Plancherel Convergence and Zeta Functions

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Contents

Chapter 1

Introduction

1.1 Summary

Two of the most studied metric invariants of a smooth closed hyperbolic surface X are the Laplace spectrum $\mathrm{Spec}_\Delta(X)$ and the length spectrum $\mathrm{Spec}_L(X)$. While the length spectrum is at least known for a few arithmetic surfaces (see e.g. [83]), the Laplace spectrum can only be worked out using numerical methods (cf. [94, 29]). For this reason, one instead tries to describe the asymptotics of the Laplace spectrum, a typical result being Weyl's law

$$\frac{N(X,x)}{\mathrm{vol}(X)} \sim \frac{1}{4\pi}x$$

for the counting function $N(X,x) = \{\lambda \in \mathrm{Spec}_\Delta(X) \mid \lambda \leq x\}$ of the Laplace operator. Now, one might ask what happens if we assume x to be fixed and instead allow the surface X to vary. Let us consider a sequence of smooth closed hyperbolic surfaces $(X_j)_{j\in\mathbb{N}}$. This sequence is called *Plancherel-convergent*, if

$$\lim_{j\to\infty} \frac{N(X_j,x)}{\mathrm{vol}(X_j)} = \mu_{\mathrm{Pl}}([0,x]) \tag{1.1}$$

holds for any $x \geq 0$. The measure, which appears on the right-hand side of (1.1) is the Plancherel measure on the unitary dual of $\mathrm{SL}_2(\mathbb{R})$. In (1.1) we have implicitely made an identification between certain representations of $\mathrm{SL}_2(\mathbb{R})$ and subsets of $[0,\infty)$, which is described in detail in Section 2.7. In Chapter 3 we will show

Theorem 1.1.1. *Any sequence of smooth closed congruence surfaces* $(X_j)_{j\in\mathbb{N}}$ *with* $\mathrm{vol}(X_j) \to \infty$ *is Plancherel-convergent.*

We want to stress here that this already has been shown by Fraczyk [43]. We will give a different proof, which significantly reduces the amount of estimates needed.

It is natural to ask in what way this form of convergence is reflected in secondary metric invariants such as the Selberg zeta function. One result in this direction by Deitmar [31, Thm. 3.2] is

Theorem 1.1.2. *Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of smooth closed hyperbolic surfaces and let Λ_j be the logarithmic derivative of the Selberg zeta function of X_j.*

1. *If the sequence $(X_j)_{j\in\mathbb{N}}$ is uniformly discrete and Plancherel convergent, then*
$$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\operatorname{vol}(X_j)} = 0$$
for $\operatorname{Re}(s) > 1$.

2. *If for* $\operatorname{Re}(s) > 1$ *we have*
$$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\operatorname{vol}(X_j)} = 0,$$
then $(X_j)_{j\in\mathbb{N}}$ is Plancherel-convergent.

Here, uniform discreteness asserts that there is a uniform lower bound for the lengths of closed geodesics on the surfaces $(X_j)_{j\in\mathbb{N}}$. This assumption was needed in the proof of the above theorem to control the accumulation rate of eigenvalues. Deitmar asks in [31, §4] whether the condition of uniform discreteness is actually needed. We will see in Section 4.3 that by a careful analysis of the accumulation rate of eigenvalues in Plancherel sequences one can establish

Theorem 1.1.3. *Let $(X_j)_{j\in\mathbb{N}}$ be a Plancherel-convergent sequence of smooth closed hyperbolic surfaces. Then there exists a constant c_A such that*
$$N(X_j, x) \leq c_A \operatorname{vol}(X_j)(1 + x). \tag{1.2}$$

This is sufficient to remove the assumption of uniform discreteness from Theorem 1.1.2. Next, Deitmar considered in [31, §4] the limit of $\operatorname{vol}(X_j)^{-1}\Lambda_j(s)$ for $s \in \mathbb{C}$ with $\operatorname{Re}(s) \leq 1$. In this range the functions $\{\Lambda_j\}_{j\in\mathbb{N}}$ may have poles for
$$s \in \mathscr{P} = \{-k \mid k \in \mathbb{N}_0\} \cup [0, 1] \cup (\tfrac{1}{2} + i\mathbb{R}),$$
which we will avoid for the moment. The functional equation of Λ_j (see e.g. [31, Prop. 3.4]) allows one to deal with those $s \in \mathbb{C} \setminus \mathscr{P}$ such that $\operatorname{Re}(s) < 0$. It remains to determine what happens for s sitting inside the critical strip
$$S = \{s \in \mathbb{C} \mid 0 \leq \operatorname{Re}(s) \leq 1\}.$$
In Section 4.4 we will employ a formula of McKean to compute the limit of $\operatorname{vol}(X_j)^{-1}\Lambda_j(s)$ for all s in
$$U_1 = \{s \in \mathbb{C} \mid \operatorname{Re}(s) > \tfrac{1}{2},\ \operatorname{Re}(s(s-1)) > -\tfrac{1}{4},\ s \notin \mathscr{P}\}.$$

If for $s \in \mathbb{C} \smallsetminus \mathscr{P}$ we let

$$F(s) = \begin{cases} 0, \ \mathrm{Re}(s) > \frac{1}{2}, \\ (s - \frac{1}{2}) \cot(\pi s), \ \mathrm{Re}(s) < \frac{1}{2}, \end{cases}$$

our final result of Chapter 4 is

Theorem 1.1.4. *Let $(X_j)_{j \in \mathbb{N}}$ be a sequence of smooth closed hyperbolic surfaces. The following two statements are equivalent:*

1. *The sequence $(X_j)_{j \in \mathbb{N}}$ is Plancherel convergent.*
2. *For each $s \in \mathbb{C} \smallsetminus \mathscr{P}$ such that either $s \in U_1$, $\mathrm{Re}(s) < 0$ or $\mathrm{Re}(s) > 1$ one has*
$$\lim_{j \to \infty} \frac{\Lambda_j(s)}{\mathrm{vol}(X_j)} = F(s).$$

The values $s \in \mathbb{C}$, for which the behaviour of the logarithmic derivative of the Selberg zeta function is known, are sketched in Figure 4.1.

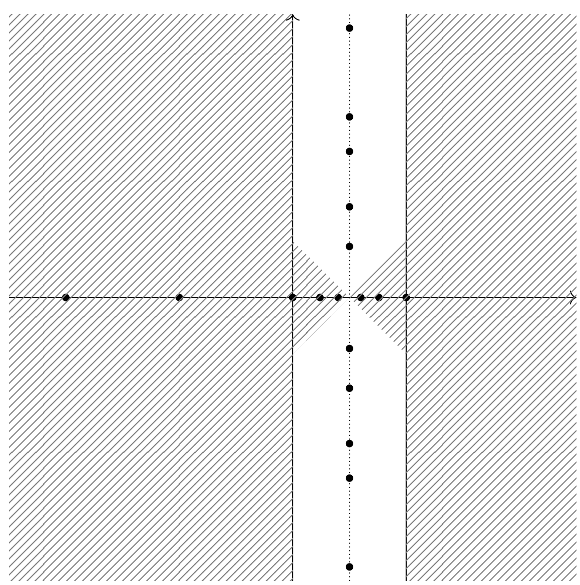

Figure 1.1: The values $s \in \mathbb{C}$ from Theorem 1.1.4.

Finally, in Chapter 5 we discuss the behaviour of $\mathrm{vol}(X_j)^{-1}\Lambda_j(s)$ for $s \in \mathscr{P}$ in a few arithmetic scenarios. We will use the Euler-Selberg constant (cf. [46])

$$\gamma_X = \lim_{s \to 1} \left(\Lambda_X(s) - \frac{1}{s-1} \right)$$

as a replacement for the logarithmic derivative of the Selberg zeta function Λ_X at $s = 1$.

Proposition 1.1.5. *Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of smooth closed congruence surfaces with* $\mathrm{vol}(X_j)\to\infty$*. Then for $s\in(\frac{39}{64},1)$ one has*

$$\lim_{j\to\infty}\frac{\Lambda_j(s)}{\mathrm{vol}(X_j)}=0$$

and

$$\lim_{j\to\infty}\frac{\gamma_{X_j}}{\mathrm{vol}(X_j)}=0.$$

Here, one utilizes a well-known uniform lower bound for the first eigenvalue of congruence surfaces (see Theorem 5.3.1). However, this may change, if there is no such bound:

Proposition 1.1.6. *There exist a Plancherel-convergent sequence of arithmetic surfaces $(X_j)_{j\in\mathbb{N}}$ such that*

$$\lim_{j\to\infty}\frac{\gamma_{X_j}}{\mathrm{vol}(X_j)}=l_0>0$$

for some positive constant l_0.

This suggests that for $s\in\mathscr{P}$ the behaviour of $\mathrm{vol}(X_j)^{-1}\Lambda_j(s)$ is not determined by $(X_j)_{j\in\mathbb{N}}$ being a Plancherel sequence and that additional information about the spectral geometry of the surfaces $(X_j)_{j\in\mathbb{N}}$ is needed.

1.2 Comparison to the Literature

The subject started with DeGeorge and Wallach [100] establishing the limit multiplicity property (see Section 2.11) for towers in a semisimple Lie group G. Here, a tower is a sequence of cocompact lattices $(\Gamma_j)_{j\in\mathbb{N}}$ in G such that each Γ_j is a normal subgroup of Γ_1 and

$$\Gamma_j\supset\Gamma_{j+1},\qquad\bigcap_{j=1}^{\infty}\Gamma_j=\{1\}\tag{1.3}$$

holds. The limit multiplicity property is a well-studied subject (see e.g. [33, 26, 40, 80, 34]). Sauvageot [84] showed that the limit multiplicity property of a sequence of lattices $(\Gamma_j)_{j\in\mathbb{N}}$ follows from establishing Plancherel convergence of $(\Gamma_j)_{j\in\mathbb{N}}$. In the more recent breakthrough [1] it was shown that a uniformly discrete sequence $(\Gamma_j)_{j\in\mathbb{N}}$ is Plancherel convergent, if it is Benjamini-Schramm convergent[1] (BS-convergent for short). The authors of [1] then established that any sequence $(\Gamma_j)_{j\in\mathbb{N}}$ of non-conjugate lattices in a semisimple Lie group of rank greater than 1 is BS-convergent. Since uniform discreteness for these lattices is a well-known consequence of the Lehmer conjecture (cf. [61, p. 322]) this essentially closes the case of higher rank Lie groups. Raimbault [78] and

[1] The definition of Benjamini-Schramm convergence can also be found in Section 2.11.

Matz [64] dealt with certain sequences of lattices $(\Gamma_j)_{j\in\mathbb{N}}$ in semisimple Lie groups of rank one, for which the degrees of the associated trace fields $F(\Gamma_j)$ are uniformly bounded. Fraczyk [43] then established Plancherel convergence for arbitrary cocompact torsion-free congruence lattices in the groups $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$. Later on, Fraczyk and Raimbault [44] removed the assumption on torsion elements. The main insight of Fraczyk was to use the Bilu equidistribution principle (cf. [8]) to establish bounds, which are uniform in the degree of the trace fields. We will also make use of these bounds, but do not need his bounds on characters of p-adic groups. The case of non-cocompact principal congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$ had already been dealt with by Sarnak [82]. Bergeron and Venkatesh [7] studied the asymptotics of analytic torsion in BS-convergent sequences and were able to establish convergence in the strongly acyclic case. Numerical data by Sengün [88] suggests that the results of Bergeron and Venkatesh should be true without assuming strong acyclicity, but, to the best knowledge of the author, this has not yet been shown (cf. [78]). Further interesting references on this topic include [5, 6, 14]. Another direction of research is the relation between Benjamini-Schramm convergence and quantum ergodicity (cf. [58]). Furthermore, BS-convergence can also be extended to more general situations (see [30]). In this thesis, we will focus on the relationship between BS-convergence and zeta functions described by Deitmar in [31].

Chapter 2

Preliminaries

2.1 The Geometry of the Upper Half-Plane

Let us recall some standard material regarding the geometry of the hyperbolic plane. Any further information can be found in [3] and [53].
The *upper half-plane* $\mathcal{H}$ is the set

$$\mathcal{H} = \{x + iy \in \mathbb{C} \mid x \in \mathbb{R},\ y \in \mathbb{R}_{>0}\}. \tag{2.1}$$

If we equip $\mathcal{H}$ with the so-called *hyperbolic metric* $ds^2 = (dx^2 + dy^2)/y^2$, it becomes a model for the hyperbolic plane. In this model, the geodesics have the following elegant description:

Theorem 2.1.1. *The geodesics in the hyperbolic plane $\mathcal{H}$ consist of straight lines and semicircles, which meet the boundary of $\mathcal{H}$ orthogonally.*

Proof. [3, Thm. 7.3.1] $\square$

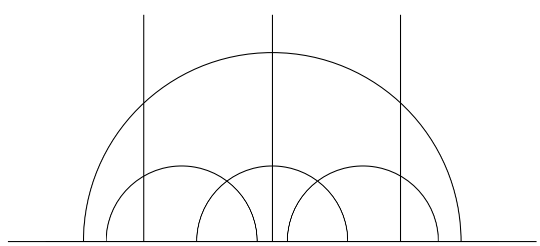

Figure 2.1: Geodesics in the upper half-plane

We have an action of the special linear group

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \middle| ad - bc = 1 \right\}$$

on the upper half-plane $\mathcal{H}$, which for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathcal{H}$ is given by

$$\phi_g(z) = \frac{az+b}{cz+d}.$$

Since the center $Z(\mathrm{SL}_2(\mathbb{R})) = \{\pm \mathbb{1}_2\}$ of $\mathrm{SL}_2(\mathbb{R})$ acts trivially, we get an action of the projective special linear group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathbb{1}_2\}$ on $\mathcal{H}$.

Remark 2.1.2. *We want to warn the reader that we will not distinguish between a matrix $g \in \mathrm{SL}_2(\mathbb{R})$ and its projection $[g] \in \mathrm{PSL}_2(\mathbb{R})$. Whenever it is necessary, we assume that a lift of $[g]$ with non-negative trace has been fixed and simply represent $[g]$ by the matrix of this lift. When speaking of the matrix $[g]$, we mean the matrix of the lift of $[g]$. This is a common abuse of notation and significantly improves the readibility of any argument involving elements $[g] \in \mathrm{PSL}_2(\mathbb{R})$.*

Any of the maps ϕ_g for $g \in \mathrm{PSL}_2(\mathbb{R})$ is an orientation-preserving isometry of the upper half-plane. The converse statement is also true:

Theorem 2.1.3. *The group of orientation-preserving isometries of the hyperbolic plane consists of the Möbius transformations $\{\phi_g \mid g \in \mathrm{PSL}_2(\mathbb{R})\}$.*

Proof. [3, Thm. 7.4.1] □

With the above theorem, one can classify isometries of the upper half-plane in terms of properties of the corresponding matrices $g \in \mathrm{PSL}_2(\mathbb{R})$. We call an element $g \in \mathrm{PSL}_2(\mathbb{R})$

- *hyperbolic*, if $|\operatorname{tr} g| > 2$,
- *elliptic*, if $|\operatorname{tr} g| < 2$,
- *parabolic*, if $|\operatorname{tr} g| = 2$.

Any hyperbolic element $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ fixes exactly two points on the extended boundary $\partial_\infty \mathcal{H} = \mathbb{R} \cup \{\infty\}$ of $\mathcal{H}$. The unique geodesic connecting these fixed points is called the *axis* of γ and denoted A_γ. Conversely, any geodesic of the hyperbolic plane appears as the axis of some hyperbolic element $\gamma \in \mathrm{PSL}_2(\mathbb{R})$. Note that an element $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ is hyperbolic, if and only if it is conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to a matrix of the form

$$\begin{pmatrix} \tau_\gamma & 0 \\ 0 & \tau_\gamma^{-1} \end{pmatrix}.$$

with $\tau_\gamma > 1$. The eigenvalues $\tau_\gamma^{\pm 1}$ are roots of the characteristic polynomial

$$p_\gamma(x) = x^2 - \operatorname{tr}(\gamma)\, x + 1, \tag{2.2}$$

and are given by

$$\tau_\gamma^{\pm 1} = \frac{\operatorname{tr}\gamma \pm \sqrt{(\operatorname{tr}\gamma)^2 - 4}}{2}. \tag{2.3}$$

The *Weyl discriminant* d_γ of γ is given by

$$d_\gamma = (1 - \tau_\gamma)(1 - \tau_\gamma^{-1}). \tag{2.4}$$

2.2 Geometry of Hyperbolic Surfaces

In this section we will use the well-understood geometry of the upper half-plane to derive results about the geometry of smooth closed hyperbolic surfaces. Our exposition is based on [3, 53, 17].
A discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ is called a *Fuchsian group*. Any elliptic element x of a Fuchsian group has finite order. We will call an element of finite order a *torsion element*. If a Fuchsian group Γ does not contain any torsion-elements, it is said to be *torsion-free*. A Fuchsian group is called *purely hyperbolic*, if every non-trivial element $\gamma \in \Gamma$ is hyperbolic. A Fuchsian group Γ is said to be *cocompact*, if the quotient space $\Gamma\backslash\mathcal{H}$ is compact.

Theorem 2.2.1. *Let Γ be a Fuchsian group.*

1. *The quotient space $\Gamma\backslash\mathcal{H}$ is compact if and only if $\mathrm{vol}(\Gamma\backslash\mathcal{H}) < \infty$ and the Fuchsian group Γ does not contain any parabolic elements.*
2. *If Γ is torsion-free, the quotient $\Gamma\backslash\mathcal{H}$ carries a uniquely determined complete hyperbolic structure such that the natural projection $\pi : \mathcal{H} \to \Gamma\backslash\mathcal{H}$ is a local isometry.*
3. *Any smooth closed hyperbolic surface X is of the form $X = \Gamma\backslash\mathcal{H}$ with Γ a purely hyperbolic group. For such a group Γ we have an isomorphism $\Gamma \cong \pi_1(X)$.*
4. *Let Γ_1, Γ_2 be purely hyperbolic groups. Then $\Gamma_1\backslash\mathcal{H}$ and $\Gamma_2\backslash\mathcal{H}$ are conformally equivalent if and only there exists $x \in \mathrm{PSL}_2(\mathbb{R})$ with $\Gamma_2 = x\Gamma_1 x^{-1}$.*

Proof. The first statement is proven in [53, Cor. 4.2.7]. For the statement regarding torsion-free Fuchsian groups we refer to [17, Thm. 1.2.4]. The last two statements[1] are given in [52, Thm. 4.19.8] and [52, Thm. 5.9.3]. □

Given a smooth closed hyperbolic surface X, we will always assume to have fixed some purely hyperbolic Fuchsian group Γ with $X \cong \Gamma\backslash\mathcal{H}$. We write $g = g(X)$ for the *genus* of X. We note that the Gauss-Bonnet theorem implies

$$\mathrm{vol}(X) = 4\pi(g-1). \tag{2.5}$$

Next we want to study closed geodesics on hyperbolic surfaces.

Theorem 2.2.2. *Let X be a smooth closed hyperbolic surface and $c : [0,1] \to X$ be a continuous, closed curve on X. In the homotopy class $[c]$ of c there exists a unique curve of minimal length. This curve is a closed geodesic.*

Proof. [17, Thm. 1.5.3] □

It is clear from the above theorem that hyperbolic elements $\gamma \in \Gamma \cong \pi_1(X)$ are closely related to closed geodesics on X. This can be made more precise (cf. e.g. [45, Prop. 9]):

[1] The third statement is not directly the statement given in [52, Thm. 4.19.8], but follows from it and the remark before [52, Thm. 5.9.3].

Proposition 2.2.3. *Let X be a smooth closed hyperbolic surface with associated Fuchsian group Γ. Then there is a bijection between Γ-conjugacy classes of hyperbolic elements $\gamma \in \Gamma$ and closed geodesics $c(\gamma)$. Under this correspondence, the length of $c(\gamma)$ is given by $l_\gamma = 2\log\tau_\gamma$, where τ_γ is the larger of the two eigenvalues of γ.*

Proof. Let $\gamma \in \Gamma$ be a hyperbolic element. After possibly conjugating the group Γ we may assume

$$\gamma = \begin{pmatrix} \tau_\gamma & 0 \\ 0 & \tau_\gamma^{-1} \end{pmatrix}.$$

Then the axis of γ is given by $A_\gamma = i\mathbb{R}_{>0}$. For any $z \in A_\gamma$ we have

$$\gamma.z = \tau_\gamma^2 z \in A_\gamma$$

so that A_γ projects down to a closed geodesic $c(\gamma)$ on $\Gamma\backslash\mathcal{H}$. A quick integration shows that the length of this geodesic is given by $l_\gamma = 2\log\tau_\gamma$.
For the converse direction, let c be a closed geodesic on $\Gamma\backslash\mathcal{H}$. By [17, Thm. 1.4.4], we can lift it to a geodesic $\widetilde{c}$ in $\mathcal{H}$. The stabilizer of the action of Γ on $\widetilde{c}$ contains a hyperbolic element γ such that $c = c(\gamma)$. □

From now on, we will not distinguish between a hyperbolic transformation γ and the corresponding geodesic $c(\gamma)$ from Proposition 2.2.3 and write γ for both objects. When speaking of the length of a hyperbolic element $\gamma \in \Gamma$, we mean the quantity $l_\gamma = 2\log\tau_\gamma$, where $\tau_\gamma > 1$ is the larger eigenvalue of γ. The following is a typical feature of closed geodesics on hyperbolic surfaces:

Theorem 2.2.4. *Let X be a smooth closed hyperbolic surface and fix some $c > 0$. Then there exist only finitely many closed geodesics on X of length $\leq c$.*

Proof. [17, Thm. 1.6.11] □

In particular, there exists a closed geodesic γ_s of shortest length l_s on X. The number $l_s = l_s(X)$ is called the *systole* (or *systolic length*) of X. Let us write $\sigma_L(X)$ for the set of lengths of closed geodesics on X. Then the above theorem shows that the multiplicity $m(l)$ for any length $l \in \sigma_L(X)$ is finite. Hence, we can define the *length spectrum*

$$\mathrm{Spec}_L(X) = \{(l, m(l)) \mid l \in \sigma_L(X)\}$$

of X. The length spectrum of $X = \Gamma\backslash\mathcal{H}$ is also denoted by $\mathrm{Spec}_L(\Gamma)$. We will sometimes write $m(l, X)$ or $m(l, \Gamma)$ instead of $m(l)$, when dealing with more than one surface.

Remark 2.2.5. *Let Γ be a cocompact torsion-free Fuchsian group. From Proposition 2.2.3 we see that all hyperbolic elements $\gamma \in \Gamma$ of fixed length l fall into finitely many Γ-conjugacy classes*

$$[\gamma_1]_\Gamma, ..., [\gamma_s]_\Gamma,$$

where the number s of these classes is equal to $m(l)$. This is a purely algebraic description of the length spectrum, which will be carried over to Fuchsian groups, which might be neither cocompact nor torsion-free.

For $\gamma \in \Gamma$ a hyperbolic element in a Fuchsian group we let

$$\Gamma_\gamma = \{x \in \Gamma \mid x\gamma x^{-1} = \gamma\}$$

be the *centralizer of* γ. The centralizer is cyclic [3, Thm. 8.1.2] and we call any element $\gamma_0 \in \Gamma$ with $\Gamma_\gamma = \langle \gamma_0 \rangle$ a *primitive element* or a *prime geodesic*. Throughout this thesis, we always let γ_0 be the prime geodesic underlying γ.
A hyperbolic element $\gamma \in \Gamma$ is called *simple*, if for all $x \in \Gamma$ either $xA_\gamma x^{-1} = A_\gamma$ or $xA_\gamma x^{-1} \cap A_\gamma = \emptyset$ holds. Otherwise, we say that γ is *non-simple*. We note that γ is simple if and only if the corresponding geodesic has no self-intersections [60, Lemma 5.3.10]. We will see in the next section that simple geodesics play an important role for the geometry of hyperbolic surfaces.

2.3 Decompositions of Hyperbolic Surfaces

Next we want to discuss certain decompositions of smooth closed hyperbolic surfaces. Our main sources for this material are [17, §3] and [17, §4].
Let X be a smooth closed hyperbolic surface and h be the Riemannian metric on X. For $p \in X$ we write

$$\exp_p : T_pX \to X$$

for the *exponential map at* $p \in X$, which is defined as follows: Every non-zero tangent vector $v \in T_pX$ defines a unique geodesic tangential to v, which starts at p. One follows that geodesic for time $||v||_h = h_p(v,v)^{1/2}$ to reach a point, which is denoted $\exp_p(v)$. Finally, one sets $\exp_p(0) = p$. The *injectivity radius* $r_p(X)$ of X at a point $p \in X$ is the supremum of all $r > 0$ such that $\exp_p$ is injective on the open ball $U_r = \{v \in T_pX : ||v||_h < r\}$. Points with small injectivity radius lie in certain subsets of X, whose geometry do not depend on the ambient surface X:

Theorem 2.3.1. *Let X be a smooth closed hyperbolic surface and γ_i, $i = 1, ..., m_X$, be the closed geodesics of length $l_i = l(\gamma_i) \leq 1$. Then the following hold:*

1. $m_X \leq 3g - 3$.
2. *The tubes*
$$T_{\gamma_i} = \{p \in X \mid \operatorname{dist}(p, \gamma_i) \leq w_i\}$$
of widths
$$w_i = \operatorname{arcsinh}(1/\sinh(l_i/2))$$
are pairwise disjoint for $i = 1, ..., m_X$.

3. *Each tube* T_{γ_i} *is isometric to a cylinder* $[-w_i, w_i] \times \mathbb{S}^1$ *equipped with the Riemannian metric*

$$ds^2 = dr^2 + l_i^2 \cosh^2 r \; dt^2.$$

4. *The injectivity radius* $r_p(X)$ *for* $p \in X \smallsetminus \bigcup_{i=1}^{m_X} T_{\gamma_i}$ *is bounded from below by* $\operatorname{arcsinh}(1)$.

5. *If* $p \in T_{\gamma_i}$ *and* $d = \operatorname{dist}(p, \partial T_{\gamma_i})$, *then*

$$\sinh r_p(X) = \cosh(l_i/2) \cosh d - \sinh d$$

Proof. All of the statements can be found in [17, §4]. □

A closed geodesic of length $l \leq 1$ is called *short*. The coordinates for the tube coming from the third statement will be called *Fermi coordinates* for the tube T_{γ_i}.

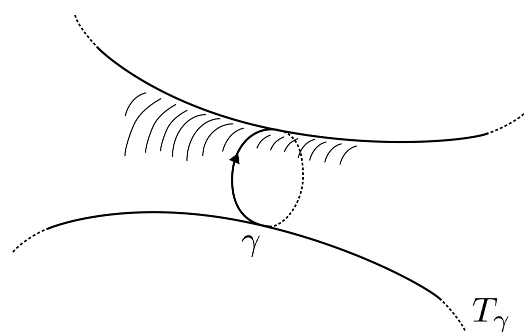

Figure 2.2: A tube T_γ around a short geodesic γ.

For technical reasons[2], we will actually work with the truncated tubes

$$T'_{\gamma_i} = \{p \in X \mid \operatorname{dist}(p, \gamma_i) \leq w_i - 1\}.$$

We will call

$$X' = \bigcup_{i=1}^{m_X} T'_{\gamma_i}$$

the *thin part* of X, the complement $X'' = X \smallsetminus X'$ the *thick part* of X and refer to $X = X' \cup X''$ as the *thick-thin decomposition* of X. While the geometry of the thin part is described by Theorem 2.3.1, points in the thick part are contained in geodesic balls of uniform size. In this sense, the thick-thin decomposition gives a description of the local geometry of hyperbolic surfaces.

[2] Roughly speaking, one needs a description of the geometry close to the boundary of the thin part (see Lemma 4.3.3).

Remark 2.3.2. *It would actually be better to speak of "a" thick-thin decomposition, as the precise meaning of the thick and thin part usually is chosen to suit the circumstances (cf. [20]).*

Next, we want to lose a few words regarding the decomposition of a closed hyperbolic surface into pairs of pants. We recall that a compact topological surface is said to have *signature* (g, n), if it is obtained from a closed topological surface by removing the interior of n disjoint closed topological disks. A compact Riemann surface of signature $(0, 3)$ is called a *Y-piece* or a *pair of pants*. For any triple of positive real numbers l_1, l_2, l_3 there exists a pair of pants Y_{l_1,l_2,l_3} with boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ of respective lengths $l_i = l_{\gamma_i}$ (see [17, Thm. 3.1.7]).

Theorem 2.3.3. *Let X be a smooth compact hyperbolic surface of genus g. Let $\gamma_1, ..., \gamma_m$ be pairwise disjoint simple closed geodesics on X. Then the following hold:*

1. $m \leq 3g - 3$.
2. *There exist simple closed geodesics $\gamma_{m+1}, ..., \gamma_{3g-3}$, which together with $\gamma_1, ..., \gamma_m$ decompose X into Y-pieces.*

Proof. [17, Thm. 4.1.1] □

2.4 The Spectrum of the Laplace Operator

Let X be a smooth closed manifold equipped with a Riemannian metric h and let μ_h be the associated measure on X. Let $L^2(X)$ be the space of square-integrable functions f on X. On $L^2(X)$ we have an inner product given by

$$(f_1, f_2)_2 = \int_X f_1 f_2 \; d\mu_h.$$

and an associated norm $||f||_2 = (f, f)^{\frac{1}{2}}$. Let $\mathfrak{X}(X)$ be the space of smooth vector fields on X. The *de Rham-differential* of a smooth function $f \in C^\infty(X)$ is the one-form on X given by $df(V) = Vf$, where $V \in \mathfrak{X}(X)$. The inner product $(\cdot, \cdot)_2$ allows us to define the adjoint $\delta = d^*$ of the de Rham-differential. The *Laplace operator* Δ acts on smooth functions via

$$\Delta f = \delta df, \qquad f \in C^\infty(X). \tag{2.6}$$

In local coordinates, the Laplacian can be written as

$$-\Delta = \frac{1}{\sqrt{\det h}} \sum_{i,j} \partial_i \left(\sqrt{\det h} \; h^{ij} \partial_j \right) \tag{2.7}$$

(see e.g. [91, §22.1]), where $\det h$ is the determinant of absolute value of the metric tensor (h_{ij}) and $h^{ij} = (h^{-1})_{ij}$ is the inverse of the metric tensor. The operator given by the right-hand side of (2.6) extends to a self-adjoint operator Δ on the *Sobolev space* $\mathscr{H}(X)$, which is the completion of $C^\infty(X)$ in the norm

$$||f||^2 = ||f||_2^2 + ||\operatorname{grad} f||_2^2,$$

where $\operatorname{grad} f$ is unique vector field on X such that

$$df(V) = h(V, \operatorname{grad} f), \qquad \text{for all } V \in \mathfrak{X}(X).$$

Theorem 2.4.1. *The operator spectrum $\sigma(\Delta)$ of Δ consists of a sequence of eigenvalues*

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \nearrow +\infty,$$

where the multiplicity $m(\lambda)$ of each eigenvalue $\lambda \in \sigma(\Delta)$ is finite.

Proof. [17, Thm. 7.2.6] □

We call the list of tuples

$$\operatorname{Spec}_\Delta(X) = \{(\lambda, m(\lambda)) \mid \lambda \in \sigma(\Delta)\}$$

the *spectrum of the Laplace operator*[3] on X or the *Laplace spectrum* of X. Occasionally, we will write $\lambda_k(X)$ in place of λ_k to emphasize the dependence on X. We will sometimes write $m(\lambda, X)$ for the multiplicity of a Laplace eigenvalue λ, when there is some danger of confusion. We call two smooth closed manifolds X_1 and X_2 *isospectral*, if $\operatorname{Spec}_\Delta(X_1) = \operatorname{Spec}_\Delta(X_2)$. The *counting function* of the Laplace operator is given by

$$N(X, x) = \sum_{\lambda \leq x} m(\lambda).$$

Let us from now on assume that X is a smooth closed hyperbolic surface of genus g. A Laplace eigenvalue λ of Δ is said to be *small*, if $\lambda < \frac{1}{4}$. There are plenty of good reasons to distinguish the eigenvalues below $\frac{1}{4}$ from those above $\frac{1}{4}$. For example, the number of small eigenvalues is bounded by the topology alone (cf. [76]),

$$\lambda_{2g-2}(X) \geq \frac{1}{4},$$

while the same is not true for the remainder of the spectrum:

Theorem 2.4.2. *Let $g \geq 2$ be fixed. For any $k \in \mathbb{N}$ and for arbitrarily small $\varepsilon > 0$ there exists a smooth closed surface X of genus g such that $\lambda_k(X) \leq \frac{1}{4} + \varepsilon$.*

Proof. [17, Thm. 8.1.2] □

[3] Note that, in our choice of convention, the spectrum of the Laplace operator is different from the operator spectrum $\sigma(\Delta)$ of Δ, as $\sigma(\Delta)$ does not keep track of the multiplicities of the eigenvalues.

Finally, we note that the first eigenvalue $\lambda_1(X)$ can be controlled by the *Cheeger constant*

$$h(X) = \inf \frac{l_u}{\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}}, \tag{2.8}$$

where u ranges over the set of all finite unions of piecewise smooth curves on X, which separate X into two disjoint subsets A and B.

Theorem 2.4.3 (Buser-Cheeger inequality)**.**

$$\frac{1}{4}h^2(X) \leq \lambda_1(X) \leq 2h(X) + 10h^2(X) \tag{2.9}$$

Proof. The left-hand side is the well-known Cheeger inequality [24], while the right-hand side is due to Buser [16]. □

2.5 Harmonic Analysis on Locally Compact Groups

Next we want to introduce some basic terminology and fundamental facts from the theory of harmonic analysis on locally compact groups. Our main references on this subject are [42] and [32].

Let G be a *locally compact group*, i.e. a topological group, whose topology is locally compact and Hausdorff. A left (respectively right) *Haar measure* on G is a non-zero Radon measure μ_G on G that satisfies $\mu_G(xE) = \mu_G(E)$ (respectively $\mu_G(Ex) = \mu_G(E)$) for every Borel set $E \subset G$ and $x \in G$. Every locally compact group possesses a Haar measure and such a measure is unique up to multiples. The group G is said to be *unimodular*, if any left Haar measure is also a right Haar measure. We assume from now on that a Haar measure μ_G has been fixed and write

$$\int_G f(x) d\mu_G(x) = \int_G f(x) dx$$

for the corresponding Haar integral. A *unitary representation* of G is a homomorphism from G into the group $\mathcal{U}(H_\pi)$ of unitary operators on some non-zero Hilbert space H_π that is continuous in the strong operator topology. If such a representation π admits a proper invariant subspace, it is said to be *reducible*. Otherwise, π is called *irreducible*. Two unitary representations $\pi_1 : G \to \mathcal{U}(H_{\pi_1})$ and $\pi_2 : G \to \mathcal{U}(H_{\pi_2})$ are called (unitarily) *equivalent*, if and only if there exists a unitary operator $U : H_{\pi_1} \to H_{\pi_2}$ such that $\pi_2(x) = U\pi_1(x)U^{-1}$ for each $x \in G$. The set of equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$ and called the *unitary dual* of G. We will always assume to have fixed some representantive π in an equivalence class $[\pi] \in \widehat{G}$ and omit the brackets from now on. We equip $\widehat{G}$ with the so-called[4] *Fell topology*. To any $f \in L^1(G)$ we associate the measurable field of operators over $\widehat{G}$

$$\widehat{f}(\pi) = \int_G f(x)\pi(x^{-1})dx,$$

[4]The definition of the Fell topology can be found in [42, §7.2].

which is referred to as the *Fourier transform* of f. There exists a Radon measure μ_{Pl} on the unitary dual $\widehat{G}$, for which under certain technical assumptions[5] a Plancherel theorem holds:

Theorem 2.5.1. *Suppose G is a second countable unimodular type I group. The Fourier transform $f \mapsto \widehat{f}$ maps $L^1(G) \cap L^2(G)$ into $\int_{\widehat{G}} H_\pi \otimes H_{\overline{\pi}} d\mu_{\mathrm{Pl}}(\pi)$ and extends to a unitary map from $L^2(G)$ onto $\int_{\widehat{G}} H_\pi \otimes H_{\bar{\pi}} d\mu_{\mathrm{Pl}}(\pi)$.*

Proof. The proof can be found in [36, §18.8]. □

The measure μ_{Pl} from Theorem 2.5.1 is called the *Plancherel measure* of $\widehat{G}$ and is unique, once the Haar measure on G has been fixed. A representation $\pi \in \widehat{G}$ is said to be *tempered*, if it lies in the support of the Plancherel measure. Let us write $\widehat{G}_{\mathrm{temp}}$ for the set of tempered representations.

2.6 Representation Theory of $\mathrm{SL}_2(\mathbb{R})$

In this section we describe the unitary dual of the group $G = \mathrm{SL}_2(\mathbb{R})$ and the corresponding Plancherel measure on $\widehat{G}$. For any further details we refer the reader to [55].
We equip $\mathrm{SL}_2(\mathbb{R})$ with the unique Haar measure μ such that $\mu(K) = 1$, where $K = \mathrm{SO}(2)$ is the maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$. The unitary dual of $\mathrm{SL}_2(\mathbb{R})$ is explicitly known:

Theorem 2.6.1. *Each irreducible unitary representation of* $\mathrm{SL}_2(\mathbb{R})$ *is, up to equivalence, contained on the following list:*

- *the trivial representation π_1,*
- *the principal series $\widehat{G}^+_{\mathrm{prin}} = \{\pi^+_{i\nu} \mid \nu \in \mathbb{R}_{\geq 0}\}$ and $\widehat{G}^-_{\mathrm{prin}} = \{\pi^-_{i\nu} \mid \nu \in \mathbb{R}_{>0}\}$,*
- *the complementary series $\widehat{G}_{\mathrm{comp}} = \{\pi^+_\nu \mid \nu \in (0, \frac{1}{2})\}$,*
- *the (limits of) discrete series $\widehat{G}_{\mathrm{disc}} = \{\delta^\pm_m \mid m \in \mathbb{N}\}$*

All of these representations are irreducible and no two irreducible representations from this list are equivalent.

Proof. [55, Thm. 16.3] □

In Figure 2.3 we have sketched the unitary dual of $\mathrm{SL}_2(\mathbb{R})$ with respect to the Fell topology (cf. [42, Fig. 7.3]). Note that the dotted lines describe non-Hausdorff points, meaning for example that the trivial representation π_1 cannot be separated from $\delta^\pm_2$ in this topology and these three points are limit points of the complementary series representation π^+_ν for $\nu \to \frac{1}{2}$.

[5]The rather technical definition of a group of type I is given on page 206 of [42]. For our purposes, it is enough to know that $\mathrm{SL}_2(\mathbb{R})$ is of type I [42, Thm. 7.8].

