{array}{l} S_{16} \quad \text{if} \quad \mathrm{d_{min}} = d_4 \\ S_{17} \quad \text{if} \quad d_4 > \mathrm{d_{min}} \end{array}\right\} \quad \text{if} \quad r_k > r_l \\
\left.\begin{array}{l} S_{18} \quad \text{if} \quad \mathrm{d_{min}} = d_4 \end{array}\right\} \quad \text{if} \quad r_k = r_l
\end{array}\right\} & \text{if} \quad \mathrm{dif_V} > 0
\end{cases} \tag{8.5}$$

$$d_2 = |r_k + r_l|, \text{distance at spatial relation EC}$$
$$d_4 = |r_k - r_l|, \text{distance at spatial relation TPP}$$

The story map of Stories-RCC (Eq. (8.2)) is a clear example of the formalization of categorization in Section 6.1. As shown in Figure 8.3, the categorization of motions in RCC by the story map $\sigma_{\mathrm{RCC}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ can be seen as a two-step process: initially, the *feature extraction function*, Φ (Eq. (8.4)), extracts the features, $\mathrm{d_{min}}$ and $\mathrm{dif_V}$, from the motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$; afterwards, the featural story map σ_{Φ} (Eq. (8.5)) assigns the story based on the features of the scenario.

8.2.2 Stories-OPRA$_1$

We generate the categorization Stories-OPRA$_1$ by applying the method for stories set generation (Sect. 8.1) to the spatial representation $\mathcal{R} = \text{OPRA}_1$ (Sect. 4.2.2). Our start point is the spatial map δ provided by OPRA$_1$; the map relates each motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ to a spatial relation.

A. Spatial Map δ_{OPRA_1}

The spatial representation OPRA$_1$ describes relative orientation of entities; for that reason OPRA$_1$ requires the orientation vectors of the entities, $\vec{o}_k$ and $\vec{o}_l$; vectors that are intrinsic to the entities. Nevertheless, we can use OPRA$_1$ in motion scenarios even when entities do not have an intrinsic orientation, e.g., simple points and circles, as we explain in the following.

In the case of moving entities, we identify the orientation vectors with the velocity vectors, $\vec{o}_* := \vec{v}_*$. This is the most common way humans, animals, and vehicles move: they move "forwards"—We leave the treatment of motions with divergent, i.e., non-aligned, orientation and velocity for future work. For that reason, the definition of the spatial map in Equations (8.6) and (8.7) uses the velocity vectors, $\vec{v}_*$, instead of the orientation vectors, $\vec{o}_*$.

In case that at least one entity is motionless $\vec{v}_* = 0$, the orientation $\vec{o}_*$ is given by the direction of its most recent past or future non-zero velocity vector $\vec{v}_*$. This solves the problem of undetermined orientation for motionless entities in the course of a trajectory. Note that when a motionless entity accelerates, it must begin the motion in the direction of its current orientation, i.e., its last velocity, so that the condition $\vec{o}_* = \vec{v}_*$ always holds. Such requirement is consistent with our ban on entities spinning (Sect. 1.2): an entity cannot stop, spin, and start again in a different direction, but it must start in the direction in which it most recently stopped.

As last case, if the entity is motionless for the whole trajectory, or we want to categorize a motion scenario alone, i.e., without trajectory, then we might question whether categorizing according to a directional representation (i.e., OPRA$_1$) is reasonable for an entity that neither has intrinsic orientation vector nor moves at any time. Of course, we can always define an arbitrary intrinsic orientation for the entity, as a workaround, but the meaningfulness of such decision is debatable.

$$\delta_{\text{OPRA}_1}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \begin{cases} \angle_a^b & \text{if} \quad \text{dif}_\text{X} > 0 \\ \angle c & \text{if} \quad \text{dif}_\text{X} = 0 \end{cases} \tag{8.6}$$

$$\text{where } \text{dif}_\text{X}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \begin{cases} \dfrac{\|\vec{x}_l - \vec{x}_k\|}{\|\vec{x}_k\| + \|\vec{x}_l\|} & \|\vec{x}_k\| \neq 0, \|\vec{x}_l\| \neq 0 \\ 0 & \|\vec{x}_k\| = \|\vec{x}_l\| = 0 \end{cases}$$

a, b, and c are defined as follows:

$$a = \begin{cases} \left.\begin{array}{lll} 0 & \text{if} & \cos(\alpha_{\Delta x v_k}) > 0 \\ 2 & \text{if} & \cos(\alpha_{\Delta x v_k}) < 0 \end{array}\right\} & \text{if} \quad \sin(\alpha_{\Delta x v_k}) = 0 \\ \begin{array}{lll} 1 & \text{if} & \sin(\alpha_{\Delta x v_k}) > 0 \\ 3 & \text{if} & \sin(\alpha_{\Delta x v_k}) < 0 \end{array} & \end{cases} \tag{8.7a}$$

$$b = \begin{cases} \left.\begin{array}{lll} 0 & \text{if} & \cos(\alpha_{v_l \Delta x}) < 0 \\ 2 & \text{if} & \cos(\alpha_{v_l \Delta x}) > 0 \end{array}\right\} & \text{if} \quad \sin(\alpha_{v_l \Delta x}) = 0 \\ \begin{array}{lll} 1 & \text{if} & \sin(\alpha_{v_l \Delta x}) > 0 \\ 3 & \text{if} & \sin(\alpha_{v_l \Delta x}) < 0 \end{array} & \end{cases} \tag{8.7b}$$

where $\Delta\vec{x} = \vec{x}_l - \vec{x}_k$

$$c = \begin{cases} \left.\begin{array}{lll} 0 & \text{if} & \cos(\alpha_{vv}) > 0 \\ 2 & \text{if} & \cos(\alpha_{vv}) < 0 \end{array}\right\} & \text{if} \quad \sin(\alpha_{vv}) = 0 \\ \begin{array}{lll} 1 & \text{if} & \sin(\alpha_{vv}) > 0 \\ 3 & \text{if} & \sin(\alpha_{vv}) < 0 \end{array} & \end{cases} \tag{8.7c}$$

When $\vec{v}_k = \vec{0}$ or $\vec{v}_l = \vec{0}$, then, instead of the zero vector, we use its most recent (past or future) non-zero velocity vector $\vec{v}_k$ or $\vec{v}_l$, assuming that the motion scenario belongs to a trajectory. Otherwise, we resort to defining an orientation vector $\vec{o}_k$ or $\vec{o}_l$.

If we examine the spatial relation δ_{OPRA_1} (Eqs. (8.6) and (8.7)), it apparently depends on four real values: dif_{X}, and the angles $\alpha_{\Delta x v_k} = \angle(\Delta\vec{x}, \vec{v}_k)$, $\alpha_{v_l \Delta x} = \angle(\vec{v}_l, \Delta\vec{x})$, $\alpha_{vv} = \angle(\vec{v}_k, \vec{v}_l)$. However, the variable α_{vv} is not independent: we see that $\alpha_{vv} = -(\alpha_{v_l \Delta x} + \alpha_{\Delta x v_k})$ because $\angle(\vec{v}_k, \vec{v}_l) = -\left(\angle(\vec{v}_l, \Delta\vec{x}) + \angle(\Delta\vec{x}, \vec{v}_k)\right)$. Thus, the total number of *featural variables* is three: dif_{X}, $\alpha_{\Delta x v_k}$, and $\alpha_{v_l \Delta x}$.

By the way, to compute δ_{OPRA_1}, we just need the *sign* of the sine and cosine of the vector angles, which can be efficiently calculated by means of dot product, $\cos(\angle(\vec{a}, \vec{b})) \propto \vec{a} \cdot \vec{b}$, and the determinant of the vectors, $\sin(\angle(\vec{a}, \vec{b})) \propto \det(\vec{a}, \vec{b})$.

B. Stories Set Σ_{OPRA_1}

Given the spatial map δ_{OPRA_1}, we only need to apply the definition of story map (Eq. (7.13)) to generate the stories—we show some examples of stories in Fig. 8.4. By applying our method, we find the whole stories set, Σ_{OPRA_1} (Table 8.2). We subdivide the stories set into six meaningful subsets Σ_*; the subsets are grouped by pairs into three superordinate groups.

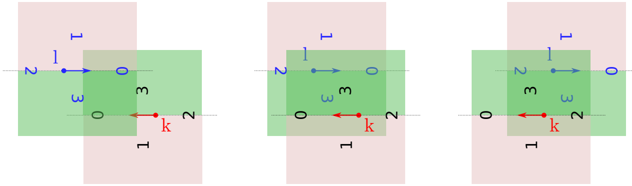
- Entities cross:
  - Σ_C both entities are moving
  - Σ_B one entity is motionless
- Entities move parallel:
  - Σ_T the entities' trajectories are superposed
  - Σ_P the entities' trajectories are not superposed

			Independent motion $\vec{v}_k \neq \vec{v}_l$			
			Entities cross $\vec{v}_k \nparallel \vec{v}_l$			
	k comes from the right of l				k comes from the left of l	
			Both entities move $\vec{v}_k \neq 0, \vec{v}_l \neq 0$			
Σ_C	$\angle_1^3 \angle_1^0 \angle_1^1 \angle_2^1 \angle_3^1$	$S_{\text{C1-1}}$	k crosses before l	$\angle_3^1 \angle_3^0 \angle_3^3 \angle_2^3 \angle_1^3$	$S_{\text{C2-1}}$	k crosses before l
	$\angle_1^3 \angle^3 \angle_3^1$	S_{C10}	entities collide	$\angle_3^1 \angle^1 \angle_1^3$	S_{C20}	entities collide
	$\angle_1^3 \angle_0^3 \angle_3^3 \angle_3^2 \angle_3^1$	S_{C11}	l crosses before k	$\angle_3^1 \angle_0^1 \angle_1^1 \angle_1^2 \angle_1^3$	S_{C21}	l crosses before k
			Entity l is motionless $\vec{v}_k \neq 0, \vec{v}_l = 0$			
Σ_B	$\angle_1^3 \angle_1^0 \angle_1^1$	$S_{\text{B1-1}}$	k crosses before l	$\angle_3^1 \angle_3^0 \angle_3^3$	$S_{\text{B2-1}}$	k crosses before l
	$\angle_0^3 \angle^3 \angle_2^1$	S_{B10}	k runs over l	$\angle_0^1 \angle^1 \angle_2^3$	S_{B20}	k runs over l
	$\angle_3^3 \angle_3^2 \angle_3^1$	S_{B11}	k crosses behind l	$\angle_1^1 \angle_1^2 \angle_1^3$	S_{B21}	k crosses behind l
			Entity k is motionless $\vec{v}_k = 0, \vec{v}_l \neq 0$			
	$\angle_1^1 \angle_2^1 \angle_3^1$	$S_{\text{B3-1}}$	l crosses behind k	$\angle_3^3 \angle_2^3 \angle_1^3$	$S_{\text{B4-1}}$	l crosses behind k
	$\angle_1^0 \angle^3 \angle_3^2$	S_{B30}	l runs over k	$\angle_3^0 \angle^1 \angle_1^2$	S_{B40}	l runs over k
	$\angle_1^3 \angle_0^3 \angle_3^3$	S_{B31}	l crosses before k	$\angle_3^1 \angle_0^1 \angle_1^1$	S_{B41}	l crosses before k
			Entities move parallel $\vec{v}_k \parallel \vec{v}_l$			
			The entities' trajectories are superposed			
Σ_T	$\angle_0^2 \angle^0 \angle_2^0$	$S_{\text{T-1}}$	k overtakes l			
	$\angle_0^0 \angle^2 \angle_2^2$	S_{T0}	k runs frontally through l			
	$\angle_2^0 \angle^0 \angle_0^2$	S_{T1}	l overtakes k			
			The entities' trajectories are not superposed			
		Entities pass each other by backwardly			Entities pass each other by frontally	
Σ_P	$\angle_1^3$	S_{P2}	l stays at the right of k k stays at the left of l	$\angle_3^3$	S_{P3}	entities stay at the right of each other
	$\angle_3^1$	$S_{\text{P-2}}$	l stays at the left of k k stays at the right of l	$\angle_1^1$	S_{P1}	entities stay at the left of each other
			Flock motion $\vec{v}_k = \vec{v}_l$			
			Non-static entities $\lVert\vec{v}_*\rVert \neq 0$			
Σ_E	$\angle_0^2$	$S_{\text{E-2}}$	k follows l	$\angle_2^0$	S_{E2}	l follows k
	$\angle_3^1$	$S_{\text{E-1}}$	k stays at the right of l	$\angle_1^3$	S_{E1}	k stays at the left of l
			$\angle^0$ S_{E0} entities coincide			
			Static scenarios $\lVert\vec{v}_*\rVert = 0$, equivalent to OPRA$_1$			
Σ_R	$\angle_0^0$ S_{R00}	$\angle_1^0$ S_{R10}	$\angle_2^0$ S_{R20}	$\angle_3^0$ S_{R30}		
	$\angle_0^1$ S_{R01}	$\angle_1^1$ S_{R11}	$\angle_2^1$ S_{R21}	$\angle_3^1$ S_{R31}		
	$\angle_0^2$ S_{R02}	$\angle_1^2$ S_{R12}	$\angle_2^2$ S_{R22}	$\angle_3^2$ S_{R32}		
	$\angle_0^3$ S_{R03}	$\angle_1^3$ S_{R13}	$\angle_2^3$ S_{R23}	$\angle_3^3$ S_{R33}		
	$\angle^0$ S_{R0}	$\angle^1$ S_{R1}	$\angle^2$ S_{R2}	$\angle^3$ S_{R3}		

Table 8.2: Stories set Σ of OPRA$_1$, divided into meaningful subsets of stories: Σ_C, Σ_B, Σ_T, Σ_P, Σ_E, Σ_R. A total of 50 stories, but 20 belong to Σ_R, i.e., fully motionless entities. The stories are described both through sequences of spatial relations and through sentences in natural language.

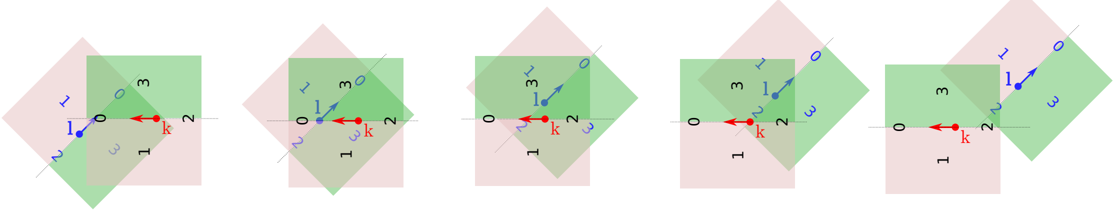
Originally based in Table 1 in Purcalla Arrufi and Kirsch (2018a), though notably modified and expanded.

Story S_{P3}



$\angle^3_3$ $\angle^3_3$ $\angle^3_3$

Story S_{C11}



$\angle^3_1$ $\angle^3_0$ $\angle^3_3$ $\angle^2_3$ $\angle^1_3$

Figure 8.4: Representation of two stories of OPRA$_1$, S_{P3} and S_{C11}. We display for each story its complete sequence of spatial relations (See Table 8.2).

Source: Purcalla Arrufi and Kirsch (2018a)

- Rigid stories:

Σ_E both entities are moving

Σ_R both entities are motionless

C. Story Map σ_{OPRA_1} and Featural Variables

As usual, we separate the description of the story map, σ_{OPRA_1}, in two cases that we treat differently: first, the non-rigid stories ($\vec{v}_k \neq \vec{v}_l$); second, the rigid stories ($\vec{v}_k = \vec{v}_l$) (See Tab. 8.2). The rigid stories behave mostly like the spatial relation OPRA$_1$, as described in Equations (8.6) and (8.7). For that reason, in this section, we only describe the story map for the first case, the non-rigid stories.

The OPRA$_1$ non-rigid stories can be fully determined using four real features:

- α_{vv}, the angle between orientations of velocities, i.e., $\angle(\vec{v}_k, \vec{v}_l)$.
  To determine the category, we compute its sign, which is equivalent to the sign of $\det(\vec{v}_k, \vec{v}_l)$.

- $\alpha_{\Delta x \Delta v}$, the angle between the positions' difference and the velocities' difference, i.e., $\angle(\Delta\vec{x}, \Delta\vec{v})$.
  To determine the category, we compute its sign, which is equivalent to the sign of $\det(\Delta\vec{x}, \Delta\vec{v})$.

- u_k and u_l, which are the velocity ratio for each entity.

$$u_k = \begin{cases} \frac{\|\vec{v}_k\|}{\max\{\|\vec{v}_k\|,\|\vec{v}_l\|\}} & \|\vec{v}_k\| \neq 0 \\ 0 & \|\vec{v}_k\| = 0 \end{cases} \quad (8.8a) \qquad u_l = \begin{cases} \frac{\|\vec{v}_l\|}{\max\{\|\vec{v}_k\|,\|\vec{v}_l\|\}} & \|\vec{v}_l\| \neq 0 \\ 0 & \|\vec{v}_l\| = 0 \end{cases} \quad (8.8b)$$

We use these ratios to know, whether and which entity is motionless, i.e., whether $u_* = 0$; and, also, to know whether one entity is faster than the other: for non-rigid scenarios $u_k < 1 \Leftrightarrow \|\vec{v}_k\| < \|\vec{v}_l\|$, or, equivalently, $u_l < 1 \Leftrightarrow \|\vec{v}_l\| < \|\vec{v}_k\|$.

Since we fully determine the OPRA$_1$ story of each scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ using the features above, we have again the categorization model displayed in Equation (6.5): first, we extract the Stories-OPRA$_1$ features from the scenario, which we can express as the feature extraction function $\varPhi$; and, second, we use the features to map the scenario into its corresponding story, which we perform by means of the featural story map of OPRA$_1$, $\sigma_{\varPhi}$—Since the featural variables are independent and each forms at most four featural regions, we can represent $\sigma_{\varPhi}$ as a tree (Fig. 8.5).

8.3 Stories-$\mathcal{R}$ of Motion Representations

The method in Section 8.1 for obtaining Stories-$\mathcal{R}$ categorizations is a general method that uses any kind of qualitative representations. Remember that, as we mentioned in Section 7.1, the only condition for using a certain qualitative representation, is that it categorizes scenarios—This condition was expressed in Equation (7.4). In the specific case of a qualitative representation *of motion* that categorizes motion scenarios, Equation (7.4) can be rewritten as follows:

$$\begin{array}{rcl} \mu : \mathcal{K} & \longrightarrow & \mathcal{M} \\ K = (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) & \longmapsto & M_i \end{array} \tag{8.9}$$

We have just made following substitutions: the qualitative representation $\mathcal{R}$ is expressed as a motion representation $\mathcal{M}$; the qualitative relation R_i is expressed as a motion relation M_i; the relational map ρ of a qualitative representation is expressed as the relational map μ of a qualitative motion representation.

The same substitutions take place in Equation (7.14) (the definition of story map), so that we obtain following:

$$\begin{array}{rcl} \sigma : \mathcal{K} & \longrightarrow & \Sigma \\ (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) & \longmapsto & S_i = (M_{i_1}, M_{i_2}, \ldots, M_{i_m}) \end{array} \tag{8.10}$$

We can see, in Eq. (8.10), that the stories of motion representations, S_i, are a sequence of relations, R_i, and, specifically, a sequence of motion relations, M_i.

Now, the construct "story of motion representations" might be confusing at first glance. If a motion representation already categorizes motion, what does a story of a motion representation categorize? The *stories of motion representations* categorize motion scenarios too. In short, we can create a new motion categorization, e.g., Stories-QTC$_{\mathrm{B21}}$ (Sect. 8.3.1), using the stories of another motion categorization, e.g., QTC$_{\mathrm{B21}}$.

Idempotence of *Stories-* Operation Notwithstanding, the process for obtaining new motion categorizations by obtaining their stories cannot be applied recursively. Indeed, the story of a certain story is by definition this same story, i.e., Stories-{Stories-$\mathcal{R}$} = Stories-$\mathcal{R}$. That is

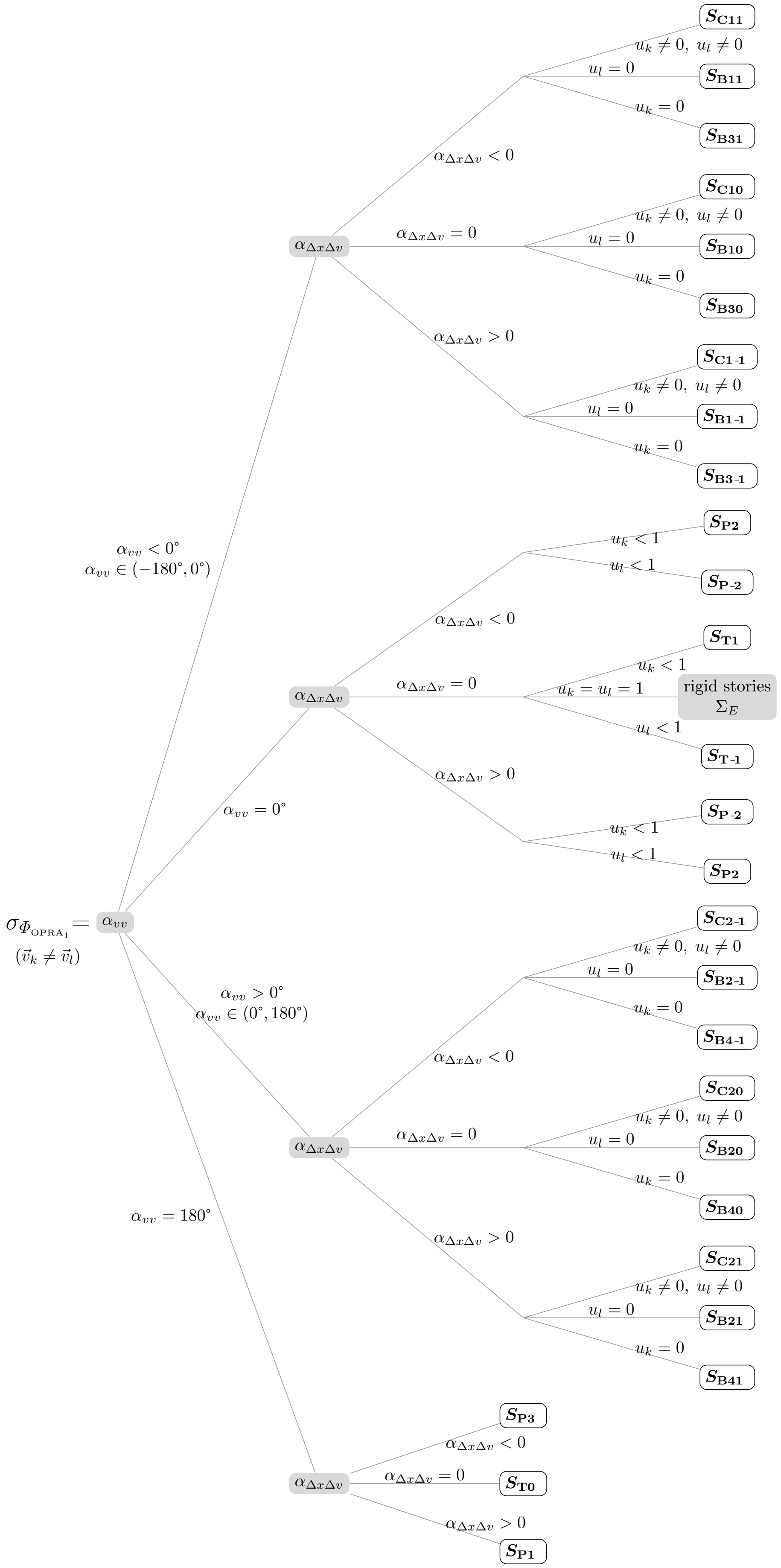


Figure 8.5: Featural story map for Stories-OPRA$_1$, σ_Φ. It maps *non-rigid* scenarios ($\vec{v}_k \neq \vec{v}_l$) into the corresponding stories by means of following four featural variables: $\alpha_{vv} = \angle(\vec{v}_k, \vec{v}_l)$; $\alpha_{\Delta x \Delta v} = \angle(\Delta\vec{x}, \Delta\vec{v})$, where $\Delta\vec{x} = \vec{x}_l - \vec{x}_k$, $\Delta\vec{v} = \vec{v}_l - \vec{v}_k$; and $u_* = \frac{\|\vec{v}_*\|}{\max\{\vec{v}_k, \vec{v}_l\}}$.

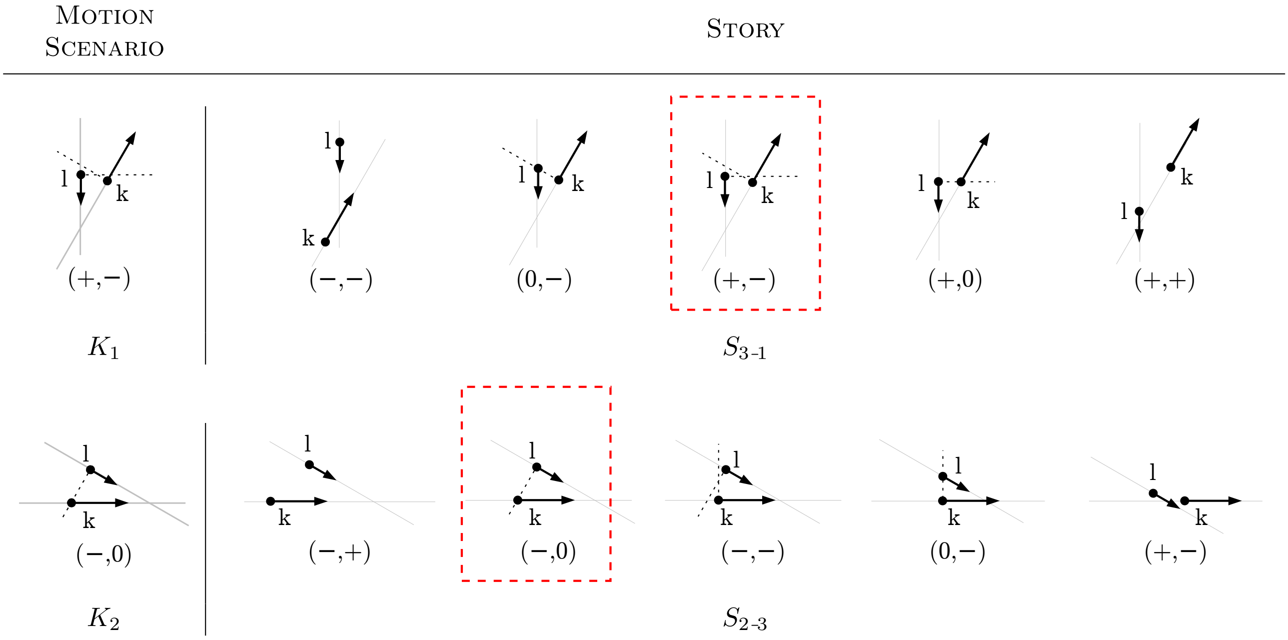


Figure 8.6: Two scenarios, K_1 and K_2, with their corresponding stories based in the motion representation $\mathrm{QTC_{B21}}$: $S_{3\text{-}1} = \sigma_{\mathrm{QTC_{B21}}}(K_1)$, and $S_{2\text{-}3} = \sigma_{\mathrm{QTC_{B21}}}(K_2)$. The position of each scenario in its own story is marked with a red dashed rectangle.

the Stories- operation that maps a qualitative representation into a motion representation is *idempotent.*

8.3.1 Stories-$\mathrm{QTC_{B21}}$

As an example of stories of motion categorization, we create Stories-$\mathrm{QTC_{B21}}$. We apply the algorithm given in Section 8.1.1 which uses the story map. The story map $\sigma_{\mathrm{QTC_{B21}}}$ is created by means of the categorization map of $\mathrm{QTC_{B21}}$, which is presented as f_μ in Section 6.3, and here we will rename μ according to the notation in Equation (8.9).

A. Story Map $\sigma_{\mathrm{QTC_{B21}}}$ and Featural Variables

The $\mathrm{QTC_{B12}}$ scenarios are mapped into 18 different stories by means of $\sigma_{\mathrm{QTC_{B21}}}$ (Tab. 8.3). For example, in Figure 8.6, we show how $\sigma_{\mathrm{QTC_{B21}}}$ maps two scenarios into their corresponding stories.

Similarly to the previous story maps, σ_{RCC} and $\sigma_{\mathrm{OPRA_1}}$, the story map $\sigma_{\mathrm{QTC_{B21}}}$ is determined by a limited number of featural variables, namely, α, τ, γ, q (Tab. 8.4). In other words, each story is determined by 4-tuples of features. Most stories are determined by a unique tuple of features, while some, such as $S_{1\text{-}2}$, consist of several tuples. An interesting cognitive question is whether stories defined with several tuples are recognized as a simple category or subjects tend to associate each single tuple to a unique category—We leave this question for future work.

We explain here, in detail, the featural variables of $\mathrm{QTC_{B21}}$:

Story name	**Story description** (as a sequence of $\mathrm{QTC_{B21}}$ relations)	**Featural variables** $(\alpha, \tau, \gamma, q)$	**Verbal description**
	Rigid stories $\vec{v}_k = \vec{v}_l$		**Flock motion**
$S_{0\text{-}1}$	$(-,+)$	$(\mathrm{P}, k^-, v_k^=, *)$	k follows after l
S_{00}	$(0,0)$	$(\mathrm{P}, k^=, v_k^=, *)$	the entities move side-by-side
S_{01}	$(+,-)$	$(\mathrm{P}, k^+, v_k^=, *)$	l follows after k
	Non-rigid stories $\vec{v}_k \neq \vec{v}_l$		**Independent motion**
$S_{1\text{-}2}$	$(-,-),(0,0),(+,+)$	$(\mathrm{A}, *, *, *)$ $(\mathrm{FC}, k^=, *, *)$ $(\mathrm{BC}, k^=, *, q^-)$	antiparallel motion front crossing, entities coincide at crossing back crossing, entities coincide at crossing, the slower entity leaves the faster behind
$S_{1\text{-}1}$	$(-,0),(0,0),(+,0)$	$(\mathrm{BC}, k^=, v_k^+, q^=)$	back crossing, entities coincide at crossing, k is faster, k ends at one side of l
S_{10}	$(-,+),(0,0),(+,-)$	$(\mathrm{P}, *, v_k^+, *)$ $(\mathrm{BC}, k^=, v_k^+, q^+)$	parallel motion, k is faster back crossing, entities coincide at crossing, k is faster, k ends ahead of l
S_{11}	$(0,-),(0,0),(0,+)$	$(\mathrm{BC}, k^=, v_k^-, q^=)$	back crossing, entities coincide at crossing, l is faster, l ends at one side of k
S_{12}	$(+,-),(0,0),(-,+)$	$(\mathrm{P}, *, v_k^-, *)$ $(\mathrm{BC}, k^=, v_k^-, q^+)$	parallel motion, l is faster back crossing, entities coincide at crossing, l is faster, l ends ahead of k
$S_{3\text{-}1}$	$(-,-),(-,0),(-,+),(0,+),(+,+)$	$(\mathrm{FC}, k^-, *, *)$ $(\mathrm{BC}, k^-, *, q^-)$	forward crossing, l crosses first back crossing, l crosses first, slower entity ends ahead of faster one
S_{31}	$(-,-),(0,-),(+,-),(+,0),(+,+)$	$(\mathrm{FC}, k^+, *, *)$ $(\mathrm{BC}, k^+, *, q^-)$	forward crossing, k crosses first back crossing, k crosses first, slower entity ends ahead of faster one
$S_{2\text{-}1}$	$[(-,0)],(-,-),(0,-),(+,-),[(+,0)]$	$(\mathrm{BC}, k^+, v_k^+, q^=)$	back crossing, k crosses first, k is faster, k ends at one side of l
$S_{2\text{-}2}$	$[(-,0)],(-,+),(0,+),(+,+),[(+,0)]$	$(\mathrm{BC}, k^-, v_k^+, q^=)$	back crossing, l crosses first, k is faster, k ends at one side of l
$S_{2\text{-}3}$	$(-,+),(-,0),(-,-),(0,-),(+,-)$	$(\mathrm{BC}, k^+, v_k^+, q^+)$	back crossing, k crosses first, k is faster, k ends ahead of l
$S_{2\text{-}4}$	$(-,+),(0,+),(+,+),(+,0),(+,-)$	$(\mathrm{BC}, k^-, v_k^+, q^+)$	back crossing, l crosses first, k is faster, k ends ahead of l
S_{21}	$[(0,-)],(-,-),(-,0),(-,+),[(0,+)]$	$(\mathrm{BC}, k^-, v_k^-, q^=)$	back crossing, l crosses first, l is faster, l ends at one side of k
S_{22}	$[(0,-)],(+,-),(+,0),(+,+),[(0,+)]$	$(\mathrm{BC}, k^+, v_k^-, q^=)$	back crossing, k crosses first, l is faster, l ends at one side of k
S_{23}	$(+,-),(0,-),(-,-),(-,0),(-,+)$	$(\mathrm{BC}, k^-, v_k^-, q^+)$	back crossing, l crosses first, l is faster, l ends ahead of k
S_{24}	$(+,-),(+,0),(+,+),(0,+),(-,+)$	$(\mathrm{BC}, k^+, v_k^-, q^+)$	back crossing, k crosses first, l is faster, l ends ahead of k

The sign '$*$' in the featural variables means that any value in the corresponding feature is allowed.
Square brackets around a relation, such as $[(-,0)]$, means that this is a limit relation either for $t \to -\infty$ or $t \to +\infty$.

Table 8.3: This table displays both the stories set of $\mathrm{QTC_{B12}}$, i.e., $\Sigma_{\mathrm{QTC_{B21}}}$, and its story map, $\sigma_{\mathrm{QTC_{B21}}}$. It contains 18 stories, and the corresponding tuple of *featural variables* $F = (\alpha, \tau, \gamma, q)$ that allows mapping each motion scenario into the corresponding story (See feat. var. Tab. 8.4). Additionally, we provide a verbal description of the story based on its featural values.
We note that some stories, such as $S_{1\text{-}2}$, are characterized by several featural tuples.

i) α, '*crossing angle*': It is the angle of the crossing trajectories, i.e., the minimum angle between the entities' velocities:

$$\alpha(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \arccos(\frac{\vec{v}_k \cdot \vec{v}_l}{\|\vec{v}_k\|\|\vec{v}_l\|}) \in [0^\circ, 180^\circ] \tag{8.11}$$

According to the values that α takes in the different stories, we identify four different featural regions that categorize stories: A, FC, BC, and P. The regions are linked to the values of α in the following equation:

$$\begin{array}{llccll} \text{A} & \textit{`antiparallel'} & \text{if} & \alpha = 180^\circ & \text{, i.e.,} & \hat{v}_k \cdot \hat{v}_l = -1 \\ \text{FC} & \textit{`front crossing'} & \text{if} & 90^\circ \leq \alpha < 180^\circ & \text{, i.e.,} & 0 \geq \hat{v}_k \cdot \hat{v}_l > -1 \\ \text{BC} & \textit{`back crossing'} & \text{if} & 0 < \alpha < 90^\circ & \text{, i.e.,} & 1 > \hat{v}_k \cdot \hat{v}_l > 0 \\ \text{P} & \textit{`parallel'} & \text{if} & \alpha = 0^\circ & \text{, i.e.,} & \hat{v}_k \cdot \hat{v}_l = +1 \end{array} \tag{8.12}$$

ii) τ, '*crossing delay*': It indicates which of the entities goes first through the crossing point. τ yields the time difference, $\tau = t_k - t_l$, between the arrival of k and l at the crossing point (See full deduction of the formulae in Propositions A.4.6 and A.4.8). Note that τ can only be computed when $\vec{v}_k \nparallel \vec{v}_l$ or $\vec{v}_k = \vec{v}_l$. For the rest of stories τ is undetermined, but it does not influences categorization.

$$\tau(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = (\vec{x}_k - \vec{x}_l) \times (\vec{v}_k - \vec{v}_l) \cdot \frac{\vec{v}_k \times \vec{v}_l}{\|\vec{v}_k \times \vec{v}_l\|^2} \qquad \text{for } \vec{v}_k \nparallel \vec{v}_l \tag{8.13a}$$

$$\tau(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = -\frac{(\vec{x}_k - \vec{x}_l) \cdot \hat{v}_k}{\|\vec{v}_k\|} \qquad \text{limit for } \vec{v}_k = \vec{v}_l \tag{8.13b}$$

According to the values that τ takes in the different stories, we identify three different featural regions that categorize stories: k^+, $k^=$, and k^-. The regions are linked to the values of τ in the following equation:

$$\begin{array}{lll} k^+ & \text{if } \tau < 0 & k \text{ crosses first} \\ k^= & \text{if } \tau = 0 & k \text{ and } l \text{ cross simultaneously, i.e., they collide} \\ k^- & \text{if } \tau > 0 & l \text{ crosses first} \end{array} \tag{8.14}$$

iii) γ, '*relative speed*': It indicates which entity moves faster.

$$\gamma(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \text{dif}_\text{V}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \tag{8.15}$$

According to the values that γ takes in the different stories, we identify three different featural regions that categorize stories: v_k^+, $v_k^=$, and v_k^-. The regions are linked to the values of γ in the following equation:

$$\begin{array}{lll} v_k^+ & \text{if } \gamma > 0, \text{ i.e., } \|\vec{v}_k\| > \|\vec{v}_l\| & k \text{ is faster} \\ v_k^= & \text{if } \gamma = 0, \text{ i.e., } \|\vec{v}_k\| = \|\vec{v}_l\| & k \text{ and } l \text{ move equally fast} \\ v_k^- & \text{if } \gamma < 0, \text{ i.e., } \|\vec{v}_k\| < \|\vec{v}_l\| & l \text{ is faster} \end{array} \tag{8.16}$$

iv) q, '*side axis limit*': This feature reveals whether in the long term, i.e., $t \to +\infty$, the slower entity has the faster entity at its *side axis* (i.e., at its perfect right or left directions), before this side axis, or behind it. More colloquially, whether in the long term the faster entity is at one side of the slower entity, the faster entity is ahead of the slower one, or the slower entity leaves the faster entity behind—The faster entity will always leave the slower entity behind. The value of the feature q is obtained in Equation (8.17a), and its interpretation in Equation (8.17b).

$$q(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \cos(\alpha)\frac{\max(\|\vec{v}_k\|, \|\vec{v}_l\|)}{\min(\|\vec{v}_k\|, \|\vec{v}_l\|)} = \begin{cases} \cos(\alpha)\dfrac{\|\vec{v}_k\|}{\|\vec{v}_l\|} & \text{if} \quad \|\vec{v}_k\| > \|\vec{v}_l\| \\ \cos(\alpha) & \text{if} \quad \|\vec{v}_k\| = \|\vec{v}_l\| \\ \cos(\alpha)\dfrac{\|\vec{v}_l\|}{\|\vec{v}_k\|} & \text{if} \quad \|\vec{v}_l\| > \|\vec{v}_k\| \end{cases} \tag{8.17a}$$

From Equation (8.11), we note that $\cos(\alpha) = \hat{v}_k \cdot \hat{v}_l$

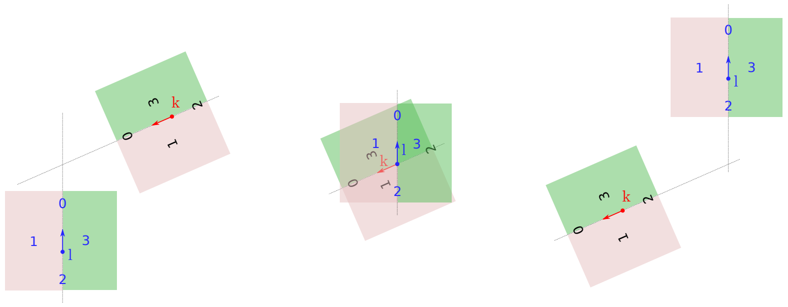
***q*-Value**	**Symbol**	**Limit relation for the slower entity $\boldsymbol{t \to +\infty}$**	**Verbal description**
$q > 1$	q^{+}	$-$	The faster entity ends ahead of the slower one
$q = 1$	$q^{=}$	0	The faster entity ends at one side of the slower entity
$q < 1$	q^{-}	$+$	The slower entity leaves the faster entity behind

(8.17b)

For example, in the story $S_{2\text{-}3}$, the faster entity is k (i.e., v_k^{+}) and the limit relation is $(+, -)$. Hence, the slower entity, l, has relation '$-$', and, consequently, has the faster entity rather before it, which we express as 'q^{+}'. Another example; in the story S_{21}, the faster entity is l (i.e., v_k^{-}) and the limit relation is $(0, +)$. Hence, the slower entity, k, has relation '0', and, consequently, has the faster entity at one side (i.e., on its side axis): we express it as '$q^{=}$'.

Feature Name (φ_i)	Trajectory Angle α	Crossing Precedence τ	Relative Speed γ	Normal Overtaking q
Feature Values	A (antiparallel) $\alpha = 180°$	▨	▨	▨
	FC (front crossing) $90° \leq \alpha < 180°$	k^+ if k crosses first k^- if l crosses first $k^=$ if k and l collide	▨	▨
	P (parallel) $\alpha = 0°$		v_k^+ if k is faster v_k^- if l is faster $v_k^=$ if k and l equally fast	▨
	BC (back crossing) $0° < \alpha < 90°$			q^+ faster behind of slower q^- faster ahead of slower $q^=$ faster at one side of slower

Table 8.4: The features φ_i that allow a full categorization of the QTC_{B21} stories. In other words, the features that define the story map $\sigma_{\text{QTC}_{\text{B21}}}$. The greyed cells illustrate that for certain trajectory angles additional features do not refine the categorization. For example, the stories with FC trajectory angle are only subcategorized by the feature *crossing precedence*, i.e., τ, while the features γ and q have no further subcategorization effect.



(a) OPRA$_1$: $\angle_1^3$
Stories-OPRA$_1$: S_{C10}

(b) OPRA$_1$: $\angle 3$
Stories-OPRA$_1$: S_{C10}

(c) OPRA$_1$: $\angle_3^1$
Stories-OPRA$_1$: S_{C10}

Figure 8.7: OPRA$_1$ story $S_{\mathrm{C10}} = (\angle_1^3, \angle 3, \angle_3^1)$; the story's sequence is illustrated with the corresponding motion scenarios. The time evolves from left to right, i.e., left picture is earlier, right picture is latter. In each scenario, we write both the OPRA$_1$ spatial relation and the Stories-OPRA$_1$ motion relation.

8.4 Motivation to Expand *Bear* into *Beaded* Categorizations

The bare story-based categorizations fulfil the initial goals and assumptions in this work (Sects. 1.1 and 1.2). Such categorizations accomplish also some basic purposes of categorization; for example, *cognitive economy*: they simplify the $\mathbb{R}^8$ kinematic space of the motion scenarios of two entities into a reduced number of categories (e.g., 16 categories in Stories-RCC, or 18 in Stories-QTC$_{\mathrm{B21}}$); and the categories, i.e., the stories, are cognitively meaningful—They correspond to certain types of motion with salient navigation features (parallelism, collision, . . .).

Nevertheless, stories alone can very limitedly be used for decision-making. In truth, a story corresponds to a whole trajectory, but, in order to take a control decision in a trajectory, we should know in which stage of the trajectory the scenario is.

For instance, consider the story S_{C10}, displayed in Figure 8.7. It is formed by the sequence of relations $(\angle_1^3, \angle 3, \angle_3^1)$, and it depicts a collision. Each scenario in the story (Figs. 8.7a to 8.7c) is categorized with the same category S_{C10} according to the categorization Stories-OPRA$_1$, although the scenarios differ from each other regarding decision-making: In scenario (a), we must control the entities to avoid collision, but in scenario (c), the collision threat is over—no action is needed.

Thus, we need some extra information besides the story in case we want to effectively use a story-based categorization for decision-making in trajectory control. The simplest solution is to append the qualitative relation of the scenario, R_i, to the story, S_j. That is, the scenario (c) would be categorized as $S_{\mathrm{C10}}(\angle_1^3)$—instead of simply S_{C10}. Similarly, the scenario (a) would be categorized as $S_{\mathrm{C10}}(\angle_3^1)$. Now the decision rule emerges lightly: collision danger exists, if the scenario belongs to category $S_{\mathrm{C10}}(\angle_1^3)$; no collision danger if the category is $S_{\mathrm{C10}}(\angle_3^1)$.

8.5 Defining beaded story-based categorizations: $S_i(R_j)$

As suggested above, we can create a new type of story-based categorizations by appending, i.e., concatenating, the qualitative relation R_j of a certain scenario to its story S_i; thus, the categories in these new categorizations are represented as $S_i(R_j)$, and we call them '*beaded categories*'. Correspondingly, we call the new categorizations '*beaded story-based categorizations*'. Each beaded story-based categorizations is named '*Motion-$\mathcal{R}$*', according to the qualitative relation $\mathcal{R}$ used to generate it. For instance, Motion-RCC and Motion-OPRA$_1$ (Sects. 8.6.1 and 8.6.2).

A *beaded* story-based categorization, or simply, a *beaded categorization* is a subset of the Cartesian product of the stories and their qualitative relations, $\Sigma \times \mathcal{R}$. Caveat, not every relation is combined with any story, but only those relations belonging to the story. Due to this asymmetry, we do not represent the pair story-relation as a tuple, (S_i, R_j), but rather in a functional way, $S_i(R_j)$.

Since the beaded categories, $S_i(R_j)$, are compound, we call R_j the '*position component*'—because it indicates the "position" of the categorized scenario in the story's sequence $(R_1, R_2, \ldots, R_{n_i})$—and we call S_i the '*story component*' or just '*story*'.

8.5.1 Story map, stories set, and categorization rule

Now, let us formalize the elements of a *beaded* categorization. Most of its elements are the same as those of a *bare* categorization. First of all, a beaded categorization categorizes the same objects a bare categorization does: *motion scenarios*. Second, once we have chosen a certain qualitative categorization $\mathcal{R}$, both types of categorizations, 'bare' (Stories-$\mathcal{R}$) and 'beaded' (Motion-$\mathcal{R}$), have the same *story map*, $\sigma_{\mathcal{R}}$, and, accordingly, the same *stories set*, $\Sigma_{\mathcal{R}}$.

On the other hand, the *categorization rule* differs between both categorization types. In the *bare* categorizations, the story map, $\sigma_{\mathcal{R}}$, is itself the categorization rule because the categories are the stories. However, in the *beaded* categorizations, the categorization rule is a Cartesian product of the story map $\sigma_{\mathcal{R}}$, and the qualitative map ρ of the qualitative representation $\mathcal{R}$ (Eq. (7.4)), i.e., $\mathrm{f}_\mu := \sigma_{\mathcal{R}} \times \rho$ (Eq. (8.18))—Remember that not all possible Cartesian combinations are possible, but a story can only be concatenated to relations that are contained in the story.

$$\begin{array}{rcl} \mathrm{f}_\mu := \sigma_{\mathcal{R}} \times \rho : \mathcal{K} & \longrightarrow & \mathcal{M} \subset \Sigma \times \mathcal{R} \\ (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) & \longmapsto & S_i(R_j) \end{array} \tag{8.18}$$

Since the categorization rule, f_μ, differs between bare and beaded categorizations, also the *categories set* $\mathcal{M}$ differs. In the bare categorizations, the categories set equals the stories set, i.e., $\mathcal{M} = \Sigma$, but in the beaded categorizations the categories set is a subset of the Cartesian product, i.e., $\mathcal{M} \subset \Sigma \times \mathcal{R}$.

A method to create a beaded story-based categorization

We outline the steps that lead to a beaded story-based categorization, Motion-$\mathcal{R}$. In the following sections, we use later the here defined method to generate the novel motion categorizations 'Motion-RCC' and 'Motion-OPRA$_1$'.

1. We have a qualitative representation $\mathcal{R}$ (either spatial, $\mathcal{D}$, or motion, $\mathcal{M}$) with a categorization rule ρ (Eq. (7.1)).

2. We determine the stories set associated with the representation, i.e., $\Sigma_{\mathcal{R}}$, and the story map, $\sigma_{\mathcal{R}}$, as we showed in Section 8.1.1—This step is equivalent to obtain the Stories-$\mathcal{R}$ categorization.

3. We immediately create the *categorization rule*, f_μ, if we combine both aforementioned maps, $\sigma_{\mathcal{R}}$ and ρ, through the Cartesian product, i.e., $\mathrm{f}_\mu := \sigma_{\mathcal{R}} \times \rho$ (Eq. (8.18)). *motion categorization* based on the *categorization rule* f_μ, as presented in Equation (6.1), Accordingly, a motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ is mapped both into a story S_i and into a qualitative spatial

relation R_j: The *motion category* of the scenario is defined as $S_i(R_j)$, i.e., the story S_i, at the spatial relation R_j.

4. The *categories set*, $\mathcal{M}$, is effortless obtained from the stories set, Σ. We obtain each *beaded category* by appending to every story $S_i = (R_1, R_2, \ldots, R_{n_i})$ each of the story relations; so we have $S_i(R_1), S_i(R_2), \ldots, S_i(R_{n_i})$. The categories set is, hence, the collection of all such *beaded categories*, $\mathcal{M} = \cup_{i=1\ldots n} \cup_{j=1\ldots n_i} S_i(R_j)$.

8.6 Motion-$\mathcal{R}$ of Spatial Representations

We show now concrete examples of beaded story-based representations (Motion-$\mathcal{R}$) obtained from the qualitative spatial representations RCC and OPRA$_1$. In other words, we expand the Stories-RCC and Stories-OPRA$_1$ representations of Section 8.2 into the representations Motion-RCC and Motion-OPRA$_1$.

8.6.1 Motion-RCC

Here, we apply the method for generating beaded motion categorizations to the spatial categorization RCC; thus, we obtain '*Motion-RCC*'.

1. We have a spatial representation $\mathcal{R}$ = RCC, which provides a map δ that relates each motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ with a spatial relation R_j.

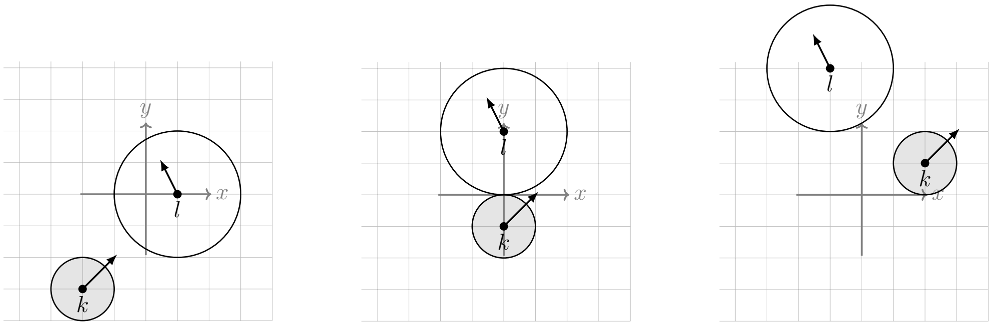
2. We already obtained the RCC stories and the story map σ_{RCC} in Section 8.2.1. $\Sigma_{\text{RCC}} = \Sigma_0 \cup \Sigma_1$. Σ_0 are the *rigid stories* and Σ_1 are the *non-rigid* stories.

$$\Sigma_0 = \{\underset{S_{01}}{(\mathrm{DC})}, \underset{S_{02}}{(\mathrm{EC})}, \underset{S_{03}}{(\mathrm{PO})}, \underset{S_{04}}{(\mathrm{TPP})}, \underset{S_{05}}{(\mathrm{NTPP})}, \underset{S_{06}}{(\mathrm{TPPI})}, \underset{S_{07}}{(\mathrm{NTPPI})}, \underset{S_{08}}{(\mathrm{EQ})}\}$$

$$\begin{aligned}\Sigma_1 = \{&\underset{S_{11}}{(\mathrm{DC})}, \underset{S_{12}}{(\mathrm{DC,EC,DC})}, \underset{S_{13}}{(\mathrm{DC,EC,PO,EC,DC})}, \underset{S_{14}}{(\mathrm{DC,EC,PO,TPP,PO,EC,DC})},\\ &\underset{S_{15}}{(\mathrm{DC,EC,PO,TPP,NTPP,TPP,PO,EC,DC})}, \underset{S_{16}}{(\mathrm{DC,EC,PO,TPPI,PO,EC,DC})},\\ &\underset{S_{17}}{(\mathrm{DC,EC,PO,TPPI,NTPPI,TPPI,PO,EC,DC})}, \underset{S_{18}}{(\mathrm{DC,EC,PO,EQ,PO,EC,DC})}\}\end{aligned} \tag{8.19}$$

3. The categorization function, f_μ, is the Cartesian product of the RCC story map, σ_{RCC} (Eqs. (8.2) to (8.5)), and the RCC spatial map, δ_{RCC} (Eq. (8.1)): $\mathrm{f}_\mu := \sigma_{\text{RCC}} \times \delta_{\text{RCC}}$

4. The categories set is obtained by expanding the stories according to the elements of their



(a) RCC: DC
Stories-RCC: S_{12}
Motion-RCC: $S_{12}(\mathrm{DC}_-)$

(b) RCC: EC
Stories-RCC: S_{12}
Motion-RCC: $S_{12}(\mathrm{EC})$

(c) RCC: DC
Stories-RCC: S_{12}
Motion-RCC: $S_{12}(\mathrm{DC}_+)$

Figure 8.8: RCC story $S_{12} = (\mathrm{DC}, \mathrm{EC}, \mathrm{DC})$. Each relation in the story's sequence is illustrated with a motion scenario. In each motion scenario, the corresponding relations, i.e., categories, of three qualitative representations, i.e., categorizations, (RCC, Stories-RCC, Motion-RCC) is shown.

sequences as follows:

$$
\begin{aligned}
&\textbf{Motion-RCC} = \{\\
&S_{01}(\mathrm{DC}); S_{02}(\mathrm{EC}); S_{03}(\mathrm{PO}); S_{04}(\mathrm{TPP}); S_{05}(\mathrm{NTPP}); S_{06}(\mathrm{TPPI}); S_{07}(\mathrm{NTPPI}); S_{08}(\mathrm{EQ});\\
&S_{11}(\mathrm{DC}); S_{12}(\mathrm{DC}_-), S_{12}(\mathrm{EC}), S_{12}(\mathrm{DC}_+); S_{13}(\mathrm{DC}_-), S_{13}(\mathrm{EC}_-), S_{13}(\mathrm{PO}), S_{13}(\mathrm{EC}_+), S_{13}(\mathrm{DC}_+);\\
&S_{14}(\mathrm{DC}_-), S_{14}(\mathrm{EC}_-), S_{14}(\mathrm{PO}_-), S_{14}(\mathrm{TPP}), S_{14}(\mathrm{PO}_+), S_{14}(\mathrm{EC}_+), S_{14}(\mathrm{DC}_+);\\
&S_{15}(\mathrm{DC}_-), S_{15}(\mathrm{EC}_-), S_{15}(\mathrm{PO}_-), S_{15}(\mathrm{TPP}_-), S_{15}(\mathrm{NTPP}), S_{15}(\mathrm{TPP}_+),\\
&S_{15}(\mathrm{PO}_+), S_{15}(\mathrm{EC}_+), S_{15}(\mathrm{DC}_+); S_{16}(\mathrm{DC}_-), S_{16}(\mathrm{EC}_-), S_{16}(\mathrm{PO}_-), S_{16}(\mathrm{TPPI}),\\
&S_{16}(\mathrm{PO}_+), S_{16}(\mathrm{EC}_+), S_{16}(\mathrm{DC}_+); S_{17}(\mathrm{DC}_-), S_{17}(\mathrm{EC}_-), S_{17}(\mathrm{PO}_-), S_{17}(\mathrm{TPPI}_-), S_{17}(\mathrm{NTPPI}),\\
&S_{17}(\mathrm{TPPI}_+), S_{17}(\mathrm{PO}_+), S_{17}(\mathrm{EC}_+), S_{17}(\mathrm{DC}_+); S_{18}(\mathrm{DC}_-), S_{18}(\mathrm{EC}_-), S_{18}(\mathrm{PO}_-), S_{18}(\mathrm{EQ}),\\
&S_{18}(\mathrm{PO}_+), S_{18}(\mathrm{EC}_+), S_{18}(\mathrm{DC}_+)\}
\end{aligned}
\tag{8.20}
$$

For example, the relation $S_{12}(\mathrm{EC})$ indicates that the entities are moving in the story S_{12} at the moment of tangency, i.e., EC (Fig. 8.8b). If the spatial relation appears multiple times in the story, such as DC in S_{12} (Fig. 8.8), we distinguish between each appearance: we distinguish chronologically adding '$-$' for the earlier appearance and '$+$' for the latter. $S_{12}(\mathrm{DC}_-)$ is the first DC (Fig. 8.8a), and $S_{13}(\mathrm{DC}_+)$, the last DC (Fig. 8.8c). In this work, we have only observed for the representation RCC this effect that relations repeat in the same story.

The total number of Motion-RCC stories is 16—as in Stories-RCC (Tab. 8.1). The total number of Motion-RCC categories is 56, from which 8 are rigid categories. Therefore, we have 48 non-rigid categories: 9 are independent of the entities' relative size; in 16 relations the first entity is smaller, $r_k < r_l$; in 16 relations the first entity is larger, $r_k > r_l$; in 7 relations both entities are equally large, $r_k = r_l$.

8.6.2 Motion-OPRA$_1$

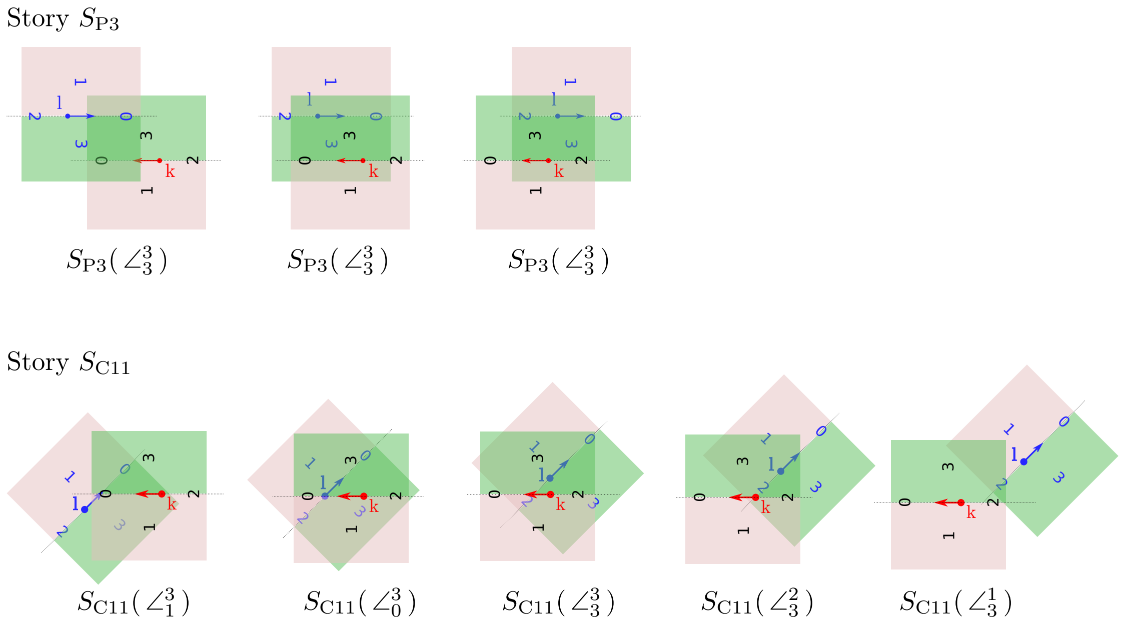
Here, we apply the method for generating beaded motion categorizations to the spatial categorization OPRA$_1$; so, we obtain '*Motion-OPRA$_1$*'.

