Microscopic derivation of Vlasov equations with singular potentials

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Abstract

The Vlasov equation is an effective equation which is used to describe the coarse-grained time evolution of a many particle system subject to Newtonian time evolution. The most interesting interaction forces one can consider for such systems are highly singular, for example Coulomb or Newton's gravitational force. Although progress has been made in proving the validity of this macroscopic model, the full Coulomb case without regularization, like a cut-off, is still an open problem. But also other highly singular forces, for example delta like forces have gained a lot of interest in the last decades.

The aim of this thesis is to make advancements in the rigorous mathematical derivation of the Vlasov-Poisson equation in regard to the cut-off size and provide a rigorous mathematical derivation of the Vlasov-Dirac-Benney equation in the large N limit of interacting particles.

In the first part of the thesis we probabilistically prove the mean-field limit and propagation of chaos of an N-particle system in three dimensions with pair potentials of the form $N^{3\beta-1}\phi(N^{\beta}x)$ for $\beta \in \left[0, \frac{1}{7}\right]$ and $\phi \in L^{\infty}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Provided that the initial positions of the N-particle trajectories are independent and identically distributed with respect to the initial density k_0, we show that under certain assumptions on k_0, the characteristics of the Vlasov-Dirac-Benney equation provide a reliable approximation of the N-particle trajectories.

In the second part we give a probabilistic proof of the mean-field limit and propagation of chaos of an N-particle system in three dimensions for a Coulomb force $f^N(q) = \pm\frac{q}{|q|^3}$ with a cut-off $|q| > N^{-\frac{5}{12}+\sigma}$, where $\sigma > 0$ can be arbitrarily small. In particular, the cut-off diameter is of a smaller order of magnitude than the average distance between the particles and their nearest neighbors.

In the third part of the thesis we give an outlook on a novel technique, which gives rise to highly significant improvements for the full Coulomb case. In order to control stronger singularities, the estimation of probabilities for extremely rare events, i.e. particles coming very close to each other, becomes crucial. However, relying solely on the information that the true and mean-field trajectories exhibit a certain distance allows for only a rough approximation. The ability to govern the extent to which a variation in the initial trajectory impacts subsequent changes will lead to better result. In other words we have to exchange the notion of convergence from a convergence in probability to a convergence in distributional sense.

We state a necessary theorem on this regard. By a probabilistic mean-field approach we show that a small displacement of a particle at time zero entails a small effect for the dynamics of the whole system, i.e. the distance between the true dynamic and the disturbed dynamic is small for later times. For that we show that the deviation remains in the order

of magnitude of the displacement. We are able to show a even stronger result for the particles which were not disturbed at the beginning, namely that the deviation decreases as the number of particles increases.

Zusammenfassung

Die Vlasov-Gleichung ist eine effektive Gleichung, die zur makroskopischen Beschreibung der zeitlichen Entwicklung von Vielteilchensystemen verwendet wird. Die interessantesten Wechselwirkungskräfte, die für ein solches System in Betracht gezogen werden können, sind hochgradig singulär, wie zum Beispiel die Coulomb-Kraft oder die newtonsche Gravitationskraft. Obwohl Fortschritte bei der Beweisführung der Gültigkeit dieses makroskopischen Modells erzielt wurden, ist der vollständige Coulomb-Fall ohne Regularisierung, wie zum Beispiel einem Cut-off, immer noch ein offenes Problem. Aber auch andere hochgradig singuläre Kräfte, zum Beispiel Delta-ähnliche Kräfte, haben in den letzten Jahrzehnten viel Interesse geweckt.

Das Ziel dieser Arbeit ist es, Fortschritte bei der rigorosen mathematischen Herleitung der Vlasov-Poisson-Gleichung in Bezug auf die Cut-off-Größe zu erzielen und eine strenge mathematische Herleitung der Vlasov-Dirac-Benney-Gleichung für eine große Anzahl von wechselwirkenden Teilchen bereitzustellen.

Im ersten Teil der Arbeit beweisen wir probabilistisch die Mittelfeldnäherung und die Ausbreitung des Chaos eines N-Teilchensystems mit Paarpotentialen der Form $N^{3\beta-1}\phi(N^\beta x)$ für $\beta \in \left[0, \frac{1}{7}\right]$ und $\phi \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Wir zeigen, dass unter bestimmten Annahmen zur Anfangsdichte k_0 die Charakteristiken der Vlasov-Dirac-Benney-Gleichung eine sehr gute Approximation des N-Teilchensystems liefern, vorausgesetzt, ihre Anfangspositionen sind unabhängig und identisch verteilt bezüglich der Dichte k_0.

Im zweiten Teil der Arbeit liefern wir einen probabilistischen Beweis für die Mittelfeldnäherung und die Ausbreitung des Chaos eines N-Teilchensystems in drei Dimensionen für eine Coulomb-Wechselwirkung $f^N(q) = \pm\frac{q}{|q|^3}$ mit einem Cut-off $|q| > N^{-\frac{5}{12}+\sigma}$, wobei $\sigma > 0$ beliebig klein sein kann. Bemerkenswert ist, dass der Durchmesser des Cut-offs im Vergleich zum durchschnittlichen Abstand zwischen den Teilchen und ihren nächsten Nachbarn von deutlich kleinerer Ordnung ist.

Im dritten Teil der Arbeit geben wir einen Ausblick auf eine neuartige Technik, die zu signifikanten Verbesserungen für den vollständigen Coulomb-Fall führt. Um stärkere Singularitäten zu kontrollieren, ist die Schätzung von Wahrscheinlichkeiten für extrem seltene Ereignisse, also Teilchen, die einander sehr nahe kommen, von entscheidender Bedeutung. Wenn man sich jedoch ausschließlich auf die Information verlässt, dass die Trajektorien des wahren und mittleren Feldes einen bestimmten Abstand aufweisen, ist nur eine grobe Annäherung möglich. Die Fähigkeit, das Ausmaß zu steuern, in dem sich eine Variation der anfänglichen Flugbahn auf nachfolgende Änderungen auswirkt, führt zu besseren Ergebnissen. Mit anderen Worten müssen wir den Begriff der Konvergenz von einer Wahrscheinlichkeitskonvergenz in eine Verteilungskonvergenz umwandeln.

Wir beweisen hierfür ein grundlegendes Theorem. Durch einen probabilistischen Mean-Field-Ansatz zeigen wir, dass eine kleine Verschiebung eines Teilchens zum Zeitpunkt Null einen kleinen Effekt auf die Dynamik des Gesamtsystems mit sich bringt, d.h. der Abstand zwischen der wahren Dynamik und der gestörten Dynamik ist für spätere Zeitpunkte gering. Dazu zeigen wir, dass die Abweichung in der Größenordnung der Verschiebung bleibt. Für die anfangs ungestörten Teilchen können wir ein noch stärkeres Ergebnis zeigen, nämlich dass die Abweichung mit zunehmender Teilchenzahl abnimmt.

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Dedication

This work is dedicated to my deceased granny, Anna Feistl.
She is the most good natured person I was allowed to meet. Her attitude to life was always positive, no matter the difficulties and trials that go with life.
She will always be an inspiration to me.

Contents

Chapter 1

Introduction

Parts of this chapter are a reprint of

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A preprint of this version is online at https://arxiv.org/abs/2307.06146.
My own contribution. This paper is joint work with my supervisor Peter Pickl. I contributed substantially to all results. In particular, I worked out the main idea and its proof. The final version was edited under guidance of my supervisor Peter Pickl.

1.1 Motivation

Understanding the dynamics of large ensembles of identical particles is a fundamental challenge in various scientific disciplines. In many physical systems, the number of particles is so vast that considering the exact dynamics of each individual particle becomes practically infeasible. To tackle this complexity, mean-field models are employed, which approximate the collective behavior of the particles in a more manageable way. The central aim of this work is to investigate and justify the mean-field approximation for systems with a large number of identical particles subject to Newtonian time evolution.
In the realm of classical mechanics, a solitary point particle moving within a 3-dimensional Euclidean space is endowed with 3 degrees of freedom. Conventionally the phase space for a single particle is in $\mathbb{R}^3 \times \mathbb{R}^3$, encompassing all conceivable positions and momenta accessible to an unconstrained solitary particle in the 3-dimensional Euclidean setting. In the context of a system comprised of N identical point particles in motion within this 3-dimensional Euclidean space, the cumulative count of degrees of freedom expands to $3N$. For example at a temperature of 273 K and a pressure of 1.01×10^5 Pa, it is observed that within a volume of 2.24×10^{-2} m^3, the number of molecules present in any ideal gas corresponds to Avogadro's number 6.02×10^{23} [31]. As the value of N approaches this magnitude, managing the entire N-particle phase space becomes unfeasible. In an astrophysical setting, N-particle calculations also quickly reach their limits. The Milky Way contains approximately 200 billion stars. With such a number of particles, the N-particle

scenario necessitates substantial technical efforts in numerical computation and often results in the emergence of inherent limitations.

Mean-field models offer a solution by simplifying the system's complexity, often reducing it to a single-particle phase space where the count of degrees of freedom remains fixed and uninfluenced by N. When the overall particle count reaches a substantial level, the system's state at a given time t can be statistically described through a distribution function $k_t(q,p)$ in the context of the one-particle phase space. This function characterizes the density of particles located at position $q \in \mathbb{R}^3$ with momentum $p \in \mathbb{R}^3$ during time t.

On this macroscopic scale, precision gives way to a level of approximation. Instead of individual components of the gas, averaged quantities in terms of macroscopic observables such as densities are investigated. At this scale, effective interactions are considered, focusing on the outcomes arising from the collective actions of the particles. This approach is well-suited for computational analysis because at this level, a partial differential equation, known as an effective equation, is commonly applicable. This equation encapsulates the behavior of the system as a whole, which is an outcome of the amalgamated influence of the particles actions.
The Vlasov equation is a classical example of an effective equation which describes the coarse-grained time evolution of such a system. While notable strides have been taken in recent times to establish a rigorous foundation for this approach, the persistence of highly singular interaction potentials that hold physical significance introduces a need for significant further advancements to attain a fully compelling outcome. The primary objective of this thesis is to drive forth additional progress in this specific regard. More precisely we delve into the exploration of two Vlasov-type mean-field equations, which find application in describing various physical systems like plasmas, molecules, and vortices in incompressible fluids. These mean-field equations come to the forefront as approximations derived from the complete set of motion equations, particularly as the number of particles rises. These equations have been established and effectively applied in the realm of physics for numerous decades, serving as a macroscopic portrayal of collisionless plasmas composed of charged or gravitating particles. They possess an intuitive quality that a proficient physicist might readily anticipate. Nonetheless, a precise mathematical derivation from foundational principles has remained unresolved. In this research, we present findings achieved through a suitable microscopic regularization, one that diminishes as the particle count grows large. The underlying mathematical frameworks employed to validate these approximations are introduced and thoroughly examined.