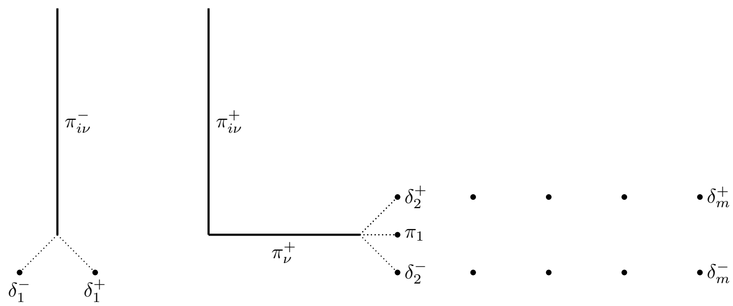

Figure 2.3: The unitary dual of $\mathrm{SL}_2(\mathbb{R})$

Let us include an explicit construction of the principal series $\widehat{G}^{\pm}_{\text{prin}}$ and complementary series $\widehat{G}_{\text{comp}}$ from [55, p. VII.1]. For this we recall the *Iwasawa decomposition* $G = ANK$ of G, where $K = \mathrm{SO}(2)$ as above and

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}, \qquad A = \left\{ \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \;\middle|\; y \in \mathbb{R}_{>0} \right\},$$

(cf. [32, §11.1]). For an element $a_y = \operatorname{diag}(y, y^{-1})$ of A we write $\log a_y = \log y$. Furthermore, let $M = \{\pm \mathbb{1}_2\}$ and $B = MAN$ be a so-called *Borel subgroup* of G. We note that M has two irreducible representations

$$\rho^+(\pm \mathbb{1}_2) = 1, \qquad \rho^-(\pm \mathbb{1}_2) = \pm 1.$$

For $\nu \in \mathbb{C}$ we get a (possibly non-unitary) representation $\rho^{\pm}_\nu$ of B by

$$\rho^{\pm}_\nu(man) = \rho^{\pm}(m) e^{\nu \log a}.$$

We will use this representation of B to induce a representation of G. Consider the space

$$V^{\pm}_\nu = \{F \in C(G) \mid \forall x \in G : F(xman) = e^{-(\nu+1/2)\log a} \rho^{\pm}(m) F(x)\}$$

equipped with the norm

$$||F||^2 = \int_K |F(k)|^2 dk.$$

G acts on this space via

$$\pi^{\pm}_\nu(g) F(x) = F(g^{-1}x).$$

Completing $V^{\pm}_\nu$ with respect to $||\cdot||$ yields a Hilbert space $H^{\pm}_\nu$ and $\pi^{\pm}_\nu$ continues to a (possibly non-unitary) representation on $H^{\pm}_\nu$. The representations $\pi^{\pm}_{i\nu}$ with $\nu \in \mathbb{R}$ are unitary and, with the exception of $\pi_0^- \cong \delta_1^+ \oplus \delta_1^-$, irreducible. The representations π^+_ν with $\nu \in (0, \frac{1}{2})$ are irreducible and unitary with respect to a suitable inner product on $H^{\pm}_\nu$.
Next we want to describe the Plancherel measure on $\widehat{\mathrm{SL}_2(\mathbb{R})}$:

Theorem 2.6.2. *The Plancherel measure on the unitary dual of* $\mathrm{SL}_2(\mathbb{R})$ *is given on the principal series* $\widehat{G}^{\pm}_{\mathrm{prin}}$ *and (limits of) discrete series* $\widehat{G}_{\mathrm{disc}}$ *by*

$$d\mu_{\mathrm{Pl}}(\pi^{+}_{i\nu}) = \nu\tanh(\pi\nu)d\nu,\ d\mu_{\mathrm{Pl}}(\pi^{-}_{i\nu}) = \nu\coth(\pi\nu)d\nu,\ \mu_{\mathrm{Pl}}(\{\delta^{\pm}_{m}\}) = m-1,$$

while the complementary series $\widehat{G}_{\mathrm{comp}}$ *and the trivial representation* π_1 *are not contained in the support of* μ_{Pl}.

Proof. [42, p.248] □

The above theorem shows that the set of tempered representations of $\mathrm{SL}_2(\mathbb{R})$ is given by

$$\widehat{G}_{\mathrm{temp}} = \widehat{G}^{+}_{\mathrm{prin}} \cup \widehat{G}^{-}_{\mathrm{prin}} \cup \{\delta^{\pm}_{m} \mid m \geq 2\}.$$

Remark 2.6.3. *Note that we have changed the normalization chosen in [42, p.248] to fit the normalization of [32, Thm. 11.3.1].*

2.7 The Selberg Trace Formula

After introducing the length spectrum $\mathrm{Spec}_L(X)$ and the Laplace spectrum $\mathrm{Spec}_\Delta(X)$ of a smooth closed hyperbolic surface X we now want to discuss the Selberg trace formula, which is a tool to compare these spectra. Our main source on this topic is [32].
Let G be a locally compact group. A subgroup $\Gamma \subset G$ is called *cocompact*, if the quotient $\Gamma\backslash G$ is compact. A discrete subgroup $\Gamma \subset G$ such that $\Gamma\backslash G$ carries a G-invariant Radon measure μ with $\mu(\Gamma\backslash G) < \infty$ is called a *lattice*.

Theorem 2.7.1. *Let* Γ *be a cocompact lattice. Then the representation*

$$R : L^2(\Gamma\backslash G) \to L^2(\Gamma\backslash G), \qquad R_y\phi(x) = \phi(xy)$$

decomposes into a direct sum of irreducible representations,

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\widehat{G}} N_\Gamma(\pi)H_\pi, \tag{2.10}$$

where each representation $\pi \in \widehat{G}$ *appears with finite multiplicity* $N_\Gamma(\pi) \in \mathbb{N}_0$ *in (2.10).*

Proof. [32, Thm. 9.2.2] □

We write $\widehat{G}_\Gamma$ for the set of representations, which appear with non-zero multiplicity in the decomposition (2.10). For $\gamma \in G$ we let

$$G_\gamma = \{x \in G \mid x\gamma x^{-1} = \gamma\}$$

be the centralizer of $\gamma \in G$.

Theorem 2.7.2 (Trace Formula)**.** *Let $\Gamma \subset G$ be a cocompact lattice and let $f \in C_c^\infty(G)$. For every $\pi \in \widehat{G}_\Gamma$ the operator $\pi(f)$ is of trace class and*

$$\sum_{\pi \in \widehat{G}_\Gamma} N_\Gamma(\pi) \operatorname{tr} \pi(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f), \tag{2.11}$$

where the sum on the right-hand side runs over all conjugacy classes $[\gamma]$ in the group Γ and $\mathcal{O}_\gamma(f)$ denotes the orbital integral

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx.$$

Proof. See [32, Thm. 9.3.2] and the notes at the end of Chapter 9 in [32]. □

Let us now restrict to the case $G = \mathrm{SL}_2(\mathbb{R})$. Using the Iwasawa decomposition from Section 2.6 we may identify the quotient G/K with the upper half-plane $\mathcal{H}$ via

$$n_x a_y = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \longleftrightarrow x + iy \in \mathcal{H}.$$

Under this identification the left action of G on G/K agrees with the action of G on $\mathcal{H}$ described in Section 2.1. Hence, we can write any smooth closed hyperbolic surface $X = \Gamma \backslash \mathcal{H}$ as a double quotient $X = \Gamma \backslash G/K$.

Remark 2.7.3. *By our choice of the normalization of the Haar measure, we have* $\operatorname{vol}(\Gamma\backslash\mathcal{H}) = \operatorname{vol}(\Gamma\backslash G)$ *for* $\Gamma \subset G$ *a cocompact lattice.*

We can use (2.10) to give a decomposition of the L^2-space

$$L^2(X) \cong \bigoplus_{\pi \in \widehat{G}} N_\Gamma(\pi) H_\pi^K, \tag{2.12}$$

where

$$H_\pi^K = \{v \in H_\pi \mid \pi(k)v = v \ \forall k \in K\}$$

is the subspace of K-invariant vectors. A representation $\pi \in \widehat{G}$ such that $H_\pi^K \neq \{0\}$ is called a *representation with a K-invariant vector.* The set of representations with a K-invariant vector is denoted $\widehat{G}_K$. These are given by

$$\widehat{G}_K = \{\pi_1\} \cup \widehat{G}_{\mathrm{comp}} \cup \widehat{G}^+_{\mathrm{prin}}.$$

(see [55, Chap. II]). Using our description for the representations π_ν^+ from Section 2.6 we quickly see that any K-invariant vector $F_\nu \in H_\nu^+$ is of the form

$$F_\nu(ank) = F_0 e^{-(\nu+1/2)\log a}$$

with $F_0 = F(\mathbb{1}_2) \in \mathbb{C}$ some constant. Since the Laplace operator Δ on the upper half-plane is given by $\Delta = y^2(\partial_x^2 + \partial_y^2)$, these K-invariant vectors are also eigenfunctions of the Laplacian[6],

$$\Delta F_\nu = \left(\frac{1}{4} - \nu^2\right) F_\nu.$$

Thus, (2.12) gives the following representation-theoretic interpretation of the Laplace-spectrum:

- The trivial representation π_1 corresponds to the eigenvalue $\lambda_0 = 0$ of multiplicity 1,
- a complementary series representation $\pi_\nu^+ \in \widehat{G}_{\text{comp}}$ corresponds to a small eigenvalue $\lambda = \frac{1}{4} - \nu^2 < \frac{1}{4}$ and the corresponding multplicities agree, i.e. $m(\lambda) = N_\Gamma(\pi_\nu^+)$,
- a principal series representation $\pi_{i\nu}^+ \in \widehat{G}_{\text{prin}}^+$ corresponds to a Laplace eigenvalue $\lambda = \frac{1}{4} + \nu^2 \geq \frac{1}{4}$ and we have $m(\lambda) = N_\Gamma(\pi_{i\nu}^+)$.

Finally, we note that the map defined by sending any representation $\pi \in \widehat{G}_K$ to the corresponding Laplace eigenvalue defines a homeomorphism $\widehat{G}_K \cong [0, \infty)$. If we let

$$L^2(\Gamma\backslash\mathcal{H}) = \mathbb{C} \oplus \bigoplus_{k=0}^{\infty} N_\Gamma(\pi_{i\nu_k}^+)(H_{i\nu_k}^+)^K$$

we get the following more explicit version of the Selberg trace formula:

Theorem 2.7.4. *Let $\varepsilon > 0$ and let h be a holomorphic function on the strip $\{|\operatorname{im}(z)| < \frac{1}{2} + \varepsilon\}$. Suppose that h is even, i.e. $h(z) = h(-z)$ and that $h(z) = O(|z|^{-2-\varepsilon})$ as $|z| \to \infty$. Then one has*

$$\sum_{k=0}^{\infty} h(\nu_k) = \frac{\operatorname{vol}(\Gamma\backslash G)}{4\pi} \int_{\mathbb{R}} r h(r) \tanh(\pi r) dr + \sum_{[\gamma] \neq 1} \frac{l_{\gamma_0}}{e^{l_\gamma/2} - e^{-l_\gamma/2}} \widehat{h}(l_\gamma). \quad (2.13)$$

Proof. [32, Thm. 11.4.1] □

If we apply (2.13) to $h_t(\nu) = e^{-t(\frac{1}{4}+\nu^2)}$, we can express the heat kernel $\theta(t) = \operatorname{tr}\exp(-t\Delta)$ in terms of the length spectrum,

$$\theta(t) = \sum_{[\gamma]} \frac{l_{\gamma_0}}{e^{l_\gamma/2} - e^{-l_\gamma/2}} \frac{e^{-t/4 - l_\gamma^2/4t}}{\sqrt{4\pi t}} + \varphi_0(t) \operatorname{vol}(X), \quad (2.14)$$

where $\varphi_0(t) = (4\pi)^{-1}\mu_{\text{Pl}}(e^{-t\lambda})$ is the fundamental solution of the heat equation at the origin. A careful analysis of (2.14) yields the following:

[6] More precisely, the Laplace operator agrees with the Casimir element Ω_G restricted to K-invariant vectors and the Casimir element acts on the K-invariant subspace of an irreducible representation by $y^2(\partial_x^2 + \partial_y^2)$.

Theorem 2.7.5 (Huber)**.** *Two smooth closed hyperbolic surfaces have the same length spectrum if and only if they have the same Laplace spectrum.*

Proof. [17, Thm. 9.2.9] □

In Section 3.5 we will see that Huber's theorem can be used to construct examples of isospectral Riemann surfaces, which are not isometric.

2.8 The Selberg Zeta Function

We will now describe some elementary properties of the Selberg zeta function. We refer the reader to [32, §11] for further details.
Recall from Section 2.2 that the number of geodesics of bounded length is finite. Hence, one may study the counting function for the length spectrum

$$\pi_\Gamma(x) = \{[\gamma] \in \Gamma \mid l_\gamma \leq x\}.$$

For the study of the asymptotics of the counting function π_Γ Selberg [87] introduced the *Selberg zeta function* ζ_Γ given by

$$\zeta_\Gamma(s) = \prod_\gamma \prod_{k \geq 0} (1 - e^{-(s+k)l_\gamma}), \tag{2.15}$$

where the first product runs over the prime geodesics of $\Gamma\backslash\mathcal{H}$.

Theorem 2.8.1. *The product representation (2.15) for $\zeta_\Gamma(s)$ converges for* $\mathrm{Re}(s) > 1$ *and extends to an entire function with the following zeroes:*

- *For each $k \in \mathbb{N}$ a zero at $s = -k$ of multiplicity $2(g-1)(2k+1)$, where g is the genus of $\Gamma\backslash\mathcal{H}$,*
- *For every eigenvalue $\lambda_k = \frac{1}{4} + \nu_k^2$ of the Laplacian Δ a zero at $s = \frac{1}{2} \pm i\nu_k$ of multiplicity $N_\Gamma(\pi^+_{i\nu_k})$.*

Proof. [32, Thm. 11.6.1] □

The holomorphic continuation of ζ_Γ is achieved by using the Selberg trace formula to express the logarithmic derivative of the Selberg zeta function

$$\Lambda_\Gamma(s) = \frac{\zeta_\Gamma'(s)}{\zeta_\Gamma(s)} = \sum_{[\gamma]} \frac{l_{\gamma_0}}{e^{l_\gamma/2} - e^{-l_\gamma/2}} e^{-(s-1/2)l_\gamma}, \ \mathrm{Re}(s) > 1, \tag{2.16}$$

in terms of the Laplace spectrum of $\Gamma\backslash\mathcal{H}$. Now, applying the standard machinery from analytic number theory to ζ_Γ allows one to derive $\pi_\Gamma(x) \sim \frac{e^{2x}}{2x}$ (see e.g. [4, §5.4.2]).

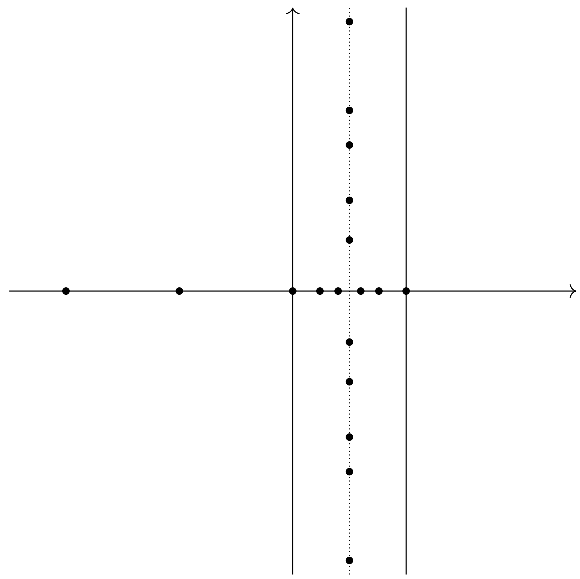

Figure 2.4: The zeroes of the Selberg zeta function ζ_Γ.

We note that the non-trivial zeroes of ζ_Γ are contained in the critical strip

$$S = \{s \in \mathbb{C} \mid 0 \leq \mathrm{Re}(s) \leq 1\}$$

and either lie in the interval $[0,1]$ or on the critical line $\frac{1}{2} + i\mathbb{R}$. Thus, the Riemann hypothesis holds for the Selberg zeta function if and only if $\Gamma\backslash\mathcal{H}$ does not have non-trivial small eigenvalues, which is expected in certain arithmetic scenarios (cf. Section 5.3).

2.9 Number-Theoretic Preliminaries

We need some standard terminology and results from algebraic number theory. The presented material can be found in most standard textbooks on algebraic number theory such as [71].

Let F be a *number field*, i.e. a field extension of $\mathbb{Q}$ of finite degree $n = [F : \mathbb{Q}]$. Let $\sigma_1, ..., \sigma_{r_1}$ be the real embeddings and $\sigma_{r_1+1}, ..., \sigma_{r_1+2r_2}$ the complex embeddings of F respectively, where the numbering is chosen so that $\overline{\sigma_{r_1+k}} = \sigma_{r_1+r_2+k}$ for $k = 1, ..., r_2$. These embeddings will be referred to as the *infinite places* of F and we write Ω_∞ for the set of infinite places. Any prime ideal $\mathfrak{p}$ of F comes with a valuation $v_\mathfrak{p}$ and an absolute value $|x|_\mathfrak{p} = p^{-v_\mathfrak{p}(x)}$, which gives an embedding $\sigma_\mathfrak{p}$ of F into a p-adic field $F_\mathfrak{p}$. We will refer to the prime ideals as the *finite places* of F. We denote the set of finite places of F by Ω_f and write $\Omega = \Omega_f \cup \Omega_\infty$ for the set of places of F. A *proper embedding* of F is a ring homomorphism $\iota : F \to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ such that the coordinate projections give all real embeddings and the complex embeddings up to complex conjugation. An *order* $\mathfrak{o}$ in F is a subring of F containing 1 such that it is a finitely generated $\mathbb{Z}$-module of rank n. We write $\mathfrak{o}^\times$ for the units in the order $\mathfrak{o}$. For ι a proper embedding of F the number

$$d(\mathfrak{o}) = \mathrm{vol}(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}/\iota(\mathfrak{o}))$$

is called the *absolute discriminant* of the order $\mathfrak{o}$. Note that $d(\mathfrak{o})$ does not depend on the chosen proper embedding. For an order $\mathfrak{o}$, we let $I(\mathfrak{o})$ be the set of all finitely generated $\mathfrak{o}$-submodules in F. Then, according to the Jordan-Zassenhaus theorem the set of isomorphism classes $[I(\mathfrak{o})]$ of $I(\mathfrak{o})$ is finite. We let $h(\mathfrak{o})$ be the cardinality of $[I(\mathfrak{o})]$ and call it the *class number* of $\mathfrak{o}$. There exists a unique maximal order in F, denoted $\mathfrak{o}_F$, which is called the *ring of integers* of F. We write $h_F = h(\mathfrak{o}_F)$ and $d_F = d(\mathfrak{o}_F)$ for the class number and the discriminant of $\mathfrak{o}_F$ respectively. We write $\mathfrak{o}_{F,\mathfrak{p}}$ for the completion of $\mathfrak{o}_F$ in $F_\mathfrak{p}$ and denote the uniformizer of $\mathfrak{o}_{F,\mathfrak{p}}$ by $\pi_\mathfrak{p}$. For an order $\mathfrak{o}$ we call the set

$$\mathfrak{f}_\mathfrak{o} = \{x \in \mathfrak{o} \mid x\mathfrak{o}_F \subset \mathfrak{o}\}$$

the *conductor* of $\mathfrak{o}$. It is an ideal in $\mathfrak{o}_F$ and $\mathfrak{o}$. The group of units $\mathfrak{o}_F^\times$ in $\mathfrak{o}_F$ is described by

Theorem 2.9.1 (Dirichlet's unit theorem)**.** *The group of units $\mathfrak{o}_F^\times$ is the direct product of the finite cyclic group of roots of unity in F and a free abelian group of rank $r_F = r_1 + r_2 - 1$, i.e.*

$$\mathfrak{o}_F^\times \cong \mathbb{Z}^{r_F} \times (\mathbb{Z}/w_F\mathbb{Z}),$$

where w_F is the number of roots of unity in F.

Proof. [71, Thm. I.7.4] □

Let $\varepsilon_1, ..., \varepsilon_{r_F}$ be a set of units, which generate the free part of $\mathfrak{o}_F^\times$. For $j = 1, ..., r_F + 1$ let $u_j = 1$, if σ_j is a real embedding and $u_j = 2$ otherwise. Then the $r_F \times (r_F + 1)$-matrix $\widetilde{R}$ given by

$$\widetilde{R}_{i,j} = u_j \log |\sigma_j(\varepsilon_i)|$$

has the property that the sum of any row is zero. Hence, the determinant of any submatrix of $\widetilde{R}$ obtained by deleting one column is independent of that column. This determinant is called the *regulator* R_F of F. The *absolute norm* $N(\mathfrak{a})$ of an ideal $\mathfrak{a}$ in $\mathfrak{o}_F$ is defined to be the cardinality of the finite quotient ring $\mathfrak{o}_F/\mathfrak{a}$. The series

$$\zeta_F(s) = \sum_{\mathfrak{a} \subset \mathfrak{o}_F} N(\mathfrak{a})^{-s},$$

where the sum runs over all ideals in $\mathfrak{o}_F$, converges absolutely for $\mathrm{Re}(s) > 1$ and extends to a meromorphic function on $\mathbb{C}$ with a single pole at $s = 1$ (see [71, Cor. 5.10]). This function ζ_F is called the *Dedekind zeta function* of the number field F. It has a zero of order r_F at $s = 0$ and the leading term of the Taylor expansion at this point is given by

$$\lim_{s \to 0} \frac{\zeta_F(s)}{s^{r_F}} = -\frac{h_F R_F}{w_F}. \tag{2.17}$$

Let L/F be a finite Galois extension of F with *Galois group*

$$\mathrm{Gal}(L/F) = \{\phi \in \mathrm{Aut}(L) \mid \forall x \in F : \phi(x) = x\}$$

and let

$$N_{L/F}(x) = \prod_{\sigma \in \mathrm{Gal}(L/F)} \sigma(x), \qquad \mathrm{Tr}_{L/F}(x) = \sum_{\sigma \in \mathrm{Gal}(L/F)} \sigma(x)$$

be the *relative norm* and *relative trace* of an element $x \in L$ respectively. The relative norm of an ideal is the ideal

$$N_{L/F}(\mathfrak{a}) = \langle N_{L/F}(x) \mid x \in \mathfrak{a} \rangle \subset F.$$

Note that we have $N_{L/\mathbb{Q}}(\mathfrak{a}) = (N(\mathfrak{a}))$ in the above notation. We write $\mathfrak{d}_{L/F}$ for the *relative discriminant* of the extension L/F and note that for a quadratic extension L/F we have the identity

$$d_L = N_{F/\mathbb{Q}}(\mathfrak{d}_{L/F}) d_F^2.$$

We say that a prime ideal $\mathfrak{P}$ of L *lies above* a prime ideal $\mathfrak{p}$ of F if $\mathfrak{p} = \mathfrak{P} \cap \mathfrak{o}_F$. The subset of prime ideals of L lying above some fixed prime ideal $\mathfrak{p}$ of $\mathfrak{o}_F$ is denoted by $\Omega_{f,\mathfrak{p}}$. The Galois group $\mathrm{Gal}(L/F)$ acts on $\Omega_{f,\mathfrak{p}}$. For a prime ideal $\mathfrak{P}$ of L we let

$$\mathrm{Gal}(L/F)_{\mathfrak{P}} = \{g \in \mathrm{Gal}(L/F) \mid g\mathfrak{P} = \mathfrak{P}\}$$

be the *decomposition group* of $\mathfrak{P}$. The action of the decomposition group on the residue class field $\mathbb{L}_{\mathfrak{P}} = \mathfrak{o}_L/\mathfrak{P}$ yields a group homomorphism

$$\phi_{\mathfrak{P}} : \mathrm{Gal}(L/F)_{\mathfrak{P}} \to \mathrm{Gal}(\mathbb{L}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}),$$

where $\mathbb{F}_{\mathfrak{p}} = \mathfrak{o}_F/\mathfrak{p}$. The kernel of this homomorphism $I_{\mathfrak{P}}$ is called the *inertia subgroup* in $\mathfrak{P}$. The homomorphism $\phi_{\mathfrak{P}}$ is surjective, so that we get an isomorphism $\mathrm{Gal}(L/F)_{\mathfrak{P}}/I_{\mathfrak{P}} \cong \mathrm{Gal}(\mathbb{L}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})$. Since $\mathbb{L}_{\mathfrak{P}}$ is a finite field, the Galois group $\mathrm{Gal}(\mathbb{L}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})$ is a finite cyclic group generated by the *Frobenius homomorphism*

$$\mathrm{Frob}_{\mathfrak{p}} : \mathbb{L}_{\mathfrak{P}} \to \mathbb{L}_{\mathfrak{P}}, \ x \mapsto x^q,$$

where $q = N(\mathfrak{p})$. If $\rho : \mathrm{Gal}(L/F) \to \mathrm{GL}(V)$ is a finite-dimensional complex representation, we get an action of $\mathrm{Gal}(L/F)_{\mathfrak{P}}/I_{\mathfrak{P}}$ on

$$V^{I_{\mathfrak{P}}} = \{v \in V : gv = v \ \forall g \in I_{\mathfrak{P}}\}.$$

and the characteristic polynomial $\det(1 - z\rho(\mathrm{Frob}_{\mathfrak{p}})|_{V^{I_{\mathfrak{P}}}})$ for $z \in \mathbb{C}$ only depends on $\mathfrak{p}$. The infinite product

$$\mathcal{L}(s, L/F, \rho) = \prod_{\mathfrak{p}} \det(1 - N(\mathfrak{p})^{-s}\rho(\mathrm{Frob}_{\mathfrak{p}})|_{V^{I_{\mathfrak{P}}}})^{-1},$$

where the product runs over all finite places of F, converges absolutely for $\mathrm{Re}(s) > 1$ and extends to a meromorphic function $\mathcal{L}(s, L/F, \rho)$ on $\mathbb{C}$ (see [71, Thm. VII.12.6]). The function $\mathcal{L}(s, L/F, \rho)$ is called the *Artin L-function* associated to the representation ρ. If the Galois extension L/F is clear from the context we will simply write $\mathcal{L}(s, \rho)$ for the Artin L-function. It is a famous conjecture of Artin (see e.g. [68]) that for any non-trivial irreducible representation ρ the Artin L-function is an entire function. Class field theory yields an answer in the abelian case:

Theorem 2.9.2. *Let L/F be a Galois extension such that $\mathrm{Gal}(L/F)$ is an abelian group. If $\rho : \mathrm{Gal}(L/F) \to \mathrm{GL}(V)$ is a non-trivial irreducible representation on a finite-dimensional complex vector space V, the Artin L-function $\mathcal{L}(s, \rho)$ is entire.*

Proof. This follows from [71, Thm.VII.10.6], as explained on page 527 in Neukirch's book [71]. □

Finally, we note that Artin L-functions appear in the factorization of Dedekind zeta functions:

Theorem 2.9.3. *For L/F a Galois extension one has*

$$\zeta_L(s) = \zeta_F(s) \prod_{\rho \neq 1} \mathcal{L}(s, \rho)^{\dim \rho},$$

where the product runs over the equivalence classes of all non-trivial irreducible representations ρ of $\mathrm{Gal}(L/F)$.

Proof. [71, Cor.VII.10.5] □

2.10 Arithmetic Fuchsian Groups

In this section we will describe the construction of arithmetic Fuchsian groups and some parts of the surrounding theory. For further details we refer the reader to standard textbooks such as [53, 60].
Let F be a field of characteristic $\neq 2$. A *quaternion algebra* $\mathcal{A}$ over F is a 4-dimensional central simple algebra over F. Each quaternion algebra is isomorphic to an algebra $\mathcal{A} = \left(\frac{a,b}{F}\right)$ over F spanned by a basis $\{1, i, j, k\}$ fulfilling the relations

$$i^2 = a,\ j^2 = b,\ k = ij = -ji,$$

where $a, b \in F^*$. On a quaternion algebra $\mathcal{A}$ there exists an involution given by

$$x = x_0 + x_1 i + x_2 j + x_3 k \mapsto \overline{x} = x_0 - x_1 i - x_2 j - x_3 k.$$

In terms of this involution one may define the *reduced trace* $\operatorname{trd}(x) = x + \bar{x}$ and the *reduced norm* $\operatorname{nrd}(x) = x \cdot \bar{x}$ of an element $x \in \mathcal{A}$. If each element in $\mathcal{A}$ has an inverse, $\mathcal{A}$ is called a *quaternion division algebra*. From now on, we let F be a number field. For any homomorphism σ of F into another field $\mathbb{K}$ define

$$\mathcal{A}^\sigma = \left(\frac{\sigma(a), \sigma(b)}{\sigma(F)}\right), \qquad \mathcal{A}^\sigma \otimes \mathbb{K} = \left(\frac{\sigma(a), \sigma(b)}{\mathbb{K}}\right).$$

The quaternion algebra $\mathcal{A}$ is said to be *ramified* at a place v of F, if $\mathcal{A}^{\sigma_v} \otimes F_v$ is a division algebra over F_v. Otherwise, $\mathcal{A}$ is said to be *unramified* at v. Let $\operatorname{Ram}(\mathcal{A})$ be the set of places, at which $\mathcal{A}$ ramifies. Let $\operatorname{Ram}_f(\mathcal{A})$ be the subset of finite places of $\operatorname{Ram}(\mathcal{A})$ and $\operatorname{Ram}_\infty(\mathcal{A})$ be the subset of infinite places in $\operatorname{Ram}(\mathcal{A})$. The *discriminant* of $\mathcal{A}$ is the ideal

$$\mathfrak{d}_\mathcal{A} = \prod_{\mathfrak{p} \in \operatorname{Ram}_f(\mathcal{A})} \mathfrak{p}.$$

of F. One can classify quaternion algebras over F according to their ramification behaviour:

Theorem 2.10.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two quaternion algebras over F. Then $\mathcal{A}_1$ is isomorphic to $\mathcal{A}_2$ as an F-algebra if and only if* $\operatorname{Ram}(\mathcal{A}_1) = \operatorname{Ram}(\mathcal{A}_2)$.

Proof. [60, Thm. 7.3.6] □

A quaternion algebra $\mathcal{A}$ is said to be *indefinite*, if there is at least one infinite place, at which $\mathcal{A}$ is unramified. We now further assume that F is a totally real field, i.e. $r_2 = 0$ and any quaternion algebra $\mathcal{A}$ will always be assumed to be unramified at exactly one infinite place. Let $\sigma_1, ..., \sigma_n$ be the n distinct embeddings of F into $\mathbb{R}$. Over $\mathbb{R}$, any quaternion algebra is either isomorphic to $M_2(\mathbb{R})$ or the Hamilton quaternions $\mathbb{H}$ and

- $\mathcal{A}^{\sigma_j} \otimes \mathbb{R} \cong M_2(\mathbb{R})$ for exactly one j,
- $\mathcal{A}^{\sigma_i} \otimes \mathbb{R} \cong \mathbb{H}$ for all $1 \leq i \leq n,\ i \neq j$.

We assume without loss of generality that σ_1 is the unique infinite place, at which $\mathcal{A}$ is unramified. We will in the following always identify F with $\sigma_1(F) \subset \mathbb{R}$ and $\mathcal{A}$ with $\mathcal{A}^{\sigma_1} \subset M_2(\mathbb{R})$, whenever this is necessary.
An *order* $\mathcal{O}$ in $\mathcal{A}$ is a subring of $\mathcal{A}$ containing 1, which is a $\mathbb{Z}$-submodule of rank $4n$. For $\mathfrak{p}$ a prime ideal of F we let $\mathcal{O}_\mathfrak{p} = \mathcal{O} \otimes \mathfrak{o}_{F,\mathfrak{p}}$ be the *localization* of the order $\mathcal{O}$ at $\mathfrak{p}$. Consider the group of units of reduced norm 1 of an order $\mathcal{O}$,

$$\mathcal{O}^1 = \{x \in \mathcal{O} \mid \operatorname{nrd}(x) = 1\}.$$

Then $\mathcal{O}^1$ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$ and the image $\Gamma(\mathcal{A}, \mathcal{O}) = P_{\sigma_1}\mathcal{O}^1$ under the projection map $P_{\sigma_1} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})/\{\pm \mathbb{1}_2\}$ is a Fuchsian group (see e.g.

[53, Thm. 5.2.7]). If $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian group, which is a subgroup of finite index in $\Gamma(\mathcal{A}, \mathcal{O})$ for some order $\mathcal{O}$ in $\mathcal{A}$, it is called a *Fuchsian group derived from the quaternion algebra* $\mathcal{A}$. If a Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is commensurable[7] with some $\Gamma(\mathcal{A}, \mathcal{O})$, it is called an *arithmetic Fuchsian group*. If Γ is an arithmetic Fuchsian group, we will call the the quotient space $\Gamma\backslash\mathcal{H}$ an *arithmetic surface*.

Theorem 2.10.2. *A subgroup* $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ *is an arithmetic Fuchsian group if and only if* $\Gamma^{(2)} = \langle \gamma^2 \mid \gamma \in \Gamma \rangle$ *is derived from a quaternion algebra.*

Proof. [53, Thm. 5.3.11] $\square$

We note that the wide commensurability[8] class of an arithmetic Fuchsian group can be classified by the *invariant trace field* $F(\Gamma) = \mathbb{Q}(\operatorname{tr} \Gamma^{(2)})$ and the *invariant quaternion algebra*

$$\mathcal{A}(\Gamma) = \left\{ \sum_{i=1}^{k} a_i \gamma_i \mid a_i \in F(\Gamma),\ \gamma_i \in \Gamma^{(2)} \right\}$$

as follows:

Theorem 2.10.3. *Let* Γ_1, Γ_2 *be arithmetic Fuchsian groups. Then* Γ_1 *and* Γ_2 *are commensurable in the wide sense, if and only if* $F(\Gamma_1) = F(\Gamma_2)$ *and there exists an* $F(\Gamma_1)$*-algebra isomorphism* $\phi : \mathcal{A}(\Gamma_1) \to \mathcal{A}(\Gamma_2)$.

Proof. [60, Thm. 8.4.6] $\square$

Arithmetic Fuchsian groups form an important source of cocompact lattices:

Theorem 2.10.4. *Suppose a Fuchsian group* Γ *is commensurable to some* $\Gamma(\mathcal{A}, \mathcal{O})$ *with* $\mathcal{O}$ *an order in a quaternion division algebra* $\mathcal{A}$. *Then the corresponding arithmetic surface* $\Gamma\backslash\mathcal{H}$ *is compact.*

Proof. [53, Thm. 5.4.1] $\square$

Arithmetic surfaces are rare among smooth closed hyperbolic surfaces, in the sense that the moduli space of hyperbolic surfaces of genus g only contains finitely many points, which correspond to arithmetic surfaces:

Theorem 2.10.5. *Let* $T > 0$ *be given. There are only finitely many conjugacy classes of arithmetic Fuchsian groups* Γ *such that* $\operatorname{vol}(\Gamma\backslash\mathcal{H}) < T$.

Proof. [60, Thm. 11.3.1] $\square$

This is a consequence of the following explicit formula for the covolumes of maximal orders:

[7] Recall that two subgroups Γ_1 and Γ_2 of a given group G are called *commensurable*, if $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2.

[8] Two subgroups Γ_1 and Γ_2 of a given group G are called *commensurable in the wide sense*, if Γ_1 and a conjugate of Γ_2 are commensurable.

Lemma 2.10.6. *Let $\mathcal{O}$ be a maximal order in a quaternion algebra $\mathcal{A}$ and $\Gamma = \Gamma(\mathcal{A}, \mathcal{O})$. Then*

$$\operatorname{vol}(\Gamma \backslash \operatorname{SL}_2(\mathbb{R})) = \frac{d_F^{3/2} \zeta_F(2) \prod_{\mathfrak{p} | \mathfrak{d}_{\mathcal{A}}} (N(\mathfrak{p}) - 1)}{(4\pi^2)^{n-1}}.$$

Proof. [60, §11.1] □

Fix a maximal order $\mathcal{O}$. A *principal congruence subgroup* (of $\mathcal{O}^1$) is a subgroup of $\mathcal{O}^1$ of the form

$$\mathcal{O}^1(\mathfrak{a}) = \{x \in \mathcal{O}^1 \mid x - 1 \in \mathfrak{a}\mathcal{O}\},$$

where $\mathfrak{a}$ is an ideal of F. A *congruence subgroup* Γ is a subgroup of $\mathcal{O}^1$, which contains some principal congruence subgroup $\mathcal{O}^1(\mathfrak{a})$. We will also call the resulting Fuchsian group $P_{\sigma_1}\Gamma \subset \operatorname{PSL}_2(\mathbb{R})$ a congruence subgroup. The quotient space $\Gamma \backslash \mathcal{H}$ will be called a *congruence surface.*

2.11 Benjamini-Schramm Convergence and Plancherel Convergence

In this section we introduce the notions of Benjamini-Schramm convergence and Plancherel convergence. For simplicity, we will restrict our discussion to cocompact torsion-free lattices in $\operatorname{SL}_2(\mathbb{R})$. For the more general setting we refer the reader to [1, 30].

As before, we let $G = \operatorname{SL}_2(\mathbb{R})$ and Γ be a cocompact lattice in G. Recall from Theorem 2.7.1 that the representation R on $L^2(\Gamma \backslash G)$ given by $R_y\phi(x) = \phi(xy)$ decomposes into a direct sum of irreducible representations

$$L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \widehat{G}} N_\Gamma(\pi) H_\pi,$$

where the multiplicities $N_\Gamma(\pi)$ are finite. The *spectral measure* μ_Γ associated with Γ is then defined by

$$\mu_\Gamma = \sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \delta_\pi,$$

where δ_π is the Dirac measure for $\pi \in \widehat{G}$.

Definition 2.11.1. Let $(\Gamma_j)_{j \in \mathbb{N}}$ be a sequence of cocompact lattices in $\operatorname{SL}_2(\mathbb{R})$. We say that $(\Gamma_j)_{j \in \mathbb{N}}$ has the *limit multiplicity property*, if

1. for any Jordan-measurable set $A \subset \widehat{G}_{\text{temp}}$ one has

$$\lim_{j \to \infty} \frac{\mu_{\Gamma_j}(A)}{\operatorname{vol}(\Gamma_j \backslash G)} = \mu_{\text{Pl}}(A),$$

2. for any bounded subset $A \subset \widehat{G} \setminus \widehat{G}_{\mathrm{temp}}$ one has

$$\lim_{j\to\infty} \frac{\mu_{\Gamma_j}(A)}{\mathrm{vol}(\Gamma_j \backslash G)} = 0.$$

It is usually easier to establish the following closely related notion:

Definition 2.11.2. A sequence $(\Gamma_j)_{j\in\mathbb{N}}$ of cocompact lattices in G is called *Plancherel convergent* (or a *Plancherel sequence*), if for every $f \in C_c^\infty(G)$ we have

$$\frac{1}{\mathrm{vol}(\Gamma_j \backslash G)} \mu_{\Gamma_j}(\widehat{f}) \to \mu_{\mathrm{Pl}}(\widehat{f})$$

as $j \to \infty$. We will call a sequence $(X_j)_{j\in\mathbb{N}}$ of smooth hyperbolic surfaces $X_j = \Gamma_j \backslash \mathcal{H}$ Plancherel-convergent (or a Plancherel-sequence), if the associated sequence of lattices $(\Gamma_j)_{j\in\mathbb{N}}$ is Plancherel-convergent.

Theorem 2.11.3. *Let* $(\Gamma_j)_{j\in\mathbb{N}}$ *be a sequence of cocompact lattices in* $\mathrm{PSL}_2(\mathbb{R})$. *If* $(\Gamma_j)_{j\in\mathbb{N}}$ *is a Plancherel sequence, it has the limit multiplicity property.*

Proof. This is a well-known consequence of Sauvageot's density principle [84]. □

There exists an even weaker notion of convergence, which implies Plancherel convergence in many scenarios (cf. [1]).

Definition 2.11.4. We say that a sequence $(X_j)_{j\in\mathbb{N}}$ of smooth closed hyperbolic surfaces is *Benjamini-Schramm convergent* (or BS-*convergent*), if for every $R > 0$ one has

$$\lim_{j\to\infty} \frac{\mathrm{vol}(\{p \in X_j : r_p(X_j) \leq R\})}{\mathrm{vol}(X_j)} = 0.$$

Alternatively, we will say that the associated sequence of cocompact torsion-free lattices $(\Gamma_j)_{j\in\mathbb{N}}$ defined by $X_j = \Gamma_j \backslash \mathcal{H}$ is Benjamini-Schramm convergent.

We call a sequence of lattices $(\Gamma_j)_{j\in\mathbb{N}}$ in G *uniformly discrete*, if there exists a unit neighborhood U in G such that $x^{-1}\Gamma_j x \cap U = \{1\}$ for all $x \in G$ and $j \in \mathbb{N}$. From the characterization of closed geodesics given in Proposition 2.2.3 one may quickly check that a sequence of cocompact torsion-free lattices is uniformly discrete if and only if the systoles $\{l_s(X_j) \mid j \in \mathbb{N}\}$ of the associated surfaces $X_j = \Gamma_j \backslash \mathcal{H}$ are uniformly bounded away from zero. We collect the following important result, which links the notions of convergence just introduced:

Theorem 2.11.5. *Let* $(\Gamma_j)_{j\in\mathbb{N}}$ *be a sequence of cocompact torsion-free lattices in* G.