1. We have a spatial representation $\mathcal{R}$ = OPRA$_1$, which provides a map δ that relates each motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ with a spatial relation R_j.
2. We already obtained the OPRA$_1$ stories and the story map in Section 8.2.2. $\Sigma_{\mathrm{OPRA}_1} = \Sigma_C \cup \Sigma_B \cup \Sigma_T \cup \Sigma_P \cup \Sigma_E \cup \Sigma_R$. The *rigid stories* are $\Sigma_E \cup \Sigma_R$; the rest are *non-rigid*.
3. The categorization function, f_μ, is the Cartesian product of the OPRA$_1$ story map, σ_{OPRA_1} (Fig. 8.5), and the OPRA$_1$ spatial map, δ_{OPRA_1} (Eqs. (8.6) and (8.7)): $\mathrm{f}_\mu \coloneqq \sigma_{\mathrm{OPRA}_1} \times \delta_{\mathrm{OPRA}_1}$
4. The categories set is obtained by expanding the stories according to the elements of their sequences as follows:

$$\begin{aligned}
&\mathbf{Motion\text{-}OPRA_1} = \{S_{\mathrm{C1\text{-}1}}(\angle^{3}_{1}), S_{\mathrm{C1\text{-}1}}(\angle^{0}_{1}), S_{\mathrm{C1\text{-}1}}(\angle^{1}_{1}), S_{\mathrm{C1\text{-}1}}(\angle^{1}_{2}), S_{\mathrm{C1\text{-}1}}(\angle^{1}_{3});\\
&S_{\mathrm{C2\text{-}1}}(\angle^{1}_{3}), S_{\mathrm{C2\text{-}1}}(\angle^{0}_{3}), S_{\mathrm{C2\text{-}1}}(\angle^{3}_{3}), S_{\mathrm{C2\text{-}1}}(\angle^{3}_{2}), S_{\mathrm{C2\text{-}1}}(\angle^{3}_{1});\\
&S_{\mathrm{C10}}(\angle^{3}_{1}), S_{\mathrm{C10}}(\angle^{3}), S_{\mathrm{C10}}(\angle^{1}_{3}); S_{\mathrm{C20}}(\angle^{1}_{3}), S_{\mathrm{C20}}(\angle^{1}), S_{\mathrm{C20}}(\angle^{3}_{1});\\
&S_{\mathrm{C11}}(\angle^{3}_{1}), S_{\mathrm{C11}}(\angle^{3}_{0}), S_{\mathrm{C11}}(\angle^{3}_{3}), S_{\mathrm{C11}}(\angle^{2}_{3}), S_{\mathrm{C11}}(\angle^{1}_{3});\\
&S_{\mathrm{C21}}(\angle^{1}_{3}), S_{\mathrm{C21}}(\angle^{1}_{0}), S_{\mathrm{C21}}(\angle^{1}_{1}), S_{\mathrm{C21}}(\angle^{2}_{1}), S_{\mathrm{C21}}(\angle^{3}_{1});\\
&S_{\mathrm{B1\text{-}1}}(\angle^{3}_{1}), S_{\mathrm{B1\text{-}1}}(\angle^{0}_{1}), S_{\mathrm{B1\text{-}1}}(\angle^{1}_{1}); S_{\mathrm{B2\text{-}1}}(\angle^{1}_{3}), S_{\mathrm{B2\text{-}1}}(\angle^{0}_{3}), S_{\mathrm{B2\text{-}1}}(\angle^{3}_{3});\\
&S_{\mathrm{B10}}(\angle^{3}_{0}), S_{\mathrm{B10}}(\angle^{3}), S_{\mathrm{B10}}(\angle^{1}_{2}); S_{\mathrm{B20}}(\angle^{1}_{0}), S_{\mathrm{B20}}(\angle^{1}), S_{\mathrm{B20}}(\angle^{3}_{2});\\
&S_{\mathrm{B11}}(\angle^{3}_{3}), S_{\mathrm{B11}}(\angle^{2}_{3}), S_{\mathrm{B11}}(\angle^{1}_{3}); S_{\mathrm{B21}}(\angle^{1}_{1}), S_{\mathrm{B21}}(\angle^{2}_{1}), S_{\mathrm{B21}}(\angle^{3}_{1});\\
&S_{\mathrm{B3\text{-}1}}(\angle^{1}_{1}), S_{\mathrm{B3\text{-}1}}(\angle^{1}_{2}), S_{\mathrm{B3\text{-}1}}(\angle^{1}_{3}); S_{\mathrm{B4\text{-}1}}(\angle^{3}_{3}), S_{\mathrm{B4\text{-}1}}(\angle^{3}_{2}), S_{\mathrm{B4\text{-}1}}(\angle^{3}_{1});\\
&S_{\mathrm{B30}}(\angle^{0}_{1}), S_{\mathrm{B30}}(\angle^{3}), S_{\mathrm{B30}}(\angle^{2}_{3}); S_{\mathrm{B40}}(\angle^{0}_{3}), S_{\mathrm{B40}}(\angle^{1}), S_{\mathrm{B40}}(\angle^{2}_{1});\\
&S_{\mathrm{B31}}(\angle^{3}_{1}), S_{\mathrm{B31}}(\angle^{3}_{0}), S_{\mathrm{B31}}(\angle^{3}_{3}); S_{\mathrm{B41}}(\angle^{1}_{3}), S_{\mathrm{B41}}(\angle^{1}_{0}), S_{\mathrm{B41}}(\angle^{1}_{1});\\
&S_{\mathrm{T\text{-}1}}(\angle^{2}_{0}), S_{\mathrm{T\text{-}1}}(\angle^{0}), S_{\mathrm{T\text{-}1}}(\angle^{0}_{2}); S_{\mathrm{T0}}(\angle^{0}_{0}), S_{\mathrm{T0}}(\angle^{2}), S_{\mathrm{T0}}(\angle^{2}_{2});\\
&S_{\mathrm{T1}}(\angle^{0}_{2}), S_{\mathrm{T1}}(\angle^{0}), S_{\mathrm{T1}}(\angle^{2}_{0}); S_{\mathrm{P2}}(\angle^{3}_{1}); S_{\mathrm{P\text{-}2}}(\angle^{1}_{3}); S_{\mathrm{P3}}(\angle^{3}_{3}); S_{\mathrm{P1}}(\angle^{1}_{1});\\
&S_{\mathrm{E\text{-}2}}(\angle^{2}_{0}); S_{\mathrm{E\text{-}1}}(\angle^{1}_{3}); S_{\mathrm{E0}}(\angle^{0}); S_{\mathrm{E1}}(\angle^{3}_{1}); S_{\mathrm{E2}}(\angle^{0}_{2});\\
&S_{\mathrm{R00}}(\angle^{0}_{0}); S_{\mathrm{R10}}(\angle^{0}_{1}); S_{\mathrm{R20}}(\angle^{0}_{2}); S_{\mathrm{R30}}(\angle^{0}_{3}); S_{\mathrm{R01}}(\angle^{1}_{0}); S_{\mathrm{R11}}(\angle^{1}_{1}); S_{\mathrm{R21}}(\angle^{1}_{2});\\
&S_{\mathrm{R31}}(\angle^{1}_{3}); S_{\mathrm{R12}}(\angle^{2}_{0}); S_{\mathrm{R12}}(\angle^{2}_{1}); S_{\mathrm{R22}}(\angle^{2}_{2}); S_{\mathrm{R32}}(\angle^{2}_{3}); S_{\mathrm{R03}}(\angle^{3}_{0}); S_{\mathrm{R13}}(\angle^{3}_{1});\\
&S_{\mathrm{R23}}(\angle^{3}_{2}); S_{\mathrm{R33}}(\angle^{3}_{3}); S_{\mathrm{R0}}(\angle^{0}); S_{\mathrm{R1}}(\angle^{1}); S_{\mathrm{R2}}(\angle^{2}); S_{\mathrm{R3}}(\angle^{3})\}
\end{aligned} \tag{8.21}$$

The total number of Motion-OPRA$_1$ stories 50. The total number of Motion-OPRA$_1$ relations is 100.

In Figure 8.9 we show examples of stories, S_{P3} and S_{C11}, and corresponding motion categories of Motion-OPRA$_1$; for example, $S_{\mathrm{P3}}(\angle^{3}_{3})$ and $S_{\mathrm{C11}}(\angle^{3}_{3})$.



Story S_{P3}

$S_{\mathrm{P3}}(\angle_3^3)$ $S_{\mathrm{P3}}(\angle_3^3)$ $S_{\mathrm{P3}}(\angle_3^3)$

Story S_{C11}

$S_{\mathrm{C11}}(\angle_1^3)$ $S_{\mathrm{C11}}(\angle_0^3)$ $S_{\mathrm{C11}}(\angle_3^3)$ $S_{\mathrm{C11}}(\angle_3^2)$ $S_{\mathrm{C11}}(\angle_3^1)$

Figure 8.9: OPRA$_1$ stories $S_{\mathrm{P3}} = (\angle_3^3)$ and $S_{\mathrm{C11}} = (\angle_1^3, \angle_0^3, \angle_3^3, \angle_3^2, \angle_3^1)$. For each scenario, we show the corresponding Motion-OPRA$_1$ category, i.e., relation, $S_i(R_j)$. Such category consists of the OPRA$_1$ story S_i (e.g., S_{C11}) and its OPRA$_1$ relation R_j (e.g., $\angle_3^2$).

Source: Purcalla Arrufi and Kirsch (2018a)

8.7 Motion-$\mathcal{R}$ of Motion Representations

We show now concrete examples of beaded story-based representations obtained from the qualitative representation of motion $\mathrm{QTC}_{\mathrm{B21}}$. In other words, we expand the bare story-based representation Stories-$\mathrm{QTC}_{\mathrm{B21}}$ in Section 8.3 into the beaded story-based representation Motion-$\mathrm{QTC}_{\mathrm{B21}}$.

8.7.1 Motion-$\mathrm{QTC}_{\mathrm{B21}}$

The very same method we applied for Motion-RCC and Motion-OPRA_1 in Sections 8.6.1 and 8.6.2 can be applied to Stories-$\mathrm{QTC}_{\mathrm{B21}}$, and, thus, we obtain the beaded motion categorization Motion-$\mathrm{QTC}_{\mathrm{B21}}$. The categorization function of Motion-$\mathrm{QTC}_{\mathrm{B21}}$, f_μ, is the Cartesian product of the $\mathrm{QTC}_{\mathrm{B21}}$ story map, $\sigma_{\mathrm{QTC}_{\mathrm{B21}}}$, and the categorization rule of $\mathrm{QTC}_{\mathrm{B21}}$, which we will call $\rho_{\mathrm{QTC}_{\mathrm{B21}}}$ (Eq. (6.10)). Accordingly, we obtain the Motion-$\mathrm{QTC}_{\mathrm{B21}}$ relations. For example, $S_{1\text{-}1}(-,0)$, $S_{1\text{-}1}(0,0)$, $S_{1\text{-}1}(+,0)$, $S_{3\text{-}1}(-,-)$, $S_{3\text{-}1}(-,0)$, $S_{3\text{-}1}(+,-)$, ...

Although we will not examine Motion-$\mathrm{QTC}_{\mathrm{B21}}$ in detail, it is important for us to mention this beaded motion categorization because Motion-$\mathrm{QTC}_{\mathrm{B21}}$ is generated from a motion categorization, namely, $\mathrm{QTC}_{\mathrm{B21}}$. In that way, we can substantiate our statement: we can both use spatial and motion categorizations to create story-based categorizations with the same methods.

8.8 Formalization and Featural Variables

Here, we describe a Motion-$\mathcal{R}$ categorization using the formalism we defined in Chapter 6. Namely, we break down its *categorization rule*, f_μ, into the *feature extraction function*, Φ, and the *featural categorization function* f_Φ, i.e., $\mathrm{f}_\mu = \mathrm{f}_\Phi \circ \Phi$. We express f_Φ and Φ using the featural functions in the component categorizations, Stories-$\mathcal{R}$ and $\mathcal{R}$, because Motion-$\mathcal{R}$ is the Cartesian product of such two categorizations.

The categorization function of Stories-$\mathcal{R}$ is the story map $\sigma_{\mathcal{R}} = \sigma_{\Phi_{\mathcal{R}}} \circ \Phi_{\sigma_{\mathcal{R}}}$; the categorization function of $\mathcal{R}$ is the qualitative map $\rho = \mathrm{f}_{\Phi_\rho} \circ \Phi_\rho$—In both cases we expressed the categorization rule by means of the feature extraction and the featural categorization rule. Since Motion-$\mathcal{R}$ = Stories-$\mathcal{R} \times \mathcal{R}$, we have that the categorization rules are also concatenated $\mathrm{f}_\mu = \sigma_{\mathcal{R}} \times \rho$. In Equation (8.22), we combine the previous formulae, and use a basic property of composing Cartesian product of functions.

$$\mathrm{f}_\mu = \sigma_{\mathcal{R}} \times \rho = (\sigma_{\Phi_{\mathcal{R}}} \circ \Phi_{\sigma_{\mathcal{R}}}) \times (\mathrm{f}_{\Phi_\rho} \circ \Phi_\rho) = (\sigma_{\Phi_{\mathcal{R}}} \times \mathrm{f}_{\Phi_\rho}) \circ (\Phi_{\sigma_{\mathcal{R}}} \times \Phi_\rho) \tag{8.22}$$

Now comparing the result of Equation (8.22) and our formalization, $\mathrm{f}_\mu = \mathrm{f}_\Phi \circ \Phi$, we can isolate the formulae for the *feature extraction function* Φ (Eq. (8.23a)), and the *featural categorization function* f_Φ (Eq. (8.23b)). We have thus obtained that each function, Φ and f_Φ, is the Cartesian product of the functions of the components, i.e., Stories-$\mathcal{R}$ and $\mathcal{R}$.

$$\Phi = \Phi_{\sigma_{\mathcal{R}}} \times \Phi_\rho \tag{8.23a}$$

$$\mathrm{f}_\Phi = \sigma_{\Phi_{\mathcal{R}}} \times \mathrm{f}_{\Phi_\rho} \tag{8.23b}$$

The Cartesian product in the feature extraction function Φ (Eq. (8.23a)) tells that the featural variables of a Motion-$\mathcal{R}$ categorization are the union of the featural variables of Stories-$\mathcal{R}$ and $\mathcal{R}$, i.e., $\mathcal{F}_{\text{Motion-}\mathcal{R}} = \mathcal{F}_{\text{Stories-}\mathcal{R}} \cup \mathcal{F}_{\mathcal{R}}$.

A. Motion-RCC Formalization

Motion-RCC is the concatenation of Stories-RCC and RCC. Concerning Stories-RCC, we thoroughly presented $\Phi_{\sigma_{\mathrm{RCC}}}$ and $\sigma_{\Phi_{\mathrm{RCC}}}$ in Section 8.2.1 (Eqs. (8.3a) to (8.5))—The featural variables are $d_{\min}$ and $\mathrm{dif}_{\mathrm{V}}$.

$$\mathrm{d}_{\min}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \|\vec{x}_l - \vec{x}_k\| |\det(\vec{x}_l - \vec{x}_k, \vec{v}_l - \vec{v}_k)| \tag{8.3a}$$

$$\mathrm{dif}_{\mathrm{V}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \frac{\|\vec{v}_l - \vec{v}_k\|}{\|\vec{v}_k\| + \|\vec{v}_l\|} \tag{8.3b}$$

Concerning RCC, we presented δ_{RCC}—the original spatial map of RCC—in Section 8.2.1. There, we noted that its only featural variable was the distance between entities $\mathrm{d}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \|\vec{x}_k - \vec{x}_l\|$. Even so, the featural categorization function of RCC used here (Eq. (8.25)) differs slightly from the original one (Eq. (8.1)), because in Stories-RCC each spatial categorization has a sign indicating its position in the story (Eq. (8.25b)). For instance, DC appears twice in the story S_{12}, and, hence, we add signs, DC_- or DC_+, to distinguish both appearances, $S_{12} = \{\mathrm{DC}_-, \mathrm{EC}, \mathrm{DC}_+\}$ (see item 4, Sect. 8.2.1)

Concluding, for the spatial representation RCC, we have following feature extraction function, Φ_{RCC}, and featural categorization function $\mathrm{f}_{\Phi_{\mathrm{RCC}}}$.

$$\Phi_{\mathrm{RCC}}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \mathrm{d}(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) = \|\vec{x}_l - \vec{x}_k\| \tag{8.24}$$

$$\mathrm{f}_{\Phi_{\mathrm{RCC}}}(d) := \begin{cases} \mathrm{DC}_* & \text{if } d > d_2 \\ \mathrm{EC}_* & \text{if } d = d_2 \\ \mathrm{PO}_* & \text{if } d_2 > d > d_4 \\ \left.\begin{array}{ll} \mathrm{TPP}_* & \text{if } d = d_4 \\ \mathrm{NTPP}_* & \text{if } d_4 > d \end{array}\right\} & \text{if } r_k \leq r_l \\ \left.\begin{array}{ll} \mathrm{TPPI}_* & \text{if } d = d_4 \\ \mathrm{NTPPI}_* & \text{if } d_4 > d \end{array}\right\} & \text{if } r_k > r_l \end{cases} \tag{8.25a}$$

$$* := \begin{cases} + & \text{if } d > d_{\min} \text{ for the whole spatial relation} \\ - & \text{if } d < d_{\min} \text{ for the whole spatial relation} \\ \varnothing & \text{otherwise} \end{cases} \tag{8.25b}$$

$$d_2 = |r_k + r_l|, \text{distance at spatial relation EC}$$
$$d_4 = |r_k - r_l|, \text{distance at spatial relation TPP}$$

Featural variables From the analysis above, we see that Motion-RCC has three featural variables: $d_{\min}$, $\mathrm{dif}_{\mathrm{V}}$, and d—The additional parameters, such as d_2, d_4, r_k, and r_l, are all scenario-independent. Since the featural space is three-dimensional, we amply verify our claim that $\dim(\mathcal{F}) < \dim(\mathcal{K})$ (Sect. 6.1.2); in fact, the featural space is dimensionally low enough that it can be graphically visualized.

B. Motion-OPRA$_1$ Formalization

Motion-OPRA$_1$ is the concatenation of Stories-OPRA$_1$ and OPRA$_1$. Concerning Stories-OPRA$_1$, we presented non-rigid Φ_σ and σ_Φ in Section 8.2.2 (Fig. 8.5); the corresponding featural variables are α_{vv}, $\alpha_{\Delta x \Delta v}$, u_k, and u_l. Concerning OPRA$_1$, we presented δ_{OPRA_1}—the spatial map of OPRA$_1$—in Section 8.2.2 (Eqs. (8.6) and (8.7)); the corresponding categorical variables are $\alpha_{\Delta x v_k}$, $\alpha_{v_l \Delta x}$, and $\mathrm{dif}_{\mathrm{X}}$.

Featural variables Apparently, the total number of featural variables in Motion-OPRA$_1$ is seven (four in Stories-OPRA$_1$ plus three in OPRA$_1$), but we observe that the variable α_{vv} is combination of $\alpha_{v_l \Delta x}$ and $\alpha_{\Delta x v_k}$ (as we proved in Section 8.2.2). Therefore, since the variable α_{vv} is not independent, we have a total number of six featural variables, i.e., $\dim(\mathcal{F}_{\text{Stories-OPRA}_1}) = 6$.

Part IV

Validating Story-Based Categorizations

Chapter 9

Experimental Evidence of Story-Based Categorization

> *The fact that it is not possible to uniquely determine cognitive structures and processes poses a clear limitation on our ability to understand the nature of human intelligence. The realization of this fact has also led to a shift in my personal goals. I am less interested in defending the exact assumptions of the theory and am more interested in evolving some theory that can account for important empirical phenomena.*
>
> J. R. Anderson (1976)

After we criticized in Section 2.4 how scanty is the experimental verification of cognitive plausibility in spatial representations, one may expect that we experimentally verify the story-based categorizations for cognitive plausibility—That is the purpose of this chapter.

This chapter bases on research we did in collaboration with Frank Papenmeier (Original paper, Papenmeier, Purcalla Arrufi, and Kirsch 2023).

9.1 About Cognitive Plausibility

We say that a certain cognitive model is '*cognitively plausible*' when it appropriately describes human cognition, concretely, human knowledge representation or reasoning (Strube 1992). This is a key concept in science: For some researchers, the very goal of Artificial Intelligence (AI) is to create machines that work in a *cognitively plausible* manner, i.e., to create machines that think like humans (See, Sweeney 2003). However, as Russell and Norvig (2014b, Sect. 1.2) advise, it is much more effective that AI concentrates on creating machines that think—and act—*rationally*. Creating *cognitive plausible* machines should be the endeavour of '*cognitive modelling*'. Hence, as we verify experimentally the story-based categorizations, we look at them from a cognitive modelling perspective.

The concept 'cognitive plausibility' has some equivalent terms: '*cognitive adequacy*' (Strube 1992; Knauff et al. 1995; Klippel et al. 2008), '*cognitive validity*' (Cohn and Renz 2008, Sect. 13.5), '*psychological validity*' (Knauff et al. 2004). From now on, we use the term 'cognitive plausibility' even when we refer to authors that use an equivalent term. Importantly, we remark that "cognitive plausibility" has a graded meaning; it expresses a broad range of nearness to human cognition: from weakly to strongly plausible (Strube 1992, p. 165).

Cognitive plausibility comprises two different but complementary aspects: *conceptual* and *inferential* cognitive plausibility (Knauff et al. 1995). '*Conceptual cognitive plausibility*' (shortened, '*conceptual plausibility*') enquires whether the categories (or classes) of a certain knowledge representation correspond to categories in human conceptual knowledge. For example, whether humans naturally categorize motion by means of the Stories-OPRA$_1$ categories (i.e., S_{C11}, $S_{\mathrm{C1\text{-}1}}$, S_{C21}, $\ldots$). '*Inferential cognitive plausibility*' (shortened, '*inferential plausibility*') enquires whether humans reason about qualitative relations similarly as qualitative reasoning does (Sects. 4.1.4 and 4.3.1); for instance, by using the *composition* operation (see example with story-based relations in Fig. 10.1)

In order to ascertain whether a qualitative representation is cognitively plausible we must resort to experimentation with humans—As M. Knauff (1999, p. 263) simply states: "The question whether an approach to [qualitative representations] can be claimed as cognitively [plausible] can be answered only on psychological experiments". In the following, we verify whether Stories-RCC and Stories-OPRA$_1$ are *conceptually* plausible by means of several experiments.

9.2 Experimental Principles

9.2.1 Pairwise Comparison

Our experimental method based on *pairwise* comparisons of stimuli. In each trial, we presented the subject with three stimuli: one was the '*reference stimulus*', and the other two were the '*comparison stimuli*'; then, we asked the subject to choose the comparison stimulus that was most similar to the reference stimulus (see trial set-ups in Fig. 9.1).

We chose a pairwise comparison task because of the nature of our stimuli: they are *motion* scenes—And this is a key difference with most categorization experiments, which use motionless stimuli. Indeed, motion limits the experimental methods we can use. For example, we find inappropriate to use a typical *free grouping* task, in which all the stimuli are presented at once and each subject groups them at will (e.g., Mast et al. 2014; Renz et al. 2000). If we present to the subjects many motion scenes at once, we might overstrain their cognitive and perceptive capacities—Not to mention that the simultaneous view of many motion scenes is dizzying (but see, Yang et al. 2015).

Another reason for pairwise comparison is that our stimuli are simultaneously categorized according to two different categorizations (Stories-RCC and Stories-OPRA$_1$). However, in a free grouping task, subjects tend to group according to just one categorization, namely, the most salient—Human laziness restrains subcategorization. Although in a *hierarchical free grouping* task (e.g., Burnett et al. 2005), it is possible to force the subjects to subcategorize, such grouping task is more demanding.

At a theoretical level, the pairwise comparison is underpinned by the work of Luce (1959) and Bradley and Terry (1952), the BTL model. By fitting the experimental data to this model, we can relate the frequency of choice of each stimuli pair to a single scale measure called '*utility ratio*'. This is useful because we can assimilate such utility ratio to the experimental similarity between each comparison stimulus and the reference stimulus.

Comparison Stimuli, Modified Stimuli, and Reference Stimulus In some trials, we also included the reference stimulus as a comparison stimulus; that is, sometimes, one comparison stimulus was identical to the reference stimulus. More concretely, we created two variants of experimental set-ups: in one variant (named 'a'), the comparison stimuli could only be '*modified stimuli*' (see later, Sect. 9.5), that is, the stimuli obtained by modifying the reference stimulus; in the other variant (named 'b'), one of the comparison stimuli could be the identical to the reference stimulus.

By including the reference stimuli in the comparison stimuli, we can verify whether the categorical effects are altered when the proportion of similar stimuli to the reference stimulus increases. Additionally, we can control more closely the experiment results, since the maximality axiom (item II., Sect. 3.4.1.A) holds for similarity: the similarity of a stimulus to itself is higher than to any other stimulus. For example, we can sort subjects out that do not predominantly choose the reference object as most similar to itself; we might conclude such subjects are not attentive to the stimuli or are performing the task carelessly.

9.2.2 Perception and Memory Tasks

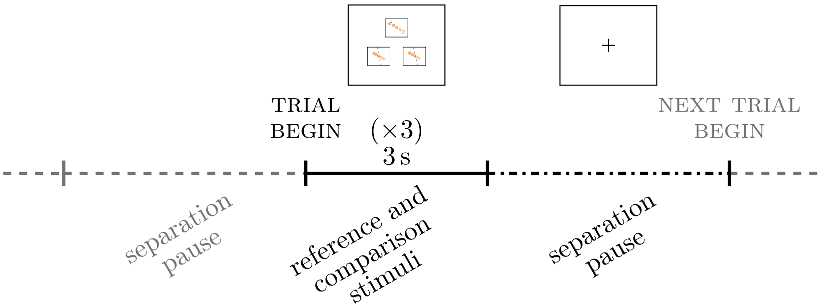
The most straightforward way to present the one reference stimulus and the two comparison stimuli is to display all three stimuli *at once*, as we do in our '*perception*' set-up (Fig. 9.1a). In such set-up, only perception is involved–there is no need to use memory. However, categorization is mostly advantageous when we store things in memory because, then, we resort to the powerful *cognitive economy* (Goldstone et al. 2012, p. 611; originally, Eleanor. Rosch 1978, p. 28). On that account, we created an additional experimental set-up, the '*memory*' set-up in which the subjects were forced to store the reference stimulus in short-term memory: we introduced a 1s pause between the display of the reference stimulus and the comparison stimuli (Fig. 9.1b).

We expect a different behaviour of the subjects between the perception and memory experiment. We know that the short-time memory has very limited storage resources for static situations (e.g., Cowan 2001; Luck and Vogel 1997); how much more limited, then, for dynamic situations (e.g., Papenmeier and Huff 2014). Therefore, we guess that, due to the cognitive load, the memory experiment might show a reduced differentiation of the categories; it might happen that only the most salient category be stored in memory, if any.

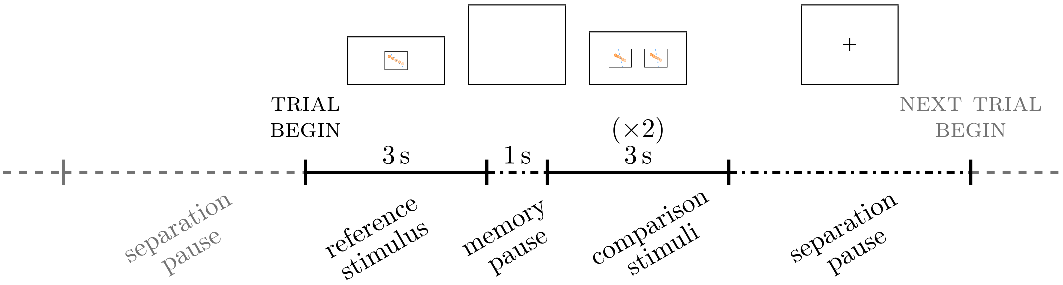
9.2.3 Experimental Hypothesis

We generated the *modified* stimuli, used as comparison stimuli, so that each had the same *feature-based* similarity to the *reference* stimulus; in other words, the modified stimuli were equidistant to the reference stimuli in the featural space. We achieved that by applying the same amount of *metric change* to the modified stimuli with respect to the reference stimulus. But we also generated them so that they had different Stories-RCC and Stories-OPRA$_1$ categories with respect to the *reference* stimulus by placing them at the other side of the category border (See later Sect. 9.3.3 for more detail). In that way, if the similarity is based *solely* on the featural values—i.e., if these story-based categorizations are irrelevant to human cognition—then, subjects should choose each modified stimulus as equally similar to the reference stimulus. Otherwise, if subjects perceive these story-based categorizations to be relevant, they should consistently choose the modified stimuli with the same Stories-RCC and Stories-OPRA$_1$ categories as the reference stimulus as most similar stimuli to the reference stimulus; conversely, subjects should choose the stimuli with different categories than the reference stimulus as less similar.

In short, if a categorization effect takes place, we shall observe a significant discrepancy between the *feature-based* similarities (which we set with the same metric value) and the *experimental* similarities obtained via fitting to the BTL model. Such discrepancy should manifest in that the experimental similarities of the modified stimuli to the reference stimulus differ amongst



(a) Trial set-up for checking categorical plausibility in *perception*.



(b) Trial set-up for checking categorical plausibility in *memory*.

Figure 9.1: We display the two trial set-ups. In the *perception* set-up (a), all three stimuli (the reference and the two comparison stimuli) are presented at once and shown up to three times (×3)—the reference stimulus is on top. In the *memory* set-up (b), the reference stimulus is presented first, isolated, and, after a 1s pause, the two comparison stimuli are presented up to two times (×2). In the separation pause between trials, we show a fixation cross.

each other (at least for a single pair of comparison stimuli). On the contrary, if these experimental similarities are practically equal—not significantly different from each other—then a categorization effect cannot be established.

Hence, our *null hypothesis* is that the experimental similarities are equal, and, consequently there is no story-based categorization effect in the subjects' choices of the stimuli. After a statistical analysis, we might choose to reject the hypothesis with a certain significance; in that case, we would state with the corresponding significance that story-based categorizations affect the subjects' choices.

9.2.4 Processing of Experimental Data

We had four experimental sets that differed from another; therefore, we processed the data of each experimental set independently of another (we describe the experimental sets below in Section 9.5). Nonetheless, the processing of experimental data was analogous for each experimental set, and we explain it in the following. We accompany the explanations with the Example 9.1: The example simulates the results of one subject's trial in the experimental set 1a (Sect. 9.5.1.A) and its processing in R.

Once the subjects had completed the experiment trials, we had a raw data matrix N_{ij} for each experimental set. Each matrix entry (i, j) contains the number of times the subjects chose stimulus i to be more similar than stimulus j to the reference stimulus r every time that the pair of comparison stimuli $\{i, j\}$ was presented. The total number of comparison stimuli, and, thus, the matrix dimensions, varied amongst experimental sets.

For each experimental set, we had more than one reference stimulus (a total of 16) with its corresponding modified stimuli (4 per reference stimulus). That is, we generated '*stimuli sets*' formed by four modified stimuli and a single reference stimulus; and we compared stimuli only within the *stimuli sets* (Sect. 9.3.3). However, the results of all 16 stimuli sets were added up into the matrix N_{ij} for each experimental set because the stimuli sets were equivalently generated regarding their categorizations.

Thanks to an R statistical package for fitting and testing the BTL model (Wickelmaier 2020), we could transform the raw data matrix N_{ij} of choices into the utility ratio u_i for every stimulus i. The utility ratios of the BTL model explain the probability of choosing a stimulus i from a certain stimuli set $A = \{i, j, \ldots, z\}$ as following

$$P(i, A) = \frac{u_i}{\sum_{a \in A} u_a} \tag{9.1}$$

The utility ratios u_* in Equation (9.1) can be assimilated to a similarity according to the Shepard's (1957) and Luce's (1963) stimulus-response choice model—compare Equation (9.1) with Equation (3.8) which models the probability of choosing a category for a certain item. We assimilated, thus, the ratio u_i to the experimental similarity of the comparison stimulus i to the reference stimulus r, i.e., $\mathrm{S}(i, r)_{exp} \equiv u_i$. Accordingly, we call u the values, '*experimental similarities*' or '*similarity ratios*'. The u values are determined up to a factor, so we normalized them in each experimental set: we set the same stimulus across experimental sets (the stimulus A) to have ratio value 1. In that way, we eased comparing the u values between experimental sets.

Finally, we obtained the confidence intervals of the experimental similarities—i.e., the confidence intervals of the u values—and so we could examine which similarities differed from each other in order to reject (or not) the null hypothesis (Sect. 9.2.3): whether the stimuli are equally similar or not.

A. Statistical significance

In our experiment, we perform two steps in which we had to assess the statistical significance. Firstly, we fitted the experimental choices' frequency N^r_{ij} to the similarity ratios u through the BTL model; thus, we had to asses the goodness of such fitting. Second, we examine whether the ratios u are statistically different in order to test our null hypothesis that the subjects' choices show no story-based categorical effect.

The goodness of the BTL fitting can be tested by means of a *non-significant* goodness of fit χ^2 test, which is provided by the same R package. A non-significant goodness of fit test works oppositely than a significant goodness of fit test. In a non-significant goodness of fit test, we accept the fitting more confidently the higher χ^2 is. Therefore, we would reject the BTL model fitting if the χ^2 value is lower than a certain probability α, i.e., if $\chi^2 < \alpha$.

Wickelmaier and Schmid (2004) suggest a value higher than $\alpha = 0.10$ to accept the BTL fitting. Nevertheless, we choose a typically reasonable value of $\alpha = 0.05$ because our goal is not to proof that the BTL model is a *very good* model to fit the results of our experiment, but rather that it is a *good enough* model so that we can use its parameters (i.e., the similarity ratios) for further data analysis.

Secondly, we test the null hypothesis that the similarity ratios u are equal, and, thus, that the subjects' similarity choices show no categorization effect. As said above, we can examine whether u are equal by comparing their 95% confidence intervals (e.g., Fig. 9.7). Nonetheless, we chose a more solid option: we apply a standard χ^2-test of significance to reject the null hypothesis with a usual rejection rule of $\chi^2 < 0.05$. If the null hypothesis were rejected, we would ascertain that there is a story-based categorization effect on the subjects' similarity choices.

Example 9.1 The matrix `choicemat` simulates the stimuli choices of a subject in Experiment 1a.

$$\texttt{choicemat} = \begin{bmatrix} 0 & 27 & 20 & 30 \\ 5 & 0 & 8 & 25 \\ 12 & 24 & 0 & 29 \\ 2 & 7 & 3 & 0 \end{bmatrix} \tag{9.2}$$

The total number of trials that the subject processed is $\sum_{\texttt{i,j}} \texttt{choicemat(i,j)} = \texttt{4} * \texttt{32} = \texttt{192}$.

Note that $\texttt{choicemat(i,j)} + \texttt{choicemat(j,i)} = \texttt{32} \quad \forall \texttt{i,j}$: Each pair of stimuli (i, j) is presented 32 times to this subject. The entries of the matrix show how many times a stimulus of the pair (i, j) was chosen over the other. For instance, $\texttt{choicemat}(1, 3)$ tell us that the stimulus *1* was chosen 20 times over the stimulus *3*, which means that the stimulus *3* was chosen 12 times over the stimulus *1*, i.e., $\texttt{choicemat}(3, 1) = 12$.

We fit the stimuli choices matrix, `choicemat`, according to the BTL model using the `eba` R package. Accordingly, we obtain the *similarity* (or *utility*) *ratios*, and the *non-significant* goodness of fit.

```
< btl_choicemat <- eba(choicemat)
< btl_choicemat
Elimination by aspects (EBA) models
Parameter estimates:
1          2          3          4
0.45443    0.09024    0.26964    0.02679
Goodness of fit (-2 log likelihood ratio):
    G2(3) = 0.0726, p = 0.9949
```

Thus, the BTL model fits the choice matrix (Eq. (9.2)) very good according to the *non-significant* goodness of fit, $p = 0.9949 > 0.05$. And the non-normalized utility ratios fitted by the model are

$u = [0.454, 0.090, 0.270, 0.027]$. We normalize the stimuli, so that we set the value of one stimulus to 1, for instance, the fourth one, $u_4 = 1.0$. We obtain also the width of the 95% confidence interval for each ratio.

```
> uscale(btlchoicemat,norm=4)
[1] 16.960392  3.368034 10.063525  1.000000
> ci <- 1.96 * sqrt(diag(cov.u(btl1,norm=4)))
[1] 4.1969331 1.6495302 3.6191012 0.6576307
```

We represent the normalized ratios with their confidence intervals in Figure 9.2. At first glance, we can reject the null hypothesis that the ratios are equal, because no ratio u_* is in the confidence interval of the other. And, thus, in our example matrix `choicemat`, the stimuli are distinguished, i.e., they have different similarity values. More formally, the `eba` package tests against equality of utility ratios: In this case, we obtain $\chi^2(3) = 91.5$, $p < 0.001$, which is consistent with the observed intervals.

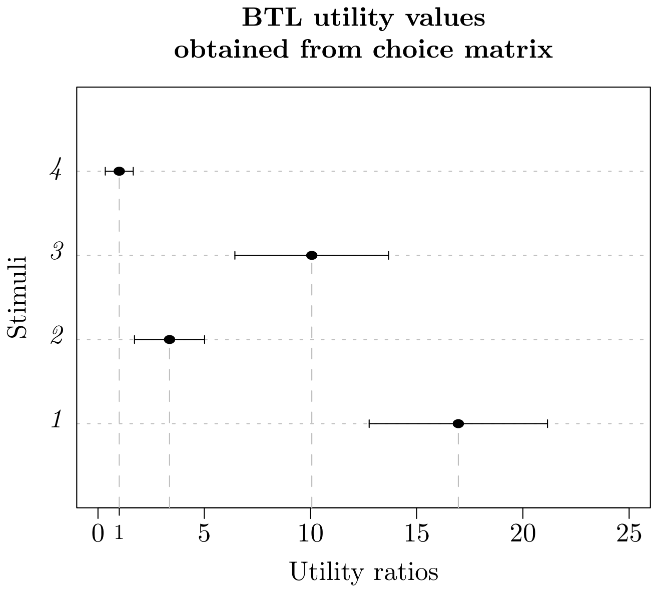


Figure 9.2: Utility ratios obtained by fitting the BTL model to the choice matrix `choicemat` (Eq. (9.2)); we show the corresponding 95% confidence intervals. The utility ratios are normalized so that the ratio of the fourth stimulus is 1.0.

9.3 Stimuli and Apparatus

We had following experimental environment. Each subject sat in a private space separated from other participants by visual shields so that he could not see the other subjects or their tablets; we tested 4 subjects simultaneously. Each subject sat in front of a Microsoft Surface Pro tablet at about 57 cm in which the stimuli were presented. The subject chose the stimuli by pressing the key '`f`' (left stimulus) or '`j`' (right stimulus) in the keyboard.

9.3.1 Stimuli Appearance and Notation

Our stimuli were motion scenes. They were displayed in a 370 pixels square with black outline; considering the resolution and distance to the tablet, the subjects saw each motion scene at a square of 7.6 degrees of visual angle. The motion scenes had a frame rate of 60 frames per second to ensure motion fluidity.

Each motion depicted two entities as two discs with different sizes and colours that moved at uniform velocity. The larger disc had 4 times the radius of the smaller disc. We decided k to be the smaller disc and l the larger, thus, in the experiment, we have $r_k < r_l$. Assigning the names k and l to the entities determines the stories in each motion scene according to the definitions in Chapter 8.

The larger disc was orangish coloured, surface RGB: 255, 119, 0, transparency 50%; edge RGB: same as the surface, transparency 25%. The smaller disc was blueish coloured, surface RGB: 3, 123, 252, transparency 20%; edge RGB: same as the surface, transparency 6%. The colours are chosen to be neutral to interpretation, and they are colour-blind distinguishable. Each disc had a black dot in the centre to ease the perception of its trajectory, and, thus, allow an easier determination of the Stories-OPRA$_1$ category.

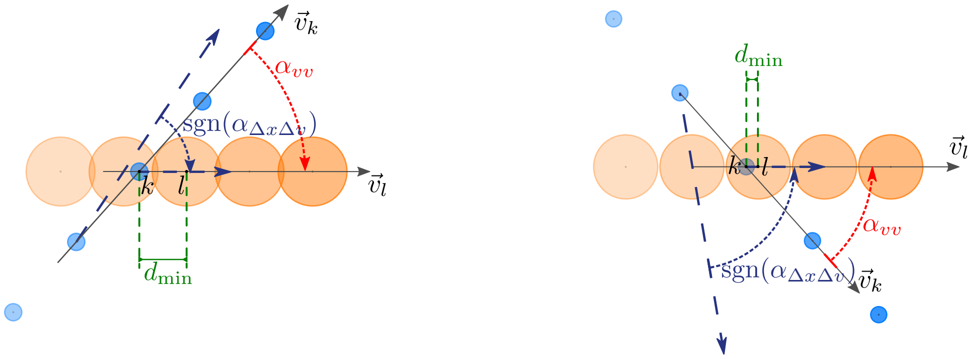
Each scene lasted 3 seconds, and we set the instant of minimum distance between discs in the middle of the motion scene, i.e., instant $t=1.5$s, in order to maximize similarity between scenes. We also determined each scene to contain all spatial relations of the stories represented; accordingly, all motion scenes began and ended with non-overlapping entities, ensuring a minimum separation distance between the entities borders, namely, at least twice the radius of the smaller disc, $2r_k$.

9.3.2 Stimuli Features

We modified our motion scenes according to three featural variables: $d_{\min}$, α_{vv}, and $\operatorname{sgn}(\alpha_{\Delta x \Delta v})$. The first variable, $d_{\min}$, is one of the two featural variables of the Stories-RCC categorization (Eq. (8.3)), and the only one we needed to set the RCC category of the motion scenes because we had no scene with *rigid stories*. By means of the other two featural variables, α_{vv}, and $\operatorname{sgn}(\alpha_{\Delta x \Delta v})$, we set the Stories-OPRA$_1$ category of the motion scenes.

In Figure 9.3 we visualize the featural variables in the motion scenes. We already explained such featural variables in Section 8.2; thus, here, we only deepen in the meaning of $\operatorname{sgn}(\alpha_{\Delta x \Delta v})$, which is the only feature with unobvious interpretation, even though its definition is plain $\operatorname{sgn}(\alpha_{\Delta x \Delta v}) = \operatorname{sgn}(\det(\Delta\vec{x}, \Delta\vec{v}))$. To visually compute the value of $\operatorname{sgn}(\alpha_{\Delta x \Delta v})$, we must set our reference frame on the entity b, and from that reference frame observe the motion of entity o: if we see o move to our right (clockwise around us) then $\operatorname{sgn}(\alpha_{\Delta x \Delta v}) = -1$, as in Fig. 9.3a; if we see o move to our left (counter-clockwise around us) then $\operatorname{sgn}(\alpha_{\Delta x \Delta v}) = +1$, as in Fig. 9.3b.

We concede, though, that $\operatorname{sgn}(\alpha_{\Delta x \Delta v})$ is not one of the featural variables of OPRA$_1$ (as we defined them in Figure 8.5) but $\alpha_{\Delta x \Delta v}$. We had defined $\alpha_{\Delta x \Delta v}$ for *motion scenarios*, but, sadly, we could not use it as featural variable for *motion scenes*. The problem is that its value is not constant during the whole scene and, therefore, the feature extraction function Φ cannot be defined for $\alpha_{\Delta x \Delta v}$; it would be ambiguous. Although the value of $\alpha_{\Delta x \Delta v}$ is not constant in a scene, its sign, i.e., $\operatorname{sgn}(\alpha_{\Delta x \Delta v})$, is. For that reason, we chose $\operatorname{sgn}(\alpha_{\Delta x \Delta v})$, as featural variable in the experiment, although it is a discrete variable, and in that sense does not fulfil the density property that we desired for a featural variable (Sect. 6.1.2). Nevertheless, we deemed $\operatorname{sgn}(\alpha_{\Delta x \Delta v})$ the best choice because it is derived from the original variable; moreover, it was constant for the stimuli in each trial.



(a) Motion scene categorized as a Stories-RCC S_{11} story ($d_{\min} > r_k + r_l$) and a Stories-OPRA$_1$ S_{C11} story ($\alpha_{vv} < 0$ and $\mathrm{sgn}(\alpha_{\Delta x \Delta v}) = -1$).

(b) Motion scene categorized as a Stories-RCC S_{15} story ($d_{\min} < |r_k - r_l|$) and a Stories-OPRA$_1$ S_{C21} story ($\alpha_{vv} > 0$ and $\mathrm{sgn}(\alpha_{\Delta x \Delta v}) = +1$).

Figure 9.3: Two motion scenes used in the experiment visualized here as a sequence of five snapshots—The snapshots opacity increases with time: the most transparent snapshot is the earliest and the most opaque is the latest. In each scene, we visualize the different featural parameters: $d_{\min}$ is the minimum distance between entities, α_{vv} is the angle between the entities velocities ($\vec{v}_k$ and $\vec{v}_l$), and $\mathrm{sgn}(\alpha_{\Delta x \Delta v})$ can be understood as how entity l moves from the viewpoint of entity k.

Source: Papenmeier, Purcalla Arrufi, and Kirsch (2023)

9.3.3 Stimuli Sets

According to our experimental principle (Sect. 9.2), in our trials we did not present a single stimulus, but pairs of stimuli (the *comparison stimuli*) that were compared with a third stimulus (the *reference stimulus*). The stimuli in a trial were generated by modifying the *reference stimulus*; hence, we decided to structure the stimuli generation in *stimuli sets* which were created from the reference stimulus.

Each '*stimuli set*' consisted of 5 stimuli: one '*reference stimulus*', called E, and four '*modified stimuli*', called A, B, C, and D. We created the modified stimuli by modifying the reference stimulus E in a symmetrical manner, that is, changing the featural parameters by the *same absolute amount*. In that way, the modified stimuli were equally similar to the reference stimulus, if we compare stimuli according to the featural similarity. Furthermore, we ensured that the modified stimuli presented all possible combinations of same–different Stories-RCC and Stories-OPRA$_1$ categories, as shown in Table 9.1. In sum, each stimuli set dealt with only two Stories-RCC categories, the Stories-RCC category of the reference stimulus, i.e., the '*reference category*', and the '*alternative category*'; analogously, it dealt with two Stories-OPRA$_1$ categories. Therefore, each *stimuli set* could be characterized by a tuple of the involved categories: *(Stories-RCC reference, Stories-OPRA$_1$ reference; Stories-RCC alternative, Stories-OPRA$_1$ alternative)*

For example, in Figure 9.4 we have the *reference categories* S_{13} and S_{C21}—the categories of the reference stimulus, E—and we have the alternative categories S_{11} and $S_{C1\text{-}1}$. We can see that the modified stimuli are symmetrically distributed around the reference stimulus because their featural parameters, α_{vv} and $d_{\min}$, are modified by the same amount. Moreover, the modified stimuli represent all possible combination of same–different categories; the reference stimulus E hat (S_{13}, S_{C21}) categories, and the modified stimuli are A (S_{13}, S_{C21}), B (S_{13}, $S_{C1\text{-}1}$), C (S_{11},

	Stimulus Name	Motion Categories As compared to the reference stimulus E		
		verbose	notation	example
Reference stimulus	E			S_{13}, $S_{\mathrm{C}21}$
Modified stimuli	A	Same Stories-RCC, Same Stories-OPRA$_1$	RccOpra	S_{13}, $S_{\mathrm{C}21}$
	B	Same Stories-RCC, Different Stories-OPRA$_1$	Rcc¬Opra	S_{13}, $S_{\mathrm{C}1\text{-}1}$
	C	Different Stories-RCC, Same Stories-OPRA$_1$	¬RccOpra	S_{11}, $S_{\mathrm{C}21}$
	D	Different Stories-RCC, Different Stories-OPRA$_1$	¬Rcc¬Opra	S_{11}, $S_{\mathrm{C}1\text{-}1}$

Table 9.1: A *stimuli set*. It consists of a *reference stimulus* and four symmetrically *modified stimuli* that share all, some, or none of its categories. The stimuli set can be characterized as *(reference categories; alternative categories)*; thus, the example can be characterized as (S_{13}, $S_{\mathrm{C}21}; S_{11}, S_{\mathrm{C}1\text{-}1}$)

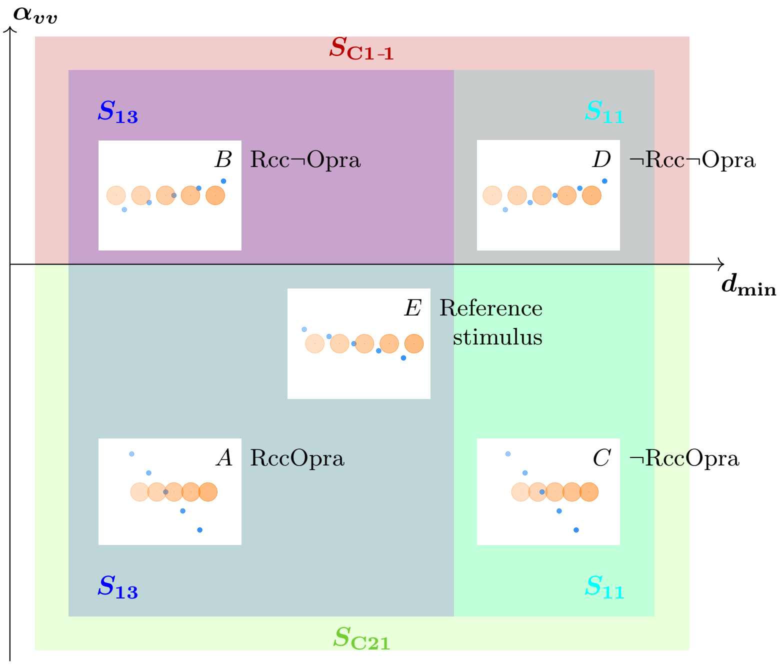


Figure 9.4: A *stimuli set* in the featural space of features α_{vv} and $d_{\min}$. The feature $\operatorname{sgn}(\alpha_{\Delta x \Delta v})$ is not represented because it is the same for all stimuli. The *modified stimuli* A, B, C, and D are generated by symmetric changes in the values of α_{vv} and $d_{\min}$ of the *reference stimulus*, E; that is, the featural distance between each stimulus and the reference stimulus is the same. The stimuli fall, however, in different categorical regions of Stories-RCC (S_{13} and S_{11}) and Stories-OPRA$_1$ ($S_{\mathrm{C}21}$ and $S_{\mathrm{C}1\text{-}1}$).

Source: Papenmeier, Purcalla Arrufi, and Kirsch (2023)

$S_{\mathrm{C}21}$), D (S_{11}, $S_{\mathrm{C}21}$). Regarding the involved categories, this stimuli set can be denoted (S_{13}, $S_{\mathrm{C}21}; S_{11}, S_{\mathrm{C}21}$).

Note that the categorization of a motion scene is invariant under rotations. That is, the featural parameters determine the motion scene up to a rotation. For example, in the scenes in Figure 9.4, the entity o moves along the positive x-axis direction, but we could rotate such scenes arbitrarily. In the trials, in order to increase variety and eliminate spurious effects, we rotated all scenes belonging to the same stimuli set by the same random angle; in that way categories, features, and similarities remained unchanged in each set. The rotation angle is different between stimuli sets and between subjects.

9.3.4 Tested categories

Due to time and budget limitations, we could not significantly test all possible combinations of motion scenes containing Stories-RCC and Stories-OPRA$_1$ stories, i.e., $16 \times 50 = 800$ combinations. Here we present the stories we chose and explain why we chose them.

One of the two critical requirements to generate a stimuli set is that we had to modify the featural variables—This requirement limits the stories we can use for the sets. The stimuli categories are defined by three featural variables: d_{min}, α_{vv}, $\mathrm{sgn}(\alpha_{\Delta x \Delta v})$. First of all, $\mathrm{sgn}(\alpha_{\Delta x \Delta v})$ is a discrete value, and our experiment works on the assumption that we can continuously modify the featural variables. For this reason, we set it constant in each stories set, that is, we only modified d_{min} and α_{vv}. A first consequence is that the *rigid stories* had to be discarded—They have α_{vv} and d_{min} constant. Also, the Stories-OPRA$_1$ stories with parallel velocities, Σ_T and Σ_C, were discarded—They have α_{vv} constant. The Stories-RCC stories S_{12}, S_{14}, and S_{18} were discarded as well because they have constant d_{min} (See Fig. 9.5).

Additionally, we decided to discard stories that had a motionless entity, Σ_B; we deemed them very particular cases of two entities motion. Thus far, we have the following stories left: S_{11}, S_{13}, and S_{15} belonging to Stories-RCC (S_{17} does not apply because in the experiment $r_k < r_l$); and the stories Σ_C of Stories-OPRA$_1$. Finally, we discarded the stories $S_{\mathrm{C}10}$ and $S_{\mathrm{C}20}$ of Σ_C because such stories are only compatible with the Stories-RCC story S_{15}, and, therefore, we could not create a whole stimuli set with them.

All in all, we have the following stories left:

- Stories-RCC: S_{11}, S_{13}, S_{15}
- Stories-OPRA$_1$: $S_{\mathrm{C}11}$, $S_{\mathrm{C}1\text{-}1}$, $S_{\mathrm{C}2\text{-}1}$, $S_{\mathrm{C}21}$

Even so, we did not create stimuli set with all possible combinations of such stories. In Stories-OPRA$_1$, we could only modify the feature α_{vv}; consequently, we could only combine the pairs $\{S_{11}, S_{2\text{-}1}\}$ and $\{S_{1\text{-}1}, S_{21}\}$—As an example, we can transition from S_{11} to $S_{1\text{-}1}$ only by changing the value of $\mathrm{sgn}(\alpha_{\Delta x \Delta v})$. In Stories-RCC, we restricted ourselves to generate sets with *neighbouring* stories. In that way, we had a lower modification of the featural parameters, and, thus, the reference and the modified stimuli have a greater similarity, which would make an observed categorization effect in our experiment more meaningful. For instance, we did not generate a stimuli set with reference Stories-RCC category S_{11} and with alternative category S_{15} because S_{15} is not neighbour of S_{11}, but S_{13} is—For that purpose we disregarded the discarded border categories S_{12} and S_{14} (see Fig. 9.5).

Summarizing, we generated stimuli set with following pairs of reference and alternative stimuli:

- Stories-RCC

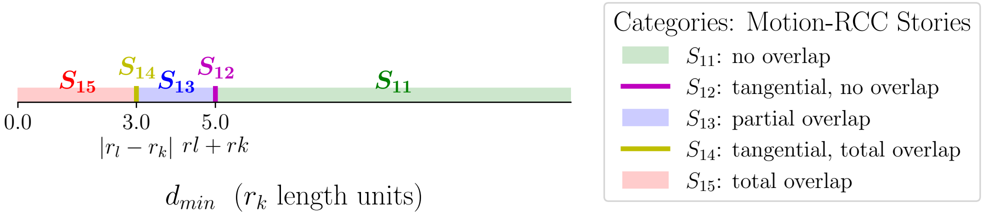


Figure 9.5: Featural space of Stories-RCC for two circular entities k and l with radii $r_k = 1$ and $r_l = 4$. The featural space contains only the variable $d_{\min}$, and, thus, it is one-dimensional. The stories are the motion categories. The regions of stories S_{11}, S_{13}, and S_{15} are one-dimensional, while stories S_{12} and S_{14} are zero-dimensional, i.e., they are points with constant $d_{\min}$ value.

  - $S_{11} \leftrightarrow S_{13}$
  - $S_{13} \leftrightarrow S_{15}$

- Stories-OPRA$_1$
  - $S_{\text{C1-1}} \leftrightarrow S_{\text{C21}}$
  - $S_{\text{C11}} \leftrightarrow S_{\text{C2-1}}$

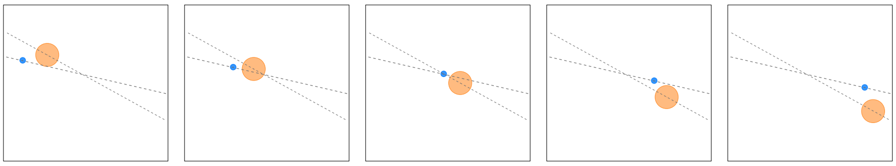
A. Total number of stimuli

The total number of stimuli sets is (*# of Stories-RCC category pairs*) × (*# of Stories-OPRA$_1$ category pairs*), that is, $4 \times 4 = 16$ stimuli sets each consisting of 5 stimuli. Hence, we generate a total of $16 \times 5 = 80$ stimuli.

9.4 Appearance of the Stories

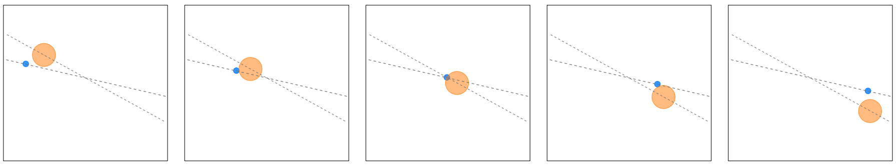
A. Stories-RCC Stories

- S_{11} DC; $d_{\min} > r_k + r_l$



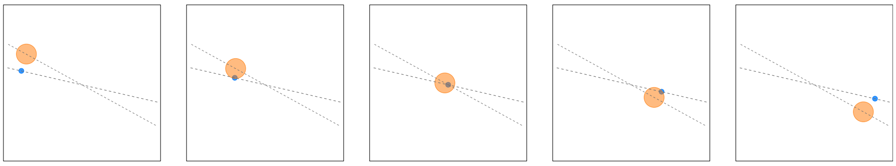
DC DC DC DC DC

- S_{13} DC, EC, PO, EC, DC; $|r_k + r_l| > d_{\min} > |r_k - r_l|$



DC EC PO EC DC

- S_{15} DC, EC, PO, TPP, NTPP, TPP, PO, EC, DC; $|r_k - r_l| > d_{\min}$

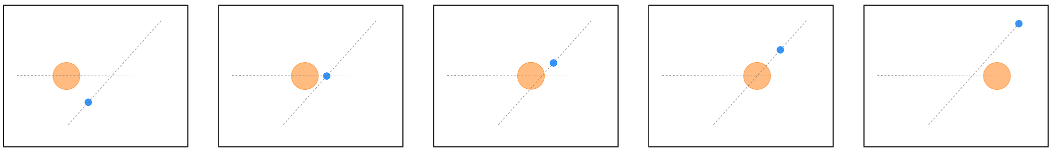


DC PO NTPP PO DC

Source: Papenmeier, Purcalla Arrufi, and Kirsch (2023)

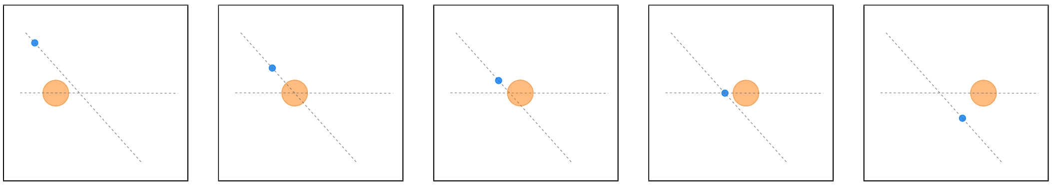
B. Stories-OPRA$_1$ Stories

- $S_{\text{C1-1}}$ $\angle_1^3\ \angle_1^0\ \angle_1^1\ \angle_2^1\ \angle_3^1$; $\alpha_{vv} < 0$, $\text{sgn}(\alpha_{\Delta x \Delta v}) > 0$



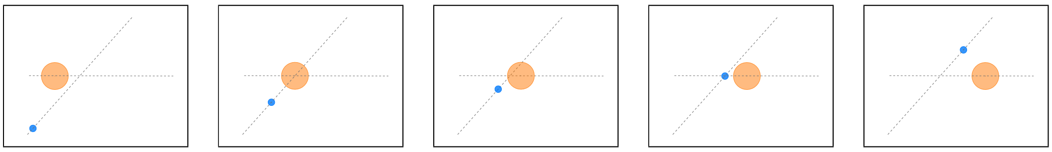
$\angle_1^3$ $\angle_1^0$ $\angle_1^1$ $\angle_2^1$ $\angle_3^1$

- S_{C21} $\angle_3^1\ \angle_0^1\ \angle_1^1\ \angle_1^2\ \angle_1^3$; $\alpha_{vv} > 0$, $\text{sgn}(\alpha_{\Delta x \Delta v}) > 0$



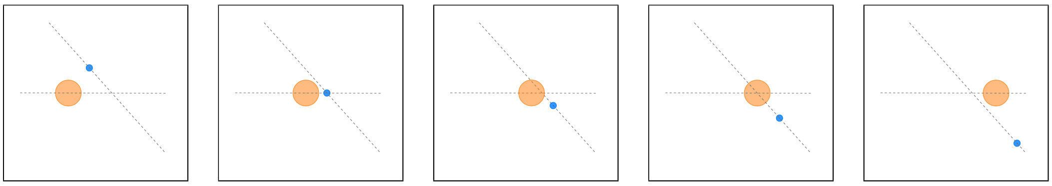
$\angle_3^1$ $\angle_0^1$ $\angle_1^1$ $\angle_1^2$ $\angle_1^3$

- S_{C11} $\angle_1^3\ \angle_0^3\ \angle_3^3\ \angle_3^2\ \angle_3^1$; $\alpha_{vv} < 0$, $\text{sgn}(\alpha_{\Delta x \Delta v}) < 0$



$\angle_1^3$ $\angle_0^3$ $\angle_3^3$ $\angle_3^2$ $\angle_3^1$

- $S_{\text{C2-1}}$ $\angle_3^1\ \angle_3^0\ \angle_3^3\ \angle_2^3\ \angle_1^3$; $\alpha_{vv} > 0$, $\text{sgn}(\alpha_{\Delta x \Delta v}) < 0$



$\angle_3^1$ $\angle_3^0$ $\angle_3^3$ $\angle_2^3$ $\angle_1^3$

Source: Papenmeier, Purcalla Arrufi, and Kirsch (ibid.)

9.5 Experimental Sets

An '*experimental set*' is each of our experimental set-ups. We have a total of four experimental sets, which originate from the combination of two independent aspects each with two possible variants.

The first aspect considers whether the subjects had to use mostly perception or to use more intensively short-time memory. One variant was called '*perception*' and noted '1'; the other variant was called '*memory*' and noted '2' (Fig. 9.1). The second aspect is the inclusion of a comparison stimulus that was *identical* to the reference stimulus. The variants were noted 'a', when the identical stimulus was not included, and 'b' when the identical stimulus was included. Accordingly, we have the following four combinations of experimental sets.