The quest to establish the validity of the mean-field approximation entails a comparative analysis between the precise dynamics of individual particles, governed by Newton's second law, and the motion equation of a representative particle influenced by its cumulative interactions with all other particles.

In summary, this work aims to explore and validate the mean-field approximation for systems composed of a large number of identical particles. By investigating the mathematical foundations and practical implications of mean-field models, a deeper understanding of the collective behavior of such systems can be achieved. In particular we focus on deterministic second order systems leading to the kinetic Vlasov equation.

1.2 Previous results

Numerous models have been proposed in the literature which reproduce the kinetic effects in effective equations describing gases or fluids. One example of such an effective equation goes back to Vlasov [44], which has been derived with mathematical rigour by Neunzert and Wick in 1974 [35]. Classical results of this kind are valid for Lipschitz-continuous forces [8, 13]. One difficulty is handling clustering of particles for singular interactions like Coulomb or Newtons gravitational force [41]. Hauray and Jabin examined singular interaction forces in three dimensions, which are scaling like $1/|q|^\lambda$ with $\lambda < 1$ [28]. They included the physically more interesting case with λ smaller but close to 2 with a lower bound on the cut-off at $q = N^{-1/6}$ a few years later [27]. They had to choose quite specific initial conditions, according to the respective N-particle law. The last deterministic result we would like to mention in the Coulomb interaction setting is [30]. It assumes no cut-off and is valid for repulsive pair-interactions, but requires a bound on the maximal forces of the microscopic system. One major difference to our work is that the results rely on deterministic initial conditions, even if some of them are formulated probabilistically. In contrast to the previous approaches Boers and Pickl [7] derive the Vlasov equations for stochastic initial conditions with interaction forces scaling like $|x|^{-3\lambda+1}$ with $(5/6 < \lambda < 1)$. They obtained a cut-off as small as the typical inter particle distance at $N^{-\frac{1}{3}}$. By exploiting the second order nature of the dynamics and introducing anisotropic scaling of the relevant metric to include the Coulomb singularity Lazarovici and Pickl [32] extended the method in [7] and obtained a microscopic derivation of the Vlasov-Poisson equation with a cut-off of $N^{-\delta}$ with $0 < \delta < \frac{1}{3}$. More recently, by examining the collisions which could occur and using the second order nature of the dynamics, the cut-off parameter was reduced to as small as $N^{-\frac{7}{18}+\sigma}$, with $\sigma > 0$ in [22]. For delta like potentials of the form $N^{3\beta-1}\phi(N^\beta)$ with $0 < \beta \leq \frac{1}{3}$ there are two note worthy results in the classical setting by Ölschläger [37], including Brownian motion, or [36] which addresses the derivation of the continuity equation in the monokinetic setting. Griffin-Pickering and Iacobelli [34] derived the Vlasov-Dirac-Benney equation without considering Brownian motion or making assumptions such as monokineticity. Their derivation is valid for a scaling parameter of order $\beta < \frac{1}{15}$.

1.3 Present result

The strategy presented in this thesis uses stochastic initial conditions, as it is based on the technique proposed in [7, 32, 22]. We present the mean-field limit for the Vlasov Dirac Benney equation and the Vlasov Poisson equation. A frequently discussed system where Vlasov type equations are used to draw the main physical features are plasmas, i.e. gases of very high temperature. At such high temperatures a significant portion of the particles are ionized, thus plasmas consist of a mixture of electrons and positively charged ions. Since the mass of the electrons is very small compared to the masses of the ions, the long-range-part of the electric field of the ions gets neutralized and the motion of the ions is effectively described by a model of particles with short rang interaction (see [24] for a more detailed discussion). At the relevant time-scales, i.e. times of an order where the interaction has an effect on the dynamics, this means short range but strong coupling for the interaction. Thus our derivation in Chapter 2 deals with potentials of the form $N^{3\beta-1}\phi(N^\beta)$ with $\beta \in [0, \frac{1}{7}]$ and $\phi \in L^\infty \mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ instead of a Coulomb potential like in Chapter 3. The usual methods for deriving effective descriptions for microscopic dynamics fail here because there is no kind of Lipschitz condition for the force. The whole interaction has a Lipschitz constant that depends on N. In all mentioned papers above, in most of the cases the potential is nice and smooth. The main part of these proofs considers cases where the force either Lipschitz-continuous or negligible due to probabilistic arguments. In our case, we have an interaction that can be felt by the leading order and additionally has a large derivative. So the force is not Lipschitz-continuous at all. With regard to our goal, the derivation of the Vlasov-Dirac-Benney equation from the microscopic Newtonian N-particle dynamics, we compare the N-particle Newtonian flow with the effective flow given by the macroscopic equation in the limit $N \to \infty$ for a pair potential of the form

$$\phi_N^\beta = N^{3\beta-1}\phi(N^\beta)$$

with $\beta \in \left[0, \frac{1}{7}\right]$ and $\phi \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Our system in Chapter 2 is between collision and mean-field behavior and describes physical situations with a long-range interaction, since the typical distance between particles is of order $N^{\frac{1}{3}}$. On the one hand, the interaction force is collision-like, so that interactions only rarely occur, i.e. one particle interacts only with a selection of particles of order $\gg 1$ but $\ll N$ and not with all particles. On the other hand, the behavior can be described with the mean-field approach. The system couples strongly but is localized so a mean-field approach can be applied. The effect of the instabilities are much more drastic. In the following we prove that the measure of the set where the maximal distance of the Newtonian trajectory and the mean-field trajectory is large gets vanishingly small as N increases.

In Chapter 3 we provide the currently most advanced derivation of the Vlasov-Poisson equation's optimal cut-off. This equation is a classical example of an effective equation describing the coarse-grained time evolution of a N-particle system with Coulomb or Newtonian pair interaction in the large N limit. Specifically, this interaction is governed by

$$f^N(q) = \pm \frac{q}{|q|^3} \text{ for } |q| > N^{-\frac{5}{12}+\sigma}$$

with cut-off at $|q| = N^{-\frac{5}{12}+\sigma}$ for arbitrarily small $\sigma > 0$. The cut-off diameter is of smaller order than the average distance of a particle to its nearest neighbour and has been significantly improved compared to the results of Grass and Pickl [22]. The underlying poof technique shows that a further improvement is possible by utilizing a finer subdivision of particle subsets.

In Chapter 4, we provide a essential stability result, i.e. we show by a probabilistic mean-field approach, that a small displacement of a particle at the beginning entails a small effect for the dynamics of the whole system, i.e. the distance between the true dynamics and the disturbed dynamic will be small at later times. This will be fundamental for a novel proof technique in which we want to improve the cut-off by changing the notion of distance by convergence in distribution.

1.4 The microscopic model

We consider a classical N-particle system subject to Newtonian dynamics interacting through a pair interaction force. Our system is distributed as a trajectory in phase space $\mathbb{R}^{6N}$. We use the notation $X = (Q, P) = (q_1, \ldots, q_N, p_1, \ldots, p_N)$, where $(Q)_j = q_j \in \mathbb{R}^3$ denotes the one-particle position and $(P)_j = p_j \in \mathbb{R}^3$ stands for its momentum. The Hamiltonian, the operator corresponding to the total energy of the system, is given by

$$H_N(X) = \sum_{j=1}^{N} \frac{p_j^2}{2m} + \sum_{1 \le j < k \le N} \phi^\beta(q_j - q_k), \tag{1.1}$$

with $\phi^\beta \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and $x_j = (q_j, p_j) \in \mathbb{R}^6$. As long as the conditions on the solution of the effective equation are valid an external potential can be added to the Hamiltonian, but it does not affect the derivation of the equation as it has the same impact on all particles regardless of their distribution. Since we consider differences between exact Newtonian dynamics and mean-field dynamics we omit the external potential without loss of generality. Setting the mass $m = 1$ leads us to the equations of motion, which determine the particle trajectories

$$\begin{cases} \dot{q}_j = p_j \\ \dot{p}_j(t) = -\sum_{k=1}^{N} \nabla_{q_j} \phi_N^\beta(q_j - q_k) = -\sum_{k=1}^{N} \frac{1}{N} f_N^\beta(q_j - q_k). \end{cases} \tag{1.2}$$

We consider the system in the mean-field scaling. An N-dependent coupling might seen unphysical on the first view. However on can arrive ant such a system by rescaling space, time and velocity coordinates accordingly, using that f^N is homogeneous (up to technical cutoff). Note that the scaling we chose is such that, as N increases, the dynamics of the cloud remains fixed (both in q and p) while its density grows linearly with N. Furthermore the scaling ensures that the total interaction per particle remains of order 1. If one chooses a much smaller scaling factor than $1/N$, the force term becomes negligible, resulting in nearly free time-evolution for large N. On the other hand for a scaling factor $\gg 1/N$, the force term becomes increasingly dominant, leading to highly complex and possibly singular behaviour that strongly depends on microscopic interaction details.

The interaction potential ϕ_N^β is defined in Chapter 2 and is of the form $\phi_N^\beta = N^{3\beta-1}\phi(N^\beta x)$ with $\beta \in [0, \frac{1}{7}]$, for some bounded spherical symmetric $\phi : \mathbb{R}^3 \to \mathbb{R}$ with $\nabla\phi(0) = 0$. In contrast, the interaction potential ϕ_N^β in Chapter 3 is a coulomb potential with a cut-off.
Furthermore f_β^N denotes the pair interaction force for the system. Analogously the total force of the system is given by $F : \mathbb{R}^{6N} \to \mathbb{R}^{3N}$, where the force exhibited on a single coordinate j is given by

$$(F(X))_j := \sum_{i \ne j} \frac{1}{N} f_\beta^N(q_i - q_j).$$

By introducing the N-particle force we can characterize the Newtonian flow as a solution of the next equation. As the vector field is Lipschitz for fixed N we have global existence and uniqueness of solutions and hence a N-particle flow.

Definition 1.1. *The Newtonian flow* $\Psi^N_{t,s}(X) = (\Psi^{1,N}_{t,s}(X), \Psi^{2,N}_{t,s}(X))$ *on* $\mathbb{R}^{6N}$ *is defined by the solution of:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi^N_{t,s}(X) = V(\Psi^N_{t,s}(X)) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N} \tag{1.3}$$

where V *is given by* $V(X) = (P, F(X))$.