1. *If* $(\Gamma_j)_{j\in\mathbb{N}}$ *is Plancherel convergent, it is* BS-*convergent.*
2. *If* $(\Gamma_j)_{j\in\mathbb{N}}$ *is* BS-*convergent and uniformly discrete, it is Plancherel convergent.*

Proof. [30, Thm. 2.6] $\square$

To investigate examples of Plancherel-convergent and BS-convergent sequences, we note the following explicit criterion for Plancherel convergence:

Proposition 2.11.6. *A sequence of smooth closed hyperbolic surfaces $(X_j)_{j\in\mathbb{N}}$ is Plancherel-convergent if and only if for each $c > 0$ one has*

$$\lim_{j\to\infty} \frac{1}{\mathrm{vol}(X_j)} \sum_{l_\gamma \le c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} = 0, \tag{2.18}$$

where the sum runs over the lengths in $\mathrm{Spec}_L(X_j)$.

Proof. We start by showing that (2.18) implies that $(X_j)_{j\in\mathbb{N}}$ is a Plancherel-sequence. Let $f \in C_c^\infty(G)$ with support sitting in some compact set K_f. Let us abbreviate $B_j = |\,\mathrm{vol}(X_j)^{-1}\mu_{\Gamma_j}(\widehat{f}) - \mu_{\mathrm{Pl}}(\widehat{f})|$. Applying the trace formula from Section 2.7 to B_j gives

$$B_j = \mathrm{vol}(X_j)^{-1} \left| \sum_{[\gamma]\neq 1} l_{\gamma_0} \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx \right|. \tag{2.19}$$

Since the orbital integral is conjugation-invariant, we may assume

$$\gamma = \begin{pmatrix} e^{l_\gamma/2} & 0 \\ 0 & e^{-l_\gamma/2} \end{pmatrix}.$$

Let $G = ANK$ be the Iwasawa decomposition from Section 2.6. Since K_f is compact and the matrix trace $\mathrm{tr} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ is continuous, there exists some constant $c = c(f) > 0$ such that

$$B_j \le \mathrm{vol}(X_j)^{-1} \sum_{l_\gamma \le c} m(l_\gamma) l_{\gamma_0} \int_{G_\gamma\backslash G} \left| f\left(x^{-1} \begin{pmatrix} e^{l_\gamma/2} & 0 \\ 0 & e^{-l_\gamma/2} \end{pmatrix} x \right) \right| dx, \tag{2.20}$$

where the sum on the right-hand side runs over the lengths in the length spectrum of X_j. A quick computation shows $G_\gamma = A$ and we get

$$\begin{aligned} \int_{G_\gamma\backslash G} |f(x^{-1}\gamma x)| dx &= \int_{\mathbb{R}} \int_0^{2\pi} |f(k_\theta^{-1} n_y^{-1} \gamma n_y k_\theta)| dy d\theta \\ &= \frac{1}{2\sinh(l_\gamma/2)} \int_{\mathbb{R}} \int_0^{2\pi} \left| f\left(k_\theta^{-1} \begin{pmatrix} e^{l_\gamma/2} & y \\ 0 & e^{-l_\gamma/2} \end{pmatrix} k_\theta \right) \right| dy d\theta. \end{aligned}$$

The integrand on the right-hand side can only be non-zero if

$$\begin{pmatrix} e^{l_\gamma/2} & y \\ 0 & e^{-l_\gamma/2} \end{pmatrix} \in KK_fK =: K_f'.$$

Since multiplication in G is continuous, the set K_f' is compact. Hence, there exists a constant M_f solely depending on f such that

$$\int_{G_\gamma \backslash G} |f(x^{-1}\gamma x)| dx \leq \frac{M_f}{\sinh(l_\gamma/2)}. \tag{2.21}$$

Plugging (2.21) into (2.20) yields

$$B_j \leq \frac{M_f}{\mathrm{vol}(X_j)} \sum_{l_\gamma \leq c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)}. \tag{2.22}$$

and thus (2.18) implies $\lim_{j\to\infty} B_j = 0$.
Let us now prove the converse direction. Let $(X_j)_{j\in\mathbb{N}}$ be a Plancherel-convergent sequence of smooth closed hyperbolic surfaces and let $c > 0$ be given. We will prove in Theorem 4.1.2 that for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ one has

$$\lim_{j\to\infty} \frac{1}{\mathrm{vol}(X_j)} \sum_{[\gamma]\neq 1} \frac{l_{\gamma_0}}{2\sinh(l_\gamma/2)} e^{-(s-1/2)l_\gamma} = 0. \tag{2.23}$$

If we take $s = 3/2$ in (2.23), we already get (2.18) by observing

$$\sum_{[\gamma]\neq 1} \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} e^{-l_\gamma} \geq e^{-c} \sum_{l_\gamma \leq c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)}.$$

This concludes the proof. □

This allows for the following interesting reformulation of BS-convergence in the uniformly discrete case:

Corollary 2.11.7. *Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a uniformly discrete sequence of torsion-free cocompact lattices in $\mathrm{SL}_2(\mathbb{R})$. Then $(\Gamma_j)_{j\in\mathbb{N}}$ is BS-convergent if and only if for any $c > 0$ one has*

$$\lim_{j\to\infty} \frac{\pi_{\Gamma_j}(c)}{\mathrm{vol}(\Gamma_j\backslash\mathcal{H})} = 0,$$

where π_{Γ_j} is the counting function of the length spectrum of $X_j = \Gamma_j\backslash\mathcal{H}$.

Proof. Let $(\Gamma_j)_{j\in\mathbb{N}}$ be uniformly discrete and let $c > 0$ be given. Then we can find constants $C_1, C_2 > 0$ such that

$$C_1 \leq \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \leq C_2$$

holds for any length $l_\gamma \leq c$ in $\mathrm{Spec}_L(X_j)$. This already given the claim in view of Proposition 2.11.6. □

Let us discuss a few examples (and non-examples) of Plancherel convergent and BS-convergent sequences to gain intuition and demonstrate structural features of the theory.

Example 2.11.8. We start with the construction of a sequence of smooth closed hyperbolic surfaces $(X_j)_{j\in\mathbb{N}}$, which is not BS-convergent. For $\varepsilon < 1$ consider the Y-piece Y_ε with all three boundary geodesics of length equal to ε. If we glue $2j-2$ copies of Y_ε along their respective boundary geodesics we get a surface X_j of volume $\mathrm{vol}(X_j) = 4\pi(j-1)$. According to Theorem 2.3.1 this sequence cannot contain any further short geodesics, so that $(X_j)_{j\in\mathbb{N}}$ is uniformly discrete. Since

$$\frac{\pi_{\Gamma_j}(\varepsilon)}{\mathrm{vol}(X_j)} = \frac{3j-3}{4\pi(j-1)} = \frac{3}{4\pi}$$

this sequence cannot be BS-convergent by Corollary 2.11.7.

Example 2.11.9. Next we construct a Plancherel sequence. Let $\mathcal{A}$ be a quaternion algebra over $\mathbb{Q}$ unramified at the infinite place such that $\mathrm{Ram}(\mathcal{A}) \neq \emptyset$. Hence, we have $\mathcal{A} \neq M_2(\mathbb{Q})$. Fix a maximal order $\mathcal{O}$ in $\mathcal{A}$ and let $(p_k)_{k\in\mathbb{N}}$ be a sequence of pairwise non-identical primes. For $j \in \mathbb{N}$ we let I_j be the ideal given by the product $I_j = (p_1)....(p_j)$. We claim that the sequence defined by $\Gamma_j = \mathcal{O}^1(I_j)$ is Plancherel-convergent. To see this let us fix some $c > 0$. The traces of elements of $\mathcal{O}$ are contained in $\mathbb{Z}$, so that there exist only finitely many traces below $2\cosh(c/2)$, say

$$\mathrm{tr}\,\gamma_1 = t_1, ..., \mathrm{tr}\,\gamma_q = t_q. \tag{2.24}$$

for $\gamma_1, ..., \gamma_q \in \mathcal{O}^1$. Now, if an element γ_s of trace t_s with $s \in \{1, ..., q\}$ is contained in $\mathcal{O}^1(I_j)$ we get from the congruence condition $\gamma_s \in 1 + I_j\mathcal{O}$ that

$$t_s - 2 = xp_1 \cdots p_j \tag{2.25}$$

for some $x \in \mathbb{Z}$. This can only happen for finitely many $j \in \{1, .., j_0\}$. Consequently, any $X_j = \Gamma_j\backslash\mathcal{H}$ with j large enough can neither contain a torsion-element nor a geodesic of length $l \leq c$ and Corollary 2.11.7 implies that $(X_j)_{j\in\mathbb{N}}$ is Plancherel-convergent.

Example 2.11.10. More generally, we will see in Chapter 3 that any sequence of cocompact torsion-free congruence subgroups $(\Gamma_j)_{j\in\mathbb{N}}$ with $\mathrm{vol}(\Gamma_j\backslash G) \to \infty$ is Plancherel-convergent.

Chapter 3

Spectra of Arithmetic Surfaces and Plancherel Convergence

3.1 Summary

We have seen at the end of the last chapter that a certain sequence of principal congruence subgroups in a fixed maximal order is Plancherel convergent. It is natural to ask, whether this also holds for arbitrary sequences of (non-conjugate) congruence subgroups. For this reason we need a more in-depth understanding of the length spectra of arithmetic surfaces. We begin this chapter by collecting information about the length spectra of arithmetic surfaces from the literature and recast them in a language suitable for our purpose. Based on this information we then go on to prove

Theorem 3.1.1. *Let* $(\Gamma_j)_{j\in\mathbb{N}}$ *be a sequence of torsion-free congruence subgroups of* $\mathrm{PSL}_2(\mathbb{R})$ *with* $\mathrm{vol}(\Gamma_j\backslash G)\to\infty$*. Then* $(\Gamma_j)_{j\in\mathbb{N}}$ *is Plancherel convergent.*

The proof of Theorem 3.1.1 will be given in Section 3.7. We note that the analogue of Theorem 3.1.1 for arbitrary sequences of arithmetic surfaces is known to be wrong (cf. [1, p.716]).

3.2 Hyperbolic Transformations and Salem Numbers

The goal of this section is to describe the lengths, which appear in the length spectrum of an arithmetic surface. The material covered here can for example be found in [60, §12] and [45].
Recall from Section 2.2 that the lengths of closed geodesics on the Riemann

surface $X = \Gamma\backslash\mathcal{H}$ are of the form $2\log\tau_\gamma$, where

$$\tau_\gamma^{\pm 1} = \frac{\operatorname{tr}\gamma \pm \sqrt{(\operatorname{tr}\gamma)^2 - 4}}{2} \tag{3.1}$$

are the eigenvalues of the hyperbolic transformation $\gamma \in \Gamma$. From now on, we assume that $\Gamma = \Gamma(\mathcal{A}, \mathcal{O})$ is always a Fuchsian group derived from a quaternion algebra $\mathcal{A}$, unless mentioned otherwise. We let $F = F(\Gamma)$ be the invariant trace field with real embeddings $\sigma_1, ..., \sigma_n$, which are chosen so that $\mathrm{Ram}_\infty(\mathcal{A}) = \{\sigma_2, ..., \sigma_n\}$.

Proposition 3.2.1. *Let $\gamma = \sigma_1(x) \in \Gamma$ be a hyperbolic element. Then $\sigma_j(\mathrm{trd}(x))$ lies in the interval $(-2, 2)$ for $j = 2, ..., n$.*

Proof. For each $j \in \{2, ..., n\}$ we have an isomorphism $\mathcal{A}^{\sigma_j} \otimes \mathbb{R} \cong \mathbb{H}$. Let us fix such an isomorphism for $j = 2, ..., n$ and write

$$\sigma_j(x) = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}.$$

Since $\mathrm{nrd}(\sigma_j(x)) = 1$, we have

$$1 = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

and thus

$$|\sigma_j(\mathrm{trd}(x))| = |\operatorname{trd}(\sigma_j(x))| = 2|x_0| \leq 2.$$

□

Because of (3.1) the above proposition shows that all conjugates of τ_γ with the exception of τ_γ^{-1} lie on the unit circle. This observation connects lengths of closed geodesics to the following class of algebraic integers:

Definition 3.2.2. A *Salem number* is a real algebraic integer $\tau > 1$ such that all conjugates of τ except τ^{-1} lie on the unit circle.

Remark 3.2.3. *One usually requires that Salem numbers have at least one conjugate on the unit circle. This restriction is relevant for some problems such as the distribution of powers of τ modulo 1 (cf. [92, §3.4]), but does not play a role in our case.*

Example 3.2.4. The root $\tau_L = 1.17628...$ of the Lehmer polynomial

$$p_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

is the smallest known Salem number.

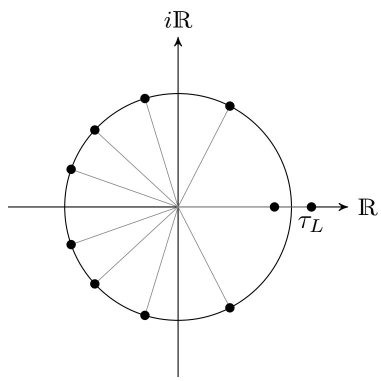

Figure 3.1: The Salem number $\tau_L = 1.17628...$ and its conjugates

On the other hand, there are plenty of examples of arithmetic surfaces, which contain a particular Salem number τ (see e.g. [60, Lemma 12.3.2]). As an immediate consequence of this discussion, we get

Theorem 3.2.5. *Let Γ be a torsion-free Fuchsian group derived from a quaternion algebra. Let l_γ be the length of a closed geodesic γ on the corresponding arithmetic surface $\Gamma\backslash\mathcal{H}$. Then $\tau_\gamma = \exp(l_\gamma/2)$ is a Salem number. Conversely, any Salem number τ is of the form $\tau = \exp(l_\gamma/2)$ for l_γ the length of a closed geodesic on an arithmetic surface X.*

It is an interesting observation by Stark [93] (and Chinburg [25]) that Salem numbers can be expressed in terms of special values of L-functions:

Theorem 3.2.6. *Let F be a totally real number field of degree $n > 1$. Let L be a relative quadratic extension of F having exactly two real places. Let $u = 2$, if L is generated by the square root of a unit in F and $u = 1$ otherwise. Let χ be the non-trivial character of $\mathrm{Gal}(L/F)$ and $\mathcal{L}(s,\chi)$ be the associated Artin L-function. Then $\mathcal{L}(s,\chi)$ vanishes to first order at $s = 0$ and*

$$\mathcal{L}'(0,\chi) = \frac{h_L 2^{n-2}\log\tau_s}{h_F u},$$

where τ_s is a unit of L, which together with $\mathfrak{o}_F^\times$ generates a subgroup of index $2u$ in $\mathfrak{o}_L^\times$.

Proof. [93, pp.63-88]. □

The algebraic integer τ_s from Theorem 3.2.6 will be referred to as *Stark unit*.

Corollary 3.2.7. *The Stark unit τ_s is a Salem number. Conversely, any Salem number τ in L is of the form $\tau = \tau_s^{k/2}$ for some $k \in \mathbb{N}$.*

Proof. [25]. □

3.3 Systoles of Arithmetic Surfaces

For Salem numbers we have the following well-known

Conjecture 3.3.1 (Salem)**.** *There exists some $\delta > 0$ such that each Salem number is larger than $1 + \delta$.*

This question has been extensively numerically investigated[1] by Boyd [11, 10], Mossinghoff [67] and others (see e.g. [41]), even though Lehmer had already found some of the smallest known Salem numbers [59]. Some of their findings are illustrated in Figure 3.2:

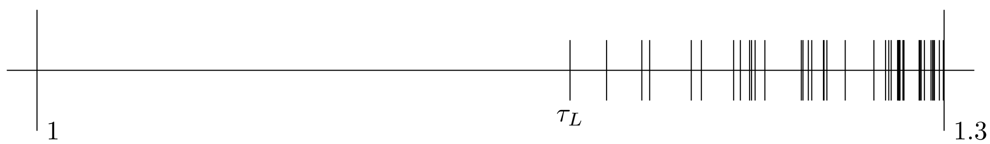

Figure 3.2: All Salem numbers $\tau < 1.3$ of degree ≤ 44.

Because of Theorem 3.2.5 and the following remark, the above conjecture is equivalent to

Conjecture 3.3.2 (Minimization problem)**.** *There exists a uniform positive constant C_0 such that any length of a closed geodesic on an arbitrary arithmetic surface is bounded from below by C_0.*

There exist partial results towards Salem's conjecture, which are sufficient for our purposes[2]:

Theorem 3.3.3. *There exists an absolute constant c_D such that for any systole l_s of an arithmetic surface $\Gamma\backslash\mathcal{H}$ with invariant trace field $F(\Gamma)$ of degree n we have*

$$\frac{1}{l_s} \leq c_D \log(n)^3.$$

Proof. [37, Theorem 1]. □

3.4 Bilu Equidistribution

When proving Plancherel convergence for congruence surfaces, we will be faced with estimating quantities such as

$$N(d_\tau) = \prod_{k=1}^{n} (\sigma_k(\tau + \tau^{-1})^2 - 4),$$

[1] We note that some of these investigations actually aimed at Lehmer's conjecture, which is more general than the above conjecture.

[2] Note that we are using a weaker version of what Dobrowolski has shown, as this suffices for our purposes.

where

$$d_\tau = (\tau + \tau^{-1})^2 - 4$$

is the discriminant of τ. Therefore, it is necessary to understand how the conjugates of a Salem number τ distribute on the unit circle $\mathbb{S}^1$. Fraczyk [43] realized that Bilu's equidistribution principle [8] can be utilized for this task. Restricted to the case of Salem numbers it states that a sequence of Salem numbers numbers $(\tau_j)_{j\in\mathbb{N}}$ with $[\mathbb{Q}(\tau_j):\mathbb{Q}] = 2n_j \to \infty$ and $\log \tau_j / n_j \to 0$ equidistributes on the circle, in the sense that for any $f \in C_c(\mathbb{C})$ one has

$$\lim_{j\to\infty} \frac{1}{2n_j} \sum_{\sigma\in\mathrm{Hom}(\mathbb{Q}(\tau_j),\mathbb{C})} f(\sigma(\tau_j)) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta.$$

Figure 3.3 visualizes this phenomenon for a certain sequence of Salem numbers $(\tau_j)_{j\in\mathbb{N}}$, which converges to the plastic number $\theta_0 \approx 1.32471$. The definition of this sequence and relevant sources are given in Appendix A.

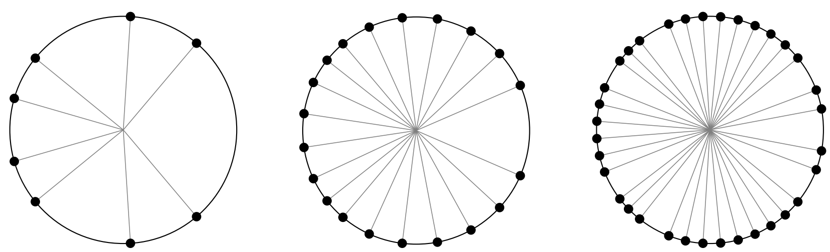

Figure 3.3: Non-real conjugates of τ_{10}, τ_{20} and τ_{40} respectively

Fraczyk made use of the estimates leading up to Bilu's equidistribution principle to derive bounds on the norm of the Weyl discriminant and the number of ideals of small norm. Since our proof will also use these estimates, we note that they form an independent and comparatively short part of his work.

Theorem 3.4.1. *Let $c > 0$ be some constant and let τ be a Salem number of degree $2n$ with $\tau \le c$. Then for each $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $\delta > 0$ there exist constants c_δ and $c_{\delta,s}$ so that*

1. $N(d_\tau) \le c_\delta(1+\delta)^n$,
2. $N(\mathfrak{d}_{L/F}) \le c_\delta(1+\delta)^n$,
3. $|\zeta_F(s)| \le c_{\delta,s}(1+\delta)^n$.

Proof. This is a direct consequence of Lemma 30 and Lemma 31 in [43]. □

3.5 Embedding Numbers of Orders

Until the end of this section, we let $\Gamma = \Gamma(\mathcal{A}, \mathcal{O})$ always be the group of norm 1 units of some order $\mathcal{O}$ sitting in a quaternion division algebra $\mathcal{A}$ over a totally real field F, such that $\mathcal{A}$ is unramified at exactly one infinite place of F. We first note that each $\gamma \in \Gamma$ gives an embedding of the relative quadratic extension $L = F(\tau_\gamma)$ into $\mathcal{A}$ via

$$\sigma_\gamma : L \to \mathcal{A},\ a + b\tau_\gamma \mapsto a + b\gamma, \qquad a, b \in F \tag{3.2}$$

One may readily verify that $\sigma_\gamma^{-1}(\sigma_\gamma(L) \cap \mathcal{O})$ defines an order in L.

Definition 3.5.1. An *optimal embedding* of an order $\mathfrak{o}$ in L into $\mathcal{O}$ is an embedding $\sigma : L \to \mathcal{A}$ such that $\sigma^{-1}(\sigma(L) \cap \mathcal{O}) = \mathfrak{o}$ holds.

The set of optimal embeddings of $\mathfrak{o}$ into $\mathcal{O}$ is denoted by $\Sigma(\mathfrak{o}, \mathcal{O})$. Any $x \in \mathcal{O}^\times$ acts on $\Sigma(\mathfrak{o}, \mathcal{O})$ by conjugation. For any subgroup $H \subset \mathcal{O}^\times$ we let $\Sigma(\mathfrak{o}, \mathcal{O})/H$ be the quotient under the action of H and write

$$m(\mathfrak{o}, \mathcal{O}; H) = |\Sigma(\mathfrak{o}, \mathcal{O})/H|$$

for the cardinality of this set. Let us also denote $\mathfrak{o}_\gamma = \mathfrak{o}_F + \tau_\gamma \mathfrak{o}_F$. One can express the multiplicity of the length of γ in terms of the embedding numbers $m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1)$:

Lemma 3.5.2. *The multiplicity $m(l_\gamma)$ of a length l_γ in the length spectrum of Γ is given by*

$$m(l_\gamma) = \sum_{\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L} m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1),$$

where the sum runs over all orders $\mathfrak{o}$ with $\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L$, which can be embedded into $\mathcal{A}$.

Proof. Any $\gamma \in \Gamma$ defines an embedding σ_γ of L into $\mathcal{A}$ through (3.2). Then $\mathfrak{o} = \sigma_\gamma^{-1}(\sigma_\gamma(L) \cap \mathcal{O})$ defines an order in L, which contains $\mathfrak{o}_\gamma$. Since any geodesic in $X = \Gamma \backslash \mathcal{H}$ is described by a Γ-conjugacy class $[\gamma']$ of some element γ' in Γ, we have

$$m(l_\gamma) \leq \sum_{\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L} m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1).$$

Now, let $\mathfrak{o}$ be an order with $\mathfrak{o}_\gamma \subset \mathfrak{o}$ and $\sigma : L \to \mathcal{A}$ be an optimal embedding of $\mathfrak{o}$. Let $\gamma' \in \mathcal{O} \cap \sigma(L)$ be the unique element with $\gamma' = \sigma(\tau_\gamma)$, i.e. $\sigma = \sigma_{\gamma'}$ in the above notation. One has

$$\gamma'^2 - \mathrm{tr}(\gamma)\gamma' + 1 = \sigma(\tau_\gamma^2 - \mathrm{tr}(\gamma)\tau_\gamma + 1) = 0,$$

which shows that γ and γ' have the same eigenvalues. Thus, γ' lies in $\mathcal{O}^1$ and the respective lengths l_γ and $l_{\gamma'}$ are equal. Hence

$$m(l_\gamma) \geq \sum_{\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L} m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1),$$

which concludes the proof. $\square$

Let us restrict our attention to maximal orders. We first observe that the embedding numbers may vanish in a few cases:

Theorem 3.5.3. *Let L be a quadratic extension of F such that L embeds into $\mathcal{A}$. Let $\mathfrak{o}$ be an order in L containing $\mathfrak{o}_F$. Then every maximal order in $\mathcal{A}$ contains a conjugate of $\mathfrak{o}$, unless both of the following conditions hold:*

1. *The extension L/F and the quaternion algebra $\mathcal{A}$ are unramified at all finite places and ramified at exactly the same set of real places.*

2. *Any prime ideal of F, which divides the relative discriminant $d(\mathfrak{o})$ of $\mathfrak{o}$, is split in L/F.*

Proof. [60, Thm. 12.4.2] □

Definition 3.5.4. An order $\mathfrak{o}$ satisfying the conditions of the above theorem is said to be *selective*. Otherwise, it is called non-selective.

Remark 3.5.5. *The first condition in Theorem 3.5.3 can only be met for our quaternion algebra $\mathcal{A}$, if $n = [F : \mathbb{Q}]$ is odd, since* $\mathrm{Ram}(\mathcal{A})$ *has even cardinality. If n is odd, there are, up to isomorphism, only n possibilities for our quaternion algebra $\mathcal{A}$ over F, for which selectivity could occur. Furthermore, there are only finitely many unramified quadratic extensions of F. In this sense, we regard selectivity as an exceptional phenomenon. Still, selectivity does occur for certain quaternion algebras (cf. Exercise 6 in [60, §12.5]).*

Remark 3.5.6. *Sometimes it may also happen that no embedding of a certain order $\mathfrak{o}$ is optimal. This depends on the ramification set of the quaternion algebra $\mathcal{A}$ and is discussed in more detail in Appendix B.*

With these considerations in mind we now express the embedding number $m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1)$ in terms of arithmetic data:

Theorem 3.5.7. *Let $\mathcal{O}$ be a maximal order. Then there exists a number $s(\mathfrak{o}, \mathcal{O}) \in \{0, 1, 2\}$ such that*

$$m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1) = \frac{s(\mathfrak{o}, \mathcal{O}) h(\mathfrak{o}) 2^{1+|\Omega_i(L)|}}{h_F [N_{L/F}(\mathfrak{o}^\times) : (\mathfrak{o}_F^\times)^2]}, \tag{3.3}$$

where $\Omega_i(L)$ is the subset of finite places in $\mathrm{Ram}(\mathcal{A})$*, which stay inert in the extension L.*

This result is essentially known in the literature and a derivation based on [98] can be found in Appendix B. For the moment we only notice that $s(\mathfrak{o}, \mathcal{O})$ does not depend on the maximal order $\mathcal{O}$, whenever $\mathfrak{o}$ is non-selective. A well-known application of (3.3), due to Vignéras [97], is the construction of pairs of isospectral Riemann surfaces, which are not isometric (see also [60, §12.4]).

3.6 Quadratic Orders

By Lemma 3.5.2 only orders $\mathfrak{o}$ in a quadratic extension $L = F(e^{l_\gamma/2})$ with $\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L$ could yield a contribution to the multiplicity of some length $l = l_\gamma$ in the length spectrum. In this section we will describe a way to parametrize these orders by studying the following more general class of orders:

Definition 3.6.1. A *quadratic $\mathfrak{o}_F$-order* $\mathfrak{o}$ is an order in a relative quadratic extension L/F such that $\mathfrak{o}$ is a module over $\mathfrak{o}_F$.

In our exposition, we closely follow [69], although we need to be more explicit in some places. Most of the results can also be found in the more recent article [13]. Let us recall some terminology and standard facts from the theory of modules over Dedekind domains. The presented material can for example be found in [70, §1.3]. In the following R is always a Dedekind domain and F its field of fractions. Any arrow between R-modules will be implicitely assumed to be a morphism of R-modules.

Definition 3.6.2. An R-module M is called *projective* if every diagram of the form

$$\begin{tikzcd} & M \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

with exact row and arbitrary R-modules A and B can be extended to a commutative diagram

$$\begin{tikzcd} & M \arrow[d] \arrow[dl] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

Proposition 3.6.3. *Any non-zero fractional ideal I of R is projective.*

Proof. [70, Prop. 1.36] □

Theorem 3.6.4. *Let M be a finitely generated R-module and let Z be the submodule of M consisting of all torsion-elements of M, i.e. of elements $x \in M$, which for some non-zero $r \in R$ satisfy $rx = 0$. Then M can be written as a direct sum*

$$M \cong R^k \oplus I \oplus Z,$$

where k is a non-negative integer and I is some ideal of R.

Proof. [70, Theorem 1.32] □

Theorem 3.6.5. *Let M_1, M_2 be finitely generated, torsion-free R-modules with*

$$M_1 = I_1 \oplus ... \oplus I_s,\ \ M_2 = J_1 \oplus ... \oplus J_t,$$

where the I_i J_j are non-zero fractional ideals of R. Then M_1 and M_2 are isomorphic if and only if $s = t$ and there exists some $a \in F$ such that

$$I_1 \cdots I_s = aJ_1 \cdots J_t$$

holds.

Proof. [70, Thm. 1.39] □

We now specialize to the case of R being the ring of integers $\mathfrak{o}_F$ of some number field F and M some quadratic $\mathfrak{o}_F$-order $\mathfrak{o}$. We have the following short exact sequence

$$0 \longrightarrow \mathfrak{o}_F \xrightarrow{\iota} \mathfrak{o} \xrightarrow{\pi} \mathfrak{o}/\mathfrak{o}_F \longrightarrow 0 \tag{3.4}$$

where ι is the inclusion map and π is the projection map. Since $\mathfrak{o}/\mathfrak{o}_F$ is a finitely-generated torsion-free $\mathfrak{o}_F$-module, Theorem 3.6.4 yields an isomorphism

$$\mathfrak{o}/\mathfrak{o}_F \cong \mathfrak{o}_F^k \oplus \mathfrak{b}$$

for some fractional ideal $\mathfrak{b}$ of $\mathfrak{o}_F$ and some $k \in \mathbb{N}$. Comparing the ranks of both modules over $\mathbb{Z}$ yields $k = 0$ and thus $\mathfrak{o}/\mathfrak{o}_F \cong \mathfrak{b}$. Theorem 3.6.5 shows that the ideal class of $\mathfrak{b}$ is uniquely determined by $\mathfrak{o}$. Let us choose $\mathfrak{b}$ such that $\mathfrak{o}_F \subset \mathfrak{b}$. With this choice we have $1 \in \mathfrak{b}$. Let us fix an isomorphism $\phi : \mathfrak{b} \to \mathfrak{o}/\mathfrak{o}_F$ and let $\pi : \mathfrak{o} \to \mathfrak{o}/\mathfrak{o}_F$ be the projection map. By Proposition 3.6.3 any non-zero fractional ideal is a projective $\mathfrak{o}_F$-module so that the inspection of the diagram

$$\begin{tikzcd} & \mathfrak{o}/\mathfrak{o}_F \arrow[dl, dashed, "\psi"'] \arrow[d, "\mathrm{Id}"] & \\ \mathfrak{o} \arrow[r, "\pi"'] & \mathfrak{o}/\mathfrak{o}_F \arrow[r] & 0 \end{tikzcd}$$

yields an $\mathfrak{o}_F$-morphism $\psi : \mathfrak{o}/\mathfrak{o}_F \to \mathfrak{o}$ with $\pi \circ \psi = \mathrm{Id}$. Hence, the exact sequence (3.4) splits and the map

$$h : \mathfrak{o}_F \oplus \mathfrak{o}/\mathfrak{o}_F \to \mathfrak{o}, \qquad (x_1, x_2) \mapsto \iota(x_1) + \psi(x_2)$$

constitutes an isomorphism. We therefore get isomorphisms

$$\mathfrak{o}_F \oplus \mathfrak{b} \xrightarrow{\mathrm{Id} \oplus \phi} \mathfrak{o}_F \oplus \mathfrak{o}/\mathfrak{o}_F \xrightarrow{h} \mathfrak{o}, \tag{3.5}$$

which yields

$$\mathfrak{o} = \mathfrak{o}_F \oplus \psi(\phi(\mathfrak{b})).$$

For $x \in \mathfrak{o}_F$ with $x\mathfrak{b} \subset \mathfrak{o}_F$ we have

$$x\psi(\phi(\mathfrak{b})) = \psi(\phi(x\mathfrak{b})) = x\mathfrak{b}\psi(\phi(1))$$

so that

$$\mathfrak{o} = \mathfrak{o}_F \oplus \mathfrak{b}\theta \tag{3.6}$$

with $\theta = \psi(\phi(1))$. Applying this procedure to the ring of integers $\mathfrak{o}_L$ of a relative quadratic extension L/F we find the decomposition $\mathfrak{o}_L = \mathfrak{o}_F \oplus \mathfrak{b}_L\theta$ for some fractional ideal $\mathfrak{b}_L$ of $\mathfrak{o}_F$.

Definition 3.6.6. The element $[\mathfrak{b}_L]$ of the ideal class group is called the *Steinitz class* of the extension L/F. We denote it by $\mathrm{st}(L/F)$.

The Steinitz classes of relative quadratic extensions are known:

Lemma 3.6.7. *Let $L = F(\sqrt{a})$ be a relative quadratic extension of F. Then $\mathfrak{d}_{L/F}a^{-1} = \mathfrak{b}^2$ for some fractional ideal $\mathfrak{b}$ of F with $[\mathfrak{b}] = \mathrm{st}(L/F)$.*

Proof. [27, Thm. 2.2]. □

For the relative quadratic extension $L = F(\sqrt{d_\gamma})$, we always choose $\mathfrak{b}_L$ such that

$$\mathfrak{b}_L^2 = \mathfrak{d}_{L/F}d_\gamma^{-1}. \tag{3.7}$$

With the Steinitz class fixed, we can parametrize quadratic $\mathfrak{o}_F$-orders in terms of integral ideals of $\mathfrak{o}_F$:

Proposition 3.6.8. *Let $\mathfrak{o}$ be a quadratic $\mathfrak{o}_F$-order sitting in a quadratic extension L/F. Then there exists an integral ideal $\mathfrak{c}_\mathfrak{o}$ of F such that*

$$\mathfrak{o} = \mathfrak{o}_F \oplus \mathfrak{c}_\mathfrak{o}\mathfrak{b}_L\theta. \tag{3.8}$$

Conversely, for any integral ideal $\mathfrak{c}$ the set $\mathfrak{o}_\mathfrak{c} = \mathfrak{o}_F \oplus \mathfrak{c}\mathfrak{b}_L\theta$ defines a quadratic $\mathfrak{o}_F$-order.

Proof. The set $\mathfrak{l} := \{x \in \mathfrak{b}_L \mid x\theta \in \mathfrak{o}\}$ defines a fractional ideal of F contained in $\mathfrak{b}_L$ and one may quickly check that $\mathfrak{o} = \mathfrak{o}_F \oplus \mathfrak{l}\theta$. By [70, Prop. 1.13], we have that $\mathfrak{l} \subset \mathfrak{b}_L$ implies the existence of a fractional ideal $\mathfrak{c}_\mathfrak{o}$ such that $\mathfrak{l} = \mathfrak{c}_\mathfrak{o}\mathfrak{b}_L$. From $\mathfrak{c}_\mathfrak{o} = \mathfrak{l}\mathfrak{b}_L^{-1}$ we get $\mathfrak{c}_\mathfrak{o} \subset \mathfrak{o}_F$ and hence that $\mathfrak{c}_\mathfrak{o}$ is integral. □

The algebraic number θ is a zero of a quadratic polynomial

$$p_\theta(x) = x^2 + a_1x + a_2 \tag{3.9}$$

with $a_1, a_2 \in F$. Let $d_\theta = a_1^2 - 4a_2$ be the discriminant of θ. We note the following formula for the discriminant of an order (cf. [69, Lemma 1.6]) :

Lemma 3.6.9. *Let $\mathfrak{o} = \mathfrak{o}_F \oplus \mathfrak{b}\theta$ be a quadratic order with $\mathfrak{b}$ some fractional ideal of F and θ an algebraic integer. Then the discriminant of $\mathfrak{o}$ is given by*

$$d(\mathfrak{o}) = d_F^2 N(\mathfrak{b}^2 d_\theta).$$

3.7 Plancherel Convergence for Congruence Surfaces

Proof of Theorem 3.1.1. For Γ_j a congruence subgroup we let $X_j = \Gamma_j\backslash\mathcal{H}$ be the associated hyperbolic surface and abbreviate $V_j = \mathrm{vol}(X_j)$. The strategy of the proof is to confirm the condition of Proposition 2.11.6, i.e. we show that for each $\varepsilon > 0$ there exists j_ε so that

$$\frac{1}{V_j}\sum_{l_\gamma \le c} m(l_\gamma, \Gamma_j)\frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} < \varepsilon \tag{3.10}$$

for each $j \ge j_\varepsilon$, where the sum runs over the lengths in $\mathrm{Spec}_L(X_j)$. The proof of (3.10) will take up the rest of this section.
Let us consider an arbitrary congruence subgroup Γ of covolume $V = \mathrm{vol}(\Gamma\backslash\mathcal{H})$ with invariant trace field F of degree n and invariant quaternion algebra $\mathcal{A}$. Let $\mathcal{O}$ be a maximal order so that $\Gamma \subset P_{\sigma_1}\mathcal{O}^1$. In view of

$$\mathrm{vol}(\Gamma\backslash\mathcal{H}) = [\Gamma : P_{\sigma_1}\mathcal{O}^1]\,\mathrm{vol}(P_{\sigma_1}\mathcal{O}^1\backslash\mathcal{H}) \tag{3.11}$$

and

$$m(l_\gamma, \Gamma) \le [\Gamma : P_{\sigma_1}\mathcal{O}^1] m(l_\gamma, P_{\sigma_1}\mathcal{O}^1), \tag{3.12}$$

we first concentrate on the case of maximal orders. From $\lim\limits_{x\to 0} x/\sinh(x) = 1$ one may derive

$$\frac{1}{\sinh(l_\gamma/2)} \le c_0 \max\{1, l_s^{-1}\} \tag{3.13}$$

for some sufficiently large constant c_0 and $l_s = l_s(X)$ the systole of $X = \Gamma\backslash\mathcal{H}$. Applying Theorem 3.3.3 to (3.13) yields

$$\frac{1}{\sinh(l_\gamma/2)} \le c_D c_0 \log(n)^3. \tag{3.14}$$

Next we want to estimate

$$\sum_{l_\gamma \le c} m(l_\gamma, \Gamma) l_{\gamma_0} = \sum_{l_\gamma \le c}\left(\sum_{\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L} m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1) l_{\gamma_0}\right). \tag{3.15}$$

According to Theorem 3.5.7 we have

$$m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1) \le 4\frac{h(\mathfrak{o})2^{|\Omega_i(L)|}}{h_F}. \tag{3.16}$$

Consider the well-known formula for the class number

$$h(\mathfrak{o}) = \frac{h_L}{[\mathfrak{o}_L^\times : \mathfrak{o}^\times]}\frac{\varphi_{\mathfrak{o}_L}(\mathfrak{f}_\mathfrak{o})}{\varphi_\mathfrak{o}(\mathfrak{f}_\mathfrak{o})}, \tag{3.17}$$

where $\varphi_{\mathfrak{o}_L}(\mathfrak{f}_\mathfrak{o})$ is the number of units in $\mathfrak{o}_L/\mathfrak{f}_\mathfrak{o}$ and $\varphi_\mathfrak{o}(\mathfrak{f}_\mathfrak{o})$ is the number of units in $\mathfrak{o}/\mathfrak{f}_\mathfrak{o}$ (see [71, Thm. 12.12]). Applying the crude estimate

$$\varphi_{\mathfrak{o}_L}(\mathfrak{f}_\mathfrak{o}) \le |\mathfrak{o}_L/\mathfrak{f}_\mathfrak{o}| = N(\mathfrak{f}_\mathfrak{o}) \tag{3.18}$$

to (3.17) shows

$$h(\mathfrak{o}) \le h_L N(\mathfrak{f}_\mathfrak{o}). \tag{3.19}$$

Plugging (3.19) into (3.16) yields

$$m(\mathfrak{o},\mathcal{O};\mathcal{O}^1) \leq 4\frac{h_L 2^{|\Omega_i(L)|}N(\mathfrak{f}_\mathfrak{o})}{h_F}. \tag{3.20}$$

We note the following

Lemma 3.7.1. *Let $\mathfrak{o}$ be a quadratic order in L with $\mathfrak{o}_\gamma \subset \mathfrak{o}$. Then one has $N_{L/\mathbb{Q}}(\mathfrak{f}_\mathfrak{o}) \leq N_{F/\mathbb{Q}}(d_\gamma)$.*

Proof. Since $\mathfrak{o}_\gamma \subset \mathfrak{o}$, one has by definition $\mathfrak{f}_{\mathfrak{o}_\gamma} \subset \mathfrak{f}_\mathfrak{o}$ and hence

$$N_{L/\mathbb{Q}}(\mathfrak{f}_\mathfrak{o}) \leq N_{L/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{o}_\gamma}). \tag{3.21}$$

It is known[3] (cf. [89, Cor. III.6.1]) that

$$(d_\gamma) = N_{L/F}(\mathfrak{f}_{\mathfrak{o}_\gamma})\mathfrak{d}_{L/F} \tag{3.22}$$

holds. Using $N_{F/\mathbb{Q}} \circ N_{L/F} = N_{L/\mathbb{Q}}$ one may conclude from (3.21) and (3.22) that

$$N_{L/\mathbb{Q}}(\mathfrak{f}_\mathfrak{o}) \leq N_{L/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{o}_\gamma}) = N_{F/\mathbb{Q}}(d_\gamma)N_{F/\mathbb{Q}}(\mathfrak{d}_{L/F})^{-1} \leq N_{F/\mathbb{Q}}(d_\gamma).$$

□

By applying the above lemma to (3.20) we get

$$m(\mathfrak{o},\mathcal{O};\mathcal{O}^1) \leq 4\frac{h_L 2^{|\Omega_i(L)|}N(d_\gamma)}{h_F}. \tag{3.23}$$

Now, regarding l_{γ_0} we note that Theorem 3.2.6 guarantees the existence of $k_0 \in \mathbb{N}$ with

$$l_{\gamma_0} = k_0 2^{2-n} u \frac{h_F}{h_L}\mathcal{L}'(0,\chi), \tag{3.24}$$

where $u \in \{1, 2\}$. We get

$$k_0 \leq \frac{c}{l_s} \leq c_D c \log(n)^3 \tag{3.25}$$

from another application of Theorem 3.3.3. The inequalities (3.23),(3.24) and (3.25) together yield

$$m(\mathfrak{o},\mathcal{O};\mathcal{O}^1)l_{\gamma_0} \leq c_1 2^{-n}\log(n)^3 N(d_\gamma)2^{|\Omega_i(L)|}\mathcal{L}'(0,\chi), \tag{3.26}$$

where $c_1 = 64c_D c$. Applying the bounds from Theorem 3.4.1 to (3.26) leads to

$$m(\mathfrak{o},\mathcal{O};\mathcal{O}^1)l_{\gamma_0} \leq c_2\frac{\log(n)^3(1+\delta)^n}{2^n}2^{|\Omega_i(L)|}\mathcal{L}'(0,\chi), \tag{3.27}$$

[3] Here, one also needs a well-known relation between the relative discriminant $\mathfrak{d}_{L/F}$ and the so-called different of an extension L/F, see e.g. [89, Prop. III.3.6].

where $c_2 = c_1 c_\delta$ and $\delta > 0$ is sufficiently small. Hence, we have

$$\sum_{\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L} m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1) l_{\gamma_0} \leq c_2 \frac{\log(n)^3 (1+\delta)^n}{2^n} 2^{|\Omega_i(L)|} \mathcal{L}'(0, \chi) \left(\sum_{\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L} 1 \right). \tag{3.28}$$

In view of (3.28) we note the following

Lemma 3.7.2. *For $\delta > 0$ sufficiently small one has*

$$\left(\sum_{\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L} 1 \right) \leq c_3 (1+\delta)^{2n},$$

where $c_3 = c_{\delta,2} c_\delta$.