1. Perception sets: (Fig. 9.1a).
   a. Set '1a': perception set without identical stimulus.
   b. Set '1b': perception set with identical stimulus.
2. Memory sets: (Fig. 9.1b).
   a. Set '2a': memory set without identical stimulus.
   b. Set '2b': memory set with identical stimulus.

Preliminaries and Subjects' Treatment

The following procedures were common to all experimental sets.

The subjects were treated in accordance with the APA standards of ethical treatment (APA 2017). At the very beginning, prior to participation, we ensured that the subjects participated in only one of the experimental sets presented in this chapter, and the subjects provided informed consent. Subsequently, they provided demographic details (age, sex, sightedness). Then, they read the instructions about their task, namely, that they had to choose between two comparison stimuli the stimulus that was more similar to a reference stimulus. We instructed the subjects to choose the comparison stimulus by pressing the key '`f`' (left comparison stimulus) or '`j`' (right comparison stimulus) on the keyboard of the tablet used.

Subjects were also informed that the comparison stimuli would be shown synchronously several times successively (three times for *perception* experiments and two times for *memory* experiments). Therefore, they would have enough time to reach a decision. But they were told to answer as soon as they had reached a decision—They had not to wait for the stimuli to be displayed several times.

Further, they were informed that following each trial, a fixation cross would be shown and they could take a self-determined pause: the next trial should begin only when the subjects pressed the space bar. After every 10% progress of trials completion, we let the subjects know about their total completion progress in the experiment.

Before beginning the trials, the subjects started with a practice stimuli set and practised by answering all possible stimuli combinations: they amounted to $6 = \binom{4}{2}$ practice trials in the 'a' variants of the experiments (where no comparison stimulus was identical to the reference stimulus), and they amount to $10 = \binom{5}{2}$ practice trials in the 'b' variants of the experiments (where one comparison stimulus could be identical to the reference stimulus)

At the end of the experiment, each subject received a reward for the participation in the experiment: either monetary compensation (about 8€) or course credit.

9.5.1 Experiment 1: Perception

In this set, we used the perception set-up (Fig. 9.1a): The reference and comparison stimuli were presented at once and synchronously in a pyramidal layout where the *reference stimulus* lay centred above the two *comparison stimuli* (Fig. 9.6).

The two variants of this experimental set, 1a and 1b, differed on the presence of a stimulus identical to the reference stimulus in the comparison stimuli.

In *experiment 1a*, the comparison stimuli of each stimuli set were the four modified stimuli (Sect. 9.3.3). Consequently, we had a total of 6 possible trials for each stimuli set, which correspond to all possible pairs of the four modified stimuli, i.e., $\binom{4}{2}$. Using the whole 16 stimuli sets, we obtained a total of $96 = 16 \times 6$ trials. To increase statistical significance, we had each subject perform twice the 96 trials, yielding a total of $192 = 96 \times 2$ trials per subject. In both the first and second group of trials, we rotated the stimuli sets randomly rotated with different values.

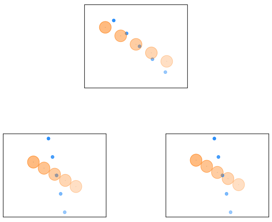


Figure 9.6: Layout of the stimuli in a perception set-up. Above, the reference stimulus; below, the comparison stimuli.

Source: Papenmeier, Purcalla Arrufi, and Kirsch (ibid.)

In *experiment 1b*, the comparison stimuli of each stimuli set were the four modified stimuli plus a stimulus identical to the reference stimulus (Sect. 9.3.3). Consequently, we had a total of 10 possible trials for each stimuli set, which correspond to all possible pairs of the five stimuli in the stimuli set, i.e., $\binom{5}{2}$. Using all the 16 stimuli sets, we obtained a total of $160 = 16 \times 10$ trials. In this experimental set, we did not run twice the whole number of stimuli sets, 320 trials (!), to prevent overstraining the subjects.

A. Experiment 1a

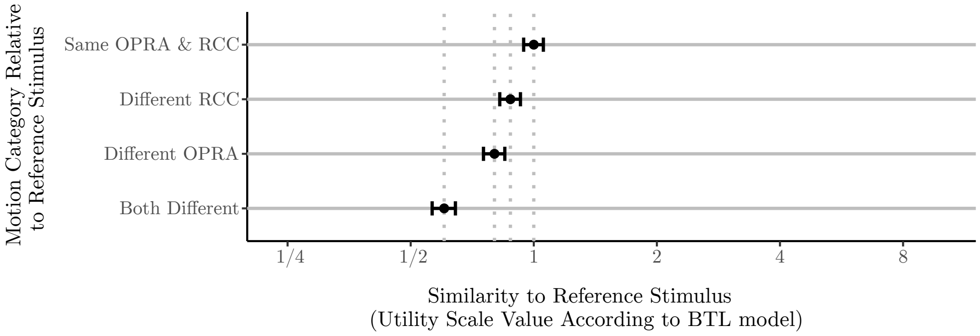
Subjects In this experiment participated 26 students from the University of Tübingen (22 female; age: 18–29 years; mean age: 23.19 years)

Results We obtained an acceptable fit of the subjects' similarity choices to the similarity ratios of the BTL model: the *non-significance* test yielded $\chi^2(3) = 7.68$, $p = 0.053$. We refuted the equality of the similarity ratios between stimuli with following significance: $\chi^2(3) = 111.69$, $p < 0.001$. Thus, the story-based categorization affected the subjects' choices.

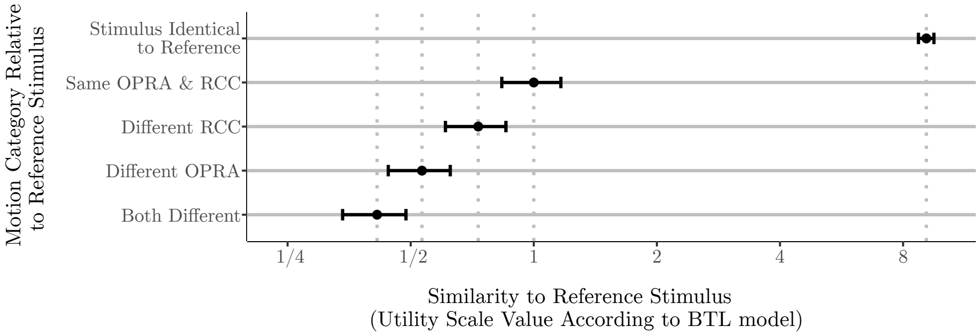
In Fig. 9.7a, we see that the similarity ratios of the stimuli were significantly different from each other (95% confidence). We also see that the different story-based categorizations had a cumulative effect: the more story-based categorizations a stimulus had in common with the reference stimulus the more similar to the reference stimulus such stimulus was perceived. However, Stories-RCC and Stories-OPRA$_1$ affected the perceived similarity with different intensity: the stimuli that shared only a Stories-OPRA$_1$ category showed a higher similarity than stimuli that shared only a Stories-RCC category.

B. Experiment 1b

Subjects In this experiment participated 26 students from the University of Tübingen (21 female; age: 19–35 years; mean age: 24.04 years)



(a) Experiment set 1a



(b) Experiment set 1b

Figure 9.7: Utility ratios according to the BTL model, i.e., experimental similarities of the comparison stimuli with respect to the reference stimulus; the similarity of the fourth stimulus, i.e., same OPRA_1 and RCC, is normalized to 1. The whiskers display the 95% confidence interval of the similarities. The comparison stimuli are sorted on the y-axis according to their common categories with the reference stimulus. The difference between both sets is that in set 1b a stimulus identical to the reference stimulus was also presented as a comparison stimulus.

Source: R code, F. Papenmeier; TikZ version J. Purcalla A.; licensed under CC BY 4.0

Results We obtained a very good fit of the subjects' similarity choices to the similarity ratios of the BTL model: the *non-significance* test yielded $\chi^2(6) = 5.44$, $p = 0.489$. We refuted the equality of the similarity ratios between stimuli with following significance: $\chi^2(4) = 1664.58$, $p < 0.001$. Thus, the story-based categorization affected the subjects' choices.

In Fig. 9.7b, we see that the similarity ratios of the stimuli were significantly different from each other (95% confidence), which means that the different story-based categorizations had a cumulative effect: the more story-based categorizations a stimulus had in common with the reference stimulus the more similar to the reference stimulus such stimulus was perceived. However, Stories-RCC and Stories-OPRA_1 affected the perceived similarity with different intensity: the stimuli that shared only a Stories-OPRA_1 category showed a higher similarity than stimuli that shared only a Stories-RCC category.

The subjects chose the identical stimulus to be extremely more similar to the reference stimulus than they chose the stimulus with both equal categories to be.

C. Discussion

Both perception experiments showed qualitatively the same results: story-based categorization affected very significantly similarity choices, the similarity effects of the story-based categorizations were cumulative, and the Stories-OPRA$_1$ categories affected much more similarity than the Stories-RCC categories.

It seems also that similarity of motion scenes were not only informed by the common categories. If only categories mattered, subjects should have chosen the identical stimulus to be so similar to the reference stimulus as they chose the stimulus with the same Stories-OPRA$_1$ and Stories-RCC category, i.e., stimulus A of the stimuli set (Tab. 9.1). However, the subjects chose the identical stimulus to be extremely more similar to the reference stimulus than they chose stimulus with both equal categories (Fig. 9.7b).

9.5.2 Experiment 2: Memory

In this set, we used the memory set-up (Fig. 9.1b): The reference stimulus and the comparison stimuli were presented separated by a pause of 1s; such pause should ensure that short-time memory intervened. First, we presented the reference stimulus at the center of the screen, and, after the pause, we presented the comparison stimuli side by side also centred (Fig. 9.8).

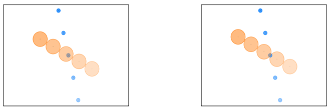


Figure 9.8: Layout of the comparison stimuli in a memory set-up, side by side. The reference stimulus is absent; it had been shown in a previous screen.

Source: Papenmeier, Purcalla Arrufi, and Kirsch (2023)

The two variants of this experimental set, 2a and 2b, differed in the presence of an identical stimulus as comparison stimulus in an analogous way as the variants 1a and 1b did. For that reason, both variants of the memory experiment had the same numbers of trials as the variants of the perception experiment: 192 trials and 160 trials, respectively.

A. Experiment 2a

Subjects In this experiment participated 26 students from the University of Tübingen (19 female; age: 18–30 years; mean age: 24.08 years)

Results We obtained a good fit of the subjects' similarity choices to the similarity ratios of the BTL model: the *non-significance* test yielded $\chi^2(3) = 5.43$, $p = 0.143$. We refuted the equality of the similarity ratios between stimuli with following significance: $\chi^2(3) = 780.17$, $p < 0.001$. Thus, the story-based categorization affected the subjects' choices.

If we analyse the similarity ratios individually (Fig. 9.9a), we see that the Stories-RCC categorization had a much lower similarity effect than Stories-OPRA$_1$. In fact, stimuli with only same Stories-RCC category were not significantly different (at 95% confidence) from stimuli having totally different categories; however, stimuli having the same categories Stories-RCC and Stories-OPRA$_1$ are significantly more similar to the reference stimulus than those having only same Stories-OPRA$_1$ category. Thus, in the latter case, the Stories-RCC category and Stories-OPRA$_1$ acted cumulatively on similarity.

B. Experiment 2b

Subjects In this experiment participated 26 students from the University of Tübingen (21 female; age: 19–31 years; mean age: 24.12 years)

Results We obtained a good fit of the subjects' similarity choices to the similarity ratios of the BTL model: the *non-significance* test yields $\chi^2(6) = 6.79$, $p = 0.341$. We refuted the equality of the similarity ratios between stimuli with following significance: $\chi^2(4) = 1515.97$, $p < 0.001$. Thus, the story-based categorization affected the subjects' choices.

If we analyse the similarity ratios individually (Fig. 9.9b), we see that the Stories-RCC categorization had no similarity effect whatsoever (at 95% confidence), while Stories-OPRA$_1$ categorization had a pronounced similarity effect. Additionally, The subjects chose the identical stimulus to be extremely more similar to the reference stimulus than they chose the stimulus with both equal categories to be similar.

C. Discussion

We observe in both variants of the memory experiment that the story-based categorizations affected the subjects' choices; however, Stories-RCC had a very limited effect compared with the perception experiments: only in one case of the experiment 2a, namely, in the stimulus with same Stories-OPRA$_1$ and same Stories-RCC, we saw that Stories-RCC categorization affected choice significantly. Hence, the categorization Stories-OPRA$_1$ was favoured over the categorization Stories-RCC through the short-memory storage.

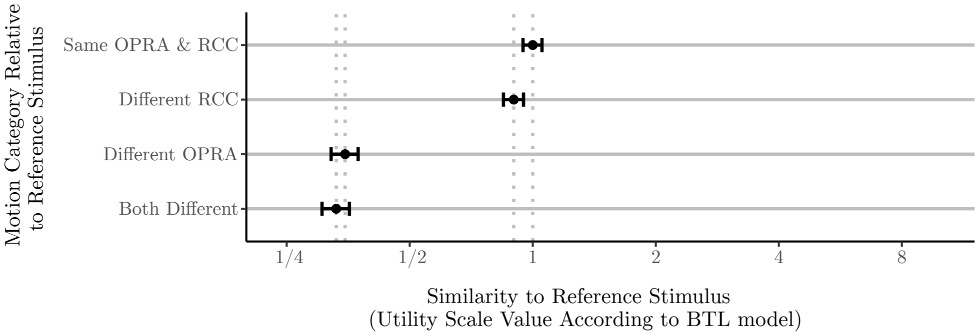
We also notice that, in the memory experiment, the identical stimulus produced the highest similarity response. Therefore, it is not categorization alone that informs similarity. In Section 9.6, we hypothesize what additional factors may inform similarity.

9.6 General Discussion

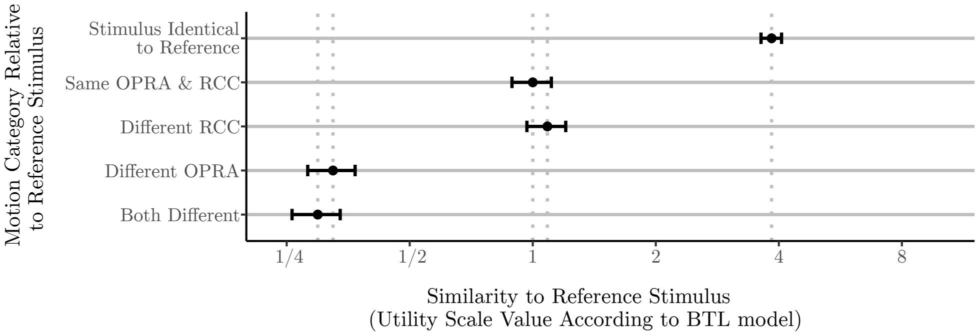
In our experimental sets, we generally established that story-based categorizations significantly influenced the similarity choice of motion scenes—An influence that could not be attributed to metric variations as we ensured that the metric changes in the stimuli were of the same amount. All experiments showed consistently that the Stories-OPRA$_1$ categorization both influenced the similarity choices and had a higher influence than the Stories-RCC categorization. Stories-RCC influenced significantly similarity choices in all experimental sets but in set 2b, in which we did not observe any significant effect of Stories-RCC categorization. In that sense, we can affirm that the story-based categorization in our experiment, RCC and Motion-OPRA$_1$, are cognitively plausible—At least in a weak manner.

We also observe, for the most part, that the different categorizations had a cumulative effect. The more categories the stimuli had in common, the higher the similarity. This cumulative effect was very weak in the memory experiments (it manifested only in two similarities of set 2a), though it was significantly established without exception in the perception experiments.

We argue that when memory intervened in the comparisons, the limited resources for storing the stimuli compelled the subjects' to favour the storage of one categorization. In fact, we believe that the same would happen in the perception set, if too many categorizations were introduced; in that case, the subjects should tend to consider only a restricted number of categorizations according to their limited cognitive resources, and use only such restricted number for similarity judgements.



(a) Experiment set 2a



(b) Experiment set 2b

Figure 9.9: Utility scale value according to the BTL model, i.e., experimental similarities of the comparison stimuli with respect to the reference stimulus; the similarity of the fourth stimulus, i.e., same OPRA_1 and RCC, is normalized to 1. The whiskers display the 95% confidence interval of the values. The comparison stimuli are sorted on the y-axis according to their common categories with the reference stimulus. The difference between both sets is that in set 2b a stimulus identical to the reference stimulus was also presented as a comparison stimulus.

Source: R code, F. Papenmeier; TikZ version J. Purcalla A.; licensed under CC BY 4.0

It is quite an open question, why the Stories-OPRA$_1$ categorization had a higher influence on similarity than Stories-RCC and not the opposite. On the one hand, it is possible that the featural variable of Stories-OPRA$_1$, α_{vv}, was more salient to the subjects than the feature of Stories-RCC, $d_{\min}$. On the other hand, Stories-RCC feature is related to the overlapping, i.e., to the collision of the entities, which we deem a central feature of motion—at least in daily life settings. This might be a possible explanation: subjects might have not valued collision as the most essential feature because we used coloured discs as entities instead of more natural entities. If we had used entities with a more relevant meaning in real life—with a higher ecological validity—for example, an arrow and a bird, subjects had possible favoured collision, i.e., Stories-RCC, over trajectory angle, i.e., Stories-OPRA$_1$.

In any case, we deem that the real cause of Stories-OPRA$_1$'s higher saliency should be experimentally determined or confirmed in future work under different—if possible more ecologically valid—conditions.

Another well established experimental fact is that the identical stimuli had an extremely higher similarity: we call it the '*identity effect*'. This effect seems obvious, considering the *maximality axiom* of similarity (item II., Sect. 3.4.1.A), which to our knowledge has consistently been confirmed experimentally: the similarity of a stimulus to itself is higher than to any other stimulus, i.e., $\mathrm{S}(A, A) \geq \mathrm{S}(A, B)$. Yet, it is not obvious the process by which the subjects could evaluate such similarity between reference and identical stimulus; we can only affirm that subjects perceived or stored more than the categories—Otherwise it would have been impossible for them to distinguish the identical stimulus from the stimulus with same Stories-OPRA$_1$ and Stories-RCC categories.

In the perception stimuli, a possible explanation to the *identity effect* is that subjects used primarily the category matching to evaluate similarity, and in case the common categories between stimuli were equal, the metric comparison was enabled. However, in the memory experiments a question arises: if some metric information was stored why did not the subjects used that information to distinguish the stimuli with different Stories-RCC categories from the ones with same Stories-RCC categories? One possible answer is that metric information was limitedly stored, and, hence, the subjects stored only metric information related to α_{vv}, the featural parameter of Stories-OPRA$_1$, but did not store the featural value $d_{\min}$, which determines the Stories-RCC categories. Here, we find also the need to deeply research under which criteria the subjects restrict the stored metric information and decide what information to store. In this work, it seems that subjects stored preferentially the metric information of Stories-OPRA$_1$ instead of Stories-RCC, but further work to corroborate this fact seems necessary.

Chapter 10

Story-Based Categorizations as Qualitative Calculi

An ordinary categorization, because of being a *set* of categories, is just endowed with the set operations, i.e., union, intersection, inclusion (See examples, Sect. 3.6). However, the story-based categorizations possess a richer list of operations (such as *converse* or *composition*) because they are not alone categorizations but also 'qualitative calculi' (Sect. 4.1.3).

In the previous chapters, we already indicated that the story-based categorizations are qualitative calculi by using alternatively the term 'motion categorization' along with the term 'motion representation'. '*Motion representation*' is a loose term for structures that range from qualitative calculi, in its strictest sense, to constructs that only resemble them. Until now, we intentionally disregarded most of the properties of story-based categorizations related to qualitative calculi: we researched *stories* simply as categories. In that respect, we spoke of story-based *categorizations*, and inspected the categorical properties of stories: their features, borders, and cognitive plausibility.

Conversely, in this chapter, we look at the story-based categorizations as 'qualitative calculi', and which extra properties they have. Now we speak of story-based *qualitative calculi*—or *qualitative representations*—and of qualitative relations, instead of story-based *categorizations* and categories; although both terminologies refer to the same reality seen from different perspectives.

In the initial sections of this chapter, we give a comprehensive description of 'qualitative calculus' for our purposes, outlining its basic characteristics and mentioning some additional properties. Subsequently, we relate qualitative calculi to general categorization models, and in particular to our categorization model. Lastly, we demonstrate that story-based categorizations are qualitative calculi.

10.1 Qualitative Calculus: Definition and Types

Since we want to prove that story-based categorization are qualitative calculi and which extra properties they have, we offer here a more detailed definition of qualitative calculus than in Section 4.3. In the following, we characterize a qualitative calculus of *binary* relations, i.e., involving only two entities, by particularizing the general definition of Dylla et al. (2017, Def. 3.3), which is originally formulated for n-ary relations.

We slightly tweak Dylla et al.'s (2017) definition of qualitative calculus to reduce unnecessary formalism load. We tighten the requirements on two operations needed in a calculus: *converse* and *composition*. Dylla et al. require that a calculus have at least *abstract* converse and composition, and, subsequently, define their *weak* counterparts by tightening the requirements on the former, however, in this work, we require from start that qualitative calculi have at least *weak* converse and composition operations.

Interestingly, Dylla et al. (2017, A:11) emphasize that, to their knowledge, no spatial calculus in the literature has only *abstract* converse and composition, but all have at least *weak* ones. Consequently, the *abstract* definitions bare negligible practical relevance. In fact, for every qualitative calculus after Dylla et al.'s (2017) definition, the *weak* operations can be readily defined (See later, Eqs. (10.2) and (10.4)).

10.1.1 Defining a Qualitative Calculus: Relations

A qualitative calculus requires a continuum domain (or '*universe*'), $\mathcal{U}$, in which the entities are described *quantitatively*. In the case of motion categorization, each entity is described in the universe $\mathcal{U} = \mathcal{X} \times \mathcal{V}$, that is, the entity's position $\mathcal{X}$ and velocity $\mathcal{V}$; for instance, $(\vec{x}_k, \vec{v}_k) \in \mathcal{U} = \mathcal{X} \times \mathcal{V}$ (See Sect. 1.2).

A chief goal of a qualitative calculus is to qualitatively describe in the universe $\mathcal{U}$ not single entities, but groups of n entities. To that end, a qualitative calculus uses mathematical '*relations*', denoted as R_i; these are subsets of the universe where the n entitities are described, i.e., $R_i \subset \mathcal{U}^n$. In the concrete case of two entities—the only we consider here—the relations R_i are binary, i.e., subsets of $\mathcal{U} \times \mathcal{U}$, or, equivalently elements of the power set $2^{\mathcal{U} \times \mathcal{U}}$.

Note that the space $\mathcal{X} \times \mathcal{V}$, let us call it '*single-motion*' space, has the space $(\mathcal{X} \times \mathcal{V}) \times (\mathcal{X} \times \mathcal{V})$ as binary relation space. This binary relation space corresponds to the kinematic space, $\mathcal{K}$, which contains the *motion scenarios*, $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$ (see def. in Sect. 6.1.1). Thus, we anticipate how relevant qualitative calculi are in our motion categorization modelling. Later, in Section 10.2, we show the full picture.

Abstract partition scheme: Base relations Concerning the relations, a qualitative calculus must have, by definition, '*base relations*'. The base relations, R_i, should form a *finite* set of relations, $\mathcal{R}$, and fulfil following: First, they are jointly exhaustive, i.e., $\mathcal{U} \times \mathcal{U} = \bigcup_{R \in \mathcal{R}} R$; second, they are pairwise disjoint, i.e., $R_i \cap R_j = \emptyset \quad \forall R_i, R_j \in \mathcal{R} \mid i \neq j$. In consequence, '*base relations*' are jointly exhaustive and pairwise disjoint (JEPD), which means that they constitute a partition of $\mathcal{U} \times \mathcal{U}$. A universe $\mathcal{U}$ that has base relations is called '*abstract partition scheme*' (2013), which is the substrate of any qualitative calculus.

An abstract partition scheme is also a fundamental construct in categorization. As we will see below (Sect. 10.2), the categorization formalism that we defined in this work is based on an abstract partition scheme. Accordingly, the finest categories of our categorization model are equivalent to the base relations of an abstract partition scheme.

Composite relations In an abstract partition scheme, we can easily create more relations, e.g., R', by union of base relations. For example, $R' = \{(u,v) \in \bigcup_{R_i \in \mathcal{R}'} R_i \mid \mathcal{R}' \subset \mathcal{R}\}$, or, simplifying notation, $R' = \bigcup_{R_i \in \mathcal{R}' \subset \mathcal{R}} R_i$. Such relations are called '*composite relations*', and form the set $2^{\mathcal{R}}$. In sum, the '*composite relations*' are all possible unions of base relations. Analogous to a base of a vector space, any composite relation, $R' \in 2^{\mathcal{R}}$, is *uniquely* determined as a union of the base relations, $R \in \mathcal{R}$. In a qualitative calculus, because it is also an abstract partition scheme, we can, as well, generate the composite relations. In sum, a qualitative calculus is sufficiently characterized by the universe $\mathcal{U}$, and the base relations, $\mathcal{R}$; hence, we note it $(\mathcal{U}, \mathcal{R})$.

The composite relations are often referred to as relations that represent *incomplete* or *coarse* information (or knowledge) because the real state of the entities can always be finer represented by means of a base relation (e.g., Freksa 1992a; Scivos and Nebel 2001, Sect. 2). For example, when two entities verify the composite relation $R' = R_i \cup R_j$, we can express it as a *disjunction*, i.e., the entities verify R_i *or* R_j. The information is incomplete, because we know that the entities can verify only one of both relations, *either* R_i *or* R_j; however, we lack enough information to decide it.

Oddly, according to this disjunction technique, the more coarse the information of certain state is, the greater the number of conjunctions we have to use. So that, as Freksa (1992a, p. 4) in his seminal paper noted, we run into a paradox: "[T]he less we know, the more complex the representation of what we know becomes. What is known is represented in terms of disjunctions of 'what could be the case'" (simple quotes added).

10.1.2 Converse and Composition

On top of the properties of an abstract partition scheme, a qualitative calculus must have two operations: *converse* (also known as *inverse*) and *composition*. These operations heavily characterize qualitative calculi. The behaviour of converse and composition determines the different types of qualitative calculi—sometimes these operations can be *strongly* defined, sometimes *weakly*—The stronger the operations can be defined, the higher the computing power of the reasoning algorithms that can be used in a qualitative calculus. '*Reasoning*', besides providing a qualitative description, is another chief goal of qualitative calculi. It consists on finding initially unknown relations in a scenario with more than two entities or check the consistency of the current relations (See Sect. 4.1.4).

Converse Every relation R in a (binary) qualitative calculus must also have a converse $\{R\}^{\smile}$, which is simply the qualitative relation of the permuted entities, in our case $k \leftrightarrow l$.

$$\{R\}^{\smile} = \{(v, u) \mid (u, v) \in R\} \tag{10.1a}$$

Adapted to motion scenarios:

$$\{R\}^{\smile} = \{(\vec{x}_l, \vec{v}_l; \vec{x}_k, \vec{v}_k) \mid (\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \in R\} \tag{10.1b}$$

Notably, the relation $\{R\}^{\smile}$ needs not be a composite relation, i.e., it can happen that $\{R\}^{\smile} \notin 2^{\mathcal{R}}$. This is inconvenient because we expect that, in a qualitative calculus, the relations set, $2^{\mathcal{R}}$, be closed under the calculus operations: Every converse relation should be a composite relation, i.e., it should be express as union of base relations $\{R\}^{\smile} = \bigcup_{R_i \in \mathcal{R}' \subset \mathcal{R}} R_i$. We can correct this inconvenience by redefining the converse relation as the minimal composite relation that contains the converse as defined in Equation (10.1). We use the symbol $R^{\smile}$ for this newly defined converse; which is called '*weak converse*' (We use a definition like Moratz and Wallgrün 2012, p. 6; See also, Dylla et al. 2017, Eq. 8).

$$R^{\smile} := \bigcap_{\substack{R \in 2^{\mathcal{R}} \\ R \supset \{R\}^{\smile}}} R \tag{10.2}$$

It is evident that $R^{\smile} \supset \{R\}^{\smile}$, and that $R^{\smile}$ is minimal. In the ideal case of equality, i.e., $R^{\smile} = \{R\}^{\smile}$, we say that $R^{\smile}$ is a '*strong converse*' (ibid., Eq. 8, 9). For the great majority of the qualitative calculi, the converse is strong (, Fig. 7), and, thus, Equations (10.1) and (10.2) are equivalent.

Composition As mentioned in Section 4.3.1, the composition of relations yields the transitive relation. As an illustration, if $(u, v) \in R_A$ and $(v, w) \in R_B$, the transitive relation R_C is the one of the entities (u, w). R_C is obtained through the composition of R_A and R_B, formally, $R_C = R_A\{\circ\}R_B$.

$$R_A\{\circ\}R_B = \{(u,w) \mid \exists v \in \mathcal{U} : (u,v) \in R_A, (v,w) \in R_B\} \tag{10.3a}$$

Adapted to motion scenarios:

$$R_A\{\circ\}R_B = \{(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \mid (\vec{x}_k, \vec{v}_k; \vec{x}_m, \vec{v}_m) \in R_A, (\vec{x}_m, \vec{v}_m; \vec{x}_l, \vec{v}_l) \in R_B\} \tag{10.3b}$$

Again, we have the inconvenience that the composition relation, as defined in Equation (10.3), need not be a composite relation, i.e., it can happen that $R_A\{\circ\}R_B \notin 2^{\mathcal{R}}$. We can correct this inconvenience by redefining the composition relation as the minimal composite relation that contains the composition as defined in Equation (10.3). We use the symbol $R_A \circ R_B$ for this newly defined composition; which is called '*weak composition*' (ibid., Eq. 8).

$$R_A \circ R_B \coloneqq \bigcap_{\substack{R \in 2^{\mathcal{R}} \\ R \supset (R_A\{\circ\}R_B)}} R \tag{10.4}$$

It is evident that $R_A \circ R_B \supset R_A\{\circ\}R_B$, and that it is minimal. In the case of equality, i.e., $R_A \circ R_B = R_A\{\circ\}R_B$, we say that $R_A \circ R_B$ is a '*strong composition*' (ibid., Eq. 10, 11). Most of the qualitative calculi have a strong composition (ibid., Fig. 7), but exceptions are, e.g., $\mathrm{QTC_C}$ and $\mathrm{OPRA_n}$, which, interestingly, are qualitative representations used to qualitatively describe motion.

Qualitative Calculus Summarizing, a '*qualitative calculus*' is an abstract partition scheme with the operations of *weak* converse and *weak* composition. More restrictive additions onto this definition make up the different types of calculi.

Importantly, the very critical property of a qualitative calculus is to be an abstract partition scheme, because based on the scheme we can always define a *weak* converse and composition by means of Equations (10.2) and (10.4).

10.1.3 Types of Qualitative Calculi

The different types of qualitative calculi originate through adding properties to the fundamental definition. There is not a proper term to allude to all types of calculi that originate through adding different properties; yet, we have terms for the most important sets of properties. For example, the qualitative calculi fulfilling most desirable properties form '*relation algebras*'; this is, for example, the case of RCC (here, we do not define the algebras, see Dylla et al. (ibid.) for a detailed description). If all properties of a relation algebra are fulfilled, except that the composition is not associative, then we have a '*semi-associative* relation algebra'; this is the case of $\mathrm{OPRA_n}$.

Another types of qualitative calculi with less desirable properties are the '*associative boolean algebras*', '*semi-associative boolean algebras with converse involution*' (e.g., $\mathrm{QTC_C}$), and the poorest case '*weakly associative boolean algebra*'.

10.2 Categorization Formalisms as Qualitative Calculi

Once qualitative calculi are defined, it is enlightening to compare the elements of a qualitative calculus with the models of categorization (Ch. 3), and, particularly, with our categorization formalism (Ch. 6).

To begin with, we draw a link between Dylla et al.'s (2013) abstract partition scheme and our categorization formalism (Sect. 6.1.1). When we formalized motion categorization, we started with a kinematic space $\mathcal{K}$ in which the motion state of our entities is *quantitatively* described as a motion scenario $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l) \in \mathcal{K}$. Note that the kinematic space, $\mathcal{K}$, can be identified with the space $\mathcal{U} \times \mathcal{U}$ of binary relations of the universe $\mathcal{U}$. Indeed, as we said above, the motion state of a single entity is represented in the *single-motion* space $\mathcal{U} = \mathcal{X} \times \mathcal{V}$; therefore, we can verify that $\mathcal{U} \times \mathcal{U} = (\mathcal{X} \times \mathcal{V}) \times (\mathcal{X} \times \mathcal{V}) = \mathcal{K}$.

Further, we defined the categorization $\mathcal{M}$ constituted of categories M_i, i.e., $\mathcal{M} = \{M_1, M_2, \dots, M_m\}$. Remarkably, such categorization is an *abstract partition scheme* of the kinematic space, $\mathcal{K}$. The categorization rule f_μ defines the categories M_i as relations in $\mathcal{U} \times \mathcal{U} = \mathcal{K}$: we showed in Equation (6.3) that each M_i corresponds to a subset $\boldsymbol{K}_i$ of $\mathcal{K}$, namely, $\boldsymbol{K}_i = {\mathrm{f}_\mu}^{-1}(M_i)$—We called such subsets *categorical regions*. Moreover, in our categorization formalism, we required that the categories M_i—actually, their associated categorical regions $\boldsymbol{K}_i$—be jointly exhaustive and partially disjoint (See Eqs. 6.4). Therefore, in Section 6.1.1, we formalized categorization, specifically, motion categorization, as an abstract partition scheme in which the categories, $M_i \in \mathcal{M}$, (or his associated categorical regions) are equivalent to base relations. Thus, we axiomatized a motion categorization as the abstract partition scheme $(\mathcal{U}, \mathcal{R}) = (\mathcal{X} \times \mathcal{V}, \mathcal{M})$.

The equivalence between our categorization formalism and an abstract partition scheme induced an equivalence of terms. Following terms are interchangeable when dealing with story-based categorizations (See also Tab. 4.1):

- 'motion categories', 'qualitative relations', and 'categorical regions'
- 'motion categorization', 'qualitative representation', 'qualitative calculus'

Incomplete information In our discussion of motion categorizations (all throughout Part III), we did not tackle the issue of 'incomplete information' when categorizing motions. When we defined a categorization $\mathcal{M}$ as a set of jointly exhaustive and partially disjoint categories, $\{M_i\}_{i=1,\dots,m}$, we did not consider the possibility of creating new motion categories as disjunctions of old motion categories, e.g., $M' = M_i \cup M_j$; that is, a motion scenario belongs to category M' when it belongs to category M_i *or* to category M_j.

To correct this omission, we show here, in the frame of qualitative calculi, how story-based categorizations deal with incomplete information. The calculi provide the native tool of *relational disjunction* to express incomplete information, and we can readily apply it to the story-based categorizations. For example, in Stories-OPRA$_1$, we can define the composite category $S_{\mathrm{C1}} = S_{\mathrm{C1\text{-}1}} \cup S_{\mathrm{C10}} \cup S_{\mathrm{C11}}$, which has a plain understandable meaning: "all motion scenarios where both entities move and the first entity (k) approaches the trajectory of the second entity (l) coming from the left". We mentioned in Section 3.6, how the union, i.e., disjunction of categories seems a cognitively plausible process; we gave the example of the concept `citrus` which can be understood as the union of all citrus fruits.

10.3 Story-Based Categorizations are Qualitative Calculi

Here, we will prove that a story-based motion categorization is a qualitative calculus. The first step is to show that a story-based motion categorization is an *abstract partition scheme* (Sect. 10.1.1), i.e., that such categorization can be seen as a universe $\mathcal{U}$ having base relations. As a particular case, it suffices to see that the categories of a story-based categorization are *base relations*. Since we have defined two types of story-based categorizations—the bare (Stories-$\mathcal{R}$) and the beaded (Motion-$\mathcal{R}$) categorizations (Ch. 9, p. 157)—we need two proofs.

Remember that the *bare* story-based representations are motion categorizations in which we identify the stories, $\{S_i\}_{i=1,\dots,n}$, with the motion categories. In the field of the qualitative calculi, we might understand such stories as the *base relations* of a story-based representation: We will prove this in the current section.

The *beaded* story-based categorizations are motion categorizations in which each motion category is a pair formed by a story, S_i, and some of its constituting relations, R_j; that is, a category is written as $S_i(R_j)$ where $R_j \in S_i$. It is clear from the definition that the *beaded* categories are a refinement of the *bare* categories: each story S_i is subdivided by means of its constituting qualitative relations $R_j \in S_i$. In this section, we will also prove that the beaded categories, i.e., $S_i(R_j)$, are the base relations of a story-based representation.

In any case, both categorizations use as foundation the stories generated by a qualitative calculus $\mathcal{R}$; therefore, a key step in our proofs is to show that the stories of any story-based categorization, $\Sigma_{\mathcal{R}}$, constitute a finite set of *base relations*, i.e., jointly exhaustive and partially disjoint relations in the kinematic space $\mathcal{K} = (\mathcal{X} \times \mathcal{V}) \times (\mathcal{X} \times \mathcal{V})$ (Lemmas 10.3.1 and 10.3.2). Once we have shown this, it is straightforward to show that both types of motion categorizations are qualitative calculi.

Lemma 10.3.1 ***Stories are partially disjoint (PD)*** *The stories $S \in \Sigma_{\mathcal{R}}$ of a story-based categorization, considered as qualitative relations in the kinematic space $\mathcal{K}$, are partially disjoint.*

Proof. Any story of a story-based categorization is a qualitative relation in the kinematic space $\mathcal{K}$. Indeed, we can see that each story is linked to a region of $\mathcal{K}$ by applying the *story map* $\sigma_{\mathcal{R}}$. This map links the stories $S \in \Sigma_{\mathcal{R}}$ and the kinematic space $\mathcal{K}$: ${\sigma_{\mathcal{R}}}^{-1}(S_i) = \boldsymbol{K}_i \subset \mathcal{K}$ (Eq. (7.13)). Since any function f verifies $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ (Monk 1969, p. 40), and the story map, $\sigma_{\mathcal{R}}$, is a function, the story map verifies following

$$ {\sigma_{\mathcal{R}}}^{-1}(S_A \cap S_B) = {\sigma_{\mathcal{R}}}^{-1}(S_A) \cap {\sigma_{\mathcal{R}}}^{-1}(S_B) \tag{10.5} $$

The stories are single, i.e., atomic, elements of the stories set $\Sigma_{\mathcal{R}}$, then, obviously, $S_A \cap S_B = \emptyset \quad \forall S_A \neq S_B \in \Sigma_{\mathcal{R}}$. Now setting this equality in Equation (10.5), we obtain that

$$ {\sigma_{\mathcal{R}}}^{-1}(\emptyset) = {\sigma_{\mathcal{R}}}^{-1}(S_A) \cap {\sigma_{\mathcal{R}}}^{-1}(S_B) \quad \forall S_A \neq S_B $$

Now considering that for a function f, $f^{-1}(\emptyset) = \emptyset$, then we have following.

$$ {\sigma_{\mathcal{R}}}^{-1}(S_A) \cap {\sigma_{\mathcal{R}}}^{-1}(S_B) = \emptyset \quad \forall S_A \neq S_B \tag{10.6} $$

And this last equation reflects exactly the condition that the stories, as qualitative relations of the kinematic space, are pairwise disjoint. □

Lemma 10.3.2 ***Stories are jointly exhaustive (JE)*** *The stories $S \in \Sigma_{\mathcal{R}}$ of a story-based categorization, considered as qualitative relations in the kinematic space $\mathcal{K}$, are jointly exhaustive.*

Proof. If the stories in $\Sigma_{\mathcal{R}}$ were not jointly exhaustive, we could find a motion scenario $K \in \mathcal{K}$ that does not belong to any story $S \in \Sigma_{\mathcal{R}}$. However, by definition, the domain of the story map $\sigma_{\mathcal{R}}$, which assigns the stories to the motion scenarios, $S_i = \sigma_{\mathcal{R}}(K)$, is the *whole* kinematic space $\mathcal{K}$ (Eq. (7.13)): each motion scenario is mapped into a story, or conversely, the story set is generated by finding all possible stories in the kinematic space (see the algorithm for $\Sigma_{\mathcal{R}}$ generation, Sect. 8.1.1). Hence, any motion scenario K must necessarily belong to a story $S \in \Sigma_{\mathcal{R}}$, and, consequently, the stories are jointly exhaustive. ☐

Corollary 10.3.2.1 ***Stories are jointly exhaustive and partially disjoint (JEPD)*** *The stories $S \in \Sigma_{\mathcal{R}}$ of a story-based categorization, considered as qualitative relations in the kinematic space $\mathcal{K}$, are jointly exhaustive and partially disjoint.*

Proof. It follows from Lemmas 10.3.1 and 10.3.2. ☐

Lemma 10.3.3 ***Stories form an abstract partition scheme*** *In the vast majority of calculi $\mathcal{R}$, the generated stories, i.e., $\Sigma_{\mathcal{R}}$, form an abstract partition scheme of the kinematic space $\mathcal{K}$.*

Proof. We have already seen (Cor. 10.3.2.1) that in every qualitative representation, $\mathcal{R}$, the generated stories, $\Sigma_{\mathcal{R}}$, are jointly exhaustive and partially disjoint in $\mathcal{K}$ because of the properties of the story map $\sigma_{\mathcal{R}}$. Thus, we just need to prove that this set is *finite*. Unfortunately, we cannot generally prove it for each and every representation, as it fully depends on the geometry of the representation. In fact, we can easily create qualitative representations with a peculiar geometry, in which infinite stories arise; for example, with a relation consisting of infinite disconnected regions. We can though prove that for the *vast majority*—possible all—of currently available qualitative representations, the stories set Σ is finite (See proofs in App. A.2.1). Summarizing, for the vast majority of calculi $\mathcal{R}$, the stories in $\Sigma_{\mathcal{R}}$ form a set of *base relations* of $\mathcal{K}$, and, therefore, they are an abstract partition scheme of $\mathcal{K}$. ☐

Proposition 10.3.1 ***Bare story-based categorization are qualitative calculi*** *In the vast majority of calculi $\mathcal{R}$, the generated stories, i.e., $\Sigma_{\mathcal{R}}$, are a qualitative calculus in the kinematic space $\mathcal{K}$. Thus, the bare story-based categorizations Stories-$\mathcal{R} = (\sigma_{\mathcal{R}}, \Sigma_{\mathcal{R}})$ are a qualitative calculus (i.e., a qualitative representation) in $\mathcal{K}$.*

Proof. In Lemma 10.3.3, we saw that the stories, $\Sigma_{\mathcal{R}}$, form an abstract partition scheme of the kinematic space $\mathcal{K}$ (in the vast majority of calculi $\mathcal{R}$). In consequence, we can upgrade the stories, $\Sigma_{\mathcal{R}}$, into a qualitative calculus if we add the operations of (weak) converse (Eq. (10.4)) and composition (Eq. (10.2)), which can always be defined upon a set of base relations. ☐

Proposition 10.3.2 ***Beaded story-based categorization are qualitative calculi*** *In the vast majority of calculi $\mathcal{R}$, the generated beaded categories of Motion-$\mathcal{R}$ categorization, i.e., $\mathcal{M}_{\mathcal{R}} = \{S_i(R_j) \mid S_i \in \Sigma_{\mathcal{R}}, R_j \in S_i\}$, are a set of base relations in the kinematic space $\mathcal{K}$. As a consequence, in such calculi $\mathcal{R}$, the beaded story-based categorizations Stories-$\mathcal{R} = (\sigma_{\mathcal{R}} \times \rho, \mathcal{M}_{\mathcal{R}})$ are a qualitative calculi (i.e., a qualitative representation) in $\mathcal{K}$.*

Proof. First, we prove that the beaded stories, Motion-$\mathcal{R}$, are an abstract partition scheme in an analogous way we proved in Lemma 10.3.3 that bare stories are an abstract partition scheme. The proof of Lemma 10.3.3 required, initially, that the stories be jointly exhaustive and partially disjoint (JEPD). We saw that the JEPD property based on two properties of $\sigma_{\mathcal{R}}$: that it is a function (used in Lem. 10.3.1), and that its domain is the whole kinematic space (used in Lem. 10.3.2). These properties are also verified by $(\sigma_{\mathcal{R}} \times \rho)$ and, therefore, the beaded stories are also JEPD in the kinematic space $\mathcal{K}$. Additionally, in Lemma 10.3.3, required that the stories be a finite set. Now, we have to prove that the beaded stories are a finite set. We prove that by means of Propositions A.2.1 and A.2.2: in the vast majority of calculi $\mathcal{R}$ the stories and their elements are finite. Accordingly, the beaded stories, Motion-$\mathcal{R}$, must be a finite set in the vast majority of calculi $\mathcal{R}$.

Second, analogously to Proposition 10.3.1, we can upgrade the beaded stories, Motion-$\mathcal{R}$, into a qualitative calculus by adding the operations of (weak) converse (Eq. (10.4)) and composition (Eq. (10.2)). □

Corollary 10.3.2.1 ***Beaded story-based categorizations are refinements of bare story-based categorizations*** *When we look at stories as relations in the kinematic space* $\mathcal{K}$*, then every story* $S_i \in$ *Stories-*$\mathcal{R}$*, i.e., every bare story-based relation, is partitioned by the beaded story-based relations,* $S_i(R) \in$ *Motion-*$\mathcal{R}$*. Equivalently,* $\forall S_i \in \Sigma_{\mathcal{R}} \quad \mathrm{f}_\mu{}^{-1}(S_i) = \bigcup_{R \in S_i} \mathrm{f}_\mu{}^{-1}(S_i(R))$ *and* $\mathrm{f}_\mu{}^{-1}(S_i(R_j)) \cap \mathrm{f}_\mu{}^{-1}(S_i(R_j))$*, where* $\mathrm{f}_\mu = \sigma_{\mathcal{R}} \times \rho$ *is the categorization function in Stories-*$\mathcal{R}$*.*

Proof. Note that, according to the categorization rule $\mathrm{f}_\mu = \sigma_{\mathcal{R}} \times \rho$ in beaded story-based categorizations, if a motion scenario belongs to a certain bare story-based category, $K \in S_i$, then it belongs also to a certain beaded story-based category $K \in S_i(R_j)$, i.e., $\mathrm{f}_\mu{}^{-1}(S_i) \subset \bigcup_{R \in S_i} \mathrm{f}_\mu{}^{-1}(S_i(R))$. Conversely, if a scenario belongs to a certain beaded story-based category, i.e., $K \in S_i(R_j)$, then the scenario must belong to the story component, i.e., $K \in S_i$, i.e., $\mathrm{f}_\mu{}^{-1}(S_i) \supset \bigcup_{R \in S_i} \mathrm{f}_\mu{}^{-1}(S_i(R))$. Therefore, we have proved that $\forall S_i \in \Sigma_{\mathcal{R}} \quad \mathrm{f}_\mu{}^{-1}(S_i) = \bigcup_{R \in S_i} \mathrm{f}_\mu{}^{-1}(S_i(R))$. That the beaded relations are disjoint was proved in Proposition 10.3.2. □

By abusing notation, we simplify the expressions in Corollary 10.3.2.1 as those in Equation (10.7) because it is clear that we are not working in $\Sigma_{\mathcal{R}}$, where stories are atomic, but in the kinematic space, where the stories are equivalent to kinematic regions $\boldsymbol{K}$.

$$\forall S_i \in \Sigma_{\mathcal{R}} \quad \begin{cases} S_i = \bigcup_{R \in S_i} S_i(R) & \text{(10.7a)} \\ S_i(R_j) \cap S_i(R_k) = \emptyset \quad \forall j \neq k & \text{(10.7b)} \end{cases}$$

Being a *refinement* is equivalent to say that a beaded categorization is a *partition* of the bare categorization. Corollary 10.3.2.1 means that the beaded categorizations add a higher granularity (higher definition) to the bare categorizations. In fact, the beaded categorization provides extra information about the temporal evolution of the entities.

Thus far, we have proved that for the vast majority of qualitative calculi $\mathcal{R}$, the generated story-based representations, bare Stories-$\mathcal{R} = (\mathcal{K}, \Sigma_{\mathcal{R}})$ and beaded Motion-$\mathcal{R} = (\mathcal{K}, \mathcal{M}_R)$, are qualitative calculi. Additionally, we proved that the the Motion-$\mathcal{R}$ calculi correspond to the Stories-$\mathcal{R}$ calculi with higher granularity. These results open the door for the generation and application (Ch. 11) of this new family of qualitative calculi.

When we proved that story-based categorizations are qualitative calculi (Props. 10.3.1 and 10.3.2), we only assumed the existence of the converse and composition operations as defined in Eqs. (10.2) to (10.4). In the next sections, we present methods to compute, or help to compute, such operations using only, as extra information, the respective tables of the generating representation $\mathcal{R}$. We can compute the converse exactly, but, regarding the composition, we can only compute an upper set bound—which we call *narrative composition*—we cannot directly compute the composition.

10.4 Converse Relations in Story-Based Categorizations

We begin by showing how to compute the converse of a beaded relation $S_i(R_j)$ because, in order to compute its converse, i.e., $S_i(R_j)^{\smile}$, we have to compute the converse of the bare relation, i.e.,

${S_i}^\smile$.

As the motion relation $S_i(R_j)$ relates the pair (k, l), computing the converse, $S_i(R_j)^\smile$, means to find the motion relation for the permuted pair, (l, k) (Sect. 4.3.1). By definition, the terms S_i and R_j stand independently, as a Cartesian product, in the relations notation; thus, $S_i(R_j)^\smile = {S_i}^\smile({R_j}^\smile)$, and we can compute the converse of each term just using the converse of the generating spatial relation $\mathcal{R}$:

- ${R_j}^\smile$ is provided by the generating representation. For example, $\mathrm{DC}^\smile = \mathrm{DC}$, $\mathrm{TPP}^\smile = \mathrm{TPPI}$, $(\angle_1^3)^\smile = \angle_3^1$, or $(\angle{\scriptstyle 2})^\smile = \angle{\scriptstyle 2}$

- ${S_i}^\smile$ is, by definition, equivalent to compute the converse on any term of the story $S_i = (R_1, \ldots, R_n)$ as sequence of relations, that is,

$$ {S_i}^\smile = ({R_1}^\smile, \ldots, {R_n}^\smile) \tag{10.8} $$

  Note, that the sequence of converses $({R_1}^\smile, \ldots, {R_n}^\smile)$ certainly correspond to a story S_j belonging to the stories set, i.e., $S_j \in \Sigma$, because the stories with permuted entities were computed as part of the standard stories. For that reason, the converse of story-based representations is a *strong converse*, and we had no need to use the weak definition (Eq. (10.2)).

  For example, in Motion-RCC, the story S_{14} corresponds to the temporal sequence $(\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPP}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$, thus ${S_{14}}^\smile = (\mathrm{DC}^\smile, \mathrm{EC}^\smile, \mathrm{PO}^\smile, \mathrm{TPP}^\smile, \mathrm{PO}^\smile, \mathrm{EC}^\smile, \mathrm{DC}^\smile) = (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPPI}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC}) = S_{16}$; consequently, ${S_{14}}^\smile = S_{16}$. Another example, the story S_{13} corresponds to the temporal sequence $(\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$, thus ${S_{13}}^\smile = (\mathrm{DC}^\smile, \mathrm{EC}^\smile, \mathrm{PO}^\smile, \mathrm{EC}^\smile, \mathrm{DC}^\smile) = (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$, which is the same story S_{13}; consequently, ${S_{13}}^\smile = S_{13}$.
  In Motion-OPRA$_1$ the story $S_{\mathrm{C}10}$ (See Table 8.2), corresponds to the temporal sequence $(\angle_1^3, \angle{\scriptstyle 3}, \angle_3^1)$; thus the converse is computed by the same procedure above: ${S_{\mathrm{C}10}}^\smile = ((\angle_1^3)^\smile, (\angle{\scriptstyle 3})^\smile, (\angle_3^1))^\smile) = (\angle_3^1, \angle{\scriptstyle 1}, \angle_1^3) = S_{\mathrm{C}20}$

Thus, finally, we obtain examples of converse for beaded relations: in Motion-RCC, $S_{12}(\mathrm{EC}_+)^\smile = S_{12}(\mathrm{EC}_+)$, $S_{14}(\mathrm{TPP})^\smile = S_{16}(\mathrm{TPPI})$; in Motion-OPRA$_1$ $S_{\mathrm{C}10}(\angle_1^3)^\smile = S_{\mathrm{C}20}(\angle_3^1)$, or, $S_{\mathrm{C}10}(\angle{\scriptstyle 3})^\smile = S_{\mathrm{C}20}(\angle{\scriptstyle 1})$.

10.5 Composition in Story-Based Categorizations

The composition of qualitative relations involves three entities k, l, and m. If we know the relation between the pair (k, l), say, R_A, and the relation between the pair (l, m), say, R_B, what would be, then, the possible relation(s) for the pair (k, m)? Such relation or relations $\widetilde{R_C} = \{R_{C_1}, \ldots, R_{C_N}\}$ is called the '*composition*' of R_A and R_B (See examples; spatial representations Sect. 4.3.1, motion representations Sect. 10.6). The composition is expressed symbolically as the operation '$\circ$', $\widetilde{R_C} = R_A \circ R_B$. In the case of beaded story-based relations we write $\widetilde{S_C(R_C)} = S_A(R_A) \circ S_B(R_B)$, which can be explicitly written as a set of beaded relations, i.e., $\widetilde{S_C(R_C)} = \{S_{C_1}(R_{C_1}), \ldots, S_{C_N}(R_{C_N})\}$. We write the *wide tilde* $\widetilde{\square}$ to denote the result of the composition, and, thus, $\widetilde{S_C(R_C)}$ refers to a set (a disjunction) of relations. In contrast, without tilde, only a single relation $S_C(R_C)$ is meant.

The result of the composition is seldom a single relation, $\widetilde{S_C(R_C)} = \{S_{C_1}(R_{C_1})\}$. For example, in RCC, if the entities (k, l) fulfil the relation PO, and (l, m) fulfil TPP, then (k, m) may fulfil one of these relations: PO, TPP, or NTPP. We express it as, $\mathrm{PO} \circ \mathrm{TPP} = \{\mathrm{PO}, \mathrm{TPP}, \mathrm{NTPP}\}$. The full table of RCC composition values are displayed in Randell, Cohn, et al. 1992, and an algorithm to find the composition in OPRA_1 is presented in Mossakowski and Moratz 2012.

Finding the composition is an arduous task that must be tailored to every representation. Though we do not exactly solve the composition for story-based relations, we take a step towards the computation of the composition by limiting its possible results. Indeed, we build an operation with story-based relations, the '*narrative composition*': $S_A(R_A) \triangledown S_B(R_B)$, for beaded relations, and $S_A \triangledown S_B$, for the bare relations. The *narrative composition* yields a small superset of the standard composition, but usually does not coincide with it (i.e., $S_A(R_A) \circ S_B(R_B) \subsetneq S_A(R_A) \triangledown S_B(R_B)$, and $S_A \circ S_B \subsetneq S_A \triangledown S_B$).

10.5.1 Defining the Narrative Composition: $S_A(R_A) \triangledown S_B(R_B)$

In this section, we describe how we end up with the narrative composition of *beaded* story-based relations, $S_A(R_A) \triangledown S_B(R_B)$. However, the narrative composition of bare story-based relations, $S_A \triangledown S_B$, will be presented in Section 10.6.3 by generalizing $S_A(R_A) \triangledown S_B(R_B)$.

The narrative composition is the result of gathering *necessary* conditions that the standard composition, $S_A(R_A) \circ S_B(R_B)$, must fulfil. Unfortunately, these conditions are *not sufficient*, i.e., the narrative composition often adds extra relations to the standard composition (as shown in the example of Section 10.6.2). Nevertheless, the narrative composition provides a good first approach to the standard composition.

For a start, we approach $\widetilde{S_C(R_C)} = S_A(R_A) \circ S_B(R_B)$ by *separately* considering its *story component* $\widetilde{S_C}$ and its *position component* $\widetilde{R_C}$. It is obvious that every relation belonging to the position component of the composition relation must belong to the composition of the position components of the composed relations, i.e., $\forall R_C \in \widetilde{R_C} \Rightarrow R_C \in R_A \circ R_B$, because the position component contains information about a single instant, and, thus, it works as a standard spatial relation. Nonetheless, remember that the result of our composition are not two separate sets, the position components $\widetilde{R_C}$ and the story components $\widetilde{S_C}$, but a set of beaded relations $\widetilde{S_C(R_C)}$: We must consider the interplay between position, R_C, and story components, S_C.

When we look at the story components, i.e., the stories, S_A and S_B, things become unexpectedly intricate. Foremost, we express the bare stories S_A and S_B as sequences of relations $R_i \in \mathcal{R}$, that is, $S_A = (R_{A_1}, R_{A_2}, \ldots, R_{A_i}, \ldots, R_{A_n})$ and $S_B = (R_{B_1}, R_{B_2}, \ldots, R_{B_j}, \ldots, R_{B_m})$. It is clear that each element R_{C_k} of a composition story $S_C = (R_{C_1}, R_{C_2}, \ldots, R_{C_k}, \ldots, R_{C_o}) \in \widetilde{S_C} = S_A \circ S_B$ is obtained by composition of relations constituting the composed stories, i.e., by $R_{A_i} \circ R_{B_j}$ (See Eq. (10.10a)). However, it is not obvious which relations $R_{A_i} \in S_A$ are composed with which relations $R_{B_j} \in S_B$, and in which order they are composed to obtain each relation $R_{C_k} \in S_C$. This is the main goal of the '*narrative composition*' $S_A(R_A) \triangledown S_B(R_B)$: to create feasible sequences of composed relations $R_{A_i} \in S_A$ and $R_{B_j} \in S_B$ that yield the composition stories $S_C \in \widetilde{S_C}$.

We show how far from trivial the composition $S_A(R_A) \circ S_B(R_B)$ is by means of a Motion-RCC example. If we want to compute the composition $\widetilde{S_C(R_C)} = S_{11}(\mathrm{DC}) \circ S_{11}(\mathrm{DC})$, where $S_{11} = (\mathrm{DC})$, then $\mathrm{DC} \circ \mathrm{DC}$ yields simply $\{(\mathrm{DC}), (\mathrm{EC}), (\mathrm{PO}), (\mathrm{TPP}), (\mathrm{NTPP}), (\mathrm{TPPI}), (\mathrm{NTPPI}), (\mathrm{EQ})\}$, which are only *one-element* sequences. This is utterly unsatisfactory: we obtain sequences that are not feasible stories, such as (EC), and, even worse, longer sequences can never be obtained this way; for example, the sequence $S_C = S_{13} = (\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{EC}, \mathrm{DC})$, which is also a possible composition story of $S_{11} \circ S_{11}$.

The composition of the story components, S_A and S_B, based on their constituting relations $R_{A_i} \in S_A$ and $R_{B_j} \in S_B$ is so peculiar because their constituting relations often occur in a *time* *interval* (t_1, t_2) and not in a single time instant t_0 alone. As a contrast, the composition of R_{A_i} and R_{B_j} in the generating representations $\mathcal{R}$ is defined for single *time instants*, or, equivalently, for static scenarios. This incompatibility between both compositions is overcome by the narrative composition $S_A(R_A) \triangledown S_B(R_B)$.

Narrative composition of beaded Stories: $\mathbf{S_A(R_A) \triangledown S_B(R_B)}$ In the following, we define the narrative composition of beaded stories, $S_A(R_A) \triangledown S_B(R_B)$, as an enumeration of necessary conditions that each story S_C belonging to the standard composition, $S_A(R_A) \circ S_B(R_B)$, must fulfil. In that way, we ensure that the narrative composition contains the standard composition, i.e.,

$$S_A(R_A) \circ S_B(R_B) \subset S_A(R_A) \triangledown S_B(R_B) \tag{10.9}$$

We denote each composition story $S_C \in S_A(R_A) \circ S_B(R_B)$ as the sequence of relations $\{R_{C_1}, \ldots, R_{C_i}, \ldots, R_{C_m}\}$; we denote the first composed story S_A as $\{R_{A_1}, \ldots, R_{A_i}, \ldots, R_{A_n}\}$, and the second composed story S_B as $\{R_{B_1}, \ldots, R_{B_i}, \ldots, R_{B_o}\}$.