The crucial observation is that the force looks like the empirical mean of the continuous function $\nabla\phi(q_j - \cdot)$ of the random variable q_j. In the limit $N \to \infty$, one might expect this to be equal to the expectation value of $\nabla\phi$ given by the convolution $f^N_\beta * \tilde{k}_t$, where $\tilde{k}_t(q,p)$ denotes the mass density at q with momentum p at time t. In the following section we explain the general strategy that we follow in this thesis based on [7, 32, 22].

1.5 Heuristics and sketch of the proof

There are two common techniques to translate the microscopic to the macroscopic system or vice versa. For $\Psi^N_{t,0}(X) = (q_i(t), p_i(t))_{i=1,..,N}$, one can define the corresponding microscopic or empirical density by

$$\mu^N_t[X] = \mu^N_0[\Psi^N_{t,0}(X)] := \frac{1}{N}\sum_{i=1}^N \delta(\cdot - q_i(t))\delta(\cdot - p_i(t)).$$

By changing the level of description one can consider an equation for a continuous mass density, which describes the same situation but from a macroscopic point of view. Furthermore, the microscopic force can be written as

$$\frac{1}{N}\sum_{j=1}^N f_N(q_i - q_j) = f_N * \mu^N_t[X](q_i).$$

This relation is often used to translate the microscopic dynamics into a Vlasov-type equation, allowing to treat $\mu^N_t[X]$ and k_t on the same footing to proof the validity of the common physical descriptions. For typical X, the empirical density $\mu^N_t[X]$ and the solution k_t of the Vlasov-type equation, which among other things is used to describes the time evolution of the distribution function of a plasma consisting of charged particles, are close to each other as $N \to \infty$. The underlying technique of this thesis operates the other way around. We translate the density k_t into a trajectory. The core concept of our strategy consists of two steps. First we sample the regularized mean-field dynamics along trajectories with random initial conditions. Then we estimate the difference between the true microscopic trajectories and the mean-field trajectories in terms of expectation. We will not translate the trajectory into a density as it is often done. Instead of summing up all iteration terms one expects a single particle to feel only the mean-field produced by all particles together. But as there is only the external force f_N, the time evolution of k_t is dictated by the continuity equation on $\mathbb{R}^6$ and by inserting the expectation value from above it leads us to the partial non-linear Vlasov type equation, which solution theory is studied for both types of singular interactions force discussed in this thesis (see Chapter 1.6).

Construction of the mean-field force

The strategy to construct the mean-field force can heuristically be explained as follows.

We split the universe into j boxes of the same volume, such that each box contains n_j particles. As the density is defined as the number of particles per volume we get $k_t(q,p) = \frac{n_j}{V_j N}$ and the force acting on one particle can be written as

$$\bar{f}(q) = \sum_j \frac{n_j}{N} f(q - q_j) = \sum_j V_j k_t(q_j, p_j) f(q - q_j).$$

This can be read as a Riemann sum and so it can be written as

$$\approx \int k_t(q,p) f(q - q_j) d^3p d^3q_j = k_t * f(q)$$

which is the convolution of k_t and f in the q-coordinate. The mean-field particles move independently, because we use the same force for every particle and we do not have pair interactions, which could lead to correlations. Thus each particle has its own force-term. In summary for fixed k_0, $N \in \mathbb{N}$ and any initial configuration $X \in \mathbb{R}^{6N}$ we consider two different time-evolutions $\Psi^N_{t,0}(X)$, given by the microscopic equations and $\Phi^N_{t,0}(X)$, given by the time-dependent mean-field force generated by f^N_t. Our goal is to show that for typical X, the two time-evolutions are close in an appropriate sense. In other words, we have a non-linear time-evolution in which $\varphi^N_{t,s}(\cdot\,; k_0)$ is the one-particle flow induced by the mean-field dynamics with initial distribution k_0, while, in turn, k_0 is transported with the flow $\varphi^N_{t,s}$.

Quantifying the accuracy of the mean-field description

We want to show that the time derivative of the distance $d_t \mathrm{d}(\Psi^N_t, \Phi^N_t)$ fulfills a Gronwall inequality. If $|f|_L < \infty$ it is easy to check, but most physically interesting cases are not Lipschitz continuous. For technical reasons it is useful to distinguish two cases $\|\Psi^N_t - \Phi^N_t\| \leq N^{-\gamma}$ and $\|\Psi^N_t - \Phi^N_t\| > N^{-\gamma}$ for $\gamma > 0$. So we introduce a stochastic process of the following form

$$J_t := \min\{1, N^{\gamma} \|\Psi^N_t - \Phi^N_t\|_\infty\}. \tag{1.4}$$

Note, that the stochastic process in Chapter 2 and 3 is slightly different from the one shown above. Furthermore in Chapter 3 a differential version of Gronwall Lemma is applied, but in the upcoming simplified form the idea of the proof stays the same. The process J_t helps us to establish a Grönwall type argument of the following kind. For all $t \in \mathbb{R}^+$ the expectation $\mathbb{E}(J_t)$ value of J_t tends to zero if $\mathbb{E}(J_0)$ tends to zero. More precisely we will estimate

$$d_t \mathbb{E}(J_t) \leq C(\mathbb{E}(J_t) + \sigma_N(1))$$

to receive

$$\mathbb{E}(J_t) \leq e^{Ct}(\mathbb{E}(J_0) + \sigma_N(1)).$$

This is useful to model the underlying problem because, if $\mathbb{E}(J_t)$ is small, then the probability to hit 1 is small, that means that the probability

$$\mathbb{P}(A) \quad \text{for} \quad A = \{|\Psi^N_{s,0}(x) - \Phi^N_{s,0}(X)| \geq N^{-\gamma}\}$$

is small, too. If $\mathbb{E}(J_0) \to 0$ and for all $t \in \mathbb{R}^+$ it holds that $d_t\mathbb{E}(J_t) \leq C(\mathbb{E}(J_t)+\sigma_N(1))$. With Gronwall Lemma it can be shown, that $\mathbb{E}(J_t) \leq e^{Ct}(\mathbb{E}(J_0) + \sigma_N(1))$ and so we get $\mathbb{E}(J_t) \to 0$. Note that we chose the same initial conditions for Ψ^N and for Φ^N, so $J_0 = 0$.

It is advantageous to do a Grönwall estimate on J_t than directly on $P(A)$, because we need some kind of smoothness for the derivative. Each probability of the set A can be translated into an expectation value of the characteristic function with $\mathbb{E}(\chi_A)$, but the stochastic process J_t starts to smoothly decline at the boundary of A. Both descriptions are basically the same, apart from the superiority of J_t in the later proof. The cut-off in the Definition of J_t has been chosen at 1, such that if J_t is smaller than 1 it is directly implied, that $|\Psi^N_{t,0}(X) - \Phi^N_{t,0}(X)|_\infty < N^{-\gamma}$.

In order to estimate the time derivative of $E(J_t)$, we note that the inequality $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(J_t) \leq C(\mathbb{E}(J_t) + o_N(1))$ is trivial because the random variable J_t has reached its maximum, the value 1. The configurations where J_t is maximal, that is $|\Psi^N_{t,0}(X) - \Phi^N_{t,0}(X)|_\infty \geq N^{-\gamma}$ are irrelevant for finding an upper bound of $\mathbb{E}_0(J_{t+dt}) - \mathbb{E}_0(J_t)$. The set of such configurations will be called $\mathcal{A}_t$ and the expectation value $\mathbb{E}(J_{t+dt} - J_t)$ restricted on the set $\mathcal{A}_t$ is less or equal 0.

Proof skeleton

The following proof techniques will be applied in Chapter 2-4. In Chapter 3 a further division of the particles into subsets depending on their relative position and speed is necessary.

In Chapter 4 we are interested in the distance between the true and the shifted system instead of the distance between the true and the mean-field system, where the proof has similarities to the upcoming sketch.

We utilize the distinction of configurations belonging to $\mathcal{A}_t$ or respectively to $\mathcal{A}^c_t$ to estimate the expectation value

$$d_t J_t = \lim_{dt \to 0} \frac{\mathbb{E}(J_{t+dt} - J_t)}{dt} \leq \mathbb{E}(|\frac{\mathrm{d}}{\mathrm{d}t} J_t|) = \mathbb{E}(|\dot{J}_t|\mathcal{A}_t) + \mathbb{E}(|\dot{J}_t|\mathcal{A}^C_t).$$

If $X \in \mathcal{A}_t$ we have $\|\Psi^N_t - \Phi^N_t\| > N^{-\gamma}$ and by the definition of the help process we get $J_t(X) = 1$ and consequently $J_{t+dt}(X) \leq 1$. This provides $\mathbb{E}(|\dot{J}_t|\mathcal{A}_t) = 0$. Furthermore for $\mathbb{E}(|\dot{J}_t||\mathcal{A}^c_t)$ one can estimate

$$\begin{aligned}
\mathbb{E}(|\dot{J}_t||A^c) &= \mathbb{E}(\|\Psi^N_t \dot{-} \Phi^N_t\|_\infty|A^c)N^\gamma \\
&\leq \mathbb{E}(\|F(\Psi^N_t) - \bar{F}(\Phi^N_t)\|_\infty|A^c)N^\gamma + \mathbb{E}(J_t|A^c) \\
&\leq \mathbb{E}(\|F(\Psi^N_t) - F(\Phi^N_t)\|_\infty)N^\gamma + \mathbb{E}(\|F(\Phi^N_t) - \bar{F}(\Phi^N_t)\|_\infty|A^c)N^\gamma \\
&\quad + \mathbb{E}(J_t|A^c).
\end{aligned}$$

The last summand $\mathbb{E}(J_t|A^c)$ is trivially bounded by $\|\Psi^N_t \dot{-} \Phi^N_t\|_\infty$ due to Newtons law.

To estimate the other terms we will introduce a version of law of large numbers and use the Markov inequality. Therefore the first addend needs some preparatory work, because we can not apply law of large numbers directly. For this we will estimate the difference by a mean value argument

$$\begin{aligned}\|F(\Psi_t^N) - F(\Phi_t^N)\|_\infty N^\gamma &= \|\frac{1}{N}\sum_{j\neq k} f(q_j^{\Psi_T^N} - q_k^{\Psi_t^N}) - f(q_j^{\Phi_t^N} - q_k^{\Phi_t^N})\|_\infty N^\gamma \\ &\leq \sum_{j\neq 1} g(q_1^{\Phi^N} - q_j^{\Phi^N}) \cdot 2\underbrace{\|\Psi^N - \Phi^N\|_\infty}_{\leq N^{-\gamma}}.\end{aligned}$$

The last term is independent from Ψ_t^N, i.e. stochastically independent. In Chapter 2 the estimation of g provided by the law of large numbers determines the choice of the parameter β due to the occurring variance term.