Proof. We first claim that

$$N(d_\theta) = N(d_\gamma) \tag{3.29}$$

holds. This readily follows from Lemma 3.6.9 and our choice of the fractional ideal $\mathfrak{b}_L$ made in (3.7), since

$$d_F^2 N(\mathfrak{d}_{L/F}) = d_{\mathfrak{o}_L} = d_F^2 N(\mathfrak{b}_L^2 d_\theta) = d_F^2 N(\mathfrak{d}_{L/F} d_\gamma^{-1} d_\theta),$$

where we used that $L = F(\sqrt{d_\gamma})$. According to Lemma 3.6.8 we may write any order with $\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L$ in the form $\mathfrak{o} = \mathfrak{o}_F + \mathfrak{c}_\mathfrak{o} \mathfrak{b}_L \theta$ with $\mathfrak{c}_\mathfrak{o}$ an integral ideal of F. Since $\mathfrak{o}_\gamma \subset \mathfrak{o}$ we have by definition of the discriminant that $d(\mathfrak{o}) \leq d(\mathfrak{o}_\gamma)$. Again applying Lemma 3.6.9 shows

$$d_F^2 N(\mathfrak{c}_\mathfrak{o}^2 \mathfrak{b}_L^2 d_\theta) = d(\mathfrak{o}) \leq d(\mathfrak{o}_\gamma) = d_F^2 N(d_\gamma), \tag{3.30}$$

which in combination with (3.29) yields

$$N(\mathfrak{c}_\mathfrak{o})^2 \leq N(\mathfrak{b}_L^{-2}) = N(\mathfrak{d}_{L/F})^{-1} N(d_\gamma) \leq N(d_\gamma). \tag{3.31}$$

We therefore get

$$\left(\sum_{\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L} 1 \right) = \sum_{\mathfrak{c}_\mathfrak{o} \subset \mathfrak{o}_F} 1 = \sum_{\mathfrak{c}_\mathfrak{o} \subset \mathfrak{o}_F} N(\mathfrak{c}_\mathfrak{o})^{-2} N(\mathfrak{c}_\mathfrak{o})^2 \leq \zeta_F(2) N(d_\gamma),$$

where the sums in the middle run over all integral ideals $\mathfrak{c}_\mathfrak{o}$ coming from a quadratic order $\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L$ as in Lemma 3.6.8. This gives the desired claim in view of the bounds from Theorem 3.4.1. □

Applying the above lemma to (3.28) shows

$$\sum_{\mathfrak{o}_\gamma \subset \mathfrak{o} \subset \mathfrak{o}_L} m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1) l_{\gamma_0} \leq c_4 \frac{\log(n)^3 (1+\delta)^{3n}}{2^n} 2^{|\Omega_i(L)|} |\mathcal{L}'(0, \chi)|. \tag{3.32}$$

with $c_4 = c_2 c_3$. Let us now deal with $|\mathcal{L}'(0, \chi)|$:

Lemma 3.7.3. *For any $\delta > 0$ sufficiently small there exists a constant c'_δ so that*

$$|\mathcal{L}'(0,\chi)| \leq c'_\delta (1+\delta)^{3n} \pi^{-n} d_F^{(1+\delta)/2}.$$

Proof. The completion $\Lambda(s,\chi)$ of $\mathcal{L}(s,\chi)$ is given by

$$\Lambda(s,\chi) = \left(\pi^{-n/2} d_F N(\mathfrak{d}_{L/F})\right)^s \Gamma\left(\tfrac{s}{2}\right) \Gamma\left(\tfrac{s+1}{2}\right)^{n-1} \mathcal{L}(s,\chi),$$

(see e.g. [56, p. 299]), where $\Gamma(s)$ is the Gamma function. We therefore have

$$\Lambda(0,\chi) = \Gamma(\tfrac{1}{2})^{n-1} \left(\lim_{s\to 0} \Gamma\left(\tfrac{s}{2}\right) \mathcal{L}(s,\chi)\right). \tag{3.33}$$

Inserting $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\lim_{s\to 0} s\Gamma(s) = 1$ into (3.33) shows

$$\Lambda(0,\chi) = 2\pi^{(n-1)/2} \left(\lim_{s\to 0} \frac{\mathcal{L}(s,\chi)}{s}\right). \tag{3.34}$$

Since

$$\lim_{s\to 0} \frac{\mathcal{L}(s,\chi)}{s} = \lim_{s\to 0} \frac{s^{-r_L}\zeta_L(s)}{s^{-r_F}\zeta_F(s)} = \frac{h_L R_L}{h_F R_F} = \mathcal{L}'(0,\chi), \tag{3.35}$$

where for the last equality we used Theorem 2.9.3 and the Taylor expansion of ζ_L and ζ_F at $s = 0$ from (2.17), we have

$$\Lambda(0,\chi) = 2\pi^{(n-1)/2} \mathcal{L}'(0,\chi). \tag{3.36}$$

Let $\delta_0 > 0$ be a real number with $0 < \delta_0 < 1$. Next, we want to apply the Phragmen-Lindelöf principle (cf. Appendix C) to $\Lambda(s,\chi)$ on the vertical strip

$$U = \{z = x + iy \in \mathbb{C} : -\delta_0 < x < 1 + \delta_0\}$$

to derive a bound for $\Lambda(0,\chi)$. We therefore have to estimate $\Lambda(s,\chi)$ on the extended boundary $\partial_\infty U = \partial U \cup \{\infty\}$ of U. Since $|\Gamma(\sigma + it)| \leq \Gamma(\sigma)$ for $\sigma > 0$ and $t \in \mathbb{R}$, we have

$$|\Lambda(1+\delta_0+it,\chi)| \leq \pi^{-n(1+\delta_0)/2} d_F^{(1+\delta_0)/2} N(\mathfrak{d}_{L/F})^{(1+\delta_0)/2} \Gamma\left(\tfrac{1+\delta_0}{2}\right) \Gamma\left(1+\tfrac{\delta_0}{2}\right)^{n-1} \zeta_F(1+\delta_0).$$

Using the bounds from Theorem 3.4.1 yields

$$|\Lambda(1+\delta_0+it)| \leq c_{\delta_0}^2 \Gamma(\tfrac{1+\delta_0}{2})(1+\delta_0)^{2n} \pi^{-n/2} d_F^{(1+\delta_0)/2} \Gamma(1+\tfrac{\delta_0}{2})^{n-1}.$$

Now, the functional equation for the completed $\Lambda(s,\chi)$ is given by

$$W(\chi)\Lambda(s,\chi) = \Lambda(1-s,\overline{\chi})$$

(see [56, Corollary XIV.8.2]) with some constant $W(\chi) \in \mathbb{C}$ such that $|W(\chi)| = 1$. Hence, we get

$$|\Lambda(-\delta_0+it,\chi)| = |\Lambda(1+\delta_0+it,\chi)| \leq c_{\delta_0}^2 \Gamma(\tfrac{1+\delta_0}{2})(1+\delta_0)^{2n} \pi^{-n/2} d_F^{(1+\delta_0)/2} \Gamma(1+\tfrac{\delta_0}{2})^{n-1}.$$

Finally, we have

$$\limsup_{s\to\infty} |\Lambda(s,\chi)| = 0,$$

as any automorphic L-functions is of finite order (cf. [50, Lemma 5.2]). Hence, the Phragmen-Lindelöf principle yields

$$|\Lambda(s,\chi)| \le c_{\delta_0}^2 \Gamma(\tfrac{1+\delta_0}{2})(1+\delta_0)^{2n}\pi^{-n/2} d_F^{(1+\delta_0)/2}\Gamma(1+\tfrac{\delta_0}{2})^{n-1}$$

for all $s \in U$. In view of (3.36) this gives

$$|\mathcal{L}'(0,\chi)| \le \frac{\sqrt{\pi}c_{\delta_0}^2\Gamma(\frac{1+\delta_0}{2})}{2}(1+\delta_0)^{2n}\pi^{-n} d_F^{(1+\delta_0)/2}\Gamma(1+\tfrac{\delta_0}{2})^{n-1}.$$

Since $\Gamma(1) = 1$, choosing δ_0 sufficiently small gives the claimed bound for $|\mathcal{L}'(0,\chi)|$ with

$$c'_\delta = \frac{\sqrt{\pi}c_{\delta_0}^2\Gamma(\frac{1+\delta_0}{2})}{2\Gamma(1+\frac{\delta_0}{2})}.$$

□

Plugging the bound from Lemma 3.7.3 into (3.32) gives

$$\sum_{\mathfrak{o}_\gamma\subset\mathfrak{o}\subset\mathfrak{o}_L} m(\mathfrak{o},\mathcal{O};\mathcal{O}^1) l_{\gamma_0} \le c_5 \frac{\log(n)^3(1+\delta)^{6n}}{(2\pi)^n} 2^{|\Omega_i(L)|} d_F^{(1+\delta)/2}, \tag{3.37}$$

where $c_5 = c_4 c'_\delta$. We clearly have $|\Omega_i(L)| \le |\operatorname{Ram}(\mathcal{A})|$ and therefore

$$\sum_{\mathfrak{o}_\gamma\subset\mathfrak{o}\subset\mathfrak{o}_L} m(\mathfrak{o},\mathcal{O};\mathcal{O}^1) l_{\gamma_0} \le c_5 \frac{\log(n)^3(1+\delta)^{6n}}{(2\pi)^n} 2^{|\operatorname{Ram}(\mathcal{A})|} d_F^{(1+\delta)/2}. \tag{3.38}$$

Note that the right-hand side of (3.38) does only depend on the invariant trace field F and the invariant quaternion algebra $\mathcal{A}$.
Returning to (3.15), we note that we still have to estimate the counting function

$$N_\Gamma(c) = \sum_{l_\gamma \le c} 1, \tag{3.39}$$

where the lengths in $\operatorname{Spec}_L(\Gamma)$ are counted without multiplicities.

Lemma 3.7.4. *For $\delta > 0$ sufficiently small, there exists a constant c''_δ so that*

$$N_\Gamma(c) \le c''_\delta \log(n)^3 2^n (1+\delta)^{2n}\zeta_F(2).$$

Proof. To any length $l = l_\gamma$ in the length spectrum of Γ we may associate[4] the principal ideal $J_l = (d_l)$ generated by $d_l = (e^{l/2} + e^{-l/2})^2 - 4$. We want to use

[4]Note that we still silently identify the number field F with its image in $\mathbb{R}$ under the embedding σ_1.

the resulting map $J : \mathrm{Spec}_L(\Gamma) \to I(\mathfrak{o}_F)$ to estimate $N_\gamma(c)$. Let us first show that

$$|J^{-1}(J_l)| \leq 4cc_D \log(n)^3 2^n. \tag{3.40}$$

Assume that $J_{l'} = J_l$ for two different lengths $l, l' \in \mathrm{Spec}_L(\Gamma)$. Then there exists a unit $v' \in \mathfrak{o}_F^\times$ with $d_l = v' d_{l'}$. By Dirichlet's unit theorem, we can choose a set of representatives $v_1, ..., v_{2^{n+1}}$ for $\mathfrak{o}_F^\times/(\mathfrak{o}_F^\times)^2$, so that we may write any unit $v \in \mathfrak{o}_F^\times$ in the form $v = v_j w^2$ with $w \in \mathfrak{o}_F^\times$. Assume that we have two different lengths l', l'' such that

$$v_j (w')^2 d_{l'} = d_l = v_j (w'')^2 d_{l''}. \tag{3.41}$$

for some $j \in \{1, ..., 2^{n+1}\}$ and $w', w'' \in \mathfrak{o}_F^\times$. Then the extensions $L' = F(\sqrt{d_{l'}})$ and $L'' = F(\sqrt{d_{l''}})$ agree. In particular, $\tau_{l''} = e^{l''/2}$ defines a Salem number in L'. By Corollary 3.2.7, we have $\tau_{l''} = \tau_s^{k/2}$ for some $k \in \mathbb{N}$ and τ_s the Stark unit in L'. By using Theorem 3.3.3, we have

$$c \geq \frac{k}{2} \log \tau_s \geq \frac{k}{2} c_D^{-1} \log(n)^{-3}$$

so that

$$k \leq 2cc_D \log(n)^3.$$

This establishes (3.40), since there are at most k possibilities for the equality (3.41) to occur. But then

$$\begin{aligned} N_\Gamma(c) &\leq 4cc_D \log(n)^3 2^n \sum_{J_l : l \leq c} 1 \\ &\leq 4cc_D \log(n)^3 2^n \sum_{J_l : l \leq c} N(J_l)^2 N(J_l)^{-2} \\ &\leq 4cc_D c_\delta^2 \log(n)^3 2^n (1+\delta)^{2n} \zeta_F(2), \end{aligned}$$

where the sums run over the principal ideals (d_l) with $l \leq c$. Note that we again utilized the bounds from Theorem 3.4.1. This yields the claimed inequality with

$$c''_\delta = 4cc_D c_\delta^2.$$

□

Using Lemma 3.7.4 together with (3.38) shows

$$\sum_{l_\gamma \leq c} m(l_\gamma) l_{\gamma_0} \leq c_6 \frac{\log(n)^6 (1+\delta)^{8n}}{\pi^n} 2^{|\,\mathrm{Ram}(\mathcal{A})|} d_F^{(1+\delta)/2} \zeta_F(2). \tag{3.42}$$

with $c_6 = c_5 c''_\delta$. But then (3.14) shows

$$\sum_{l_\gamma \leq c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \leq c_7 \frac{\log(n)^9 (1+\delta)^{8n}}{\pi^n} 2^{|\,\mathrm{Ram}(\mathcal{A})|} d_F^{(1+\delta)/2} \zeta_F(2) \tag{3.43}$$

with $c_7 = c_6 c_D c_0$. Using the expression for the volume of the maximal order from Lemma 2.10.6, we arrive at

$$\frac{1}{V} \sum_{l_\gamma \leq c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \leq c_8 \frac{\log(n)^9 (1+\delta)^{8n} (4\pi)^n}{\prod_{\mathfrak{p} \mid \mathfrak{d}_{\mathcal{A}}} \left(\frac{N(\mathfrak{p})-1}{2} \right) d_F^{1-\frac{\delta}{2}}} \tag{3.44}$$

with $c_8 = c_7/(4\pi^2)$. We note that $N(\mathfrak{p}) - 1 = 1$ is only possible, if $\mathfrak{p} \mid 2$. Hence, if we let $\mathfrak{d}_{\mathcal{A}}^*$ be the product over all prime ideals in $\mathrm{Ram}(\mathcal{A})$, which do not divide 2, we get

$$\frac{1}{V} \sum_{l_\gamma \leq c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \leq c_8 \frac{\log(n)^9 (1+\delta)^{8n} (8\pi)^n}{\prod_{\mathfrak{p} \mid \mathfrak{d}_{\mathcal{A}}^*} \left(\frac{N(\mathfrak{p})-1}{2} \right) d_F^{1-\frac{\delta}{2}}}. \tag{3.45}$$

Let us now fix some $\delta > 0$ with $\delta \leq \frac{1}{500}$. Using Odlyzko's bound [75, eq. (2.5)] for the discriminant,

$$d_F \geq 60.8^n$$

for n large enough, we see that

$$\frac{(1+\frac{1}{500})^{8n} (8\pi)^n}{d_F^{1-1/1000}} \leq \left(\frac{(1+\frac{1}{500})^8 (8\pi)}{(60.8)^{5/6-1/1000}} \right)^n \frac{1}{d_F^{1/6}} \leq 0.84^n \frac{1}{d_F^{1/6}}.$$

This guarantees the existence of a constant c_9 such that

$$\frac{\log(n)^9 (1+\frac{1}{500})^{8n} (8\pi)^n}{d_F^{1-1/1000}} \leq c_9 \frac{1}{d_F^{1/6}} \tag{3.46}$$

for any totally real number field F. Plugging this into (3.45), we finally end up with

$$\frac{1}{V} \sum_{l_\gamma \leq c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \leq c_{10} \frac{1}{d_F^{1/6} \prod_{\mathfrak{p} \mid \mathfrak{d}_{\mathcal{A}}^*} \left(\frac{N(\mathfrak{p})-1}{2} \right)}, \tag{3.47}$$

where $c_{10} = c_8 c_9$. Now, let $(\Gamma_j)_{j \in \mathbb{N}}$ be a sequence of torsion-free non-conjugate congruence subgroups and let $\varepsilon > 0$ be given. Let $F_j = F(\Gamma_j)$ and $\mathcal{A}_j = \mathcal{A}(\Gamma_j)$ be the respective invariant trace fields and invariant quaternion algebras. It is a well-known result that there exist only finitely many number fields of bounded discriminant, so that

$$\frac{1}{V} \sum_{l_\gamma \leq c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} > \varepsilon \tag{3.48}$$

is only possible for finitely many invariant trace fields $F_1, ..., F_s$. According to the classification of quaternion algebras (see e.g. [60, Thm. 7.3.6]), inequality (3.47) also shows that there only finitely many invariant quaternion algebras $\mathcal{A}_1, ..., \mathcal{A}_t$ over these fields $F_1, ..., F_s$, for which we possibly could have (3.48). In these invariant quaternion algebras, there are, up to conjugacy, only finitely many maximal orders $\mathcal{O}_1, ..., \mathcal{O}_u$ (cf. [60, §6.7]). Hence, all congruence subgroups, for which (3.48) could hold, lie in finitely many arithmetic Fuchsian groups $P_{\sigma_1}\mathcal{O}_1^1, ..., P_{\sigma_1}\mathcal{O}_u^1$. Let $(\Gamma_{j_k})_{k\in\mathbb{N}}$ be the subsequence of $(\Gamma_j)_{j\in}$ consisting of those congruence subgroups contained in $P_{\sigma_1}\mathcal{O}_1^1, ..., P_{\sigma_1}\mathcal{O}_u^1$. This sequence is uniformly discrete and known to be Benjamini-Schramm convergent (see [1, Thm. 1.12]). In particular, $(\Gamma_{j_k})_{k\in\mathbb{N}}$ is Plancherel convergent by Theorem 2.11.5 and we can find k_0 such that

$$\frac{1}{V_{j_k}} \sum_{l_\gamma \leq c} m(l_\gamma, \Gamma_{j_k}) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \leq \varepsilon \tag{3.49}$$

for all $k \geq k_0$. This shows

$$\frac{1}{V_j} \sum_{l_\gamma \leq c} m(l_\gamma, \Gamma_j) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \leq \varepsilon \tag{3.50}$$

for $j \geq j_{k_0}$, which concludes the proof. $\square$

Chapter 4

Plancherel Convergence and Zeta Functions

4.1 Summary

In the previous sections we have discussed basic properties of Plancherel-convergent sequences and have many examples. Now, a natural question is how this form of convergence is reflected by secondary metric invariants such as the Selberg zeta function. One first result in this direction stems from Deitmar (see [31, Thm. 3.2]):

Theorem 4.1.1. *Let* $(\Gamma_j)_{j\in\mathbb{N}}$ *be a sequence of torsion-free cocompact lattices in* $\mathrm{SL}_2(\mathbb{R})$ *and* Λ_j *be the logarithmic derivative of the Selberg zeta function for* $X_j = \Gamma_j\backslash\mathcal{H}$.

1. *If the sequence* $(\Gamma_j)_{j\in\mathbb{N}}$ *is uniformly discrete and Plancherel, then*

$$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\mathrm{vol}(\Gamma_j\backslash G)} = 0$$

   for $\mathrm{Re}(s) > 1$.

2. *If for* $\mathrm{Re}(s) > 1$ *one has*

$$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\mathrm{vol}(\Gamma_j\backslash G)} = 0$$

   then the sequence $(\Gamma_j)_{j\in\mathbb{N}}$ *is Plancherel.*

In [31, §4] it is asked whether the condition of uniform discreteness is actually necessary in the first part of the statement. This condition was needed to avoid the possible accumulation of eigenvalues in fixed intervals caused by short geodesics (cf. [17, §8.4]). We will see in Section 4.3 that a careful analysis of the accumulation rate of eigenvalues in Plancherel sequences allows one to remove

the assumption of uniform discreteness from Theorem 4.1.1.
Next, Deitmar [31, §4] also considered the limits $\mathrm{vol}(\Gamma_j\backslash G)^{-1}\Lambda_j(s)$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \leq 1$. The functional equation [31, Prop. 3.4] for the logarithmic derivative of the Selberg zeta function allows one to deal with $s \in \mathbb{C} \smallsetminus \mathscr{P}$ such that $\mathrm{Re}(s) < 0$ (cf. Section 4.4). It remains to determine what happens for s sitting inside the critical strip

$$S = \{s \in \mathbb{C} \mid 0 \leq \mathrm{Re}(s) \leq 1\}.$$

In Section 4.4 we will use a formula of McKean for the logarithmic derivative of the Selberg zeta function to compute the limit of $\mathrm{vol}(\Gamma_j\backslash G)^{-1}\Lambda_j(s)$ for all s sitting in

$$U_1 = \{s \in \mathbb{C} \ \mid \mathrm{Re}(s) > 1/2,\ \mathrm{Re}(s(s-1)) > -\frac{1}{4},\ s \notin \mathscr{P}\}.$$

For $s \in \mathbb{C} \smallsetminus \mathscr{P}$ we let

$$F(s) = \begin{cases} 0,\ \mathrm{Re}(s) > \frac{1}{2}, \\ (s - \frac{1}{2})\cot(\pi s),\ \mathrm{Re}(s) < \frac{1}{2}. \end{cases}$$

Then the final result is

Theorem 4.1.2. *Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a sequence of torsion-free cocompact lattices in $G = \mathrm{SL}_2(\mathbb{R})$. The following two statements are equivalent:*

1. *The sequence $(\Gamma_j)_{j\in\mathbb{N}}$ is Plancherel.*
2. *For each $s \in \mathbb{C} \smallsetminus \mathscr{P}$ such that either $s \in U_1$, $\mathrm{Re}(s) < 0$ or $\mathrm{Re}(s) > 1$ one has*
$$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\mathrm{vol}(\Gamma_j\backslash G)} = F(s).$$

The values $s \in \mathbb{C}$, for which the behaviour of the logarithmic derivative of the Selberg zeta function is known, are sketched in Figure 4.1.

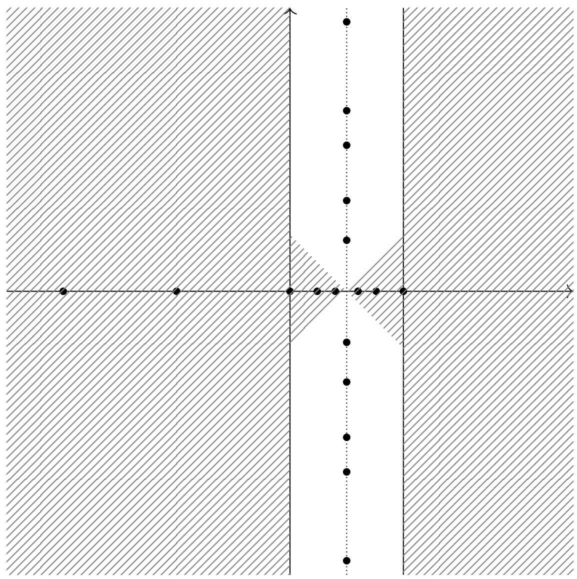

Figure 4.1: The values $s \in \mathbb{C}$ from Theorem 4.1.2

4.2 Tools from Spectral Geometry

In this section we extend our discussion of the Laplace operator to manifolds with boundary and collect some standard tools from spectral geometry, which can be used to estimate the counting function of the Laplace operator. Further details can be found in [22, §1.5].

Let X be a smooth compact manifold (possibly with boundary) equipped with a Riemannian metric h and let Δ be the corresponding Laplace operator. One can study solutions $\phi \in C^2(X)$ of the eigenvalue equation

$$\Delta\phi = \lambda\phi \tag{4.1}$$

by introducing the Sobolev space $\mathscr{H}(X)$. On the Sobolev space $\mathscr{H}(X)$ one may use the Dirichlet integral

$$D[f, h] = (\operatorname{grad} f, \operatorname{grad} h)$$

to give the following weak formulation of (4.1)

$$D[\phi, f] = -\lambda(\phi, f), \qquad \phi \in C^2(\overline{X}), \tag{4.2}$$

valid for certain boundary conditions on X and f sitting in certain closed subspaces of $\mathscr{H}(X)$:

- **Closed eigenvalue problem:** In this case we assume that X has no boundary. For fixed $\phi \in C^2(X)$ the functional $F_\phi = D[\phi, \cdot]$, initially defined on $C^\infty(X)$, can be extended to the whole space $\mathscr{H}(X)$ by using standard arguments from functional analysis. If in this case (4.2) is valid for some $\lambda \in \mathbb{C}$ and all $f \in \mathscr{H}(X)$, we call λ a *(closed) eigenvalue.*

- **Neumann eigenvalue problem:** Here, one considers (4.2) under the assumption that $\phi \in C^2(\overline{X})$ and $\nu\phi = 0$ on ∂X, where ν is the outward unit normal vector field on the boundary of X. Then F_ϕ can again be extended to $\mathscr{H}(X)$ and the solutions λ of (4.2) will be called *Neumann eigenvalues.*

- **Dirichlet eigenvalue problem** In this case one considers $\phi \in C^2(\overline{X})$ satisfying $\phi = 0$ on ∂X. Then F_ϕ can be extended to the completion of $C_c^\infty(X)$ in $\mathscr{H}(X)$ and solutions λ of (4.2) will be referred to as *Dirichlet eigenvalues.*

Given each of the above eigenvalue problems we define the space of *admissible functions* $\mathfrak{H}(X)$ to be $\mathscr{H}(X)$ in the case of closed and Neumann eigenvalues and to be the completion of $C_c^\infty(X)$ in $\mathscr{H}(X)$ in the case of Dirichlet eigenvalues. It is well-known (see e.g. [22, Thm. 1.3.1]) that for each of the above eigenvalue problems the solutions λ of (4.2) form an increasing sequence

$$0 \le \lambda_1 \le \lambda_2 \le .. \le \lambda_k \to \infty \text{ as } k \to \infty,$$

where each eigenvalue appears with finite multiplicity. The function, which counts the eigenvalues in the interval $[0, x]$ will be denoted $N(X, x)$ in the case of closed eigenvalues and $N_N(X, x)$ or $N_D(X, x)$ for Neumann eigenvalues and Dirichlet eigenvalues respectively. The characterization (4.2) allows for a variational formulation of the eigenvalue problem:

Theorem 4.2.1 (Max-Min principle)**.** *Let $v_1, ..., v_{k-1} \in L^2(X)$ and for non-zero $f \in \mathfrak{H}(X)$ let*

$$\mathcal{R}(f) = \frac{D[f, f]}{||f||^2}$$

be the so-called Rayleigh-Ritz quotient. Then for any of the above eigenvalue problems one has

$$\inf \mathcal{R}(f) \le \lambda_k, \tag{4.3}$$

where the infimum varies over non-zero functions f orthogonal to the span of $v_1, ..., v_{k-1}$ in $L^2(M)$. If $v_1, ..., v_{k-1}$ form an orthonormal basis of eigenfunctions for the eigenvalues $\lambda_1, ..., \lambda_{k-1}$, we have equality in (4.3).

Proof. [22, §1.5] □

From the Max-Min principle one immediately gets various monotonicity properties of eigenvalues (cf. [51, §3]):

- **Domain monotonicity:** Let $\Omega_1, ..., \Omega_m$ be pairwise disjoint regular[1] domains in X with $X = \overline{\Omega_1} \cup ... \cup \overline{\Omega_m}$, whose boundaries, when intersecting

[1] A subset of Ω of a smooth manifold is called a regular domain, if Ω is connected, has compact closure and non-empty smooth boundary.

∂X, do so transversally[2]. Given an eigenvalue problem on X with respective eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$, consider for each $q = 1, ..., m$ the eigenvalue problem on Ω_q obtained by requiring either Dirichlet or Neumann boundary conditions for all $q = 1, ..., m$ and leaving original data on $\partial\Omega_q \cap \partial X$ unchanged. Arrange all the eigenvalues of $\Omega_1, ..., \Omega_m$ in an increasing sequence, where each eigenvalue is repeated according to its multiplicity. We will denote this sequence by $(\omega_k)_{k\in\mathbb{N}}$ in the case of Dirichlet boundary conditions and $(\kappa_k)_{k\in\mathbb{N}}$ for Neumann boundary conditions.

Lemma 4.2.2. *With the notation as above, one has*

$$\kappa_k \leq \lambda_k \leq \omega_k, \qquad \textit{for all } k \in \mathbb{N}.$$

Proof. [22, §1.5] □

- **Boundary condition monotonicity:** Let P be an operator in divergence form on an interval $[a, b]$,

  $$P = \partial_r \left(f(r)\partial_r\right) - V(r), \;\; a \leq r \leq b$$

  with $f \in C^\infty([a, b])$ such that $f(r) > 0$. Then one can compare eigenvalues for Dirichlet and Neumann boundary conditions as follows:

  Lemma 4.2.3. *Let $(\omega_k)_{k\in\mathbb{N}}$ and $(\kappa_k)_{k\in\mathbb{N}}$ be the solutions of*

  $$Pu(r) + \lambda u(r) = 0$$

  with respect to Dirichlet and Neumann boundary conditions respectively. Then for all $k \in \mathbb{N}$ one has

  $$\kappa_k \leq \omega_k \leq \kappa_{k+2}.$$

  Proof. This is a direct consequence of Corollary 1 in [101, §3]. □

- **Potential monotonicity:** Let $V : [a, b] \to \mathbb{R}$ be a continuous function[3]. We collect the following result for the operator $\partial_r^2 - V(r)$ acting on functions on some interval $[a, b]$:

  Lemma 4.2.4. *Let $P_1 = -\partial_r^2 + V_1(r)$ and $P_2 = -\partial_r^2 + V_2(r)$ be two operators over a compact interval $[a, b]$ with $V_1, V_2 \in C([a, b])$ so that $V_1(r) \geq V_2(r)$ for all $r \in [a, b]$. Then the eigenvalues of P_1 are larger than the corresponding eigenvalues of P_2 with respect to either Dirichlet or Neumann boundary conditions.*

  Proof. [51, Lemma 3.3] □

[2] Two regular domains Ω and Ω' intersect transversally in a point p if the tangent space T_pX is generated by the vectors in $T_p\Omega$ and $T_p\Omega'$.

[3] In this context, V is usually called a potential in formal analogy with the Hamilton operator from quantum mechanics.

4.3 Bounds for the Accumulation Rate

For X a smooth closed hyperbolic surface there are, roughly speaking, two geometric quantities, which control the number of Laplace eigenvalues lying in a fixed interval $[0, x]$: These are the lengths of short geodesics in X and the volume of X. In [31] Deitmar imposed the condition of uniform discreteness to avoid the accumulation of Laplace eigenvalues caused by short geodesics. More precisely, he used uniform discreteness to derive the bound

$$N(\Gamma_j \backslash \mathcal{H}, x) \leq C \operatorname{vol}(\Gamma_j \backslash \mathcal{H}) x \tag{4.4}$$

with $C > 0$ some absolute constant. We will show that the bound (4.4) automatically holds for any Plancherel sequence $(\Gamma_j)_{j \in \mathbb{N}}$. Since the bound (4.4) is the only reason, why uniform discreteness was needed in the proof of [31, Thm. 3.2], one can remove this condition.

Theorem 4.3.1. *For each Plancherel sequence $(\Gamma_j)_{j \in \mathbb{N}}$ of torsion-free cocompact lattices in $\mathrm{PSL}_2(\mathbb{R})$ there exists a positive constant c_A such that*

$$N(\Gamma_j \backslash \mathcal{H}, x) \leq c_A \operatorname{vol}(\Gamma_j \backslash \mathcal{H})(1 + x). \tag{4.5}$$

The proof of this theorem will take up the rest of this section. We note that our approach of proving Theorem 4.3.1 is heavily inspired by the works [23, 51] on degenerating sequences of hyperbolic surfaces and three-manifolds and we will use many of their ideas.
Let $X = \Gamma \backslash \mathcal{H}$ be a smooth closed hyperbolic surface with m_X closed geodesics of length ≤ 1. Recall the thick-thin decomposition $X = X' \cup X''$ from Section 2.3, where

$$X' = \bigcup_{i=1}^{m_X} T'_{\gamma_i}, \qquad X'' = X \smallsetminus X'.$$

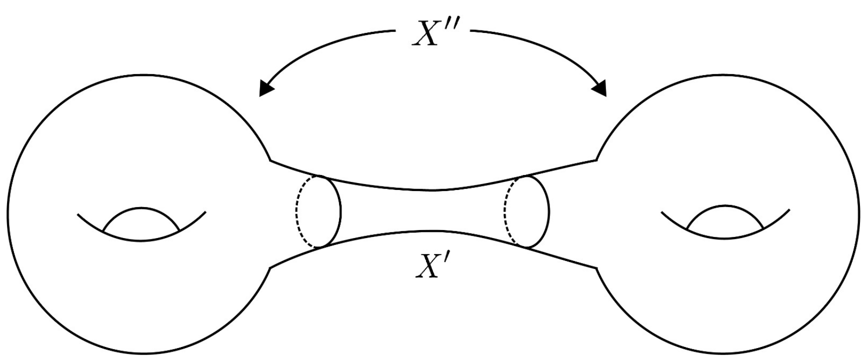

Figure 4.2: The thick-thin decomposition of a surface with a single short geodesic.

Then domain monotonicity for Neumann boundary conditions implies

$$N(X,x) \leq N_N(X',x) + N_N(X'',x). \tag{4.6}$$

We will estimate the counting functions on the right-hand side of (4.6) separately. We start with the thin part X'. Note that the next proof is particularly close to the proof of [51, Thm. 1.4].

Lemma 4.3.2. *There exists a constant $c_{12} > 0$, which does not depend on X, such that for each $x > \frac{1}{4}$ one has*

$$N_N(X',x) \leq c_{12} \left(\sum_{i=1}^{m_X} w_i \right) (1+x),$$

where w_i is the width of the tube T_{γ_i}.