I. Each element of a composition story, $R_{C_i} \in S_C$, must belong to the standard composition of two elements of the composed stories, S_A and S_B.

$$\forall R_{C_i} \in S_C \quad \exists R_{A_j} \in S_A \text{ and } \exists R_{B_k} \in S_B \mid R_{C_i} \in R_{A_j} \circ R_{B_k} \tag{10.10a}$$

II. The first element of the composition story, R_{C_1}, must belong to the composition of the first elements of the composed stories, and the last element of the composition story, R_{C_m}, must necessarily belong to the last elements of the composed stories.

$$S_C = \{R_{C_1}, \ldots, R_{C_i}, \ldots, R_{C_n}\} \Rightarrow R_{C_1} \in R_{A_1} \circ R_{B_1} \text{ and } R_{C_m} \in R_{A_n} \circ R_{B_o} \tag{10.10b}$$

III. At least one element of the composition story, R_{C_i}, must belong to the composition of the position components of the composed beaded stories.

$$\exists R_{C_i} \in S_C \mid R_{C_i} \in R_A \circ R_B \tag{10.10c}$$

IV. Successive elements of the composition story, S_C, must belong to the composition of successive or to the same elements of the composed stories, S_A and S_B.

$$\begin{aligned} &\forall R_{C_i}, R_{C_{i+1}} \in S_C \quad \exists j, k, l, m \mid R_{C_i} = R_{A_j} \circ R_{B_k}; R_{C_{i+1}} = R_{A_l} \circ R_{B_m} \\ &\qquad\qquad\qquad\qquad \text{where } j \leq l \leq j+1 \text{ and } k \leq m \leq k+1 \end{aligned} \tag{10.10d}$$

Hence, the '*narrative composition*' of beaded stories, $S_A(R_A) \triangledown S_B(R_B)$, is the set of beaded stories $S_C(R_C)$ that fulfil the conditions in Items I. to IV.. We did not rigorously proved these conditions as they can be intuitively verified.

10.5.2 Computing the Narrative Composition

In this section, we provide a more friendly method to compute the *narrative composition* than just giving the conditions it fulfils: below, we express Items I. to IV. as an algorithm. Furthermore, we define some concepts—the *substrings composition*, *composition path*, and *narrative composition matrix*—that simplify the execution of the algorithm.

Algorithm for narrative composition $\mathbf{S_A(R_A)} \triangledown \mathbf{S_B(R_B)}$

0. Initially, the narrative composition is empty; it contains no relation, $\widetilde{S_C(R_C)} = \emptyset$

1. We begin with the composition of the first relations in each story, i.e., with $R_{A_1} \circ R_{B_1}$, and we build *all* possible sequences of relations of this composition that are *prefixes*, i.e., that are beginning substrings, of stories in $\Sigma_{\mathcal{R}}$ (the empty sequence '( )' is not allowed). We call this '*set of prefixes*' P.

2. We choose a composition of relations that is neighbour of the previous composition used. More concretely, if the previous composition was $R_{A_i} \circ R_{B_j}$, we choose $R_{A_i} \circ R_{B_{j+1}}$, or $R_{A_{i+1}} \circ R_{B_j}$, or $R_{A_{i+1}} \circ R_{B_{j+1}}$. Let us say we choose $R_{A_i} \circ R_{B_{j+1}}$. We build all possible prefixes of stories in $\Sigma_{\mathcal{R}}$ obtained by concatenating the substrings of P and all possible substrings of stories generated with the relations resulting of the chosen composition, e.g., $R_{A_i} \circ R_{B_{j+1}}$, (we fuse the repeated relations on subsequent positions). We redefine the prefix set, P, as this new set of prefixes.

3. We iterate step 2 until we arrive at the composition of the end relations in S_A and S_B, i.e., until we arrive at $R_{A_n} \circ R_{B_o}$. And we also build all possible prefixes of stories in $\Sigma_{\mathcal{R}}$ obtained by concatenating the sequences of P and all possible stories substrings generated with the relations of $R_{A_n} \circ R_{B_o}$. We redefine P as this new set of prefixes.

   As we iterate step 2, one of the compositions we choose has to be $R_A \circ R_B$, since we are computing $S_A(R_A) \triangledown S_B(R_B)$.

4. The elements of P that form stories $S_C \in \Sigma_{\mathcal{R}}$ are joined with their respective position components, R_C, obtained by the composition of $R_A \circ R_B$ and added to the set $\widetilde{S_C(R_C)}$

5. We repeat steps 1 to 4 until we have chosen all possible paths of neighbouring compositions beginning at the start composition, $R_{A_1} \circ R_{B_1}$, and finishing at the end composition, $R_{A_n} \circ R_{B_o}$. In that way, the narrative composition $\widetilde{S_C(R_C)}$ contains, at last, all possible relations $S_C(R_C)$ fulfilling the conditions in Items I. to IV.. In other words, $\widetilde{S_C(R_C)}$ is the narrative composition $S_A(R_A) \triangledown S_B(R_B)$.

We realize that the algorithm above bases chiefly in two combinatorial operations: the generation of story *prefixes* (see items 1 to 3) and the generation of *composition paths* (see item 5). Accordingly, in the following, we devise tools that simplify the computation of these operations: we define the '*substring composition*' which simplifies the generation of prefixes, and we define the '*narrative composition matrix*' which facilitates the generation of composition paths.

Substrings composition: $\mathbf{R_A} \triangledown \mathbf{R_B}$ We define the '*substring composition*', $R_A \triangledown R_B$ of two relations R_A and R_B belonging to $\mathcal{R}$ as all possible *substrings of stories* in $\Sigma_{\mathcal{R}}$ that contain exclusively relations that belong to the standard composition $R_A \circ R_B$.

Alternatively, we can define $R_A \triangledown R_B$ as all possible *substrings of stories* in $\Sigma_{\mathcal{R}}$ that can be formed exclusively with relations belonging to $R_A \circ R_B$.

Note that we have one operator, the composition '$\triangledown$', with two different definitions depending on the nature its operands: when it acts upon the relations in $\mathcal{R}$, $R_j \in \mathcal{R}$, it is the 'substrings composition'; when it acts upon story-based relations—$S_i \in \text{Stories-}\mathcal{R}$ or $S_i(R_j) \in \text{Motion-}\mathcal{R}$—it is the 'narrative composition'.

We find it appropriate to overload the operator '$\triangledown$' because we use the substrings composition $R_A \triangledown R_B$ as a building block for the narrative composition $S_A(R_A) \triangledown S_B(R_B)$.

Example 10.1 (Substrings composition) The standard composition of DC and EC yields five possible relations: $\mathrm{DC} \circ \mathrm{EC} = \{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPPI}, \mathrm{NTPPI}\}$. Consequently, $\mathrm{DC} \triangledown \mathrm{EC}$ is formed by all the combinations of relations in $\{\mathrm{DC}, \mathrm{EC}, \mathrm{PO}, \mathrm{TPPI}, \mathrm{NTPPI}\}$ that form substrings of stories in Σ_{RCC}, i.e., $\mathrm{DC} \triangledown \mathrm{EC} = \{(\mathrm{DC}), (\mathrm{EC}), \ldots, (\mathrm{DC}, \mathrm{EC}, \mathrm{DC}), \ldots, (\mathrm{EC}, \mathrm{PO}, \mathrm{EC}), \ldots, (\mathrm{EC}, \mathrm{PO}, \mathrm{TPPI}, \mathrm{PO}), \ldots\}$. Indeed, for example, (DC) is substring of story S_{11} (See Tab. 8.1)—in fact, it is the full story—(EC, PO, EC) is substring of S_{13}, and so on. However, a sequence such as (EC, TPPI, EC), though it is a combination of relations from the composition $\mathrm{DC} \circ \mathrm{EC}$, does not belong to the narrative composition $\mathrm{DC} \triangledown \mathrm{EC}$, because it is not the substring of any story in Σ_{RCC}.

The computation of the narrative composition—or at least its formulation—is greatly simplified by using the substrings composition. For example, in step 2, the expression "all possible substrings of stories generated with the relations of the chosen composition, e.g., $R_{A_i} \circ R_{B_{j+1}}$" can be reformulated as simply as "$R_{A_i} \triangledown R_{B_{j+1}}$". For that reason, henceforth, we will mainly refer to the *substring composition* instead to the *standard composition*, $R_{A_i} \circ R_{B_{j+1}}$.

Composition path A '*composition path*' is a sequence of substrings compositions as defined in steps 2 and 3. A sequence that begins with $R_{A_1} \triangledown R_{B_1}$; ends with $R_{A_n} \triangledown R_{B_o}$; and after each element $R_{A_i} \triangledown R_{B_j}$ follows either $R_{A_{i+1}} \triangledown R_{B_j}$, or $R_{A_i} \triangledown R_{B_{j+1}}$, or $R_{A_{i+1}} \triangledown R_{B_{j+1}}$. Moreover, since a composition path stems from computing a narrative composition $S_A(R_A) \triangledown S_B(R_B)$, the narrative composition of the position components, $R_A \triangledown R_B$, must be present in the path.

In Equation (10.11), we represent a composition path in a most general way. The narrative composition of the position components R_A and R_B stands out grey-boxed. In Example 10.2, we generate three concrete examples of composition paths.

$$(R_{A_1} \triangledown R_{B_1}, \ldots, R_{A_i} \triangledown R_{B_j}, \ldots, \boxed{R_A \triangledown R_B}, \ldots, R_{A_n} \triangledown R_{B_o}) \tag{10.11}$$

Notably, a *composition path*, such as in Equation (10.11), represents not a single story, but a set of stories: the set of all stories belonging to $\Sigma_{\mathcal{R}}$ that are formed by combinatorial concatenation of a substring of each $R_{A_i} \triangledown R_{B_j}$.

Example 10.2 (Composition paths) Suppose we want to compute the narrative composition $S_A(R_{A_1}) \triangledown S_B(R_{B_3})$, where the stories are $S_A = (R_{A_1}, R_{A_2}, R_{A_3})$ and $S_B = (R_{B_1}, R_{B_2}, R_{B_3}, R_{B_4}, R_{B_5})$. Then some possible *composition paths* are shown in Equations (10.12a) to (10.12c). The grey boxes contain the narrative composition of the position components $R_{A_1} \triangledown R_{B_3}$.

$$\begin{aligned}(R_{A_1} \triangledown R_{B_1}, R_{A_1} \triangledown R_{B_2}, \boxed{R_{A_1} \triangledown R_{B_3}}, R_{A_1} \triangledown R_{B_4},\\ R_{A_1} \triangledown R_{B_5}, R_{A_2} \triangledown R_{B_5}, R_{A_3} \triangledown R_{B_5})\end{aligned} \tag{10.12a}$$

$$\begin{aligned}(R_{A_1} \triangledown R_{B_1}, R_{A_1} \triangledown R_{B_2}, \boxed{R_{A_1} \triangledown R_{B_3}}, R_{A_2} \triangledown R_{B_4},\\ R_{A_3} \triangledown R_{B_5})\end{aligned} \tag{10.12b}$$

$$\begin{aligned}(R_{A_1} \triangledown R_{B_1}, R_{A_1} \triangledown R_{B_2}, \boxed{R_{A_1} \triangledown R_{B_3}}, R_{A_2} \triangledown R_{B_3},\\ R_{A_2} \triangledown R_{B_4}, R_{A_3} \triangledown R_{B_4}, R_{A_3} \triangledown R_{B_5})\end{aligned} \tag{10.12c}$$

Finding *all* composition paths is the last step (item 5) in our way to obtain the narrative composition $\widetilde{S_C(R_C)} = S_A(R_A) \triangledown S_B(R_B)$. Actually, each composition path produces relations $S_C(R_C)$ belonging to the narrative composition $\widetilde{S_C(R_C)}$, so that the union of all the $S_C(R_C)$ relations in the composition paths yields the narrative composition.

Narrative composition matrix Even if finding all composition paths is no dire computational challenge, we can make this task easier by using a tabular representation: the 'narrative composition matrix'.

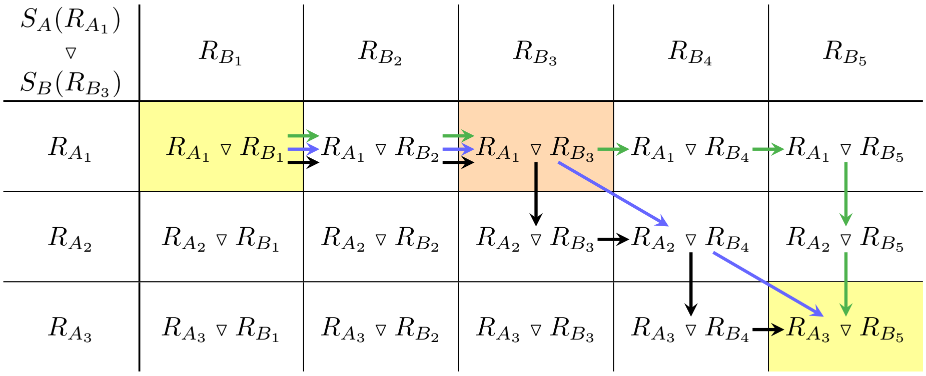
The '*narrative composition matrix*' of a narrative composition, e.g., $S_A(R_{A_1}) \triangledown S_B(R_{B_3})$, is the two-dimensional matrix formed by all possible substring compositions of the relations that constitute the story components, i.e., it is formed by all $R_{A_i} \triangledown R_{B_j}$ so that $R_{A_i} \in S_A, R_{B_j} \in S_B$. On this matrix, the *composition paths* can be clearly visualized. For example, in Table 10.1, we represent the composition matrix $S_A(R_{A_1}) \triangledown S_B(R_{B_3})$ along with the composition paths of Equation (10.12).

The conditions that each composition path must fulfil, as we stated them in the definition, can be reformulated as drawing rules for paths in the composition matrix:

i. Every path begins in the upper left corner and ends in the lower right corner (yellow coloured cells in Tables 10.1 and 10.2)

ii. Every path can only be generated by moving from every cell either rightwards, downwards or diagonally rightwards downwards.

iii. Every path must pass through the cell containing the narrative composition of the position components, i.e., $R_A \triangledown R_B$; (orange coloured cell in Tables 10.1 and 10.2). For example, for the narrative composition $S_A(R_{A_1}) \triangledown S_B(R_{B_3})$, in Table 10.1, it is $R_{A_1} \triangledown R_{B_3}$.

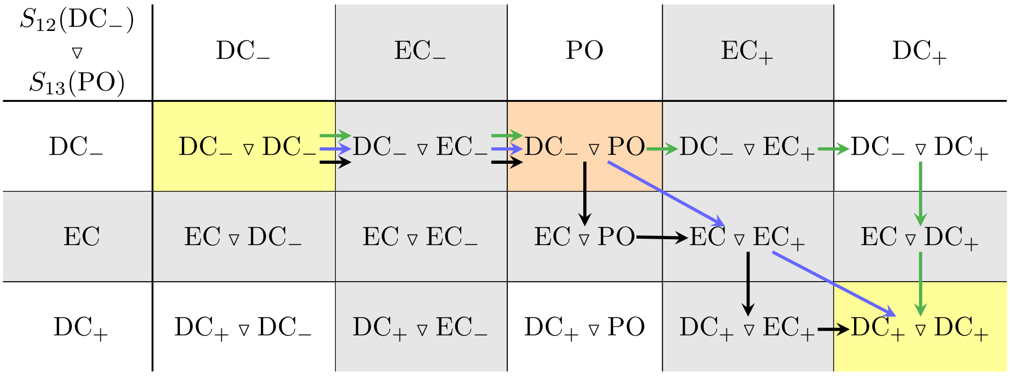


$S_A(R_{A_1})$ $\triangledown$ $S_B(R_{B_3})$	R_{B_1}	R_{B_2}	R_{B_3}	R_{B_4}	R_{B_5}
R_{A_1}	$R_{A_1} \triangledown R_{B_1}$	$R_{A_1} \triangledown R_{B_2}$	$R_{A_1} \triangledown R_{B_3}$	$R_{A_1} \triangledown R_{B_4}$	$R_{A_1} \triangledown R_{B_5}$
R_{A_2}	$R_{A_2} \triangledown R_{B_1}$	$R_{A_2} \triangledown R_{B_2}$	$R_{A_2} \triangledown R_{B_3}$	$R_{A_2} \triangledown R_{B_4}$	$R_{A_2} \triangledown R_{B_5}$
R_{A_3}	$R_{A_3} \triangledown R_{B_1}$	$R_{A_3} \triangledown R_{B_2}$	$R_{A_3} \triangledown R_{B_3}$	$R_{A_3} \triangledown R_{B_4}$	$R_{A_3} \triangledown R_{B_5}$

Table 10.1: *Narrative composition matrix* of the beaded motion relations $S_A(R_{A_1})$ and $S_B(R_{B_3})$. The corresponding stories are $S_A = (R_{A_1}, R_{A_2}, R_{A_3})$ and $S_B = (R_{B_1}, R_{B_2}, R_{B_3}, R_{B_4}, R_{B_5})$. We show the three examples of valid paths in Equation (10.12): The *green* path corresponds to $(R_{A_1} \triangledown R_{B_1}, R_{A_1} \triangledown R_{B_2}, R_{A_1} \triangledown R_{B_3}, R_{A_1} \triangledown R_{B_4}, R_{A_1} \triangledown R_{B_5}, R_{A_2} \triangledown R_{B_5}, R_{A_3} \triangledown R_{B_5})$ (Eq. (10.12a)); the *blue* path to $(R_{A_1} \triangledown R_{B_1}, R_{A_1} \triangledown R_{B_2}, R_{A_1} \triangledown R_{B_3}, R_{A_2} \triangledown R_{B_4}, R_{A_3} \triangledown R_{B_5})$ (Eq. (10.12b)); and the *black* corresponds to $(R_{A_1} \triangledown R_{B_1}, R_{A_1} \triangledown R_{B_2}, R_{A_1} \triangledown R_{B_3}, R_{A_2} \triangledown R_{B_3}, R_{A_2} \triangledown R_{B_4}, R_{A_3} \triangledown R_{B_4}, R_{A_3} \triangledown R_{B_5})$ (Eq. (10.12c)). According to Items i. to iii. the paths must go through the coloured cells.

Source: Purcalla Arrufi and Kirsch (2018b)

Additional constraints in the composition paths We can reduce the large combinations of possible composition paths to compute the narrative composition, if it happens that some



$S_{12}(\mathrm{DC}_-)$ $\triangledown$ $S_{13}(\mathrm{PO})$	DC_-	EC_-	PO	EC_+	DC_+
DC_-	$\mathrm{DC}_- \triangledown \mathrm{DC}_-$	$\mathrm{DC}_- \triangledown \mathrm{EC}_-$	$\mathrm{DC}_- \triangledown \mathrm{PO}$	$\mathrm{DC}_- \triangledown \mathrm{EC}_+$	$\mathrm{DC}_- \triangledown \mathrm{DC}_+$
EC	$\mathrm{EC} \triangledown \mathrm{DC}_-$	$\mathrm{EC} \triangledown \mathrm{EC}_-$	$\mathrm{EC} \triangledown \mathrm{PO}$	$\mathrm{EC} \triangledown \mathrm{EC}_+$	$\mathrm{EC} \triangledown \mathrm{DC}_+$
DC_+	$\mathrm{DC}_+ \triangledown \mathrm{DC}_-$	$\mathrm{DC}_+ \triangledown \mathrm{EC}_-$	$\mathrm{DC}_+ \triangledown \mathrm{PO}$	$\mathrm{DC}_+ \triangledown \mathrm{EC}_+$	$\mathrm{DC}_+ \triangledown \mathrm{DC}_+$

Table 10.2: This table exemplifies a real Motion-RCC case of the general case in Table 10.1. We compose narratively the motion relations $S_{12}(\mathrm{DC}_-)$ and $S_{13}(\mathrm{PO})$; where the stories components are $S_{12} = (\mathrm{DC}_-, \mathrm{EC}, \mathrm{DC}_+)$, $S_{13} = (\mathrm{DC}_-, \mathrm{EC}_-, \mathrm{PO}, \mathrm{EC}_+, \mathrm{DC}_+)$. The difference are the *grey cells*—they correspond to *punctual relations*—therefore, we cannot directly step from grey cell into grey cell; this makes the black path an invalid path in this representation. However, the narrative compositions given by the blue and green are perfectly valid.

Source: Purcalla Arrufi and Kirsch (2018b)

relations R_{A_i} or R_{B_j} of the composed stories, S_A and S_B, occur only in a *single time instant* t_0—We call them '*punctual relations*'. A decisive property is that each punctual relation, e.g., R_{A_i}, can only belong to a single substring composition, e.g., $R_{A_i} \triangledown R_{B_j}$. Indeed, if a punctual relation would appear consecutively in two substring compositions, e.g., $\ldots, \underline{R_{A_i}} \triangledown R_{B_j}, \underline{R_{A_i}} \triangledown R_{B_{j+1}}, \ldots$, then it would necessarily last more than a single time instant.

We can add this new constraint to the rules for creating paths in the narrative composition matrix (items i. to iii.):

iv. *(dominance constraint)* In every path, a *punctual relation* cannot appear in two consecutive substring compositions.

As an example, consider the narrative composition matrix in test Table 10.2. It involves two relations in Motion-RCC, namely, $S_{12}(\mathrm{DC}_-)$ and $S_{13}(\mathrm{PO})$. Now, remarkably, in the RCC stories, the relations EC, TPP, and TPPI are *punctual relations* when the involved entities are strictly convex (e.g., circles but not squares), as we assumed in this work (Sect. 8.2.1). Thus, a path in the narrative composition matrix cannot have two consecutive substring compositions with the EC relation; for instance, the black path in Table 10.2, ($\mathrm{DC}_- \triangledown \mathrm{DC}_-$, $\mathrm{DC}_- \triangledown \mathrm{EC}_-$, $\mathrm{DC}_- \triangledown \mathrm{PO}$, $\mathrm{EC} \triangledown \mathrm{PO}$, $\mathrm{EC} \triangledown \mathrm{EC}_+$, $\mathrm{DC}_+ \triangledown \mathrm{EC}_+$, $\mathrm{DC}_+ \triangledown \mathrm{DC}_+$), is unfeasible because the *punctual* relation EC appears twice *consecutively*, namely, in '$\ldots, \underline{\mathrm{EC}} \triangledown \mathrm{PO}, \underline{\mathrm{EC}} \triangledown \mathrm{EC}_+, \ldots$'. This is not the only consecutive appearance of the relation EC, it appears again twice *consecutively* in '$\ldots, \mathrm{EC} \triangledown \underline{\mathrm{EC}_+}, \mathrm{DC}_+ \triangledown \underline{\mathrm{EC}_+}, \ldots$'.

We can visually apply this constraint by grey-colouring the rows and columns of the narrative composition matrix (as in Tab. 10.2) and enforcing that paths do not go along the grey rows and columns. That is, a path can contain grey cells but not consecutively.

We set apart this rule, iv., from the other ones, i.–iii., because the information about the time intervals of each relation $R_i \in \mathcal{R}$ is not always available. In fact, as we saw in Section 7.4, information about time intervals of relations is equivalent to information about their *dominance* *properties*—For that reason we call Item iv. 'dominance constraint'. Certainly, only relations that are *position states* in the dominance sense can—but they do not have to—be a *punctual state*. Moreover, the geometry of the entities determines which position states are punctual, so that, for example, in RCC the relation EC is punctual for certain geometries of the entities, e.g., strictly convex regions, but for others it is not.

Interestingly, according to our definitions of the entities in Section 8.2, RCC and OPRA$_1$ both have *punctual relations*; what is more, all position states are punctual relations. These '*punctual relations*' are EC, TPP, and EQ, in RCC; and, in OPRA$_1$, every relation $\angle^y_x$ that contains 0 or 2 (e.g., $\angle^3_0$, $\angle^2_2$, or $\angle 0$).

10.6 Examples of Narrative and Standard Composition in Story-Based Representations

In this Section we present two full examples of *narrative* composition in the story-based relations of Motion-OPRA$_1$. Subsequently, we check the narrative results to obtain the *standard* composition between such relations; these examples show the peculiarity that both compositions are equal. We apply the algorithm in Section 10.5.2 assisted by the narrative composition matrix to obtain the composition paths.

10.6.1 Compositions $S_{\mathrm{C21}}(\angle^3_1) \triangledown S_{\mathrm{T\text{-}1}}(\angle^2_0)$ and $S_{\mathrm{C21}}(\angle^3_1) \circ S_{\mathrm{T\text{-}1}}(\angle^2_0)$

We aim to compute the narrative composition $S_{\mathrm{C21}}(\angle^3_1) \triangledown S_{\mathrm{T\text{-}1}}(\angle^2_0)$ (See Figs. 10.1a and 10.1b).

Firstly, we express the stories as sequence of spatial relations: $S_{\mathrm{C21}} = (\angle^1_3, \angle^1_0, \angle^1_1, \angle^2_1, \angle^3_1)$ and $S_{\mathrm{T\text{-}1}} = (\angle^2_0, \angle 0, \angle^0_2)$. Secondly, we compute the narrative composition of stories by means of the narrative composition matrix (Table 10.3).

The only possible composition path in the matrix that passes through the composed position components, i.e., $\angle^3_1 \triangledown \angle^2_0$ (blue cell), is the blue path (See Tab. 10.3a). By resolving the substring compositions (Tab. 10.3b), we obtain all temporal sequences of relations generated by the blue path, for example, $(\angle^1_0, \angle^1_1, \angle^1_2, \angle^2_1, \angle^3_0, \angle^3_1, \angle^3_3)$, and $(\angle^1_1, \angle^2_1, \angle^3_1)$. Only two of all generated sequences are Motion-OPRA$_1$ stories (see Table 8.2): S_{C21} and $S_{\mathrm{B21}} = (\angle^1_1, \angle^2_1, \angle^3_1)$. The relation of $\angle^3_1 \circ \angle^2_0$ that is part of the respective resultant stories corresponds to the position components of the resultant relation. Accordingly, we obtain the narrative composition

$$S_{\mathrm{C21}}(\angle^3_1) \triangledown S_{\mathrm{T\text{-}1}}(\angle^2_0) = \{S_{\mathrm{C21}}(\angle^3_1), S_{\mathrm{B21}}(\angle^3_1)\}$$

If we look at the resultant relations of the narrative composition, we see that all are feasible. Particularly, S_{B21} is feasible even though the second entity in the relation is motionless, i.e., $\vec{v}_m = 0$, because the composed story $S_{\mathrm{T\text{-}1}}$ is possible for $\vec{v}_m = 0$. Therefore, finally, both the standard and narrative composition coincide,

$$S_{\mathrm{C21}}(\angle^3_1) \circ S_{\mathrm{T\text{-}1}}(\angle^2_0) = \{S_{\mathrm{C21}}(\angle^3_1), S_{\mathrm{B21}}(\angle^3_1)\} \quad \subseteq \quad S_{\mathrm{C21}}(\angle^3_1) \triangledown S_{\mathrm{T\text{-}1}}(\angle^2_0) = \{S_{\mathrm{C21}}(\angle^3_1), S_{\mathrm{B21}}(\angle^3_1)\}$$

10.6.2 Compositions $S_{\mathrm{C21}}(\angle^1_3) \triangledown S_{\mathrm{T\text{-}1}}(\angle^0_2)$ and $S_{\mathrm{C21}}(\angle^1_3) \circ S_{\mathrm{T\text{-}1}}(\angle^0_2)$

Proceeding as in Section 10.6.1, we compute the narrative composition $S_{\mathrm{C21}}(\angle^1_3) \triangledown S_{\mathrm{T\text{-}1}}(\angle^0_2)$ using the narrative composition matrix (Table 10.3). In this case, the only possible path in the matrix that passes through the composed position components, i.e., $\angle^1_3 \triangledown \angle^0_2$ (red cell), is the red path.

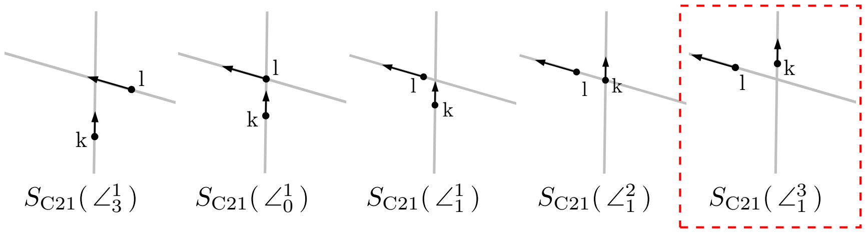
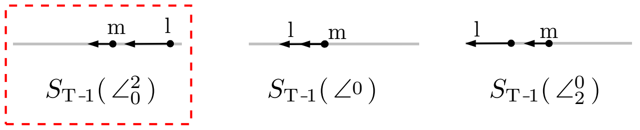
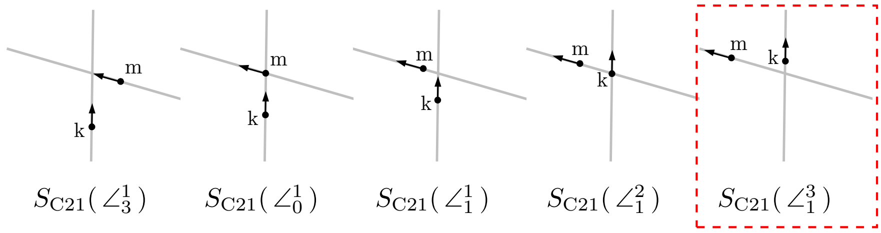
	MOTION RELATION	ENTITIES	STORY SEQUENCE	VERBALIZATION OF THE COMPOSITION
(a)	$S_{\mathrm{C21}}(\angle^{3}_{1})$	k, l	 $S_{\mathrm{C21}}(\angle^{1}_{3})$ $S_{\mathrm{C21}}(\angle^{1}_{0})$ $S_{\mathrm{C21}}(\angle^{1}_{1})$ $S_{\mathrm{C21}}(\angle^{2}_{1})$ $S_{\mathrm{C21}}(\angle^{3}_{1})$	The entity l came from the right of k and crossed first. The entity k crossed later.
(b)	$S_{\mathrm{T\text{-}1}}(\angle^{2}_{0})$	l, m	 $S_{\mathrm{T\text{-}1}}(\angle^{2}_{0})$ $S_{\mathrm{T\text{-}1}}(\angle 0)$ $S_{\mathrm{T\text{-}1}}(\angle^{0}_{2})$	The entity l goes after m on the same trajectory path, and it will overtake m.
(c)	$S_{\mathrm{C21}}(\angle^{3}_{1})$	k, m	 $S_{\mathrm{C21}}(\angle^{1}_{3})$ $S_{\mathrm{C21}}(\angle^{1}_{0})$ $S_{\mathrm{C21}}(\angle^{1}_{1})$ $S_{\mathrm{C21}}(\angle^{2}_{1})$ $S_{\mathrm{C21}}(\angle^{3}_{1})$	Based on the current motion relations in (a) and (b), i.e., composing (a) and (b), we conclude that the entity m came from the right of k and crossed first. The entity k crossed later.

Figure 10.1: We both visualize and verbally describe the computation of the composition $S_{\mathrm{C21}}(\angle^{3}_{1}) \circ S_{\mathrm{T\text{-}1}}(\angle^{2}_{0})$ (relations in row (a) and row (b), respectively) which yields only the relation $S_{\mathrm{C21}}(\angle^{3}_{1})$ (row (c)).

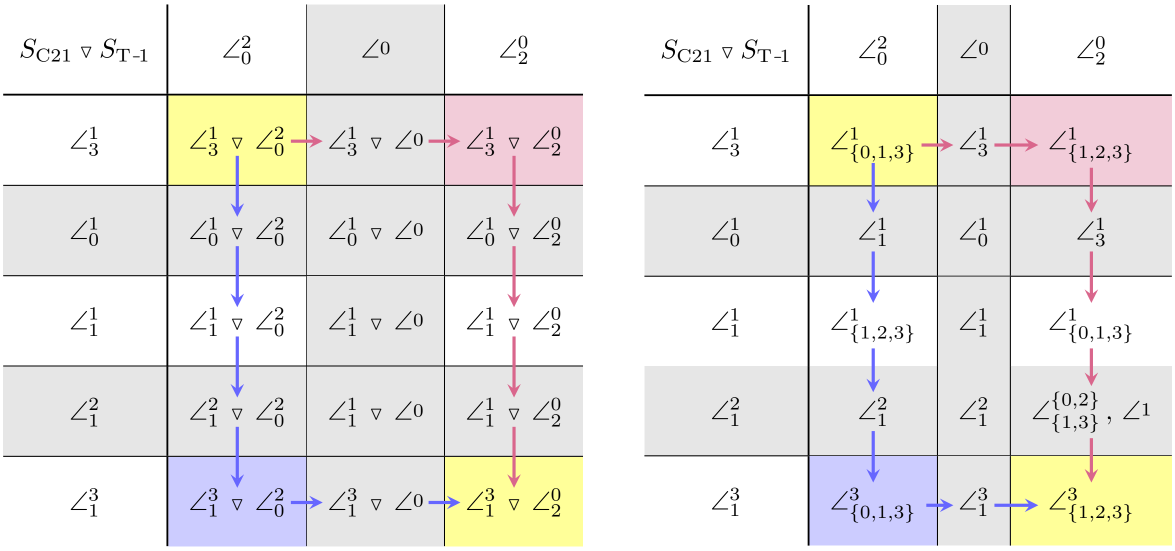
Source: heavily modified from Purcalla Arrufi and Kirsch (2018b); licensed under CC BY 4.0

If we analyse all possible sequences of relations that the red path generates by narrative composition, we obtain only four stories, which correspond to four beaded motion relations:

$$S_{\mathrm{C21}}(\angle^{1}_{3}) \triangledown S_{\mathrm{T\text{-}1}}(\angle^{0}_{2}) = \{S_{\mathrm{C2\text{-}1}}(\angle^{1}_{3}), S_{\mathrm{C20}}(\angle^{1}_{3}), S_{\mathrm{C21}}(\angle^{1}_{3}), S_{\mathrm{B2\text{-}1}}(\angle^{1}_{3})\}$$

Analogous as in Section 10.6.1, we see that all relations are feasible. Particularly, the relation $S_{\mathrm{B2\text{-}1}}(\angle^{1}_{3})$ is feasible even though the second entity in the relation is motionless, i.e., $\vec{v}_m = 0$, because the composed story $S_{\mathrm{T\text{-}1}}$ is possible for $\vec{v}_m = 0$. Therefore, finally, both the standard and narrative composition coincide,

$$S_{\mathrm{C21}}(\angle^{1}_{3}) \circ S_{\mathrm{T\text{-}1}}(\angle^{0}_{2}) = \{S_{\mathrm{C2\text{-}1}}(\angle^{1}_{3}), S_{\mathrm{C20}}(\angle^{1}_{3}), S_{\mathrm{C21}}(\angle^{1}_{3}), S_{\mathrm{B2\text{-}1}}(\angle^{1}_{3})\} \quad \subseteq \quad S_{\mathrm{C21}}(\angle^{1}_{3}) \triangledown S_{\mathrm{T\text{-}1}}(\angle^{0}_{2})$$



$S_{\mathrm{C21}} \triangledown S_{\mathrm{T\text{-}1}}$	$\angle^2_0$	$\angle 0$	$\angle^0_2$
$\angle^1_3$	$\angle^1_3 \triangledown \angle^2_0$	$\angle^1_3 \triangledown \angle 0$	$\angle^1_3 \triangledown \angle^0_2$
$\angle^1_0$	$\angle^1_0 \triangledown \angle^2_0$	$\angle^1_0 \triangledown \angle 0$	$\angle^1_0 \triangledown \angle^0_2$
$\angle^1_1$	$\angle^1_1 \triangledown \angle^2_0$	$\angle^1_1 \triangledown \angle 0$	$\angle^1_1 \triangledown \angle^0_2$
$\angle^2_1$	$\angle^2_1 \triangledown \angle^2_0$	$\angle^1_1 \triangledown \angle 0$	$\angle^1_1 \triangledown \angle^0_2$
$\angle^3_1$	$\angle^3_1 \triangledown \angle^2_0$	$\angle^3_1 \triangledown \angle 0$	$\angle^3_1 \triangledown \angle^0_2$

(a) *Narrative composition matrix* of the stories $S_{\mathrm{C21}} = (\angle^1_3, \angle^1_0, \angle^1_1, \angle^2_1, \angle^3_1)$ and $S_{\mathrm{T\text{-}1}} = (\angle^2_0, \angle 0, \angle^0_2)$.

$S_{\mathrm{C21}} \triangledown S_{\mathrm{T\text{-}1}}$	$\angle^2_0$	$\angle 0$	$\angle^0_2$
$\angle^1_3$	$\angle^1_{\{0,1,3\}}$	$\angle^1_3$	$\angle^1_{\{1,2,3\}}$
$\angle^1_0$	$\angle^1_1$	$\angle^1_0$	$\angle^1_3$
$\angle^1_1$	$\angle^1_{\{1,2,3\}}$	$\angle^1_1$	$\angle^1_{\{0,1,3\}}$
$\angle^2_1$	$\angle^2_1$	$\angle^2_1$	$\angle^{\{0,2\}}_{\{1,3\}}, \angle 1$
$\angle^3_1$	$\angle^3_{\{0,1,3\}}$	$\angle^3_1$	$\angle^3_{\{1,2,3\}}$

(b) We have computed the standard composition of relations in each cell of Table (a) (See the algorithm for OPRA$_1$ composition in Mossakowski and Moratz 2012). The substring composition on each cell is the combination of the relations in the cell.

Table 10.3: In these narrative composition matrices, we compute two examples of compositions: $S_{\mathrm{C21}}(\angle^3_1) \triangledown S_{\mathrm{T\text{-}1}}(\angle^2_0)$, blue path passing through $\angle^3_1 \triangledown \angle^2_0$; and $S_{\mathrm{C21}}(\angle^1_3) \triangledown S_{\mathrm{T\text{-}1}}(\angle^0_2)$, red path passing through $\angle^1_3 \triangledown \angle^0_2$.
We can use the same matrix for both compositions, because the story components of the relations, i.e., S_{C21} and $S_{\mathrm{T\text{-}1}}$, are the same. The start and end cell for both examples are the same, they are yellow coloured. The cell of the current position component is coloured according to the path.

Source: Purcalla Arrufi and Kirsch (2018b)

10.6.3 The Narrative Composition $S_A \triangledown S_B$

We define the narrative composition for *bare* relations, i.e., only stories, $S_A \triangledown S_B$, in Equation (10.13). We base the definition on the narrative compositions *beaded* stories, i.e., $S_A(R_A)$, and $S_B(R_B)$.

Definition 10.6.1 ***Narrative Composition of Bare Relations $S_A \triangledown S_B$*** The *bare* narrative composition is defined as the union of all stories obtained in the *beaded* narrative composition.

$$S_A \triangledown S_B = \{S_C \quad | \quad S_C(R_{C_i}) \in \bigcup_{\substack{\forall R_A \in S_A \\ \forall R_B \in S_B}} S_A(R_A) \triangledown S_B(R_B)\} \tag{10.13}$$

Proposition 10.6.1 *The narrative composition of bare stories $S_A \triangledown S_B$ is an upper bound of the composition of bare stories $S_A \circ S_B$.*

$$S_A \circ S_B \subset S_A \triangledown S_B$$

Proof. When looking at the stories as relations, we can express each bare story as the union of all the beaded stories that have the bare story as story component, i.e., $S = \bigcup_{R \in S} S(R)$ (Eq. (10.7a)). And by slightly working on the expression we obtain the result to prove.

$$S_A \circ S_B \overset{\substack{S = \bigcup_{R \in S} S(R) \\ \text{(Eq. (10.7a))}}}{=} \left(\bigcup_{R_A \in S_A} S_A(R_A) \right) \circ \left(\bigcup_{R_B \in S_B} S_B(R_B) \right)$$

$\downarrow$ Because of the distributivity property of the binary qualitative calculi arising from the definition of composition (Eq. (10.3a))
$(R_1 \cup R_2) \circ R_3 = (R_1 \circ R_3) \cup (R_2 \circ R_3)$
$R_3 \circ (R_1 \cup R_2) = (R_3 \circ R_1) \cup (R_3 \circ R_2)$

$$S_A \circ S_B = \bigcup_{\substack{\forall R_A \in S_A \\ \forall R_B \in S_B}} S_A(R_A) \circ S_B(R_B)$$

$\downarrow$ $\forall S_A(R_A), S_B(R_B)$
$S_A(R_A) \circ S_B(R_B) \subset S_A(R_A) \triangledown S_B(R_B)$ (Eq. (10.9))

$$S_A \circ S_B \subset \bigcup_{\substack{\forall R_A \in S_A \\ \forall R_B \in S_B}} S_A(R_A) \triangledown S_B(R_B) \overset{\substack{\text{By Def.} \\ \text{Eq. (10.13)}}}{=} S_A \triangledown S_B$$

□

Chapter 11

Additional Applications and Properties of Story-Based Categorizations

Throughout our work, and more specifically in Chapters 6 to 10, we have presented the foundations and basic uses of the story-based categorizations. In this chapter, we present additional properties and applications that are not so fundamental, but are derived from the first ones.

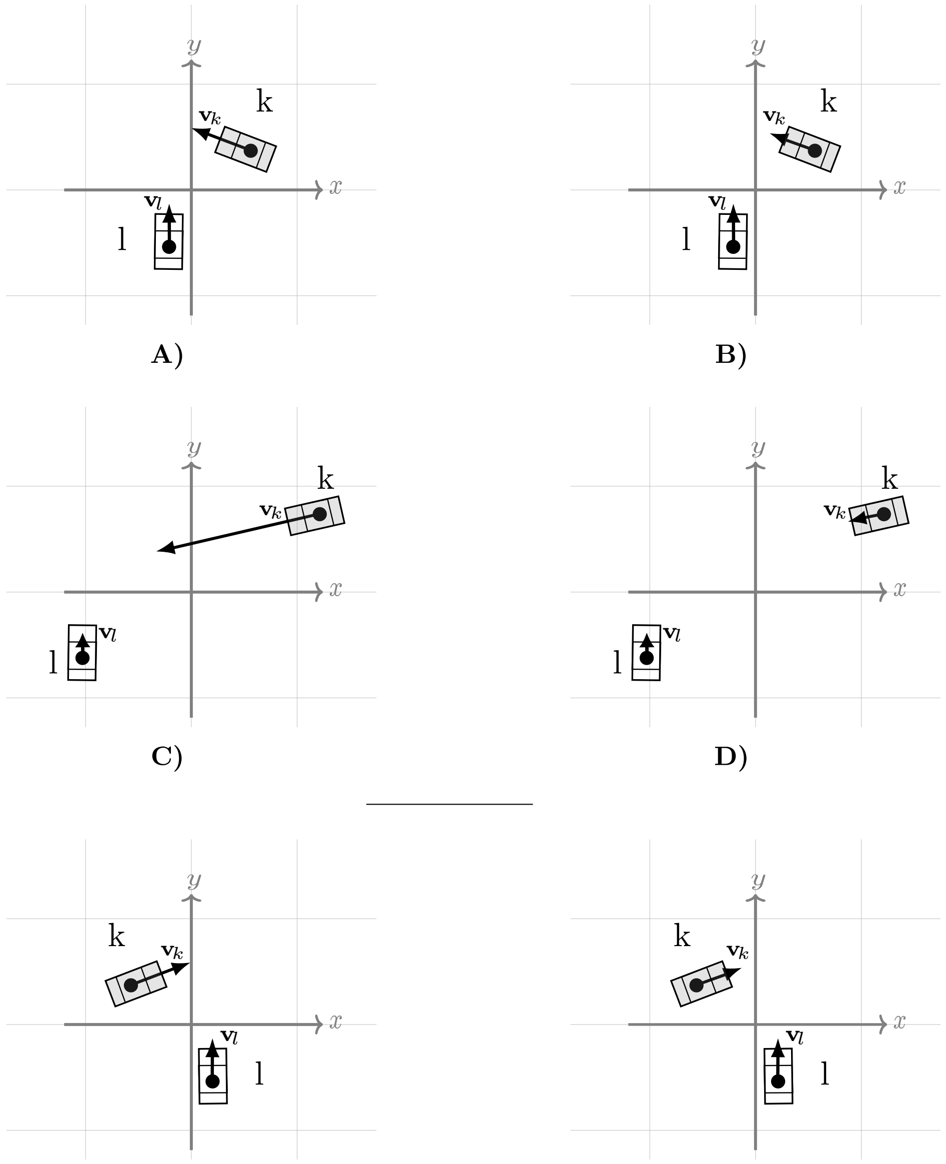
Any categorization is worthless in itself, unless it proves to have useful or attractive properties. Indeed, a categorization, as we understand it in its most basic sense, is just a partition of a certain space of states. Specifically, a motion categorization partitions a kinematic space (Sect. 6.1.1), leading to at least an uncountable variety of motion categories, an infinite number of categories higher than the cardinality of $\mathbb{R}$. There must be a very strong reason for choosing certain categorizations over the rest in the overwhelming landscape of motion categorizations. We believe that the properties we present here are such a reason.

Some properties and applications we present here are not thoroughly developed—they are sketched or just mentioned—they should be fully implemented in further work. Thus, in this chapter, we present also a great deal of future work on story-based categorization.

11.1 Creating a Variety of Categorization Criteria

In Section 2.2, we provided a glimpse into the variety of criteria in motion categorization; this is analogous to the variety of categorizations in natural objects (See Sect. 3.4.4). As with natural objects, the relevance of a categorization is linked to its purpose: without knowing the purpose, we cannot say beforehand which categorization is most relevant. Therefore, we believe that it is essential for a categorization model or method to possess the capacity to handle a wide range of categorizations: A categorization model should be highly adaptable, able to generate a variety of categorizations criteria—The greater the variety of categorizations in a method the higher number of purposes that a method can serve.

Following, we itemize three methods to create a fulsome variety of motion categorizations using the tools presented in previous chapters. Afterwards, in Sections 11.1.1 to 11.1.3, we illustrate these methods with the six scenarios in Figure 11.1; four of these scenarios were already presented as a categorization challenge in Chapter 2.



A) **B)** **C)** **D)** **E)** **F)**

Figure 11.1: These are six motion scenarios to illustrate the variety of motion categorizations. The first four motion scenarios (A, B, C, D) were introduced in Figure 2.1; here, we add scenarios E and F, which are, respectively, the scenarios A and B reflected about the y-axis.

Source: derivative of Purcalla Arrufi and Kirsch (2018a)

- We can create different story-based motion categorizations by using *different generating representations*. For instance, using two different generating representations, RCC and OPRA$_1$, we can create four different story-based categorizations: Stories-RCC, Stories-OPRA$_1$, Motion-RCC, and Motion-OPRA$_1$; each of them providing different categories for motion scenarios.

- We can create different story-based motion categorizations by combining them through *Cartesian product*. For instance, starting with Motion-RCC, Motion-OPRA$_1$, and Motion-QTC$_{\mathrm{B21}}$, we can generate Motion-RCC $\times$ Motion-OPRA$_1$, or Motion-RCC $\times$ Motion-QTC$_{\mathrm{B21}}$, or also, Motion-RCC $\times$ Motion-OPRA$_1$ $\times$ Motion-QTC$_{\mathrm{B21}}$, and so on.

- We can also create different motion categorizations by *combining the features* of the story-based categorizations. This method relies on our categorization model (Ch. 6). Instead of dealing with the story-based categorizations as a whole, we deal with their features. For example, we can pick one feature of Motion-RCC, e.g., $d_{\min}$, and one feature of Motion-OPRA$_1$, e.g., α_{vv}, and build a new motion categorization $\{d_{\min}, \alpha_{vv}\}$ that categorizes by means of these two features.

11.1.1 Variety based on the generating spatial representation

Here we exemplify how choosing a different qualitative representation (RCC or OPRA$_1$) as a generator of the story-based categorization yields a variety of categorizations.

Using *Motion-RCC* In that way, the scenarios of Figure 11.1 are categorized as follows: the scenarios A, C, D, E belong to the relation $S_{11}(\mathrm{DC}_-)$, and, thus, we group them in a category $(\mathrm{A}, \mathrm{C}, \mathrm{D}, \mathrm{E})$. The scenarios B and F belong to the relation $S_{13}(\mathrm{DC}_-)$, and, thus, we group them in a category (B, F). Motion-RCC partitions the scenarios in two categories.

These categories originate through the feature $d_{\min}$, i.e., the degree of maximum overlap in the trajectory. Hence, we can interpret these categories as $(\mathrm{A}, \mathrm{C}, \mathrm{D}, \mathrm{E})$ being the category of scenarios that currently are and remain disconnected in the future, i.e., they do not evolve into collision; while (B, F) is the category of scenarios that currently are disconnected, but the entities evolve into a partial collision (unless velocity is modified). We summarize this result in a table to easily compare with the results in Sections 11.1.2 and 11.1.3.

MOTION-RCC RELATION	VERBAL INTERPRETATION	BELONGING SCENARIOS
$S_{11}(\mathrm{DC}_-)$	Entities are and *remain disconnected* in the future.	A, C, D, E
$S_{13}(\mathrm{DC}_-)$	Entities are currently disconnected, but evolve into a *partial collision*.	B, F

Using *Motion-OPRA$_1$* In that way, the scenarios of Figure 11.1 are categorized as follows: the scenarios A, B, C belong to the relation $S_{\mathrm{C1\text{-}1}}(\angle_1^3)$, and, thus, form a category $(\mathrm{A}, \mathrm{B}, \mathrm{C})$. The scenario D belongs to the relation $S_{\mathrm{C11}}(\angle_1^3)$, and, thus, forms its own category (D). Finally, the scenarios E and F belong to the relation $S_{\mathrm{C2\text{-}1}}(\angle_3^1)$, and, accordingly, they form a category (E, F). Motion-OPRA$_1$ partitions the scenarios in three categories.

These categories originate through the features $\alpha_{\Delta x \Delta v}$ and α_{vv}. Hence, this categorization distinguishes which entity crosses first, and from which side—left or right—the entity k approaches l. In category $(\mathrm{A}, \mathrm{B}, \mathrm{C})$, entity k crosses before l and approaches l from the right. In (D), k crosses after l and approaches it from the right. In (E, F), k crosses before l and approaches it from the left (unless velocity is modified). Again, we summarize the result in a table:

MOTION-OPRA$_1$ RELATION	VERBAL INTERPRETATION	BELONGING SCENARIOS
$S_{\text{C1-1}}(\angle_1^3)$	Entity k approaches l from the *right* and will cross *before* it.	A, B, C
$S_{\text{C11}}(\angle_1^3)$	Entity k approaches l from the *right* and will cross *after* it.	D
$S_{\text{C2-1}}(\angle_3^1)$	Entity k approaches l from the *left* and will cross *before* it.	E, F

Let us estimate the total number of story-based categorizations that we can create in this way. In the literature, we find about 60 qualitative representations, if we count only the main subtypes (Dylla et al. 2017). Since for each representation $\mathcal{R}$ we can create Stories-$\mathcal{R}$ and Motion-$\mathcal{R}$, then we can obtain a total of about 120 ($= 60 \times 2$) story-based motion representations. In this work, we have only studied four of them, i.e., Stories-RCC, Stories-OPRA$_1$, Motion-RCC, and Motion-OPRA$_1$.

Remarkably, if we consider *all* possible variations of qualitative representations in Dylla et al.'s (2017) survey, we can obtain an infinite (countable) number of story-based representations. Some qualitative representations have variations that are indexed by a natural value, i.e., $n = 1, 2, \ldots$ Such index corresponds to their dimensions or granularity; for example, OPRA$_\text{n}$, STAR$_n$, or PC$_\text{n}$ (Tab. 4.2). Accordingly, we have infinite of those representations, one for each index value, and for each of them we can generate two different story-based representations, Stories-$\mathcal{R}$ and Motion-$\mathcal{R}$, for instance, Stories-OPRA$_\text{n}$ and Motion-OPRA$_\text{n}$.

11.1.2 Variety based on combining whole categorizations

Another possibility for creating new motion categorizations is to combine them by the Cartesian product (Sect. 5.4). Thus, we can create the extended motion representation '*Motion-RCC* × *Motion-OPRA$_1$*', the Cartesian product of Motion-RCC and Motion-OPRA$_1$.

As we explained in Section 5.4, the resulting categories of the Cartesian product are refinement of the original categories, i.e., the original categories we presented in Section 11.1.1 are partitioned in smaller categories.

In the example (Fig. 11.1) the Cartesian product would yield the following categories:

MOTION-RCC × MOTION-OPRA$_1$ RELATION	VERBAL INTERPRETATION	BELONGING SCENARIOS
$(S_{11}(\text{DC}_{-}), S_{\text{C1-1}}(\angle_1^3))$	Entities are and *remain disconnected* in the future. Entity k approaches l from the *right* and will cross *before* it.	A,C
$(S_{13}(\text{DC}_{-}), S_{\text{C1-1}}(\angle_1^3))$	Entities are currently disconnected, but evolve into a *partial collision*. Entity k approaches l from the *right* and will cross *before* it.	B
$(S_{11}(\text{DC}_{-}), S_{\text{C11}}(\angle_1^3))$	Entities are and *remain disconnected* in the future. Entity k approaches l from the *right* and will cross *after* it.	D
$(S_{11}(\text{DC}_{-}), S_{\text{C2-1}}(\angle_3^1))$	Entities are and *remain disconnected* in the future. Entity k approaches l from the *left* and will cross *before* it.	E
$(S_{13}(\text{DC}_{-}), S_{\text{C2-1}}(\angle_3^1))$	Entities are currently disconnected, but evolve into a *partial collision*. Entity k approaches l from the *left* and will cross *before* it.	F

These categories originate by considering at the same time features of both Motion-RCC and Motion-OPRA$_1$, i.e., d_{min}, $\alpha_{\Delta x \Delta v}$, and α_{vv}. That is, we distinguish the scenarios regarding whether the entities collide (and how), which entity crosses first, and from which side entity k approaches entity l.

11.1.3 Feature-based Generation of Categorizations

The methods used above (Sects. 11.1.1 and 11.1.2) to generate a variety of categorizations are only part of the story[1]. We can work out a more general method—possibly, the most general one—by regarding features instead of entire categorizations. Instead of combining the entire categorizations (e.g., Motion-RCC and Motion-OPRA$_1$), we combine the features of the story-based categorizations, i.e., the featural variables.

For instance, we saw in Chapter 8 that Motion-RCC has three featural variables, $d_{\min}$, dif_{V}, and d; we also saw that Motion-OPRA$_1$ has six featural variables α_{vv}, $\alpha_{\Delta x \Delta v}$, u_k, u_l, dif_{X}, $\alpha_{v_l \Delta x}$: This makes up a total of nine variables—but note that the feature dif_{X} is implied by d; that means, they are not equivalent, but whenever they appear together, dif_{X} is superfluous. Anyway, we can create whatever motion categorization by combining these nine features; For example, we can combine both $d_{\min}$ and α_{vv}, or even regard one feature alone, such as α_{vv}.

We use the six motion scenarios of Figure 11.1 as test examples. For a start, we review the methods we used above, which regard the story-based categorizations as a whole. Accordingly, we obtained 3 possible categorizations:

1. Categorizing with Motion-RCC, we obtain the category $(\text{A}, \text{C}, \text{D}, \text{E})$, i.e., the relation S_{11}; and category (B,F), i.e., the relation S_{13}.
2. Categorizing with Motion-OPRA$_1$, we obtain the category $(\text{A}, \text{B}, \text{C})$, i.e., the relation $S_{\text{C1-1}}$; the category (D), i.e., the relation S_{C11}; and the category (E, F), i.e., the relation $S_{\text{C2-1}}$.
3. Categorizing with Motion-RCC × Motion-OPRA$_1$, we obtain the categories (A, C), (B), (D), (E), and (F); which are the refinement of the first two categorizations.

Now we apply the categorization by features to the motion scenarios in Figure 11.1. From all possible features combinations we explain two as examples:

a. We take two features: $\boldsymbol{\{d_{\min}, \alpha_{vv}\}}$ as defined by Stories-RCC and Stories-OPRA$_1$ in Eq. (8.5) and Fig. 8.5. Based on such features, we obtain following categorization:

FEATURES		VERBAL INTERPRETATION	BELONGING SCENARIOS
$d_{\min}$	α_{vv}		
$d_{\min} > d_2$	$\alpha_{vv} < 0°$	Entities are and *remain disconnected* in the future. Entity k approaches l from the *right.*	A, C, D
$d_{\min} > d_2$	$\alpha_{vv} > 0°$	Entities are and *remain disconnected* in the future. Entity k approaches l from the *left.*	E
$d_2 > d_{\min} > d_4$	$\alpha_{vv} < 0°$	Entities are currently disconnected, but evolve into a *partial collision.* Entity k approaches l from the *right.*	B
$d_2 > d_{\min} > d_4$	$\alpha_{vv} > 0°$	Entities are and *remain disconnected* in the future. Entity k approaches l from the *left.*	F

The value d_2 is the distance at which the entities are tangent, i.e., the relation is EC.
The value d_4 is the distance at which the entities are tangent with overlap, i.e., the relation is TPP.

b. We take one feature $\boldsymbol{\{\alpha_{vv}\}}$, the minimal categorization amount of features. This feature has very few categorical regions: it has only four categories which are constituted by two

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[1] pun intended

angular regions, $\alpha_{vv} > 0°$ and $\alpha_{vv} < 0°$, and their boundaries $\alpha_{vv} = 0°$, $\alpha_{vv} = 180°$. Accordingly, the scenarios in Eq. (8.5) and Fig. 8.5 form two categories—the boundaries are empty because, in our example, there are no scenarios with parallel or antiparallel velocities:

Features α_{vv}	Verbal interpretation	Belonging scenarios
$\alpha_{vv} < 0°$	Entity k approaches l from the *right*.	A, B, C, D
$\alpha_{vv} = 0°$	Entities move parallel.	∅
$\alpha_{vv} > 0°$	Entity k approaches l from the *left*.	E, F
$\alpha_{vv} = 180°$	Entities move antiparallel	∅

We see that the new categories generated by combining selected story-based features in Items a. and b. are different of those generated by combining entire story-based categorizations in Items 1 to 3. We can generate a higher number of categorizations through all possible combinations of features than through combinations of entire story-based categorizations. What is more, the categorizations created by combining entire categorizations are a particular case of those created by combining features. Certainly, categorizing with Motion-RCC alone (item 1) is equivalent to categorize with the features $\{d_{\min}, \mathrm{dif}_{\mathrm{V}}, d\}$; categorizing with Motion-OPRA$_1$ alone (item 2) is equivalent to categorize with the features $\{\alpha_{vv}, \alpha_{\Delta x \Delta v}, u_k, u_l, \mathrm{dif}_{\mathrm{X}}, \alpha_{v_l \Delta x}\}$, and categorizing with Motion-RCC × Motion-OPRA$_1$ (item 3) is equivalent to categorize with the whole features, i.e., $\{\alpha_{vv}, \alpha_{\Delta x \Delta v}, u_k, u_l, \alpha_{v_l \Delta x}; d_{\min}, \mathrm{dif}_{\mathrm{V}}, d\}$. In sum, we have validated the feature-based generation of motion categorizations as a most general method.

We must give an essential caveat when creating motion categorizations by combining features: Use at least *one motion feature*! This caveat applies only when we deal with features of Motion-$\mathcal{R}$ categorizations (i.e., beaded categorizations) because a Motion-$\mathcal{R}$ categorization is a combination of both a motion categorization Stories-$\mathcal{R}$ and its generating categorization $\mathcal{R}$. Consequently, some features of Motion-$\mathcal{R}$ belong to Stories-$\mathcal{R}$ and some others to $\mathcal{R}$. The problem arises when $\mathcal{R}$ is a *spatial* categorization, such as RCC, and we choose *only* features from $\mathcal{R}$: in that case the categorization created from those chosen features is not a motion categorization, it is just a *spatial* categorization.

For example, Motion-RCC has three featural variables, $d_{\min}$, $\mathrm{dif}_{\mathrm{V}}$, and d; the variables $d_{\min}$ and $\mathrm{dif}_{\mathrm{V}}$ stem from Stories-RCC, while the variable d stems from RCC. Therefore, we cannot choose d as unique featural variable, because the generated categorization $\{d\}$ is a *spatial* categorization. We need at least one motion feature, e.g., $\mathrm{dif}_{\mathrm{V}}$. The categorization generated by $\{\mathrm{dif}_{\mathrm{V}}, d\}$ is indeed a motion categorization.

We achieve an extraordinary variety of motion categorizations by combining features. As an illustration, we take Motion-RCC and Motion-OPRA$_1$, which have a total of nine featural variables: three of them (d, $\mathrm{dif}_{\mathrm{X}}$, $\alpha_{v_l \Delta x}$) stem from the generating spatial representation, i.e., they cannot appear alone in a categorization; moreover, $\mathrm{dif}_{\mathrm{X}}$ is implied by d. This amounts to 186 $(=(2^6 - 1)(2 \cdot 3))$ different motion categorizations.

Concluding, our modelling of motion categorization (Ch. 6) reveals that the categories originate ultimately from scenarios sharing common features (i.e., featural variables). Any story-based categorization is a bundle of features, which, according to their different values, form the categories. We can pick, as we need, features from different story-based categorizations in order to build new categorizations; in that case, we see that the possible number of categorizations grows exponentially $\sim 2^{|\text{Features}|}$. For example, if we are interested on collisions, we would pick the feature $d_{\min}$, which determines the degree of overlap of entities at crossing; if we also want to discriminate which entity crosses first, we would pick a combination of the features $\alpha_{\Delta x \Delta v}$ and α_{vv}. In that way, we obtain categorizations tailored to the features that are relevant for us.

The creation of categorizations through combination of features is a recurrent topic in the literature, and the basis for the generation of stimuli in categorization experiments (e.g., Tversky and Gati 1982, pp. 130–142; Nosofsky 1986, p. 43). The novelty here is that we do not elaborate or devise the features ourselves, but we extract the features from the generated story-based categorizations in what we call a top-down categorization model (Fig. 6.1).