1.6 Vlasov-type equations

Looking for a macroscopic law of motion for the particle density leads us to the Vlasov equation. It is a differential equation describing time evolution of the distribution function of plasma. Depending on the choice of interaction potential we will consider two types of Vlasov equations in this thesis.

1.6.1 Vlasov-Poisson equation

The Vlasov-Poisson equation describes the time evolution of a plasma consisting of charged particles with gravitational or electrostatic force. It reads as follows

$$\begin{cases}\partial_t k + p\cdot\nabla_q k + E(t,q)\cdot\nabla_p k = 0,\\ E(t,q) = f * \tilde{k}_t = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\tilde{k}_t(y)(q-y)}{|q-y|^3}\,d^3y,\\ \tilde{k}_t(y) = \int_{\mathbb{R}^3} k_t(y,p)\,d^3p,\end{cases} \tag{1.5}$$

with the initial density $k_0(q,p)$. The enduring relevance of the Poisson kernel, an ancient concept tracing its origins to Newton's theory of gravitation, persists in the fields of cosmology and astrophysics. Its continued utilization is primarily attributed to its applicability on a large scale, where relativistic effects can often be considered negligible. Within this framework, each individual particle within the system symbolizes either a star or a more extensive celestial structure.

In situations involving repulsive interactions, the Poisson kernel is employed to describe electrostatic interactions among particles, frequently in the realm of plasma physics, where it encompasses the consideration of various particle species or components.

Moreover, the Poisson kernel plays a pivotal role in first-order models, notably in contexts like chemotaxis, which revolves around the motion of bacteria or cells induced by a chemical potential. In the context of such biological systems the force field E_N can be interpreted as the gradient of concentration for a chemical substance generated by each individual particle.

In the case of the regularized interaction force f^N defined in Definition 3.1 the solution theory is known by Braun and Hepp [8]. In this case the force is Lipschitz continuous. The Cauchy problem for the Vlasov-Poisson system for the non-regularized, singular force f^∞ has been a subject of extensive research in recent decades. Global existence and uniqueness of classical solutions in two dimensions were achieved by Ukai and Okabe [43] and later global weak solutions with finite energy were first constructed by Arsenev [1] in dimension $d = 3$.

Subsequently, global existence and uniqueness for specified cases of more regular solutions were separately established by Lion and Perthame [33] and by Pfaffelmoser [38] using different techniques. To achieve this they had to control the plasma velocities over all time. Lion and Perthame [33] constructed weak solutions with finite velocity moments expressed as $\int\int_{\mathbb{R}^3\times\mathbb{R}^3} |p|^m k_t(q,p)\, d^3q\, d^3p < \infty$ for $m > 3$. Di Perna and Lions [12] proved that such solutions remain constant along the trajectories in a weakly sense.

Pfaffelmosers approach [38] hinges on a meticulous examination of the characteristics to control the expansion of velocity support. This approach results in the achievement of global existence and uniqueness of classical, compactly supported solutions. Moreover, these solutions propagate the regularity of the initial density due to the work of L. Desvillettes, E. Miot and C. Saffirio [11].

For further enhancements and developments, one may refer to the works of Schaeffer [16], Wollman [17], Castella [9], Loeper [18], Chen and Zhang [10]. Furthermore, Gasser, Jabin, and Perthame [19] have established the propagation of velocity moments for $m > 2$, with an additional assumption regarding space moments. In [40] Salort demonstrated the existence and uniqueness of weak solutions for small moments.

For our purposes, a result established by Horst [26] is sufficient, as it provides global existence of classical solutions (uniquely) under conditions that closely align with the assumptions required for the proof of our Theorem 3.2 in Chapter 3. He specifically shows that there is a continuously differentiable function $k : [0,T]\times\mathbb{R}^6 \to [0,\infty)$ for any $T > 0$ that satisfies the Vlasov-Poisson equation for any initial condition $k(0,\cdot) = k_0 \in L^1(\mathbb{R}^6)$, which is non-negative, continuously differentiable, and satisfies the following conditions for a suitable constant $C > 0$, some $\delta > 0$, and all $(q,p) \in \mathbb{R}^6$:

$$
\begin{aligned}
(i) &\quad k_0(q,p) \leq \frac{C}{(1+|p|)^{3+\delta}} \\
(ii) &\quad |\nabla k_0(q,p)| \leq \frac{C}{(1+|p|)^{3+\delta}} \\
(iii) &\quad \int_{\mathbb{R}^6} |p|^2 k_0(q,p) d^3q d^3p < \infty.
\end{aligned}
$$

Essentially, for each time interval $[0,T)$, there exists a constant $C > 0$, depending on k_0 and T, such that

$$
\sup_{0\leq s<T} |\widetilde{k}_s|_\infty < C(T,k_0).
$$

Then for any time interval $[0,T] \subseteq [0,\infty)$ there exists a unique solution to the Vlasov-Poisson equation with initial data k_0.

1.6.2 Vlasov-Dirac-Benney equation

The Vlasov-Dirac-Benney equation is a Vlasov type equation with the interaction potential replaced by a Dirac mass. It reads as follows

$$\begin{cases} \partial_t k \ + \ p \cdot \nabla_q k \ + \ E(t,q) \cdot \nabla_p k \ = \ 0, \\ E(t,q) = f * \tilde{k}_t, \\ \tilde{k}_t(y) = \displaystyle\int_{\mathbb{R}^d} k_t(y,p) \, d^d p \end{cases} \tag{1.6}$$

with the initial density $k|_{t=0} = k_0(q,p)$. The nomenclature goes back to the fact that in one of the most important configurations, i.e. initial data near a one-bump profile, it is equivalent to the Benney-equation [2]. The Vlasov-Dirac-Benney equation describes a plasma, i.e. gases of very high temperature. At such high temperatures a significant portion of the particles are ionized, thus plasmas consist of a mixture of electrons and positively charged ions. Since the mass of the electrons is very small compared to the masses of the ions, the long-range-part of the electric field of the ions gets neutralized and the motion of the ions is effectively described by a model of particles with short rang interaction (see [24] for a more detailed discussion). At the relevant time-scales, i.e. times of an order where the interaction has an effect on the dynamics, this means short range but strong coupling for the interaction. Thus our derivation deals with potentials of the form $N^{3\beta-1}\phi(N^\beta)$ for some positive potentials ϕ. But due to its highly singular nature, the solution theory is not trivial. One existence result we want to mention here is [6]. The authors consider so called water bags, which are piecewise constant functions, as initial data. Another result is [15] which proves the existence for short times of analytical solutions in dimension one. Bardos and Besse [4, 3] could show, in dimension one, that the problem is wellposed for functions that for all q have the shape of one bump.

Another approach is to use the Penrose stability condition, introduced in [39] for homogeneous, i.e. q-independent equilibria $k(p)$

Definition 1.2. *For $p \mapsto k(p)$ the Penrose function is defined by*

$$\mathcal{P}(\gamma,\tau,\eta,k) = 1 - \int_0^{+\infty} e^{-(\gamma+i\tau)s} \, \frac{i\eta}{1+|\eta|^2} \cdot (\mathcal{F}_p \nabla_p k)(\eta s) \, ds, \ \gamma > 0, \ \tau \in \mathbb{R}, \ \eta \in \mathbb{R}^d \backslash \{0\}$$

where $\mathcal{F}_p$ denotes the Fourier transform in momentum coordinate p. The profile k satisfies the c_0 Penrose stability condition if

$$\inf_{(\gamma,\tau,\eta) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}} |\mathcal{P}(\gamma,\tau,\eta,k)| \geq c_0. \tag{1.7}$$

These assumptions are for example satisfied in a small data regime, for "one bump" profiles in $d = 1$ and also for any radial non-increasing functions in any dimension.

Han-Kwan and Rousset [25] used Penrose stability in the q-dependent case. They assumed that $k_0(q,p)$ satisfies the $\frac{c_0}{2}$ Penrose stability condition for any $q \in \mathbb{R}^d$. Furthermore they required that the initial density $k_0(q,p) \in \mathcal{H}^{2m}_{2r}$ with $2m >$

$4+\frac{d}{2}+\lfloor\frac{d}{2}\rfloor$, $2r>\max(d,2+\frac{d}{2})$, where the weighted Sobolev norms for $k\in\mathbb{N}, r\in\mathbb{R}$ are given by

$$\|k\|_{\mathcal{H}^k_r} := \left(\sum_{|\alpha|+|\beta|\le k}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(1+|p|^2)^r|\partial_q^\alpha\partial_p^\beta k|^2\, d^dp d^dq\right)^{1/2},$$

where $\alpha=(\alpha,\cdots,\alpha_d),\beta=(\beta_1,\cdots,\beta_d)\in\mathbb{N}^d$, $|\alpha|=\sum_{i=1}^d\alpha, |\beta|=\sum_{i=1}^d\beta_i$, and $\partial^\alpha:=\partial_{q_1}^{\alpha_1}\cdots\partial_{q_d}^{\alpha_d}, \partial_p^\beta:=\partial_{p_1}^{\beta_1}\cdots\partial_{p_d}^{\beta_d}$.

Under these assumptions, Han-Kwan and Rousset proved in any dimension existence and uniqueness of solutions of the Vlasov-Dirac-Benney equation (1.6) on a compact time interval.

In Chapter 2 of this thesis we will assume existence of a C^∞ solution of the Vlasov-Dirac-Benney equation and derive it from the microscopic N-particle dynamics.

1.7 The mean-field model

To compare the microscopic system and the macroscopic system we translate the density k_t of the Vlasov-type equations into a trajectory. We will now define the characteristic flow in alignment with the heuristics (Section 1.5) previously introduced. The characteristics of Vlasov equation similar to Definition 1.1 are given by the following system of Newtonian differential equations

$$\begin{cases} \frac{d\bar{q}}{dt}=\bar{p} \\ \frac{d\bar{p}}{dt}=f^N*\widetilde{k}_t(\bar{q}):=\bar{f}_t^N \end{cases} \tag{1.8}$$

where $\widetilde{k}_t$ denotes the previously introduced 'spatial density'. We now introduce the effective one-particle flow $(\varphi_{t,s}^N)_{t\le s}$ for any probability density $k_0:\mathbb{R}^6\to\mathbb{R}_0^+$ and lift it up to the N-particle phase space.