Proof. By domain monotonicity for Neumann boundary conditions one has

$$N_N(X',x) \leq \sum_{i=1}^{m_X} N_N(T'_{\gamma_i},x).$$

To prove the lemma it therefore suffices to show that for a closed geodesic γ in X of length $l \leq 1$ there exists some constant $c_{12} > 0$ such that

$$N_N(T'_\gamma,x) \leq c_{12} w_l (1+x), \tag{4.7}$$

where w_l is the width of the tube T_γ. According to Theorem 2.3.1 we can represent the truncated tube T'_γ by Fermi coordinates

$$\{(r,t) \in \mathbb{R}^2 \mid 1 - w_l \leq r \leq w_l - 1,\ 0 \leq t \leq 1\}/(r,0) \sim (r,1),$$

in which the metric takes the form

$$ds^2 = dr^2 + l^2 \cosh^2 r \ dt^2.$$

In these coordinates the Laplace operator reads[4]

$$-\Delta = \partial_r^2 + \tanh r \ \partial_r + \frac{1}{l^2 \cosh^2 r} \ \partial_t^2. \tag{4.8}$$

To estimate the Neumann spectrum of Δ on the truncated tube T'_γ, we want to make use of its rotational symmetry. Thus, consider the Δ-invariant decomposition

$$L^2(T'_\gamma) = \bigoplus_{n \in \mathbb{Z}} L^2_n(T'_\gamma),$$

[4]We note that in equation (2.2) on page 272 of [51] there is a misprint. There should be a "+" in front of $\tanh r\partial_r$.

where $L^2_n(T'_\gamma)$ consists of smooth, square-integrable functions $v : T'_\gamma \to \mathbb{C}$ of the form $v(r,t) = u(r)e^{2\pi int}$. Let Δ_n be the restriction of Δ to $L^2_n(T'_\gamma)$. This operator is given by

$$\Delta_n = -\partial_r^2 - \tanh r\ \partial_r + \frac{4\pi^2 n^2}{l^2 \cosh^2 r}. \tag{4.9}$$

Let κ_n be an eigenvalue of Δ_n for Neumann boundary conditions and ϕ_n be the corresponding eigenfunction, i.e.

$$\frac{1}{l\cosh r}\ \partial_r(l\cosh r\ \partial_r\phi_n) - \frac{4\pi^2 n^2}{l^2\cosh^2 r}\phi_n + \kappa_n\phi_n = 0 \tag{4.10}$$

and

$$\partial_r\phi_n(1-w_l,t) = \partial_r\phi_n(w_l-1,t) = 0. \tag{4.11}$$

If we multiply (4.10) by $l\cosh r\ \phi_n(r,t)$ and integrate the resulting expression over $[1-w_l, w_l-1]$, we get by using (4.11) that

$$\begin{aligned}
&\kappa_n \int_{1-w_l}^{w_l-1} l\cosh r\ \phi_n^2(r,t)dr - \int_{1-w_l}^{w_l-1} l\cosh r\ (\partial_r\phi_n(r,t))^2 dr \\
&= 4\pi^2n^2 \int_{1-w_l}^{w_l-1} \frac{1}{l^2\cosh^2 r} l\cosh r\ \phi_n^2(r,t)dr.
\end{aligned}$$

Since the function

$$f_l : [-w_l, w_l] \to \mathbb{R},\ f_l(r) = (l\cosh r)^{-2}$$

assumes its minima for $r \in \{-w_l, w_l\}$ and

$$\lim_{l\to 0} f_l(w_l) = \frac{1}{4},$$

there exists an absolute constant $c_{11} > 0$ such that

$$(l\cosh r)^{-2} \geq c_{11}, \qquad \text{for all } l \in (0,1], r \in [1-w_l, w_l-1].$$

Plugging this into the above equality yields

$$\begin{aligned}
&\kappa_n \int_{1-w_l}^{w_l-1} l\cosh r\ \phi_n^2(r,t)dr - \int_{1-w_l}^{w_l-1} l\cosh r\ (\partial_r\phi_n(r,t))^2 dr \\
&\geq 4\pi^2n^2c_{11} \int_{1-w_l}^{w_l-1} l\cosh r\ \phi_n^2(r,t)dr,
\end{aligned}$$

which shows

$$\kappa_n \geq 4\pi^2 n^2 c_{11}. \tag{4.12}$$

Thus, only those $n \in \mathbb{Z}$ with

$$|n| \leq \sqrt{\frac{x}{4\pi^2 c_{11}}} \tag{4.13}$$

have to be accounted for to estimate $N_N(T'_\gamma, x)$. Assume n to be fixed. We will abuse notation and also write Δ_n for the differential operator given by the right-hand side of (4.9) acting on smooth functions $u : [1 - w_l, w_l - 1] \to \mathbb{C}$. We are left with estimating the number of solutions of the (one-dimensional) problem

$$\begin{cases} \Delta_n u = \lambda u, \\ u'(1 - w_l) = u'(w_l - 1) = 0. \end{cases} \tag{4.14}$$

For this, we will consider the conjugate operator

$$\begin{aligned} \widetilde{\Delta}_n &= \cosh^{1/2} r \; \Delta_n \cosh^{-1/2} r \\ &= -\partial_r^2 + \left(\frac{1}{2} - \frac{1}{4} \tanh^2 r + \frac{4\pi^2 n^2}{l^2 \cosh^2 r} \right) \\ &= -\partial_r^2 + \left(\frac{1}{4} + \frac{1}{4\cosh^2 r} + \frac{4\pi^2 n^2}{l^2 \cosh^2 r} \right). \end{aligned}$$

Unfortunately, this conjugation does not respect Neumann boundary conditions. However, it does preserve Dirichlet boundary conditions. For this reason, we switch to Dirichlet boundary conditions by applying boundary condition monotonicity and now consider the Dirichlet problem

$$\begin{cases} u'' - \left(\frac{1}{4} + \frac{1}{4\cosh^2 r} + \frac{4\pi^2 n^2}{l^2 \cosh^2 r} \right) u + \lambda u = 0, \\ u(1 - w_l) = u(w_l - 1) = 0. \end{cases} \tag{4.15}$$

Potential monotonicity shows that the counting function of (4.15) is bounded by the counting function of the problem

$$\begin{cases} u'' - \frac{1}{4} u + \lambda u = 0, \\ u(1 - w_l) = u(w_l - 1) = 0. \end{cases} \tag{4.16}$$

Problem (4.16) can be exactly solved and its solutions are given by

$$u_k(r) = a \sin\left(\frac{k\pi(r - 1 + w_l)}{2w_l - 2} \right), \; \lambda_k = \frac{1}{4} + \left(\frac{k\pi}{2w_l - 2} \right)^2, \qquad \text{for } k \in \mathbb{Z}, \; a \in \mathbb{C}.$$

Since

$$|\{\lambda_k \mid \lambda_k \leq x\}| \leq \frac{4(w_l - 1)}{\pi} \sqrt{x - \frac{1}{4}},$$

we get in view of (4.13) and the earlier mentioned boundary condition monotonicity that

$$N_N(T'_\gamma, x) \leq 2\sqrt{\frac{x}{4\pi^2 c_{11}}} \left(\frac{4(w_l - 1)}{\pi} \sqrt{x - \frac{1}{4}} + 2 \right).$$

This shows

$$N_N(T'_\gamma, x) \leq c_{12} w_l (1 + x)$$

with

$$c_{12} = \frac{4}{\pi\sqrt{c_{11}}}.$$

□

We continue with the thick part. The proof of the following lemma is an adaptation of [23, Lemma 3.6]:

Lemma 4.3.3. *There exists an absolute constant* c_{16} *such that for any* $x > 1/4$ *one has*

$$N_N(X'', x) \leq c_{16} \operatorname{vol}(X)(1 + x). \tag{4.17}$$

Proof. Let us fix[5] $\delta = \frac{1}{2}$ and set $X_1 = X \setminus \bigcup_{i=1}^{m_X} T_{\gamma_i}$. By the fourth statement of Theorem 2.3.1 each point $p \in X_1$ satisfies $r_p(X) > \delta$. Take a maximal set $\mathcal{P}$ of points $p \in X_1$ of pairwise distances greater than or equal to δ. Since balls of radius $\delta/2$ with center at $p \in \mathcal{P}$ are disjoint, we have

$$|\mathcal{P}| \leq \operatorname{vol}(B_{\delta/2}(p))^{-1} \operatorname{vol}(X), \tag{4.18}$$

where $p \in \mathcal{P}$ is arbitrary[6]. Consider the covering $\mathcal{U}$ of X'' consisting of δ-balls $B_\delta(p)$ centered at $p \in \mathcal{P}$ and annuli $\Omega_i = T_{\gamma_i} \setminus T'_{\gamma_i}$, $i = 1, ..., m_X$. We claim that there exists an absolute constant $c_{13} > 0$ such that for every $p_0 \in X''$ the number m_0 of sets $U \in \mathcal{U}$ containing p_0 is at most c_{13}: By the second statement of Theorem 2.3.1 the tubes T_{γ_i}, $i = 1, ..., m_X$ are disjoint, so that p_0 can sit in at most one tube at the same time. If p_0 is contained in some δ-ball $B_\delta(p_1)$ with $p_1 \in \mathcal{P}$, we already have $\operatorname{dist}(p_0, p_1) = \delta$ by the choice of $\mathcal{P}$ and therefore $B_\delta(p_1) \subset B_{2\delta}(p_0)$. Again, the balls of radius $\delta/2$ with center at $p \in \mathcal{P}$ are disjoint, so that

$$m_0 \leq \frac{\operatorname{vol}(B_{2\delta}(p_0))}{\operatorname{vol}(B_{\delta/2}(p_1))} + 1. \tag{4.19}$$

One may check from the fifth statement of Theorem 2.3.1 that

$$r_{p_0}(X) > \operatorname{arcsinh}(\cosh(1/2)\cosh(1) - \sinh(1)) > 0.53 > \frac{\delta}{2}$$

and hence $\operatorname{vol}(B_{\delta/2}(p_0)) = \operatorname{vol}(B_{\delta/2}(p_1))$. Plugging this into (4.19) yields

$$m_0 \leq \frac{\operatorname{vol}(B_{2\delta}(p_0))}{\operatorname{vol}(B_{\delta/2}(p_0))} + 1. \tag{4.20}$$

A standard volume comparison estimate (see e.g. [47, eq. (2.2.2)]) then shows

$$\frac{\operatorname{vol}(B_{2\delta}(p_0))}{\operatorname{vol}(B_{\delta/2}(p_0))} \leq 16e^{2\delta}\delta^2 \tag{4.21}$$

[5] We use the notation $\delta = \frac{1}{2}$ for conceptual clarity and to aid future generalizations of the above argument to any semisimple Lie group G of non-compact type, where the value of δ depends on the Margulis constant of G.

[6] We are using here that $\operatorname{vol}(B_\delta(p))$ does not depend on p, since $r_p(X) > \delta$.

and hence $m_0 \leq c_{13}$ with

$$c_{13} = 1 + 16e^{2\delta}\delta^2. \tag{4.22}$$

We now prove the desired inequality (4.17). Let $(\kappa_k)_{k\in\mathbb{N}}$ be the sequence obtained by arranging all Neumann eigenvalues of all sets $U \in \mathcal{U}$ in increasing order into a single sequence and let $(\kappa'_k)_{k\in\mathbb{N}}$ be the sequence of Neumann eigenvalues of X''. Let us show that

$$\kappa'_k \geq \kappa_k / c_{13}. \tag{4.23}$$

For this consider the Hilbert spaces $\mathscr{H}(X'')$ and $H = \bigoplus_{U\in\mathcal{U}} H^1(U)$ and the restriction map $F : f \mapsto (f|_U)_{U\in\mathcal{U}}$. Then the eigenvalues $(\kappa'_k)_{k\in\mathbb{N}}$ have a Max-Min characterization in terms of the Rayleigh-Ritz quotient

$$\mathcal{R}(f) = \frac{\int_{X''} |\operatorname{grad}(f)|^2}{\int_{X''} |f|^2}.$$

Similarly, the $(\kappa_k)_{k\in\mathbb{N}}$ are critical values of

$$\mathcal{R}_1(f) = \frac{\sum_{U\in\mathcal{U}} \int_U |\operatorname{grad}(f_U)|^2}{\sum_{U\in\mathcal{U}} \cdot \int_U |f_U|^2}.$$

One verifies easily

$$(1/c_{13})\mathcal{R}_1(F(f)) \leq \mathcal{R}(f),$$

which shows (4.23) in view of the Max-Min principle. From (4.23) we get

$$N_N(X'', x) \leq |\mathcal{P}| N_N(B_\delta(p_0), c_{13}x) + \sum_{i=1}^{m_X} N_N(\Omega_i, c_{13}x), \tag{4.24}$$

where $p_0 \in \mathcal{P}$ is arbitrary. Arguing exactly as in Lemma 4.3.2, one may derive for any $x > 1/4$ the bound

$$N_N(\Omega_i, x) \leq c_{14}(1 + x), \qquad i = 1, ..., m_X, \tag{4.25}$$

for some absolute constant $c_{14} > 0$. Furthermore, [47, Thm. 1.2.6] shows for any $x > 0$ that

$$N_N(B_\delta(p_0), x) \leq c_{15}(1 + x) \tag{4.26}$$

with $c_{15} > 0$ some absolute constant. Plugging inequalities (4.18), (4.25) and (4.26) into (4.24) gives

$$N_N(X'', x) \leq \frac{c_{15}}{\operatorname{vol}(B_{\delta/2}(p_0))} \operatorname{vol}(X)(1 + c_{13}x) + c_{14} m_X (1 + c_{13}x).$$

By the first statement of Theorem 2.3.1 we have

$$m_X \leq 3g - 3 < 4\pi(g - 1) = \operatorname{vol}(X) \tag{4.27}$$

and therefore

$$N_N(X'', x) \leq c_{16} \operatorname{vol}(X)(1 + x)$$

with

$$c_{16} = \frac{c_{13}c_{15}}{\operatorname{vol}(B_{\delta/2}(p_0))} + c_{13}c_{14}.$$

□

Finally, we also need a lower bound for the counting function:

Lemma 4.3.4. *For $x > 1/2$ one has*

$$N(X,x) \geq \frac{4}{\pi}\left(\sum_{i=1}^{m_X}(w_i - 1)\right)\sqrt{x - \frac{1}{2}} - 2m_X. \tag{4.28}$$

Proof. By domain monotonicity for Dirichlet boundary conditions one has

$$N(X,x) \geq N_D(X'',x) + \sum_{i=1}^{m_X} N_D(T'_{\gamma_i},x). \tag{4.29}$$

Let the notation be as in the proof of Lemma 4.3.2. It suffices to consider the operator

$$\widetilde{\Delta}_0 = \partial_r^2 - \left(\frac{1}{4} + \frac{1}{4\cosh^2 r}\right). \tag{4.30}$$

Since $\cosh^{-2} r \leq 1$, an application of potential monotonicity to the Dirichlet problem for the operator (4.30) shows that $N_D(T'_\gamma, x)$ is bounded from below by the counting function of the problem

$$\begin{cases} u'' - \frac{1}{2}u + \lambda u = 0, \\ u(1-w_l) = u(w_l - 1) = 0. \end{cases} \tag{4.31}$$

Problem (4.31) can exactly be solved and the solutions are given by

$$u_k(r) = a\sin\left(\frac{k\pi(r-1)}{2w_l - 2}\right),\ \lambda_k = \frac{1}{2} + \left(\frac{k\pi}{2w_l - 2}\right)^2,\ \text{for } k \in \mathbb{Z},\ a \in \mathbb{C}.$$

Hence, for $x > 1/2$ we have

$$N_D(T'_\gamma, x) \geq \frac{4}{\pi}(w_l - 1)\sqrt{x - \frac{1}{2}} - 2,$$

which in view of (4.29) gives the desired inequality (4.28). □

Proof of Theorem 4.3.1. Let m_j be the number of closed geodesics of length smaller than or equal to 1 in $X_j = \Gamma_j\backslash\mathcal{H}$. From (4.6) and the upper bounds for the counting function given in Lemma 4.3.2 and Lemma 4.3.3, we get

$$N(X_j, x) \leq \left[c_{12}\left(\sum_{i=1}^{m_j} w_i\right) + c_{16}\operatorname{vol}(X_j)\right](1+x) \tag{4.32}$$

Now, fix some $x_0 > 1/2$ and let 1_I be the characteristic function of the interval $I = [-x_0, x_0]$. By [82, Lemma 1] there exists an even function h on $\mathbb{R}$, whose Fourier transform is of compact support and such that $1_I \leq h$. Fix some $\varepsilon > 0$. Since $(\Gamma_j)_{j\in\mathbb{N}}$ is Plancherel-convergent, we have for j large enough that

$$|\mu_{\Gamma_j}(h) - \operatorname{vol}(X_j)\mu_{\mathrm{Pl}}(h)| \leq \varepsilon\operatorname{vol}(X_j). \tag{4.33}$$

Combining Lemma 4.3.4 and (4.33) shows

$$\begin{aligned} \frac{4}{\pi}\left(\sum_{i=1}^{m_j}(w_i-1)\right)\sqrt{x_0-\frac{1}{2}}-2m_j &\leq N(X_j,x_0) \\ &\leq \mu_{\Gamma_j}(h) \\ &\leq (\mu_{\mathrm{Pl}}(h)+\varepsilon)\operatorname{vol}(\Gamma_j\backslash\mathcal{H}). \end{aligned}$$

As before, we have $m_j \leq \operatorname{vol}(X_j)$ (cf. (4.27)), so that we may derive from the above inequality that

$$\sum_{i=1}^{m_j} w_i \leq c_{17}\operatorname{vol}(X_j) \tag{4.34}$$

holds with

$$c_{17} = \left(1+\frac{\pi}{4\sqrt{x_0-\frac{1}{2}}}(\mu_{\mathrm{Pl}}(h)+\varepsilon+2)\right).$$

After plugging (4.34) into (4.32), we end up with

$$N(X_j,x) \leq c_A\operatorname{vol}(X_j)(1+x),$$

where

$$c_A = c_{12}c_{17}+c_{16}.$$

This concludes the proof.

□

As a corollary to Theorem 4.3.1 we obtain

Theorem 4.3.5. *Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a sequence of torsion-free cocompact lattices in $\mathrm{SL}_2(\mathbb{R})$ and Λ_j be the logarithmic derivative of the Selberg zeta function of $X_j=\Gamma_j\backslash\mathcal{H}$. The sequence $(\Gamma_j)_{j\in\mathbb{N}}$ is Plancherel convergent if and only if*

$$\lim_{j\to\infty}\frac{\Lambda_j(s)}{\operatorname{vol}(\Gamma_j\backslash G)} = 0$$

for all $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$.

4.4 Convergence inside the Critical Strip

Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a Plancherel-convergent sequence of torsion-free cocompact lattices in $\mathrm{SL}_2(\mathbb{R})$. Theorem 4.1.1 deals with the behaviour of $\operatorname{vol}(\Gamma_j\backslash G)^{-1}\Lambda_j(s)$ in the half-plane $\{s\in\mathbb{C}\mid \mathrm{Re}(s)>1\}$. The restriction to this particular domain was necessary in the proof to ensure the convergence of the Euler product representing the Selberg zeta functions $\zeta_{\Gamma_j}(s)$, $j\in\mathbb{N}$. It is natural to ask what happens for other values of s. One may use the functional equation of ζ_{Γ_j} to deal with $s\in\mathbb{C}\smallsetminus\mathscr{P}$ such that $\mathrm{Re}(s)<0$:

Proposition 4.4.1. *Let Γ be a torsion-free cocompact lattice. For $s \in \mathbb{C} \setminus \mathscr{P}$ one has*
$$\Lambda_\Gamma(s) + \Lambda_\Gamma(1-s) = \left(s - \frac{1}{2}\right) \operatorname{vol}(\Gamma \backslash G) \cot(\pi s).$$

Proof. [31, Prop. 3.4] □

Corollary 4.4.2. *Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a Plancherel-convergent sequence of torsion-free cocompact lattices. Then for $s \in \mathbb{C} \setminus \mathscr{P}$ with $\mathrm{Re}(s) < 0$ we have*
$$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\operatorname{vol}(\Gamma_j \backslash G)} = \left(s - \frac{1}{2}\right) \cot(\pi s).$$

Proof. This directly follows from Theorem 4.1.1 and Proposition 4.4.1. □

It remains to work out the case of $s \in \mathbb{C}$ lying in the critical strip
$$S = \{s \in \mathbb{C} \mid 0 \leq \mathrm{Re}(s) \leq 1\}.$$
If $s \in \mathscr{P}$, we may have that s is a pole of Λ_j for some $j \in \mathbb{N}$, in which case $\operatorname{vol}(\Gamma_j \backslash G)^{-1}\Lambda_j(s)$ would not even be defined. We therefore restrict our attention to those s sitting outside $\mathscr{P}$. Central to our treatment of these values will be McKean's formula [65, p.239] for the logarithmic derivative of the Selberg zeta function:

Proposition 4.4.3 (McKean's formula)**.** *Let $X = \Gamma\backslash\mathcal{H}$ be a smooth closed hyperbolic surface, $(\lambda_k)_{k\in\mathbb{N}}$ the Laplace eigenvalues of X and Λ_Γ be the logarithmic derivative of the Selberg zeta function. Let s be a complex number sitting in*
$$U_0 = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1,\ \mathrm{Re}(s(s-1)) > 0\}.$$
Then one has
$$\Lambda_\Gamma(s) = (2s-1)\int_0^\infty e^{-t\xi}(\theta(t) - \varphi_0(t)\operatorname{vol}(X))dt, \tag{4.35}$$
where $\theta(t) = \sum_{k=0}^\infty e^{-t\lambda_k}$ is the trace of the heat kernel, $\varphi_0(t) = (4\pi)^{-1}\mu_{\mathrm{Pl}}(e^{-t\lambda})$ is the fundamental solution of the heat equation at the origin and $\xi = s(s-1)$.

Proof. Fix some $s \in U_0$ and let $\xi = s(s-1)$. Let us first check that the integral on the right-hand side of (4.35) is absolutely convergent. Since both $\theta(t)$ and $\varphi_0(t)$ are monotonically decreasing positive functions, the integrand is $O(e^{-t\,\mathrm{Re}(\xi)})$ for $t \to \infty$. For $t \to 0$ one has the known small-time asymptotics (see e.g. [66])
$$\theta(t) \sim \frac{\operatorname{vol}(X)}{4\pi t} + O(1),\ \ \varphi_0(t) \sim \frac{1}{4\pi t} + O(1), \tag{4.36}$$
which shows that $\theta(t) - \varphi_0(t)\operatorname{vol}(X)$ stays bounded as $t \to 0$. Thus, we have
$$\int_0^\infty |e^{-t\xi}(\theta(t) - \varphi_0(t)\operatorname{vol}(X))|dt < \infty. \tag{4.37}$$

Let us evaluate this integral. Recall from (2.14) that

$$\sum_{[\gamma]} \frac{l_{\gamma_0}}{e^{l_\gamma/2} - e^{-l_\gamma/2}} \frac{e^{-t/4 - l_\gamma^2/4t}}{\sqrt{4\pi t}} = \theta(t) - \varphi_0(t)\,\mathrm{vol}(X) \tag{4.38}$$

holds. We also need the well-known Laplace transform [74, p.41]

$$\int_0^\infty e^{-tz} \frac{e^{-a^2/4t}}{\sqrt{4\pi t}} dt = \frac{1}{2} \frac{e^{-a\sqrt{z}}}{\sqrt{z}}, \qquad \text{for } \mathrm{Re}(z) > 0,\ \mathrm{Re}(a) \geq 0,$$

where for $\sqrt{z} = \exp(\frac{1}{2}\log(z))$ one takes the logarithm with branch cut at the non-positive real numbers. Multiplying (4.38) with $e^{-t\xi}$ and integrating over $[0, \infty)$ yields

$$\begin{aligned}
\int_0^\infty e^{-t\xi}(\theta(t) - \varphi_0(t)\,\mathrm{vol}(X))dt &= \sum_{[\gamma]} \frac{l_{\gamma_0}}{e^{l_\gamma/2} - e^{-l_\gamma/2}} \int_0^\infty e^{-t(\xi+1/4)} \frac{e^{-l_\gamma^2/4t}}{\sqrt{4\pi t}} \\
&= \frac{1}{2s-1} \sum_{[\gamma]} \frac{l_{\gamma_0}}{e^{l_\gamma/2} - e^{-l_\gamma/2}} e^{-l_\gamma(s-1/2)} \\
&= \frac{1}{2s-1} \Lambda_\Gamma(s)
\end{aligned}$$

where we used the expression (2.16) for the logarithmic derivative of the Selberg zeta function, which is valid for $\mathrm{Re}(s) > 1$. Note that interchanging the integral and the infinite sum in the first step is allowed, since (4.37) permits us to use Fubini's theorem. This concludes the proof. □

McKean's formula expresses Λ_Γ as a Laplace transform in the variable ξ. The poles in the range $-\frac{1}{4} < \xi \leq 0$ correspond to the small eigenvalues $0 \leq \lambda_k < \frac{1}{4}$ of $\Gamma\backslash\mathcal{H}$. Wolpert [102, p.285] realized that after subtracting the contributions coming from small eigenvalues,

$$\frac{\Lambda_\Gamma(s)}{2s-1} - \sum_{0 \leq \lambda_k < \frac{1}{4}} \frac{1}{s(s-1) + \lambda_k} = \int_0^\infty e^{-t\xi}(\theta^*(t) - \varphi_0(t)\,\mathrm{vol}(X))dt, \tag{4.39}$$

where $\theta^*(t) = \sum_{\lambda_k \geq \frac{1}{4}} e^{-t\lambda_k}$, the remaining integral on the right-hand side of (4.39) stays finite on the larger domain $\mathrm{Re}(\xi) > -1/4$. It is essentially this observation, which allows us to extend Theorem 4.1.1 into the critical strip S:

Theorem 4.4.4. *Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a Plancherel sequence. For s sitting in*

$$U_1 = \{s \in \mathbb{C} \ \mid \mathrm{Re}(s) > 1/2,\ \mathrm{Re}(s(s-1)) > -\frac{1}{4},\ s \notin \mathscr{P}\}.$$

one has

$$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\mathrm{vol}(\Gamma_j\backslash\mathcal{H})} = 0.$$

Proof. For $j \in \mathbb{N}$ let $X_j = \Gamma_j \backslash \mathcal{H}$ be the smooth closed hyperbolic surface associated to the lattice Γ_j and $(\lambda_{j,k})_{k\in\mathbb{N}}$ be the Laplace eigenvalues of X_j. For $s \in U_0$ we have by McKean's formula

$$\begin{aligned}
\frac{\Lambda_j(s)}{\mathrm{vol}(X_j)} &= (2s-1)\int_0^1 e^{-t\xi}\left(\frac{\theta_j(t)}{\mathrm{vol}(X_j)} - \varphi_0(t)\right)dt \\
&+ (2s-1)\int_1^\infty e^{-t\xi}\left(\frac{\theta_j^*(t)}{\mathrm{vol}(X_j)} - \varphi_0(t)\right)dt \\
&+ \frac{(2s-1)}{\mathrm{vol}(X_j)}\int_1^\infty e^{-t\xi}\left(\sum_{0\le\lambda_{j,k}<\frac{1}{4}1} e^{-t\lambda_{j,k}}\right)dt \\
&=: I_{1,j}(s) + I_{2,j}(s) + I_{3,j}(s),
\end{aligned}$$

where $\theta_j^*(t) = \sum_{\lambda_{j,k}\ge\frac{1}{4}} e^{-t\lambda_{j,k}}$. Let us first check that the equality

$$\frac{\Lambda_j(s)}{\mathrm{vol}(X_j)} = I_{1,j}(s) + I_{2,j}(s) + I_{3,j}(s) \tag{4.40}$$

can be extended to U_1. Starting with $I_{3,j}$ we observe that

$$I_{3,j}(s) = \frac{2s-1}{\mathrm{vol}(X_j)} \sum_{0\le\lambda_{j,k}<\frac{1}{4}} \frac{e^{-(\xi+\lambda_{j,k})}}{\xi+\lambda_{j,k}}, \tag{4.41}$$

which shows that $I_{3,j}$ extends to a holomorphic function on U_1. Regarding $I_{2,j}(s)$ we claim that the integral defining $I_{2,j}(s)$ converges absolutely for $s \in U_1$. To see this we write $T_k = \lambda_{j,k} - \frac{1}{4}$ and let k_0 be the smallest natural number so that $T_{k_0} \ge 0$. Then we have

$$\begin{aligned}
\frac{\theta_j^*(t)}{\mathrm{vol}(X_j)} &= \frac{1}{\mathrm{vol}(X_j)}\sum_{k=k_0}^\infty m(X_j,\lambda_{j,k})e^{-t\lambda_{j,k}} \\
&= \frac{e^{-t/4}}{\mathrm{vol}(X_j)}\sum_{k=k_0}^\infty m(X_j, T_k+1/4)e^{-tT_k} \\
&\le \frac{e^{-t/4}}{\mathrm{vol}(X_j)}\sum_{T=0}^\infty e^{-tT}\left(\sum_{T\le T_k\le T+1} m(X_j,T_k+1/4)\right) \\
&\le \frac{e^{-t/4}}{\mathrm{vol}(X_j)}\sum_{T=0}^\infty e^{-tT}N(X_j,T+5/4) \\
&\le c_A\left(\sum_{T=0}^\infty e^{-tT}(T+9/4)\right)e^{-t/4}.
\end{aligned}$$

Since

$$\phi_0(t) = (4\pi)^{-1}\left(\int_{-\infty}^\infty e^{-tr^2} r\tanh(\pi r)dr\right)e^{-t/4},$$

and $b(t) = \int_{-\infty}^{\infty} e^{-tr^2} r \tanh^2(\pi r) dr$ is monotonically decreasing on $[1, \infty)$, we get for $t \geq 1$

$$\left| \frac{\theta_j^*(t)}{\mathrm{vol}(X_j)} - \varphi_0(t) \right| \leq c_{18} e^{-t/4}, \tag{4.42}$$

where

$$c_{18} = c_A \left(\sum_{T=0} e^{-T}(T + 9/4) \right) + \frac{b(1)}{4\pi} < \infty.$$

This shows that the integral representing $I_{2,j}(s)$ converges absolutely for $s \in U_1$ and by a standard result on parameter-valued integrals (see e.g. [38, Satz IV.5.6]) we have that the resulting function $I_{2,j} : U_1 \to \mathbb{C}$ is continuous. Next we will prove that $I_{2,j}$ is holomorphic on U_1 by using Morera's theorem. Hence, consider a closed piecewise smooth curve η in U_1 and let

$$f_j(t) = \frac{\theta_j^*(t)}{\mathrm{vol}(X_j)} - \varphi_0(t).$$

Then

$$\begin{aligned} \int_\eta I_{2,j}(s) ds &= \int_\eta \int_1^\infty (2s-1) e^{-ts(s-1)} f_j(t) dt ds \\ &= \int_1^\infty \left(\int_\eta (2s-1) e^{-ts(s-1)} ds \right) f_j(t) dt = 0, \end{aligned}$$

and the holomorphy of $I_{2,j} : U_1 \to \mathbb{C}$ follows. In the above computation the application of Fubini's theorem in the second step is justified due to the integrand being absolutely integrable. In the last step we used Cauchy's integral theorem for the analytic function $f(s) = (2s-1)e^{-ts(s-1)}$ on the simply-connected domain U_1. For $I_{1,j}(s)$ one notes that the small-time asymptotics (4.36) gives

$$\left| \frac{\theta_j(t)}{\mathrm{vol}(X_j)} - \varphi_0(t) \right| \leq c(X_j)$$

for $c(X_j) > 0$ some constant depending on X_j and then proceeds as before to show that $I_{1,j}$ defines a holomorphic function on U_1. Hence, we have shown that $I_{1,j}(s) + I_{2,j}(s) + I_{3,j}(s)$ is a holomorphic function on U_1, which by McKean's formula agrees with $\mathrm{vol}(X_j)^{-1}\Lambda_j$ on a subset containing an accumulation point. By the identity theorem for holomorphic function this implies (4.40) for $s \in U_1$. Now, we will estimate each of the three contributions in (4.40) separately. Let us start with $I_{3,j}(s)$. We have

$$|I_{3,j}(s)| = \frac{1}{\mathrm{vol}(X_j)} \left| \sum_{0 \leq \lambda_{j,k} < \frac{1}{4}} \frac{2s-1}{s(s-1) + \lambda_{j,k}} \right| \leq C_s \frac{N(X_j, \frac{1}{4})}{\mathrm{vol}(X_j)}, \tag{4.43}$$

where

$$C_s = \sup_{y \in [0, \frac{1}{4}]} \left| \frac{2s-1}{s(s-1) + y} \right| < \infty.$$

Since the small eigenvalues $\{\lambda_{j,k} \mid \lambda_{j,k} < \frac{1}{4}\}$ correspond to either the complementary series representations $\widehat{G}_{\mathrm{comp}}$ or the trivial representation π_1, we get

$$|I_{3,j}(s)| \le C_s \frac{\mu_{\Gamma_j}(\{\pi_1\} \cup \widehat{G}_{\mathrm{comp}} \cup \{\pi_0^+\})}{\mathrm{vol}(X_j)}. \tag{4.44}$$

Neither the trivial representation nor the complementary series representations are contained in the support of the Plancherel measure, so that Theorem 2.11.3 shows that $I_{3,j}(s)$ converges to zero for $j \to \infty$. Let us turn to $I_{1,j}(s)$. We have for any $x_0 > 0$ that

$$\begin{aligned}
|I_{1,j}(s)| &= \left|(2s-1)\int_0^1 e^{-t\xi}\left(\frac{\theta_j(t)}{\mathrm{vol}(X_j)} - \varphi_0(t)\right)dt\right| \\
&\le |2s-1| \int_0^1 e^{-t\,\mathrm{Re}(\xi)}\left(\frac{\theta_j(t)}{\mathrm{vol}(X_j)} - \varphi_0(t)\right)dt \\
&\le |2s-1|e^{x_0} \int_0^1 e^{-t(\mathrm{Re}(\xi)+x_0)}\left(\frac{\theta_j(t)}{\mathrm{vol}(X_j)} - \varphi_0(t)\right)dt \\
&\le |2s-1|e^{x_0} \int_0^\infty e^{-t(\mathrm{Re}(\xi)+x_0)}\left(\frac{\theta_j(t)}{\mathrm{vol}(X_j)} - \varphi_0(t)\right)dt
\end{aligned}$$

where we have used that $\theta_j(t) - \mathrm{vol}(X_j)\varphi_0(t)$ is positive for $t > 0$, which follows from (4.38). By choosing x_0 so that $\mathrm{Re}(\xi) + x_0 \ge 2$ holds, we get from another application of McKean's formula that

$$|I_{1,j}(s)| \le \left(\frac{|2s-1|e^{x_0}}{5}\right)\frac{\Lambda_j(2)}{\mathrm{vol}(X_j)}. \tag{4.45}$$

and the right-hand side of (4.45) converges to 0 for $j \to \infty$ by Theorem 4.3.5. It remains to check that $I_{2,j}(s)$ vanishes in the limit $j \to \infty$. Consider the functions

$$\phi_j(t) = e^{-t\xi}\left(\frac{\theta_j^*(t)}{\mathrm{vol}(X_j)} - \varphi_0(t)\right), \qquad t \in [1,\infty).$$

We want to apply Lebesgue's theorem to derive

$$\lim_{j\to\infty} \int_1^\infty \phi_j(t)dt = 0. \tag{4.46}$$

From (4.42) we see that $\phi(t) = c_{18}e^{-t\delta}$ with $\delta = \mathrm{Re}(\xi) + 1/4 > 0$ is an integrable majorant for the $\phi_j,\ j \in \mathbb{N}$. We are left with showing that $\phi_j(t) \to 0$ as $j \to \infty$. Let $\varepsilon > 0$ be some positive real number and

$$h_t(\lambda) = \begin{cases} e^{-t\lambda}, & \lambda \ge \frac{1}{4}, \\ 0, & \text{else}. \end{cases}$$

Since the trivial representation and the complementary series do not lie in the support of the Plancherel measure, we have

$$\phi_j(t) = e^{-t\xi}\left(\frac{\mu_{\Gamma_j}(h_t)}{\mathrm{vol}(X_j)} - \mu_{\mathrm{Pl}}(h_t)\right).$$

Let $J \subset [0,\infty)$ be a relatively open, bounded interval and 1_J the characteristic function of J. By Sauvageot's density principle [84] one has

$$\left|\frac{\mu_{\Gamma_j}(1_J)}{\mathrm{vol}(X_j)} - \mu_{\mathrm{Pl}}(1_J)\right| \leq \varepsilon, \tag{4.47}$$

for j large enough, which clearly extends to any linear combination of such characteristic functions. Standard measure theory [38, Satz III.4.13] allows us to find a sequence of linear combinations $(g_i)_{i\in\mathbb{N}}$ of such functions with

$$0 \leq g_i \nearrow h_t, \text{ as } i \to \infty$$

outside a set of Plancherel measure zero, and therefore by the Theorem of monotone convergence

$$|\mu_{\mathrm{Pl}}(g_i) - \mu_{\mathrm{Pl}}(h_t)| \leq \varepsilon \tag{4.48}$$

for i large enough. The proof of [38, Satz III.4.13] shows that for the function h_t we even have that

$$\Phi_i(T) = \sup_{\lambda\in[T,T+1)} |h_t(\lambda) - g_i(\lambda)| \tag{4.49}$$

tends to zero as $i \to \infty$. Furthermore,

$$\begin{aligned} 0 \leq \mu_{\Gamma_j}(h_t - g_i) &= \sum_{k=0}^{\infty} h_t(\lambda_{j,k}) - g_i(\lambda_{j,k}) \\ &\leq \sum_{T=0}^{\infty} \Phi_i(T)\left(N(X_j, T+1) - N(X_j, T)\right) \\ &\leq c_A \,\mathrm{vol}(X_j) \sum_{T=0}^{\infty} \Phi_i(T)(2T+3), \end{aligned}$$

where we made use of Theorem 4.3.1. Since $\Phi_i(T) \leq 2e^{-T}$, we can apply dominated convergence to the last sum to derive

$$|\mu_{\Gamma_j}(h_t) - \mu_{\Gamma_j}(g_i)| \leq \varepsilon \,\mathrm{vol}(X_j) \tag{4.50}$$

for i large enough. Combining the estimates (4.47), (4.48) and (4.50) we end up with

$$\begin{aligned} \left|\frac{\mu_{\Gamma_j}(h_t)}{\mathrm{vol}(X_j)} - \mu_{\mathrm{Pl}}(h_t)\right| &\leq \frac{1}{\mathrm{vol}(X_j)}|\mu_{\Gamma_j}(h_t) - \mu_{\Gamma_j}(g_i)| \\ &+ \left|\frac{\mu_{\Gamma_j}(g_i)}{\mathrm{vol}(X_j)} - \mu_{\mathrm{Pl}}(g_i)\right| \\ &+ |\mu_{\mathrm{Pl}}(g_i) - \mu_{\mathrm{Pl}}(h_t)| \leq 3\varepsilon, \end{aligned}$$

for i and j sufficiently large. This implies $\lim_{j\to\infty} \phi_j(t) = 0$ for $t \geq 1$. □

Chapter 5

Spectral Geometry of Congruence Surfaces and Zeta Functions

5.1 Summary

When describing the behaviour of zeta functions of a Plancherel sequence $(X_j)_{j\in\mathbb{N}}$, we have, until now, always excluded the set

$$\mathscr{P} = \{-k \mid k \in \mathbb{N}\} \cup [0,1] \cup (\tfrac{1}{2} + i\mathbb{R})$$

from our considerations, as any point in this set could be a pole of the logarithmic derivative of the Selberg zeta function. The goal of this chapter is to close this gap by discussing the expected behaviour of the zeta functions for $s \in \mathscr{P}$.
Let us start with considering the critical line $\frac{1}{2}+i\mathbb{R}$. The poles of Λ_j will become dense in $\frac{1}{2} + i\mathbb{R}$ for $j \to \infty$, in the sense that any fixed open subset of $\frac{1}{2} + i\mathbb{R}$ contains a pole of Λ_j for j sufficiently large. If we are given a point $s \in \frac{1}{2} + i\mathbb{R}$, which is not a pole of any of the Λ_j, the literature unfortunately does not offer a way to bound the rate with which the poles are approaching the point s. The author expects the behaviour of $\mathrm{vol}(\Gamma_j\backslash G)^{-1}\Lambda_j(s)$ to be chaotic, even though he is unable to offer any precise results in this direction.
Regarding the set $\{-k \mid k \in \mathbb{N}\}$ we note that according to Theorem 2.8.1 any $s = -k$ with $k \in \mathbb{N}$ is a pole of Λ_j for each $j \in \mathbb{N}$, so that one either completely ignores these points or removes the poles, in which case the asymptotics follow from Theorem 4.1.2 and the functional equation of the Selberg zeta function.
This leaves us with the interval $[0,1]$. Any pole in $[0,1]$ comes from a small eigenvalue $\lambda \leq \frac{1}{4}$ and these can be controlled in terms of the spectral geometry of the surface in question (cf. [86]). In some arithmetic scenarios, there conjecturally are no non-zero small eigenvalues and using partial results towards this conjecture, we can show the following

Proposition 5.1.1. *Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of smooth closed congruence surfaces with* $\mathrm{vol}(X_j) \to \infty$. *Then for $s \in (\frac{39}{64}, 1)$ one has*

$$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\mathrm{vol}(X_j)} = 0.$$

Furthermore, for the Euler-Selberg constants one has

$$\lim_{j\to\infty} \frac{\gamma_{X_j}}{\mathrm{vol}(X_j)} = 0.$$

The above proposition fails for arbitrary Plancherel sequences:

Proposition 5.1.2. *There exist a Plancherel-convergent sequence of smooth closed arithmetic surfaces $(X_j)_{j\in\mathbb{N}}$ such that*

$$\lim_{j\to\infty} \frac{\gamma_{X_j}}{\mathrm{vol}(X_j)} = l_0$$

for some positive constant $l_0 > 0$.

This suggests that, while one still may get convergence of the (possibly renormalized) zeta functions, the limit is not uniquely fixed by the property of being a Plancherel sequence. The construction of the arithmetic surfaces from Proposition 5.1.2 will be the main content of the following sections. On our way we will demonstrate a technique to construct congruence surfaces containing prescribed Y-pieces. The proofs of the above propositions can be found in Section 5.3.