11.2 High Dimensional Categorizations of Motion

As we mentioned in Section 1.2, the story-based method is applicable in any spatial dimensions, particularly, in three or more dimensions, where very little research has been done (Sect. 5.2).

To obtain a motion categorization that operates in more than two dimensions, we simply resort to a spatial representation that also operates in more than two dimensions. We find nine such representations in Dylla et al.'s (2017) survey (Table 4.2): $9^{(+)}$-Int, Block Algebra, Lines of Sight, Occlusion Calculus, 3D-Orientation Model, RCC, Point Calculus, Region Occlusion Calculus, Visibility Relations.

For example, if we wanted a motion representation with n-dimensional block entities in a n-dimensional space, we would take the Block Algebra, $\mathrm{BA_n}$ and create Stories-$\mathrm{BA_n}$ and Motion-$\mathrm{BA_n}$, which are story-based motion representations that describe the motion of n-dimensional block entities in a n-dimensional space. Another representations that we can use to generate high dimensional story-based representations are the Point Calculus ($\mathrm{PC_n}$), which represents points in n-dimensions, and thus generating Stories-$\mathrm{PC_3}$ and Motion-$\mathrm{PC_3}$ we obtain motion representations for points in three dimensions.

We choose RCC to fully illustrate a three-dimensional story-based representation. RCC can be trivially extended to three-dimensional regions: we call it RCC-3d; the RCC qualitative relations (Fig. 7.2) that we presented for discs in two dimensions—DC, EC, PO, TPP, NTPP, EQ, TPPI, NTPPI—are the same for 3-balls in three dimensions (Zlatanova 2000). Consequently, the bare (Stories-RCC-3d) and the beaded relations (Motion-RCC-3d) in three dimensions are the same as those in two dimensions (Eq. (8.20)), but with a three-dimensional meaning.

$$
\begin{aligned}
&\textbf{Motion-RCC-3d} = \{\\
&S_{01}(\mathrm{DC}), S_{02}(\mathrm{EC}), S_{03}(\mathrm{PO}), S_{04}(\mathrm{TPP}), S_{05}(\mathrm{NTPP}),\\
&S_{06}(\mathrm{TPPI}), S_{07}(\mathrm{NTPPI}), S_{08}(\mathrm{EQ}),\\
&S_{11}(\mathrm{DC}), S_{12}(\mathrm{DC}_-), S_{12}(\mathrm{EC}), S_{12}(\mathrm{DC}_+),\\
&S_{13}(\mathrm{DC}_-), S_{13}(\mathrm{EC}_-), S_{13}(\mathrm{PO}), S_{13}(\mathrm{EC}_+), S_{13}(\mathrm{DC}_+), \qquad (11.1)\\
&S_{14}(\mathrm{DC}_-), S_{14}(\mathrm{EC}_-), S_{14}(\mathrm{PO}_-), S_{14}(\mathrm{TPP}),\\
&S_{14}(\mathrm{PO}_+), S_{14}(\mathrm{EC}_+), S_{14}(\mathrm{DC}_+),\\
&S_{15}(\mathrm{DC}_-), S_{15}(\mathrm{EC}_-), S_{15}(\mathrm{PO}_-), S_{15}(\mathrm{TPP}_-), S_{15}(\mathrm{NTPP}),\\
&S_{15}(\mathrm{TPP}_+), S_{15}(\mathrm{PO}_+), S_{15}(\mathrm{EC}_+), S_{15}(\mathrm{DC}_+)\\
&S_{16}(\mathrm{DC}_-), S_{16}(\mathrm{EC}_-), S_{16}(\mathrm{PO}_-), S_{16}(\mathrm{TPPI}),\\
&S_{16}(\mathrm{PO}_+), S_{16}(\mathrm{EC}_+), S_{16}(\mathrm{DC}_+),\\
&S_{17}(\mathrm{DC}_-), S_{17}(\mathrm{EC}_-), S_{17}(\mathrm{PO}_-), S_{17}(\mathrm{TPPI}_-), S_{17}(\mathrm{NTPPI}),\\
&S_{17}(\mathrm{TPPI}_+), S_{17}(\mathrm{PO}_+), S_{17}(\mathrm{EC}_+), S_{17}(\mathrm{DC}_+),\\
&S_{18}(\mathrm{DC}_-), S_{18}(\mathrm{EC}_-), S_{18}(\mathrm{PO}_-), S_{18}(\mathrm{EQ}),
\end{aligned}
$$

$$S_{18}(\mathrm{PO}_+), S_{18}(\mathrm{EC}_+), S_{18}(\mathrm{DC}_+)\}$$

As an example, $S_{13}(\mathrm{DC}_-)$ corresponds to the scenario where two spheres are disconnected, but they move towards one another, so that their uniform trajectory goes through a partial overlapping (PO), i.e., a partial inclusion of the spheres.

11.3 Categorization of Motionless Entities

Story-based categorizations of motion categorize motion scenarios when entities—one or both entities—are motionless, in contrast to some motion categorizations (Sect. 5.2). As examples, see Figure 11.2, and also Figure 11.4, where the trajectory of entity k with respect to l is described by Motion-RCC, although k is motionless. There is also no need of resorting to orientation vectors as in OPRA_1 or QRPC.

Accordingly, these representations are not only valid for dynamic environments, but also valid for static ones. For example, if a robot navigates through an empty office (a *static environment* because no object moves but the robot), and suddenly a person appears walking on the robot's way (now a *dynamic environment*), there is no need to switch representation: The story-based representation allows for further motion description and control of the robot.

Figure 11.2: Two motion scenarios, both having the same category, i.e., $S_{14}(\mathrm{DC}_-)$, in the motion representation Motion-RCC. The fact that in the second scenario the entity k is motionless is no hindrance for a scenario categorization.

Source: Purcalla Arrufi and Kirsch (2018a)

11.4 Qualitative Description of General Trajectories

A property of the story-based representations of motion—and of any categorization of motion scenarios—is the ability to qualitatively describe any kind of *two-entities trajectories*, that is, pairs of trajectories. This is mainly used in recognition of trajectories (i.e., motion patterns): Trajectories that are encoded through qualitative relations can be easily categorized as a certain type of motion, e.g., an 'avoidance manoeuvre', and, therefore, facilitates the development of interactive navigation routines (e.g., Delafontaine et al. 2011; Hanheide et al. 2012).

Qualitative representations for motion scenarios describe the trajectories of two entities as a finite sequence of qualitative relations (Van De Weghe et al. 2005; 2006). In story-based motion representations, we must differentiate though between two sorts of *sequences*: First, the *stories*, which are *sequences of relations* that belong to the generating representation $\mathcal{R}$; such sequences are created upon (hypothetical) time evolution of scenarios under uniform motion, e.g., the Stories-RCC story $S_{12} = (\mathrm{DC}_-, \mathrm{EC}, \mathrm{DC}_+)$ is a sequence of RCC relations. Second, the *sequences of stories*; such sequences describe the pairs of trajectories, e.g., the trajectories of entities l (trajectory T_2) and n (motionless) in Figure 11.3, is described as a sequence of Stories-RCC relations, namely, $T_2 = [S_{15}, S_{14}, S_{13}, S_{12}, S_{11}]$. Note that we use different delimiters: parenthesis

'(...)' for any sequence that defines a *story*, and square brackets '[...]' for any sequences that defines a *trajectories pair*.

We exemplify how the story-based representations describe pairs of trajectories by using the four entities in Figure 11.3: k, l, m, and n. Note that k, l, and m, move in trajectories T_1, T_2, and T_3, respectively, while we leave n motionless, i.e., its trajectory is just a point. We create three pairs of trajectories, (k, n), (l, n), and (m, n) that we describe below. All three pairs of trajectories in our example involve the motionless particle n because it is much easier both for visualizing and for computing the motion relations; in any case, the generality of our conclusions is preserved for any kind of motion trajectories.

In order to describe the pairs of trajectories, we employ the spatial qualitative representation RCC, and its generated story-based motion representations Stories-RCC, and Motion-RCC. The trajectory descriptions are the temporal sequences of the relations that occur along the trajectory. In Figure 11.4, we show in detail how two pairs of trajectories, (k, n) and (l, n), are described by means of Motion-RCC.

(k, n): the trajectories of entities k (trajectory T_1) and n (motionless) is described as follows,

$$\begin{aligned}\text{RCC:}\ & [\text{DC}]\\ \text{Stories-RCC:}\ & [S_{11}]\\ \text{Motion-RCC:}\ & [S_{11}(\text{DC})]\end{aligned}$$

(l, n): the trajectories of entities l (trajectory T_2) and n (motionless) is described as follows,

$$\begin{aligned}\text{RCC:}\ & [\text{DC}]\\ \text{Stories-RCC:}\ & [S_{15}, S_{14}, S_{13}, S_{12}, S_{11}]\\ \text{Motion-RCC:}\ & [S_{15}(\text{DC}_-), S_{14}(\text{DC}_-), S_{13}(\text{DC}_-), S_{12}(\text{DC}), S_{11}(\text{DC})]\end{aligned}$$

(m, n): the trajectories of entities m (trajectory T_3) and n (motionless) is described as follows,

$$\begin{aligned}\text{RCC:}\ & [\text{DC}, \text{EC}, \text{PO}, \text{TPP}, \text{NTPP}, \text{TPP}, \text{PO}, \text{EC}, \text{DC}]\\ \text{Stories-RCC:}\ & [S_{15}, S_{14}, S_{13}, S_{12}, S_{11}]\\ \text{Motion-RCC:}\ & [S_{15}(\text{DC}_-), S_{15}(\text{EC}_-), S_{15}(\text{PO}_-), S_{15}(\text{TPP}_-), S_{15}(\text{NTPP}), S_{15}(\text{TPP}_+), S_{15}(\text{PO}_+),\\ & S_{15}(\text{EC}_+), S_{15}(\text{DC}_+), S_{14}(\text{DC}_+), S_{13}(\text{DC}_+), S_{12}(\text{DC}_+), S_{11}(\text{DC})]\end{aligned}$$

Since we deal with RCC related representations, we will focus on how the representations describe the trajectories concerning the collision risk. First of all, we see that RCC does not distinguish the trajectories of entities k (T_1) and l (T_2) with respect to n: RCC describes both pairs trajectories, (k, n) and (l, n) ,with the same sequence, [DC]. However, the RCC story-based representations, Stories-RCC and Motion-RCC, describe the two-entities trajectories differently: in (k, n) a single relation, S_{11} and S_{11}(DC), shows that the entity k passes n by, and there is *no* collision risk throughout the whole trajectory; while in (l, n) the first relation of the descriptive sequence is S_{15} and $S_{15}(\text{DC}_-)$, which shows that along the trajectories does exist a collision risk.

Second, we see that Stories-RCC fails to differentiate between actual collisions, as seen with entities (m, n), i.e., T_3, and collision risks without actual collisions, as seen with entities (l, n), i.e., T_2. Both pairs of trajectories (m, n) and (l, n) are described by Stories-RCC as the same sequence $[S_{15}, S_{14}, S_{13}, S_{12}, S_{11}]$. This arises because each story is defined as the sequence of a scenario in uniform motion that extends in the unbounded time interval $(-\infty, \infty)$ (Sect. 7.5.2). Consequently, when a collision story, e.g., S_{15}, appears in the description of a trajectory we

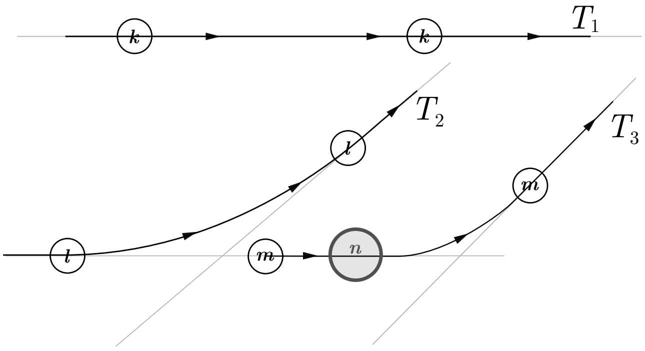


Figure 11.3: We show four entities with their respective trajectories: k moves along T_1, l along T_2, m along T_3, and n is motionless, i.e., it is at a fixed point. In Section 11.4, we describe three pairs of trajectories, namely, (k, n), (l, n), and (m, n).

Source: derivative of Purcalla Arrufi and Kirsch (2018a)

cannot tell whether the trajectory is following a possible collision in the past, or heading towards a future collision. However, the beaded relations resolve this ambiguity: the relation $S_{15}(\mathrm{DC}_-)$ in a trajectory sequence, as in (m, n), means that the entities are heading towards a future collision, whereas the relation $S_{15}(\mathrm{DC}_+)$ means that the entities might have had a collision in the past.

Wrapping up, a spatial representation alone, such as RCC, can describe two-entities trajectories as sequences of relations, but it fails to distinguish certain trajectories because their relations lack predictive power; for example, pairs of trajectories, (k, n) and (l, n) have the same RCC description, namely, [DC]. In contrast, the story-based representations can distinguish the trajectories that a spatial representation cannot; for example, Stories-RCC distinguishes the trajectories pairs (k, n) and (l, n): the first one is collision free without any collision threat, which is indicated by S_{11} as the unique relation, and the second one has a collision threat, which is indicated by the presence of relation S_{15}.

Nevertheless, the *bare* story-based representations, such as Stories-RCC, have severe shortcomings. A bare story-based representation cannot unambiguously record a collision that occurs along the trajectory; we illustrate this with an example: the trajectories pair that has experienced a collision, (m, n), has the same Stories-RCC sequence as the trajectories pair that has only experienced a collision threat, (l, n). In fact, the *bare* story-based representation fails to distinguish 'prediction' and 'reality': the relation S_{15} in the trajectory description indicates either a collision threat, as in (l, n), or a collision that has occurred along the trajectory, as in (m, n).

All considered, the *beaded* story-based representations, such as Motion-RCC, overcome the shortcomings of both *spatial* representations, such as RCC, and *bare* story-based representations, such as Stories-RCC, when it comes to describe pairs of trajectories. First, their relations have a full-fledged predictive power. As an illustration, in Figure 11.3, Motion-RCC signals a collision threat at the beginning of the trajectories pair (k, n), it encodes the threat as the relation $S_{15}(\mathrm{DC}_-)$, while Stories-RCC shows merely S_{15}, which is ambiguous as to whether it is only a threat or an actual collision. Second, the beaded categorizations register the relevant events that occurred along the trajectories pair, such as an actual collision in the pair (m, n), which is encoded by Motion-RCC in each of the relations between $S_{15}(\mathrm{EC}_-)$ and $S_{15}(\mathrm{EC}_+)$.

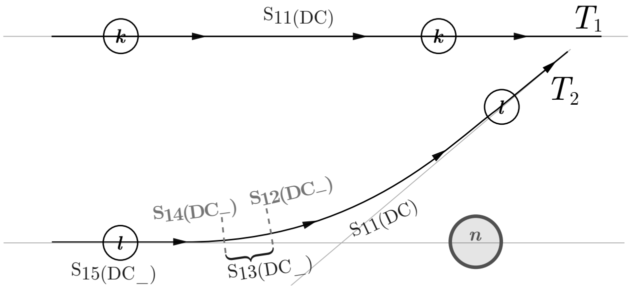


Figure 11.4: The pairs of trajectories (k, n) and (l, n) of Fig. 11.3 are described here in detail according to Motion-RCC: we show the sections of the trajectories where the relations occur. (k, n) is described as $[S_{11}(\mathrm{DC})]$; thus, it has only one relation. (l, n) is described as $[S_{15}(\mathrm{DC}_{-}), S_{14}(\mathrm{DC}_{-}), S_{13}(\mathrm{DC}_{-}), S_{12}(\mathrm{DC}), S_{11}(\mathrm{DC})]$; the relations $S_{14}(\mathrm{DC}_{-})$ and $S_{12}(\mathrm{DC}_{-})$ occur at single trajectory points marked with a dashed line.

Source: derivative of Purcalla Arrufi and Kirsch (2018a)

11.5 Decision-Making and Control

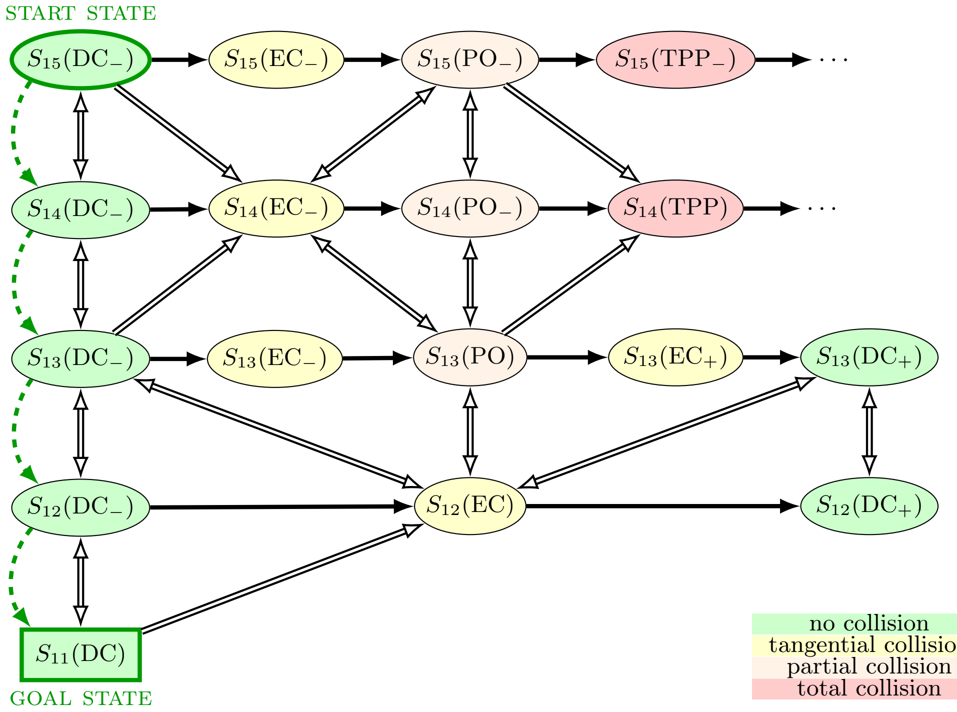
Beyond describing trajectories qualitatively, the story-based representations of motion can be used for decision-making and control of trajectories. In truth, any qualitative representation can, if we employ conceptual neighbourhood diagrams (general analysis, Dylla and Wallgrün 2007; examples, Dylla et al. 2007; Bellotto et al. 2013).

As we explained in Section 4.1.4, *decision-making* differs from *reasoning*. '*Decision-making*' refers to finding the actions to steer an entity, whose motion with respect to other entities is described qualitatively, while '*reasoning*' refers to finding the missing qualitative relations in a scenario with several entities, or checking the consistency of the given relations; moreover, reasoning it is founded on the operations of qualitative calculi, namely, converse and composition (Sect. 10.1.2). If we want to *make decisions* about the trajectory of moving entities, such as avoid collision, we have to determine which *control actions* implement such desired decision. On this account, 'decision-making' and 'control' are two sides of the same coin; thus, we will use predominantly the term 'control' to refer to any of them.

The *conceptual neighbourhood diagram* (Sect. 7.3) is the basis for control of entities in a motion scenario. Due to the nature of story-based representations (Ch. 7), their conceptual neighbourhood diagram provides two important tools: it shows how the scenario evolves in time if the velocities remain unchanged, and it shows how acceleration affects the motion state of the entities, i.e., into which state we transition when we change velocity. For example, by visual inspection, we realize that the only state that does not "naturally" evolve into a collision (i.e., that has no ➔ arrow leading to a collision state) is the state $S_{11}(\mathrm{DC}_{-})$.

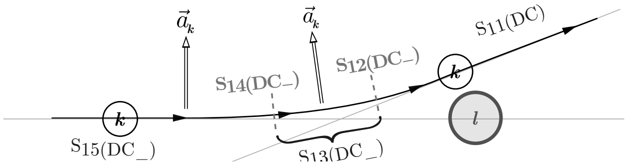
As an illustration, in Figure 11.5, we display the conceptual neighbourhood diagram of a story-based representation, namely, Motion-RCC (Fig. 11.5a). We use the diagram to plan the control actions that lead to the avoidance manoeuvre of two entities (Fig. 11.5b).

At the onset of the motion in Figure 11.5b, entity k is in a collision-threatened motion state with respect to the motionless entity l; this is registered as the relation $S_{15}(\mathrm{DC}_{-})$ (bold green ellipse) in the conceptual neighbourhood diagram (Fig. 11.5a). We make the decision to manoeuvre into a collision free motion state, namely, $S_{11}(\mathrm{DC})$ (bold green rectangle). We seek then a path in the diagram from the start state, $S_{15}(\mathrm{DC}_{-})$, to the goal state, $S_{11}(\mathrm{DC})$, because the two entities' manoeuvre is qualitatively described as a path connecting the *start state* with



(a) Fragment of the conceptual neighbourhood diagram of Motion-RCC; by means of arrows, we show the transitions between motion states, i.e., between motion relations. In the transitions, we set as constraints that the entities must perform continuous velocity changes, and they are not allowed to stop; otherwise, all transitions would be possible. The simple arrows, →, show the transitions that are possible when velocities remain unchanged (i.e., transitions with preserved story). The double arrows, ⇒, show the transitions between states that entail velocity changes, i.e., at least one entity must accelerate (i.e., the story changes). The green dashed arrows, - ▶, display an example of trajectory control. Departing from a collision-threatened state, $S_{15}(\mathrm{DC}_-)$ (green ellipse), these arrows describe the path of collision-free transitions leading to a state with collision-free evolution, $S_{11}(\mathrm{DC})$ (green rectangle).

Source: derivative of Purcalla Arrufi and Kirsch (2018a)



(b) An example of the trajectory control implemented in Figure 11.5a. The entity k avoids collision with entity l, which is motionless; that is, the trajectory description evolves from relation $S_{15}(\mathrm{DC}_-)$ to relation $S_{11}(\mathrm{DC})$ according to Motion-RCC. The control action, the acceleration $\vec{a}_k$, is determined by the transitions in the conceptual neighbourhood diagram.

Figure 11.5: An example of decision-making and control of motion scenarios. In the figure (a), we perform the decision-making and determine the control; in the figure (b), we show how the result of the figure (a) is implemented in a motion scenario.

the *goal state*. Since infinitely many paths connect our start and goal state, we have to impose further conditions; in fact, a necessary condition to avoid collision is that no state in the path is a collision state. Accordingly, the only possible solution is the path $[S_{15}(\mathrm{DC}_-), S_{14}(\mathrm{DC}_-), S_{13}(\mathrm{DC}_-), S_{12}(\mathrm{DC}_-), S_{11}(\mathrm{DC})]$ (dashed arrows ⇢), which is materialized as the avoidance trajectory of entity k in Figure 11.5b. Notably, this path also fulfils the optimal condition of being the shortest path between the states.

The actions needed to reach the goal state, $S_{11}(\mathrm{DC})$, are given by the transitions in the solution path. In our diagram, we have only distinguished between two types of transitions: those which are possible without changing velocity (simple arrows, ➞), and those for which acceleration is need (double arrows, ⇒). In consequence, we see that our solution trajectory requires of a permanent acceleration: that is what we observe in the trajectory of entity k (Fig. 11.5b), a permanent normal acceleration $\vec{a}_k$.

The transitions in the conceptual neighbourhood diagram can be more precisely specified as we did: for example, we can specify which accelerations (tangential or normal) render the transition possible, the lowest upper bounds for those accelerations, and, analogously for velocities.

To conclude, we summarize the control properties of the conceptual neighbourhood diagram of the story-based qualitative relations:

i. We can predict how a motion scenario evolves, i.e., which are the future motion states, if velocities remain unchanged. This is indicated by the simple arrows, ➞.

ii. We can plan motions by setting the start and goal states; even for complex motions we might set an arbitrary number of middle goal states.

iii. We can obtain the required control decisions that bring the motion scenario from a certain start state into a certain goal state by finding the path in the conceptual neighbourhood diagram. As there are, usually, many paths connecting two scenarios, we must set conditions on the chosen path; for instance, absence of collisions, minimum path length, or avoiding normal accelerations, i.e., changes in direction.

iv. The allowed transitions that can only be obtained by acceleration are indicated by double arrows, ⇒. In these transitions, we can specify the type of acceleration required, either normal or tangential, and also their lowest upper bounds.

11.6 Motion Categorization for Multiple Entities

'*Multiple entities*' refers to scenarios with *more than two* entities. In this section, we show how to generalize the story-based categorizations, which are defined in a two entities scenario, so that they categorize scenarios with multiple entities.

First of all, we define the qualitative relations that describe a scenario with three entities. As with the motion scenarios, we begin with a qualitative representation $\mathcal{R}$ that relates two entities, for example, through the qualitative relation R_{kl}. Based on this binary relation, the qualitative relation describing a scenario with three entities (k, l, and m) is the tuple of all possible binary relations of the three entities, which amount to three relations, that is, (R_{kl}, R_{km}, R_{lm}). We call $3\mathcal{R}$ this '*three-entities representation*' based on the binary representation $\mathcal{R}$.

As an illustration, in Figure 11.6, we display many scenarios consisting of three entities (k, l, and m) which are described by $3\mathrm{OPRA}_1$relations. For instance, the first scenario of Fig. 11.6a has the relation $(\angle_3^1, \angle_1^1, \angle_0^2)$: $\angle_3^1$ is the OPRA$_1$ relation of k and l, $\angle_1^1$ is the OPRA$_1$ relation of k and m, and $\angle_0^2$ the OPRA$_1$ relation of l and m.

Secondly, in the same fashion that we created the story-based representations Stories-$\mathcal{R}$ and Motion-$\mathcal{R}$, based on a qualitative representation $\mathcal{R}$ (Ch. 8), we can create the story-based representations Stories-3$\mathcal{R}$ and Motion-3$\mathcal{R}$ based on the qualitative representation 3$\mathcal{R}$ which relates three entities in a motion scenario. Indeed, we can find the stories of three-entities scenarios; in other words, we can find the complete temporal sequence of relations in uniform motion for any motion scenario of three entities.

As an example, in Figure 11.6, we picture two stories each of three entities: S_M, in Figure (a) and S_U in Figure (b)—We display only three relations belonging to each story, not the full sequence of relations. These stories belong to the representation Stories-3OPRA$_1$, whose full stories set $\Sigma_{3\text{OPRA}_1}$ we leave for future work. Hence, according to Stories-3OPRA$_1$, all scenarios of Figure (a) are categorized as S_M, and all scenarios of Figure (b) as S_U. If we now consider Motion-3OPRA$_1$, we categorize each scenario with a beaded relation, $S_i(R_j)$; for instance, the leftmost scenarios in Figures 11.6a and 11.6b are respectively categorized as $S_M(\angle_3^1, \angle_1^1, \angle_0^2)$ and $S_U(\angle_3^1, \angle_1^1, \angle_0^2)$ according to Motion-3OPRA$_1$: they have the same 3OPRA$_1$ relation, $(\angle_3^1, \angle_1^1, \angle_0^2)$, but different story Stories-3OPRA$_1$.

11.7 Inspirational Cognitive and Mathematical Properties

In the sections above, we have described properties of story-based categorizations that are immediately relevant to their application. For instance, the property in Section 11.2 is crucial if we want to use story-based categorizations in aircraft navigation: they can operate in a three-dimensional space.

In this section, we present properties that take our understanding of the story-based categorizations to a higher theoretical level than immediate applications. In this sense, the properties are much more thought-provoking ideas that we hope will stimulate further research.

11.7.1 Not just categories, but rather a process

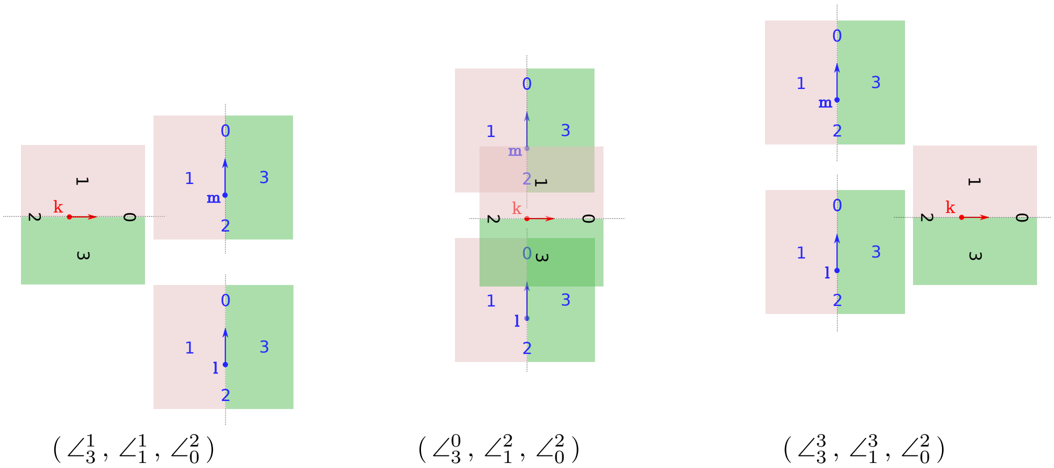
As Cohen and Lefebvre (2005, p. 13) observe, scientists are becoming more interested in *the process of categorization* than in the categories themselves. The story-based method for creating qualitative representations of motion joins this trend, because it is more than just a "method" for obtaining categories: it describes a process by which an agent might categorize motion (We generalize the process to a 'cognitive agent', avoiding the terms 'human' and 'animal'). The method, which is outlined in Sect. 8.5, can be translated into following plausible cognitive process with its corresponding assumptions.

Assumptions:

a. When perceiving an instantaneous motion (i.e., a motion scenario), an agent integrates it into a verisimilar trajectory: the agent extrapolates the present into a probable past and a expected future.

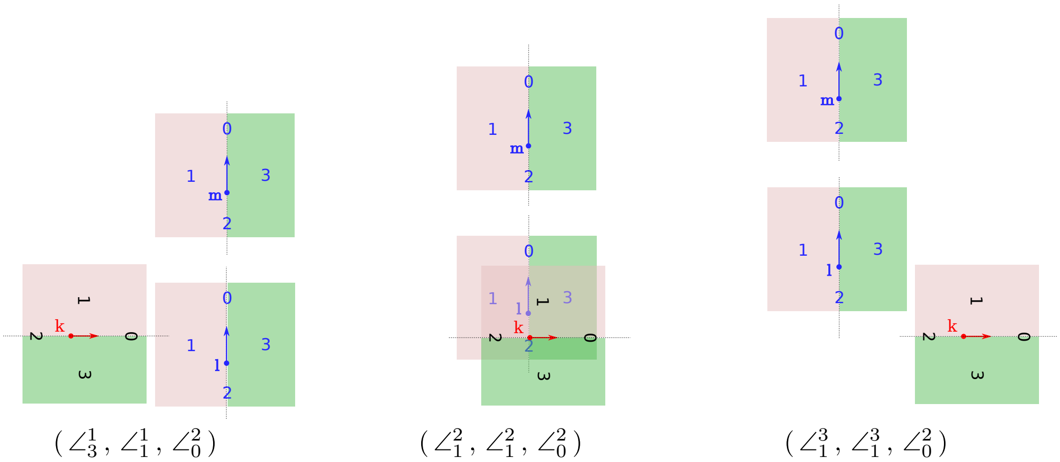
b. A probable condition that an agent uses to extrapolate instantaneous motion into a trajectory is the uniform motion, that is, a trajectory in which no acceleration has acted or will act on the entities.

c. Spatial categorizations determine motion categorizations. Depending on the context, or the goal, the agent chooses certain spatial categorization(s). An agent stores a



$(\angle_3^1 , \angle_1^1 , \angle_0^2)$ $(\angle_3^0 , \angle_1^2 , \angle_0^2)$ $(\angle_3^3 , \angle_1^3 , \angle_0^2)$

(a) A story of three entities that we call S_M. In this story, entity k moves through between l and m, while l follows after m at constant distance. We picture some relations of this story and its corresponding scenarios: $(\angle_3^1 , \angle_1^1 , \angle_0^2)$, $(\angle_1^2 , \angle_1^2 , \angle_0^2)$, and $(\angle_1^3 , \angle_1^3 , \angle_0^2)$.



$(\angle_3^1 , \angle_1^1 , \angle_0^2)$ $(\angle_1^2 , \angle_1^2 , \angle_0^2)$ $(\angle_1^3 , \angle_1^3 , \angle_0^2)$

(b) A story of three entities that we call S_U. In this story, entity k passes behind l and m, while l follows after m at constant distance. We picture some relations of this story and its corresponding scenarios: $(\angle_3^1 , \angle_1^1 , \angle_0^2)$, $(\angle_1^2 , \angle_1^2 , \angle_0^2)$, and $(\angle_1^3 , \angle_1^3 , \angle_0^2)$.

Figure 11.6: Two stories, S_M in Figure (a) and S_U Figure (b), each with three moving entities, k, l, and m. Each motion scenario is categorized according to the qualitative representation for three entities 3OPRA$_1$: the spatial relation for each scenario is given by a 3-tuple containing its three binary OPRA$_1$ qualitative relations, (R_{kl}, R_{km}, R_{lm}).

Source: Purcalla Arrufi and Kirsch (2018a)

trajectory as a sequence of spatial categories or relations—This last statement is consistent with the results of our experiment

Process:

1. After perceiving a motion scenario, an agent automatically integrates it into the corresponding trajectory extrapolated by uniform motion: past, present and future.

2. The extrapolated trajectory is translated into a 'story', that is, into a sequence(s) of spatial relations, i.e., into a *story*, according to the relevant spatial representation(s) for the task at hand.
3. The original motion scenario (item 1) is categorized by means of the story (item 2).

11.7.2 Acquisition of New Concepts and Explainability

We have provided methods to build motion categorizations, Stories-$\mathcal{R}$ and Motion-$\mathcal{R}$, based on sequences of relations in $\mathcal{R}$, i.e., based on the *stories*. Now, we realize that the stories capture concepts of motion; for example, 'collision', 'parallelism', 'crossing precedence', 'by-passing', . . . The fact that stories capture these concepts is remarkable in the measure that those concepts were not present in the original spatial relations.

For example, the stories of Motion-RCC correspond to diverse degrees of collision: S_{11} corresponds to no collision; S_{12} to tangential collisions; S_{13} to partial collisions; S_{14} and S_{15} to total collisions.

In Motion-OPRA$_1$ the number of concepts is higher. We see that each subset of stories Σ_* corresponds to a simple motion concept.

- Σ_C correspond to stories of both entities moving and crossing
- Σ_B correspond to stories where one entity is still and the other crossing.
- Σ_T corresponds to superposed trajectories.
- Σ_P corresponds to parallel trajectories (not superposed)
- Σ_E corresponds to trajectories of two entities moving in alignment and maintaining distance from one another.
- Σ_R are still stories: both entities are still.

Within each subset Σ_* we find a further variety of concepts; for instance, the crossing stories Σ_C are subcategorized according to which entity crosses first, i.e., reaches the crossing point first: in $S_{\mathrm{C}*\text{-}1}$, k crosses first, and, in $S_{\mathrm{C}*1}$, l crosses first. Remarkably, the concepts are intimately related to the features; even more, each concept in motion categorization can be defined as a set of features (For instance the, Σ_* sets in Figure 11.7): as extreme case a single feature is already a concept, e.g., $\alpha_{vv} = 180°$ equates to 'antiparallel'.

Explainability is a consequence of stories relating to concepts of motion. The story-based categorizations provide categories that can be understood by a 'human expert'. As opposed to artificial intelligence methods, such as neural networks, where the categorization criteria used are often opaque.

Natural Language Interface Another consequence of stories relating to concepts is that they can be effectively linked to language, and, consequently, story-based representations can serve as an interface to natural language communication. In other words, we can formulate as story-based relations the intermediate step between a command or information in natural language (such as "follow the other vehicle at a constant distance") and its implementation in low level navigation commands (such as "keep the angle between your velocity vector and the relative position of the vehicle at zero and keep the norm of its relative position constant"): These commands can be expressed compactly in Motion-OPRA$_1$ as "mantain the relation $S_{\mathrm{E}\text{-}2}(\angle_0^2)$."

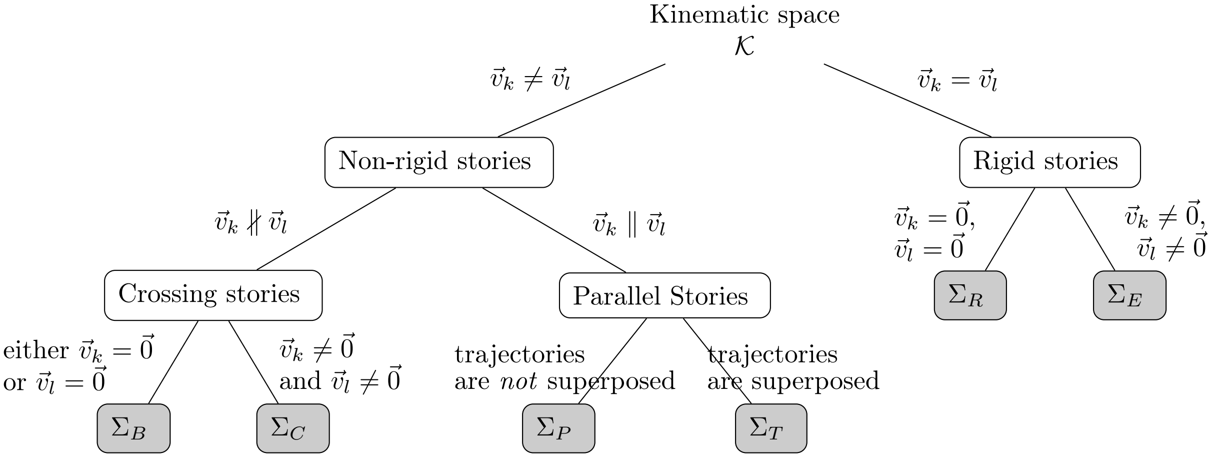


Figure 11.7: A hierarchical taxonomy of Stories-OPRA$_1$. The taxonomy stops at the Σ_* categories. Each edge contains the criterium that refines a higher category into a lower; for instance, we refine the 'non-rigid stories' into the 'crossing stories' when we consider the subset of non-rigid stories with non-parallel velocities, i.e., $\vec{v}_k \nparallel \vec{v}_l$.

11.7.3 Hierarchical Taxonomy

In Section 3.6, we mentioned how humans organize categories hierarchically. Accordingly, if story-based categorizations are cognitively plausible they should be able to display also a hierarchical structure in which the lowest, most detailed level is constituted by either the bare, S_i, or the beaded, $S_i(R_j)$, categories; the highest level is the kinematic space $\mathcal{K}$; and the middle levels are variable, depending on many factors, such as purpose or expertise.

We implicitly presented a hierarchical taxonomy of the categorization Stories-OPRA$_1$ in Figure 8.5 when we displayed the *featural story map* of Stories-OPRA$_1$ as a tree: each hierarchical level corresponded to a new feature until we reached the lowest level, the stories. We began with the feature α_{vv}, that is, the largest supercategories were the four sets having values $\alpha_{vv} < 0°$, $\alpha_{vv} = 0°$, $\alpha_{vv} > 0°$, and $\alpha_{vv} = 180°$. Next, we subdivided those supercategories by iteratively adding $\alpha_{\Delta x \Delta v}$, $\hat{v}_k$, and $\hat{v}_l$ until we reached the Stories-OPRA$_1$ stories.

Here, in Figure 11.7, we present a more real-life taxonomy. At each hierarchical level, common knowledge features (e.g., parallelism or motionlessness) refine the supercategories in subcategories until we reach the Σ_* sets. We might further refine the Σ_* sets until we reach the stories, S_i. Notably, the features we use to create this taxonomy can be formulated in terms of the Stories-OPRA$_1$ features. For example, $\vec{v}_k = \vec{v}_l$ is equivalent to $\alpha_{vv} = 0°$ and $u_k = u_l$; or $\vec{v}_k \nparallel \vec{v}_l$ is equivalent to $\alpha_{vv} \neq 0°$ and $\alpha_{vv} \neq 180°$.

With this example of hierarchical taxonomy, we show how story-based categorizations integrate in the landscape of human categorization. In addition, we want to motivate future work on story-based representations because the precise structure of the middle categories in such a hierarchy should be the object of experimental research. Also, an interesting research question is which level in the hierarchy is the *basic level* (Sect. 3.6.2) for motion categorization.

11.7.4 Cognitive Plausibility

The results of our experiment (Ch. 9) are our main support argument for the cognitive plausibility of story-based representations. Here, though, we present further arguments supporting cognitive plausibility of story-based representations.

A. Relation to Optical Invariants

An important step in cognition is to extract information from the sensory input. If we consider the visual stimuli of a moving human, it is known that we use *optical invariants* (Fajen and Phillips 2013, p. 74), i.e., “optical variables whose values remain invariant whenever the actor is in a state that if maintained, will bring about a successful outcome”. For example, if the relative angle between a human’s direction and the direction of a moving entity remains constant, the human will—if nothing changes—collide with the moving entity.

Optical variables provide, therefore, information about the *current future* (D. N. Lee et al. 2009), that is, what will eventually happen if one’s current state (deceleration, running speed) remains constant.

In a similar vein, story-based relations are founded in *invariants*. Certainly, each story is uniquely characterized by the values of the *featural variables*; such values are constant throughout a story. Since the motion categories are directly built upon the stories, we conclude that our featural variables are the invariants in story-based categorization. Exactly as the optical invariants in visual perception, the featural variables remain constant in time and determine the state of the entities, i.e., their motion category, thus, providing information about the *current future*.

For example, consider these three motion scenarios described by three different Motion-OPRA$_1$ relations: $S_{\text{C1-1}}(\angle_1^3)$, $S_{\text{C10}}(\angle_1^3)$, $S_{\text{P12}}(\angle_1^3)$. All three motion relations describe two entities whose spatial OPRA$_1$ relation is $\angle_1^3$, but the story components of the motion relation, i.e., $S_{\text{C1-1}}$, S_{C10}, and S_{P12}, show clearly that the evolution in each case is different. The story components are distinguished by the featural variables (Tab. 8.2), which are constant in time, and give information about the future evolution of the corresponding motion scenarios:

- Regarding the featural variable α_{vv}, we see that
  - $S_{\text{C1-1}}$ and S_{C10} evolve into an acute crossing motion, $\alpha_{vv} < 0$.
  - S_{P12} is a parallel motion, $\alpha_{vv} = 0$.
- Regarding the featural variable $\alpha_{\Delta x \Delta v}$, we see that for punctual entities
  - S_{C10} leads to a collision state, $\alpha_{\Delta x \Delta v} = 0$.
  - $S_{\text{C1-1}}$, $\alpha_{\Delta x \Delta v} > 0$, is collision free and entity k crosses first.
  - S_{P12}, is collision free $\alpha_{\Delta x \Delta v} \neq 0$, and we cannot determine which entity crosses first (obviously, also, because they move parallel).

B. Motion Encoding and Anticipatory Behaviour

From a cognitive viewpoint, we can see the story-based representations of motion as a way to encode sensory information of two entities instantaneously moving. In that sense, we would expect these encodings to validate the principle of anticipatory behaviour and the need to represent interaction goals; in the words of M. V. Butz and E. F. Kutter: “[. . .] the brain does not represent space for its own sake, but rather the internal representations develop to be able to convert sensory information in such a way that motor behaviour can be executed effectively.” (Butz and Kutter (2017, Sect. 10.2.1))

Certainly, as argued in Section A, the story-based representations encode a motion scenario with information about the current future, which facilitate such anticipatory behaviour. What is more, we can determine the anticipatory behaviour, so that we lead the entities into the desired goal state: as we saw in Section 11.5, the Motion-$\mathcal{R}$ story-based representations allow for trajectory control.

11.7.5 Qualitative Generalization of the Derivative: A Qualitative Velocity

In this section, we want to bring near the story-based categorizations to the classical discipline of differential geometry and its description of trajectories. We relate our *qualitative* concepts in the story-based categorizations with the *quantitative* concepts of trajectories in differential geometry.

Our purpose is to make the qualitative concepts more accessible to scientist with a strong "quantitative basis of knowledge", e.g., engineers, physicists. Eventually, in case our analogy can be extended to further quantitative concepts, researchers could achieve new results in qualitative motion categorization.

We draw an analogy between the *stories* S_i, in qualitative spaces, and the *directed tangent line* of a trajectory, in quantitative spaces. Taking the analogy further: we identify a beaded story-based relation, $S_i(R_j)$, in a qualitative space, with the current position and current velocity of the trajectory, $\vec{v}_k(t) = \frac{d\gamma}{dt}(t)$, in a quantitative space.

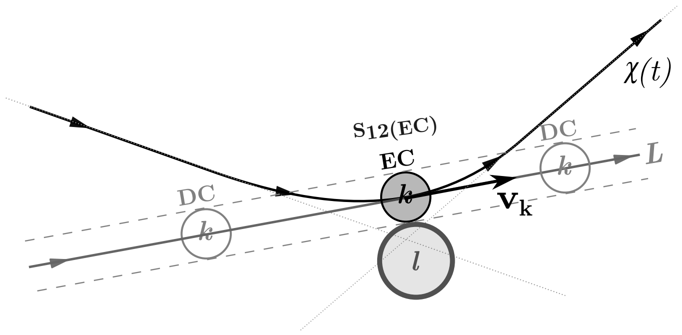
The analogy can be fully visualized if the second entity, l, is motionless, as in our example (Fig. 11.8). In case that the second entity moves with velocity $\vec{v}_l$, we should apply a velocity shift, $-\vec{v}_l$, to the whole motion scenario, and so we would have an equivalent scenario in which l velocity is $\vec{0}$.

In kinematics, the '*tangent line*' of trajectory $\gamma(t)$ of an entity k is the line defined by the derivative $\vec{v}_k(t_0) = \frac{d\gamma}{dt}(t)\Big|_{t=t_0}$ and the current position $\vec{x}_k(t_0) = \gamma(t_0)$. The '*directed tangent line*' is the tangent line oriented as the velocity vector $\vec{v}_k(t_0)$.

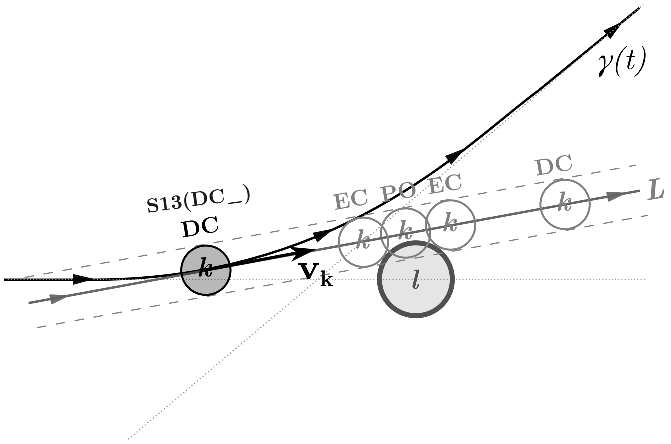
For example, in Figure 11.8b, we see the trajectory $\gamma(t)$ with a directed tangent line, L, at the current position of entity k; the line L has arrows that show its direction. Now, if we had the directed tangent line L but without the current position of k, our best assumption for the position of k would be that it lies "somewhere" on the directed tangent line L and moves along the line direction. This is analogous to the information provided by a story $S_i = \{R_1, R_2, \dots, R_n\}$: if we say that a motion scenario belongs to story S_i, we know that the entities belong to one of the story relations, i.e., R_1 to R_n, and that the motion evolves from R_j into R_{j+1} $\forall j \neq n$ (R_n, the last one, is a steady state), but we do not know exactly which relation currently takes place. In that sense, the categorization by stories is analogous to describe quantitative motion by means of directed tangent lines.

From a *quantitative* standpoint, if we additionally provide the current position $\vec{x}_k(t_0)$ besides the tangent line L and the velocity vector, then we have a complete quantitative information about the motion of entity k at the current instant, t_0. An analogous *qualitative* step would be to provide the current qualitative relation R_j of the story S_i that entity k and entity l (motionless) fulfil at the current instant t_0. Thus, the beaded relation $S_i(R_j)$ in a certain instant t_0 is the qualitative analogous to the current position $\vec{x}_k(t_0)$ and current directed tangent line L of the motion trajectory of entity k, where S_i is analogous to the current directed tangent line, and R_j is analogous to the current position.

Summarizing, in the qualitative space of story-based representations, the story S_i plays the role that the velocity, i.e., the derivative of the trajectory, plays in a quantitative space, while the relation R_j plays the role of a position.



(a) A two entities motion. Entity k moves along trajectory $\chi(t)$, and entity l remains motionless. We single out an instant in the trajectory where the Motion-RCC relation is S_{12}(EC). In light grey, we show the relations that make up S_{12}, namely, DC, EC, DC.



(b) A two entities motion. Entity k moves along trajectory $\gamma(t)$, and entity l remains motionless. We single out an instant in the trajectory where the Motion-RCC relation is S_{13}(DC$_-$). In light grey, we show the relations that make up S_{13}, namely, DC, EC, PO, EC, DC.

Source: Purcalla Arrufi and Kirsch (2018a)

Figure 11.8: We display two pictures, (a) and (b), containing two motions of the entities pairs k and l. The entity k moves along trajectories $\chi(t)$ and $\gamma(t)$, respectively, while entity l remains motionless. In each trajectory, we single out a certain instant by colouring entity k darker and displaying its current velocity vector $\vec{v}_k$, its current Motion-RCC relation $S_i(R_j)$, and its current tangent line L. The tangent line L help us visualize the story at the chosen instant: we show in lighter gray the sequences of relations that make up the story.

Part V

Conclusion

Chapter 12

Addressed Challenges and Shortcomings

Complementary to our contributions (Sect. 1.3), we have explained what makes motion categorization difficult (Ch. 2), and what aspects of current motion categorizations (as qualitative representations of motion) are deficient (Sect. 5.2). Here we show how the story-based representations tackle those challenges and overcome those shortcomings. We are confident that this provides a great insight into how to solve the motion categorization problem and how to create new qualitative representations of motion.

12.1 Tackled Challenges in Motion Categorization

In Chapter 2, we presented the challenges that the categorization of motion pose. Here, we check how the story-based categorizations perform in those challenges.

12.1.1 Avoiding High Dimensional Spaces

In Sect. 2.1, we explained how awkward is to deal with moving entities because their states are represented in very high-dimensional spaces (e.g., $\mathbb{R}^8$ or $\mathbb{R}^{18}$), namely the *kinematic spaces* (Sect. 6.1.1), which contain the position and velocity vectors of the entities. Blessedly, the story-based method to obtain the categorizations (Chs. 7 and 8) works in the spatial coordinate space of the entities. The stories are obtained and represented in the position space ($\mathbb{R}^2$ or $\mathbb{R}^3$) without needing to add extra coordinates, which makes easier for humans both mental and graphical representation.

In Sect. 5.5.2, we find a linguistic cognitive argument endorsing our story-based method of creating motion categorizations that we expound as follows. Representing the motion state of an entity in a high dimensional space where the velocities are coordinates amounts to represent motion using the tools of space. A *coordinates space* is just that, ‘a space’. If according to Section 5.5.2 our cognitive treatment of space and motion is noticeably different, we are discouraged to use coordinate spaces to represent motion, i.e., velocities. We certainly avoid a high dimensional treatment of motion, and rely essentially on the spatial representations of the entities: Motion is represented as a sequence of spatial relations, that is, as a story.

12.1.2 Providing a Variety of Categorization Criteria

In Section 2.2, we set the requirement that any model of motion categorizations should be able to generate a variety of categorizations criteria. In Section 11.1, we addressed this requirement by showing three methods to create a manifold of motion categorizations: i. We can create different motion categorizations by choosing different generating representations $\mathcal{R}$ from which we obtain Stories-$\mathcal{R}$ and Motion-$\mathcal{R}$; ii. We can create different motion categorizations by Cartesian product of motion categorizations, that is, $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n$; iii. We can create different motion categorizations by combining features of the story-based categorizations, $\mathcal{M} = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_n$.

The above methods provide a combinatoric growth of the possible story-based representations of motion, as we show by taking as generating representations the about 60 qualitative representations of Dylla et al.'s (2017).

With the first method (item i.), we can obtain about 120 ($= 60 \times 2$) story-based representations, besides the naturally indexed representations, such as $\mathrm{OPRA_n}$, which yield an infinity of story-based representations, e.g., Motion-$\mathrm{OPRA_n}$ for $n = 1, 2, \ldots$ Further, using the second method (item ii.), we make Cartesian product combinations of the story-based representations obtained with the first method. In that case, provided that each generating representation is independent of the rest, we would obtain at least about 2^{60} new motion representations—For the sake of simplicity we have only considered the Motion-$\mathcal{R}$ representations. Finally, using the third method (item iii.), we can combine the features of each story-based representation obtained with the first method. In that case, if the total number of *independent* features is |Features|, then we can generate about $2^{|\mathrm{Features}|}$ which is $(2^{60})^{N_F}$ where N_F is the average number of independent features per representation. Whether $N_F \geq 1$ or not depends on the properties of the features: For motion categorization in a certain dimension, do we have a limited number of motion features? Does every story-based representation Motion-$\mathcal{R}$ (or Stories-$\mathcal{R}$) have at least one independent feature? These are profound questions that spark interesting lines of future research. Anyway, the number of motion categorizations obtained by the above methods seems overwhelming.

12.1.3 Overcoming a Loose Cognitive Structure

We pointed out in Section 2.3 how different is the *spatial* and the *motion* categorization in humans. While humans categorize systematically the space by partitioning it, humans do not systematically partition the motion space; in fact, it is questionable that the 'motion space' is part of human cognition, i.e., it is not cognitively plausible. This is intimately related to the first challenge (Sect. 12.1.1), the high dimensionality of motion spaces: if motion spaces have such a high dimensionality both a mental representation and, even more, a systematic partition is discouraged, and only certain motions (certain regions of the motion space) are singled out, or cognitively considered—for such regions alone we have verbs that describe motion. Therefore, motion categorization in humans is sparse, or cognitively loose.

We overcame this sparseness in human motion categorization in one respect: we *exhaustively* categorized the space of motion scenarios, $\mathcal{K}$, by means of the story-based method (Ch. 7). Indeed, we showed in Section 10.3 that the story-based categorizations partition the space of motion scenarios. There is a caveat: we restricted our categorization to motion scenarios. Nevertheless, once restricted to motion scenarios, the categorization was exhaustive.

For the categorization of fully general motions, we suggested the use of sequences of story-based relations (Sect. 11.4), as most natural method. Thus, at the end, we deliver a method to partition the motion space of two moving entities, although this was not our main purpose.

12.1.4 Relieving Disciplinary Fragmentation and Tension

In this work, we tried hard to relieve the disciplinary fragmentation, and, hence, the tension, in the research of spatial and motion categorization which we exposed in Section 2.4. We believe we have contributed to bridge the understanding of motion categorization between the AI disciplines dealing with qualitative representations and psychology-related disciplines.

As first contribution, we detailed the terminology in psychology for categorizations (Ch. 3) and the comparable terminology in AI (Ch. 4); and we related both terminologies (e.g., Tab. 4.1) all throughout Chapters 3 to 6.

More than simply terminology, we studied the qualitative representations as categorizations from the standpoint of psychology: in Section 6.4, we sought the categorization model that best describes a qualitative representation of motion. A crucial step in relating qualitative representation to psychological models was to formalize it according to our categorization formalism (Ch. 6); our formalism is customized for qualitative representations, but described both from a psychological and an AI perspective (Sect. 6.2).

Moreover, in our endeavour to bridge different fields of cognition, we have also digressed into the field of linguistics (Sections 4.1.2 and 5.5). Our main purpose was to underpin some statements and cognitive peculiarities about motion and spatial categorizations. As a result, we could relate categorization elements between linguistics and AI (e.g., Tab. 5.1).

As second contribution, we have tested experimentally how our story-based categorizations, which are AI inspired, are cognitively plausible, i.e., how they inform human cognition (Ch. 9). Our results show a patent significant influence of the story-based categorizations in human categorization of motion scenes. This opens the door for the application of story-based categorizations in psychology or any field in human cognition. By publishing our results (Papenmeier et al. 2023), we expect to draw the attention of cognitive scientists in fields other than AI to the story-based representations and more generally to qualitative representations.

12.2 Overcome Shortcomings in Motion Representations

The challenges in Section 12.1 concern motion categorization in general, and motion scenarios categorization in particular. Nonetheless, we used the qualitative representations to tackle all those challenges. Here, we review how we also dealt with the shortcomings that qualitative representations of motion exhibit (Sect. 5.2).

12.2.1 Boosting Number of Motion Representations

We discerned an unusually sparse work on qualitative representations of motion, as compared to the vast research in spatial qualitative representations. We solved this spareness by offering several methods that create a variety of qualitative motion representations: These methods are those expounded in Section 12.1.2, which, at the same time, generate a variety of categorization criteria. It is clear that we obtain a variety of categorization criteria by, among other things, creating a variety of qualitative representations of motion that provide such criteria.

The first method (item i.) creates story-based representations, and so achieves a considerable amount of novel qualitative motion representations that the other methods (items ii. and iii.) amplify exponentially, because they are applicable to any qualitative representation—even only categorization—of motion

As an illustration, we compare the variety of qualitative representations of motion generated in Section 12.1.2 with the qualitative representations of motion in the literature (See our survey, Sect. 5.1). In the literature, we found only three *genuine* qualitative representations of motion

QTC, QRPC, and RCC-d. If we consider all types of QTC variants, we count less than ten motion representations; even if we add the orientation representations, namely, Dipole Calculus, DIA, and OPRA as motion representations, we achieve less than 15 motion representations. In contrast, we can generate 120 story-based motion representations using the current (spatial or motion) representations in the literature: a factor greater than ×8.

12.2.2 Three-dimensional and Beyond

Most motion representations have limited dimensionality in two different senses: the entities involved are mainly one-dimensional, and they move in a one- or two-dimensional space. We solved this shortcoming by taking spatial representations with higher dimensionality and creating the corresponding story-based motion representations, in which entities are both high dimensional entities in a high dimensional space.

In Section 11.2, we listed the qualitative spatial representations that operate in three or more dimensions—A total of nine representations, some even indexed by dimension, such as $\mathrm{PC_n}$ or $\mathrm{BA_n}$. So, we can use them to create the corresponding Stories-$\mathcal{R}$ and Motion-$\mathcal{R}$ representations that operate, as well, in three or more dimensions. As an example, we easily created the story-based categorization Motion-RCC-3d (Eq. (11.1)), which is the story-based representation generated by RCC-3d.

12.2.3 Categorizing motionless entities

For entities without intrinsic orientation, such as simple points or circles, some motion representations—and every directional spatial representation—are ill-defined when one or both entities are motionless; because these representations use the direction of the velocity vector as orientation, and the zero vector, $\vec{0}$, has no defined direction (See Sect. 8.2.2.A).

Most story-based representations are independent of the entities' orientation, in fact, only the story-based representations that are generated from a directional representation, such as OPRA or DIA, rely on the intrinsic orientation of entities; therefore, the rest of story-based representations can be smoothly applied to motionless entities. For example, in Figure 11.8, we used Motion-RCC to describe the trajectories of an entities pair in which one entity, l, is motionless throughout the trajectory.

12.2.4 Helping to Compute the Composition

As we remarked in Sections 4.1.4 and 5.2, one of the most laborious task in a qualitative representation is to find the composition table, or, at least, to provide effective algorithms to compute it. For that reason, some spatial representations and most motion representations either ignore composition or deal rather shallowly with it. On the contrary, we have worked out a method that helps to compute the composition in any story-based categorization (Sect. 10.5). Although, admittedly, the method does not exactly yield the composition, but a superset of relations containing the composition: what we call the 'narrative composition'; a further refining step is required to discard the relations that belong *narrative composition* and do not belong to the standard composition.

Chapter 13

Summary and Conclusion

At this point, we have already explained and demonstrated all the contributions mentioned at the introduction, in Section 1.3. Thus, we conclude this work by hinting at future work in this field of story-based categorizations: first, very concrete research topics, that, because of time, we could not develop in this work; second, the general long term research lines in this area.

Epilogue: Future Work

After we have shown and explained how story-based representations work, and we have detailed their most attractive advantages, we hope that scientists will be strongly motivated to use them. Based on their properties, we are confident that story-based representations are a solid foundation to process motion machinally, specifically, to implement human-computer interaction routines and achieve a human-aware navigation of robots.

Both human-aware navigation and standard autonomous navigation already benefit from navigation algorithms based on qualitative representations. An extra asset of story-based representation, is that, due to its huge variety of motion features, they can be much more finely tailored to each navigation situation.