Definition 1.3. *Let $k_0:\mathbb{R}^6\to\mathbb{R}_0^+$ be a probability density and $k:\mathbb{R}\times\mathbb{R}^6\to\mathbb{R}_0^+$, which gives for each time t the effective distribution function time-evolved with respect to $\varphi_{t,s}^N : k(0,\cdot)=k_0$ and*

$$k_t^N(x):=k^N(t,x)=k_0(\varphi_{0,t}^N(x)).$$

For $x=(q,p)$, the effective flow $\varphi_{t,s}^N$ itself is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{t,s}^N(x)=v^t(\varphi_{t,s}^N(x))$$

where v^t is given by $v^t(x)=(p,\bar{f}_t^N(q))$. Here the mean-field force $\bar{f}_t^N$ is defined as $\bar{f}_t^N=f_\beta^N*\tilde{k}_t^N$ and $\tilde{k}_t^N:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}_0^+$ is given by

$$\tilde{k}_t^N(q):=\int k_t^N(p,q)d^3p.$$

By using this approach, a new trajectory is obtained that is influenced by the mean-field force instead of the pair interaction force like in the Newtonian system (1.2). Now we have two trajectories which we will compare and show later that they are close to each other.
To this end, we consider the lift of $\varphi^N_{t,s}(\cdot)$ to the N-particle phase-space, which we denote by $\Phi^N_{t,s}$. To lift the effective one-particle flow to the N-particle space we define the mean-field flow by:

Definition 1.4. *The respective* $\Phi^N_{t,s} = (\Phi^{1,N}_{t,s}, \Phi^{2,N}_{t,s}) = (\varphi^N_{t,s})^{\otimes N}$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^N_{t,s}(X) = \bar{V}_t(X), \tag{1.9}$$

with $\bar{V}_t(X) = (P, \bar{F}_t(Q))$ *and* $\bar{F}_t$ *given by* $(\bar{F}_t(Q))_j := \bar{f}^N_t(q_j)$.

The mean-field particles move independent because the same force acts on every particle and we do not have pair interactions, which lead to correlations. For fixed k_0, $N \in \mathbb{N}$ and for any initial configuration $X \in \mathbb{R}^{6N}$, we considere the two time-evolutions from Definitions 1.4 and 1.1. On the one hand the Newtonian flow $\Psi^N_{t,0}(X)$, given by the microscopic equations and on the other hand the mean-field flow $\Phi^N_{t,0}(X)$, given by the time-dependent mean-field force generated by f^N_t. Our goal is to show that for typical X, these two time-evolutions are close in an appropriate sense. In other words, we have non-linear time-evolution in which $\varphi^N_{t,s}(\cdot\,; k_0)$ is the one-particle flow induced by the mean-field dynamics with initial distribution k_0, while, in turn, k_0 is transported with the flow $\varphi^N_{t,s}$. It generally suffices to consider the initial time $s = 0$, since $\varphi^N_{t,s}$ fullfills the semi-group property $\varphi^N_{t,s'} \circ \varphi^N_{s',s} = \varphi^N_{t,s}$ [32].

In the following two sections we show that the two flows defined in Definition 1.4 and 1.1 are close to each other in the context of the Vlasov-Poisson for Coulomb interaction (f^N defined in Definition 3.1) and Vlasov-Dirac-Benney equation for delta like interactions (f^N defined in Definition 2.1) and hence the microscopic and the macroscopic approach describe the same system.

Therefore we will show that the mean-field limit and propagation of chaos of an N-particle system with the respective pair potential. Furthermore we will show that the characteristics of the considered Vlasov type equation provide a reliable approximation of the N-particle trajectories, provided their initial positions are independent and identically distributed with respect to density k_0.

Chapter 2

Microscopic derivation of the Vlasov-Dirac-Benney equation with with a strong short range force

The main parts of this chapter are a reprint of

> Pickl. P and Feistl, M. (2023). Microscopic derivation of Vlasov equation with compactly supported pair potentials.

A preprint of this version is online at https://arxiv.org/abs/2307.06146.
My own contribution. This paper is joint work with my supervisor Peter Pickl. I contributed substantially to all results. In particular, I worked out the main idea and its proof. The final version was edited under guidance of my supervisor Peter Pickl.

In the following we consider a classical N-particle system, distributed as a trajectory in phase space $\mathbb{R}^{6N}$ and subject to Newtonian dynamics like introduced in Chapter 1. The interaction potential ϕ_N^β in dimension three that underlies this chapter is given by

$$\phi_N^\beta = N^{3\beta-1}\phi(N^\beta x), \qquad \beta \in \left[0, \frac{1}{7}\right],$$

for some bounded spherical symmetric $\phi : \mathbb{R}^3 \to \mathbb{R}$ with $\nabla\phi(0) = 0$. We show that the N-particle trajectory Ψ_t starting from Ψ_0 (i.i.d. with the common density k_0) remains close to the mean-field trajectory Φ_t with the same initial configuration $\Psi_0 = \Phi_0$ during any finite time $[0, T]$ and so the microscopic and the macroscopic approach describe the same system. Throughout this Chapter C denotes a positive finite constant which may vary from place to place but most importantly it will be independent of N. The factor $\beta \in \mathbb{R}$ determines the scaling behavior of the interaction and depending on how one chooses β one gets another hydrodynamic equation. Usually ϕ_N^β scales with the particle number such that the total interaction energy scales in the same way as the total kinetic energy of the N particles, so that the L^1-norm of ϕ_N^β is proportional to N^{-1} with $\phi_N^\beta(x) = N^{-1+3\beta}\phi(N^\beta x)$ for $\phi \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Note that we assume that ϕ and thus the pair interaction force f will be independent of the momentum.

The case $\beta = 0$ was studied in [8]. The strength of the interaction is of order $\frac{1}{N}$ and hence the equations of motion consider a weakly interaction gas. For positive β the support of the potential shrinks and the strength of the interaction increases with N and $\lim_{N\to\infty} \phi_N = \delta_0$. Thus the case $\beta = \frac{1}{3}$ describes in contrast a strong interaction process. The interaction strength is of order 1 but two particles only interact when their distance is of order of the typical inter particle distance in $\mathbb{R}^3$. As long as $\beta < 1/3$ the mean-field approximation is from the heuristical point of view not surprising because the interaction potentials overlap as the typical particle distance has approximately the size of $N^{\frac{1}{3}}$. This mean inter-particle distance is consequently smaller than the range of the interaction. Hence, on average, every particle interacts with many other particles, and the interactions are weak since $N^{-1}N^{3\beta} \to 0$ as $N \to \infty$. As long as the correlations are sufficiently mild, the law of large numbers gives that the interaction can be replaced by its expectation value, the so-called mean-field.

The potential gradient $\nabla_q \phi_N^\beta = f_\beta^N$ determines the pair interaction function, which is given by

Definition 2.1. *For $N \in \mathbb{N} \cup \{\infty\}$ and a smooth function $l \in W^{1,\infty}(\mathbb{R}^3) \cap W^{1,1}(\mathbb{R}^3)$, vanishing at infinity with bounded derivatives, the interaction force $f_\beta^N : \mathbb{R}^3 \to \mathbb{R}^3$ is given by*

$$f_\beta^N(q) = N^{4\beta} l(N^\beta q)$$

with $0 < \beta \leq \frac{1}{7}$.

Analogously the total force of the system is given by $F : \mathbb{R}^{6N} \to \mathbb{R}^{3N}$, where the force exhibited on a single coordinate j is given by

$$(F(X))_j := \sum_{i \neq j} \frac{1}{N} f_\beta^N(q_i - q_j).$$

By introducing this N-particle force we can characterize the Newtonian flow by Definition 1.1. As the vector field is Lipschitz for fixed N we have global existence and uniqueness of solutions and hence a N-particle flow.

2.1 Statement of the results

Theorem 2.2. *Assume that the parameter β, describing the scaling of the interaction, satisfies $\beta \leq \frac{1}{7}$. Let $T > 0$ be such that a solution $k \in \mathcal{C}([0,T], \mathcal{H}_{2r}^{2m-1})$ of the Vlasov-Dirac-Benney equation (1.6) exists. Moreover, let $(\Phi_{t,s}^\infty)_{t,s\in\mathbb{R}}$ be the related lifted effective flow defined in Definition 1.4 as well as $(\Psi_{t,s}^N)_{t,s\in\mathbb{R}}$ the N-particle flow defined in Definition 1.1. If $0 < \alpha < \beta \leq \frac{1}{7}$, Then for any $\gamma > 0$ there exists a $C_\gamma > 0$ such that for any $0 < \alpha < \beta$ and any $N \in \mathbb{N}$ with $N \geq N_0(T, \beta, k_0)$ it holds that*

$$\mathbb{P}\big(X \in \mathbb{R}^{6N} : \sup_{0\leq s\leq T} |\Psi_{s,0}^N(X) - \Phi_{s,0}^\infty(X)|_\infty > N^{-\alpha}\big) \leq C_\gamma N^{-\gamma}. \tag{2.1}$$

It is intuitively clear that the coarse grained effective description gets more appropriate as the number of particles increases and becomes exact in the limit $N \to \infty$. This Theorem implies Propagation of Chaos and thus convergence of the marginals of the N-particle density towards products of solutions of the mean-field equation.

2.1.1 Notation and preliminary studies

The solution of the Vlasov-Dirac-Benney equation $k_t^N : \mathbb{R}^6 \to \mathbb{R}_0^+$ can be read as a one-particle probability density. The probabilities and expectation values in this and the following section are meant with respect to the product measure given at a certain time. For any random variable $R : \mathbb{R}^{6N} \to \mathbb{R}$ and any element B of the Borell σ-algebra we have

$$\mathbb{P}_t(R \in B) = \int_{R^{-1}(B)} \prod_{j=1}^{N} k_t^N(x_j)dX \quad \text{and} \quad \mathbb{E}_t(R) = \int_{\mathbb{R}^{6N}} R(X) \prod_{j=1}^{N} k_t^N(x_j)dX.$$

Since the measure is invariant under $\Phi_{t,s}^N$ it follows that

$$\mathbb{E}_s(R \circ \Phi_{t,s}^N) = \int_{\mathbb{R}^{6N}} R(\Phi_{t,s}^N(X)) \prod_{j=1}^{N} k_s^N(x_j)dX = \int_{\mathbb{R}^{6N}} R(X) \prod_{j=1}^{N} k_s^N(\varphi_{s,t}^N(x_j))dX$$

and since $k_s^N(\varphi_{s,t}^N(x_j)) = k_t^N(x_j)$ we get $\mathbb{E}_s(R \circ \Phi_{t,s}^N) = \mathbb{E}_t(R)$. During the proof it is helpful to deviate into cases and therefore we will restrict ourselves to certain configurations. For this purpose we introduce the restricted expectation value.