5.2 Arithmetic and Geometry of Two-Generator Subgroups

The goal of this section is to study a few arithmetic and geometric aspects of two-generator subgroups of $\mathrm{PSL}_2(\mathbb{R})$. The first part of this discussion is based on works of Maskit (cf. [62, 63]).

Let us first consider the relationship between certain two-generator subgroups and Y-pieces. Let Y_0 be a Y-piece. Since the signature of Y_0 is $(0, 3)$, the associated Fuchsian group Γ with $\Gamma\backslash\mathcal{H} \cong Y_0$ is of the form

$$\Gamma = \langle \alpha, \beta, \gamma \mid \alpha\beta\gamma = 1 \rangle \tag{5.1}$$

(cf. [72]), where α, β and γ correspond to loops in Y_0, which are freely homotopic to the boundary geodesics of Y_0. Clearly, our presentation of Γ in (5.1) is redundant and we can write $\Gamma = \langle \alpha, \beta \rangle$ for two hyperbolic transformations $\alpha, \beta \in \mathrm{PSL}_2(\mathbb{R})$. The axes A_α and A_β are disjoint and an elementary computation (see [54, Lemma 1]) shows that[1]

$$\mathrm{tr}\,\alpha\,\mathrm{tr}\,\beta\,\mathrm{tr}\,\alpha\beta < 0. \tag{5.2}$$

[1]Note that the sign of the left-hand side of (5.2) does not depend on the choice of lifts for α, β.

Now, we consider the converse direction and start with a two-generator subgroup $\Gamma = \langle \alpha, \beta \rangle$ of $\mathrm{PSL}_2(\mathbb{R})$ and try to find the right obstructions on α and β so that $\Gamma \backslash \mathcal{H}$ is a surface of signature $(0,3)$ with boundary geodesics given by α, β and $(\alpha\beta)^{-1}$. Let Y_0 be a Y-piece, whose boundary geodesics are given by a, b and c with respective lengths l_a, l_b and l_c. If we cut Y_0 open along the common orthogonals of the boundary geodesics, we get a geodesic hexagon, which can be isometrically embedded into the upper half-plane:

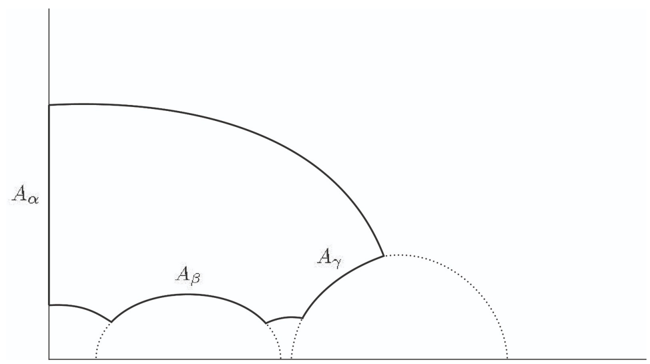

Figure 5.1: A geodesic hexagon in the upper half-plane.

In view of Figure 5.1, it is natural (cf. [63, §3]) to make the following

Definition 5.2.1. A triple $(\alpha, \beta, \gamma) \in \mathrm{PSL}_2(\mathbb{R})$ of transformations are called *geometric generators for a pants group* if the following hold:

1. The transformations α, β and γ are hyperbolic and the three axes A_α, A_β and A_γ are disjoint.
2. The three axes A_α, A_β and A_γ bound a common region D.
3. When traversing any of these axes from the repelling fixed point to the attracting point, the region D always lies on the right.
4. The group $\Gamma = \langle \alpha, \beta, \gamma \rangle$ has a single defining relation, namely $\alpha\beta\gamma = 1$.

We call $\Gamma = \langle \alpha, \beta \rangle$ a *pants group.*

Theorem 5.2.2. *Let $\alpha, \beta \in \mathrm{PSL}_2(\mathbb{R})$ be two hyperbolic transformations. If the axes A_α, A_β are disjoint and (5.2) holds, the transformations α, β and $(\alpha\beta)^{-1}$ are geometric generators of a pants group.*

Proof. [62, Thm. 4.1] $\square$

Let us explicitly write down matrices fulfilling the conditions of the above definition. Let us fix $x, y, \mu \in \mathbb{R}_{>0}$. If we make the choice (cf. [63, §4])

$$\alpha = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}, \qquad \beta = \frac{1}{\sinh \mu} \begin{pmatrix} \sinh(\mu - y) & \sinh y \\ -\sinh y & \sinh(\mu + y) \end{pmatrix}, \tag{5.3}$$

the attracting and repelling fixed point of α are given by 0 and ∞ respectively, while for β the attracting and repelling fixed point are given by $e^{-\mu}$ and e^{μ} respectively. We have

$$\gamma = (\alpha\beta)^{-1} = \frac{1}{\sinh\mu}\begin{pmatrix} e^{-x}\sinh(\mu+y) & -e^{x}\sinh y \\ e^{-x}\sinh y & e^{x}\sinh(\mu-y) \end{pmatrix}. \tag{5.4}$$

It is an elementary exercise to show that one can choose the three parameters x, y and μ in such a way that

$$\operatorname{tr}\alpha = 2\cosh(l_a/2), \qquad \operatorname{tr}\beta = 2\cosh(l_b/2), \qquad \operatorname{tr}\gamma = -2\cosh(l_c/2)$$

holds. By applying Theorem 5.2.2 to α, β we see that $\Gamma = \langle\alpha, \beta\rangle$ is a pants group.

Remark 5.2.3. *For the interested reader we note that the quantity μ is related to the distance d between A_α and A_β via $\coth\mu = \cosh d$.*

In summary, we have a bijection between conjugacy classes of pants groups and isometry classes of Y-pieces, where a pants group $\Gamma = \langle\alpha, \beta\rangle$ is mapped to the quotient surface $Y = \Gamma\backslash\mathcal{H}$ of signature $(0, 3)$.
Now, assume we are given a pants group $\Gamma = \langle\alpha, \beta\rangle$ contained in a cocompact torsion-free Fuchsian group Γ'. The next proposition gives a criterion to decide, when the geodesics α, β and $\gamma = (\alpha\beta)^{-1}$ appear as the boundary geodesics of some Y-piece Y_0 in the hyperbolic surface $X = \Gamma'\backslash\mathcal{H}$:

Proposition 5.2.4. *Let $X = \Gamma'\backslash\mathcal{H}$ be a smooth closed hyperbolic surface. Let $\alpha, \beta \in \Gamma'$ be geometric generators of a pants group $\Gamma = \langle\alpha, \beta\rangle$. Assume that the geodesics α and β are simple and disjoint. Then the geodesic $\gamma = (\alpha\beta)^{-1}$ is simple and disjoint to both α and β. Furthermore, the three geodesics α, β and γ form the boundary geodesics of a pair of pants Y_0 in X.*

Proof. Let us choose a point $z_0 \in X$ and fix an isomorphism $\Gamma' \cong \pi_1(X, z_0)$. A small free homotopy, which is sketched in Figure 5.2, deforms $\gamma = (\alpha\beta)^{-1}$ into a curve μ, which is simple and disjoint to α, β.

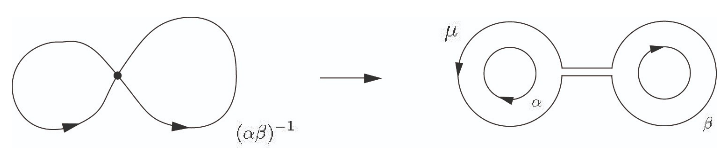

Figure 5.2: A small homotopy, which deforms $\gamma = (\alpha\beta)^{-1}$ into a curve disjoint to α, β.

Since γ is the unique geodesic in the free homotopy class of μ, we get from [17, Thm. 1.6.6] that γ is simple and from [17, Thm. 1.6.7] that γ is disjoint to α and β. Now, the Theorem of Baer-Zieschang [17, Thm. A.3] guarantees the existence of a homeomorphism $\phi : X \to X$ such that $\phi(\alpha) = \alpha$, $\phi(\beta) = \beta$ and

$\phi(\mu) = \gamma$. Since cutting the surface along α, β and μ disconnects the surface X into a surface Y of signature $(0,3)$ and some remainder $X'' = X \setminus Y$, the same must be true for their respective images under ϕ. Therefore, $\phi(Y)$ has signature $(0,3)$. $\square$

Remark 5.2.5. *The author expects that Proposition 5.2.4 is known to experts, but has not yet seen it in the literature.*

Let us turn to the arithmetic side. Let as before $\Gamma = \langle \alpha, \beta \rangle$ be a two-generator subgroup of $\mathrm{PSL}_2(\mathbb{R})$. We call Γ an *arithmetic* two-generator subgroup, if there exists a Fuchsian group Γ_a derived from a quaternion algebra such that $\Gamma \subset \Gamma_a$. If Γ is in addition a pants group, we call it an *arithmetic pants group.* We have the following neat criterion for the arithmeticity of two-generator subgroups:

Lemma 5.2.6. *Let Γ be a two-generator subgroup of $\mathrm{PSL}_2(\mathbb{R})$ with invariant trace field $F = F(\Gamma)$ and invariant quaternion algebra $\mathcal{A} = \mathcal{A}(\Gamma)$. Then Γ is arithmetic if the following two conditions are fulfilled:*

1. *F is a totally real number field and the traces $\operatorname{tr}\alpha, \operatorname{tr}\beta$ and $\operatorname{tr}\alpha\beta$ lie in the ring of integers $\mathfrak{o}_F$ of F.*
2. *The invariant quaternion algebra $\mathcal{A}$ is unramified at exactly one infinite place of F.*

Proof. Let $w(\alpha, \beta)$ be a word in the letters α, β. Then it is a well-known fact (cf. [95]) that there exists a polynomial P_w with integral coefficients such that

$$\operatorname{tr} w(\alpha, \beta) = P_w(\operatorname{tr}\alpha, \operatorname{tr}\beta, \operatorname{tr}\alpha\beta).$$

Therefore, the first condition implies $\operatorname{tr} w(\alpha, \beta) \in \mathfrak{o}_F$ and

$$\{\operatorname{tr}\gamma : \gamma \in \Gamma\} \subset \mathfrak{o}_F \tag{5.5}$$

follows. Using (5.5) one may quickly check that

$$\mathcal{O} = \left\{ \sum a_i \gamma_i \mid a_i \in \mathfrak{o}_F, \gamma_i \in \Gamma \right\} \tag{5.6}$$

defines an order in $\mathcal{A}$, where the sums appearing on the right-hand side of (5.6) are assumed to be finite. Since $\mathcal{A}$ is unramified at exactly one infinite place σ_1 of F, we have that $P_{\sigma_1}\mathcal{O}^1$ is a Fuchsian group derived from a quaternion algebra. By construction we have $\Gamma \subset P_{\sigma_1}\mathcal{O}^1$. $\square$

Our next goal is to find congruence subgroups, which contain certain prescribed arithmetic two-generator subgroups. For this we recall that for each natural number p one has the well-known homomorphism

$$\Phi_p : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$$

given by reducing the matrix entries modulo p, which can be used to classify congruence subgroups[2] of $\mathrm{PSL}_2(\mathbb{Z})$. There exist maps with analogous properties for maximal orders in (most) quaternion algebras:

Proposition 5.2.7. *Let $\mathcal{O}$ be a maximal order in a quaternion algebra $\mathcal{A}$ over a field F. Let σ_1 be an infinite place of F with $\sigma_1 \notin \mathrm{Ram}(\mathcal{A})$. Let $\mathfrak{p}$ be a prime of F of norm $N(\mathfrak{p}) = q$ so that $\mathfrak{p} \notin \mathrm{Ram}(\mathcal{A})$. Then there exists a group epimorphism $\varphi_{\mathfrak{p}} : \mathcal{O}^1 \to \mathrm{SL}_2(\mathbb{F}_q)$ such that*

$$\operatorname{tr} \varphi_{\mathfrak{p}}(x) \equiv \operatorname{trd}(x) \mod \mathfrak{p} \tag{5.7}$$

for each $x \in \mathcal{O}^1$. Furthermore, the ensuing diagram

$$\begin{array}{ccc} \mathcal{O}^1 & \longrightarrow & \mathrm{SL}_2(\mathbb{F}_q) \\ \downarrow & & \downarrow \\ P_{\sigma_1}\mathcal{O}^1 & \dashrightarrow & \mathrm{PSL}_2(\mathbb{F}_q) \end{array}$$

can be completed to a commutative diagram by a unique group epimorphism $\Phi_{\mathfrak{p}} : P_{\sigma_1}\mathcal{O}^1 \to \mathrm{PSL}_2(\mathbb{F}_q)$ such that $\ker \Phi_{\mathfrak{p}} = P_{\sigma_1}(\mathcal{O}^1(\mathfrak{p}))$ is a principal congruence subgroup.

Proof. Let $\mathrm{pr}_{\mathfrak{p}} : \mathcal{O} \to \mathcal{O}/\mathfrak{p}\mathcal{O}$ be the projection map and let us write $[x]_1$ for the equivalence class of $x \in \mathcal{O}$ in $\mathcal{O}/\mathfrak{p}\mathcal{O}$. Let $\pi_{\mathfrak{p}}$ be a uniformizer in $F_{\mathfrak{p}}$. For $y \in \mathcal{O}_{\mathfrak{p}}$ we let $[y]_2$ be the equivalence class of y in $\mathcal{O}_{\mathfrak{p}}/\pi_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$. We claim that

$$f_{\mathfrak{p}} : \mathcal{O}/\mathfrak{p}\mathcal{O} \to \mathcal{O}_{\mathfrak{p}}/\pi_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}, \qquad [x]_1 \mapsto [x]_2$$

defines a vector space isomorphism[3]. Let us first check that $f_{\mathfrak{p}}$ is well-defined. Let $x' \in \mathcal{O}$ with $x' = x + ry$, where $r \in \mathfrak{p}$ and $y \in \mathcal{O}$. We may write $r = \pi_{\mathfrak{p}} u$ for $u \in \mathfrak{o}_{F,\mathfrak{p}}$. Then we have $ry = \pi_{\mathfrak{p}}(uy) \in \pi_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$ and $[x]_2 = [x']_2$ follows. Thus, the map $f_{\mathfrak{p}}$ is well-defined. The proof of linearity is straightforward and left to the reader. Next, let $x, x' \in \mathcal{O}$ with $[x]_2 = [x']_2$. This implies the existence of $y \in \mathcal{O}_{\mathfrak{p}}$ with $x = x' + \pi_{\mathfrak{p}} y$ and hence

$$x - x' = \pi_{\mathfrak{p}} y \in \mathcal{O} \cap (\pi_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}) = \mathfrak{p}\mathcal{O}.$$

This implies $[x]_1 = [x']_1$. Regarding the surjectivity of $f_{\mathfrak{p}}$ we consider an arbitrary element $y = ry' \in \mathcal{O}_{\mathfrak{p}}$ with $r \in \mathfrak{o}_{F,\mathfrak{p}}$ and $y' \in \mathcal{O}$. We can represent r as a power series $r = \sum_j r_j \pi_{\mathfrak{p}}^j$ with $r_j \in \mathfrak{o}_F$. For $x = r_0 y' \in \mathcal{O}$ we get $[x]_2 = [y]_2$. Now, one can utilize linearity of $f_{\mathfrak{p}}$ to prove that $f_{\mathfrak{p}}$ is surjective.
Since $\mathcal{A}$ is unramified at $\mathfrak{p}$, we have $\mathcal{A}^{\mathfrak{p}} \otimes F_{\mathfrak{p}} \cong M_2(F_{\mathfrak{p}})$. Let us fix this isomorphism. Then $\mathcal{O}_{\mathfrak{p}}$ is a maximal order in $M_2(F_{\mathfrak{p}})$ (see [60, Lemma 6.2.8]), which by [60, Thm. 6.5.3] implies the existence of $g \in M_2(F_{\mathfrak{p}})$ such that

[2]To classify all congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ one clearly needs to consider the homomorphisms Φ_N with N a natural number. For simplicity we restrict ourselves to the case of primes.

[3]We silently identify the finite fields $\mathfrak{o}_F/\mathfrak{p}$ and $\mathfrak{o}_{F,\mathfrak{p}}/\pi_{\mathfrak{p}}\mathfrak{o}_{F,\mathfrak{p}}$.

$g\mathcal{O}_\mathfrak{p}g^{-1} = M_2(\mathfrak{o}_{F,\mathfrak{p}})$. Writing $[x]_3$ for the equivalence class of $x \in M_2(\mathfrak{o}_{F,\mathfrak{p}})$ in $M_2(\mathfrak{o}_{F,\mathfrak{p}})/\pi_\mathfrak{p} M_2(\mathfrak{o}_{F,\mathfrak{p}})$, we have the well-defined bijection

$$g_\mathfrak{p} : \mathcal{O}_\mathfrak{p}/\pi_\mathfrak{p}\mathcal{O}_\mathfrak{p} \to M_2(\mathfrak{o}_{F,\mathfrak{p}})/\pi_\mathfrak{p} M_2(\mathfrak{o}_{F,\mathfrak{p}}),\ [x]_2 \mapsto [gxg^{-1}]_3.$$

Since $M_2(\mathbb{F}_q)$ is isomorphic to $M_2(\mathfrak{o}_{F,\mathfrak{p}})/\pi_\mathfrak{p} M_2(\mathfrak{o}_{F,\mathfrak{p}})$, we can compose the three maps $\mathrm{pr}_\mathfrak{p}, f_\mathfrak{p}$ and $g_\mathfrak{p}$ to get a map from $\mathcal{O}$ to $M_2(\mathbb{F}_q)$. It is straightforward to check that any of these three maps preserves reduced traces and reduced norms modulo $\mathfrak{p}$. Since these maps also preserve the unit element and products, we get a group homorphism $\varphi_\mathfrak{p} : \mathcal{O}^1 \to \mathrm{SL}_2(\mathbb{F}_q)$ such that (5.7) holds. The surjectivity of $\varphi_\mathfrak{p}$ is a well-known consequence of strong approximation for quaternion algebras (cf. [98, Lemma 28.5.14]). Note that it is possible to apply strong approximation here, because $\sigma_1 \notin \mathrm{Ram}(\mathcal{A})$ holds.
Let $P_\pm : \mathrm{SL}_2(\mathbb{F}_q) \to \mathrm{PSL}_2(\mathbb{F}_q)$ be the map given by dividing out $\pm\mathbb{1}_2$. We want to show the existence of a group epimorphism $\Phi_\mathfrak{p} : P_{\sigma_1}\mathcal{O}^1 \to \mathrm{PSL}_2(\mathbb{F}_q)$ such that

$$\Phi_\mathfrak{p} \circ P_{\sigma_1} = P_\pm \circ \varphi_\mathfrak{p}. \tag{5.8}$$

For any $x \in P_{\sigma_1}\mathcal{O}^1$ we fix a lift $\widetilde{x} \in \mathcal{O}^1$ and define

$$\Phi_\mathfrak{p}(x) = P_\pm(\varphi_\mathfrak{p}(\widetilde{x})).$$

This map does not depend on the chosen lift, since $\varphi_\mathfrak{p}$ preserves $\pm\mathbb{1}_2$. For another element $y \in P_{\sigma_1}\mathcal{O}^1$ with lift $\widetilde{y} \in \mathcal{O}^1$ we see that $\widetilde{x}\widetilde{y}$ is a lift of the product xy. Hence, the map $\Phi_\mathfrak{p}$ is a group homomorphism. Assume that there exists another group homomorphism $\Phi'_\mathfrak{p} : P_{\sigma_1}\mathcal{O}^1 \to \mathrm{PSL}_2(\mathbb{F}_q)$ such that (5.8) holds. Then for any $x \in P_{\sigma_1}\mathcal{O}^1$ we have

$$\Phi'_\mathfrak{p}(x) = \Phi'_\mathfrak{p}(P_{\sigma_1}(\widetilde{x})) = P_\pm(\varphi_\mathfrak{p}(\widetilde{x})) = \Phi_\mathfrak{p}(P_{\sigma_1}(\widetilde{x})) = \Phi_\mathfrak{p}(x)$$

and $\Phi'_\mathfrak{p} = \Phi_\mathfrak{p}$ follows. The surjectivity of $\Phi_\mathfrak{p}$ follows from the fact that any element $\widetilde{x} \in \mathcal{O}^1$ appears as a lift of some $x \in P_{\sigma_1}\mathcal{O}^1$ and the surjectivity of $\varphi_\mathfrak{p}$. Finally, the identity $\ker\Phi_\mathfrak{p} = P_{\sigma_1}(\mathcal{O}^1(\mathfrak{p}))$ follows from $\mathcal{O}^1(\mathfrak{p}) = \mathrm{pr}_\mathfrak{p}^{-1}([\mathbb{1}_2]_1)$. $\square$

Remark 5.2.8. *Proposition 5.2.7 should be regarded as well-known, even though the author has not yet seen it in the above form in the literature.*

In the following we will sometimes abbreviate $\Phi_\mathfrak{p}(x) = \overline{x}$, when the prime ideal $\mathfrak{p}$ is clear from the context. The map $\Phi_\mathfrak{p}$ gives a bijection between congruence subgroups containing $P_{\sigma_1}(\mathcal{O}^1(\mathfrak{p}))$ and subgroups of $\mathrm{PSL}_2(\mathbb{F}_q)$. The subgroups of $\mathrm{PSL}_2(\mathbb{F}_q)$ are completely classified:

Theorem 5.2.9. *Let p be a rational prime and $q = p^f$ with $f \in \mathbb{N}$. Any subgroup of $\mathrm{PSL}_2(\mathbb{F}_q)$ is contained on the following list:*

- *Cyclic groups: $\mathcal{C}_k = \langle x \mid x^k = 1\rangle$ for $k \geq 1$,*
- *Elementary abelian p-groups: $E_{p^l} = (\mathbb{Z}/p\mathbb{Z})^l$ for $l \in \mathbb{N}$,*
- *Dihedral groups: $D_m = \langle r, s \mid r^m = s^2 = (rs)^2 = 1\rangle$ for $m \geq 2$,*

- *Classical matrix groups:* $\mathrm{PSL}_2(\mathbb{F}_{p^n})$ *and* $\mathrm{PGL}_2(\mathbb{F}_{p^n})$ *for* $n \in \mathbb{N}$*,*
- *Permutation groups:* A_4, S_4 *and* A_5*,*
- *Semidirect products of cyclic groups with elementary abelian* p*-groups.*

Proof. [49, Thm. 8.27] □

Remark 5.2.10. *One can even describe the possible parameters* k, l, m *and* n *in the above theorem in terms of* q*. For our purposes it is more than enough to only list the possible types of groups.*

Let us now describe the generic image of an arithmetic pants group under the reduction map $\Phi_{\mathfrak{p}}$:

Proposition 5.2.11. *Let* $\Gamma = \langle \alpha, \beta \rangle$ *be an arithmetic pants group. For all but finitely many prime ideals the group* $\Phi_{\mathfrak{p}}(\Gamma)$ *is a classical matrix group.*

Proof. The above proposition essentially follows from observing that Γ_0 is a purely hyperbolic group and that $x \in P_{\sigma_1}(\mathcal{O}^1(\mathfrak{p}))$ implies $\mathrm{trd}(x) \equiv \pm 2 \mod \mathfrak{p}$, or in other words, $\mathfrak{p}$ either divides $\mathrm{trd}(x) - 2$ or $\mathrm{trd}(x) + 2$. Let us go through all possible cases. If $\Phi_{\mathfrak{p}}(\Gamma_0)$ is a cyclic group, the commutator $[\alpha, \beta]$ has trace ± 2 modulo $\mathfrak{p}$, which can only be the case for finitely many prime ideals. If $\Phi_{\mathfrak{p}}(\Gamma_0)$ is an elementary abelian p-group, we can, after possibly conjugating, assume that $\Phi_{\mathfrak{p}}(\Gamma_0)$ is contained in $P_{\pm}(N)$, where

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \; x \in \mathbb{F}_q \right\} \subset \mathrm{SL}_2(\mathbb{F}_q)$$

and $P_{\pm} : \mathrm{SL}_2(\mathbb{F}_q) \to \mathrm{PSL}_2(\mathbb{F}_q)$ is the usual projection. This implies that $\mathrm{trd}(\alpha) \equiv \pm 2 \mod \mathfrak{p}$, which is only possible for finitely many prime ideals. Next we assume that $\Phi_{\mathfrak{p}}(\Gamma_0)$ is a dihedral group D_m for some $m \geq 2$. We note that D_m can be written in the form

$$D_m = \{1, r, r^2, ..., r^{m-1}, s, rs, r^2 s, ..., r^{m-1} s\},$$

where the last m elements all have order 2. There are only finitely many prime ideals, for which the order of α, β could be 2, so that $\overline{\alpha} = r^{k_1}$ and $\overline{\beta} = r^{k_2}$ for some $k_1, k_2 \in \mathbb{N}$ in all but finitely many cases. But then the trace of the commutator $[\alpha, \beta]$ reduces to ± 2 modulo $\mathfrak{p}$, which is only possible in finitely many cases. Regarding the permutation groups A_4, S_4 and A_5, we note that there are only finitely many prime ideals, for which the order of $\overline{\alpha}$ is smaller than or equal to $|A_5| = 60$. Finally, let us assume that $\Phi_{\mathfrak{p}}(\Gamma_0)$ is given by a semidirect product of a cyclic group with a elementary abelian p-group. After possibly conjugating, we can assume that $\Phi_{\mathfrak{p}}(\Gamma_0)$ is contained in $P_{\pm}(B)$, where

$$B = \left\{ \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix} \middle| \; x \in \mathbb{F}_q, \; y \in \mathbb{F}_q^{\times} \right\}.$$

This again implies that the trace of $[\alpha, \beta]$ reduces to ± 2 modulo $\mathfrak{p}$ and we are done. □

In the next section we will see an example of an arithmetic two-generator subgroup Γ_0 such that $\Phi_{\mathfrak{p}}(\Gamma_0)$ is a dihedral group.

5.3 Arithmetic Surfaces and Zeta Functions

In this section we first prove Proposition 5.1.1 and then construct the Plancherel sequence needed for Proposition 5.1.2. We begin with the following well-known lower bound for the first non-zero eigenvalue of a congruence surface:

Theorem 5.3.1. *Let X_c be a congruence surface. Then one has*

$$\lambda_1(X_c) \geq \frac{1}{4} - \left(\frac{7}{64}\right)^2 = 0.238... \tag{5.9}$$

Proof. This lower bound follows from bounds towards the generalized Ramanujan conjecture [9], see e.g. [96] for the case $F = \mathbb{Q}$. □

The lower bound from Theorem 5.3.1 guarantees large pole-free regions of the logarithmic derivative of the Selberg zeta function:

Proof of Proposition 5.1.1. Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of congruence surfaces with $\mathrm{vol}(X_j) \to \infty$ and let Λ_j be the logarithmic derivative of the Selberg zeta function of X_j. Using the notation and results from the proof of Theorem 4.4.4, we have

$$\frac{\Lambda_j(s)}{\mathrm{vol}(X_j)} = I_{1,j}(s) + I_{2,j}(s) + I_{3,j}(s).$$

for any $s \in (\frac{1}{2}, 1)$, which is not a pole of Λ_j. By Theorem 4.4.4 any sequence of smooth closed congruence surfaces $(X_j)_{j\in\mathbb{N}}$ with $\mathrm{vol}(X_j) \to \infty$ is Plancherel-convergent, so that we can repeat the proof of Theorem 4.4.4 to show that $I_{1,j}(s)$ and $I_{2,j}(s)$ converge to zero for $s \in (\frac{1}{2}, 1)$. Now, fix a point $s \in (\frac{39}{64}, 1)$ and let $\varepsilon > 0$ such that $s = \frac{39}{64} + \varepsilon$. Since

$$I_{3,j}(s) = \frac{2s-1}{\mathrm{vol}(X_j)} \sum_{0 \leq \lambda_{j,k} < \frac{1}{4}} \frac{e^{-(\xi + \lambda_{j,k})}}{\xi + \lambda_{j,k}},$$

with $\xi = s(s-1)$ we can use the bound from Theorem 5.3.1 to derive

$$|I_{3,j}(s)| \leq \frac{1}{\frac{7}{32}\varepsilon + \varepsilon^2} \frac{N(X_j, \frac{1}{4})}{\mathrm{vol}(X_j)}. \tag{5.10}$$

The right-hand side of (5.10) converges to zero by the limit multiplicity property of $(X_j)_{j\in\mathbb{N}}$ and

$$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\mathrm{vol}(X_j)} = 0$$

for $s \in (\frac{39}{64}, 1)$ follows. The claim regarding the Euler-Selberg constants can be proved analogously. □

While congruence surfaces conjecturally do not have any non-trivial small eigenvalues, this is not true for arithmetic surfaces. For the proof of Proposition 5.1.2 we will construct a Plancherel-convergent sequence of arithmetic surfaces $(X_j)_{j\in\mathbb{N}}$, which have a single non-zero eigenvalue converging to $\lambda_0 = 0$. This single eigenvalue will yield a non-trivial contribution to the limit of $\mathrm{vol}(X_j)^{-1}\gamma_{X_j}$. The construction of this sequence will take up the rest of this section.
Let $\tau = e^{l_\tau/2}$ be a Salem number of degree $2n$ with $n \geq 2$ so that $\tau < 1.2$. We additionally assume that τ has been chosen such that it is the smallest Salem number in its degree. Let $L = \mathbb{Q}(\tau)$ be the associated number field. Let $F = \mathbb{Q}(\omega)$ be the totally real subfield of L generated by $\omega = \tau + \tau^{-1}$. Let $\sigma_1, ..., \sigma_n$ be the real embeddings of F, chosen in such a way that $\sigma_1(\omega) > 2$. Let $x, y \in \mathfrak{o}_L$ be non-zero algebraic integers and $m \in \mathbb{N}$. A prime ideal $\mathfrak{P}$ is called a *primitive divisor* of $x^m - y^m$, if $\mathfrak{P} \mid x^m - y^m$ and $\mathfrak{P} \nmid x^k - y^k$ holds for $k < m$. For primitive divisors, we have the following result by Schinzel and Postnikova [77, Thm. 1]:

Theorem 5.3.2. *Let x, y be relatively prime algebraic integers in a number field L such that $\frac{x}{y}$ is not a root of unity. Then there exists a constant $m_0 = m_0(x, y)$ such that $x^m - y^m$ has a primitive divisor for $m \geq m_0$.*

Hence, we can find $m_0 = m_0(\tau, 1) \in \mathbb{N}$ such that $\tau^m - 1$ has a primitive divisor for $m \geq m_0$. Let p_0 be a rational prime with $m = 4p_0 > m_0$ and $\mathfrak{P}_0$ be a primitive divisor of $\tau^{4p_0} - 1$,

$$\mathfrak{P}_0 \mid \tau^{4p_0} - 1, \qquad \mathfrak{P}_0 \nmid \tau^k - 1 \text{ for } k < 4p_0. \tag{5.11}$$

Let $\mathfrak{p}_0 = \mathfrak{P}_0 \cap \mathfrak{o}_F$ be the prime ideal of F lying below $\mathfrak{P}_0$. After possibly replacing p_0 by a larger prime, we can assume that $\mathfrak{p}_0$ is a non-dyadic[4] prime ideal so that

$$\mathfrak{p}_0 \nmid 4, \qquad \mathfrak{p}_0 \nmid \omega^2 - 4. \tag{5.12}$$

We now want to make a few assumptions on $\mathfrak{p}_0$ to be able to deal with torsion elements later on. First of all, there are only finitely many cyclotomic extensions $K_1, ..., K_s$ of F with $[K_j : F] = 2$. Furthermore, in each cyclotomic extension K_j there are only finitely many elements $x_{j_1}, ..., x_{j_t}$ of finite order. Let us write m_{j_k} for the order of x_{j_k} and $t_{j_k} = \mathrm{Tr}_{L/F}(x_{j_k})$ for the respective traces. After possibly replacing p_0 we can assume that

$$p_0 \nmid m_{j_k}, \qquad \mathfrak{p}_0 \nmid t_{j_k}^2 - 4, \qquad j = 1, ..., s, \; k = 1, ..., t. \tag{5.13}$$

Next, we recall the strong approximation theorem for number fields:

Theorem 5.3.3. *Let S_0 be a finite set of places of F and $\mathfrak{q} \in \Omega_F \smallsetminus S_0$. Assume that $x_\mathfrak{p}$ is given for $\mathfrak{p} \in S_0$. Then for each $\varepsilon > 0$ there exists $x \in F$ such that*

- *$|x - x_\mathfrak{p}|_\mathfrak{p} < \varepsilon$ for each $\mathfrak{p} \in S_0$,*
- *$|x_\mathfrak{p}|_\mathfrak{p} \leq 1$ for each $\mathfrak{p} \in \Omega \smallsetminus (S_0 \cup \mathfrak{q})$.*

[4] A prime ideal $\mathfrak{p}$ is called dyadic, if it lies over (2) and called non-dyadic otherwise.

Proof. [73, §33G] $\square$

According to Theorem 5.3.3 we can find $\theta \in \mathfrak{o}_F$ so that

$$\sigma_1(\theta) > 2, \qquad \sigma_j(\theta^2 + \omega^2 - 4) < 0,\ j = 2, ..., n, \qquad v_{\mathfrak{p}_0}(\theta) = 1. \tag{5.14}$$

Let us choose matrices α_0, β as in Section 5.2 with

$$\operatorname{tr} \alpha_0 = \omega, \qquad \operatorname{tr} \beta = \theta^2 - 2, \qquad \operatorname{tr} \alpha_0 \beta = -\omega. \tag{5.15}$$

and let β_0 be the unique in element in the stabilizer G_β of β with $\beta = \beta_0^2$. With this choice we have

$$\operatorname{tr} \alpha_0 = \omega, \qquad \operatorname{tr} \beta_0 = \theta, \qquad \operatorname{tr} \alpha_0 \beta_0 = 0. \tag{5.16}$$

Let $\Gamma_0 = \langle \alpha_0, \beta_0 \rangle$ be the subgroup generated by α_0, β_0.

Lemma 5.3.4. *The group $\Gamma_0 = \langle \alpha_0, \beta_0 \rangle$ is an arithmetic two-generator subgroup. For the corresponding invariant quaternion algebra $\mathcal{A} = \mathcal{A}(\Gamma_0)$ we have $\sigma_2, ..., \sigma_n \in \operatorname{Ram}(\mathcal{A})$ and $\sigma_1, \mathfrak{p}_0 \notin \operatorname{Ram}(\mathcal{A})$.*

Proof. According to [60, Thm. 3.6.1] we can express the Hilbert symbol of $\mathcal{A}$ by

$$\mathcal{A} = \left(\frac{\theta^2 - 4, \theta^2 + \omega^2 - 4}{F} \right).$$

Since a real quaternion algebra $\mathcal{B} = \left(\frac{a,b}{\mathbb{R}}\right)$ ramifies if and only if $a, b < 0$, we get from (5.14) that $\sigma_1 \notin \operatorname{Ram}(\mathcal{A})$ and $\sigma_2, ..., \sigma_n \in \operatorname{Ram}(\mathcal{A})$. Hence, Lemma 5.2.6 implies that Γ_0 is an arithmetic two-generator subgroup. Regarding the finite place $\mathfrak{p}_0$ we observe that a quaternion algebra $\mathcal{D} = \left(\frac{a,b}{F_\mathfrak{p}}\right)$ ramifies at $\mathfrak{p}_0$ if and only if for $a = a_0 \pi_\mathfrak{p}^{v_\mathfrak{p}(a)}$ and $b = b_0 \pi_\mathfrak{p}^{v_\mathfrak{p}(b)}$ one has

$$(-1)^{v_\mathfrak{p}(a) v_\mathfrak{p}(b)(q_0 - 1)/2} \left(\frac{a_0}{\mathfrak{p}} \right)^{v_\mathfrak{p}(b)} \left(\frac{b_0}{\mathfrak{p}} \right)^{v_\mathfrak{p}(a)} = -1,$$

(see [98, Eq. 12.4.10]), where q_0 is the cardinality of the residue class field $k = \mathfrak{o}_F / \mathfrak{p}_0$ and

$$\left(\frac{a_0}{\mathfrak{p}_0} \right) = \begin{cases} +1, & \text{if } a_0 \in k^{\times 2}, \\ -1, & \text{if } a_0 \in k^\times \smallsetminus k^{\times 2} \end{cases}$$

is the Legendre symbol. By (5.12) and (5.14), we have $v_{\mathfrak{p}_0}(\theta^2 - 4) = 0$ and $v_{\mathfrak{p}_0}(\theta^2 + \omega^2 - 4) = 0$, which shows $\mathfrak{p}_0 \notin \operatorname{Ram}(\mathcal{A})$. $\square$

Next, we work out the image of Γ_0 under the group homomorphism $\Phi_{\mathfrak{p}_0}$:

Lemma 5.3.5. *The group $\Phi_{\mathfrak{p}_0}(\Gamma_0)$ is isomorphic to the dihedral group*

$$D_{2p_0} = \langle r, s \mid r^{2p_0} = s^2 = (rs)^2 = 1 \rangle$$

and a group isomorphism $\psi : D_{2p_0} \to \Phi_{\mathfrak{p}_0}(\Gamma_0)$ is given by sending $r \mapsto \overline{\alpha_0}$ and $s \mapsto \overline{\beta_0}$.