Abundantly throughout this work, we have remarked open research lines as "future work". Amongst all, we deem that the most promising ones are following. In first place, an exhaustive elaboration of the Stories-$\mathcal{R}$ and Motion-$\mathcal{R}$ for all available spatial representations, and an analysis of their motion features, discerning which are most relevant in which situations. Secondly, to determine the precise relation of the story-based representation to human cognition; at the moment, we have proved a rather general statement: story-based representations are cognitively plausible (in the weakest sense). Deeper research is needed to know which story-based motion features are most salient to humans, and to which extent humans categorize, i.e., encode, the motion information of a motion scenario or a uniform trajectory as story-based relations; in other words, to determine how strong is their cognitive plausibility. We also believe that the psychological attributes of story-based categorization should be comprehensively researched: specify the mathematical form of the similarity function in the featural spaces, quantify the gradation effects in category membership and border categories, identify the taxonomical hierarchies and their basic levels. Finally, a very exciting topic, is to relate the story-based representations to natural language: can we, and how do we verbalize story-based relations in natural language?—There is still enough to do!

We expect to see soon the first real applications of story-based representations in navigation, and that research groups other than ours take further the theoretical and experimental studies in story-based representations.

Part VI

Appendices

Appendix A

Mathematical Formulae and Proofs

A.1 Multiplicativity measure

Proposition A.1.1 *We have two finite qualitative motion representations, namely,* $\mathcal{M}_A$*, having* $|\mathcal{M}_A|$ *relations, and* $\mathcal{M}_B$*, having* $|\mathcal{M}_B|$ *relations that partition the same kinematic space,* $\mathcal{K}$*. Then, the number of relations in the product representation* $\mathcal{M}_A \times \mathcal{M}_B$*, i.e.,* $|\mathcal{M}_A \times \mathcal{M}_B|$*, is bounded according to following inequality.*

$$\max(\mathcal{M}_A, \mathcal{M}_B) \leq |\mathcal{M}_A \times \mathcal{M}_B| \leq |\mathcal{M}_A| \cdot |\mathcal{M}_B| \tag{A.1}$$

Proof.

lower bound From the definition of product representation, $\mathcal{M}_A \times \mathcal{M}_B$, we will proof that both inequalities $|\mathcal{M}_A| \leq |\mathcal{M}_A \times \mathcal{M}_B|$ and $|\mathcal{M}_B| \leq |\mathcal{M}_A \times \mathcal{M}_B|$ apply. If so, it immediately implies that $\max(\mathcal{M}_A, \mathcal{M}_B) \leq |\mathcal{M}_A \times \mathcal{M}_B|$.

Indeed, for every relation, i.e., category, $M_i \in \mathcal{M}_A$ we find a corresponding motion state $K_i \in \mathcal{K}$ that is categorized in such category, i.e., $M_i = \mathrm{f}_{\mu\, \mathcal{M}_A}(K_i)$. Now, each of those motion states K_i is categorized as a motion relation $(M_i, *)$ in the product representation $\mathcal{M}_A \times \mathcal{M}_B$; and none of these relations are equal, i.e., $(M_i, *) \neq (M_j, *) \quad \forall i \neq j$. Thus, for every relation M_i of $\mathcal{M}_A$, we find at least one relation, $(M_i, *)$ in the product representation $\mathcal{M}_A \times \mathcal{M}_B$; which implies that the total number of relations in the product representation must be at least equal to the number of representations in $\mathcal{M}_A$, that is $|\mathcal{M}_A| \leq |\mathcal{M}_A \times \mathcal{M}_B|$. Analoguously, we proof the inequality for $\mathcal{M}_B$.

upper bound The motion relations of the product representation $\mathcal{M}_A \times \mathcal{M}_B$ have the form (M_i^A, M_j^B) where $M_i^A \in \mathcal{M}_A$ and $M_j^B \in \mathcal{M}_B$. The maximum number of product relations occurs when every combinations of relations of $\mathcal{M}_A$ and $\mathcal{M}_B$ is realizable. In this case, we have a total of $|\mathcal{M}_A| \cdot |\mathcal{M}_B|$ relations.

□

Proposition A.1.2 *We can define following function* $\mu : \mathcal{M}_A \times \mathcal{M}_B \to \mathbb{R}$ *for any motion representation with more than one relation*[*]*, i.e.,* $|\mathcal{M}_A|, |\mathcal{M}_B| > 1$*.*

[*] *Note that representations with a single element have no categorizing effect, since any motion state has the same only category. Therefore, we justifiably disregard them.*

$$\mu\left(\mathcal{M}_A, \mathcal{M}_B\right) = \frac{\frac{|\mathcal{M}_A \times \mathcal{M}_B|}{\max(|\mathcal{M}_A|, |\mathcal{M}_B|)} - 1}{\min(|\mathcal{M}_A|, |\mathcal{M}_B|) - 1} \tag{A.2}$$

This function has following properties:

1. *It is symmetric.* $\mu\left(\mathcal{M}_A, \mathcal{M}_B\right) = \mu\left(\mathcal{M}_B, \mathcal{M}_A\right)$
2. *Its image is* $[0, 1]$
   - *i.* $\mu\left(\mathcal{M}_A, \mathcal{M}_B\right) = 0 \quad \Leftrightarrow \quad |\mathcal{M}_A \times \mathcal{M}_B| = \max(\mathcal{M}_A, \mathcal{M}_B)$
   - *ii.* $\mu\left(\mathcal{M}_A, \mathcal{M}_B\right) = 1 \quad \Leftrightarrow \quad |\mathcal{M}_A \times \mathcal{M}_B| = |\mathcal{M}_A| \cdot |\mathcal{M}_B|$
3. *Given three motion representations* $\mathcal{M}_A$*,* $\mathcal{M}_{B_1}$*, and* $\mathcal{M}_{B_2}$*, so that* $|\mathcal{M}_{B_1}| = |\mathcal{M}_{B_2}|$*. Then* μ *is strictly monotone on* $|\mathcal{M}_A \times \mathcal{M}_{B_i}|$
   - *i.* $\mu\left(\mathcal{M}_A, \mathcal{M}_{B_1}\right) < \mu\left(\mathcal{M}_A, \mathcal{M}_{B_2}\right) \quad \Leftrightarrow \quad |\mathcal{M}_A \times \mathcal{M}_{B_1}| < |\mathcal{M}_A \times \mathcal{M}_{B_2}|$
   - *ii.* $\mu\left(\mathcal{M}_A, \mathcal{M}_{B_1}\right) = \mu\left(\mathcal{M}_A, \mathcal{M}_{B_2}\right) \quad \Leftrightarrow \quad |\mathcal{M}_A \times \mathcal{M}_{B_1}| = |\mathcal{M}_A \times \mathcal{M}_{B_2}|$

Proof. We take as start point Equation (A.1).

$$\max(|\mathcal{M}_A|, |\mathcal{M}_B|) \leq \quad |\mathcal{M}_A \times \mathcal{M}_B| \quad \leq |\mathcal{M}_A| \cdot |\mathcal{M}_B|$$

$$\Big\downarrow \quad - \max(|\mathcal{M}_A|, |\mathcal{M}_B|)$$

$$0 \leq \quad |\mathcal{M}_A \times \mathcal{M}_B| - \max(|\mathcal{M}_A|, |\mathcal{M}_B|) \quad \leq |\mathcal{M}_A| \cdot |\mathcal{M}_B| - \max(\mathcal{M}_A, \mathcal{M}_B)$$

$$\Big\downarrow \quad |\mathcal{M}_A| \cdot |\mathcal{M}_B| = \max(|\mathcal{M}_A|, |\mathcal{M}_B|) \min(|\mathcal{M}_A|, |\mathcal{M}_B|)$$

$$0 \leq \quad |\mathcal{M}_A \times \mathcal{M}_B| - \max(|\mathcal{M}_A|, |\mathcal{M}_B|) \quad \leq \max(|\mathcal{M}_A|, |\mathcal{M}_B|)\big(\min(|\mathcal{M}_A|, |\mathcal{M}_B|) - 1\big)$$

$$\Big\downarrow \quad \cdot\, 1/(\max(|\mathcal{M}_A|, |\mathcal{M}_B|)\big(\min(|\mathcal{M}_A|, |\mathcal{M}_B|) - 1\big)$$

$$0 \leq \quad \underbrace{\frac{\frac{|\mathcal{M}_A \times \mathcal{M}_B|}{\max(|\mathcal{M}_A|, |\mathcal{M}_B|)} - 1}{\min(|\mathcal{M}_A|, |\mathcal{M}_B|) - 1}}_{:=\mu(\mathcal{M}_A, \mathcal{M}_B)} \quad \leq 1$$

a. The symmetry of $\mu\left(X, Y\right)$ (proposition's item 1) is direct consequence of the symmetry of $|X \times Y|$, $\max(|X|, |Y|)$, and $\min(|X|, |Y|)$.

b. In the transformation above, we have not only proved Item 2 of the proposition, but we have also simultaneously proved the equalities in Items i. and ii..

c. The strict monotonicity of $\mu\left(\mathcal{M}_A, \mathcal{M}_B\right)$ with respect to $|\mathcal{M}_A \times \mathcal{M}_B|$ keeping $|\mathcal{M}_A|$ and $|\mathcal{M}_B|$ constant is evident in the definition of μ.

□

A.2 General Properties of Temporal Sequences of Relations

A.2.1 Finitude of Stories based on Convex Relations

Proposition A.2.1 *In a qualitative representation* $(\mathcal{U}, \mathcal{R})$ *whose base relations have convex borders, and each relation* R_i *is formed by a finite number of disconnected regions* n_i*, the number of transitions in a story must be finite, and consequently the number of base relations constituting a story must be finite, and also the number of stories must be finite. More concretely:*

a. *The maximum number of qualitative relations constituting a story is*

$$1 + 2\sum_{i=1}^{|\mathcal{R}|} n_i$$

Proof. Each story is created by a uniform motion, that is, it is created by a straight line in the universe space $\mathcal{U}$. If the borders of the base relations are convex, any straight line can only cross the borders at a maximum of two points ('in' and 'out' points). Since each relation R_i is formed by a finite number of disconnected regions, which we call n_i, a straight line can at most perform $2n_i$ border crossings for each relation R_i. And, thus, adding for all relations, we obtain a maximum of $2\sum_{i=1}^{|\mathcal{R}|} n_i$ transitions in a story. Finally, the total number of relations in a story are the number of transitions plus one. So, we end up with the formula.

Further, if the number of relations in a story has an upper bound, and we have a finite number of relations, then we can only generate a finite number of stories. ☐

A particular case of Proposition A.2.1 is when each relation in a representation is constituted by a convex region. We treat this case in the following Proposition. In such case, we can give bounds for the length and number of stories.

Proposition A.2.2 *In a qualitative representation $(\mathcal{U}, \mathcal{R})$ whose base relations are convex, the base relations constituting a story cannot be repeated, i.e., $\forall R_i, R_j \in S = (R_1, \ldots, R_n) \mid i \neq j$ then $R_i \neq R_j$.*

This implies following:

a. *The maximum number of base relations constituting a story is the total number of base relations, i.e., $|\mathcal{R}|$.*

b. *The maximum number of stories, i.e., the absolute upper bound of $|\Sigma|$, is given by the following formula of which we gave also simplified upper and lower bounds.*

$$|\mathcal{R}||\mathcal{R}|! \geq \sum_{i=1}^{|\mathcal{R}|-1} \frac{|\mathcal{R}|!}{(|\mathcal{R}|-i)!} = e\,\Gamma(|\mathcal{R}|+1,1) - (|\mathcal{R}|!+1) \geq |\mathcal{R}|!$$

Proof. We give an intuitive proof based on geometrical principles. First of all, a story describes, by means of base qualitative relations, the trajectory in the kinematic space, $\mathcal{K}$, generated by the temporal evolution of a motion scenario, $(\vec{x}_k, \vec{v}_k; \vec{x}_l, \vec{v}_l)$, in uniform motion. Second, by definition, the base relations are a finite partition of the kinematic space and according to the preposition the base relations are convex. Since the uniform motion trajectory of the motion scenario in the kinematic space is a straight line, and the base relations form a finite convex partition, we deduce that each base relation can occur maximum once on the trajectory of the motion scenario: a straight line intersects with a convex region at most in one simple segment, a point, or null. Consequently, the base relations that the lineal trajectory crosses can appear only once in the story sequence. ☐

Arguably, most qualitative representations have base relations with convex borders (e.g., RCC). Furthermore, in many of them the base relations are themselves convex (e.g., OPRA_n and QTC). Hence, the finitude of stories has an extreme wide reach in qualitative relations.

A.2.2 Finitude of Stories based on Extreme Relations

Definition A.2.1 ***Extreme Relations*** The *extreme relations* are those relations of a story that remain unchanged when $t \to -\infty$ or $t \to +\infty$. That is, a relation R_a is extreme in $t \to -\infty$, if and only if $\exists t_a$, so that in the time interval $(-\infty, t_a)$ the relation between entities is R_a. Analogously, a relation R_b is extreme in $t \to +\infty$ if and only if $\exists t_b$, so that in the time interval $(t_b, +\infty)$ the relation between entities is R_b.

Lemma A.2.1 ***Existence of extreme relations for two entities in uniform motion*** *Two* regular enough[1] *entities that move in uniform motion and are described by a qualitative representation based on overlapping, intersection, or orientation, have a story with extreme relations both for* $t \to -\infty$ *and* $t \to +\infty$*.*

Proof. We name the entities k and l and they have constant velocities $\vec{v}_k$ and $\vec{v}_l$.

1. In the case $\vec{v}_k = \vec{v}_l$ the relation between both entities, R_i, remains constant — this relation is the whole story —, therefore, trivially, R_i is the extreme relation for both $t \to -\infty$ and $t \to +\infty$.

2. In the case $\vec{v}_k \neq \vec{v}_l$ we distinguish two subcases regarding what feature the representation bases on: overlapping-intersection of entities, or relative orientation.

   a. Representations based on *overlapping-intersection* of finite entities have either one or two qualitative relations for the case of 'no overlapping-intersection', e.g., the relation DC in RCC (Fig. 7.2); the relation *disjoint* in 9-Int (Egenhofer 1991); or the relations '<' and '>' in Allen's Algebra (Allen 1983). The mentioned relations must be the extreme relations for each representation, because the distance between two entities that move at different velocities tends to infinity for $t \to \pm\infty$; and consequently the entities do not overlap-intersect any more.

   b. Representations based on *relative orientation* between entities use the *connecting unit vector* between them, i.e., $\hat{kl}(t) = \frac{\vec{x}_l(t)-\vec{x}_k(t)}{\|\vec{x}_l(t)-\vec{x}_k(t)\|}$, for which in uniform motion, i.e., $\vec{x}_k(t) = \vec{v}_k t + \vec{x}_{k0}$ and $\vec{x}_l(t) = \vec{v}_l t + \vec{x}_{l0}$, we obtain both limits :

$$\hat{kl}(t) = \frac{(\vec{v}_l - \vec{v}_k)t + (\vec{x}_l - \vec{x}_k)}{\sqrt{\|\vec{v}_l - \vec{v}_k\|^2 t^2 + 2(\vec{v}_l - \vec{v}_k))((\vec{x}_l - \vec{x}_k) + \|\vec{x}_l - \vec{x}_k\|^2}}$$

$$\lim_{t\to+\infty} \hat{kl}(t) = \frac{\vec{v}_l - \vec{v}_k}{\|\vec{v}_l - \vec{v}_k\|} \qquad \text{(A.3a)} \qquad\qquad \lim_{t\to-\infty} \hat{kl}(t) = -\lim_{t\to+\infty} \hat{kl}(t) \qquad \text{(A.3b)}$$

   Because both limits for the connecting vector exist, the extreme relations of any story exist; they are the relations neighbouring each limit.

□

Lemma A.2.2 ***Finitude of the Temporal Sequences of Relations in Finite Time Intervals*** *In uniform motion, for regular enough*[1] *entities, a temporal sequence of relations in a finite time interval is also finite.*

Proof. A qualitative representation partitions the phase space of two *regular enough* finite entities in a finite number of regions, i.e., the qualitative relations. Therefore by moving in uniform motion in a finite time interval the system goes through a finite number of such regions, i.e., the resultant temporal sequence of relations must be finite. □

Proposition A.2.3 ***Finitude of the Stories*** *We can reasonably show that for two* regular enough[1] *entities the stories are finite in length.*

We build the proof on two properties: first, stories in uniform motion have *extreme relations* (Lem. A.2.1); second, temporal sequences of relations in uniform motion are finite over a finite time interval (Lem. A.2.2).

Proof. According to Lemma A.2.1 two regular enough entities in uniform motion have extreme relations. That is, we can find two time instants t_a and t_b, with $t_a < t_b$, so that in the time interval $(-\infty, t_a)$ the entities' relation remains constant—we call it R_a—and in the time interval $(t_b, +\infty)$ the entities' relation remains constant, we call it R_b.

Now, According to Lemma A.2.2, these regular enough entities moving in uniform motion have a finite temporal sequence of relations in the interval $[t_a, t_b]$, say $(R_1, \dots, R_n)$.

Consequently the story of the two entities, i.e., the temporal sequence of relations in the interval $(-\infty, t_a) \bigcup [t_a, t_b] \bigcup (t_b, \infty)$, would be finite, as it is obtained by concatenating the two extreme relations and the temporal sequence: $(R_a, R_1, \dots, R_n, R_b)$. In case any extreme relation coincides with its border relation, i.e., $R_a = R_1$ or $R_b = R_n$, we merge the repeated ones. ☐

Lemma A.2.3 ***The longest story*** *The stories set is finite, if and only if it exists a longest story, i.e., a story that has more or equal relations than any other.*

Proposition A.2.4 ***Finitude of the Stories Set*** *The set of stories in uniform motion, i.e., the stories set, is finite.*

Proof. We cannot rigorously prove that the stories set is finite, but Lemma A.2.3 gives an equivalent condition that help us to see that the number of possible stories must be finite in most qualitative representations: if we prove that there is a story with more or an equal number of relations than any other, then the stories set must be finite. This is the case in RCC (Tab. 8.1), where the longest story is S_{14}. ☐

A.3 Generation of QTC$_B$ Stories

A.4 Kinematics of the 2-D Uniform Motion of two Entities

A.4.1 Basic kinematic formulae

The position formulae of two entities moving in uniform motion are determined by their velocities, $\vec{v}_k$ and $\vec{v}_l$ (which are time independent), and by the position known in a certain instant t_0, $\vec{x}_k(t_0)$ and $\vec{x}_l(t_0)$:

$$\vec{x}_k(t) = \vec{x}_k(t_0) + \vec{v}_k(t - t_0) \tag{A.4}$$
$$\vec{x}_l(t) = \vec{x}_l(t_0) + \vec{v}_l(t - t_0) \tag{A.5}$$

and, substracting the first to the last one, (A.5) – (A.4)

$$\Delta\vec{x}(t) = \Delta\vec{x}(t_0) + \Delta\vec{v}(t - t_0) \tag{A.6}$$

---

[1] *Enough regular* entities are those finite in size with a finite number of features, i.e., a finite number of vertices, edges, concavities, holes, . . .

Note that following definitions apply for the whole Appendix A.4

$$\Delta\vec{x} := \vec{x}_l - \vec{x}_k \tag{A.7}$$

$$\Delta\vec{v} := \vec{v}_l - \vec{v}_k \tag{A.8}$$

A.4.2 Distance between entities

Proposition A.4.1 *The distance $d\big(\vec{x}_k(t), \vec{x}_l(t)\big) = \|\Delta\vec{x}(t)\|$ between two entities k and l moving in uniform motion is given by*

$$d^2\big(\vec{x}_k(t), \vec{x}_l(t)\big) = \|\Delta\vec{x}(t_0)\|^2 + \|\Delta\vec{v}\|^2(t-t_0)^2 + 2\Delta\vec{x}(t_0)\Delta\vec{v}(t-t_0) \tag{A.9}$$

Proof.

$$d^2\big(\vec{x}_k(t), \vec{x}_l(t)\big) = \|\vec{x}_l(t) - \vec{x}_k(t)\|^2 \overset{\textit{Eqs. (A.4) and (A.5)}}{=} \big\|\big(\vec{x}_l(t_0) - \vec{x}_k(t_0)\big) + (\vec{v}_l - \vec{v}_k)(t-t_0)\big\|^2$$

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v} \quad \downarrow$$

$$d^2\big(\vec{x}_k(t), \vec{x}_l(t)\big) = \|\vec{x}_l(t_0) - \vec{x}_k(t_0)\|^2 + \|\vec{v}_l - \vec{v}_k\|^2(t-t_0)^2 + 2\big(\vec{x}_l(t_0) - \vec{x}_k(t_0)\big)(\vec{v}_l - \vec{v}_k)(t-t_0)$$

□

Corollary A.4.1.1 *The time t_{min} at the minimum distance between entities is following:*

$$\begin{aligned} t_{\text{min}} &= -\frac{\Delta\vec{x}(t_0)\Delta\vec{v}}{\|\Delta\vec{v}\|^2} + t_0 \\ &\textit{or, equivalently,} \\ t_{\text{min}} &= -\frac{\|\Delta\vec{x}(t_0)\|}{\|\Delta\vec{v}\|}\cos(\Delta\vec{x}(t_0), \Delta\vec{v}) + t_0 \end{aligned} \tag{A.10}$$

Proof. We find the vertex of the quadratic equation of $d^2(t)$ in Equation (A.9). Since $f(x) = x^2$ for non-negative values such as $d(t)$ is increasing monotone, the minimum in $d^2(t)$ is the minimum in $d(t)$.

$$\begin{aligned} d^2\big(\vec{x}_k(t), \vec{x}_l(t)\big) &= \overbrace{\|\Delta\vec{x}\|^2}^{c} + \overbrace{\|\Delta\vec{v}\|^2}^{a}(t-t_0)^2 + \overbrace{2\Delta\vec{x}\Delta\vec{v}}^{b}(t-t_0) \\ t_{\text{min}} &= \frac{-b}{2a} + t_0 \text{ is the t value at the minimum of the parabola} \end{aligned}$$

□

Corollary A.4.1.2 *The minimum reached distance between entities is following:*

$$d_{\text{min}} = \|\Delta\vec{x}(t_0)\||\sin(\Delta\vec{x}(t_0), \Delta\vec{v})| \ \forall t_0 \tag{A.11}$$

Proof. We use, as in Cor. A.4.1.1, the parabolic equation in Equation (A.9). We know that the value at the vertex is $d^2_{\text{min}} = c - \frac{b^2}{4a}$ and, thus, substituting we obtain following:

$$d^2_{\text{min}} = \|\Delta\vec{x}(t_0)\|^2 - \frac{\cancel{4}\big(\Delta\vec{x}(t_0)\Delta\vec{v}\big)^2}{\cancel{4}\|\Delta\vec{v}\|^2}$$

$$\Big\downarrow \quad \vec{a}\vec{b} = \|a\|\|b\|\cos(\vec{a},\vec{b})$$

$$\begin{aligned}
d^2_{\text{min}} &= \|\Delta\vec{x}(t_0)\|^2 - \frac{\|\Delta\vec{x}(t_0)\|^2 \cancel{\|\Delta\vec{v}\|^2} \cos^2(\Delta\vec{x}(t_0)\Delta\vec{v})}{\cancel{\|\Delta\vec{v}\|^2}} \\
d^2_{\text{min}} &= \|\Delta\vec{x}(t_0)\|^2 \big(1 - \cos^2(\Delta\vec{x}(t_0)\Delta\vec{v})\big) \\
d^2_{\text{min}} &= \|\Delta\vec{x}(t_0)\|^2 \sin^2(\Delta\vec{x}(t_0)\Delta\vec{v})
\end{aligned}$$

$$\text{Taking the root at both sides} \quad \Big\downarrow \quad \sqrt{x^2} = |x| \text{ and } |d| = d$$

$$d^2_{\text{min}} = \|\Delta\vec{x}(t_0)\||\sin(\Delta\vec{x}(t_0)\Delta\vec{v})|$$

□

A.4.3 Angle between $\Delta\vec{x}(t)$ and $\Delta\vec{v}$

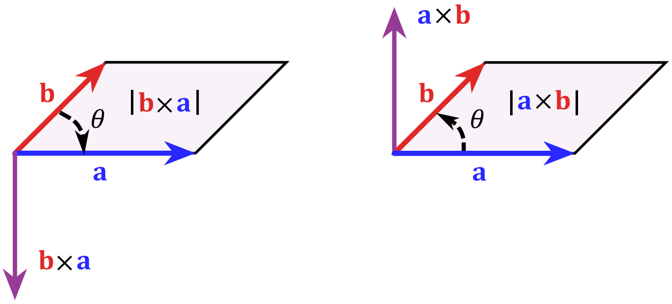
Proposition A.4.2 *In uniform motion, the angle between* $\Delta\vec{x}(t) = \vec{x}_l(t) - \vec{x}_k(t)$ *and* $\Delta\vec{v} = \vec{v}_l - \vec{v}_k$ *at any time* t *has the same sign as the angle between* $\Delta\vec{x}(t_0)$ *and* $\Delta\vec{x}(t_1)$*, where* $t_0 < t_1$*.*

Proof. We will assume without loss of generality that $\Delta\vec{x}(t) = \vec{x}_l(t) - \vec{x}_k(t)$ is computed at $t = 0$ (we can always shift the times to that end). Since the motion is uniform, $\Delta\vec{v}$ is time independent, and, additionally, we have following formula for the entities' positions:

$$\vec{x}_k(t_0) = \vec{x}_k + \vec{v}_k t_0 \qquad \vec{x}_l(t_0) = \vec{x}_l + \vec{v}_l t_0 \tag{A.12}$$

$$\vec{x}_k(t_1) = \vec{x}_k(t_0) + \vec{v}_k(t_1 - t_0) \qquad \vec{x}_l(t_1) = \vec{x}_l(t_0) + \vec{v}_l(t_1 - t_0) \tag{A.13}$$

The entities in uniform motion move on a plane. For that reason, the sign of the angle between vectors equals the sign of the cross product. The cross product has positive sign when is parallel to $\vec{z}$ and has negative sign when is parallel to $-\vec{z}$; $\vec{z}$ is any vector that is perpendicular to the plane and positive oriented to the plane base.



Now we compute the position increments between t_0 and t_1:

$$\Delta\vec{x}(t_0) = \vec{x}_l(t_0) - \vec{x}_k(t_0) \overset{Eq.\ (A.12)}{=} \Delta\vec{x} + \Delta\vec{v}t_0 \tag{A.14}$$

$$\Delta\vec{x}(t_1) = \vec{x}_l(t_1) - \vec{x}_k(t_1) \overset{Eq.\ (A.13)}{=} \Delta\vec{x}(t_0) + \Delta\vec{v}(t_1 - t_0) \tag{A.15}$$

On computing the cross product, we see that $\Delta\vec{x}(t_0) \times \Delta\vec{x}(t_1)$ (Eqs. (A.14) and (A.15)) is parallel to the cross product $\Delta\vec{x} \times \Delta\vec{v}$. Therefore, both vector pairs, $(\Delta\vec{x}(t_0), \Delta\vec{x}(t_1))$ and $(\Delta\vec{x}, \Delta\vec{v})$ have the same angle sign.

$$
\begin{aligned}
\Delta\vec{x}(t_0) \times \Delta\vec{x}(t_1) &= \Delta\vec{x}(t_0) \times \big(\cancelto{0}{\Delta\vec{x}(t_0)} + \Delta\vec{v}(t_1 - t_0)\big) \\
\Delta\vec{x}(t_0) \times \Delta\vec{x}(t_1) &= (t_1 - t_0)\,\Delta\vec{x}(t_0) \times \Delta\vec{v} \overset{Eq.\ (A.14)}{=} (t_1 - t_0)\big(\Delta\vec{x} + \cancelto{0}{\Delta\vec{v}}\ t_0\big) \times \Delta\vec{v} \\
\Delta\vec{x}(t_0) \times \Delta\vec{x}(t_1) &= \underbrace{(t_1 - t_0)}_{>0}\,\Delta\vec{x} \times \Delta\vec{v}
\end{aligned}
$$

Note that neither t_1 nor t_0 need be greater than 0. $\square$

Corollary A.4.2.1 *The sign of* $\sin(\Delta\vec{x}(t), \Delta\vec{v})$ *is constant for each uniform motion.* $\square$

Proposition A.4.3

$$\Delta\vec{x}(t) \cdot \Delta\vec{v} = \Delta\vec{x}(t_0) \cdot \Delta\vec{v} + \|\Delta\vec{v}\|^2 (t - t_0) \tag{A.16}$$

Proof.

$$\Delta\vec{x}(t) \cdot \Delta\vec{v} \overset{Eq.\ (A.6)}{=} (\Delta\vec{x}(t_0) + \Delta\vec{v}(t - t_0)) \cdot \Delta\vec{v}$$

$\square$

Corollary A.4.3.1 *At the point of maximum approach (i.e., minimum distance), the vector* $\Delta\vec{x}$ *and* $\Delta\vec{v}$ *are perpendicular.*

$$\Delta\vec{x}(t_{\min}) \cdot \Delta\vec{v} = 0 \tag{A.17}$$

Proof with scalar product. We seek the zeros of $\Delta\vec{x}(t) \cdot \Delta\vec{v}$, i.e., the $t = t_\perp$ that $\Delta\vec{x}(t_\perp) \cdot \Delta\vec{v} = 0$.

$$\Delta\vec{x}(t_\perp) \cdot \Delta\vec{v} = 0$$

$$\Big\downarrow \ Eq.\ (A.16)$$

$$
\begin{aligned}
\Delta\vec{x}(t_0) \cdot \Delta\vec{v} + \|\Delta\vec{v}\|^2 (t_\perp - t_0) &= 0 \\
t_\perp &= -\frac{\Delta\vec{x}(t_0) \cdot \Delta\vec{v}}{\|\Delta\vec{v}\|^2} + t_0
\end{aligned}
$$

Comparing with Equation (A.10), we see that $t_\perp = t_{\min}$.

$\square$

Proof with $d_{\min}$. Starting with Equation (A.11), we take $t_0 = t_{\min}$:

$$
\begin{aligned}
d_{\min} &= \overbrace{\|\Delta\vec{x}(t_{\min})\|}^{=d_{\min}} |\sin(\Delta\vec{x}(t_{\min}), \Delta\vec{v})| \\
1 &= |\sin(\Delta\vec{x}(t_{\min}), \Delta\vec{v})|
\end{aligned}
$$

Consequently, $\cos(\Delta\vec{x}(t_{\min}), \Delta\vec{v}) = 0$

$\square$

Corollary A.4.3.2 *A compact form of Equation (A.16) is achieved at* $t_0 = t_{\min}$*:*

$$\Delta\vec{x}(t)\cdot\Delta\vec{v} = \|\Delta\vec{v}\|^2(t - t_{\min}) \tag{A.18}$$

Proof.

$$\Delta\vec{x}(t)\cdot\Delta\vec{v} = \cancelto{0\ (Eq.\ (A.17))}{\Delta\vec{x}(t_{\min})\cdot\Delta\vec{v}} + \|\Delta\vec{v}\|^2(t - t_{\min}) \tag{A.19}$$

□

Corollary A.4.3.3

$$\Delta\vec{x}(t)\cdot\Delta\vec{v} < 0 \quad \forall t < t_{\min} \tag{A.20}$$

$$\Delta\vec{x}(t)\cdot\Delta\vec{v} > 0 \quad \forall t > t_{\min} \tag{A.21}$$

Proof. According to Equation (A.18) $\Delta\vec{x}(t)\cdot\Delta\vec{v}$ is strictly monotone increasing and the only zero value is atained at $t = t_{\min}$ □

Corollary A.4.3.4 *The distance between entities* $\|\Delta\vec{x}(t)\|$ *can be expressed at* $t_{\min}$ *in following simplified form:*

$$\|\Delta\vec{x}(t)\| = \sqrt{d_{\min}^2 + \|\Delta\vec{v}\|^2(t - t_{\min})^2} \tag{A.22}$$

Proof. We substitute $t_0 = t_{\min}$ in Equation (A.9) and use Equation (A.17). □

A.4.4 Crossing precedence

Definition A.4.1 ***Precedence Vector*** For two moving entities k and l with trajectories $\vec{x}_k(t)$ and $\vec{x}_l(t)$, and velocities $\vec{v}_k(t)$ and $\vec{v}_l(t)$, we define the *'precedence vector'* as

$$\vec{M}(t) = \Delta\vec{x}(t)\times\Delta\vec{v}(t) = (\vec{x}_l(t) - \vec{x}_k(t))\times(\vec{v}_l(t) - \vec{v}_k(t)) \tag{A.23}$$

If the velocities are constant, the *precedence vector* can discern the '*crossing precedence*', that is, the first entity to go through the crossing point.

Note that $\vec{M}$ can be interpreted as twice the '*areal velocity*' of l with respect to k since $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}(t) = (\vec{r}(t)\times\vec{v}(t))/2$.

Proposition A.4.4 *For any moving entities k and l with constant velocities $\vec{v}_k$ and $\vec{v}_l$, the* precedence vector $\vec{M}(t)$ *(Def. A.4.1) is time independent.*

We can give three different proofs for this fact.

Proof 1: Areal velocity. $\vec{M}$ is proportional to the areal velocity $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}(t)$ of l motion seen from k reference system (as k be motionless). The areal velocity is conserved when no forces act upon the "moving" entity l, also $\vec{M}$ is conserved. □

Proof 2: Kinematic equations.

$$\vec{M}(t) = \Delta\vec{x}(t)\times\Delta\vec{v} \overset{Eqs.\ (A.4)\ and\ (A.5)}{=} \big(\Delta\vec{x}(t_0) + \cancelto{0}{\Delta\vec{v}}(t - t_0)\big)\times\Delta\vec{v}$$

$$\vec{M}(t) = \Delta\vec{x}(t_0)\times\Delta\vec{v} \quad \forall t_0$$

□

Proof 3: Time derivative. It follows directly from taking time derivative of $\vec{M}(t)$.

$$\dot{\vec{M}}(t) = \underbrace{(\dot{\vec{x}}_l(t) - \dot{\vec{x}}_k(t)) \times (\vec{v}_l(t) - \vec{v}_k(t))}_{\substack{= \vec{0} \\ \text{Product of vector by itself,} \\ \vec{v}(t) := \dot{\vec{x}}(t)}} + \underbrace{(\vec{x}_l(t) - \vec{x}_k(t)) \times (\dot{\vec{v}}_l(t) - \dot{\vec{v}}_k(t))}_{\substack{= \vec{0} \\ \text{Velocities are constant,} \\ \dot{\vec{v}}(t) = \vec{0}}} = \vec{0}$$

$\square$

Corollary A.4.4.1 *If $\vec{M}(t) = \Delta\vec{x}(t) \times \Delta\vec{v}(t)$ is time independent in uniform motion, then its value is the same $\forall\ t$. Thus, since the velocities are constant, we have following expression:*

$$\vec{M} = \Delta\vec{x}(t_0) \times \Delta\vec{v} = \Delta\vec{x}(t) \times \Delta\vec{v} \quad \forall\ t \tag{A.24}$$

Corollary A.4.4.2 *In uniform motion, the norm of the precedence vector is $M(t) := \|\vec{M}(t)\| = \|\Delta\vec{v}\| d_{\min}$.*

Proof.

$$\|\vec{M}(t)\| \overset{Eq.\ (A.24)}{=} \|\Delta\vec{x}(t_0) \times \Delta\vec{v}\|$$

$$\Big\downarrow \quad \|\vec{a} \times \vec{b}\| = \|a\|\|b\||\sin(\vec{a}, \vec{b})|$$

$$\|\vec{M}\| = \|\Delta\vec{x}(t_0)\|\|\Delta\vec{v}\||\sin(\Delta\vec{x}(t_0), \Delta\vec{v})| \overset{Cor.\ A.4.1.2}{=} \|\Delta\vec{v}\| d_{\min}$$

$\square$

Proposition A.4.5 ***Crossing Precedence*** *If two crossing entities k and l move each at constant velocity $\vec{v}_k$ and $\vec{v}_l$, then the* precedence vector *$\vec{M}(t)$ is time independent and takes following values depending of which entity goes first through the crossing point C.*

$$\vec{M} = \begin{cases} -\alpha(\vec{v}_k \times \vec{v}_l),\ \alpha > 0 & \textit{if } k \textit{ crosses first, i.e., arrives first at the crossing point} \\ \vec{0} & \textit{if } k \textit{ and } l \textit{ cross simultaneously, arrive at the same} \\ & \quad \textit{time at the crossing point} \\ \beta(\vec{v}_k \times \vec{v}_l),\ \beta > 0 & \textit{if } l \textit{ crosses first, i.e., arrives first at the crossing point} \end{cases} \tag{A.25}$$

Proof. We will split the proof into three cases depending on which entity arrives first at the crossing point C (which has coordinates $\vec{C}$). Since for constant velocities the *precedence vector* $\vec{M}(t)$ is time independent (Prop. A.4.4), it suffices to compute it at some instant t_*, i.e., $\vec{M} = \vec{M}(t_*)$. In each case below we will choose t_*, so that the computations are simplified.

Case A.4.5.1 (k arrives first at C). We compute $\vec{M}(t_*)$ at instant $t_* = t_k$, when k is at C, i.e., $\vec{x}_k(t_k) = \vec{C}$. At this instant, k is aligned with l's velocity (as l is heading to C), then, the difference of positions (l's minus k's) is negatively proportional to l's velocity, i.e.,

$$\vec{x}_l(t_k) - \vec{x}_k(t_k) = \vec{x}_l(t_k) - \vec{C} = -\alpha\vec{v}_l \quad \text{where } \alpha > 0. \tag{A.26}$$

Substituting Equation (A.26) in (A.23) we get

$$\vec{M}(t_k) = -\alpha\vec{v}_l \times (\vec{v}_l - \vec{v}_k) = \alpha(\vec{v}_l \times \vec{v}_k) = -\alpha(\vec{v}_k \times \vec{v}_l)$$

Case A.4.5.2 (k and l superpose—arrive simultaneously—at C). We compute $\vec{M}(t_*)$ at instant $t_* = t_c$, when k and l meet at C. At this instant k and l have the same position, i.e., $\vec{C}$, therefore, $\vec{x}_l(t_c) - \vec{x}_k(t_c) = \vec{0}$. Now, substituting it in Equation (A.23) we get

$$\vec{M}(t_c) = \vec{0} \times (\vec{v}_l - \vec{v}_k) = \vec{0}$$

Case A.4.5.3 (l arrives first at C). The argument is the same as in Case A.4.5.1, but swapping k and l. Accordingly, we have that the difference of positions (l's minus k's) when l is at the crossing point, $\vec{C}$, is positively proportional to k's velocity:

$$\vec{x}_l(t_l) - \vec{x}_k(t_l) = \vec{C} - \vec{x}_k(t_l) = \beta\vec{v}_k \quad \text{where } \beta > 0. \tag{A.27}$$

And substituting (A.27) in Equation (A.23) we get

$$\vec{M}(t_l) = \beta\vec{v}_k \times (\vec{v}_l - \vec{v}_k) = \beta(\vec{v}_k \times \vec{v}_l)$$

□

Corollary A.4.5.1 *The sign of $T = \vec{M} \cdot (\vec{v}_k \times \vec{v}_l)$ determines the* crossing precedence. *If $T < 0$ then k crosses first; if $T = 0$ then k and l cross simultaneously; if $T > 0$ l crosses first.*

Definition A.4.2 ***Crossing Delay*** For two moving entities k and l with trajectories $\vec{x}_k(t)$ and $\vec{x}_l(t)$, and velocities $\vec{v}_k(t)$ and $\vec{v}_l(t)$, we define a scalar, the *'crossing delay'*, as

$$\tau(t) = \vec{M}(t) \cdot \frac{\vec{v}_k(t) \times \vec{v}_l(t)}{\|\vec{v}_k(t) \times \vec{v}_l(t)\|^2} \tag{A.28}$$

where $\vec{M}(t) = \Delta\vec{x}(t) \times \Delta\vec{v}(t)$ is the *precedence vector* (Eq. (A.23)).

Proposition A.4.6 *If two entities k and l, crossing at point C (position $\vec{C}$), move each at constant velocity $\vec{v}_k$ and $\vec{v}_l$, then the* crossing delay *τ (Def. A.4.2) is time independent and yields the signed time difference between the arrivals of k and l to C. Thus, it yields k's crossing delay related to l:*

$$\tau = t_k - t_l \quad \textit{where } t_k \textit{ and } t_l \textit{ fulfil } \vec{x}_k(t_k) = \vec{x}_l(t_l) = \vec{C} \tag{A.29}$$

Proof. Since velocities are constant, we take the precedence vector $\vec{M}$ from Equation (A.25) and substitute it into the definition of crossing delay $\tau(t)$ (Eq. (A.28)). We get straight away that τ is time independent:

$$\tau = \vec{M} \cdot \frac{\vec{v}_k \times \vec{v}_l}{\|\vec{v}_k \times \vec{v}_l\|^2} = \begin{cases} -\alpha & \text{if } k \text{ arrives first at } C \\ 0 & \text{if } k \text{ and } l \text{ superpose at } C \\ \beta & \text{if } l \text{ arrives first at } C \end{cases} \tag{A.30}$$

where α and β are the positive constants in Equation (A.25)

Now we interpret the meanings of α and β.

α. If we look at Equation (A.26), we see that α is the difference between the instant of l's arrival to C, i.e., t_l, and k's arrival to C, i.e., t_k. Indeed, taking norms on Eq. (A.26), we have $\alpha = \|\vec{C} - \vec{x}_l(t_k)\| / \|\vec{v}_l\|$. That is, α is the quotient between l's distance to crossing point C (when k is at it) and l's velocity. Formally,

$$\alpha = t_l - t_k \quad \text{where } t_k \text{ and } t_l \text{ fulfil } \vec{x}_k(t_k) = \vec{x}_l(t_l) = \vec{C} \tag{A.31}$$

β. An analogous reasoning, but using Equation (A.27), shows that β is the difference between the instant of k's arrival and l's arrival to C. Formally,

$$\beta = t_k - t_l \quad \text{where } t_k \text{ and } t_l \text{ fulfil } \vec{x}_k(t_k) = \vec{x}_l(t_l) = \vec{C} \tag{A.32}$$

Substituting Equations (A.31) and (A.32) into Equation (A.30), and applying that when k and l superpose they reach C at the same time $t_k = t_l$, we get Equation (A.29). ☐

Corollary A.4.6.1 *From the Propositions A.4.5 and A.4.6, we obtain for uniform motion that*

$$\vec{M} = \tau(\vec{v}_k \times \vec{v}_l) \tag{A.33}$$

Proposition A.4.7 *The crossing delay can be computed using the equation:*

$$\tau = \frac{\|\Delta\vec{x}\|\|\Delta\vec{v}\| \sin(\Delta\vec{x}, \Delta\vec{v})}{\|\vec{v}_k\|\|\vec{v}_l\| \sin(\vec{v}_k, \vec{v}_l)} \tag{A.34}$$

or, equivalently,

$$\tau = \frac{d_{\min}\|\Delta\vec{v}\|}{\|\vec{v}_k\|\|\vec{v}_l\||\sin(\vec{v}_k, \vec{v}_l)|} \operatorname{sgn}(\sin(\Delta\vec{x}, \Delta\vec{v}) \sin(\vec{v}_k, \vec{v}_l)) \tag{A.35}$$

Proof. We have two equalities for $\vec{M}$, namely, Equations (A.24) and (A.33), and we bring them together.

$$\begin{aligned} \tau(\vec{v}_k \times \vec{v}_l) &= \Delta\vec{x}(t_0) \times \Delta\vec{v} \\ &\Big\downarrow \cdot (\vec{v}_k \times \vec{v}_l) \\ \tau\|\vec{v}_k \times \vec{v}_l\|^2 &= (\vec{v}_k \times \vec{v}_l) \cdot (\Delta\vec{x}(t_0) \times \Delta\vec{v}) \end{aligned} \tag{A.36}$$

Without loss of generality, we assume that the positive defined vector normal to the plain where the entities move is $\hat{z}$. In that case, the cross products are proportional to $\hat{z}$:

$$\begin{aligned} \vec{v}_k \times \vec{v}_l &= \|\vec{v}_k\|\|\vec{v}_l\| \sin(\vec{v}_k, \vec{v}_l)\hat{z} \\ \Delta\vec{x}(t_0) \times \Delta\vec{v} &= \|\Delta\vec{x}(t_0)\|\|\Delta\vec{v}\| \sin(\Delta\vec{x}(t_0), \Delta\vec{v})\hat{z} \end{aligned}$$

We substitute the expressions of the cross products in the previous equation, and, by isolating τ, we obtain Equation (A.34)

$$\begin{aligned} \tau\|\vec{v}_k\|^{\cancel{2}}\|\vec{v}_l\|^{\cancel{2}} \sin^{\cancel{2}}(\vec{v}_k, \vec{v}_l) &= \cancel{\|\vec{v}_k\|}\,\cancel{\|\vec{v}_l\|}\,\cancel{\sin(\vec{v}_k, \vec{v}_l)}\,\cancel{\hat{z}} \cdot \|\Delta\vec{x}(t_0)\|\|\Delta\vec{v}\| \sin(\Delta\vec{x}(t_0), \Delta\vec{v})\cancel{\hat{z}} \\ \tau &= \frac{\|\Delta\vec{x}\|\|\Delta\vec{v}\| \sin(\Delta\vec{x}, \Delta\vec{v})}{\|\vec{v}_k\|\|\vec{v}_l\| \sin(\vec{v}_k, \vec{v}_l)} \\ &\Big\downarrow \textit{Eq. } (\text{A.11}) \\ \tau &= \frac{d_{\min}\|\Delta\vec{v}\|}{\|\vec{v}_k\|\|\vec{v}_l\||\sin(\vec{v}_k, \vec{v}_l)|} \operatorname{sgn}(\sin(\Delta\vec{x}, \Delta\vec{v}) \sin(\vec{v}_k, \vec{v}_l)) \end{aligned}$$

☐

Corollary A.4.7.1 *The sign of the crossing delay τ is the multiplication of the signs of $\angle(\Delta\vec{x},\Delta\vec{v})$ and $\angle(\vec{v}_k,\vec{v}_l)$*

$$\mathrm{sgn}(\tau)=\mathrm{sgn}(\angle(\Delta\vec{x},\Delta\vec{v}))\,\mathrm{sgn}(\angle(\vec{v}_k,\vec{v}_l)) \tag{A.37}$$

Proposition A.4.8 ***Crossing Delay for Parallel Trajectories*** *The* crossing delay, τ *(Eq. (A.28)), is undetermined for parallel velocities, i.e., $\vec{v}_k \parallel \vec{v}_l$, excepting the case of identical velocities $\vec{v}_k=\vec{v}_l$, in which we obtain following limit.*

$$\tau=-\frac{(\vec{x}_k-\vec{x}_l)\cdot\hat{v}_k}{\|\vec{v}_k\|} \tag{A.38}$$

Proof. Entity k has velocity $\vec{v}_k=\|\vec{v}_k\|\hat{v}_k$, and entity l has velocity $\vec{v}_l(\alpha)=\|\vec{v}_l\|(\cos\alpha\,\hat{v}_k+\sin\alpha\,\hat{v}_k^{\perp})$, where $\hat{v}_k^{\perp}$ fulfils $\hat{v}_k\cdot\hat{v}_k^{\perp}=0$ and $\hat{v}_k\times\hat{v}_k^{\perp}=\hat{z}$, being $\hat{z}$ the unit vector of the positive oriented coordinate system $\{\hat{x},\hat{y},\hat{z}\}$. The entities k and l move on the plane $\{\hat{x},\hat{y}\}$, however the third coordinate $\hat{z}$ is useful for computations. Note that the definition of $\vec{v}_l(\alpha)$ is a fully general velocity definition; nevertheless, we use α in order to obtain the limit $\lim_{\alpha\to 0}\vec{v}_l(\alpha)=\|\vec{v}_l\|\hat{v}_k$ in which k and l have parallel velocities.

Our onset equation is the definition of the crossing delay (Eq. (A.28)), in which we have also substituted the definition of $\vec{M}(t)$ (Eq. (A.23)).

$$\tau(\alpha)=\overbrace{(\Delta\vec{x}(t;\alpha)\times\Delta\vec{v}(\alpha))}^{(A)}\cdot\frac{\overbrace{\vec{v}_k\times\vec{v}_l(\alpha)}^{(B)}}{\underbrace{\|\vec{v}_k\times\vec{v}_l(\alpha)\|^2}_{(C)}} \tag{A.39}$$

(A) Here we compute the expression $(-\Delta\vec{x}(t;\alpha)\times-\Delta\vec{v}(\alpha))$, which is equivalent to $(\Delta\vec{x}(t;\alpha)\times\Delta\vec{v}(\alpha))$. We compute first the expression $-\Delta\vec{x}(t;\alpha)$:

$$\begin{array}{rl} & \vec{x}_k(t)=\vec{x}_k(0)+\|\vec{v}_k\|\hat{v}_k t \\ - & \\ & \vec{x}_l(t;\alpha)=\vec{x}_l(0)+\|\vec{v}_l\|(\cos\alpha\hat{v}_k t+\|\vec{v}_l\|\sin\alpha\hat{v}_k^{\perp}t) \\ \hline \end{array}$$

$$\vec{x}_k(t)-\vec{x}_l(t;\alpha)=\vec{x}_k(0)-\vec{x}_l(0)+(\|\vec{v}_k\|-\|\vec{v}_l\|\cos\alpha)\hat{v}_k t-\|\vec{v}_l\|\sin\alpha\hat{v}_k^{\perp}t \tag{A.40}$$

Second, we compute $-\Delta\vec{v}(\alpha)$:

$$\begin{aligned}\vec{v}_k-\vec{v}_l(\alpha)&=\|\vec{v}_k\|\hat{v}_k-\|\vec{v}_l\|\left(\cos\alpha\hat{v}_k+\sin\alpha\hat{v}_k^{\perp}\right)\\&=(\|\vec{v}_k\|-\|\vec{v}_l\|\cos\alpha)\hat{v}_k-\|\vec{v}_l\|\sin\alpha\hat{v}_k^{\perp}\end{aligned} \tag{A.41}$$

We can now perform the cross product between Equation (A.40) and Equation (A.41) to obtain:

$$\begin{aligned}-\Delta\vec{x}(t;\alpha)\times-\Delta\vec{v}(\alpha)=(\|\vec{v}_k\|-\|\vec{v}_l\|\cos\alpha)(\vec{x}_k(0)-\vec{x}_l(0))\times\hat{v}_k-&\\ \|\vec{v}_l\|\sin\alpha(\vec{x}_k(0)-\vec{x}_l(0))\times\hat{v}_k^{\perp}&\end{aligned} \tag{A.42}$$

(B) Expressing $\vec{v}_l(\alpha)$ as shown above $\|\vec{v}_l\|(\cos\alpha\,\hat{v}_k+\sin\alpha\,\hat{v}_k^{\perp})$, we can obtain following compact result for the cross product $\vec{v}_k\times\vec{v}_l(\alpha)$:

$$\vec{v}_k\times\vec{v}_l(\alpha)=\|\vec{v}_k\|\|\vec{v}_l\|\sin\alpha\hat{z} \tag{A.43}$$

(C) Based on the previous item the norm of the scalar product $\|\vec{v}_k \times \vec{v}_l(\alpha)\|$ is following:

$$\|\vec{v}_k \times \vec{v}_l(\alpha)\| = \|\vec{v}_k\|\|\vec{v}_l\||\sin\alpha| \tag{A.44}$$

Now we introduce the results for each expression in Items (A) to (C) into the corresponding expressions of onset Equation (A.39). This yields the following intermediate result:

$$\tau(\alpha) = \frac{(\|\vec{v}_k\| - \|\vec{v}_l\|\cos\alpha)(\vec{x}_k(0) - \vec{x}_l(0)) \times \hat{v}_k - \|\vec{v}_l\|\sin\alpha(\vec{x}_k(0) - \vec{x}_l(0)) \times \hat{v}_k^{\perp}}{\|\vec{v}_k\|\|\vec{v}_l\||\sin\alpha|} \cdot \frac{\sin\alpha}{|\sin\alpha|}\hat{z} \tag{A.45}$$

Now, simplifying, and considering that $\forall\alpha \ \frac{\sin\alpha}{|\sin\alpha|^2} = \frac{1}{\sin\alpha}$ and $\frac{\sin^2\alpha}{|\sin\alpha|^2} = 1$ also for $\lim_{\alpha\to 0}$, and defining $\Delta\vec{x}(0) = \vec{x}_k(0) - \vec{x}_l(0)$ we obtain following:

$$\tau(\alpha) = \left[\left(\frac{\|\vec{v}_k\| - \|\vec{v}_l\|\cos\alpha}{\|\vec{v}_k\|\|\vec{v}_l\|\sin\alpha}\right)\Delta\vec{x}(0) \times \hat{v}_k - \frac{\Delta\vec{x}(0)}{\|\vec{v}_k\|} \times \hat{v}_k^{\perp}\right] \cdot \hat{z} \tag{A.46}$$

Finally, we express $\Delta\vec{x}(0)$ in the orthonormal base $\{\hat{v}_k, \hat{v}_k^{\perp}\}$:

$$\Delta\vec{x}(0) = \Delta\vec{x}(0)_{\hat{v}_k}\hat{v}_k + \Delta\vec{x}(0)_{\hat{v}_k^{\perp}}\hat{v}_k^{\perp} \tag{A.47}$$

And substitute Equation (A.47) into Equation (A.46), we obtain the very final expression of the crossing delay $\tau(\alpha)$.

$$\tau(\alpha) = \frac{\|\vec{v}_k\| - \|\vec{v}_l\|\cos\alpha}{\|\vec{v}_k\|\|\vec{v}_l\|\sin\alpha}\Delta\vec{x}(0)_{\hat{v}_k^{\perp}} - \frac{1}{\|\vec{v}_k\|}\Delta\vec{x}(0)_{\hat{v}_k} \tag{A.48}$$

First note that, as expected, the crossing delay $\tau(\alpha)$ is a scalar. Second, that it is time independent, as it should be according to Proposition A.4.6. Now, if we search the limit $\lim_{\alpha\to 0}\tau(\alpha)$ in Equation (A.48), it is easy to obtain the final result:

i. if $\|\vec{v}_k\| = \|\vec{v}_l\|$, the limit exists and is following:

$$\lim_{\alpha\to 0}\tau(\alpha) = -\frac{\Delta\vec{x}(0)_{\hat{v}_k}}{\|\vec{v}_k\|} \tag{A.49}$$

ii. if $\|\vec{v}_k\| \neq \|\vec{v}_l\|$, the limit does not exist. We have $\lim_{\alpha\to 0^+} = \operatorname{sgn}(\|\vec{v}_k\| - \|\vec{v}_l\|)\infty$ and $\lim_{\alpha\to 0^-} = -\operatorname{sgn}(\|\vec{v}_k\| - \|\vec{v}_l\|)\infty$

□

A.5 Similarity

Proposition A.5.1 ***Triangle Inequality in Similarity Metric Models*** *Given a metric model of similarity, that is, a model where the similarity is computed using a distance* $\mathrm{d}(A, B)$ *through* $\mathrm{S}(A, B) = e^{-\mathrm{d}^{\alpha}(A,B)}$, $0 \leq \alpha \leq 1$, *then the* triangular inequality *in the distance,* $\mathrm{d}(A, B) + \mathrm{d}(B, C) \geq \mathrm{d}(A, C) \quad \forall$*items* A, B, C, *implies following multiplicative inequality in the similarities*

$$\mathrm{S}(A, C) \geq \mathrm{S}(A, B)\ \mathrm{S}(B, C) \qquad \forall \textit{items } A, B, C$$

Proof. if $\mathrm{d}(A,B)+\mathrm{d}(B,C)\geq\mathrm{d}(A,C)$, then, because $\forall\alpha>0$, $e^{-x^{\alpha}}$ is a decreasing function, we have

$$e^{-[\mathrm{d}(A,B)+\mathrm{d}(B,C)]^{\alpha}}\leq e^{-\mathrm{d}^{\alpha}(A,C)}=\mathrm{S}(A,C) \tag{A.50}$$

Now, for $0\leq\alpha\leq1$ and $x,y\geq0$, we have that $x^{\alpha}+y^{\alpha}\geq(x+y)^{\alpha}$. It follows that

$$\mathrm{S}(A,B)\ \mathrm{S}(B,C)=e^{-\mathrm{d}^{\alpha}(A,B)}e^{-\mathrm{d}^{\alpha}(B,C)}=e^{-\left[\mathrm{d}^{\alpha}(A,B)+\mathrm{d}^{\alpha}(B,C)\right]}\leq e^{-[\mathrm{d}(A,B)+\mathrm{d}(B,C)]^{\alpha}} \tag{A.51}$$

Consequently, from Equations (A.50) and (A.51) we obtain

$$\mathrm{S}(A,B)\ \mathrm{S}(B,C)=e^{-\mathrm{d}^{\alpha}(A,B)}e^{-\mathrm{d}^{\alpha}(B,C)}\leq e^{-\mathrm{d}^{\alpha}(A,C)}=\mathrm{S}(A,C)$$

□

Corollary A.5.1.1 *For $\alpha\geq1$ and $x,y\geq0$, we have $(x+y)^{\alpha}\geq x^{\alpha}+y^{\alpha}$ and, therefore, inequality in Equation (A.51) is reversed*

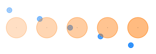
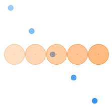
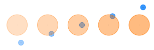
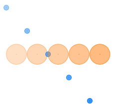
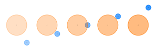
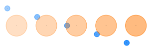
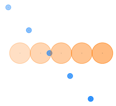
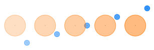
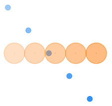
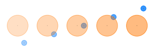
$$e^{-[\mathrm{d}(A,B)+\mathrm{d}(B,C)]^{\alpha}}\leq e^{-\left[\mathrm{d}^{\alpha}(A,B)+\mathrm{d}^{\alpha}(B,C)\right]}=\mathrm{S}(A,B)\ \mathrm{S}(B,C)$$

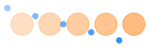
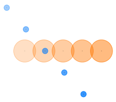
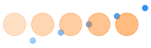
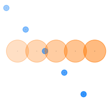
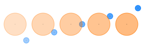
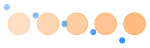
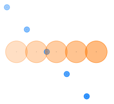
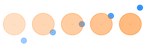
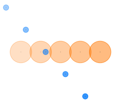
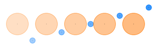
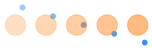
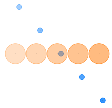
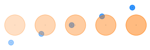
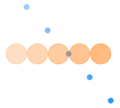
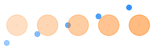
And consequently for $\alpha\geq1$ no inequality for $\mathrm{S}(A,B)$, $\mathrm{S}(B,C)$, and $\mathrm{S}(A,C)$ can be derived using Equation (A.50).