Definition 2.3. *Let X be a random variable and $A \subset \mathbb{R}^{6N}$ a set, then the restricted expectation value is given by*

$$\mathbb{E}(X|A) := \mathbb{E}(X^A) \qquad \textit{with} \qquad X^A(\omega) = \begin{cases} X(\omega) & \omega \in A \\ 0 & \omega \notin A, \end{cases}$$

$$\mathbb{E}(X) = \sum_{\omega \in \mathbb{R}^{6N}} X(\omega)\mathbb{P}(\omega) \qquad \textit{and} \qquad \mathbb{E}(X|A) = \sum_{\omega \in A} X(\omega)\mathbb{P}(\omega).$$

Now we introduce a suitable notation of distance on $\mathbb{R}^6$, which enables us to prove that for finite time $\Psi_{t,0}^N$ and $\Phi_{t,0}^N$ will typically be close with respect to that notation of this distance. Since we are dealing with probabilistic initial conditions we introduce a stochastic process J_t which is such that a small expectation value of J_t implies that $\Psi_{t,0}^N(X)$ an $\Phi_{t,0}^N(X)$ are close as described in the previous chapter. In view of Theorem 2.2, our aim is to show that $\mathbb{P}_0(\sup_{0 \le s \le t} |\Psi_{s,0}^N - \Phi_{s,0}^N|_\infty > N^{-\gamma})$ tends to zero faster than any inverse power of N. This will be implemented modifying the stochastic process J_t (1.4) in order to separate the error terms coming from the law of large numbers from other sources of errors. This can be done by defining J_t in the following way:

Definition 2.4. *Let $\Phi_{s,0}^N(X)$ be the mean-field flow defined in Definition 1.4 and $\Psi_{s,0}^N(X)$ the microscopic flow defined in Definition 1.1. We denote the projection* *onto the spatial or respectively the momentum coordinates by* $\Phi^{N,1}_{s,0}(X)$ *and* $\Phi^{N,2}_{s,0}(X)$. *Let for* $T > 0$ *and without loose of generality* $N > 1$ *the auxiliary process be defined as follows*

$$J^N_t(X) := \min\left\{1, \sup_{0\le s\le t}\left\{\sigma_{N,t}N^{\alpha}\left(\sqrt{\ln(N)}\left|\Psi^{1,N}_{t,0}(X) - \Phi^{1,N}_{t,0}(X)\right|_\infty + \left|\Psi^{2,N}_{t,0}(X) - \Phi^{2,N}_{t,0}(X)\right|_\infty + N^{5\beta-1}\right)\right\}\right\}$$

for $0 \le t \le T$ *with scaling factor* $\sigma_{N,s} = e^{\lambda\sqrt{\ln(N)}(T-s)}$. *Here* $|\cdot|_\infty$ *denotes the supremum norm on* $\mathbb{R}^{6N}$.

The metric $|\Psi^N_{t,0}(X) - \Phi^N_{t,0}(X)|_\infty$ allows better stability estimates as it is much stronger than usual weak distances between probability measures.

The spatial and momentum coordinates are weighed differently to take advantage of the system's second-order nature when comparing microscopic trajectories to the mean-field equation's characteristic curves. The growth of the spatial distance is trivially bounded by the difference of the respective momenta due to Newton law. The idea behind this weighted norm is to be a little more 'strict' on deviations in space and to obtain better control on fluctuations of the force. Moreover the scaling factor $e^{\lambda\sqrt{\ln(N)}(T-s)}$ optimizes the rate of convergence as compensates the time dependent natural fluctuations.

We now want to estimate the time derivative of $E(J^N_t)$. As already described in Chapter 1.5, whenever the random variable J_t has the value 1, it has reached its maximum so the inequality $\partial^+_t\mathbb{E}(J^N_t) \le C(\mathbb{E}(J^N_t) + o_N(1)))$ is trivial. The configurations where J_t is maximal, that is $|\Psi^N_{t,0}(X)-\Phi^N_{t,0}(X)|_\infty \ge N^{-\alpha}$ are irrelevant for finding an upper bound of $\partial^+_t\mathbb{E}(J_t)$. The set of such configurations will be called $\mathcal{A}_t$. We will show that the expectations value $\partial^+_t\mathbb{E}(J^N_t)$ restricted on the set $\mathcal{A}_t$ is less or equal 0.

We only have to consider the cases where J^N_t is smaller than 1 since $\partial^+_t J^N_t = 0$ for $J^N_t = 1$, but then we have the boundary condition by definition of the random variable, i.e $|\Psi^{2,N}_{t,0}(X) - \Phi^{2,N}_{t,0}(X)|_\infty < N^{-\alpha}$. Due to the pre-factor $\sqrt{\ln(N)}$ in Definition 2.4, the particular anisotropic scaling of our metric will allow us to 'trade' part of this divergence for a tighter control on spatial fluctuations. Note that $e^{\sqrt{\ln(N)}}$ grows slower than N^{ε} for any $\varepsilon > 0$. This will suffice to establish the desired convergence ([32] implemented the same idea).

By the definition of J^N_t we get the boundary condition for free and the construction of J^N_t motivates the following lemma.

Lemma 2.5. *For* $t > 0$ *there exists a constant* $C_\gamma < \infty$, *under the assumptions of Theorem 2.2, such that*

$$\mathbb{E}_0(J^N_t) \le C_\gamma N^{-\gamma}.$$

for $\gamma > 0$.

Theorem 2.2 follows directly from Lemma 2.5, since the following probability can be estimated according to the description above.

$$\mathbb{P}_0(\sup_{0\le s\le t}|\Psi^N_{s,0}(X) - \Phi^N_{s,0}(X)|) \ge N^{-\alpha}) = \mathbb{P}_0(J^N_t = 1) \le \mathbb{E}_0(J^N_t).$$

The proof of Lemma 2.5 is based on a Gronwall argument and therefore we will give an upper bound on $\partial_t^+ \mathbb{E}_0(J_t)$ by introducing a suitable partition of the phase space $\mathbb{R}^{6N}$. A first observation is that the growth of $\partial_t^+ \mathbb{E}_0(J_t)$ stems from the fluctuation in the force, which itself can be estimated by

$$\begin{aligned}\left|F(\Psi^N_{t,0}(X)) - \bar{F}(\Phi^N_{t,0}(X))\right|_\infty \leq& \left|F(\Psi^N_{t,0}(X)) - F(\Phi^N_{t,0}(X))\right|_\infty \\ &+ \left|F(\Phi^N_{t,0}(X)) - \bar{F}(\Phi^N_{t,0}(X))\right|_\infty .\end{aligned}$$

To control theses two addends we will introduce unlikely sets. The sets of configurations X for which the second term $|F(\Phi^N_{t,0}(X)) - \bar{F}(\Phi^N_{t,0}(X))|_\infty$ is large will be denoted by $\mathcal{B}_t$. Large means in our case larger than $N^{5\beta-1}\ln(N)$. Since any difference in the force is directly translatable into a growth in the difference $|\Psi^N_{t,0}(X) - \Phi^N_{t,0}(X)|_\infty$ which is multiplied by $N^{-\alpha}$ in the definition of J^N_t. We will see, that the probability to be in $\mathcal{B}_t$ is indeed small. If F was globally Lipschitz continuous, the first term $|F(\Psi^N_{t,0}(X)) - F(\Phi^N_{t,0}(X))|_\infty$ would directly translate in the difference $|\Psi^N_{t,0}(X) - \Phi^N_{t,0}(X)|_\infty$ as $|F(\Psi^N_{t,0}(X)) - F(\Phi^N_{t,0}(X))|_\infty \leq L_{Lip}|\Psi^N_{t,0}(X) - \Phi^N_{t,0}(X)|_\infty$ and the result would be proven. The forces we consider are singular and so unfortunately there is no global Lipschitz constant. Although there are configurations for which the force becomes singular in the limit $N \to \infty$, for example when all particles have the same position, these configurations are not very likely. To control the first addend and to implement this argument we will introduce a function g in Definition 2.7 which controls the difference $|f^N(x) - f^N(x+\delta)|_\infty$ for a $2N^{-\alpha} > \delta \in \mathbb{R}^3$. This will be proven in Lemma 2.8 observing that we only need to take into account fluctuations smaller than $N^{-\alpha}$ by Definition of J^N_t. In a further step we will control $G = \frac{1}{N}\sum_{j=1}^N g(q_j$
large, denoted by C_t, is very unlikely. For the configurations which are left we use the fact that the force term is short range and the associated scaling behavior is in our favour. Consequently we will get a good estimate on $|F(\Psi^N_{t,0}(X)) - F(\Phi^N_{t,0}(X))|_\infty$.

Controlling the growth of the force

In the following section we overcome the problem that forces f^N_β become singular in the limit $N \to \infty$ and hence do not satisfy a uniform Lipschitz bound. The function $l : \mathbb{R}^3 \to \mathbb{R}$ occurring in the definition of f_N was defined such that $D_1 l, D_2 l$ and $D_3 l$ are defined everywhere and are bounded functions. Its easy to check that $l^N(q)$ satisfy a Lipschitz condition by a mean value argument.

Lemma 2.6. *Let $l \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) : ||D^\alpha l||_\infty \leq C_\alpha$ be a smooth vanishing at infinity function with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Then there is a $L > 0$ such that*

$$|l(a) - l(b)|_\infty \leq L|a-b|_\infty$$

Proof. Since $D_i l$ is bounded for $i \in \{1,2,3\}$ there is a $S_1 = \sup(||D_1 f(x)|| : X \in \mathbb{R}^3), S_2 = \sup(||D_2 l(x)|| : X \in \mathbb{R}^3)$ and $S_3 = \sup(||D_1 l(x)|| : X \in \mathbb{R}^3)$. For $a, b \in \mathbb{R}^3$ we can estimate the difference $l(a) - l(b)$ by the triangle inequality

$$\begin{aligned}|l(a) - l(b)| \leq& |l(a_1,a_2,a_3) - l(a_1,a_2,b_3)| + |l(a_1,a_2,b_3) - l(a_1,b_2,b_3)| \\ &+ |l(a_1,b_2,b_3) - l(b_1,b_2,b_3)| .\end{aligned} \tag{2.2}$$

Since the partial derivatives exist everywhere in $\mathbb{R}^3$, we can use the one-dimensional mean Value Theorem to show that there is some ξ such that:

$$\frac{l(a_1,a_2,a_3)-l(a_1,b_2,a_3)}{a_2-b_2} = D_2 l(a_1,\xi,a_3).$$

By the definition of S_2, it follows that $|l(a_1,a_2,a_3)-l(a_1,b_2,a_3)| \leq S_2|a_2-b_2|$. And similarly for the other addends of estimate (2.2). Additionally using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|l(a)-l(b)| &\leq S_1|a_1-b_1|+S_2|a_2-b_2|+S_3|a_3-b_3| \\ &\leq \sqrt{S_1^2+S_2^2+S_3^2}\cdot\left((a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2\right) \\ &= \sqrt{S_1^2+S_2^2+S_3^2}\cdot||a-b||\end{aligned}$$

So l is indeed Lipschitz continuous with $L=\sqrt{S_1^2+S_2^2+S_3^2}$. □

Next we define a function g_N^β, which provides a bound for fluctuations of f_N^β.