Proof. First, we check that $\operatorname{ord}(\overline{\beta_0}) = 2$. By the choice made in (5.16) we have

$$\operatorname{tr}\beta_0 \equiv 0 \mod \mathfrak{p}_0. \tag{5.17}$$

Combining (5.17) with the Cayley-Hamilton theorem gives $\overline{\beta_0}^2 = 1$. Since (5.17) also implies that $\overline{\beta_0}$ cannot have order 1, we get $\operatorname{ord}(\overline{\beta_0}) = 2$. In the same way one may derive $\operatorname{ord}(\overline{\beta_0\alpha_0}) = 2$. Next, we show $\operatorname{ord}(\overline{\alpha_0}) = 2p_0$. Recall from (5.11) that

$$\mathfrak{P}_0 \mid (\tau^{2p_0} - 1)(\tau^{2p_0} + 1).$$

Since $\mathfrak{P}_0$ is a primitive divisor, we get $\mathfrak{P}_0 \mid (\tau^{2p_0} + 1)$. Furthermore, we also have $\mathfrak{P}_0 \mid \tau^{-2p_0} + 1$, as we can write

$$(\tau^{-2p_0} + 1) = \tau^{-2p_0}(\tau^{2p_0} + 1)$$

and τ^{-2p_0} is a unit. This implies

$$\operatorname{Tr}_{L/F}(\tau^{2p_0} + 1) = (\tau^{2p_0} + 1) + (\tau^{-2p_0} + 1) \in \mathfrak{P}_0 \cap F = \mathfrak{p}_0$$

and hence

$$\operatorname{tr}\alpha^{2p_0} \equiv \pm 2 \mod \mathfrak{p}_0. \tag{5.18}$$

Therefore, $\overline{\alpha_0}^{2p_0}$ lies in some elementary abelian p_0-group $E_{\mathfrak{p}_0} \subset \mathrm{PSL}_2(\mathbb{F}_{\mathfrak{p}_0})$. By (5.12) the group $H = \langle\overline{\alpha_0}\rangle$ is not an elementary abelian p_0-group, so that the Partition Lemma [49, Thm. 8.5] yields

$$\overline{\alpha_0}^{2p_0} \in H \cap E_{\mathfrak{p}_0} = \{1\}.$$

Thus, the order of $\overline{\alpha_0}$ is either $1, 2, p_0$ or $2p_0$. The possibilities $\operatorname{ord}(\overline{\alpha_0}) = 1$ and $\operatorname{ord}(\overline{\alpha_0}) = 2$ can be excluded using (5.12). If $\operatorname{ord}(\overline{\alpha_0}) = p_0$, one either gets $\tau^{p_0} - 1 \in \mathfrak{P}_0$ or $\tau^{p_0} + 1 \in \mathfrak{P}_0$. In both cases we get a contradiction to (5.11) and $\operatorname{ord}(\overline{\alpha_0}) = 2p_0$ follows. Now, it is guaranteed that the assignment

$$r \mapsto \overline{\alpha_0}, \qquad s \mapsto \overline{\beta_0}$$

can be extended to a surjective group homomorphism $\psi : D_{2p_0} \to \Phi_{\mathfrak{p}_0}(\Gamma_0)$. We will prove the injectivity of ψ by showing

$$|\Phi_{\mathfrak{p}_0}(\Gamma_0)| = 4p_0 = |D_{2p_0}|. \tag{5.19}$$

We claim that two elements $\overline{\beta_0}^{\varepsilon_1}\overline{\alpha_0}^{k_1}$ and $\overline{\beta_0}^{\varepsilon_2}\overline{\alpha_0}^{k_2}$ with $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ and $k_1, k_2 \in \{0, 1, ..., 2p_0 - 1\}$ can only be equal if $\varepsilon_1 = \varepsilon_2$ and $k_1 = k_2$ holds. Assume that this is not case, i.e.

$$1 = (\overline{\beta_0}^{\varepsilon_2}\overline{\alpha_0}^{k_2})^{-1}\overline{\beta_0}^{\varepsilon_1}\overline{\alpha_0}^{k_1} = \overline{\alpha_0}^{-k_2}\overline{\beta_0}^{\varepsilon_1-\varepsilon_2}\overline{\alpha_0}^{k_1} \tag{5.20}$$

with either $k_1 \neq k_2$ or $\varepsilon_1 \neq \varepsilon_2$. If $\varepsilon_1 - \varepsilon_2 \equiv 0 \mod 2$, equation (5.20) reads $\overline{\alpha_0}^{k_1-k_2} = 1$. If $k_1 \neq k_2$, we get a contradiction to $\operatorname{ord}(\overline{\alpha_0}) = 2p_0$, since

$|k_1 - k_2| < 2p_0$. In the case $\varepsilon_1 - \varepsilon_2 \equiv 1 \mod 2$ we may apply the relation $\overline{\beta_0}\overline{\alpha_0} = \overline{\alpha_0}^{-1}\overline{\beta_0}$, which follows from $\mathrm{ord}(\overline{\beta_0}\overline{\alpha_0}) = 2$ and $\mathrm{ord}(\overline{\beta_0}) = 2$, to derive

$$\overline{\alpha_0}^{k_1+k_2} = \overline{\beta_0}. \tag{5.21}$$

This again yields a contradiction, since

$$\overline{\beta_0}\overline{\alpha_0}\overline{\beta_0}^{-1} = \overline{\alpha_0}^{-1} \neq \overline{\alpha_0} = \overline{\alpha_0}^{k_1+k_2}\overline{\alpha_0}\ \overline{\alpha_0}^{-(k_1+k_2)}. \tag{5.22}$$

This shows $|\Phi_{\mathfrak{p}_0}(\Gamma_0)| \geq 4p_0$. The surjectivity of ψ implies $|\Phi_{\mathfrak{p}_0}(\Gamma_0)| \leq 4p_0$ and (5.19) follows. This concludes the proof. $\square$

From now on, we fix the isomorphism ψ and write $D_{2p_0} = \Phi_{\mathfrak{p}_0}(\Gamma_0)$. Let us consider the normal subgroup $N = \langle r^2 \rangle$ of D_{2p_0}. Writing $\Gamma_c = \Phi_{\mathfrak{p}_0}^{-1}(N)$ and $\Gamma_1 = \Phi_{\mathfrak{p}_0}^{-1}(D_{2p_0})$, we clearly have $\Gamma_c \trianglelefteq \Gamma_1$. The quotient group $\Gamma_c \backslash \Gamma_1$ is the Klein four group and can be presented as

$$V_4 = \{1, [\alpha_0], [\beta_0], [\alpha_0\beta_0]\}.$$

We have an action of $V_4 = \Gamma_c\backslash\Gamma_1$ on the quotient space $X_c = \Gamma_c\backslash\mathcal{H}$ via

$$[\gamma].(\Gamma_c z) = \Gamma_c(\gamma.z)$$

Let us denote

$$\alpha = \alpha_0^2, \qquad \delta = \beta_0\alpha\beta_0^{-1}, \qquad \eta = \delta^{-1}\alpha^{-1} = (\beta_0\alpha_0^{-1})^2, \qquad \rho = \alpha_0\eta\alpha_0^{-1}.$$

Let us also write $\eta_0 = \beta_0\alpha_0^{-1}$. We make the following observations:

- The involution $[\alpha_0]$ preserves the geodesics α, δ and interchanges η, ρ.
- The involution $[\eta_0]$ preserves the geodesics η, ρ and interchanges α, δ.

Our next goal is to utilize the symmetry group V_4 to study the geometry of our surface:

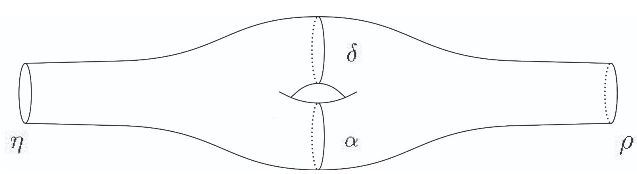

Figure 5.3: The geometric configuration of the four geodesics α, δ, η and ρ.

Lemma 5.3.6. *The quotient space $X_c = \Gamma_c\backslash\mathcal{H}$ is a smooth closed congruence surface. The four geodesics corresponding to α, β, η and ρ are simple and pairwise disjoint. Let Y_η be the Y-piece with boundary geodesics α, δ, η and Y_ρ be the Y-piece with boundary geodesics α, δ, ρ. Then we have $Y_\eta^\circ \cap Y_\rho^\circ = \emptyset$.*

Proof. The group Γ_c is a congruence subgroup, since by Proposition 5.2.7 it contains $P_{\sigma_1}(\mathcal{O}^1(\mathfrak{p}_0))$. From $F \neq \mathbb{Q}$ it follows that X_c is closed (cf. [53, Thm. 5.4.1] and [53, Thm. 5.2.6]). Regarding the smoothness of X_c we need to prove that Γ_c is torsion-free. Assume to the contrary that there exists a torsion-element $x_t \in \Gamma_c$ of order $m_t > 1$. By the second assumption made in (5.13) the element $\overline{x_t} \in N$ has order m_t. But any non-trivial element in N has order p_0, so that we get a contradiction to the first assumption made in (5.13). Thus, Γ_c is torsion-free.
Next, we show that the geodesic α is simple. Since its length $l_\alpha = 4\log\tau$ is smaller than 1, we get from [17, Thm. 4.2.1] that α is the power of some simple geodesic α', i.e. $\alpha = (\alpha')^{k_0}$ for some $k_0 \in \mathbb{Z}$. If we let $\tau' = e^{l_{\alpha'}/2}$ and $L' = F[\tau']$, then $\tau = (\tau')^{k_0}$ and $L \subset L'$ follows. Since $[L : \mathbb{Q}] = 2n = [L' : \mathbb{Q}]$, we have $L = L'$ and the minimality assumption on τ implies $|k_0| = 1$. In particular, the geodesic α is simple. The same argument proves that δ is simple.
Let us check that α and δ are disjoint. Assume to the contrary that δ intersects α. As the width w_δ of the tube T_δ around δ is greater than 1 and $l_\alpha < 1$, the geodesic α is completely contained in T_δ. This tube contains exactly two simple closed geodesics, namely δ and δ^{-1}. Then Proposition 2.2.3 guarantees the existence of an element $x_0 \in \Gamma_c$ with $\delta = x_0\alpha x_0^{-1}$. From $\delta = \beta_0\alpha\beta_0^{-1}$ we get

$$x_0^{-1}\beta_0 \in G_\alpha \cap P_{\sigma_1}\mathcal{O}^1 = \{\alpha_0^k \mid k \in \mathbb{Z}\}.$$

If we let $x_0^{-1}\beta_0 = \alpha_0^{k_0}$ and $\overline{x_0} = \overline{\alpha_0}^{2k_1}$ for some $k_0, k_1 \in \mathbb{Z}$, we get

$$\overline{\beta_0} = \Phi_{\mathfrak{p}_0}(x_0\alpha_0^{k_0}) = \overline{\alpha_0}^{k_0+2k_1}$$

in contradiction to Lemma 5.3.5. Hence, the geodesics α and δ are disjoint. From the choice of the matrices representing α and δ one can check that $\Gamma = \langle\alpha, \delta\rangle$ is a pants group. Therefore, we can employ Proposition 5.2.4 to conclude that η and ρ are simple geodesics and that they are disjoint to α and δ.
It remains to show that η and ρ are disjoint. First, we claim that η and ρ intersect in at most finitely many points. Assume that this is not the case. Then η and ρ agree as point sets and applying Proposition 2.2.3 guarantees the existence of $x_0 \in \Gamma_c$ with $\rho = x_0\eta x_0^{-1}$. From $\rho = \alpha_0\eta\alpha_0^{-1}$ we get

$$x_0^{-1}\alpha_0 \in G_\eta \cap \Gamma_1.$$

We claim that

$$G_\eta \cap \Gamma_1 = \{\eta_0^k \mid k \in \mathbb{Z}\}. \tag{5.23}$$

Otherwise, there would be an element $\eta_0' \in \Gamma_1$ with $\eta_0 = (\eta_0')^k$ for some $k \in \mathbb{Z}$ with $|k| > 1$. In consequence, $[\eta_0']$ would be an element in $\Gamma_c\backslash\Gamma_1$ of order greater than 2, which is a contradiction to $\Gamma_c\backslash\Gamma_1$ being the Klein four group. From (5.23) we get

$$x_0^{-1}\alpha_0 = \eta_0^{k_0} \tag{5.24}$$

for some $k_0 \in \mathbb{Z}$. Let $\overline{x_0} = \overline{\alpha_0}^{2k_1}$ for some $k_1 \in \mathbb{Z}$. We can apply the map $\Phi_{\mathfrak{p}_0}$ to (5.24) to derive

$$\overline{\alpha_0}^{1-2k_1} = (\overline{\beta_0}\,\overline{\alpha_0}^{-1})^{k_0}. \tag{5.25}$$

If in (5.24) the number k_0 is even, say $k_0 = 2k_2$ for $k_2 \in \mathbb{N}$, we get

$$\alpha_0 = x_0 \eta^{k_1} \in \Gamma_c,$$

which is a contradiction. If k_0 is odd, we see that the element on the left-hand side of (5.24) has order p_0, while the one on the right-hand of (5.24) side either has order 1 or 2. This is a contradiction. Thus, η and ρ differ as point sets and can only intersect in finitely many points.
Now, assume that η and ρ intersect in the points $z_1, ..., z_{n_0}$. We note that these points come in pairs $(z, [\alpha_0\beta_0].z)$, since the symmetry $[\eta_0]$ preserves η, ρ and acts non-trivially on η. Let us first assume that we have at least 4 (non-identical) points of intersection $z_1, ..., z_4, ...$, where the ordering is chosen according to the order in which η passes through ρ. Let r be the common orthogonal in Y_ρ between α and δ. We let η_{12} be the subarc of η in Y_ρ between z_1 and z_2 and η_{34} be the subarc of η in Y_ρ between z_3 and z_4. We will lead this to a contradiction by sketching a few homotopies. For the construction of these it is important to note that we know the local geometry around points on the boundary of Y_ρ, as one of the half-tubes $T_{\rho,h}$ is not contained in Y_ρ (see Figure 5.4).

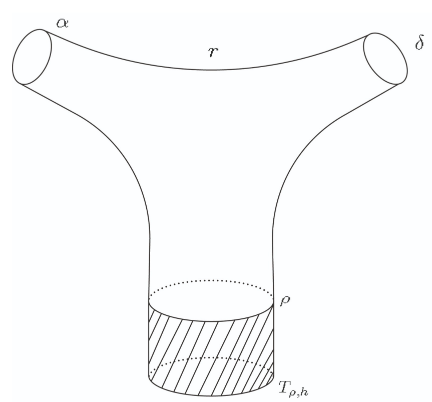

Figure 5.4: The Y-piece Y_ρ.

Now, both of the subarcs η_{12} and η_{34} pass through r, as otherwise we can find a loop η' in the free homotopy class of η such that η' does not intersect ρ in either z_1, z_2 or z_3, z_4 (cf. Figure 5.5). This is a contradiction, since by [17, Thm. 1.6.7] this would imply that η and ρ intersect in at most $n_0 - 2$ points.

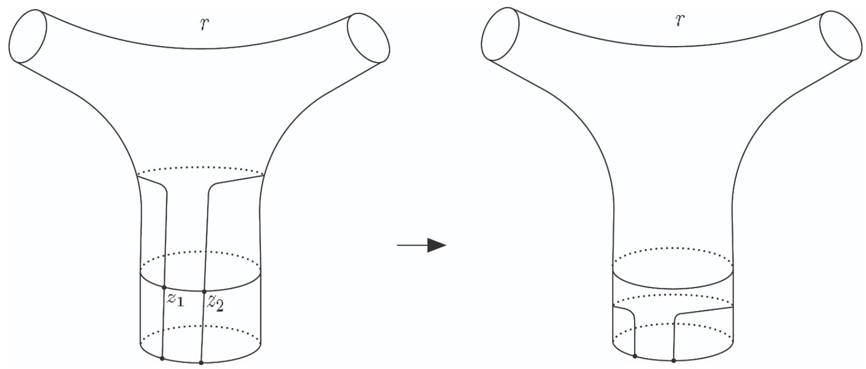

Figure 5.5: A homotopy of η, which decreases the intersection number of η and ρ.

Now, there exists a free homotopy (sketched in Figure 5.6), which moves η_{34} on top of η_{12}.

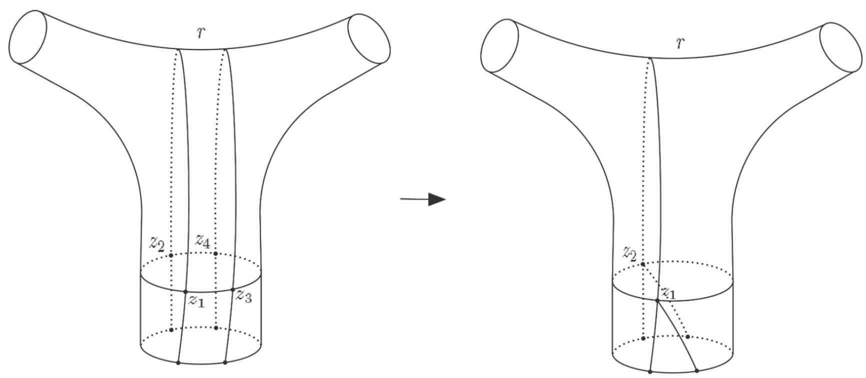

Figure 5.6: A homotopy of η, which moves η_{34} on top of η_{12}.

Again, we can employ [17, Thm. 1.6.7] to derive that η and ρ intersect in at most $n_0 - 2$ points, which is a contradiction. Hence, there exists at most one pair of intersection points (z_1, z_2). Cutting Y_ρ along η_{12} disconnects it into two connected components Y_ρ^+ and Y_ρ^-. We claim that at least one of these components is contained in Y_η. To see this we note that removing α, δ and η disconnects X_c into two connected components, one of which is Y_η. The Y-piece Y_η contains one of the half-tubes $T_{\eta,h}$. Since Y_ρ^+ touches one of the half-tubes around η, while Y_ρ^- touches the other half of the tube, one of the components Y_ρ^+ and Y_ρ^- is contained in Y_η. Without loss of generality we assume that $Y_\rho^+ \subset Y_\eta$. Now, we claim that Y_ρ^- is the image of Y_ρ^+ under the involution $[\eta_0]$. To see this we first note that Y_ρ is fixed by the involution $[\eta_0]$, i.e.

$$[\eta_0].Y_\rho = Y_\rho. \tag{5.26}$$

This follows from observing that $[\eta_0]$ maps the triple of geodesics (α, δ, ρ) to the triple (δ, α, ρ) and hence leaves the decomposition of X_c into Y_ρ and $X_c \setminus Y_\rho$ invariant. The geodesic arc $[\eta_0].\eta_{12}$ runs from z_2 and z_1 and by (5.26) is contained in Y_ρ. If it were different from η_{12}, the geodesic η would be non-simple. Hence, we get

$$[\eta_0].\eta_{12} = \eta_{12}. \tag{5.27}$$

Now, by (5.26) and (5.27) the component Y_ρ^+ is mapped to a connected component of $Y_\rho \setminus \eta$. Hence, either Y_ρ^+ is mapped to itself or to Y_ρ^+. We can exclude the former case by noting that under $[\eta_0]$ the geodesic α is mapped to δ. This implies

$$[\eta_0].Y_\rho^+ = Y_\rho^-. \tag{5.28}$$

Arguing as before we see that $[\eta_0]$ also maps Y_η to itself. This implies $Y_\rho^- \subset Y_\eta$ and $Y_\rho \subset Y_\eta$ follows. Since Y_η does not contain a simple geodesic in its interior, we see by comparing lengths that η and ρ need to be equal as point sets. We have already seen that this is not a case and we arrive at a contradiction
It remains to show that Y_η and Y_ρ only intersect in the boundary. By [17, Thm. 4.1.1] we can find simple closed geodesics $\gamma_5, ..., \gamma_{3g-3}$ such that the geodesics $\alpha, \delta, \eta, \rho, \gamma_5, ..., \gamma_{3g-3}$ are pairwise disjoint and removing them disconnects X_c into $2g-2$ pairs of pants $Y_1, ..., Y_{2g-2}$, so that $Y_j^\circ \cap Y_k^\circ = \emptyset$. After removing the first three geodesics α, δ and η we get from Proposition 5.2.4 that one of the connected components of $X_c \setminus (\alpha \cup \delta \cup \eta)$ is given by Y_η. Since Y_η does not contain any further simple geodesics, which could possibly disconnect it, it has to appear in the list $Y_1, ..., Y_{2g-2}$. Without loss of generality we may assume $Y_1 = Y_\eta$. Repeating the same argument with ρ in place of η shows that $Y_\rho = Y_k$ for some $k = 1, ..., 2g-2$. Since Y_η and Y_ρ have different boundary geodesics, we have $k \neq 1$ and $Y_\eta^\circ \cap Y_\rho^\circ = \emptyset$ follows. □

Let us write $Z = Y_\eta \cup Y_\rho$. Let ξ be the unique simple geodesic running around the hole of Z, which intersects α in one point. Let us abbreviate $g = g(X_c)$ and fix some point $z_0 \in X_c$. We choose a canonical dissection of X_c (cf. [17, §6.7] and the references given there) consisting of loops $a_1, b_1, ..., a_g, b_g$ with base point z_0 such that the unique geodesic in the free homotopy class of a_1 is ξ and the unique geodesic in the free homotopy class of b_1 is α. Then Γ_c can be written in terms of generators and relations as (cf. [48, p.51])

$$\Gamma_c = \langle a_1, b_1, ..., a_g, b_g \mid [a_1, b_1]...[a_g, b_g] = 1 \rangle, \tag{5.29}$$

where we have fixed an isomorphism between Γ_c and $\pi_1(X_c, z_0)$. For $j = 1, ..., 2g$ we write e_j for the vector in $\mathbb{R}^{2g}$ with 1 in the jth place and zeroes elsewhere. Then we have the well-known group homomorphism

$$\varphi_H : \Gamma_c \to H_1(X_c) \cong \mathbb{Z}^{2g}, \tag{5.30}$$

into the first homology group of X_c defined by $\varphi_H(a_j) = e_{2j-1}$ and $\varphi_H(b_j) = e_{2j}$. We let $\Gamma_a = \varphi_H^{-1}(2\mathbb{Z} \times \mathbb{Z}^{2g-1})$ and $X_a = \Gamma_a \backslash \mathcal{H}$ be the corresponding arithmetic surface. We claim that X_a gets disconnected by removing δ and $\xi\delta\xi^{-1}$. This follows from the following

Lemma 5.3.7. *Let X_c^+ be the surface of signature $(g-1,2)$ obtained by cutting X_c along δ and $p^+ : X_c^+ \to X_c$ be the projection map. Let X_c^- be an isometric copy of X_c^+ with projection map $p^- : X_c^- \to X_c$. Let the boundary curve of X_c^-, which under the aforementioned isometry is mapped to δ, be denoted by δ^-. Then X_a is isometric to a surface obtained by gluing X_c^+ and X_c^- along their boundary curves in such a way that δ^- is identified with $\xi\delta\xi^{-1}$.*

Proof. Let $p : X_a \to X_c$ be the covering map of degree 2 given by mapping $\Gamma_a z \mapsto \Gamma_c z$ and let $X_a' = \Gamma_a' \backslash \mathcal{H}$ be the surface obtained by gluing X_c^+ and X_c^- along their boundaries without twisting such that δ^- is identified with $\xi\delta\xi^{-1}$. Then we have a covering map $p' : X_a' \to X_c$ of degree 2 given by mapping any point $z \in X_c^{\pm}$ to $p^{\pm}(z)$. Fix points $\widetilde{z_0} \in X_a$ and $\widetilde{z_0}' \in X_a'$ with $p(\widetilde{z_0}) = p'(\widetilde{z_0}) = z_0$. Furthermore, fix isomorphisms $\Gamma_a \cong \pi_1(X_a, \widetilde{z_0})$ and $\Gamma_a' \cong \pi_1(X_a', \widetilde{z_0}')$. Let $p_* : \Gamma_a \to \Gamma_c$ and $p_*' : \Gamma_a' \to \Gamma_c$ be the push-forwards of the covering maps p and p' respectively. By a standard result of algebraic topology (see [48, Prop. 1.32]) we have that $p_*'(\Gamma_a')$ is a subgroup of Γ_c of index 2. We claim that $p_*'(\Gamma_a') \subset \Gamma_a$. For this we observe that the loops $a_1, b_1^2, a_2, b_2, ..., a_g, b_g$ all lie in the image of p_*', while a quick proof by contradiction shows that b_1 is not contained in $p_*'(\Gamma_a')$. Hence, we have $\varphi_H(p_*'(\Gamma_a')) = 2\mathbb{Z} \times \mathbb{Z}^{2g-1}$ and $p_*'(\Gamma_a') \subset \Gamma_a$ follows. Since both groups Γ_a and $p_*'(\Gamma_a')$ are of index 2 in Γ_c we get $\Gamma_a = p_*'(\Gamma_a')$. In an analogous manner, one may derive $p_*(\Gamma_a) = \Gamma_a$. Hence, we arrive at $p_*(\Gamma_a) = p_*'(\Gamma_a')$. By the Galois theory of coverings this guarantees the existence of a homeomorphism $f : X_a \to X_a'$ such that $p = p' \circ f$ (see [48, Prop. 1.37]). We claim that f is a local isometry. Consider an arbitrary point $\widetilde{z}$ in X_a and let $z = p(\widetilde{z})$ be the corresponding point in X_c. Since p' is a local isometry there exists a ball $B_r(z)$ of radius $r > 0$ around z such that $(p')^{-1} : B_r(z) \to X_a'$ exists and is a local isometry. Hence, we have $f = (p')^{-1} \circ p$ on $B_r(\widetilde{z})$ and f is a local isometry. But any homeomorphism of Riemannian manifolds, which is a local isometry, is a (global) isometry. Hence, X_a and X_a' are isometric. □

This arithmetic surface has the interesting feature that it has a single Laplace eigenvalue close to zero, while the remaining non-zero eigenvalues stay uniformly away from $\lambda_0 = 0$:

Lemma 5.3.8. *The first eigenvalue $\lambda_1(X_a)$ of the arithmetic surface X_a can be bounded from above and below by*

$$\frac{c_{19} l_\tau}{\mathrm{vol}(X_a)} \leq \lambda_1(X_a) \leq \frac{c_{20}\ l_\tau}{\mathrm{vol}(X_a)}, \tag{5.31}$$

where c_{19} and c_{20} are absolute positive constants. Furthermore, there exists an absolute positive constant c_{21} such that

$$\lambda_k(X_a) \geq c_{21} \tag{5.32}$$

whenever $k \geq 2$.

Proof. For the proof of the upper bound in (5.31) we recall the Buser-Cheeger inequality

$$\lambda_1(X_a) \le 2h(X_a) + 10h^2(X_a). \tag{5.33}$$

From Lemma 5.3.7 we see that X_a gets disconnected by removing δ and $\xi\delta\xi^{-1}$, where each of the resulting connected components has two-dimensional volume given by $\mathrm{vol}(X_c) = \frac{1}{2}\,\mathrm{vol}(X_a)$. This implies

$$h(X_a) \le \frac{8l_\tau}{\mathrm{vol}(X_a)}. \tag{5.34}$$

Combining (5.33) and (5.34) gives the right-hand side of (5.31) with

$$c_{20} = 656.$$

For the left hand-side of (5.31) we recall from [35, Thm. 1.3] the strengthened Cheeger inequality for 2-covers

$$\lambda_1(X_a) \ge \frac{1}{4}\sqrt{\lambda_1(X_c)}\; h(X_a). \tag{5.35}$$

Since τ is the smallest Salem number of degree $2n$, the length of any geodesic in X_a is bounded from below by l_τ. This implies

$$h(X_a) \ge \frac{2l_\tau}{\mathrm{vol}(X_a)}. \tag{5.36}$$

Plugging (5.36) into (5.35) gives the left-hand side of (5.31) with

$$c_{19} = \frac{1}{2}\sqrt{\frac{1}{4} - \left(\frac{7}{64}\right)^2}.$$

Regarding the lower bound (5.32) we apply domain monotonicity (cf. Section 4.2) to $X_a = X_c^+ \cup X_c^-$ to derive

$$\lambda_2(X_a) \ge \kappa_1(X_c^+), \tag{5.37}$$

where $\kappa_1(X_c^+)$ is the first Neumann eigenvalue of X_c^+. The Cheeger inequality (cf. Section 2.4) also holds for the first Neumann eigenvalue (see [18, Thm. 1.6]),

$$\kappa_1(X_c^+) \ge \tfrac{1}{4} h_N(X_c^+) \tag{5.38}$$

with

$$h_N(X_c^+) = \inf_u \frac{l(u)}{\min\{\mathrm{vol}(A_u), \mathrm{vol}(B_u)\}},$$

where u ranges over the set of all finite unions of piecewise smooth curves on X_c^+, which separate X_c^+ into two disjoint subsets A_u and B_u. We claim that

$$h_N(X_c^+) > 0.01, \tag{5.39}$$

which implies

$$\lambda_2(X_a) \geq c_{21}$$

with

$$c_{21} = 0.000025.$$

Assume to the contrary that

$$h_N(X_c^+) \leq 0.01. \tag{5.40}$$

Then we can find a finite union of piecewise smooth curves $u = u_1 \cup ... \cup u_s$, which decomposes X_c^+ into two disjoint subsets A_u and B_u with $\mathrm{vol}(A_u) \leq \mathrm{vol}(B_u)$ such that

$$\frac{l_u}{\mathrm{vol}(A_u)} \leq 0.02. \tag{5.41}$$

For the sake of simplicity we will only deal with the case that u is a single curve, but the careful reader may check that the full argument extends to finite unions of curves. Applying the Buser-Cheeger inequality to the closed surface X_c we may derive

$$h(X_c) \geq 0.08. \tag{5.42}$$

Now, if we cut the closed surface X_c along u and δ, it gets disconnected into two disjoint subsets, which are isometric to A_u and B_u. Thus, using (5.42) we get

$$\frac{l_u}{\mathrm{vol}(A_u)} + \frac{2l_\tau}{\mathrm{vol}(A_u)} \geq 0.08 \tag{5.43}$$

Combining (5.41) and (5.43) gives

$$\frac{1}{\mathrm{vol}(A_u)} \geq \frac{0.03}{l_\tau}. \tag{5.44}$$

If u is a loop, we have $l_u \geq l_\tau$ by Theorem 3.2.5 and the minimality of τ. If we plug this into (5.44), we arrive at

$$\frac{l_u}{\mathrm{vol}(A_u)} \geq 0.03,$$

which is a contradiction to (5.41). If u is not a closed curve, it touches the boundary of X_c^+ at both endpoints. If u is contained in a single half-tube around one of the boundary geodesics, we can use the argument from the second case in [86, p.281] to derive

$$h_N(X_c^+) \geq 0.5.$$

If it leaves the half-tubes around the boundary, we have by Theorem 2.3.1 that $l_u \geq 1$ and hence

$$\frac{1}{\mathrm{vol}(A_u)} \leq \frac{l_u}{\mathrm{vol}(A_u)} \leq 0.02. \tag{5.45}$$

Combining (5.45) with (5.43) and noting that $2l_\tau \leq 1$, we get

$$\frac{l_u}{\mathrm{vol}(A_u)} \geq 0.06,$$

which is a contradiction to (5.41). □

We can finally turn our attention to Proposition 5.1.2:

Proof of Proposition 5.1.2. Let $(\mathfrak{p}_j)_{j\in\mathbb{N}}$ be a sequence of prime ideals in F with $N(\mathfrak{p}_j)\to\infty$ as $j\to\infty$, which fulfill the assumptions made for $\mathfrak{p}_0$. Replacing $\mathfrak{p}_0$ by any of these prime ideals gives a sequence of congruence subgroups $\Gamma_{c,j}$ and arithmetic Fuchsian groups $\Gamma_{a,j}\subset\Gamma_{c,j}$ such that the estimates from Lemma 5.3.8 hold for any of the arithmetic surfaces $X_j=\Gamma_{a,j}\backslash\mathcal{H}$. We claim that $(X_j)_{j\in\mathbb{N}}$ is Plancherel convergent. We first observe that by Theorem 3.1.1 we have that $(\Gamma_{c,j})_{j\in\mathbb{N}}$ is a Plancherel sequence. Now, the bound

$$|\Lambda_{\Gamma_{a,j}}(s)|\leq 2\Lambda_{\Gamma_{c,j}}(\mathrm{Re}(s)),$$

valid for $\mathrm{Re}(s)>1$, together with Theorem 1.1.2 implies Plancherel convergence of $(X_j)_{j\in\mathbb{N}}$.
Using the notation of the proof of Theorem 4.4.4 we write

$$\frac{\gamma_{X_j}}{\mathrm{vol}(X_j)}=I_{1,j}(1)+I_{2,j}(1)+\frac{1}{\mathrm{vol}(X_j)}\sum_{0<\lambda_{j,k}<\frac{1}{4}}\frac{e^{-\lambda_{j,k}}}{\lambda_{j,k}}. \tag{5.46}$$

We note that, as before, the first two summands on the right-hand side of (5.46) converge to zero for $j\to\infty$, so that we are left with estimating

$$\frac{1}{\mathrm{vol}(X_j)}\sum_{0<\lambda_{j,k}<\frac{1}{4}}\frac{e^{-\lambda_{j,k}}}{\lambda_{j,k}}=\frac{e^{-\lambda_{j,1}}}{\lambda_{j,1}\,\mathrm{vol}(X_j)}+\frac{1}{\mathrm{vol}(X_j)}\sum_{\lambda_{j,1}<\lambda_{j,k}<\frac{1}{4}}\frac{e^{-\lambda_{j,k}}}{\lambda_{j,k}} \tag{5.47}$$

The second term on the right-hand side of (5.47) converges to zero by the same argument as in the proof of Proposition 5.1.1. From Lemma 5.3.8 we get the existence of positive constants C_0 and C_1 such that

$$C_0\leq\frac{\lambda_{j,1}^{-1}}{\mathrm{vol}(X_j)}\leq C_1. \tag{5.48}$$

Thus, after possibly replacing $(X_j)_{j\in\mathbb{N}}$ by a subsequence, we can assume that

$$\lim_{j\to\infty}\frac{\lambda_{j,1}^{-1}}{\mathrm{vol}(X_j)}=l_0 \tag{5.49}$$

for some constant l_0 with $C_0\leq l_0\leq C_1$. Hence, we end up with

$$\lim_{j\to\infty}\frac{\gamma_{X_j}}{\mathrm{vol}(X_j)}=l_0>0.$$

□

Chapter 6

Discussion and Further Projects

6.1 Plancherel Convergence of Non-compact Congruence Subgroups

In Chapter 3 we showed that any sequence of cocompact torsion-free congruence subgroups $(\Gamma_j)_{j\in\mathbb{N}}$ in $G = \mathrm{PSL}_2(\mathbb{R})$ with $\mathrm{vol}(\Gamma_j\backslash G) \to \infty$ is Plancherel convergent. Here the author only excluded torsion elements for simplicity. Raimbault and Fraczyk showed in [44] that there is no need to exclude congruence subgroups with torsion. The author expects that the methods from Chapter 3 should also be sufficient to deal with torsion elements. We now want to consider congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$, which are not cocompact. For Γ a cofinite lattice in $\mathrm{PSL}_2(\mathbb{R})$ we have a decomposition of the form (see [57])

$$L^2(\Gamma\backslash G) = L^2_{\mathrm{disc}}(\Gamma\backslash G) \oplus L^2_{\mathrm{ac}}(\Gamma\backslash G)$$

where $L^2_{\mathrm{disc}}(\Gamma\backslash G)$ is the maximal subspace of $L^2(\Gamma\backslash G)$ on which the right regular representation R decomposes discretely. Again, we let

$$L^2_{\mathrm{disc}}(\Gamma\backslash G) = \bigoplus_{\pi\in\widehat{G}} N_\Gamma(\pi) H_\pi,$$

and consider the spectral measure

$$\mu_\Gamma = \sum_{\pi\in\widehat{G}} N_\Gamma(\pi)\delta_\pi.$$

This allows one to extend the definition of Plancherel convergence to cofinite lattices. Now, any non-cocompact congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ is contained in (a conjugate of) $\mathrm{SL}_2(\mathbb{Z})$. In particular, any non-cocompact principal congruence subgroup is of the form

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

for some $N \in \mathbb{N}$ and congruence subgroups are in bijective correspondence with subgroups of the groups $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, $N \in \mathbb{N}$. The only new ingredient in the trace formula for these groups is the appearance of poles of the determinant $\phi_\Gamma(s) = \det C_\Gamma(s)$ of the scattering matrix C_Γ. For principal congruence subgroups Sarnak [82] was able to link the poles of $\phi_{\Gamma(N)}$ to zeroes of certain L-functions and then estimate the number of these zeroes by standard methods from number theory. One possible starting point for the non-principal congruence subgroups is the work of Reznikov [79], where the determinant of the scattering matrix is expressed as a ratio of automorphic L-functions.

6.2 Plancherel Convergence and Zeta Functions

In the preceding chapters we analyzed, in what way Plancherel convergence of a sequence $(\Gamma_j)_{j\in\mathbb{N}}$ of lattices in $\mathrm{PSL}_2(\mathbb{R})$ is reflected by their respective zeta functions. There still remain a few open questions (cf. [31, §4]):

- **Convergence inside the critical strip:** Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a sequence of cocompact torsion-free lattices in $G = \mathrm{PSL}_2(\mathbb{R})$. In Section 4.4, we did show that $\mathrm{vol}(\Gamma_j\backslash G)^{-1}\Lambda_j(s)$ converges for s sitting inside a certain subset U_1 of the critical strip. At the same time the examples constructed in Section 5.3 suggest that there should be no uniform behaviour for $s \in \mathscr{P}$. These problems are caused by the poles the logarithmic derivative of the zeta functions and we expect they can be avoided by staying sufficiently far away from the poles. Recalling the notation from Theorem 4.1.2 we therefore make the following

  Conjecture 6.2.1. *Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a sequence of torsion-free cocompact lattices in $G = \mathrm{PSL}_2(\mathbb{R})$. The following two statements are equivalent:*

  1. *The sequence $(\Gamma_j)_{j\in\mathbb{N}}$ is Plancherel convergent.*
  2. *For each $s \in \mathbb{C} \smallsetminus \mathscr{P}$ one has*

  $$\lim_{j\to\infty} \frac{\Lambda_j(s)}{\mathrm{vol}(\Gamma_j\backslash G)} = F(s)$$

- **General rank one groups:** In our discussion we exclusively considered lattices sitting in $G = \mathrm{PSL}_2(\mathbb{R})$. Now, let G be a semisimple Lie group of rank one with Iwasawa decomposition $G = ANK$. Let M be the centralizer of A in K. For $\sigma \in \widehat{M}$ and $\Gamma \subset G$ a cocompact torsion-free lattice one can define a (twisted) Selberg zeta function $\zeta_\Gamma(s,\sigma)$ (cf. [15]). It is natural to try to extend Theorem 4.1.2 to these zeta functions. By generalizing the arguments of [31] one can prove[1]

[1] Here we have normalized the zeta functions in such a way that the axis of absolute convergence lies at $\mathrm{Re}(s) = 1$.

Theorem 6.2.2. *Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a sequence of torsion-free cocompact lattices in a semisimple Lie group $G = ANK$ of rank one and $M = Z_K(A)$. Let $\Lambda_j(s,\sigma)$ be the logarithmic derivative of the Selberg zeta function $\zeta_{\Gamma_j}(s,\sigma)$. Then the following holds:*

1. *If the sequence $(\Gamma_j)_{j\in\mathbb{N}}$ is uniformly discrete and Plancherel, then*
$$\lim_{j\to\infty} \frac{\Lambda_j(s,\sigma)}{\mathrm{vol}(\Gamma_j\backslash G)} = 0$$
holds for any $\sigma \in \widehat{M}$ and any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.
2. *If*
$$\lim_{j\to\infty} \frac{\Lambda_j(s,\sigma)}{\mathrm{vol}(\Gamma_j\backslash G)} = 0$$
holds for any $\sigma \in \widehat{M}$ and any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the sequence $(\Gamma_j)_{j\in\mathbb{N}}$ is Plancherel-convergent.

The proof is completely analogous to the proof of [31, Thm. 3.2]. Again, one might try to remove the condition of uniform discreteness from Theorem 6.2.2 using the methods from Section 4.3. While the author believes that these methods should also work for semisimple Lie groups of rank one, there are the following two problems, which need to be adressed:

1. In Section 4.3 we applied methods from spectral geometry to the thick-thin decomposition of a hyperbolic surface to estimate the counting function of the Laplace operator. For these methods to work it is necessary that the boundary of the thin part is smooth. This can fail for any rank one group G, which is not a cover of either $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$ (cf. [20]). A possible way of approaching this problem is to apply quasi-isometries as in [20] to the thin part to get a sufficiently well-behaved boundary, while changing the Laplace spectrum only by a controlled amount.
2. In the case of a hyperbolic surface X, we controlled the thin part $X' = \cup_{i=1}^{m_X} T'_{\gamma_i}$ by using the explicit formulae for the Riemannian metric
$$ds^2 = dr^2 + l_i^2 \cosh^2 r dt^2 \tag{6.1}$$
on the tubes T'_{γ_i} and the asymptotic formula
$$w_i \sim \log\left(\frac{2}{l_i}\right) \tag{6.2}$$
for the widths of the tubes T_{γ_i}. While the analogue of (6.1) for semisimple Lie groups of rank one is either known (see e.g. [18, eq. (4.17)] for the group $G = \mathrm{SO}(n,1)$) or can be quickly worked out, the analogue of (6.2) seems to require some effort. We only note here that for $G = \mathrm{PSL}_2(\mathbb{C})$ one can find useful bounds in [39, p.50], which allow one to prove the analogue of Theorem 4.3.1 for the group $\mathrm{PSL}_2(\mathbb{C})$.

6.3 Benjamini-Schramm Convergence and Limit Multiplicities

We have seen in Section 2.11 that any Plancherel convergent sequence $(\Gamma_j)_{j\in\mathbb{N}}$ has the limit multiplicity property. Now, if the sequence $(\Gamma_j)_{j\in\mathbb{N}}$ is only Benjamini-Schramm convergent, do we still get the limit multiplicity property for certain subsets $M \subset \widehat{G}$, i.e. do we have

$$\lim_{j\to\infty} \frac{\mu_j(A)}{\mathrm{vol}(\Gamma_j\backslash G)} = \mu_{\mathrm{Pl}}(A) \tag{6.3}$$

for any Jordan-measurable subset $A \subset M$? For the sake of simplicity we will restrict the following discussion to the group $\mathrm{SL}_2(\mathbb{R})$. Recall the classification of unitary representations of $\mathrm{SL}_2(\mathbb{R})$ from Theorem 2.6.1:

- **Discrete series representations:** The multiplicities for (limits of) discrete series representations are explicitly known (cf. e.g. [99, p.174]),

  $$N_\Gamma(\delta_n^\pm) = \mathrm{vol}(\Gamma\backslash G)\ \mu_{\mathrm{Pl}}(\{\delta_n^\pm\}), \qquad n \geq 3, \tag{6.4}$$

  and

  $$N_\Gamma(\delta_2^\pm) = \mathrm{vol}(\Gamma\backslash G)\ \mu_{\mathrm{Pl}}(\{\delta_2^\pm\}) + 1, \qquad N_\Gamma(\{\delta_1^\pm\}) = 0. \tag{6.5}$$

  From (6.4) and (6.5) it is clear that we have the limit multiplicity property for $\widehat{G}_{\mathrm{disc}}$ as long as $\mathrm{vol}(\Gamma_j\backslash G) \to \infty$.