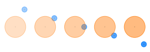
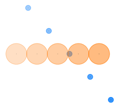
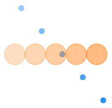
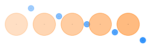
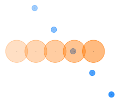
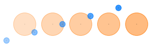
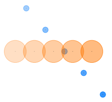
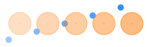
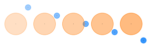
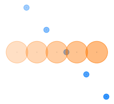
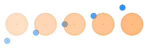
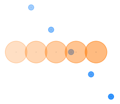
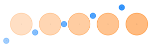
Appendix B

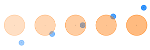
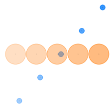
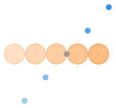
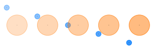
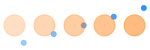
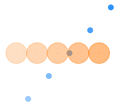
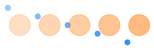
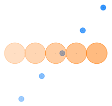
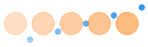
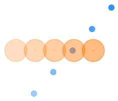
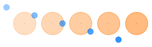
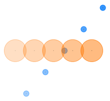
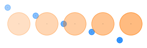
Experiment Data

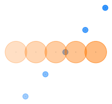
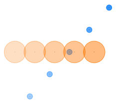
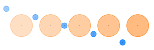
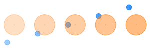
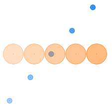
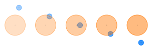
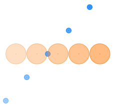
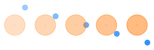
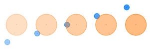
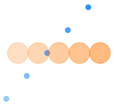
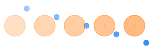
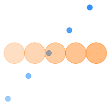
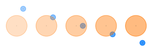
B.1 Stimuli Traces

Group Nr.	Reference stimulus	Transformed stimuli			
	E	*A* RccOpra	*B* Rcc¬Opra	*C* ¬RccOpra	*D* ¬Rcc¬Opra
001	 S_{15} S_{C21} mr-co mo-bp $d_{\min} = 2.75$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.17\,\mathrm{s}$ $\vec{x}_k = (-1.38, 2.94)$ $\vec{v}_k = (15.9, -4.56)$ $\vec{x}_l = (1.38, -2.94)$ $\vec{v}_l = (15.9, -0)$	 S_{15} S_{C21} mr-co mo-bp $d_{\min} = 1.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.14\,\mathrm{s}$ $\vec{x}_k = (-0.75, 3.16)$ $\vec{v}_k = (11, -12.3)$ $\vec{x}_l = (0.75, -3.16)$ $\vec{v}_l = (11, -0)$	 S_{15} $S_{C1\text{-}1}$ mr-co mo-op $d_{\min} = 1.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.09\,\mathrm{s}$ $\vec{x}_k = (0.75, -3.16)$ $\vec{v}_k = (15.9, 4.56)$ $\vec{x}_l = (-0.75, 3.16)$ $\vec{v}_l = (15.9, -0)$	 S_{13} S_{C21} mr-po mo-bp $d_{\min} = 4.00$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.36\,\mathrm{s}$ $\vec{x}_k = (-2, 2.56)$ $\vec{v}_k = (11, -12.3)$ $\vec{x}_l = (2, -2.56)$ $\vec{v}_l = (11, -0)$	 S_{13} $S_{C1\text{-}1}$ mr-po mo-op $d_{\min} = 4.00$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.25\,\mathrm{s}$ $\vec{x}_k = (2, -2.56)$ $\vec{v}_k = (15.9, 4.56)$ $\vec{x}_l = (-2, 2.56)$ $\vec{v}_l = (15.9, -0)$
002					

	S_{13} S_{C21} mr-po mo-bp $d_{\min} = 3.50$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.23\,\mathrm{s}$ $\vec{x}_k = (-1.75, 3.02)$ $\vec{v}_k = (15.2, -4.36)$ $\vec{x}_l = (1.75, -3.02)$ $\vec{v}_l = (15.2, 0)$	S_{13} S_{C21} mr-po mo-bp $d_{\min} = 4.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.43\,\mathrm{s}$ $\vec{x}_k = (-2.25, 2.67)$ $\vec{v}_k = (10.5, -11.7)$ $\vec{x}_l = (2.25, -2.67)$ $\vec{v}_l = (10.5, -0)$	S_{13} $S_{\mathrm{C1\text{-}1}}$ mr-po mo-op $d_{\min} = 4.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.30\,\mathrm{s}$ $\vec{x}_k = (2.25, -2.67)$ $\vec{v}_k = (15.2, 4.36)$ $\vec{x}_l = (-2.25, 2.67)$ $\vec{v}_l = (15.2, 0)$	S_{15} S_{C21} mr-co mo-bp $d_{\min} = 2.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.24\,\mathrm{s}$ $\vec{x}_k = (-1.25, 3.26)$ $\vec{v}_k = (10.5, -11.7)$ $\vec{x}_l = (1.25, -3.26)$ $\vec{v}_l = (10.5, 0)$	S_{15} $S_{\mathrm{C1\text{-}1}}$ mr-co mo-op $d_{\min} = 2.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.16\,\mathrm{s}$ $\vec{x}_k = (1.25, -3.26)$ $\vec{v}_k = (15.2, 4.36)$ $\vec{x}_l = (-1.25, 3.26)$ $\vec{v}_l = (15.2, -0)$
003	 S_{11} S_{C21} mr-no mo-bp $d_{\min} = 5.25$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.39\,\mathrm{s}$ $\vec{x}_k = (-2.62, 3.35)$ $\vec{v}_k = (13.3, -3.83)$ $\vec{x}_l = (2.62, -3.35)$ $\vec{v}_l = (13.3, 0)$	 S_{11} S_{C21} mr-no mo-bp $d_{\min} = 6.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.70\,\mathrm{s}$ $\vec{x}_k = (-3.25, 2.75)$ $\vec{v}_k = (9.25, -10.3)$ $\vec{x}_l = (3.25, -2.75)$ $\vec{v}_l = (9.25, 0)$	 S_{11} $S_{\mathrm{C1\text{-}1}}$ mr-no mo-op $d_{\min} = 6.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.49\,\mathrm{s}$ $\vec{x}_k = (3.25, -2.75)$ $\vec{v}_k = (13.3, 3.83)$ $\vec{x}_l = (-3.25, 2.75)$ $\vec{v}_l = (13.3, 0)$	 S_{13} S_{C21} mr-po mo-bp $d_{\min} = 4.00$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.43\,\mathrm{s}$ $\vec{x}_k = (-2, 3.76)$ $\vec{v}_k = (9.25, -10.3)$ $\vec{x}_l = (2, -3.76)$ $\vec{v}_l = (9.25, -0)$	 S_{13} $S_{\mathrm{C1\text{-}1}}$ mr-po mo-op $d_{\min} = 4.00$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.30\,\mathrm{s}$ $\vec{x}_k = (2, -3.76)$ $\vec{v}_k = (13.3, 3.83)$ $\vec{x}_l = (-2, 3.76)$ $\vec{v}_l = (13.3, -0)$
004	 S_{13} S_{C21} mr-po mo-bp $d_{\min} = 4.50$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.32\,\mathrm{s}$ $\vec{x}_k = (-2.25, 2.69)$ $\vec{v}_k = (14.1, -4.04)$ $\vec{x}_l = (2.25, -2.69)$ $\vec{v}_l = (14.1, 0)$	 S_{13} S_{C21} mr-po mo-bp $d_{\min} = 3.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.36\,\mathrm{s}$ $\vec{x}_k = (-1.75, 3.04)$ $\vec{v}_k = (9.76, -10.9)$ $\vec{x}_l = (1.75, -3.04)$ $\vec{v}_l = (9.76, 0)$	 S_{13} $S_{\mathrm{C1\text{-}1}}$ mr-po mo-op $d_{\min} = 3.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.25\,\mathrm{s}$ $\vec{x}_k = (1.75, -3.04)$ $\vec{v}_k = (14.1, 4.04)$ $\vec{x}_l = (-1.75, 3.04)$ $\vec{v}_l = (14.1, -0)$	 S_{11} S_{C21} mr-no mo-bp $d_{\min} = 5.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.56\,\mathrm{s}$ $\vec{x}_k = (-2.75, 2.18)$ $\vec{v}_k = (9.76, -10.9)$ $\vec{x}_l = (2.75, -2.18)$ $\vec{v}_l = (9.76, 0)$	 S_{11} $S_{\mathrm{C1\text{-}1}}$ mr-no mo-op $d_{\min} = 5.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.39\,\mathrm{s}$ $\vec{x}_k = (2.75, -2.18)$ $\vec{v}_k = (14.1, 4.04)$ $\vec{x}_l = (-2.75, 2.18)$ $\vec{v}_l = (14.1, -0)$
005					

	$S_{15}\ S_{\text{C2-1}}$ mr-co mo-bn $d_{\min} = 2.75$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.17\,\mathrm{s}$ $\vec{x}_k = (1.38, 2.94)$ $\vec{v}_k = (15.9, -4.56)$ $\vec{x}_l = (-1.38, -2.94)$ $\vec{v}_l = (15.9, 0)$	$S_{15}\ S_{\text{C2-1}}$ mr-co mo-bn $d_{\min} = 1.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.14\,\mathrm{s}$ $\vec{x}_k = (0.75, 3.16)$ $\vec{v}_k = (11, -12.3)$ $\vec{x}_l = (-0.75, -3.16)$ $\vec{v}_l = (11, -0)$	$S_{15}\ S_{\text{C11}}$ mr-co mo-on $d_{\min} = 1.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.09\,\mathrm{s}$ $\vec{x}_k = (-0.75, -3.16)$ $\vec{v}_k = (15.9, 4.56)$ $\vec{x}_l = (0.75, 3.16)$ $\vec{v}_l = (15.9, 0)$	$S_{13}\ S_{\text{C2-1}}$ mr-po mo-bn $d_{\min} = 4.00$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.36\,\mathrm{s}$ $\vec{x}_k = (2, 2.56)$ $\vec{v}_k = (11, -12.3)$ $\vec{x}_l = (-2, -2.56)$ $\vec{v}_l = (11, -0)$	$S_{13}\ S_{\text{C11}}$ mr-po mo-on $d_{\min} = 4.00$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.25\,\mathrm{s}$ $\vec{x}_k = (-2, -2.56)$ $\vec{v}_k = (15.9, 4.56)$ $\vec{x}_l = (2, 2.56)$ $\vec{v}_l = (15.9, 0)$
006	 $S_{13}\ S_{\text{C2-1}}$ mr-po mo-bn $d_{\min} = 3.50$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.23\,\mathrm{s}$ $\vec{x}_k = (1.75, 3.02)$ $\vec{v}_k = (15.2, -4.36)$ $\vec{x}_l = (-1.75, -3.02)$ $\vec{v}_l = (15.2, 0)$	 $S_{13}\ S_{\text{C2-1}}$ mr-po mo-bn $d_{\min} = 4.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.43\,\mathrm{s}$ $\vec{x}_k = (2.25, 2.67)$ $\vec{v}_k = (10.5, -11.7)$ $\vec{x}_l = (-2.25, -2.67)$ $\vec{v}_l = (10.5, 0)$	 $S_{13}\ S_{\text{C11}}$ mr-po mo-on $d_{\min} = 4.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.30\,\mathrm{s}$ $\vec{x}_k = (-2.25, -2.67)$ $\vec{v}_k = (15.2, 4.36)$ $\vec{x}_l = (2.25, 2.67)$ $\vec{v}_l = (15.2, -0)$	 $S_{15}\ S_{\text{C2-1}}$ mr-co mo-bn $d_{\min} = 2.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.24\,\mathrm{s}$ $\vec{x}_k = (1.25, 3.26)$ $\vec{v}_k = (10.5, -11.7)$ $\vec{x}_l = (-1.25, -3.26)$ $\vec{v}_l = (10.5, 0)$	 $S_{15}\ S_{\text{C11}}$ mr-co mo-on $d_{\min} = 2.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.16\,\mathrm{s}$ $\vec{x}_k = (-1.25, -3.26)$ $\vec{v}_k = (15.2, 4.36)$ $\vec{x}_l = (1.25, 3.26)$ $\vec{v}_l = (15.2, 0)$
007	 $S_{11}\ S_{\text{C2-1}}$ mr-no mo-bn $d_{\min} = 5.25$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.39\,\mathrm{s}$ $\vec{x}_k = (2.62, 3.35)$ $\vec{v}_k = (13.3, -3.83)$ $\vec{x}_l = (-2.62, -3.35)$ $\vec{v}_l = (13.3, 0)$	 $S_{11}\ S_{\text{C2-1}}$ mr-no mo-bn $d_{\min} = 6.50$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.70\,\mathrm{s}$ $\vec{x}_k = (3.25, 2.75)$ $\vec{v}_k = (9.25, -10.3)$ $\vec{x}_l = (-3.25, -2.75)$ $\vec{v}_l = (9.25, 0)$	 $S_{11}\ S_{\text{C11}}$ mr-no mo-on $d_{\min} = 6.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.49\,\mathrm{s}$ $\vec{x}_k = (-3.25, -2.75)$ $\vec{v}_k = (13.3, 3.83)$ $\vec{x}_l = (3.25, 2.75)$ $\vec{v}_l = (13.3, 0)$	 $S_{13}\ S_{\text{C2-1}}$ mr-po mo-bn $d_{\min} = 4.00$ $\alpha_{vv} = 48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.43\,\mathrm{s}$ $\vec{x}_k = (2, 3.76)$ $\vec{v}_k = (9.25, -10.3)$ $\vec{x}_l = (-2, -3.76)$ $\vec{v}_l = (9.25, -0)$	 $S_{13}\ S_{\text{C11}}$ mr-po mo-on $d_{\min} = 4.00$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.30\,\mathrm{s}$ $\vec{x}_k = (-2, -3.76)$ $\vec{v}_k = (13.3, 3.83)$ $\vec{x}_l = (2, 3.76)$ $\vec{v}_l = (13.3, 0)$
008					

	S_{13} $S_{\mathrm{C2\text{-}1}}$ mr-po mo-bn $d_{\min} = 4.50$ $\alpha_{vv} = 16.04°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.32\,\mathrm{s}$ $\vec{x}_k = (2.25, 2.69)$ $\vec{v}_k = (14.1, -4.04)$ $\vec{x}_l = (-2.25, -2.69)$ $\vec{v}_l = (14.1, 0)$	S_{13} $S_{\mathrm{C2\text{-}1}}$ mr-po mo-bn $d_{\min} = 3.50$ $\alpha_{vv} = 48.13°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.36\,\mathrm{s}$ $\vec{x}_k = (1.75, 3.04)$ $\vec{v}_k = (9.76, -10.9)$ $\vec{x}_l = (-1.75, -3.04)$ $\vec{v}_l = (9.76, 0)$	S_{13} S_{C11} mr-po mo-on $d_{\min} = 3.50$ $\alpha_{vv} = -16.04°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.25\,\mathrm{s}$ $\vec{x}_k = (-1.75, -3.04)$ $\vec{v}_k = (14.1, 4.04)$ $\vec{x}_l = (1.75, 3.04)$ $\vec{v}_l = (14.1, 0)$	S_{11} $S_{\mathrm{C2\text{-}1}}$ mr-no mo-bn $d_{\min} = 5.50$ $\alpha_{vv} = 48.13°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.56\,\mathrm{s}$ $\vec{x}_k = (2.75, 2.18)$ $\vec{v}_k = (9.76, -10.9)$ $\vec{x}_l = (-2.75, -2.18)$ $\vec{v}_l = (9.76, -0)$	S_{11} S_{C11} mr-no mo-on $d_{\min} = 5.50$ $\alpha_{vv} = -16.04°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.39\,\mathrm{s}$ $\vec{x}_k = (-2.75, -2.18)$ $\vec{v}_k = (14.1, 4.04)$ $\vec{x}_l = (2.75, 2.18)$ $\vec{v}_l = (14.1, 0)$
009	 S_{15} $S_{\mathrm{C1\text{-}1}}$ mr-co mo-op $d_{\min} = 2.75$ $\alpha_{vv} = -16.04°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.17\,\mathrm{s}$ $\vec{x}_k = (1.38, -2.94)$ $\vec{v}_k = (15.9, 4.56)$ $\vec{x}_l = (-1.38, 2.94)$ $\vec{v}_l = (15.9, -0)$	 S_{15} $S_{\mathrm{C1\text{-}1}}$ mr-co mo-op $d_{\min} = 1.50$ $\alpha_{vv} = -48.13°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.14\,\mathrm{s}$ $\vec{x}_k = (0.75, -3.16)$ $\vec{v}_k = (11, 12.3)$ $\vec{x}_l = (-0.75, 3.16)$ $\vec{v}_l = (11, 0)$	 S_{15} S_{C21} mr-co mo-bp $d_{\min} = 1.50$ $\alpha_{vv} = 16.04°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.09\,\mathrm{s}$ $\vec{x}_k = (-0.75, 3.16)$ $\vec{v}_k = (15.9, -4.56)$ $\vec{x}_l = (0.75, -3.16)$ $\vec{v}_l = (15.9, 0)$	 S_{13} $S_{\mathrm{C1\text{-}1}}$ mr-po mo-op $d_{\min} = 4.00$ $\alpha_{vv} = -48.13°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.36\,\mathrm{s}$ $\vec{x}_k = (2, -2.56)$ $\vec{v}_k = (11, 12.3)$ $\vec{x}_l = (-2, 2.56)$ $\vec{v}_l = (11, 0)$	 S_{13} S_{C21} mr-po mo-bp $d_{\min} = 4.00$ $\alpha_{vv} = 16.04°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.25\,\mathrm{s}$ $\vec{x}_k = (-2, 2.56)$ $\vec{v}_k = (15.9, -4.56)$ $\vec{x}_l = (2, -2.56)$ $\vec{v}_l = (15.9, -0)$
010	 S_{13} $S_{\mathrm{C1\text{-}1}}$ mr-po mo-op $d_{\min} = 3.50$ $\alpha_{vv} = -16.04°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.23\,\mathrm{s}$ $\vec{x}_k = (1.75, -3.02)$ $\vec{v}_k = (15.2, 4.36)$ $\vec{x}_l = (-1.75, 3.02)$ $\vec{v}_l = (15.2, -0)$	 S_{13} $S_{\mathrm{C1\text{-}1}}$ mr-po mo-op $d_{\min} = 4.50$ $\alpha_{vv} = -48.13°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.43\,\mathrm{s}$ $\vec{x}_k = (2.25, -2.67)$ $\vec{v}_k = (10.5, 11.7)$ $\vec{x}_l = (-2.25, 2.67)$ $\vec{v}_l = (10.5, -0)$	 S_{13} S_{C21} mr-po mo-bp $d_{\min} = 4.50$ $\alpha_{vv} = 16.04°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.30\,\mathrm{s}$ $\vec{x}_k = (-2.25, 2.67)$ $\vec{v}_k = (15.2, -4.36)$ $\vec{x}_l = (2.25, -2.67)$ $\vec{v}_l = (15.2, 0)$	 S_{15} $S_{\mathrm{C1\text{-}1}}$ mr-co mo-op $d_{\min} = 2.50$ $\alpha_{vv} = -48.13°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.24\,\mathrm{s}$ $\vec{x}_k = (1.25, -3.26)$ $\vec{v}_k = (10.5, 11.7)$ $\vec{x}_l = (-1.25, 3.26)$ $\vec{v}_l = (10.5, -0)$	 S_{15} S_{C21} mr-co mo-bp $d_{\min} = 2.50$ $\alpha_{vv} = 16.04°$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.16\,\mathrm{s}$ $\vec{x}_k = (-1.25, 3.26)$ $\vec{v}_k = (15.2, -4.36)$ $\vec{x}_l = (1.25, -3.26)$ $\vec{v}_l = (15.2, -0)$
011					

	S_{11} $S_{\text{C1-1}}$ mr-no mo-op $d_{\min} = 5.25$ $\alpha_{vv} = -16.04^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.39\,\text{s}$ $\vec{x}_k = (2.62, -3.35)$ $\vec{v}_k = (13.3, 3.83)$ $\vec{x}_l = (-2.62, 3.35)$ $\vec{v}_l = (13.3, -0)$	S_{11} $S_{\text{C1-1}}$ mr-no mo-op $d_{\min} = 6.50$ $\alpha_{vv} = -48.13^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.70\,\text{s}$ $\vec{x}_k = (3.25, -2.75)$ $\vec{v}_k = (9.25, 10.3)$ $\vec{x}_l = (-3.25, 2.75)$ $\vec{v}_l = (9.25, 0)$	S_{11} S_{C21} mr-no mo-bp $d_{\min} = 6.50$ $\alpha_{vv} = 16.04^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.49\,\text{s}$ $\vec{x}_k = (-3.25, 2.75)$ $\vec{v}_k = (13.3, -3.83)$ $\vec{x}_l = (3.25, -2.75)$ $\vec{v}_l = (13.3, -0)$	S_{13} $S_{\text{C1-1}}$ mr-po mo-op $d_{\min} = 4.00$ $\alpha_{vv} = -48.13^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.43\,\text{s}$ $\vec{x}_k = (2, -3.76)$ $\vec{v}_k = (9.25, 10.3)$ $\vec{x}_l = (-2, 3.76)$ $\vec{v}_l = (9.25, 0)$	S_{13} S_{C21} mr-po mo-bp $d_{\min} = 4.00$ $\alpha_{vv} = 16.04^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.30\,\text{s}$ $\vec{x}_k = (-2, 3.76)$ $\vec{v}_k = (13.3, -3.83)$ $\vec{x}_l = (2, -3.76)$ $\vec{v}_l = (13.3, -0)$
012	 S_{13} $S_{\text{C1-1}}$ mr-po mo-op $d_{\min} = 4.50$ $\alpha_{vv} = -16.04^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.32\,\text{s}$ $\vec{x}_k = (2.25, -2.69)$ $\vec{v}_k = (14.1, 4.04)$ $\vec{x}_l = (-2.25, 2.69)$ $\vec{v}_l = (14.1, -0)$	 S_{13} $S_{\text{C1-1}}$ mr-po mo-op $d_{\min} = 3.50$ $\alpha_{vv} = -48.13^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.36\,\text{s}$ $\vec{x}_k = (1.75, -3.04)$ $\vec{v}_k = (9.76, 10.9)$ $\vec{x}_l = (-1.75, 3.04)$ $\vec{v}_l = (9.76, 0)$	 S_{13} S_{C21} mr-po mo-bp $d_{\min} = 3.50$ $\alpha_{vv} = 16.04^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.25\,\text{s}$ $\vec{x}_k = (-1.75, 3.04)$ $\vec{v}_k = (14.1, -4.04)$ $\vec{x}_l = (1.75, -3.04)$ $\vec{v}_l = (14.1, -0)$	 S_{11} $S_{\text{C1-1}}$ mr-no mo-op $d_{\min} = 5.50$ $\alpha_{vv} = -48.13^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = -0.56\,\text{s}$ $\vec{x}_k = (2.75, -2.18)$ $\vec{v}_k = (9.76, 10.9)$ $\vec{x}_l = (-2.75, 2.18)$ $\vec{v}_l = (9.76, 0)$	 S_{11} S_{C21} mr-no mo-bp $d_{\min} = 5.50$ $\alpha_{vv} = 16.04^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = +1$ $\tau = 0.39\,\text{s}$ $\vec{x}_k = (-2.75, 2.18)$ $\vec{v}_k = (14.1, -4.04)$ $\vec{x}_l = (2.75, -2.18)$ $\vec{v}_l = (14.1, -0)$
013	 S_{15} S_{C11} mr-co mo-on $d_{\min} = 2.75$ $\alpha_{vv} = -16.04^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.17\,\text{s}$ $\vec{x}_k = (-1.38, -2.94)$ $\vec{v}_k = (15.9, 4.56)$ $\vec{x}_l = (1.38, 2.94)$ $\vec{v}_l = (15.9, 0)$	 S_{15} S_{C11} mr-co mo-on $d_{\min} = 1.50$ $\alpha_{vv} = -48.13^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.14\,\text{s}$ $\vec{x}_k = (-0.75, -3.16)$ $\vec{v}_k = (11, 12.3)$ $\vec{x}_l = (0.75, 3.16)$ $\vec{v}_l = (11, 0)$	 S_{15} $S_{\text{C2-1}}$ mr-co mo-bn $d_{\min} = 1.50$ $\alpha_{vv} = 16.04^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.09\,\text{s}$ $\vec{x}_k = (0.75, 3.16)$ $\vec{v}_k = (15.9, -4.56)$ $\vec{x}_l = (-0.75, -3.16)$ $\vec{v}_l = (15.9, 0)$	 S_{13} S_{C11} mr-po mo-on $d_{\min} = 4.00$ $\alpha_{vv} = -48.13^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.36\,\text{s}$ $\vec{x}_k = (-2, -2.56)$ $\vec{v}_k = (11, 12.3)$ $\vec{x}_l = (2, 2.56)$ $\vec{v}_l = (11, 0)$	 S_{13} $S_{\text{C2-1}}$ mr-po mo-bn $d_{\min} = 4.00$ $\alpha_{vv} = 16.04^\circ$ $\text{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.25\,\text{s}$ $\vec{x}_k = (2, 2.56)$ $\vec{v}_k = (15.9, -4.56)$ $\vec{x}_l = (-2, -2.56)$ $\vec{v}_l = (15.9, 0)$
014					

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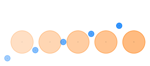
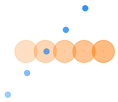
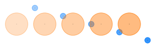
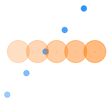
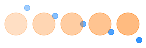
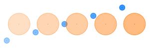
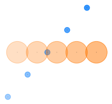
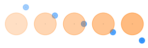
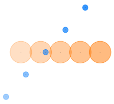
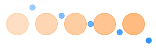
015	 S_{11} S_{C11} mr-no mo-on $d_{\mathrm{min}} = 5.25$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.39\,\mathrm{s}$ $\vec{x}_k = (-2.62, -3.35)$ $\vec{v}_k = (13.3, 3.83)$ $\vec{x}_l = (2.62, 3.35)$ $\vec{v}_l = (13.3, -0)$	 S_{11} S_{C11} mr-no mo-on $d_{\mathrm{min}} = 6.50$ $\alpha_{vv} = -48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.70\,\mathrm{s}$ $\vec{x}_k = (-3.25, -2.75)$ $\vec{v}_k = (9.25, 10.3)$ $\vec{x}_l = (3.25, 2.75)$ $\vec{v}_l = (9.25, 0)$	 S_{11} $S_{\text{C2-1}}$ mr-no mo-bn $d_{\mathrm{min}} = 6.50$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.49\,\mathrm{s}$ $\vec{x}_k = (3.25, 2.75)$ $\vec{v}_k = (13.3, -3.83)$ $\vec{x}_l = (-3.25, -2.75)$ $\vec{v}_l = (13.3, -0)$	 S_{13} S_{C11} mr-po mo-on $d_{\mathrm{min}} = 4.00$ $\alpha_{vv} = -48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.43\,\mathrm{s}$ $\vec{x}_k = (-2, -3.76)$ $\vec{v}_k = (9.25, 10.3)$ $\vec{x}_l = (2, 3.76)$ $\vec{v}_l = (9.25, 0)$	 S_{13} $S_{\text{C2-1}}$ mr-po mo-bn $d_{\mathrm{min}} = 4.00$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.30\,\mathrm{s}$ $\vec{x}_k = (2, 3.76)$ $\vec{v}_k = (13.3, -3.83)$ $\vec{x}_l = (-2, -3.76)$ $\vec{v}_l = (13.3, 0)$
016	 S_{13} S_{C11} mr-po mo-on $d_{\mathrm{min}} = 4.50$ $\alpha_{vv} = -16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.32\,\mathrm{s}$ $\vec{x}_k = (-2.25, -2.69)$ $\vec{v}_k = (14.1, 4.04)$ $\vec{x}_l = (2.25, 2.69)$ $\vec{v}_l = (14.1, -0)$	 S_{13} S_{C11} mr-po mo-on $d_{\mathrm{min}} = 3.50$ $\alpha_{vv} = -48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.36\,\mathrm{s}$ $\vec{x}_k = (-1.75, -3.04)$ $\vec{v}_k = (9.76, 10.9)$ $\vec{x}_l = (1.75, 3.04)$ $\vec{v}_l = (9.76, -0)$	 S_{13} $S_{\text{C2-1}}$ mr-po mo-bn $d_{\mathrm{min}} = 3.50$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.25\,\mathrm{s}$ $\vec{x}_k = (1.75, 3.04)$ $\vec{v}_k = (14.1, -4.04)$ $\vec{x}_l = (-1.75, -3.04)$ $\vec{v}_l = (14.1, 0)$	 S_{11} S_{C11} mr-no mo-on $d_{\mathrm{min}} = 5.50$ $\alpha_{vv} = -48.13^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = 0.56\,\mathrm{s}$ $\vec{x}_k = (-2.75, -2.18)$ $\vec{v}_k = (9.76, 10.9)$ $\vec{x}_l = (2.75, 2.18)$ $\vec{v}_l = (9.76, -0)$	 S_{11} $S_{\text{C2-1}}$ mr-no mo-bn $d_{\mathrm{min}} = 5.50$ $\alpha_{vv} = 16.04^\circ$ $\operatorname{sgn}(\alpha_{\Delta\vec{x}\Delta\vec{v}}) = -1$ $\tau = -0.39\,\mathrm{s}$ $\vec{x}_k = (2.75, 2.18)$ $\vec{v}_k = (14.1, -4.04)$ $\vec{x}_l = (-2.75, -2.18)$ $\vec{v}_l = (14.1, 0)$

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Bibliography

Albath, J., Leopold, J. L., and Maglia, A. M. (2010). “RCC-3D : Qualitative Spatial Reasoning in 3D”. In: *CAINE-2010, 23nd International Conference on Computer Applications in Industry and Engineering, Sponsored by the International Society for Computers and Their Applications (ISCA)*. Las Vegas, Nevada, USA, pp. 74–79. URL: `http://web.mst.edu/$sim$chaman/home/pubs/2010CAINE_LasVegas.pdf` (cit. on p. 73).

Allen, J. F. (1983). “Maintaining Knowledge about Temporal Intervals”. In: *Communications of the ACM* 26.11, pp. 832–843. ISSN: 0001-0782. DOI: `10.1145/182.358434`. HDL: `1802/10574` (cit. on pp. 30, 230).

Alvarez Bravo, J. V. and Glez-Cabrera, F. (2022). “Qualitative Reasoning in Qrpc”. In: *SSRN Electronic Journal*. ISSN: 1556-5068. DOI: `10.2139/ssrn.4173341`. URL: `https://www.ssrn.com/abstract=4173341` (visited on 01/23/2024) (cit. on p. 73).

Anderson, J. R. (1976). *Language, Memory, and Thought*. Hillsdale, NJ: Erlbaum, pp. xiii, 546. ISBN: 0-470-15187-0 (cit. on pp. 88, 157).

APA (2017). *Ethical Principles of Psychologists and Code of Conduct*. URL: `https://www.apa.org/ethics/code/` (visited on 05/04/2022) (cit. on p. 170).

Apostle, H. G. (1980). *Aristotle’s Categories and Propositions : (De Interpretatione)*. Grinnell: The Peripatetic Press. ISBN: 978-0-9602870-5-5 (cit. on p. 24).

Appelman, I. B. and Mayzner, M. S. (1982). “Application of Geometric Models to Letter Recognition: Distance and Density”. In: *Journal of Experimental Psychology: General* 111.1, pp. 60–100. ISSN: 0096-3445. DOI: `10.1037/0096-3445.111.1.60`. PMID: `6460835` (cit. on p. 47).

Ashby, F. G., ed. (1992). *Multidimensional Models of Perception and Cognition*. Scientific Psychology Series. Hillsdale, NJ: Psychology Press, p. 544. ISBN: 978-1-315-80760-7. DOI: `10.4324/9781315807607` (cit. on p. 44).

Ashby, F. G. and Maddox, W. T. (2005). “Human Category Learning”. In: *Annual Review of Psychology* 56, pp. 149–178. ISSN: 0066-4308. DOI: `10.1146/annurev.psych.56.091103.070217`. PMID: `15709932` (cit. on p. 52).

Attneave, F. (1950). “Dimensions of Similarity”. In: *The American Journal of Psychology* 63.4, pp. 516–556. ISSN: 00029556. DOI: `10.2307/1418869`. JSTOR: `1418869`. URL: `http://www.jstor.org/stable/1418869` (cit. on pp. 42, 45 sq.).

Bahrick, L. E., Gogate, L. J., and Ruiz, I. (2002). “Attention and Memory for Faces and Actions in Infancy: The Salience of Actions over Faces in Dynamic Events”. In: *Child Development* 73.6, pp. 1629–1643. DOI: `10.1111/1467-8624.00495` (cit. on p. 25).

Balbiani, P., Condotta, J.-F., and Fariñas del Cerro, L. (1998). “A Model for Reasoning about Bidimensional Temporal Relations”. In: *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR-1998)*. Morgan Kaufmann Publishers Inc., pp. 124–130. URL: `http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.33.1029` (cit. on p. 70).

Balbiani, P. and Osmani, A. (11, 2000). "A Model for Reasoning about Topologic Relations between Cyclic Intervals". In: *Proceedings of the Seventh International Conference on Principles of Knowledge Representation and Reasoning.* KR'00. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., pp. 378–385. URL: `https://dl.acm.org/doi/10.5555/3087011.3087052` (visited on 02/24/2021) (cit. on p. 70).

Barsalou, L. W. (1983). "Ad Hoc Categories". In: *Memory and Cognition* 11.3, pp. 211–227. ISSN: 1532-5946. DOI: `10.3758/BF03196968` (cit. on pp. 39, 41).

— (1985). "Ideals, Central Tendency, and Frequency of Instantiation as Determinants of Graded Structure in Categories". In: *Journal of Experimental Psychology: Learning, Memory, and Cognition* 11.4, pp. 629–654. ISSN: 0278-7393. DOI: `10.1037/0278-7393.11.1-4.629` (cit. on pp. 37 sqq.).

Bartoshuk, L. M. (2019). "Sensation and Perception". In: *The Cambridge Handbook of the Intellectual History of Psychology.* Ed. by Sternberg, R. J. and Pickren, W. E. Cambridge Handbooks in Psychology. New York: Cambridge University Press. Chap. 4, pp. 88–110. ISBN: 978-1-108-29087-6. DOI: `10.1017/9781108290876.005` (cit. on p. 43).

Battig, W. F. and Montague, W. E. (1969). "Category Norms of Verbal Items in 56 Categories: A Replication and Extension of the Connecticut Category Norms." In: *Journal of Experimental Psychology* 80 (3, Pt.2), pp. 1–46. ISSN: 0022-1015. DOI: `10.1037/h0027577` (cit. on p. 37).

Beals, R., Krantz, D. H., and Tversky, A. (1968). "Foundations of Multidimensional Scaling". In: *Psychological Review* 75.2, pp. 127–142. ISSN: 0033-295X. DOI: `10.1037/h0025470` (cit. on p. 44).

Bellotto, N. (2012). "Robot Control Based on Qualitative Representation of Human Trajectories". In: *2012 AAAI Spring Symposium Series.* ISBN: 978-1-57735-551-9. URL: `http://eprints.lincoln.ac.uk/4780/` (cit. on p. 17).

Bellotto, N., Hanheide, M., and de Weghe, N. (2013). "Qualitative Design and Implementation of Human-Robot Spatial Interactions". In: *Social Robotics.* Ed. by Herrmann, G., Pearson, MartinJ., Lenz, A., Bremner, P., Spiers, A., and Leonards, U. Vol. 8239. Lecture Notes in Computer Science. Springer International Publishing, pp. 331–340. ISBN: 978-3-319-02674-9. DOI: `10.1007/978-3-319-02675-6_33`. URL: `http://webpages.lincoln.ac.uk/nbellotto/doc/Bellotto2013a.pdf` (cit. on p. 207).

Berlin, B. (1978). "Ethnobiological Classification". In: *Cognition and Categorization.* Ed. by Rosch, E. Lawrence Erlbaum Associates, pp. 9–26 (cit. on p. 59).

— (2014). *Ethnobiological Classification : Principles of Categorization of Plants and Animals in Traditional Societies.* Princeton Legacy Library. Princeton: Princeton University Press, pp. xviii, 335. ISBN: 978-0-691-09469-4 (cit. on p. 59).

Blair, M. and Homa, D. (2003). "As Easy to Memorize as They Are to Classify: The 5-4 Categories and the Category Advantage". In: *Memory & Cognition* 31.8, pp. 1293–1301. ISSN: 0090-502X. DOI: `10.3758/bf03195812`. PMID: `15058690` (cit. on p. 51).

Borg, Ingwer., Groenen, P. J. F., and Mair, Patrick. (2018). *Applied Multidimensional Scaling and Unfolding.* 2nd ed. Springer, Cham. ISBN: 978-3-319-73471-2. DOI: `10.1007/978-3-319-73471-2` (cit. on p. 41).

Bower, G. H. and Clapper, J. P. (1989). "Experimental Methods in Cognitive Science". In: *Foundations of Cognitive Science.* Ed. by Posner, M. I. Cambridge, MA, US: MIT press Cambridge, MA. Chap. 7, pp. 245–300 (cit. on p. 36).

Bradley, R. A. and Terry, M. E. (1952). "Rank Analysis of Incomplete Block Designs: I. The Method of Paired Comparisons". In: *Biometrika* 39.3/4, p. 324. ISSN: 00063444. DOI: `10.2307/2334029`. JSTOR: `2334029`. URL: `https://www.jstor.org/stable/2334029?origin=crossref` (visited on 04/28/2022) (cit. on p. 158).

Braisby, N. (2012). “Concepts”. In: *Cognitive Psychology*. Ed. by Gellatly, A. and Braisby, N. 2nd ed. Oxford: OUP Oxford. Chap. 5. ISBN: 978-0-19-923699-2 (cit. on pp. 87 sq.).

Bruner, J. S., Austin, G. A., and Goodnow, J. J. (1956). *A Study of Thinking*. New York; London; Sydney: John Wiley and Sons. ISBN: 978-0-471-11415-4 (cit. on p. 23).

Bunnin, N. and Yu, J. (2004). *The Blackwell Dictionary of Western Philosophy*. Wiley Online Books. Blackwell Publishing. ISBN: 978-0-470-99637-9. DOI: `10.1002/9780470996379` (cit. on p. 60).

Burnett, R. C., Medin, D. L., Ross, N. O., and Blok, S. V. (2005). “Ideal Is Typical.” In: *Canadian journal of experimental psychology = Revue canadienne de psychologie experimentale* 59.1, pp. 3–10. ISSN: 1196-1961 (Print). PMID: `15832626` (cit. on p. 158).

Butz, M. V. and Kutter, E. F. (1, 2017). *How the Mind Comes into Being: An Introduction to Cognitive Science from a Functional and Computational Perspective*. Oxford University Press. ISBN: 978-0-19-873969-2. DOI: `10.1093/acprof:oso/9780198739692.001.0001` (cit. on pp. 17, 214).

Carothers, N. L. (2000). *Real Analysis*. Cambridge, UK: Cambridge University Press. ISBN: 978-0-521-49749-7 (cit. on p. 46).

Carroll, J. D. (1976). “Spatial, Non-Spatial and Hybrid Models for Scaling”. In: *Psychometrika* 41.4, pp. 439–463. ISSN: 1860-0980. DOI: `10.1007/BF02296969` (cit. on p. 47).

Chavoshi, S. H., De Baets, B., Neutens, T., De Tré, G., and de Weghe, N. (2015). “Exploring Dance Movement Data Using Sequence Alignment Methods”. In: *PLoS ONE* 10.7, e0132452. ISSN: 1932-6203. DOI: `10.1371/journal.pone.0132452` (cit. on p. 106).

Chen, J., Cohn, A. G., Liu, D., Wang, S., Ouyang, J., and Yu, Q. (2015). “A Survey of Qualitative Spatial Representations”. In: *The Knowledge Engineering Review* 30.01, pp. 106–136. ISSN: 0269-8889. DOI: `10.1017/S0269888913000350`. arXiv: `1312.0049v1` (cit. on pp. 17, 63, 71 sq.).

Chisholm, R. M. (1992). “The Basic Ontological Categories”. In: *Language, Truth and Ontology*. Ed. by Mulligan, K. Vol. 51. Philosophical Studies Series. Dordrecht: Springer, pp. 1–13. ISBN: 978-94-011-2602-1. DOI: `10.1007/978-94-011-2602-1_1` (cit. on p. 60).

Clementini, E. and Billen, R. (2006). “Modeling and Computing Ternary Projective Relations between Regions”. In: *IEEE Transactions on Knowledge and Data Engineering* 18.6, pp. 799–814. ISSN: 1558-2191. DOI: `10.1109/TKDE.2006.102` (cit. on p. 70).

Clementini, E., Di Felice, P., and van Oosterom, P. (1993). “A Small Set of Formal Topological Relationships Suitable for End-User Interaction”. In: *Advances in Spatial Databases*. Ed. by Abel, D. and Chin Ooi, B. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, pp. 277–295. ISBN: 978-3-540-47765-5. DOI: `10.1007/3-540-56869-7_16` (cit. on p. 70).

Cohen, H. and Lefebvre, C. (2005). *Handbook of Categorization in Cognitive Science*. Amsterdam: Elsevier Science, p. 1136. ISBN: 978-0-08-044612-7. DOI: `10.1016/B978-0-08-044612-7.X5053-7` (cit. on pp. 23, 210).

Cohn, A. G. and Hazarika, S. M. (2001a). “Qualitative Spatial Representations and Reasoning: An Overview”. In: *Fundamenta Informaticae* 46, pp. 1–29 (cit. on pp. 63, 73).

Cohn, A. G. and Hazarika, S. M. (20–May 22, 2001b). “Continuous Transitions in Mereotopology”. In: *Commonsense-2001: 5th Symposium on Logical Formalizations of Commonsense Reasoning*. New York, NY, USA. URL: `http://commonsensereasoning.org/2001/final/cohnsmh.pdf` (visited on 09/07/2020) (cit. on p. 113).

Cohn, A. G. and Renz, J. (2008). “Qualitative Spatial Representation and Reasoning”. In: *Handbook of Knowledge Representation*. Ed. by van Harmelen, V. L. and Porter, B. Vol. 3. Foundations of Artificial Intelligence. Elsevier. Chap. 13, pp. 551–596. DOI: `10.1016/S1574-6526(07)03013-1`. URL: `http://www.comp.leeds.ac.uk/qsr/pub/chapter13n.pdf` (cit. on pp. 63, 157).

Cohn, A. G. (1995). “The Challenge of Qualitative Spatial Reasoning”. In: *ACM Computing Surveys (CSUR)* 44.113, pp. 1–5. URL: `http://dl.acm.org/citation.cfm?id=212112` (cit. on p. 31).

Cohn, A. G., Bennett, B., Gooday, J., and Gotts, N. M. (1, 1997). “Qualitative Spatial Representation and Reasoning with the Region Connection Calculus”. In: *GeoInformatica* 1.3, pp. 275–316. ISSN: 1573-7624. DOI: `10.1023/A:1009712514511` (cit. on pp. 64 sq., 113).

Cohn, A. G., Gooday, J. M., and Bennett, B. (1994). “A Comparison of Structures in Spatial and Temporal Logics”. In: *Philosophy and the Cognitive Sciencies, 16th International Wittgenstein Symposium, 15 - 22 August 1993, Kirchberg Am Wechsel (Austria)*. 16th International Wittgenstein Symposium, 15 - 22 August 1993, Kirchberg Am Wechsel (Austria). Ed. by Casati, R., Smith, B., and White, G. Österreichische Ludwig-Wittgenstein-Gesellschaft: Schriftenreihe Der Wittgenstein-Gesellschaft ; 21. Vienna: Hölder-Pichler-Tempsky. ISBN: 978-3-209-01747-5. URL: `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.35.7815&rep=rep1&type=pdf` (cit. on p. 113).

Coletti, G. and Bouchon-Meunier, B. (2019). “A Study of Similarity Measures through the Paradigm of Measurement Theory: The Classic Case”. In: *Soft Computing* 23.16, pp. 6827–6845. ISSN: 1433-7479. DOI: `10.1007/s00500-018-03724-3` (cit. on p. 49).

Collins, A. M. and Quillian, M. R. (1969). “Retrieval Time from Semantic Memory”. In: *Journal of Verbal Learning and Verbal Behavior* 8.2, pp. 240–247. ISSN: 0022-5371. DOI: `10.1016/S0022-5371(69)80069-1` (cit. on p. 56).

Colman, A. M. (22, 2015). *A Dictionary of Psychology*. Fourth Edition. Oxford Quick Reference. Oxford, New York: Oxford University Press. 896 pp. ISBN: 978-0-19-965768-1. DOI: `10.1093/acref/9780199657681.001.0001`. Google Books: `UDnvBQAAQBAJ` (cit. on pp. 94, 125).

Corter, J. E. (1987). “Similarity, Confusability, and the Density Hypothesis”. In: *Journal of experimental psychology. General* 116.3, pp. 238–249. ISSN: 0096-3445. DOI: `10.1037//0096-3445.116.3.238`. PMID: `2957458` (cit. on p. 47).

— (1988). “Testing the Density Hypothesis: Reply to Krumhansl”. In: *Journal of Experimental Psychology: General* 117.1, pp. 105–106. ISSN: 0096-3445. DOI: `10.1037/0096-3445.117.1.105` (cit. on p. 47).

Cowan, N. (2001). “The Magical Number 4 in Short-Term Memory: A Reconsideration of Mental Storage Capacity”. In: *The Behavioral and Brain Sciences* 24.1, 87–114, discussion 114–185. ISSN: 0140-525X. DOI: `10.1017/s0140525x01003922`. PMID: `11515286` (cit. on p. 159).

Cristani, M. (20, 1999). “The Complexity of Reasoning about Spatial Congruence”. In: *Journal of Artificial Intelligence Research* 11, pp. 361–390. ISSN: 1076-9757. DOI: `10.1613/jair.641`. arXiv: `1106.0664`. URL: `http://arxiv.org/abs/1106.0664` (visited on 02/24/2021) (cit. on p. 70).

Cumpa, J. and Tegtmeier, E. (2011). *Ontological Categories*. Eide: Foundations of Ontology v. 3. Frankfurt: De Gruyter. ISBN: 978-3-11-032940-7 (cit. on p. 60).

Dawson, M. R. W. (2013). *Mind, Body, World : Foundations of Cognitive Science*. OPEL (Open Paths to Enriched Learning) Vol. 1. Edmonton, Canada: Athabasca University Press. ISBN: 978-1-927356-17-3. URL: `http://www.aupress.ca/index.php/books/120227` (cit. on p. 30).

Delafontaine, M., Cohn, A. G., and Van De Weghe, N. (2011). “Implementing a Qualitative Calculus to Analyse Moving Point Objects”. In: *Expert Systems with Applications* 38.5, pp. 5187–5196. ISSN: 09574174. DOI: `10.1016/j.eswa.2010.10.042` (cit. on pp. 25, 71, 73 sq., 106, 204).

Douven, I. (2020). “Fuzzy Concept Combination: An Empirical Study”. In: *Fuzzy Sets and Systems. An International Journal in Information Science and Engineering*. ISSN: 0165-0114. DOI: `10.1016/j.fss.2020.03.004` (cit. on p. 40).

Dunn-Rankin, P., Knezek, G. A., Wallace, S. R., and Zhang, S. (2004). *Scaling Methods*. Lawrence Erlbaum Associates, Inc., p. 254. ISBN: 0-8058-1802-2 (cit. on pp. 42, 52).

Dylla, F., Frommberger, L., Wallgrün, J. O., Wolter, D., Nebel, B., and Wölfl, S. (2007). "SailAway: Formalizing Navigation Rules". In: *Proceedings of the Artificial and Ambient Intelligence Symposium on Spatial Reasoning and Communication, AISB'07*, pp. 1–5. URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.64.6638&rep=rep1&type=pdf (cit. on pp. 29, 63, 72, 207).

Dylla, F. (2008). *An Agent Control Perspective on Qualitative Spatial Reasoning: Towards More Intuitive Spatial Agent Development*. Vol. 320. IOS Press (cit. on p. 113).

Dylla, F., Coors, M., and Bhatt, M. (2012). "Socially Compliant Navigation in Crowded Environments". In: *Spatio-Temporal Dynamics*, p. 9 (cit. on p. 113).

Dylla, F., Lee, J. H., Mossakowski, T., Schneider, T., Delden, A. V., Ven, J. V. D., and Wolter, D. (2017). "A Survey of Qualitative Spatial and Temporal Calculi: Algebraic and Computational Properties". In: *ACM Computing Surveys (CSUR)* 50.1, p. 7. arXiv: 1606.00133 (cit. on pp. 19, 31, 63, 67, 69–72, 106, 177–180, 200, 203, 220).

Dylla, F. and Moratz, R. (2005). "Exploiting Qualitative Spatial Neighborhoods in the Situation Calculus". In: *Spatial Cognition IV. Reasoning, Action, Interaction: International Conference Spatial Cognition 2004, Frauenchiemsee, Germany, October 11-13, 2004, Revised Selected Papers*. Ed. by Freksa, C., Knauff, M., Krieg-Brückner, B., Nebel, B., and Barkowsky, T. configuration 1. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 304–322. ISBN: 978-3-540-32255-9. DOI: 10.1007/978-3-540-32255-9_18 (cit. on p. 63).

Dylla, F., Mossakowski, T., Schneider, T., and Wolter, D. (2013). "Algebraic Properties of Qualitative Spatio-Temporal Calculi". In: *Spatial Information Theory*. Ed. by Tenbrink, T., Stell, J., Galton, A., and Wood, Z. Vol. 8116. Lecture Notes in Computer Science. Springer International Publishing, pp. 516–536. ISBN: 978-3-319-01789-1. DOI: 10.1007/978-3-319-01790-7_28. URL: http://www.informatik.uni-bremen.de/tdki/research/papers/2013/DMSW-COSIT13.pdf (cit. on pp. 178, 181).

Dylla, F. and Wallgrün, J. O. (2007). "Qualitative Spatial Reasoning with Conceptual Neighborhoods for Agent Control". In: *Journal of Intelligent and Robotic Systems* 48.1, pp. 55–78. ISSN: 0921-0296. DOI: 10.1007/s10846-006-9099-4 (cit. on pp. 63, 113, 207).

Egenhofer, M. J. (1991). "Reasoning about Binary Topological Relations". In: *Advances in Spatial Databases: 2nd Symposium, SSD '91 Zurich, Switzerland, August 28–30, 1991 Proceedings*. Ed. by Günther, O. and Schek, H.-J. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 141–160. ISBN: 978-3-540-47615-3. DOI: 10.1007/3-540-54414-3_36 (cit. on pp. 63, 70, 73, 230).

Egenhofer, M. J. and Sharma, J. (1993). "Assessing the Consistency of Complete and Incomplete Topological Information". In: *Geographical Systems* 1.1, pp. 47–68. URL: https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.16.1253 (cit. on p. 70).

Egenhofer, M. J. and Al-Taha, K. K. (1992). "Reasoning about Gradual Changes of Topological Relationships". In: *Theories and Methods of Spatio-Temporal Reasoning in Geographic Space*. Ed. by Frank, A. U., Campari, I., and Formentini, U. Red. by Goos, G. and Hartmanis, J. Vol. 639. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 196–219. ISBN: 978-3-540-55966-5. DOI: 10.1007/3-540-55966-3_12 (cit. on p. 113).

Ennis, D. M. (1992). "Modeling Similarity and Identification When There Are Momentary Fluctuations in Psychological Magnitudes". In: *Multidimensional Models of Perception and Cognition*. Ed. by Ashby, F. G. Scientific Psychology Series. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc. Chap. 11, pp. 279–298. ISBN: 0-8058-0577-X (cit. on p. 44).

Fajen, B. R. and Phillips, F. (2013). "Spatial Perception and Action". In: *Handbook of Spatial Cognition*. Ed. by Waller, D. and Nadel, L. Washington: American Psychological Association. Chap. 3, pp. 67–80. ISBN: 978-1-4338-1204-0. DOI: 10.1037/13936-004 (cit. on p. 214).

Fantz, R. L. and Nevis, S. (1967). “Pattern Preferences and Perceptual-Cognitive Development in Early Infancy”. In: *Merrill-Palmer Quarterly of Behavior and Development* 13.1, pp. 77–108. ISSN: 0026-0150. JSTOR: 23082720. URL: http://www.jstor.org/stable/23082720 (cit. on p. 25).

Frank, A. U. (1991). “Qualitative Spatial Reasoning with Cardinal Directions”. In: *7. Österreichische Artificial-Intelligence-Tagung / Seventh Austrian Conference on Artificial Intelligence*. Ed. by Brauer, W. and Kaindl, H. Vol. 287. Informatik-Fachberichte. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 157–167. ISBN: 978-3-540-54567-5. DOI: 10.1007/978-3-642-46752-3_17 (cit. on p. 70).

Freksa, C. and Zimmermann, K. (1992). “On the Utilization of Spatial Structures for Cognitively Plausible and Efficient Reasoning”. In: 1992 IEEE International Conference on Systems, Man, and Cybernetics, 261–266 vol.1 (cit. on p. 70).

Freksa, C. (1992a). “Temporal Reasoning Based on Semi-Intervals”. In: *Artificial Intelligence* 54, pp. 199–227. ISSN: 0004-3702. DOI: 10.1016/0004-3702(92)90090-K. URL: http://www.icsi.berkeley.edu/ftp/global/global/pub/techreports/1990/tr-90-016.pdf (cit. on pp. 30, 63, 113, 179).

— (1992b). “Using Orientation Information for Qualitative Spatial Reasoning”. In: *Theories and Methods of Spatio-Temporal Reasoning in Geographic Space* 639.1, pp. 162–178. ISSN: 0302-9743. DOI: 10.1007/3-540-55966-3_10. URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.2.3577&rep=rep1&type=pdf (cit. on pp. 72, 74).

Galton, A. (1995). “Towards a Qualitative Theory of Movement”. In: *Spatial Information Theory a Theoretical Basis for GIS*. Ed. by Frank, AndrewU. and Kuhn, W. Vol. 988. Lecture Notes in Computer Science. Springer Berlin Heidelberg, pp. 377–396. ISBN: 978-3-540-60392-4. DOI: 10.1007/3-540-60392-1_25 (cit. on p. 113).

— (2000). *Qualitative Spatial Change*. Spatial Information Systems. Oxford ; New York: Oxford University Press. 409 pp. ISBN: 978-0-19-823397-8. Google Books: KFqAAAAAMAAJ (cit. on p. 105).

— (2001). “Dominance Diagrams: A Tool for Qualitative Reasoning about Continuous Systems”. In: *Fundamenta Informaticae* 46.1, pp. 55–70. URL: http://iospress.metapress.com/content/330X1J7L63MTXVB9 (cit. on pp. 113 sqq.).

Galton, A. P. (8–Sept. 9, 1994). “Lines of Sight”. In: *Proceedings of the Seventh Annual Irish Conference on Artificial Intelligence and Cognitive Science (AICS’94)*. Ed. by Keane, M., Cunningham, P., Brady, M., and Byrne, R. Trinity College Dublin: Dublin University Press, pp. 103–113. URL: http://www.secamlocal.ex.ac.uk/people/staff/apgalton/papers/sightlines.ps (cit. on p. 70).

Gärdenfors, P. (2004). *Conceptual Spaces: The Geometry of Thought*. London, UK: MIT Press, p. 307. ISBN: 978-0-262-57219-4. URL: https://mitpress.mit.edu/books/conceptual-spaces (cit. on pp. 22, 31, 88).

— (2014). *The Geometry of Meaning: Semantics Based on Conceptual Spaces*. MIT Press. URL: https://mitpress.mit.edu/books/geometry-meaning (cit. on p. 88).

Gati, I. and Tversky, A. (1984). “Weighting Common and Distinctive Features in Perceptual and Conceptual Judgments”. In: *Cognitive Psychology* 16.3, pp. 341–370. ISSN: 0010-0285. DOI: 10.1016/0010-0285(84)90013-6 (cit. on pp. 42, 49).

Gerrig, R. J. and Zimbardo, P. G. (2005). *Psychology and Life*. 17th ed. Boston : Pearson-/Allen and Bacon. 710 pp. ISBN: 978-0-205-41799-5. URL: http://archive.org/details/psychologylife00rich (visited on 06/10/2021) (cit. on pp. 23, 36).

Glazer, R. and Nakamoto, K. (1991). “Cognitive Geometry: An Analysis of Structure Underlying Representations of Similarity”. In: *Marketing Science* 10.3, pp. 205–228. DOI: 10.1287/mksc.10.3.205. JSTOR: 183942. URL: https://www.jstor.org/stable/183942 (cit. on p. 44).

Glez-Cabrera, F. J., Álvarez-Bravo, J. V., and Díaz, F. (2013). “QRPC: A New Qualitative Model for Representing Motion Patterns”. In: *Expert Systems with Applications* 40.11, pp. 4547–4561. ISSN: 09574174. DOI: 10.1016/j.eswa.2013.01.058 (cit. on pp. 23, 71, 74).

Goldstone, R. L., Kersten, A., and Carvalho, P. F. (2012). “Concepts and Categorization”. In: *Handbook of Psychology, Second Edition*. Ed. by Healy, A. F., Proctor, R. W., and Weiner, I. B. 2nd ed. Vol. 4. New Jersey: John Wiley & Sons Inc. Chap. 22, pp. 607–630. ISBN: 978-1-118-13388-0. DOI: 10.1002/9781118133880.hop204022. URL: https://pcl.sitehost.iu.edu/papers/concepts2012.pdf (cit. on pp. 54, 159).

Goldstone, R. L. (1994). “The Role of Similarity in Categorization: Providing a Groundwork”. In: *Cognition* 52.2, pp. 125–157. ISSN: 00100277. DOI: 10.1016/0010-0277(94)90065-5. PMID: 7924201. URL: https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.588.4485 (visited on 06/24/2021) (cit. on p. 41).

— (1, 1996). “Isolated and Interrelated Concepts”. In: *Memory and Cognition* 24.5, pp. 608–628. ISSN: 1532-5946. DOI: 10.3758/BF03201087 (cit. on pp. 52, 94).

Goodman, N. (1972a). “Likeness”. In: *Problems and Projects*. Indianapolis: Bobbs-Merrill. Chap. IX, pp. 421–447. ISBN: 978-0-915144-37-2 (cit. on p. 53).

— (1972b). “Seven Strictures on Similarity”. In: *Problems and Projects*. Bobs-Merril. Chap. IX, 2, pp. 437–447 (cit. on p. 60).

Gottfried, B. (2004). “Reasoning about Intervals in Two Dimensions”. In: *2004 IEEE International Conference on Systems, Man and Cybernetics (IEEE Cat. No.04CH37583)*. 2004 IEEE International Conference on Systems, Man and Cybernetics (IEEE Cat. No.04CH37583). Vol. 6. The Hague, Netherlands: IEEE, pp. 5324–5332. ISBN: 978-0-7803-8567-2. DOI: 10.1109/ICSMC.2004.1401040 (cit. on p. 70).

Gregson, R. A. M. (A. M. (1975). *Psychometrics of Similarity*. New York : Academic Press. ix, 262. ISBN: 978-0-12-301550-1. URL: http://archive.org/details/psychometricsofs0000greg (visited on 04/01/2021) (cit. on p. 49).

Hahn, U. and Heit, E. (2001). “Semantic Similarity, Cognitive Psychology Of”. In: *International Encyclopedia of the Social and Behavioral Sciences*. Ed. by Smelser, N. J. and Baltes, P. B. 1979. Oxford: Pergamon, pp. 13878–13881. ISBN: 978-0-08-043076-8. DOI: 10.1016/B0-08-043076-7/01548-5. URL: https://archive.org/details/internationalenc0020unse (cit. on pp. 43, 49).

Hahn, U. and Chater, N. (1997). “Concepts and Similarity”. In: *Knowledge, Concepts, and Categories*. Ed. by Lamberts, K. and Shanks, D. Studies in Cognition. Cambridge, MA: The MIT Press. Chap. 2, pp. 43–92. DOI: 10.4324/9780203765418 (cit. on p. 45).

Haith, M. M. (1980). *Rules That Babies Look by: The Organization of Newborn Visual Activity*. 1st ed. Lawrence Erlbaum Associates, p. 160. ISBN: 978-0-89859-033-3 (cit. on p. 25).

Hampton, J. A. (1979). “Polymorphous Concepts in Semantic Memory”. In: *Journal of Verbal Learning and Verbal Behavior* 18.4, pp. 441–461. ISSN: 0022-5371. DOI: 10.1016/S0022-5371(79)90246-9. URL: https://www.researchgate.net/publication/222455995_Polymorphous_concepts_in_semantic_memory (cit. on p. 94).

— (1995). “Testing the Prototype Theory of Concepts”. In: *Journal of Memory and Language* 34.5, pp. 686–708. ISSN: 0749-596X. DOI: 10.1006/jmla.1995.1031 (cit. on p. 51).

— (2011). “Conceptual Combinations and Fuzzy Logic”. In: *Concepts and Fuzzy Logic*. MIT Press Cambridge, MA, pp. 209–232. ISBN: 978-0-262-01647-6. URL: https://mitpress.mit.edu/books/concepts-and-fuzzy-logic (cit. on p. 89).

Hanheide, M., Peters, A., and Bellotto, N. (2012). “Analysis of Human-Robot Spatial Behaviour Applying a Qualitative Trajectory Calculus”. In: *Proceedings - IEEE International Workshop on Robot and Human Interactive Communication*, pp. 689–694. DOI: 10.1109/ROMAN.2012.

6343831. URL: http://webpages.lincoln.ac.uk/nbellotto/doc/Hanheide2012.pdf (cit. on pp. 74, 106 sq., 204).

Harnad, S. (2017). “To Cognize Is to Categorize: Cognition Is Categorization”. In: *Handbook of Categorization in Cognitive Science (Second Edition)*. Ed. by Cohen, H. and Lefebvre, C. Second Edi. San Diego: Elsevier. Chap. 2, pp. 21–54. ISBN: 978-0-08-101107-2. DOI: 10.1016/B978-0-08-101107-2.00002-6 (cit. on p. 23).

Hazarika, S. M. and Cohn, A. G. (2001). “Qualitative Spatio-Temporal Continuity”. In: *Spatial Information Theory*. Ed. by Montello, D. R. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, pp. 92–107. ISBN: 978-3-540-45424-3. DOI: 10.1007/3-540-45424-1_7 (cit. on p. 113).

Heider, E. R. (1972). “Universals in Color Naming and Memory”. In: *Journal of Experimental Psychology* 93.1, pp. 10–20. DOI: 10.1037/h0032606. PMID: 5013326 (cit. on p. 37).

Hernández, D. (1994). *Qualitative Representation of Spatial Knowledge*. Ed. by Hernández, D. Vol. 804. Lecture Notes in Computer Science 0302-9743. Berlin Heidelberg: Springer-Verlag. ISBN: 978-3-540-58058-4. DOI: 10.1007/BFb0020328 (cit. on pp. 27, 30, 62 sq.).

Hespos, S. J. and Baillargeon, R. (2001a). “Infants’ Knowledge about Occlusion and Containment Events: A Surprising Discrepancy”. In: *Psychological Science* 12.2, pp. 141–147. ISSN: 0956-7976. DOI: 10.1111/1467-9280.00324 (cit. on p. 62).

— (2001b). “Reasoning about Containment Events in Very Young Infants”. In: *Cognition* 78.3, pp. 207–245. ISSN: 0010-0277. DOI: 10.1016/S0010-0277(00)00118-9 (cit. on p. 62).

Hobbs, J. R. and Moore, R. C. (1985). *Formal Theories of the Commonsense World*. Norwood, NJ: Ablex Pub. xxii, 455. ISBN: 978-0-89391-213-0. URL: https://archive.org/details/formaltheoriesof0000unse/ (cit. on p. 30).