Definition 2.7. *Let $g:\mathbb{R}^3\to\mathbb{R}^3$ with*

$$g_N^\beta(q) := L\cdot N^{5\beta}\mathbb{1}_{\{supp_l\}}(N^\beta q).$$

and the total fluctuation G be defined by $(G(X))_j := \sum_{j=1}^N \frac{1}{N} g_N^\beta(q_i-q_j)$. Furthermore $\bar{G}_t$ is given by $(\bar{G}_t(X))_j := \bar{g}_t(q_j)$ with $\bar{g}_t(q) = g_N^\beta * \tilde{k}_t^N(q)$.

To show that the difference $|f_N^\beta(x)-f_N^\beta(x+\delta)|_\infty$ can be controlled by g_N^β we prove the following lemma.

Lemma 2.8. *For any $\delta\in\mathbb{R}^3$ it follows that*

$$|f_N^\beta(x)-f_N^\beta(x+\delta)|_\infty \leq g_N^\beta(q)|\delta|_\infty.$$

Proof. We recall that f_N^β was defined by

$$f_N^\beta(q) = N^{4\beta} l(N^\beta q)$$

and that l is Lipschitz continuous by Lemma 2.6. Hence we get

$$|f_N^\beta(x)-f_N^\beta(x+\delta)|_\infty \leq LN^{4\beta}|N^\beta\delta|_\infty \leq LN^{5\beta}|\delta|_\infty = g_N^\beta(q)|\delta|_\infty.$$

□

Notice that in the case where we use G to control the fluctuation we know by the construction of J_t^N that $|\Psi-\Phi| < N^{-\alpha}$. Furthermore the following observations of the force f_N^β and the fluctuation g_N^β turn out to be very helpful in the sequel. One crucial consequence of the bounded density is that the mean-field force remains bounded, as well.

Lemma 2.9. *Let $g^N(x)$ be defined in Definition 2.7 and $\tilde{k} \in W^{2,1}(\mathbb{R}^3) \cap W^{2,\infty}(\mathbb{R}^3)$. Then there exists a constant $C > 0$ independent of N such that*

$$\|f_N^\beta * \tilde{k}\|_\infty \leq C\|\nabla\tilde{k}\|_\infty \tag{2.3}$$

and

$$\|g_N^\beta * \tilde{k}\|_\infty \leq C\|\Delta\tilde{k}\|_\infty. \tag{2.4}$$

Proof. The function $\phi : \mathbb{R}^3 \to \mathbb{R}$ is rotationally symmetric. It holds that $\phi(x) = h(||x||)$ for some measurable functions $h : \mathbb{R}_0^+ \to \mathbb{R}$ and hence $\int_{K_{a,b}} \phi(x)dx = \omega_n \int_a^b h(r)r^2 dr$ for $K_{a,b} := \{X \in \mathbb{R}^3 : a < ||x|| < b\}$. For $\phi \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ the theorem on coordinate transformation provides

$$\|\phi^N\|_1 = \int_{\mathbb{R}^3} |N^{3\beta}\phi(N^\beta x)dx| \leq C \int_{\mathbb{R}^3} |\phi(y)|dy \leq C$$

and for $f_N^\beta \in C^\infty(\mathbb{R}^3)$ Leibniz integral rule implies

$$\|f_N^\beta * \tilde{k}\|_\infty = \|\nabla\phi^N * \tilde{k}\|_\infty = \|\phi^N * \nabla\tilde{k}\|_\infty \leq C\|\phi^N\|_1\|\nabla\tilde{k}\|_\infty \leq C\|\nabla\tilde{k}\|_\infty.$$

Analogously we can estimate

$$\|g_N^\beta * \tilde{k}\|_\infty = \|\Delta\phi^N * \tilde{k}\|_\infty = \|\phi^N * \Delta\tilde{k}\|_\infty \leq C\|\phi^N\|_1\|\Delta\tilde{k}\|_\infty \leq C\|\Delta\tilde{k}\|_\infty.$$

By applying the Vlasov property $k_t(q,p) = k_0(\bar{X}(q,p))$ and the assumptions on k according to the solution theory we can see that $\|f^N * \tilde{k}\|_\infty$ and $\|g^N * \tilde{k}\|_\infty$ are bounded by a constant not depending on N.

□

2.1.2 The evolution of $E(J_t^N)$

Since $\frac{\mathrm{d}}{\mathrm{d}t}J_t^N(X) \leq 0$ if $\sup_{0\leq s\leq t} |\Psi_{s,0}^N(X) - \Phi_{s,0}^N(X)|_\infty \geq N^{-\alpha}$ we only have to consider situations in which mean-field trajectories and microscopic trajectories are close. In order to control the evolution of $E(J_t^N)$ we will partition the phase space as described in Section 1.5.

Definition 2.10. *Let for any $t \in \mathbb{R}$ the sets $\mathcal{A}_t, \mathcal{B}_t, \mathcal{C}_t$ be given by*

$$\begin{aligned}
X \in \mathcal{A}_t &\Leftrightarrow |J_t^N| = 1 \\
X \in \mathcal{B}_t &\Leftrightarrow |F(\Phi_{t,0}^N(X)) - \bar{F}(\Phi_{t,0}^N(X))|_\infty > N^{-1+5\beta}\ln(N) \\
X \in \mathcal{C}_t &\Leftrightarrow |G(\Phi_{t,0}^N(X)) - \bar{G}(\Phi_{t,0}^N(X))|_\infty > N^{-1+7\beta}\ln(N).
\end{aligned}$$

For estimating the probability of configurations $X \in \mathcal{B}_t$ and $X \in \mathcal{C}_t$ we will use a law of large numbers argument. It will turn out, that these configurations are very unlikely.

Law of large numbers

The underlying proof technique is designed for stochastic initial conditions, thus allowing for law of large number estimates that turn out to be very powerful. Note that the particles evolving with the mean-field flow remain statistically independent at all times. We use the following Lemma to provide the probability bounds of random variables.

Lemma 2.11. *Let $Z_1, \cdots, Z_N$ be i.i.d. random variables with $\mathbb{E}[Z_i] = 0$, $\mathbb{E}[Z_j^2] \leq r(N)$ and $|Z_i| \leq C\sqrt{Nr(N)}$. Then for any $\gamma > 0$, the sample mean $\bar{Z} = \frac{1}{N}\sum_{i=1}^N Z_i$ satisfies*

$$\mathbb{P}\left(|\bar{Z}| \geq \frac{C_\gamma \sqrt{r(N)}\ln(N)}{\sqrt{N}}\right) \leq N^{-\gamma},$$

where C_γ depends only on C and γ.

The proof can be seen in [21, Lemma 1]. It is a direct result of Taylor's expansion and Markov's inequality. Furthermore it is a direct consequence of the following Lemma.

Lemma 2.12. *For $N \in \mathbb{N}$ let $Z_1, \ldots, Z_N$ be independent and identically distributed random variables on $\mathbb{R}^3$ with $||Z_j||_\infty \leq C$, $\mathbb{E}(Z_j) = 0$, $\mathbb{E}(Z_j^2) \leq \frac{C}{N}$ for all $i \in \{1, \ldots, N\}$, then the finite sum of the random variables $S_N := \sum_{i=1}^N Z_i$ full fills*

$$\mathbb{E}(e^{|S_N|}) \leq C.$$

Proof. From the Taylor series expansion we have $e^x = 1 + x + x^2 e^\mu$ with $|\mu| < |x|$. As the expectation value is linear and using the properties of the random variable we get

$$\mathbb{E}(e^Z) = 1 + \mathbb{E}(Z) + \mathbb{E}(Z^2 \cdot e^\mu) \leq 1 + 0 + \mathbb{E}(Z^2) \cdot C \leq 1 + \frac{C}{N}.$$

For the (positive or negative) sum of the independent and identically distributed random variables a similar inequality follows

$$\mathbb{E}(e^{\pm S_n}) = \mathbb{E}(e^{\pm \sum_{j=1}^N Z_j}) \overset{\text{iid}}{=} (\mathbb{E}(e^{\pm Z_j}))^N \leq (1 \pm \frac{C}{N})^N \overset{N \to \infty}{\longrightarrow} e^{\pm C}.$$

So in total we get $\mathbb{E}(e^{|S_n|}) \leq C$ as $e^{|x|} \leq e^x + e^{-x}$ for all $X \in \mathbb{R}$. □

By Markov inequality we get for $\mu > 0$

$$\mathbb{P}(e^{|S_N|} \geq \mu) \leq \frac{C}{\mu} \Rightarrow \mathbb{P}(|S_N| \geq \ln(\mu)) \leq \frac{C}{\ln(\mu)}$$

and Lemma 2.11 is a direct consequence.
Now we will estimate the probability of the unlikely sets defined in Definition 2.10. Therefore we recall the notation

$$(\overline{F}^N(\overline{X}_t))_i := \int_{\mathbb{R}^3} f_\beta^N(\overline{q}_i^t - q) k_t^N(q) d^3q \quad \text{and} \quad (\overline{G}^N(\overline{X}_t))_i := \int_{\mathbb{R}^3} g_N^\beta(\overline{q}_i^t - q) k_t^N(q) d^3q \tag{2.5}$$

and introduce the underlying version of the Law of Large Numbers for the thesis.