- **Principal series representations:** The author expects that the limit multiplicity property for principal series representations is what distinguishes Plancherel convergent sequences from those sequences, which are only BS-convergent. Unfortunately, the author is not aware of a BS-convergent sequence in $\mathrm{PSL}_2(\mathbb{R})$, which is not Plancherel-convergent. However, in Appendix D we construct a sequence of smooth closed hyperbolic surfaces, which might be a candidate.

- **Complementary series representations:** In contrast to principal series representations, the number of complementary series representations in the decomposition $L^2(\Gamma\backslash G)$ can be bounded in terms of the topology of the surface $\Gamma\backslash\mathcal{H}$ alone (cf. [17, Thm. 8.1.1]), i.e. we have an absolute constant c_B such that

  $$\mu_\Gamma(\widehat{G}_{\mathrm{comp}}) \leq c_B \,\mathrm{vol}(\Gamma\backslash G). \tag{6.6}$$

  Now, there are sequences of smooth closed hyperbolic surfaces for which the number of complementary series representations grows linear in the volume (cf. [19, §8.4]), but these are not Benjamini-Schramm convergent (see Example 2.11.8). The intuition behind these examples is that for each small eigenvalue of a surface X there should exist a subdomain of X with small isoperimetric quotient (cf. [17, §8.1]). Benjamini-Schramm convergence may be sufficient to show that the number of these subdomains can only grow sublinear in the volume. As a starting point one may consider [86], where the relationship between small eigenvalues and the geometry of the surface is discussed in more detail.

Appendix

A Limit Points of Salem Numbers

Let $\mathcal{T}$ be the set of Salem numbers. In this section we briefly discuss the set of limit points $\mathcal{T}^{(1)}$ of $\mathcal{T}$ and review what is known about it. Our main reference is [12].
Let us introduce the following class of algebraic integers:

Definition A.1. A Pisot-Vijayaraghavan number is a real algebraic integer $\theta > 1$ such that all conjugates of θ lie in the unit disk $\{z \mid |z| < 1\}$.

We denote the set of Pisot-Vijayaraghavan numbers by $\mathcal{S}$. An example of a Pisot-Vijayaraghavan number is the plastic number $\theta_0 = 1.32471...$, which is the unique real root of the polynomial $p_0(x) = x^3 - x - 1$. For a polynomial $p(x) = a_0 + ... + a_n x^n$ we let $p^*(x) = a_n + ... + a_0 x^n$ be its reciprocal polynomial. One can show that $\mathcal{S} \subset \mathcal{T}^{(1)}$ by the following construction (see [81, §7]): Start with a Pisot-Vijayaraghavan number θ and let P_θ be the minimal polynomial. Then for large enough $n \in \mathbb{N}$, the polynomial

$$R_n(x) = x^n P_\theta(x) \pm P_\theta^*(x) \tag{A.1}$$

splits into a product of cyclotomic polynomials and a Salem polynomial[2]. This defines a sequence $(\tau_n)_{n\in\mathbb{N}}$ of Salem numbers, which converges to θ for $n \to \infty$. In Section 3.4.1 we used the sequence of Salem numbers defined by taking the minimal polynomial $P_\theta(x) = x^3 - x - 1$ of the plastic number[3]. For example, we produced τ_{10} by factorizing

$$R_{10}(x) = x^{10}(x^3 - x - 1) + (x^3 + x^2 - 1). \tag{A.2}$$

into

$$R_{10}(x) = (x-1)(x^2 + x + 1)(x^{10} - x^8 - x^5 - x^2 + 1), \tag{A.3}$$

where the last factor on the right-hand side of (A.3) gives a Salem polynomial. For the interested reader we note that Boyd speculates in [12] that one might have $\mathcal{T}^{(1)} = \mathcal{S}$. But, to the best knowledge of the author, this has not yet been shown.

[2]We call a polynomial a Salem polynomial, if it is the minimal polynomial of a Salem number.

[3]Note that for small $n \in \mathbb{N}$ the polynomial from (A.1) does not produce Salem numbers. Still, to avoid any confusion we do not change the indexing of the sequence.

B Length Multiplicity

Throughout this section F is a totally real field, $\mathcal{A}$ is a quaternion algebra over F unramified at exactly one infinite place of F and $\mathcal{O}$ is a maximal order in $\mathcal{A}$. Furthermore, we let $\mathfrak{o}$ be some order in a relative quadratic extension L/F, such that L embeds into $\mathcal{A}$. For $\mathfrak{p} \in \Omega$ we let $\mathcal{O}_{\mathfrak{p}} = \mathcal{O} \otimes \mathfrak{o}_{F,\mathfrak{p}}$ and $\mathfrak{o}_{\mathfrak{p}} = \mathfrak{o} \otimes \mathfrak{o}_{F,\mathfrak{p}}$ the localizations of $\mathcal{O}$ and $\mathfrak{o}$ at $\mathfrak{p}$. The central goal of this section is to prove

Theorem B.1. *The embedding numbers $m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1)$ can be expressed as*

$$m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1) = \frac{s(\mathfrak{o}, \mathcal{O}) h(\mathfrak{o}) 2^{1+|\Omega_i(L)|}}{h_F [N_{L/F}(\mathfrak{o}^\times) : (\mathfrak{o}_F^\times)^2]}, \tag{B.1}$$

where $\Omega_i(L)$ is the subset of finite places in $\mathrm{Ram}(\mathcal{A})$*, which stay inert in the extension L.*

Further definitions and facts can be found in [98] (see in particular Sections §17, §30 and §31). We first remark that we can reduce the computation of $m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1)$ to the computation of $m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^\times)$:

Lemma B.2. *If Γ is a group with $\mathcal{O}^1 \subset \Gamma \subset \mathcal{O}^\times$, then*

$$m(\mathfrak{o}, \mathcal{O}; \Gamma) = m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^\times)[\mathrm{nrd}(\mathcal{O}^\times) : \mathrm{nrd}(\Gamma)\,\mathrm{nrd}(\mathfrak{o}^\times)].$$

Proof. [98, Lemma 30.3.14] □

The embedding numbers $m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^\times)$ will be determined using local-global methods. We collect the following definitions from [98]:
An *ideal* $I \subset \mathcal{A}$ is an $\mathfrak{o}_F$-module such that $FI = \mathcal{A}$. An ideal I comes with two orders

$$\mathcal{O}_L(I) = \{\alpha \in \mathcal{A} : \alpha I \subset I\}, \qquad \mathcal{O}_R(I) = \{\alpha \in \mathcal{A} : I\alpha \subset I\}.$$

An ideal I is said to be *invertible*, if there exists an ideal $J \subset \mathcal{A}$ such that

$$IJ = \mathcal{O}_L(I) = \mathcal{O}_R(J),\ JI = \mathcal{O}_L(J) = \mathcal{O}_R(I).$$

I is called *two-sided*, if $\mathcal{O}_L(I) = \mathcal{O}_R(I)$. The set of all two-sided ideals I sitting in $\mathcal{O}$ is denoted by $\mathrm{Idl}(\mathcal{O})$. If I is of the form $I = \alpha\mathcal{O}$ for some $\alpha \in \mathcal{A}^\times$, then I is called a *principal two-sided ideal* and the set of principal two-sided ideals in $\mathcal{O}$ is denoted by $\mathrm{PIdl}(\mathcal{O})$. We say that two ideals $I, J \subset \mathcal{A}$ are *in the same right class* and write $I \sim_R J$, if there exists $\alpha \in \mathcal{A}^\times$ such that $\alpha I = J$. The right class of an ideal I is written as $[I]_R$. The *right class set* of $\mathcal{O}$ is given by

$$\mathrm{Cls}_R\,\mathcal{O} = \{[I]_R : I \subset \mathcal{A} \text{ invertible and } \mathcal{O}_R(I) = \mathcal{O}\}.$$

One may define the left class set $\mathrm{Cls}_L\,\mathcal{O}$ analogously. Since the standard involution on $\mathcal{A}$ induces a bijection between the right and left class set, we will simply write $\mathrm{Cls}\,\mathcal{O}$ instead of $\mathrm{Cls}_R\,\mathcal{O}$. This set is always finite [98, Thm. 17.7.1] and its cardinality $h(\mathcal{O})$ is called the *class number* of $\mathcal{O}$. Next, consider two orders $\mathcal{O}'$ and $\mathcal{O}''$ in $\mathcal{A}$. We say that $\mathcal{O}'$ and $\mathcal{O}''$ are *of the same type*, if there exists $\alpha \in \mathcal{A}^\times$ such that $\mathcal{O}' = \alpha^{-1}\mathcal{O}\alpha$. According to the Skolem-Noether theorem two orders $\mathcal{O}'$ and $\mathcal{O}''$ are of the same type if and only if they are isomorphic as rings. We say that $\mathcal{O}'$ is *connected* to $\mathcal{O}''$, if there exists an invertible ideal I with $\mathcal{O}_L(I) = \mathcal{O}'$ and $\mathcal{O}_R(I) = \mathcal{O}''$. Since invertible ideals are locally principal [98, Thm. 16.6.1], two orders will be connected if and only if they are *locally of the same type* (i.e. locally isomorphic as rings). The *genus* of $\mathcal{O}$ is the set $\operatorname{Gen}\mathcal{O}$ of orders in $\mathcal{A}$ locally isomorphic to $\mathcal{O}$. The *type set* $\operatorname{Typ}\mathcal{O}$ of $\mathcal{O}$ is the set of $\mathfrak{o}_F$-isomorphism classes of orders in the genus of $\mathcal{O}$. Now, consider the localizations $\mathfrak{o}_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ at a finite place $\mathfrak{p}$ of F. Let us denote

$$s_1(\mathfrak{o}, \mathfrak{p}) = \begin{cases} 1, & \text{if } \mathfrak{o}_\mathfrak{p} \text{ is integrally closed} \\ 0, & \text{else.} \end{cases}$$

and

$$s_1(\mathfrak{o}) = \prod_{\mathfrak{p} \in \operatorname{Ram}(\mathcal{A})} s_1(\mathfrak{o}, \mathfrak{p}).$$

We define the local embedding number $m(\mathfrak{o}_\mathfrak{p}, \mathcal{O}_\mathfrak{p}; \mathcal{O}_\mathfrak{p}^\times)$ as the number of optimal embeddings of $\mathfrak{o}_\mathfrak{p}$ into $\mathcal{O}_\mathfrak{p}$ modulo the action of $\mathcal{O}_\mathfrak{p}^\times$. These local embedding numbers are explicitly known for maximal orders $\mathcal{O}$:

Proposition B.3. *One has*

$$m(\mathfrak{o}_\mathfrak{p}, \mathcal{O}_\mathfrak{p}; \mathcal{O}_\mathfrak{p}^\times) = \begin{cases} s_1(\mathfrak{o}, \mathfrak{p}) \left(1 - \left(\frac{L}{\mathfrak{p}}\right)\right), & \text{if } \mathfrak{p} \in \operatorname{Ram}(\mathcal{A}), \\ 1, & \text{else.} \end{cases}$$

Proof. [98, Prop. 30.5.3] □

The above proposition shows in particular that the product

$$m(\widehat{\mathfrak{o}}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times) := \prod_{\mathfrak{p} \in \Omega_f} m(\mathfrak{o}_\mathfrak{p}, \mathcal{O}_\mathfrak{p}; \mathcal{O}_\mathfrak{p}^\times) = s_1(\mathfrak{o}) 2^{|\Omega_i(L)|} \tag{B.2}$$

is finite. The relevant result linking global embedding numbers to their local counterpart is given by

Theorem B.4. *We have*

$$\sum_{[I] \in \operatorname{Cls}\mathcal{O}} m(\mathfrak{o}, \mathcal{O}_L(I); \mathcal{O}_L(I)^\times) = h(\mathfrak{o}) m(\widehat{\mathfrak{o}}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times). \tag{B.3}$$

Proof. [98, Thm. 30.4.7] □

The above theorem allows the evaluation of $m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^\times)$ by the next result:

Theorem B.5. *Assume that $m(\widehat{\mathfrak{o}}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times) \neq 0$. Then the following holds:*

1. *If $\mathfrak{o}$ does not embed optimally into $\mathcal{O}$, we have $\Sigma(\mathfrak{o}, \mathcal{O}') \neq \emptyset$ for precisely half of the types $[\mathcal{O}'] \in \operatorname{Typ} \mathcal{O}$.*

2. *One has*
$$m(\mathfrak{o}, \mathcal{O}'; \mathcal{O}'^{\times}) = m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^{\times})$$
for $\mathcal{O}' \in \operatorname{Gen} \mathcal{O}$, whenever both sides are non-zero.

Proof. [98, Thm. 31.1.7][4] □

We note that the vanishing of $m(\widehat{\mathfrak{o}}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^{\times})$ already implies by (B.3) that $m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^{\times}) = 0$. Hence, from now on we will only consider the case $m(\widehat{\mathfrak{o}}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^{\times}) \neq 0$, which is equivalent to $s_1(\mathfrak{o}) = 1$.

Corollary B.6. *Suppose that $m(\mathfrak{o}, \mathcal{O}'; \mathcal{O}'^{\times}) \neq 0$ for all $\mathcal{O}' \in \operatorname{Gen} \mathcal{O}$. Then*

$$m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^{\times}) = \frac{h(\mathfrak{o})}{h(\mathcal{O})} m(\widehat{\mathfrak{o}}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^{\times}).$$

Proof. If I is an ideal in $\mathcal{A}$ with $[I]_R \in \operatorname{Cls} \mathcal{O}$, we have by definition $\mathcal{O}_L(I) \in \operatorname{Gen} \mathcal{O}$. One may then apply the second statement of Theorem B.5 to Theorem B.4 and use (B.2). □

If $\mathfrak{o}$ does embed optimally into $\mathcal{O}$, but $m(\mathfrak{o}, \mathcal{O}'; \mathcal{O}'^{\times})$ vanishes for some other order $\mathcal{O}' \in \operatorname{Gen} \mathcal{O}$, we still have that $\mathfrak{o}$ does embed into precisely half of the types $[\mathcal{O}'] \in \operatorname{Typ} \mathcal{O}$. If we want to apply (B.3) in this case, we need to understand the fibers of the surjective map

$$\phi : \operatorname{Cls} \mathcal{O} \to \operatorname{Typ} \mathcal{O}, \qquad [I] \mapsto \mathcal{O}_L(I).$$

Proposition B.7. *The map $I \mapsto [I]$ induces a bijection*

$$\operatorname{PIdl}(\mathcal{O}) \backslash \operatorname{Idl}(\mathcal{O}) \to \{[I] \in \operatorname{Cls} \mathcal{O} \mid \mathcal{O}_L(I) \cong \mathcal{O}\}$$

Proof. [98, Prop. 18.5.10] □

Hence the number of elements in a single fiber of the map ϕ is equal to the cardinality $|\operatorname{PIdl}(\mathcal{O}) \backslash \operatorname{Idl}(\mathcal{O})|$. This number luckily does not depend on $\mathcal{O}$:

Proposition B.8. *Let $\mathcal{O}$ and $\mathcal{O}'$ be locally isomorphic orders in $\mathcal{A}$. Then*

$$|\operatorname{PIdl}(\mathcal{O}) \backslash \operatorname{Idl}(\mathcal{O})| = |\operatorname{PIdl}(\mathcal{O}') \backslash \operatorname{Idl}(\mathcal{O}')|$$

holds.

Proof. [98, Prop. 28.9.7] □

Let us define a number $s_2(\mathfrak{o}, \mathcal{O})$ by

[4] For the careful reader we note that the assumption $m(\widehat{\mathfrak{o}}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^{\times}) \neq 0$ is not contained in the statement of Theorem 31.1.7 itself, but has been made in Paragraph 31.1.4.

- $s_2(\mathfrak{o}, \mathcal{O}) = 0$, if $\mathfrak{o}$ does not embed optimally into $\mathcal{O}$,
- $s_2(\mathfrak{o}, \mathcal{O}) = 1$, if $m(\mathfrak{o}, \mathcal{O}'; \mathcal{O}'^{\times}) \neq 0$ for all $\mathcal{O}' \in \operatorname{Gen} \mathcal{O}$,
- $s_2(\mathfrak{o}, \mathcal{O}) = 2$, else.

and let $s(\mathfrak{o}, \mathcal{O}) = s_1(\mathfrak{o}) s_2(\mathfrak{o}, \mathcal{O})$.

Corollary B.9. *One has*

$$m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^{\times}) = s(\mathfrak{o}, \mathcal{O}) \frac{h(\mathfrak{o})}{h(\mathcal{O})} 2^{|\Omega_i(L)|}.$$

Proof. The case $s(\mathfrak{o}, \mathcal{O}) = 0$ is trivial and the case $s(\mathfrak{o}, \mathcal{O}) = 1$ has already been dealed with in Corollary B.6. Let us now assume that $\mathfrak{o}$ embeds optimally into $\mathcal{O}$, but there exists $[\mathcal{O}''] \in \operatorname{Gen} \mathcal{O}$ such that $\mathfrak{o}$ does not embed optimally into $\mathcal{O}''$. We first note that

$$\sum_{[I] \in \operatorname{Cls} \mathcal{O}} m(\mathfrak{o}, \mathcal{O}_L(I); \mathcal{O}_L(I)^{\times}) = \sum_{\mathcal{O}' \in \operatorname{Typ} \mathcal{O}} |\phi^{-1}(\mathcal{O}')| m(\mathfrak{o}, \mathcal{O}'; \mathcal{O}'^{\times}). \tag{B.4}$$

If we let $(\operatorname{Typ} \mathcal{O})_*$ be the subset of $\operatorname{Typ} \mathcal{O}$ consisting of those $[\mathcal{O}']$ such that $\mathfrak{o}$ embeds into $\mathcal{O}'$, we can apply the second statement of Theorem B.5 to (B.4) to deduce

$$\sum_{\mathcal{O}' \in \operatorname{Typ} \mathcal{O}} |\phi^{-1}(\mathcal{O}')| m(\mathfrak{o}, \mathcal{O}'; \mathcal{O}'^{\times}) = m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^{\times}) \sum_{\mathcal{O}' \in (\operatorname{Typ} \mathcal{O})_*} |\phi^{-1}(\mathcal{O}')|. \tag{B.5}$$

Combining Proposition B.7 and Proposition B.8 yields $|\phi^{-1}(\mathcal{O}')| = |\phi^{-1}(\mathcal{O})|$ for $\mathcal{O}' \in \operatorname{Typ} \mathcal{O}$ and therefore we can apply Theorem B.5 to (B.5) to derive

$$m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^{\times}) \sum_{\mathcal{O}' \in (\operatorname{Typ} \mathcal{O})_*} |\phi^{-1}(\mathcal{O}')| = m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^{\times}) |\phi^{-1}(\mathcal{O})| \frac{|\operatorname{Typ} \mathcal{O}|}{2}.$$

Since $h(\mathcal{O}) = |\phi^{-1}(\mathcal{O})| |\operatorname{Typ} \mathcal{O}|$, we get

$$\sum_{[I] \in \operatorname{Cls} \mathcal{O}} m(\mathfrak{o}, \mathcal{O}_L(I); \mathcal{O}_L(I)^{\times}) = \frac{1}{2} m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^{\times}) h(\mathcal{O}),$$

which concludes the proof in view of Theorem B.4. $\square$

Proof of Theorem B.1. In view of Corollary B.9 and Lemma B.2 we have

$$m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1) = s(\mathfrak{o}, \mathcal{O}) h(\mathfrak{o}) 2^{|\Omega_i(L)|} \frac{[\operatorname{nrd}(\mathcal{O}^{\times}) : \operatorname{nrd}(\mathfrak{o}^{\times})]}{h(\mathcal{O})} \tag{B.6}$$

By an application of the third group isomorphism theorem we get

$$[\operatorname{nrd}(\mathcal{O}^{\times}) : (\mathfrak{o}_F^{\times})^2] = [\operatorname{nrd}(\mathcal{O}^{\times}) : \operatorname{nrd}(\mathfrak{o}^{\times})][\operatorname{nrd}(\mathfrak{o}^{\times}) : (\mathfrak{o}_F^{\times})^2]. \tag{B.7}$$

Next, combining [98, Cor. 31.1.11], [98, Lemma 39.4.7] and [98, Lemma 28.5.17] shows

$$[\mathrm{nrd}(\mathcal{O}^{\times}) : (\mathfrak{o}_F^{\times})^2] = 2\frac{h(\mathcal{O})}{h_F}. \tag{B.8}$$

Plugging equations (B.7) and (B.8) into (B.6) finally gives

$$m(\mathfrak{o}, \mathcal{O}; \mathcal{O}^1) = \frac{s(\mathfrak{o}, \mathcal{O})h(\mathfrak{o})2^{1+|\Omega_i(L)|}}{h_F[N_{L/F}(\mathfrak{o}^{\times}) : (\mathfrak{o}_F^{\times})^2]},$$

where we used that the reduced norm on $\mathcal{A}$ agrees with the relative norm $N_{L/F}$, when restricted to L. $\square$

C The Phragmen-Lindelöf Principle

The Phragmen-Lindelöf principle is a substitute for the maximum modulus theorem for analytic functions $f : U \to \mathbb{C}$ on an unbounded domain U. There exist different versions of it. We will use it in the form stated in [28]. Any further details and proofs can be found in [28].
The *extended complex plane* $\mathbb{C}_\infty$ is the one-point compactification of $\mathbb{C}$, i.e. it is the topological space $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, whose topology consists of open sets $U \subset \mathbb{C}$ together with all sets $V = (\mathbb{C} \setminus K) \cup \{\infty\}$, where K is a compact subset of $\mathbb{C}$. For a subset $U \subset \mathbb{C}$ we let $\partial_\infty U$ be the boundary of U in $\mathbb{C}_\infty$ and call it the *extended boundary* of U. We have that $\partial_\infty U = \partial U$ if U is bounded and $\partial_\infty U = \partial U \cup \{\infty\}$ if U is unbounded.

Definition C.1. Let U be an open subset of $\mathbb{C}$ and $\overline{U}$ be its closure in the standard topology of $\mathbb{C}$. If $f : U \to \mathbb{R}$ and $x \in \overline{U}$ or $x = \infty$, then the *limit superior* of $f(s)$ as s approaches x, denoted by $\limsup_{s\to x} f(s)$ is defined by

$$\limsup_{s\to x} f(s) = \lim_{r\to 0^+} \sup\{f(s) : s \in U \cap B(x, r)\},$$

where $B(x, r)$ is the ball of radius r centered around x with respect to the standard metric of $\mathbb{C}$.

Theorem C.2. *Let $U \subset \mathbb{C}$ be a simply connected region and let f be an analytic function on U. Suppose there is an analytic function $\varphi : U \to \mathbb{C}$, which never vanishes and is bounded on U. If M is a constant and $\partial_\infty U = X \cup Y$ such that*

1. *for every x in X, $\limsup_{s\to x} |f(s)| \le M$;*
2. *for every y in Y and $\eta > 0$, $\limsup_{s\to y} |f(s)||\varphi(s)|^\eta \le M$;*

then $|f(s)| \le M$ for all $s \in U$.

Proof. [28, Thm. 4.1] $\square$

D A Degenerating Plancherel sequence

Before starting the construction of a Plancherel-convergent sequence, which is not uniformly discrete, we need to extent our discussion of Y-pieces from Section 2.3 to possibly non-compact surfaces. The material can be found for example in [17, §4.4].
One can paste together two degenerate hexagons, which are hexagons with either one, two or three points at infinity to get a *degenerate Y-piece* (or *degenerate pair of pants*). We will refer to the points at infinity as *punctures.* We will extend the notation Y_{l_1,l_2,l_3} to degenerate Y-pieces by writing $l_i = 0$ for any boundary component, which is a puncture. A degenerate Y-piece contains around each puncture a neighborhood $\mathcal{C}$, which is isometric to $(-\infty, \log 2] \times \mathbb{S}^1$ equipped with the Riemannian metric

$$ds^2 = dr^2 + e^{2r} dt^2.$$

Such a neighborhood is called a *cusp* and depicted in Figure 1.

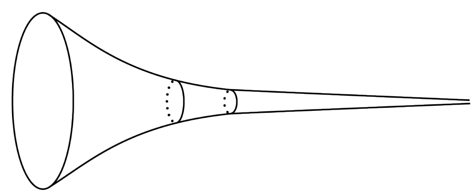

Figure 1: A cusp around a puncture.

A Y-piece has signature $(0, p; q)$ if it has p boundary geodesics and q cusps. A cofinite hyperbolic surface of genus g is said to have *signature* $(g, p; q)$, if it has p boundary geodesics and q cusps.

Theorem D.1. *Let X be a (possibly) non-compact smooth hyperbolic surface of signature $(g, 0; q)$. Let $\gamma_1, ..., \gamma_m$ be pairwise disjoint simple closed geodesics on X. Then the following hold*

1. *$m \leq 3g - 3 + q$,*
2. *There exist simple closed geodesics $\gamma_{m+1}, ..., \gamma_{3g-3+q}$, which together with $\gamma_1, ..., \gamma_m$ decompose X into Y-pieces.*
3. *The tubes T_{γ_i}, $i = 1, ..., 3g - 3 + q$ and the cusps $\mathcal{C}^1, ..., \mathcal{C}^q$ are all pairwise disjoint.*

Proof. [17, Thm. 4.4.6] □

Now, if we take a smooth hyperbolic surface X and pinch a simple closed geodesic γ on X, i.e. we let $l_\gamma \to 0$, then we intuitively expect that the tube T_γ around γ converges in a suitable sense to (two copies of) a cusp $\mathcal{C}$. To give this a precise meaning we need some additional terminology (cf. [21]). Let Y_{l_1,l_2,l_3} be a Y-piece and let $r_0 \in (0,\infty)$. The *horocycles*

$$h_{r_0} = \{p \in \mathcal{C} \mid \mathrm{dist}(p, \partial\mathcal{C}) = r_0\}$$

for a cusp $\mathcal{C}$ in Y_{l_1,l_2,l_3} and the curves

$$\gamma^{r_0} = \{p \in X \mid \mathrm{dist}(p, \gamma) = r_0\}$$

for γ a boundary geodesic of Y_{l_1,l_2,l_3} and $0 < r_0 < w_\gamma$ are called *equidistant curves*. Now, select in each half-tube or cusp an equidistant curve β_i of length λ_i. Then the closure of the connected component of $Y_{l_1,l_2,l_3} \setminus (\beta_1 \cup \beta_2 \cup \beta_3)$ not containing any of the boundary geodesics of Y_{l_1,l_2,l_3} or punctures is called a *restricted Y-piece* and denoted $Y^{\lambda_1,\lambda_2,\lambda_3}_{l_1,l_2,l_3}$. Let us also write $Y^{c}_{l_1,l_2}$ instead of $Y^{l_1,l_2,c}_{l_1,l_2,0}$. A homeomorphism $\phi : Y \to Y'$ of possibly restricted Y-pieces is called *boundary-coherent*, if for corresponding boundary curves α_i of Y and α'_i of Y' in standard parametrization one has

$$\phi(\alpha_i(t)) = \alpha'_i(t), \qquad \forall t \in [0,1].$$

For each boundary length $l_i \geq 0$ we let

$$P_i = \{p \in Y_{l_1,l_2,l_3} \mid \mathrm{dist}(p, \gamma_i) < \log(\tfrac{2}{l_i})\} \tag{D.1}$$

if $0 < l_i < 2$ and $P_i = \emptyset$ for $l_i \geq 2$. In the degenerate case, we let P_i be the set of points that lie outside of the horocycle of length 1. Then

$$\widehat{Y}_{l_1,l_2,l_3} = Y_{l_1,l_2,l_3} \setminus (P_1 \cup P_2 \cup P_3)$$

is called a *reduced Y-piece*. Finally, let us recall that a piecewise smooth[5] mapping $\Psi : M \to N$ of Riemannian manifolds M and N is called a *quasi-isometry*, if there exists $d > 0$ such that for any tangent vector v of M we have

$$\frac{1}{d}||v||_M \leq ||D\Psi(v)||_N \leq d||v||_M \tag{D.2}$$

The infimum over all the d such that (D.2) holds is called the *length distortion* F and denoted d_Ψ.

Theorem D.2. *Let $0 \leq l_1, l_2$ and $0 < \varepsilon < \frac{1}{2}$. Set $\varepsilon^* = \frac{2}{\pi}\varepsilon$. Then there exists a boundary-coherent homeomorphism*

$$\phi : Y_{l_1,l_2,\varepsilon} \to Y^{\varepsilon^*}_{l_1,l_2}$$

such that

1. $\phi(\widehat{Y}_{l_1,l_2,\varepsilon}) = \widehat{Y}_{l_1,l_2,0}$

[5]By a piecewise smooth mapping we mean a homeomorphism, which is smooth on the complement of a finite number of curves.

2. *The restriction of ϕ to $\widehat{Y}_{l_1,l_2,\varepsilon}$ is boundary-coherent and has length distortion $d_\phi \leq 1 + \frac{5}{4}\varepsilon^2$.*

Proof. [21, Thm. 5.1] □

Remark D.3. *Theorem D.2 can be extended in an obvious manner to Y-pieces with more than one degenerating boundary geodesic.*

Now, we can finally construct a Plancherel-convergent sequence of smooth hyperbolic surfaces $(X_j)_{j\in\mathbb{N}}$, which is not uniformly discrete. For this we adapt an example from [19]. Let us recall a few facts about the principal congruence subgroups

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

We denote by $X(N) = \Gamma(N)\backslash\mathcal{H}$ the congruence surface of level N. Let us write g_N for the genus of $X(N)$ and b_N for the number of boundary components of $X(N)$.

Theorem D.4. *The principal congruence subgroup $\Gamma(N)$ is torsion-free for $N \geq 3$. All boundary components of $X(N)$ are punctures and we have*

$$g_N = 1 + \frac{d_N(N-6)}{24N}, \qquad b_N = \frac{d_N}{2N}, \tag{D.3}$$

where d_N is given by $d_2 = 12$ and $d_N = N^3 \prod_{p|N}(1 - 1/p^2)$ for $N \geq 3$.

Proof. [90] □

Note that the number of cusps of $X(N)$ always is even for $N \geq 3$. The systole $l_{s,N}$ of $X(N)$ is given by (cf. [85, Lemma 2])

$$2\cosh(l_{s,N}/2) = (N^2 - 2). \tag{D.4}$$

Now, decompose $X(N)$ into pairs of pants. The boundary components of the pants are either geodesics or punctures. We keep the boundary geodesics and replace each puncture by a geodesic of length t. Let us reassemble these pieces using the old identifications. Since the number of cusps of $X(N)$ is even, we can identify the remaining geodesics in pairs. This yields a smooth closed hyperbolic surface $X_t(N)$. By counting the number of Y-pieces involved one can show $g(X_t(N)) \geq g_N$. We also note that the surface $X_t(N)$ contains $b_N/2$ disjoint simple closed geodesics $\gamma_i,\ i = 1, ..., b_N/2$ of length t. Now, let $(N_j)_{j\in\mathbb{N}}$ be a sequence of natural numbers $N_j \geq 3$ with $N_j \to \infty$ as $j \to \infty$ and $(t_j)_{j\in\mathbb{N}}$ be a sequence of positive real numbers converging towards zero. Let us write $b_j = b_{N_j}/2$ and $g_j = g_{N_j}$. Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of smooth closed hyperbolic surfaces defined by $X_j = X_{t_j}(N_j)$.

Lemma D.5. *The sequence $(X_j)_{j\in\mathbb{N}}$ is Plancherel convergent if and only if t_j^{-1} grows subexponentially in N_j.*

Proof. Let us fix some $c > 0$. We will first describe all closed geodesics in X_j of length smaller than or equal to c for j sufficiently large. Let γ be a simple closed geodesic in some X_j, which is not freely homotopic to some power of one of the geodesics $\gamma_1, ..., \gamma_{b_j}$. If γ intersects any of the geodesics $\gamma_i,\ i = 1, ..., b_j$ we get from [17, Cor. 4.1.2] that

$$\sinh(l_\gamma/2) \geq \frac{1}{\sinh(t_j/2)}.$$

Hence, γ can be dismissed for j large enough. If γ does not intersect any of the $\gamma_i,\ i = 1, ..., b_j$, we have by [17, Thm. 4.1.1] that γ lies outside of the tubes T_{γ_i} with $i = 1, ..., b_j$. There exists a boundary-coherent quasi-isometry

$$\phi : X_j \setminus \bigcup_{i=1}^{b_j} P_i \to X(N_j) \setminus \bigcup_{i=1}^{b_j} P'_i$$

given by the identity on any Y-piece, where no boundary geodesic has been replaced in the above process and the map from Theorem D.2 in the remaining cases. Its length distortion is bounded by $d_\phi \leq 1 + \frac{5}{4}t_j^2$. Therefore, $\phi(\gamma)$ defines an element $[\phi(\gamma)]$ in $\Gamma(N_j)$ and

$$l(\phi(\gamma)) \leq (1 + \tfrac{5}{4}t_j^2)\ l(\gamma). \tag{D.5}$$

We claim that $[\phi(\gamma)] \in \Gamma(N_j)$ is covered by a hyperbolic transformation, i.e. there exist a closed geodesic in the free homotopy class of $\phi(\gamma)$. Assume otherwise that $[\phi(\gamma)]$ is covered by a parabolic transformation. Then by [2, p.72] the curve $\phi(\gamma)$ can be homotoped into the power of a simple loop around a puncture of $X(N_j)$. Now, applying ϕ^{-1} gives a homotopy of γ into the tube around some geodesic γ_{i_0} for $i_0 \in \{1, ..., b_j\}$. Hence, γ is homotopic to some power of γ_{i_0}, which is a contradiction to our assumption on γ. Consequently, there exists a hyperbolic transformation $\eta_\gamma \in \Gamma(N_j)$, which covers $[\phi(\gamma)]$ and Theorem 2.2.2 implies

$$2\,\mathrm{arcosh}((N_j^2 - 2)/2) \leq l(\eta_\gamma) \leq l(\phi(\gamma)) \leq (1 + \tfrac{5}{4}t_j^2) l(\gamma). \tag{D.6}$$

Since t_j is bounded from above, inequality (D.6) shows that for N_j large enough there are no simple closed geodesics of length $\leq c$ apart from $\gamma_1, ..., \gamma_{b_j}$. Next, let γ be a non-simple closed geodesic different from a power of one of the $\gamma_1, ..., \gamma_{b_j}$. According to [17, Thm. 4.2.4] any non-simple primitive geodesic of smallest length is a figure-eight geodesic[6] δ embedded into a Y-piece. Any Y-piece contains at least one boundary geodesic not belonging to $\gamma_1, ..., \gamma_{b_j}$, since otherwise $X(N_j)$ would not be connected. Then the length formula for δ (see [17, eq. (4.2.3)]) yields

$$l(\gamma) \geq l(\delta) \geq 2(1 + \tfrac{5}{4}t_j^2)^{-1}\,\mathrm{arcosh}((N_j^2 - 2)/2).$$

This proves that for j large enough any geodesic in X_j of length $\leq c$ is a power of some of the geodesics $\gamma_1, ..., \gamma_{b_j}$.

[6]A figure-eight geodesic is a closed geodesic with exactly one self-intersection.

Finally, we want to apply the criterion for Plancherel-convergence from Proposition 2.11.6. We compute

$$\sum_{l_\gamma \le c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} = b_j \sum_{k \in \mathbb{N}: kt_j \le c} \frac{t_j}{\sinh(kt_j/2)}. \tag{D.7}$$

Since $\lim_{x\to 0} x/\sinh(x) = 1$ there exists for sufficiently small $\varepsilon > 0$ positive constants C_0 and C_1 such that

$$C_0 \le \frac{x}{\sinh(x)} \le C_1$$

for $x \in (0, \varepsilon]$. Applying this to (D.7) gives

$$\sum_{l_\gamma \le c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \le 2b_j C_1 \left(\sum_{k \in \mathbb{N}: kt_j \le \varepsilon} \frac{1}{k} \right) + \frac{b_j t_j}{\sinh(\varepsilon/2)}. \tag{D.8}$$

Using the asymptotic expansion of the harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma_E + O\left(\frac{1}{n}\right),$$

where γ_E is the Euler-Mascheroni constant, gives

$$\sum_{l_\gamma \le c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \le C_2 b_j |\log t_j| \tag{D.9}$$

for C_2 some sufficiently large constant. Plugging in the values for b_j and g_j from (D.3) yields

$$\frac{1}{\mathrm{vol}(X_j)} \sum_{l_\gamma \le c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \le \frac{3C_2 |\log t_j|}{\pi(N_j - 6)}. \tag{D.10}$$

In particular, the right-hand side of (D.10) goes to zero, if t_j^{-1} grows subexponentially in N_j, in which case Proposition 2.11.6 implies the Plancherel convergence of $(X_j)_{j\in\mathbb{N}}$. In an analogous manner, one can derive the lower bound

$$\frac{1}{\mathrm{vol}(X_j)} \sum_{l_\gamma \le c} m(l_\gamma) \frac{l_{\gamma_0}}{\sinh(l_\gamma/2)} \ge C_3 \frac{|\log t_j|}{\pi(N_j - 6)}. \tag{D.11}$$

for C_3 some positive constant and j sufficiently large. Hence, we can employ Proposition 2.11.6 to show that $(X_j)_{j\in\mathbb{N}}$ is not Plancherel-convergent, if t_j^{-1} grows at least exponentially in N_j. □

E Notations and Conventions

In this section we collect, for the sake of completeness, a few conventions and notations, which are used throughout this thesis.

- **Isomorphisms:** If we speak of an isomorphism between two objects A and B, we always mean an isomorphism in the respective category. For example, if A, B are groups, an isomorphism $f : A \to B$ is a group isomorphism, or if A, B are smooth manifolds any isomorphism $f : A \to B$ is a diffeomorphism.

- **Generators of a subgroup:** If G is a group and $g_1, ..., g_n \in G$ are elements of G, we denote the subgroup H generated by $g_1, ..., g_n$ by

$$H = \langle g_1, ..., g_n \rangle.$$

- **Vinogradov notation:** When writing down any bounds, we usually write out the constants, whenever possible. While this introduces some amount of bookkeeping, the author thinks that it makes it easier to follow the computations. Hence, the author has decided against using Vinogradov notation.

- **Subsets with unit removed:** If S is a subset of group G and 1 is the unit in G, we write $S^* = S \setminus \{1\}$.

- **Cardinality:** If S is a finite set, we write $|S|$ for the cardinality of S.

- **Big O-Notation:** In a few places, we use the so-called Big O-Notation to write down asymptotics.

- **Properties of manifolds:** We always assume that any manifold appearing in this thesis is connected and orientable without explicitly mentioning it.

- **Finite fields:** Let p be a rational prime and $n \in \mathbb{N}$ an integer. For $q = p^n$ a power of p we write $\mathbb{F}_q$ for the finite field of cardinality q.

- **Order of an element:** Let $x \in G$ be an element of a group G. If $n \in \mathbb{N}$ is the smallest natural number such that $x^n = 1$, we let $\mathrm{ord}(x) = n$ be the order of x.

- **Interior points:** If X is a topological space and $A \subset X$ a subset of X, we write A° for the set of interior points of A.

- **Curves as point sets:** For a curve $c : [0, 1] \to X$ in a topological space X we also write c for the point set

$$\{c(t) \mid t \in [0, 1]\}.$$

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