Holman, E. W. (1979). “Monotonic Models for Asymmetric Proximities”. In: *Journal of Mathematical Psychology* 20.1, pp. 1–15. ISSN: 0022-2496. DOI: 10.1016/0022-2496(79)90031-2 (cit. on p. 47).

Houdé, O. (2004). *Dictionary of Cognitive Science : Neuroscience, Psychology, Artificial Intelligence, Linguistics, and Philosophy*. New York: Routledge. ISBN: 978-0-203-61952-0 (cit. on p. 89).

Irwing, P., Booth, T., and Hughes, D. J. (2018). “The Wiley Handbook of Psychometric Testing”. In: *The Wiley Handbook of Psychometric Testing*. Wiley Online Books. John Wiley & Sons Ltd. ISBN: 978-1-118-48977-2. DOI: 10.1002/9781118489772 (cit. on p. 53).

Isli, A. and Cohn, A. G. (2000). “A New Approach to Cyclic Ordering of 2D Orientations Using Ternary Relation Algebras”. In: *Artificial Intelligence* 122.1-2, pp. 137–187. ISSN: 0004-3702. DOI: 10.1016/S0004-3702(00)00044-8. URL: http://linkinghub.elsevier.com/retrieve/pii/S0004370200000448 (cit. on p. 70).

Jaccard, P. (1901). “Distribution de La Flore Alpine Dans Le Bassin Des Dranses et Dans Quelques Régions Voisines”. In: *Bulletin de la Société Vaudoise des Sciences Naturelles* 37.140, p. 241. ISSN: 0037-9603. DOI: 10.5169/seals-266440 (cit. on p. 49).

— (1902). “Lois de Distribution Florale Dans La Zone Alpine”. In: *Bulletin de la Société Vaudoise des Sciences Naturelles* 38.144, pp. 69–130. ISSN: 0037-9603. DOI: 10.5169/seals-266762 (cit. on p. 49).

Katsikopoulos, K. (2019). “Kirsch’s, and Everyone’s, Bind: How to Build Models for the Wild?” In: *Cognitive Processing* 20.2, pp. 269–272. ISSN: 1612-4790. DOI: 10.1007/s10339-019-00914-1 (cit. on p. 31).

Keren, G. B. and Baggen, S. (1981). “Recognition Models of Alphanumeric Characters”. In: *Perception and Psychophysics* 29.3, pp. 234–246. ISSN: 0031-5117. DOI: 10.3758/BF03207290. URL: https://research.tue.nl/en/publications/recognition-models-of-alphanumeric-characters (cit. on pp. 47, 49).

Kess, J. F. (1992). *Psycholinguistics : Psychology, Linguistics, and the Study of Natural Language*. Amsterdam Studies in the Theory and History of Linguistic Science. Series IV, Current Issues in Linguistic Theory v. 86. Amsterdam: John Benjamins Publishing Company, pp. xiv, 383. ISBN: 978-90-272-3583-1 (cit. on p. 29).

Kirsch, A. (2019). "A Unifying Computational Model of Decision Making". In: *Cognitive Processing* 20.2, pp. 243–259. ISSN: 1612-4790. DOI: `10.1007/s10339-019-00904-3` (cit. on p. 31).

Klippel, A., Worboys, M., and Duckham, M. (2008). "Identifying Factors of Geographic Event Conceptualisation". In: *International Journal of Geographical Information Science* 22.2, pp. 183–204. DOI: `10.1080/13658810701405607` (cit. on p. 157).

Klippel, A., Li, R., Yang, J., Hardisty, F., and Xu, S. (2013). "The Egenhofer–Cohn Hypothesis or, Topological Relativity?" In: *Cognitive and Linguistic Aspects of Geographic Space: New Perspectives on Geographic Information Research*. Ed. by Raubal, M., Mark, D. M., and Frank, A. U. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 195–215. ISBN: 978-3-642-34359-9. DOI: `10.1007/978-3-642-34359-9_11`. URL: `https://citeseerx.ist.psu.edu/pdf/507e95a8108ce700265840cbc39f07d821007b04` (cit. on p. 31).

Klippel, A., Yang, J., Wallgrün, J. O., Dylla, F., and Li, R. (2012). "Assessing Similarities of Qualitative Spatio-Temporal Relations". In: *Spatial Cognition VIII*. Ed. by Stachniss, C., Schill, K., and Uttal, D. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 242–261. ISBN: 978-3-642-32732-2. DOI: `10.1007/978-3-642-32732-2_17`. Google Books: `6AG7BQAAQBAJ` (cit. on p. 41).

Klir, G. J. and Bělohlávek, R., eds. (2011). *Concepts and Fuzzy Logic*. Cambridge, Mass: The MIT Press. xii, 274. ISBN: 978-0-262-01647-6. URL: `https://mitpress.mit.edu/books/concepts-and-fuzzy-logic` (cit. on p. 40).

Knauff, M. (1999). "The Cognitive Adequacy of Allen's Interval Calculus for Qualitative Spatial Representation and Reasoning". In: *Spatial Cognition and Computation* 1.3, pp. 261–290. ISSN: 13875868. DOI: `10.1023/A:1010097601575` (cit. on p. 158).

Knauff, M., Rauh, R., and Renz, J. (1997). "A Cognitive Assessment of Topological Spatial Relations: Results from an Empirical Investigation". In: *Spatial Information Theory A Theoretical Basis for GIS*, pp. 193–206 (cit. on p. 30).

Knauff, M., Rauh, R., and Schlieder, C. (1995). "Preferred Mental Models in Qualitative Spatial Reasoning: A Cognitive Assessment of Allen's Calculus". In: *Proceedings of the Seventeenth Annual Conference of the Cognitive Science Society*. Mahwah, NJ: Lawrence Erlbaum Associates, pp. 200–205 (cit. on pp. 30, 157 sq.).

Knauff, M., Strube, G., Jola, C., Rauh, R., and Schlieder, C. (2004). "The Psychological Validity of Qualitative Spatial Reasoning in One Dimension". In: *Spatial Cognition & Computation* 4.2, pp. 167–188. ISSN: 1387-5868. DOI: `10.1207/s15427633scc0402_3` (cit. on pp. 31, 157).

Köhler, C. (2002). "The Occlusion Calculus". In: *In Proceedings of the Cognitive Vision Workshop*. Zurich, Switzerland (cit. on p. 70).

Krantz, D. H. and Tversky, A. (1975). "Similarity of Rectangles: An Analysis of Subjective Dimensions". In: *Journal of Mathematical Psychology* 12.1, pp. 4–34. ISSN: 0022-2496. DOI: `10.1016/0022-2496(75)90047-4` (cit. on p. 41).

Krumhansl, C. L. (1988). "Testing the Density Hypothesis: Comment on Corter". In: *Journal of Experimental Psychology: General* 117.1, pp. 101–104. ISSN: 0096-3445. DOI: `10.1037/0096-3445.117.1.101` (cit. on p. 47).

Krumhansl, C. L. (1978). "Concerning the Applicability of Geometric Models to Similarity Data: The Interrelationship between Similarity and Spatial Density". In: *Psychological Review* 85.5, pp. 445–463. ISSN: 0033-295X. DOI: `10.1037/0033-295X.85.5.445` (cit. on p. 47).

Kruschke, J. K. (2001). “Categorization and Similarity Models”. In: *International Encyclopedia of the Social and Behavioral Sciences*. Ed. by Smelser, N. J. and Baltes, P. B. Oxford: Pergamon, pp. 1532–1535. ISBN: 978-0-08-043076-8. DOI: 10.1016/B0-08-043076-7/00622-7. URL: https://archive.org/details/internationalenc0003unse_2001 (cit. on pp. 41, 43 sq.).

Kruse, T., Pandey, A. K., Alami, R., and Kirsch, A. (2013). “Human-Aware Robot Navigation: A Survey”. In: 61.12, pp. 1726–1743. ISSN: 0921-8890. DOI: 10.1016/j.robot.2013.05.007 (cit. on p. 17).

Kurata, Y. (2, 2010). “9+-Intersection Calculi for Spatial Reasoning on the Topological Relations between Heterogeneous Objects”. In: *Proceedings of the 18th SIGSPATIAL International Conference on Advances in Geographic Information Systems*. GIS ’10. New York, NY, USA: Association for Computing Machinery, pp. 390–393. ISBN: 978-1-4503-0428-3. DOI: 10.1145/1869790.1869844 (cit. on p. 70).

Kurata, Y. and Shi, H. (2008a). “Interpreting Motion Expressions in Route Instructions Using Two Projection-Based Spatial Models”. In: *KI 2008: Advances in Artificial Intelligence (LNCS)*. Ed. by R. Dengel, A., Berns, K., Breuel, T. M., Bomarius, F., and Roth-Berghofer, T. R. Vol. 5243. Springer, Berlin, Heidelberg. DOI: 10.1007/978-3-540-85845-4_32. URL: http://www.comp.tmu.ac.jp/kurata/research/YKurataHShi-KI08-poster.pdf (cit. on pp. 24, 70 sq., 74).

— (2008b). “Rfdl: Models for Capturing Directional and Topological Characteristics of Path-Landmark Arrangements”. In: *Proceedings of the GIScience International Workshop “Moving Objects: From Natural to Formal Language"*. Ed. by Van de Weghe, N., Billen, R., Kuijpers, B., and Bogaert, P. Park City, Utah, USA. URL: http://www.comp.tmu.ac.jp/kurata/research/YKurataHShi-MO08.pdf (cit. on p. 74).

Lakoff, G. (1973). “Hedges: A Study in Meaning Criteria and the Logic of Fuzzy Concepts”. In: *Journal of Philosophical Logic* 2.4, pp. 458–508. ISSN: 1573-0433. DOI: 10.1007/BF00262952 (cit. on pp. 39 sq.).

Landau, B. and Jackendoff, R. (1993). ““What” and “Where” in Spatial Language and Spatial Cognition”. In: *Behavioral and Brain Sciences* 16.2, pp. 217–238. ISSN: 1469-1825, 0140-525X. DOI: 10.1017/S0140525X00029733 (cit. on pp. 62, 81).

Lee, D. N., Bootsma, R. J., Land, M., Regan, D., and Gray, R. (2009). “Lee’s 1976 Paper”. In: *Perception* 38.6, pp. 837–858. DOI: 10.1068/pmklee. PMID: 19806967 (cit. on p. 214).

Lee, J. H., Renz, J., and Wolter, D. (2013). “StarVars-effective Reasoning about Relative Directions”. In: *Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence IJCAI*. Ed. by Rossi, F. 1. AAAI Press / International Joint Conferences on Artificial Intelligence, pp. 976–982. ISBN: 978-1-57735-633-2. URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.375.846&rep=rep1&type=pdf (cit. on p. 70).

Lee, M. D. (2001). “Determining the Dimensionality of Multidimensional Scaling Representations for Cognitive Modeling”. In: *Journal of Mathematical Psychology* 45.1, pp. 149–166. ISSN: 00222496. DOI: 10.1006/jmps.1999.1300 (cit. on p. 52).

Lichtenthäler, C., Peters, A., Griffiths, S., and Kirsch, A. (2013). “Social Navigation - Identifying Robot Navigation Patterns in a Path Crossing Scenario”. In: *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)* 8239 LNAI (Exc 277), pp. 84–93. ISSN: 03029743. DOI: 10.1007/978-3-319-02675-6_9. URL: http://www6.in.tum.de/Main/Publications/lichtenthaeler2013social.pdf (cit. on p. 17).

Ligozat, G. (2012). *Qualitative Spatial and Temporal Reasoning*. London: ISTE Ltd, p. 527. ISBN: 978-1-84821-252-7 (cit. on p. 63).

Ligozat, G. and Renz, J. (2004). “What Is a Qualitative Calculus? A General Framework”. In: *PRICAI 2004: Trends in Artificial Intelligence*. Ed. by Zhang, C., W. Guesgen, H., and Yeap, W.-K. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 53–64. ISBN: 978-3-540-28633-2. URL: `https://hal.archives-ouvertes.fr/hal-01487502/document` (cit. on p. 106).

López, A., Atran, S., Coley, J. D., Medin, D. L., and Smith, E. E. (1997). "The Tree of Life: Universal and Cultural Features of Folkbiological Taxonomies and Inductions". In: *Cognitive Psychology* 32.3, pp. 251–295. ISSN: 0010-0285. DOI: `10.1006/cogp.1997.0651` (cit. on p. 45).

Luce, R. D. (1963). "Detection and Recognition". In: *Handbook of Mathematical Psychology.* Ed. by Luce, R. D., Bush, R. R., and Galanter, E. Vol. 1. John Wiley & Sons., pp. 1–103 (cit. on pp. 50, 161).

Luce, R. D. (1959). *Individual Choice Behavior a Theoretical Analysis.* New York: Wiley. XII, 153 S. URL: `https://rds-tue.ibs-bw.de/link?kid=107583791X` (cit. on p. 158).

Luck, S. J. and Vogel, E. K. (20, 1997). "The Capacity of Visual Working Memory for Features and Conjunctions". In: *Nature* 390.6657, pp. 279–281. ISSN: 0028-0836. DOI: `10.1038/36846`. PMID: `9384378` (cit. on p. 159).

Lynch, E. B., Coley, J. D., and Medin, D. L. (1, 2000). "Tall Is Typical: Central Tendency, Ideal Dimensions, and Graded Category Structure among Tree Experts and Novices". In: *Memory & Cognition* 28.1, pp. 41–50. ISSN: 1532-5946. DOI: `10.3758/BF03211575` (cit. on p. 38).

Macdonald, C. (2005). *Varieties of Things: Foundations of Contemporary Metaphysics.* Wiley Online Books, pp. x, 278. ISBN: 978-0-470-77568-4. DOI: `10.1002/9780470775684` (cit. on p. 60).

Maddox, W. T. (1992). "Perceptual and Decisional Separability". In: *Multidimensional Models of Perception and Cognition.* Ed. by Ashby, F. G. Scientific Psychology Series. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc. Chap. 7, pp. 147–180. ISBN: 0-8058-0577-X (cit. on p. 45).

Mair, P., Groenen, P. J. F., and Leeuw, J. de (13, 2022). "More on Multidimensional Scaling and Unfolding in R: Smacof Version 2". In: *Journal of Statistical Software* 102, pp. 1–47. ISSN: 1548-7660. DOI: `10.18637/jss.v102.i10` (cit. on p. 41).

Mandler, J. M. (2012). "On the Spatial Foundations of the Conceptual System and Its Enrichment". In: *Cognitive Science* 36.3, pp. 421–451. ISSN: 1551-6709. DOI: `10.1111/j.1551-6709.2012.01241.x`. PMID: `22435402` (cit. on p. 25).

Mandler, J. M. (2004). *The Foundations of Mind : Origins of Conceptual Thought.* Cary, UNITED STATES: Oxford University Press, Incorporated. ISBN: 978-0-19-803839-9. URL: `http://ebookcentral.proquest.com/lib/bsb/detail.action?docID=3052027` (cit. on p. 62).

Mark, D. M. and Egenhofer, M. J. (1994). "Modeling Spatial Relations Between Lines and Regions: Combining Formal Mathematical Models and Human Subjects Testing". In: *Cartography and Geographic Information Systems* 21.4, pp. 195–212. ISSN: 1050-9844. DOI: `10.1559/152304094782540637`. URL: `https://www.tandfonline.com/doi/abs/10.1559/152304094782540637` (cit. on pp. 30 sq.).

— (1995). "Topology of Prototypical Spatial Relations Between Lines and Regions in English and Spanish". In: *In: Proceedings of the Twelfth International Symposium on Computer- Assisted Cartography. Volume 4*, pp. 245–254 (cit. on pp. 30 sq.).

Mast, V., Wolter, D., Klippel, A., Wallgrün, J., and Tenbrink, T. (2014). "Boundaries and Prototypes in Categorizing Direction". In: *Spatial Cognition IX.* Ed. by Freksa, C., Nebel, B., Hegarty, M., and Barkowsky, T. Vol. 8684. Lecture Notes in Computer Science. Springer International Publishing, pp. 92–107. ISBN: 978-3-319-11214-5. DOI: `10.1007/978-3-319-11215-2_7` (cit. on p. 158).

Mavridis, N., Bellotto, N., Iliopoulos, K., and Van de Weghe, N. (2015). "QTC3D: Extending the Qualitative Trajectory Calculus to Three Dimensions". In: *Information Sciences* 322, pp. 20–30. ISSN: 00200255. DOI: `10.1016/j.ins.2015.06.002`. arXiv: `1402.3779v1 [cs.OH]` (cit. on p. 72).

McCloskey, M. E. and Glucksberg, S. (1978). “Natural Categories: Well Defined or Fuzzy Sets?” In: *Memory and Cognition* 6.4, pp. 462–472. ISSN: 1532-5946. DOI: `10.3758/BF03197480` (cit. on pp. 24, 36 sq., 39 sq.).

McDonnell, J. V. and Gureckis, T. M. (2011). “Adaptive Clustering Models of Categorization”. In: *Formal Approaches in Categorization*. Ed. by Pothos, E. M. and Wills, A. J. Cambridge University Press, pp. 220–252. DOI: `10.1017/CBO9780511921322.010` (cit. on p. 51).

Medin, D. L. (2011). “Comments on Models and Categorization Theories: The Razor’s Edge”. In: *Formal Approaches in Categorization*. Ed. by Pothos, E. M. and Wills, A. J. Cambridge University Press, pp. 325–331. DOI: `10.1017/CBO9780511921322.015` (cit. on pp. 38, 51).

Medin, D. L. and Coley, J. D. (1998). “Concepts and Categorization”. In: *Perception and Cognition at Century’s End*. Ed. by Hochberg, J. Handbook of Perception and Cognition (2nd Ed.) San Diego: Academic Press. Chap. 13, pp. 403–439. ISBN: 978-0-12-102570-0. DOI: `10.1016/B978-012102570-0/50016-1` (cit. on p. 36).

Medin, D. L. and Heit, E. (1999). “Categorization”. In: *Cognitive Science*. Ed. by Bly, B. M. and Rumelhart, D. E. B. T. C. S. Handbook of Perception and Cognition (2nd Ed.) San Diego: Academic Press. Chap. 3, pp. 99–143. ISBN: 978-0-12-601730-4. DOI: `10.1016/B978-012601730-4/50005-6` (cit. on pp. 23 sq., 41).

Medin, D. L. and Schaffer, M. M. (1978). “Context Theory of Classification Learning.” In: *Psychological review* 85.3, pp. 207–238. ISSN: 1939-1471, 0033-295X. DOI: `10.1037/0033-295X.85.3.207`. URL: `http://groups.psych.northwestern.edu/medin/documents/MedinSchaffer1978PsychRev.pdf` (cit. on p. 50).

Mervis, C. B., Catlin, J., and Rosch, E. (1975). “Development of the Structure of Color Categories.” In: *Developmental Psychology* 11.1, pp. 54–60. ISSN: 0012-1649. DOI: `10.1037/h0076118` (cit. on p. 37).

— (1976). “Relationships among Goodness-of-Example, Category Norms, and Word Frequency”. In: *Bulletin of the Psychonomic Society* 7.3, pp. 283–284. ISSN: 0090-5054. DOI: `10.3758/BF03337190` (cit. on pp. 24, 37).

Mervis, C. B. and Crisafi, M. A. (1982). “Order of Acquisition of Subordinate-, Basic-, and Superordinate-Level Categories.” In: *Child Development* 53, pp. 258–266. ISSN: 00093920 (cit. on p. 59).

Minda, J. P. and Smith, J. D. (2011). “Prototype Models of Categorization: Basic Formulation, Predictions, and Limitations”. In: *Formal Approaches in Categorization*. Ed. by Pothos, E. M. and Wills, A. J. Cambridge University Press, pp. 40–64. DOI: `10.1017/CBO9780511921322.003` (cit. on pp. 50 sq.).

Monk, J. D. (1969). *Introduction to Set Theory*. International Series in Pure and Applied Mathematics. New York: McGraw-Hill. xii, 193. ISBN: 978-0-07-042715-0. URL: `http://archive.org/details/introductiontose0000monk` (cit. on p. 182).

Moratz, R. (2006). “Representing Relative Direction as a Binary Relation of Oriented Points”. In: *ECAI*. Ed. by Brewka, G., Coradeschi, S., Perini, A., and Traverso, P., pp. 407–411. ISBN: 978-1-60750-189-3. URL: `http://ebooks.iospress.nl/volumearticle/2721` (cit. on pp. 64 sqq., 70, 72).

Moratz, R. and Ragni, M. (2008). “Qualitative Spatial Reasoning about Relative Point Position”. In: *Journal of Visual Languages and Computing* 19.1, pp. 75–98. ISSN: 1045-926X. DOI: `10.1016/j.jvlc.2006.11.001` (cit. on p. 70).

Moratz, R., Renz, J., and Wolter, D. (2000). “Qualitative Spatial Reasoning about Line Segments”. In: *In Proceedings of the 14th European Conference on Artificial Intelligence (ECAI 2000)*, pp. 234–238. URL: `http://www.informatik.uni-bremen.de/kogrob/papers/ecai2000_dipol.pdf` (cit. on pp. 70, 72, 74).

Moratz, R. and Wallgrün, J. O. (19, 2012). "Spatial Reasoning with Augmented Points: Extending Cardinal Directions with Local Distances". In: *Journal of Spatial Information Science* 5, pp. 1–30. ISSN: 1948-660X. DOI: `10.5311/JOSIS.2012.5.84`. URL: `http://josis.org/index.php/josis/article/view/84` (visited on 02/24/2021) (cit. on pp. 70, 179).

Mossakowski, T. and Moratz, R. (2010). "Qualitative Reasoning about Relative Direction on Adjustable Levels of Granularity" (cit. on pp. 64, 74).

— (2012). "Qualitative Reasoning about Relative Direction of Oriented Points". In: *Artificial Intelligence* 180–181, pp. 34–45. ISSN: 0004-3702. DOI: `10.1016/j.artint.2011.10.003` (cit. on pp. 186, 194).

Murphy, G. L. (2002). "The Big Book of Concepts." In: *The Big Book of Concepts.* Cambridge, MA, US: MIT Press, p. 555. ISBN: 0-262-13409-8 (Hardcover). DOI: `10.7551/mitpress/1602.001.0001` (cit. on pp. 24, 36 sqq., 42, 50 sq., 59, 87).

— (2011). "The Contribution (and Drawbacks) of Models to the Study of Concepts". In: *Formal Approaches in Categorization.* Ed. by Wills, A. J. and Pothos, E. M. Cambridge: Cambridge University Press. Chap. 13, pp. 299–312. ISBN: 978-0-521-19048-0 (cit. on p. 88).

Murphy, G. L. and Brownell, H. H. (1985). "Category Differentiation in Object Recognition: Typicality Constraints on the Basic Category Advantage." In: *Journal of Experimental Psychology. Learning, Memory and Cognition* 11, pp. 70–84. ISSN: 02787393. DOI: `10.0.4.13/0278-7393.11.1.70` (cit. on p. 58).

Murphy, G. L. and Lassaline, M. E. (1997). "Hierarchical Structure in Concepts and the Basic Level of Categorization". In: *Knowledge, Concepts and Categories.* Ed. by Lamberts, K. and Shanks, D. R. Studies in Cognition. Cambridge, MA: The MIT Press. Chap. 3, pp. 93–131. ISBN: 978-0-86377-491-1. DOI: `10.4324/9780203765418` (cit. on pp. 56, 58 sq.).

Murphy, G. L. and Medin, D. L. (1985). "The Role of Theories in Conceptual Coherence". In: *Psychological Review* 92.3, pp. 289–316. ISSN: 0033-295X. DOI: `10.1037/0033-295X.92.3.289`. PMID: `4023146` (cit. on pp. 41, 60, 126).

Musto, A., Stein, K., Eisenkolb, A., Schill, K., and Brauer, W. (1998). "Generalization, Segmentation, and Classification of Qualitative Motion Data". In: *Proc. of the 13th European Conference on Artificial Intelligence (ECAI-98)*, pp. 180–184. URL: `https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.43.2520` (cit. on p. 72).

Musto, A., Stein, K., Eisenkolb, A., and Rofer, T. (1999). "Qualitative and Quantitative Representations of Locomotion and Their Application in Robot Navigation". In: *IJCAI International Joint Conference on Artificial Intelligence.* Vol. 2, pp. 1067–1072 (cit. on p. 72).

Navarrete, I., Morales, A., Sciavicco, G., and Cardenas-Viedma, M. A. (2013). "Spatial Reasoning with Rectangular Cardinal Relations: The Convex Tractable Subalgebra". In: *Annals of Mathematics and Artificial Intelligence* 67.1, pp. 31–70. ISSN: 1012-2443, 1573-7470. DOI: `10.1007/s10472-012-9327-5` (cit. on p. 70).

Nosofsky, R. M. (1986). "Attention, Similarity, and the Identification–Categorization Relationship". In: *Journal of Experimental Psychology: General* 115.1, pp. 39–57. ISSN: 0096-3445. DOI: `10.1037/0096-3445.115.1.39` (cit. on pp. 45, 50, 203).

— (1991a). "Stimulus Bias, Asymmetric Similarity, and Classification". In: *Cognitive Psychology* 23.1, pp. 94–140. ISSN: 0010-0285. DOI: `10.1016/0010-0285(91)90004-8` (cit. on p. 47).

— (1992a). "Exemplar-Based Approach to Relating Categorization, Identification, and Recognition". In: *Multidimensional Models of Perception and Cognition.* Ed. by Ashby, F. G. Scientific Psychology Series. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc. Chap. 14, pp. 363–393. ISBN: 0-8058-0577-X (cit. on pp. 44, 53).

— (1992b). "Exemplars, Prototypes, and Similarity Rules". In: *Essays in Honor of William k. Estes.* Ed. by Healy, A. F., Kosslyn, S. M., and Shiffrin, R. M. Vol. 1. Hillsdale, NJ:

Lawrence Erlbaum Associates, Inc. Chap. 8, pp. 149–167. ISBN: 0-8058-1097-8. DOI: `10.4324/9780203763162` (cit. on pp. 45, 51).

Nosofsky, R. M. (2011). "The Generalized Context Model: An Exemplar Model of Classification". In: *Formal Approaches in Categorization*. Ed. by Pothos, E. M. and Wills, A. J. Cambridge University Press, pp. 18–39. DOI: `10.1017/CBO9780511921322.002` (cit. on pp. 44, 50 sq.).

— (1984). "Choice, Similarity, and the Context Theory of Classification". In: *Journal of Experimental Psychology: Learning, Memory, and Cognition* 10.1, pp. 104–114. ISSN: 1939-1285(Electronic),0278-7393(Print). DOI: `10.1037/0278-7393.10.1.104`. PMID: `6242730`. URL: `http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.169.3688` (cit. on p. 51).

— (1991b). "Tests of an Exemplar Model for Relating Perceptual Classification and Recognition Memory." In: *Journal of experimental psychology. Human perception and performance* 17.1, pp. 3–27. ISSN: 0096-1523. DOI: `10.1037//0096-1523.17.1.3`. PMID: `1826320`. URL: `http://apps.usd.edu/coglab/psyc792/pdf/Nosofsky1991.pdf` (cit. on p. 53).

— (2000). "Exemplar Representation without Generalization? Comment on Smith and Minda's (2000) "Thirty Categorization Results in Search of a Model."" In: *Journal of Experimental Psychology: Learning, Memory, and Cognition* 26.6, pp. 1735–1743. ISSN: 1939-1285(Electronic),0278-7393(Print). DOI: `10.1037/0278-7393.26.6.1735`. URL: `https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.169.4426` (cit. on p. 52).

OuYang, J., Fu, Q., and Liu, D. (2007). "A Model for Representing Topological Relations Between Simple Concave Regions". In: *Computational Science – ICCS 2007*. Ed. by Shi, Y., van Albada, G. D., Dongarra, J., and Sloot, P. M. A. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, pp. 160–167. ISBN: 978-3-540-72584-8. DOI: `10.1007/978-3-540-72584-8_21` (cit. on p. 65).

Oxford English Dictionary (n.d.). *Christian, Adj. and n.* URL: `https://oed.com/view/Entry/32448?rskey=WkGYUj&result=1&isAdvanced=false#eid9366701` (cit. on p. 38).

Pacheco, J., Escrig, M. T., and Toledo, F. (2001). "Representing and Reasoning on Three-Dimensional Qualitative Orientation Point Objects". In: *Progress in Artificial Intelligence*. Ed. by Brazdil, P. and Jorge, A. Red. by Goos, G., Hartmanis, J., and van Leeuwen, J. Vol. 2258. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 298–305. ISBN: 978-3-540-43030-8. DOI: `10.1007/3-540-45329-6_30` (cit. on p. 70).

Papenmeier, F. and Huff, M. (1, 2014). "Viewpoint-Dependent Representation of Contextual Information in Visual Working Memory". In: *Attention, Perception, & Psychophysics* 76.3, pp. 663–668. ISSN: 1943-393X. DOI: `10.3758/s13414-014-0632-4`. URL: `https://doi.org/10.3758/s13414-014-0632-4` (visited on 04/28/2022) (cit. on p. 159).

Papenmeier, F., Purcalla Arrufi, J., and Kirsch, A. (2023). "Stories in the Mind? The Role of Story-Based Categorizations in Motion Classification". In: *Cognitive Science* 47.9, e13332. ISSN: 0364-0213, 1551-6709. DOI: `10.1111/cogs.13332`. URL: `https://onlinelibrary.wiley.com/doi/10.1111/cogs.13332` (visited on 09/09/2023) (cit. on pp. 3, 157, 165 sq., 169, 171, 173, 221).

Perone, S., Madole, K. L., Ross-Sheehy, S., Carey, M., and Oakes, L. M. (2008). "The Relation between Infants' Activity with Objects and Attention to Object Appearance." In: *Developmental Psychology* 44.5, pp. 1242–1248. DOI: `10.1037/0012-1649.44.5.1242`. PMCID: `PMC2596282`. PMID: `18793058` (cit. on p. 25).

Perrin, N. A. (1992). "Uniting Identification, Similarity and Preference: General Recognition Theory". In: *Multidimensional Models of Perception and Cognition*. Ed. by Ashby, F. G. Scientific Psychology Series. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc. Chap. 6, pp. 123–146. ISBN: 0-8058-0577-X (cit. on p. 44).

Popper, K. R. (1935). *Logik Der Forschung Zur Erkenntnistheorie Der Modernen Naturwissenschaft.* Wien: Springer. VI, 248 S. DOI: 10.1524/9783050063782. URL: https://rds-tue.ibs-bw.de/link?kid=1145888909 (cit. on p. 53).

Posner, M. I. and Keele, S. W. (1968). "On the Genesis of Abstract Ideas". In: *Journal of experimental psychology* 77 (3, Pt. 1), pp. 353–363. DOI: 10.1037/h0025953. PMID: 5665566 (cit. on p. 38).

Pothos, E. M. and Wills, A. J., eds. (2011a). *Formal Approaches in Categorization.* Cambridge, UK: Cambridge University Press, pp. xii, 336. ISBN: 978-0-521-19048-0. DOI: 10.1017/CBO9780511921322 (cit. on p. 87).

— (2011b). "Introduction". In: *Formal Approaches in Categorization.* Ed. by Pothos, E. M. and Wills, A. J. Cambridge University Press, pp. 1–17. DOI: 10.1017/CBO9780511921322.001 (cit. on p. 45).

Pruden, S. M., Roseberry, S., Göksun, T., Hirsh-Pasek, K., and Golinkoff, R. M. (2013). "Infant Categorization of Path Relations during Dynamic Events". In: *Child Development* 84.1, pp. 331–345. DOI: 10.1111/j.1467-8624.2012.01843.x (cit. on p. 19).

Pulverman, R., Golinkoff, R. M., Hirsh-Pasek, K., and Buresh, J. S. (2008). "Infants Discriminate Manners and Paths in Non-Linguistic Dynamic Events". In: *Cognition* 108.3, pp. 825–830. ISSN: 0010-0277. DOI: 10.1016/j.cognition.2008.04.009 (cit. on p. 19).

Purcalla Arrufi, J. and Kirsch, A. (2017). "Using Stories to Create Qualitative Representations of Motion". In: *ProSocrates 2017, 2nd Symposium on Problem-Solving, Creativity and Spatial Reasoning in Cognitive Systems.* Ed. by Olteteanu, A. M. and Falomir, Z. July. Delmenhorst, Germany: CEUR Workshop Proceedings, pp. 19–28. URL: http://ceur-ws.org/Vol-1869/paper-3.pdf (cit. on pp. 3, 123, 131).

— (2018a). "Motion Categorisation: Representing Velocity Qualitatively". In: *Cognitive Systems Research* 52 (Special Issue on Problem-solving, Creativity and Spatial Reasoning in Cognitive Systems (ProSocrates)), pp. 117–131. DOI: 10.1016/j.cogsys.2018.06.005. URL: https://www.researchgate.net/publication/325897499_Motion_Categorisation_Representing_Velocity_Qualitatively (cit. on pp. 3, 28, 65 sq., 74, 125, 129, 136 sq., 150, 198, 204, 206 sqq., 211, 216).

— (2018b). "Qualitative Reasoning with Story-Based Motion Representations: Inverse and Composition". In: *QR '18: 31st International Workshop on Qualitative Reasoning (Co-Located at IJCAI'18).* 31st International Workshop on Qualitative Reasoning, Co-Located at Int'l Joint Conference on Artificial Intelligence (IJCAI'18). Ed. by Falomir, Z., Coghill, G. M., and Pang, W. Stockholm, pp. 16–23. URL: https://homepages.abdn.ac.uk/pang.wei/pages/QR2018/QR18-Proceedings.pdf (cit. on pp. 3, 73, 190 sq., 193 sq.).

Quine, W. V. O. (1969). "Natural Kinds". In: *Essays in Honor of Carl g. Hempel.* Ed. by Rescher, N. Dordrecht: Springer, pp. 5–23. ISBN: 978-94-017-1466-2. DOI: 10.1007/978-94-017-1466-2_2 (cit. on p. 60).

Quinn, P. C. (1994). "The Categorization of Above and Below Spatial Relations by Young Infants". In: *Child Development* 65.1, pp. 58–69. ISSN: 1467-8624. DOI: 10.1111/j.1467-8624.1994.tb00734.x (cit. on p. 62).

Quinn, P. C., Cummins, M., Kase, J., Martin, E., and Weissman, S. (1996). "Development of Categorical Representations for above and below Spatial Relations in 3- to 7-Month-Old Infants". In: *Developmental Psychology* 32.5, pp. 942–950. ISSN: 0012-1649. DOI: 10.1037/0012-1649.32.5.942 (cit. on p. 62).

Randell, D. A., Cohn, A. G., and Cui, Z. (1992). "Computing Transitivity Tables : A Challenge for Automated Theorem Provers Constructing Transitivity Tables". In: *Automated Deduction - CADE-11, 11th International Conference on Automated Deduction, Saratoga Springs, NY,* *USA, June 15-18, 1992, Proceedings*. Springer Berlin Heidelberg, pp. 786–790. DOI: 10.1007/3-540-55602-8_225 (cit. on pp. 64, 68, 186).

Randell, D. A., Cui, Z., and Cohn, A. G. (1992a). "An Interval Logic for Space Based on 'Connection'". In: *Proceedings of the 10th European Conference on Artificial Intelligence*. ECAI '92. New York, NY, USA: John Wiley & Sons, Inc., pp. 394–398. ISBN: 0-471-93608-1. URL: http://dl.acm.org/citation.cfm?id=145448.146741 (cit. on p. 70).

Randell, D. A., Witkowski, M., and Shanahan, M. (2001). "From Images to Bodies: Modelling and Exploiting Spatial Occlusion and Motion Parallax". In: *Proceedings of the 17th International Joint Conference on Artificial Intelligence-Volume 1*, pp. 57–66 (cit. on p. 70).

Randell, D. A., Cui, Z., and Cohn, A. G. (25, 1992b). "A Spatial Logic Based on Regions and Connection". In: *Proceedings of the Third International Conference on Principles of Knowledge Representation and Reasoning*. KR'92. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., pp. 165–176. ISBN: 978-1-55860-262-5. URL: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.39.486 (visited on 06/08/2022) (cit. on pp. 31, 64).

Reed, S. K. (1, 1972). "Pattern Recognition and Categorization". In: *Cognitive Psychology* 3.3, pp. 382–407. ISSN: 0010-0285. DOI: 10.1016/0010-0285(72)90014-X (cit. on p. 38).

Renz, J. (2001). "A Spatial Odyssey of the Interval Algebra: 1. Directed Intervals". In: *IJCAI International Joint Conference on Artificial Intelligence*. August. Morgan Kaufmann Publishers Inc., pp. 51–56. URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.366.1364&rep=rep1&type=pdf (cit. on p. 72).

Renz, J. and Mitra, D. (2004). "Qualitative Direction Calculi with Arbitrary Granularity". In: *PRICAI 2004: Trends in Artificial Intelligence*. Ed. by Zhang, C., W. Guesgen, H., and Yeap, W.-K. Vol. 3157. Lecture Notes in Computer Science. Springer Berlin Heidelberg, pp. 65–74. ISBN: 978-3-540-22817-2. DOI: 10.1007/978-3-540-28633-2_9. URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.85.9910&rep=rep1&type=pdf (cit. on p. 70).

Renz, J. and Nebel, B. (2007). "Qualitative Spatial Reasoning Using Constraint Calculi". In: *Handbook of Spatial Logics*. Ed. by Aiello, M., Pratt-Hartmann, I., and Van Benthem, J. Springer Netherlands. Chap. 4, pp. 161–215. ISBN: 978-1-4020-5586-7. DOI: 10.1007/978-1-4020-5587-4_4. URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.336.8873&rep=rep1&type=pdf (cit. on pp. 63 sq.).

Renz, J., Rauh, R., and Knauff, M. (2000). "Towards Cognitive Adequacy of Topological Spatial Relations". In: *Spatial Cognition II*. Ed. by Freksa, C., Habel, C., Brauer, W., and Wender, KarlF. Vol. 1849. Lecture Notes in Computer Science. Springer Berlin Heidelberg, pp. 184–197. ISBN: 978-3-540-67584-6. DOI: 10.1007/3-540-45460-8_14 (cit. on p. 158).

Rips, L. J., Shoben, E. J., and Smith, E. E. (1973). "Semantic Distance and the Verification of Semantic Relations." In: *Journal of Verbal Learning and Verbal Behavior* 12.1, pp. 1–20. ISSN: 0022-5371. DOI: 10.1016/S0022-5371(73)80056-8 (cit. on pp. 24, 37 sq., 45 sq., 52, 62).

Rogers, T. T. and McClelland, J. L. (2011). "Semantics without Categorization". In: *Formal Approaches in Categorization*. Ed. by Pothos, E. M. and Wills, A. J. Cambridge University Press, pp. 88–119. DOI: 10.1017/CBO9780511921322.005 (cit. on p. 44).

Roor, R. (2018). "Fusion von Tageskontext Und Mobilitätsgewohnheiten Als Enablersystem Für Mobilitätsassistenten". Disserta Verlag / University of Tübingen, p. 186. ISBN: 9783959354547 (cit. on p. 25).

Rosch, E. (1977). "Human Categorization". In: *Studies in Cross-Cultural Psychology*. Ed. by Warren, N. Vol. 1. London ; New York: Academic Press, pp. 1–49. ISBN: 978-0-12-609201-1. URL: https://archive.org/details/studiesincrosscu0001unse/page/n21/mode/2up (cit. on p. 38).

Rosch, E. and Mervis, C. B. (1975). "Family Resemblances: Studies in the Internal Structure of Categories". In: *Cognitive Psychology* 7.4, pp. 573–605. ISSN: 0010-0285. DOI: `10.1016/0010-0285(75)90024-9` (cit. on pp. 24, 37, 39).

Rosch, E., Mervis, C. B., Gray, W. D., Johnson, D. M., and Boyes-Braem, P. (1976). "Basic Objects in Natural Categories". In: *Cognitive Psychology* 8.3, pp. 382–439. ISSN: 0010-0285. DOI: `10.1016/0010-0285(76)90013-X` (cit. on pp. 59 sq.).

Rosch, E., Simpson, C., and Miller, R. S. (1976). "Structural Bases of Typicality Effects". In: *Journal of Experimental Psychology: Human Perception and Performance* 2.4, pp. 491–502. ISSN: 0096-1523. DOI: `10.1037/0096-1523.2.4.491`. URL: `https://www.academia.edu/20446780/Structural_bases_of_typicality_effects` (cit. on p. 37).

Rosch, E. H. (1973). "On the Internal Structure of Perceptual and Semantic Categories". In: *Cognitive Development and the Acquisition of Language*. Ed. by Moore, T. E. New York: Academic Press, pp. 111–144. DOI: `10.1016/B978-0-12-505850-6.50010-4`. URL: `https://www.academia.edu/24474386/On_the_internal_structure_of_perceptual_and_semantic_categories` (cit. on pp. 24, 37, 39).

Rosch, Eleanor. (1978). "Principles of Categorisation". In: *Cognition and Categorization*. Ed. by Rosch, Eleanor. and Lloyd, B. B. Hillsdale, N.J.; New York: L. Erlbaum Associates, Inc. Chap. 2, pp. 27–48. ISBN: 978-0-470-26377-8. DOI: `10.1016/B978-1-4832-1446-7.50028-5`. URL: `https://escholarship.org/uc/item/0sz9c8qh` (cit. on pp. 23 sq., 37, 43, 159).

Rosch, Eleanor. and Lloyd, B. B. (1978). *Cognition and Categorization*. Hillsdale, N.J.; New York: Lawrence Erlbaum Associates, Inc., viii, 328 pages. ISBN: 978-0-470-26377-8 (cit. on pp. 23 sq.).

Ross, D. (2019). "Empiricism, Sciences, and Engineering: Cognitive Science as a Zone of Integration". In: *Cognitive Processing* 20.2, pp. 261–267. ISSN: 1612-4790. DOI: `10.1007/s10339-019-00916-z` (cit. on p. 31).

Russell, S. J. and Norvig, P. (2014a). "Constraint Satisfaction Problems". In: *Artificial Intelligence*. 3rd. Pearson Education Limited. Chap. 7. ISBN: 978-1-292-02420-2 (cit. on p. 63).

— (2014b). "Introduction". In: *Artificial Intelligence*. 3rd. Pearson Education Limited. Chap. 1. ISBN: 978-1-292-02420-2 (cit. on pp. 25, 157).

— (2014c). "Knowledge Representation". In: *Artificial Intelligence*. 3rd. Pearson Education Limited. Chap. 12. ISBN: 978-1-292-02420-2 (cit. on pp. 23, 36, 58).

— (2014d). "Learning from Examples". In: *Artificial Intelligence*. 3rd. Pearson Education Limited. Chap. 18. ISBN: 978-1-292-02420-2 (cit. on p. 52).

Sabharwal, C. and Leopold, J. L. (2014). "Evolution of Region Connection Calculus to VRCC-3D+". In: *New Mathematics and Natural Computation* 10, pp. 103–141. URL: `https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.827.3017&rep=rep1&type=pdf` (cit. on p. 70).

Schwartz, C. A. (1997). "The Rise and Fall of Uncitedness". In: *College & Research Libraries* 58.1, pp. 19–29. ISSN: 0010-0870. DOI: `10.5860/crl.58.1.19` (cit. on p. 30).

Scivos, A. and Nebel, B. (2001). "Double-Crossing: Decidability and Computational Complexity of a Qualitative Calculus for Navigation". In: *Spatial Information Theory*. Springer, pp. 431–446. URL: `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.15.5799&rep=rep1&type=pdf` (cit. on pp. 70, 179).

Shanks, D. R. (2001). "Concept Learning and Representation: Models". In: *International Encyclopedia of the Social and Behavioral Sciences*. Ed. by Smelser, N. J. and Baltes, P. B. Oxford: Pergamon, pp. 2491–2495. ISBN: 978-0-08-043076-8. DOI: `10.1016/B0-08-043076-7/00536-2`. URL: `https://archive.org/details/internationalenc0004unse_d4r3` (cit. on p. 54).

Shepard, R. N. (1957). “Stimulus and Response Generalization: A Stochastic Model Relating Generalization to Distance in Psychological Space”. In: *Psychometrika* 22.4, pp. 325–345. ISSN: 1860-0980. DOI: `10.1007/BF02288967` (cit. on pp. 44 sq., 50, 161).

— (1987). “Toward a Universal Law of Generalization for Psychological Science”. In: *Science* 237.4820, pp. 1317–1323. ISSN: 0036-8075. DOI: `10.1126/science.3629243` (cit. on p. 44).

Shoben, E. J. (1976). “The Verification of Semantic Relations in a Same-Different Paradigm: An Asymmetry in Semantic Memory”. In: *Journal of Verbal Learning and Verbal Behavior* 15.4, pp. 365–379. ISSN: 0022-5371. DOI: `10.1016/S0022-5371(76)90032-3` (cit. on p. 46).

Skansi, S. (2018). *Introduction to Deep Learning: From Logical Calculus to Artificial Intelligence*. Cham, Switzerland: Springer International Publishing AG, part of Springer Nature 201, pp. viii, 140. ISBN: 978-3-319-73004-2. DOI: `10.1007/978-3-319-73004-2` (cit. on p. 94).

Skew, Adj. and Adv. (2020). *Skew, Adj. and Adv. :* in: *Oxford English Dictionary*. URL: `https://oed.com/view/Entry/180791?` (visited on 12/11/2020) (cit. on p. 54).

Skiadopoulos, S. and Koubarakis, M. (2004). “Composing Cardinal Direction Relations”. In: *Artificial Intelligence* 152.2, pp. 143–171. ISSN: 00043702. DOI: `10.1016/S0004-3702(03)00137-1` (cit. on p. 70).

Sloman, S. A., Love, B. C., and Ahn, W.-k. (1998). “Feature Centrality and Conceptual Coherence”. In: *Cognitive Science* 22.2, pp. 189–228. ISSN: 0364-0213. DOI: `10.1207/s15516709cog2202_2` (cit. on p. 36).

Smith, E. E. and Medin, D. L. (1981). *Categories and Concepts*. Cambridge, Mass.: Harvard University Press, p. 203. ISBN: 978-0-674-86627-0. DOI: `10.4159/harvard.9780674866270` (cit. on pp. 44 sq., 50).

Smith, E. E., Shoben, E. J., and Rips, L. J. (1974). “Structure and Process in Semantic Memory: A Featural Model for Semantic Decisions.” In: *Psychological Review* 81.3, pp. 214–241. ISSN: 0033-295X. DOI: `10.1037/h0036351` (cit. on p. 24).

Smith, J. D., Murray Jr., M. J., and Minda, J. P. (1997). “Straight Talk about Linear Separability”. In: *Journal of Experimental Psychology: Learning, Memory, and Cognition* 23.3, pp. 659–680. ISSN: 1939-1285, 0278-7393. DOI: `10.1037/0278-7393.23.3.659` (cit. on pp. 51 sq.).

Smith, J. D., Redford, J. S., and Haas, S. M. (2008). “Prototype Abstraction by Monkeys (Macaca Mulatta)”. In: *Journal of Experimental Psychology: General* 137.2, pp. 390–401. ISSN: 1939-2222(Electronic),0096-3445(Print). DOI: `10.1037/0096-3445.137.2.390` (cit. on p. 51).

Smith, J. D. and Minda, J. P. (1998). “Prototypes in the Mist: The Early Epochs of Category Learning”. In: *Journal of Experimental Psychology: Learning, Memory, and Cognition* 24.6, pp. 1411–1436. ISSN: 1939-1285(Electronic),0278-7393(Print). DOI: `10.1037/0278-7393.24.6.1411`. URL: `http://matt.colorado.edu/teaching/categories/sm98.pdf` (cit. on pp. 51 sq.).

— (2000). “Thirty Categorization Results in Search of a Model”. In: *Journal of Experimental Psychology. Learning, Memory, and Cognition* 26.1, pp. 3–27. ISSN: 0278-7393. DOI: `10.1037//0278-7393.26.1.3`. PMID: `10682288` (cit. on p. 52).

Sternberg, R. J. and Pickren, W. E., eds. (2019). *The Cambridge Handbook of the Intellectual History of Psychology*. Cambridge Handbooks in Psychology. New York: Cambridge University Press. ISBN: 978-1-108-29087-6. DOI: `10.1017/9781108290876` (cit. on p. 41).

Stevens, S. S. (1946). “On the Theory of Scales of Measurement”. In: *Science* 103.2684, pp. 677–680. ISSN: 0036-8075. JSTOR: `1671815`. URL: `http://www.jstor.org/stable/1671815` (cit. on p. 41).

Strube, G. (1992). “The Role of Cognitive Science in Knowledge Engineering”. In: *Contemporary Knowledge Engineering and Cognition*. Ed. by Schmalhofer, F., Strube, G., and Wetter, T.

Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, pp. 159–174. ISBN: 978-3-540-47277-3. DOI: 10.1007/BFb0045689 (cit. on pp. 30, 157 sq.).

Sweeney, L. (2003). "That's AI? : A History and Critique of the Field". In: DOI: 10.1184/R1/6610337 (cit. on p. 157).

Talmy, L. (1983). "How Language Structures Space". In: *Spatial Orientation*. Ed. by Pick, H. L. and Acredolo, L. P. Boston, MA: Springer US, pp. 225–282. ISBN: 978-1-4615-9327-0. DOI: 10.1007/978-1-4615-9325-6_11. URL: http://doursat.free.fr/docs/HSS512F_F09/Talmy_1983_space_language.pdf (visited on 03/22/2021) (cit. on pp. 62, 83).

— (2000a). *Toward a Cognitive Semantics*. 2 vols. Language, Speech, and Communication. Cambridge, Mass: A Bradford Book. ISBN: 978-0-262-20120-9. DOI: 10.7551/mitpress/6847.001.0001. URL: https://www.acsu.buffalo.edu/~talmy/talmyweb/TCS.html (cit. on pp. 29, 80).

— (11, 2000b). *Toward a Cognitive Semantics: Concept Structuring Systems*. Vol. 1. 2 vols. The MIT Press. ISBN: 978-0-262-28466-0. DOI: 10.7551/mitpress/6847.001.0001 (cit. on pp. 80, 82).

Tarquini, F., De Felice, G., Fogliaroni, P., and Clementini, E. (2007). "A Qualitative Model for Visibility Relations". In: *KI 2007: Advances in Artificial Intelligence*. Ed. by Hertzberg, J., Beetz, M., and Englert, R. Vol. 4667. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 510–513. ISBN: 978-3-540-74564-8. DOI: 10.1007/978-3-540-74565-5_52 (cit. on p. 70).

Tversky, A. (1977). "Features of Similarity". In: *Psychological Review* 84.4, pp. 327–352. ISSN: 0033-295X. DOI: 10.1037/0033-295X.84.4.327 (cit. on pp. 43 sq., 47 sqq., 53).

Tversky, A. and Gati, I. (1978). "Studies of Similarity". In: *Cognition and Categorization*. Ed. by Rosch, E. and Lloyd, B. B. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc. Chap. 4, pp. 79–98 (cit. on pp. 42, 47).

— (1982). "Similarity, Separability, and the Triangle Inequality". In: *Psychological Review* 89.2, pp. 123–154. ISSN: 0033-295X. DOI: 10.1037/0033-295X.89.2.123. PMID: 7089125 (cit. on pp. 44, 46 sq., 203).

Tversky, A. and Hutchinson, J. W. (1986). "Nearest Neighbor Analysis of Psychological Spaces". In: *Psychological Review* 93, pp. 3–22. ISSN: 0033295X. DOI: 10.1037/0033-295X.93.1.3 (cit. on pp. 41, 44).

Tversky, A. and Krantz, D. H. (1970). "The Dimensional Representation and the Metric Structure of Similarity Data". In: *Journal of Mathematical Psychology* 7.3, pp. 572–596. ISSN: 0022-2496. DOI: 10.1016/0022-2496(70)90041-6 (cit. on p. 44).

Ullman, S. (1979). *The Interpretation of Visual Motion*. The MIT Press Series in Artificial Intelligence. Cambridge, Mass.: The MIT Press, pp. xi, 229. ISBN: 978-0-262-25712-1. DOI: 10.7551/mitpress/3877.001.0001 (cit. on p. 126).

Van de Weghe, N. (2004). "Representing and Reasoning about Moving Objects: A Qualitative Approach". Ghent University, p. 168. URL: http://hdl.handle.net/1854/LU-668977 (cit. on pp. 17, 71, 74 sq., 114).

Van de Weghe, N., Bogaert, P., Cohn, A., Delafontaine, M., De Temmerman, L., Neutens, T., De Maeyer, P., and Witlox, F. (2007). "How to Handle Incomplete Knowledge Concerning Moving Objects". In: *Proceedings of the Workshop on Behaviour Monitoring and Interpretation BMI'07*. Behaviour Monitoring and Interpretation (BMI '07). Ed. by Gottfried, B. Centre for Computing Technologies (TZI), University of Bremen, Bremen, Germany: CEUR Workshop Proceedings, pp. 91–101. URL: http://ceur-ws.org/Vol-296/paper07vandeweghe.pdf (cit. on p. 75).

Van De Weghe, N., Cohn, A. G., De Tré, G., and De Maeyer, P. (2006). "A Qualitative Trajectory Calculus as a Basis for Representing Moving Objects in Geographical Information Systems". In: *Control and Cybernetics* 35.1, pp. 97–119. ISSN: 03248569. URL: http://citeseerx.ist.

psu.edu/viewdoc/download?doi=10.1.1.379.3744&rep=rep1&type=pdf (cit. on pp. 87, 95, 204).

Van De Weghe, N., Cohn, A. G., De Maeyer, P., and Witlox, F. (2005). "Representing Moving Objects in Computer-Based Expert Systems: The Overtake Event Example". In: *Expert Systems with Applications* 29.4, pp. 977–983. ISSN: 0957-4174. DOI: 10.1016/j.eswa.2005.06.022 (cit. on pp. 23, 63, 204).

Van de Weghe, N. and De Maeyer, P. (2005). "Conceptual Neighbourhood Diagrams for Representing Moving Objects". In: *Perspectives in Conceptual Modeling. ER 2005. Lecture Notes in Computer Science*. Ed. by Akoka, J., Liddle, S. W., Song, I.-Y., Bertolotto, M., Comyn-Wattiau, I., van den Heuvel, W.-J., Kolp, M., Trujillo, J., Kop, C., and Mayr, H. C. Vol. 3770. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 228–238. ISBN: 978-3-540-32239-9. DOI: 10.1007/11568346_25 (cit. on pp. 63, 113).

Van de Weghe, N., Kuijpers, B., Bogaert, P., and De Maeyer, P. (2005). "A Qualitative Trajectory Calculus and the Composition of Its Relations". In: *GeoSpatial Semantics*. Ed. by Rodríguez, M. A., Cruz, I., Levashkin, S., and Egenhofer, M. J. Vol. 3799. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 60–76. ISBN: 978-3-540-30288-9. DOI: 10.1007/11586180_5 (cit. on pp. 64, 73).

VandenBos, G. R., ed. (2007). *APA Dictionary of Psychology*. 1st ed. Washington, DC: American Psychological Association. xvi, 1024. ISBN: 978-1-59147-380-0. URL: http://archive.org/details/apadictionaryofp2007unse (visited on 08/19/2020) (cit. on pp. 94, 125).

Vilain, M. B. and Kautz, H. A. (1986). "Constraint Propagation Algorithms for Temporal Reasoning". In: *AAAI*. URL: http://www.aaai.org/Papers/AAAI/1986/AAAI86-063.pdf (cit. on p. 70).

Volkert, A., Müller, S., and Kirsch, A. (2018). "Human-like Prototypes for Psychologically Inspired Knowledge Representation". In: *Procedia Computer Science* 123, pp. 501–506. ISSN: 1877-0509. DOI: 10.1016/j.procs.2018.01.076 (cit. on pp. 44, 53).

Volkert, A., Schröder, J., and Kirsch, A. (2019). "Categorization in Real-World Tasks". In: *Proceedings of the Seventh Annual Conference on Advances in Cognitive Systems*. Ed. by Cox, M. T. Cambridge, MA, US, pp. 37–54 (cit. on p. 53).

Wallgrün, J. O., Wolter, D., and Richter, K.-F. (2, 2010). "Qualitative Matching of Spatial Information". In: *Proceedings of the 18th SIGSPATIAL International Conference on Advances in Geographic Information Systems*. GIS '10. New York, NY, USA: Association for Computing Machinery, pp. 300–309. ISBN: 978-1-4503-0428-3. DOI: 10.1145/1869790.1869833. URL: https://doi.org/10.1145/1869790.1869833 (visited on 02/24/2021) (cit. on p. 70).

Warglien, M., Gärdenfors, P., and Westera, M. (2012). "Event Structure, Conceptual Spaces and the Semantics of Verbs". In: *Theoretical Linguistics* 38.3–4, pp. 159–193. ISSN: 1613-4060. DOI: 10.1515/tl-2012-0010. URL: https://iris.unive.it/bitstream/10278/37082/1/semantics_of_verbs.pdf (cit. on p. 111).

Wertheimer, M. (1923). "Untersuchungen Zur Lehre von Der Gestalt II". In: *Psychologische Forschung: Zeitschrift für Psychologie und ihre Grenzwissenschaften* 4, pp. 301–350. DOI: 10.1007/BF00410640. URL: https://digitalesammlungen.uni-weimar.de/viewer/image/lit38308/1/ (visited on 08/21/2020) (cit. on p. 125).

Wickelmaier, F. (20, 2020). *Eba: Elimination-by-Aspects Models*. Version 1.10-0. URL: https://CRAN.R-project.org/package=eba (visited on 04/29/2022) (cit. on p. 161).

Wickelmaier, F. and Schmid, C. (1, 2004). "A Matlab Function to Estimate Choice Model Parameters from Paired-Comparison Data". In: *Behavior Research Methods, Instruments, & Computers* 36.1, pp. 29–40. ISSN: 1532-5970. DOI: 10.3758/BF03195547 (cit. on p. 162).

Wilkins, A. J. (1971). "Conjoint Frequency, Category Size, and Categorization Time". In: *Journal of Verbal Learning and Verbal Behavior* 10.4, pp. 382–385. ISSN: 0022-5371. DOI: `10.1016/S0022-5371(71)80036-1` (cit. on pp. 24, 37).

Wisniewski, E. (2001). "Feature Representations in Cognitive Psychology". In: *International Encyclopedia of the Social & Behavioral Sciences*. Ed. by Smelser, N. J. and Baltes, P. B. Vol. 8. Oxford: Pergamon, pp. 5433–5437. ISBN: 978-0-08-043076-8. DOI: `10.1016/B0-08-043076-7/01482-0`. URL: `https://archive.org/details/internationalenc0008unse_s0v4` (cit. on p. 43).

Wisniewski, E. J. (15, 2002). "Concepts and Categorization". In: *Stevens' Handbook of Experimental Psychology*. Ed. by Pashler, H. Hoboken, NJ, USA: John Wiley & Sons, Inc., pas0211. ISBN: 978-0-471-44333-9. DOI: `10.1002/0471214426.pas0211` (cit. on pp. 41, 43, 45, 49, 51, 56).

Wolter, D. and Wallgrün, J. O. (2012). "Qualitative Spatial Reasoning for Applications: New Challenges and the SparQ Toolbox". In: *Geographic Information Systems: Concepts, Methodologies, Tools, and Applications*. Vol. Ch. 10. IGI Global, pp. 1639–1664 (cit. on p. 31).

Wu, J., Claramunt, C., and Deng, M. (2014). "Towards a Qualitative Representation of Movement". In: *Advances in Conceptual Modeling*. Ed. by Indulska, M. and Purao, S. Vol. 8823. Lecture Notes in Computer Science. Springer International Publishing, pp. 191–200. ISBN: 978-3-319-12255-7. DOI: `10.1007/978-3-319-12256-4_20` (cit. on pp. 29, 72, 74, 78 sq.).

Yang, J., Klippel, A., and Li, R. (2015). "The Cognition of Change: Scaling Deformations in Mind and Spatial Theories". In: *Cartography and Geographic Information Science* 42.3, pp. 224–234. ISSN: 1523-0406. DOI: `10.1080/15230406.2014.991425`. URL: `https://scholar.archive.org/work/yxznrt7bvnfl7pewduhht4z4jq/access/wayback/http://www.cognitivegiscience.psu.edu/pdfs/yang2014cognition-prefinal.pdf` (cit. on pp. 31, 158).

Zadeh, L. A. (1972). "A Fuzzy-Set-Theoretic Interpretation of Linguistic Hedges". In: *Journal of Cybernetics* 2.3, pp. 4–34. DOI: `10.1080/01969727208542910` (cit. on p. 40).

Zimmermann, K. and Freksa, C. (1996). "Qualitative Spatial Reasoning Using Orientation, Distance, and Path Knowledge". In: *Applied Intelligence* 6.1, pp. 49–58. ISSN: 0924-669X. DOI: `10.1007/BF00117601`. URL: `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.2.4261&rep=rep1&type=pdf` (cit. on pp. 72, 74).

Zlatanova, S. (2000). "On 3D Topological Relationships". In: *Database and Expert Systems Applications, 2000. Proceedings. 11th International Workshop On*. IEEE, pp. 913–919. URL: `http://zlatanova.xyz/thesis/html/refer/ps/sz_asdm.pdf` (cit. on p. 203).