Lemma 2.13. *At any fixed time $t \in [0,T]$, suppose that $\overline{X}_t$ satisfies the mean-field dynamics, introduced in Definition 1.4, then F^N and $\overline{F}^N$, defined in Definition 2.1 and (2.5) and respectively G^N and $\overline{G}^N$, introduced in Definition 2.8, fulfill the following statement. For any $\gamma > 0$ and $0 \le \beta \le \frac{1}{7}$, there is a constant $C_\gamma > 0$ depending only on γ, T and k_0 such that*

$$\mathbb{P}\left(\left\|F^N(\overline{X}_t) - \overline{F}^N(\overline{X}_t)\right\|_\infty \ge C_\gamma N^{5\beta-1}\ln(N)\right) \le N^{-\gamma}, \tag{2.6}$$

and

$$\mathbb{P}\left(\left\|G^N(\overline{X}_t) - \overline{G}^N(\overline{X}_t)\right\|_\infty \ge C_\gamma N^{7\beta-1}\ln(N)\right) \le N^{-\gamma}. \tag{2.7}$$

Proof. We prove Lemma 2.13 by applying Lemma 2.11 and the following generalized version of Young's inequality for convolutions for $p, r \in L^1$.

$$\|p * r\|_\infty \le \|p\|_1 \|r\|_\infty$$

Since $\|\tilde{k}^N\|_1 = 1$ and $\|\tilde{k}^N\|_\infty$ are bounded, it holds due to Lemma 2.9 that

$$\|\tilde{k}_t^N(q_1) * f_N^\beta\|_\infty \le \|\nabla \tilde{k}_t^N(q_1)\|_1 \le C\|\nabla \tilde{k}\|_1 \tag{2.8}$$

$$\|\tilde{k}_t^N(q_1) * g_N^\beta\|_\infty \le C\|\Delta \tilde{k}_t^N(q_1)\|_1. \tag{2.9}$$

Hence we get

$$\left\|f_N^\beta \ * \tilde{k}_t^N(q_1)\right\|_\infty \le C \qquad \text{and} \qquad \left\|g_N^\beta \ * \tilde{k}_t^N(q_1)\right\|_\infty \le C.$$

Using inequality (2.8) and (2.9) we get $\left|f_N^\beta(q_1 - q_j) - f_N^\beta * \tilde{k}_t^N(q_1)\right| \le C$. Analogously in the case of the fluctuation we get $\left|g_N^\beta(q_1 - q_j) - g_N^\beta * \tilde{k}_t^N(q_1)\right| \le C$. The expectation values of $\left|f_\beta^N(q_1 - q_j)\right|^2$ and $\left|g_N^\beta(q_1 - q_j)\right|^2$ can be estimated using the theorem on coordinate transformation by

$$\begin{aligned}\int \tilde{k}_t^N(q_j)\left|f_\beta^N(q_1 - q_j)\right|^2 d^3q_j &\le C\int \left(N^{4\beta} l(N^\beta q)\right)^2 d^3q \\ &\le CN^{8\beta}N^{-3\beta} \le CN^{5\beta}\end{aligned}$$

and analogously

$$\begin{aligned}\int \tilde{k}_t^N(q_j)\left|g_N^\beta(q_1 - q_j)\right|^2 d^3q_j &\le C\int \left(CN^{5\beta}\mathbb{1}_{\{\mathrm{supp}_l\}}(N^\beta q)\right)^2 d^3q \\ &\le CN^{10\beta-3\beta} \le CN^{7\beta}.\end{aligned}$$

Due to the exchangeability of particles, we can estimate

$$(F^N(\overline{X}_t))_1 - (\overline{F}^N(\overline{X}_t))_1 = \frac{1}{N}\sum_{j=2}^N f_\beta^N(\overline{q}_1^t - \overline{q}_j^t) - \int_{\mathbb{R}^3} f_\beta^N(\overline{q}_1^t - q)\tilde{k}_t^N(q)d^3q = \frac{1}{N}\sum_{j=2}^N Z_j,$$

for the random variable

$$Z_j := f_\beta^N(\overline{q}_1^t - \overline{q}_j^t) - \int_{\mathbb{R}^3} f_\beta^N(\overline{q}_1^t - q)\tilde{k}_t^N(q)d^3q.$$

Since $\overline{q}_1^t$ and $\overline{q}_j^t$ are independent if $j \neq 1$ and $f_\beta^N(0) = 0$, let us consider $\overline{q}_1^t$ as given and denote $\mathbb{E}'[\cdot] = \mathbb{E}[\cdot|\overline{q}_1^t]$. The condition $\mathbb{E}'[Z_j] = 0$ of Lemma 2.11 holds since

$$\mathbb{E}'\left[f_\beta^N(\overline{q}_1^t - \overline{q}_j^t)\right] = \int_{\mathbb{R}^6} f_\beta^N(\overline{q}_1^t - q)k_t^N(q,p)d^3qd^3p = \int_{\mathbb{R}^3} f_\beta^N(\overline{q}_1^t - q)\tilde{k}_t^N(q)d^3q.$$

Additionally we need a bound for the variance

$$\mathbb{E}'\left[|Z_j|^2\right] = \mathbb{E}'\left[\left|f_\beta^N(\overline{q}_1^t - \overline{q}_j^t) - \int_{\mathbb{R}^3} f_\beta^N(\overline{q}_1^t - q)\tilde{k}_t^N(q)d^3q\right|^2\right].$$

We know by Lemma 2.9 that

$$\int_{\mathbb{R}^3} f_\beta^N(\overline{q}_1^t - q)\tilde{k}_t^N(q)dq \leq C(\|\tilde{k}^N\|_1 + \|\tilde{k}^N\|_\infty),$$

which suffices to estimate expectation value

$$\mathbb{E}'\left[f_\beta^N(\overline{q}_1^t - \overline{q}_j^t)\right] = \int_{\mathbb{R}^3} f_\beta^N(\overline{q}_1^t - q)\tilde{k}_t^N(q)dq \leq C(\|\tilde{k}^N\|_1 + \|\tilde{k}^N\|_\infty)N^\beta \leq C$$

and the variance

$$\mathbb{E}'\left[f_\beta^N(\overline{q}_1^t - \overline{q}_j^t)^2\right] = \int_{\mathbb{R}^3} f_\beta^N(\overline{q}_1^t - q)^2\tilde{k}_t^N(q)dq \leq \|\tilde{k}^N\|_\infty\|\tilde{k}^N\|_2^2 N^{5\beta} \leq CN^{5\beta}.$$

Hence we get

$$\mathbb{E}'\left[|Z_j|^2\right] \leq CN^{5\beta}.$$

For $r(N) = CN^{5\beta}$ it follows that $|Z_j| \leq C\sqrt{Nr(N)}$. Using Lemma 2.11, we have the probability bound

$$\mathbb{P}\left(\left|(F^N(\overline{X}_t))_1 - (\overline{F}^N(\overline{X}_t))_1\right| \geq CN^{5\beta-1}\ln(N)\right) \leq N^{-\gamma}.$$

Similarly, the same bound also holds for all other indexes $i = 2, \dots, N$, which leads to

$$\mathbb{P}\left(\left\|F^N(\overline{X}_t) - \overline{F}^N(\overline{X}_t)\right\|_\infty \geq CN^{5\beta-1}\ln(N)\right) \leq N^{1-\gamma}. \tag{2.10}$$

Let C_γ be the constant C in (2.10) which is only depending on γ, T and k_0, then we conclude the proof of bound (2.6). To prove the second bound (2.7), we follow the same procedure as above

$$(G^N(\overline{X}_t))_1 - (\overline{G}^N(\overline{X}_t))_1 = \frac{1}{N}\sum_{j=2}^N g_N^\beta(\overline{q}_1^t - \overline{q}_j^t) - \int_{\mathbb{R}^3} g_N^\beta(\overline{q}_1^t - q)\tilde{k}_t^N(q)d^3q = \frac{1}{N}\sum_{j=2}^N Z_j,$$

with the random variable

$$Z_j = g_N^\beta(\overline{q}_1^t - \overline{q}_j^t) - \int_{\mathbb{R}^3} g_N^\beta(\overline{q}_1^t - q)\tilde{k}_t^N(q)d^3q.$$

It holds that $\mathbb{E}'[Z_j] = 0$. By using the definition of g_N^β and the fact that the integration by substitution yields the rescaling factor $N^{-3\beta}$ in dimension $d = 3$, the expectation value and variance is bounded by

$$\mathbb{E}'\left[g_N^\beta(\overline{q}_1^t - \overline{q}_j^t)\right] = \int_{\mathbb{R}^3} g_N^\beta(\overline{q}_1^t - q)\tilde{k}_t^N(q)d^3q \le N^{2\beta}C(\|\tilde{k}^N\|_1 + \|\tilde{k}^N\|_\infty) \le C,$$
$$\mathbb{E}'\left[g_N^\beta(\overline{q}_1^t - \overline{q}_j^t)^2\right] = \int_{\mathbb{R}^3} g_N^\beta(\overline{q}_1^t - q)^2\tilde{k}_t^N(q)d^3q \le CN^{7\beta}(\|\tilde{k}^N\|_1 + \|\tilde{k}^N\|_\infty) \le CN^{7\beta}.$$

Hence we get

$$\mathbb{E}'\left[|Z_j|^2\right] \le CN^{7\beta}.$$

For $r(N) = CN^{7\beta}$ it follows that $|Z_j| \le C\sqrt{Nr(N)}$. With Lemma 2.11 we can derive the probability bound

$$\mathbb{P}\left(\left|(G^N(\overline{q}_t))_1 - (\overline{G}^N(\overline{q}_t))_1\right| \ge CN^{7\beta-1}\ln(N)\right) \le N^{-\gamma},$$

which leads to

$$\mathbb{P}\left(\left\|G^N(\overline{X}_t) - \overline{G}^N(\overline{X}_t)\right\|_\infty \ge CN^{7\beta-1}\ln(N)\right) \le N^{1-\gamma}. \tag{2.11}$$

Thus, inequality (2.7) follows from inequality (2.11). □

So far we could show, that the probability of X being in one of the unlikely set defined in Definition 2.10 decreases faster than any negative power of N. For any time $0 < t < T$, initial conditions in $(B_t \cap C_t)^c$ are typical with respect to the product measure $K_0 := \otimes^N k_0$ on $\mathbb{R}^{6N}$.

Controlling the Expectation value of J_t^N

We are left to estimate the expectation $E_0(J_t^N)$ and remember that it was split into

$$E_0(J_t^N) = \mathbf{E}_0(J_t^N \mid \mathcal{A}_t) + \mathbf{E}_0\left(J_t^N \mid \mathcal{A}_t^c \setminus (\mathcal{B}_t^c \cap C_t^c)\right) + \mathbf{E}_0\left(J_t^N \mid (\mathcal{A}_t \cup \mathcal{B}_t \cup \mathcal{C}_t)\right)^c.$$

As we already know that, on the set $\mathcal{A}_t$ the process $J_t^N(X)$ is already maximal and we have $\frac{\mathrm{d}}{\mathrm{d}t}J_t^N(X) = 0$ and thus also $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_t(J_t^N \mid \mathcal{A}_t) = 0$. To estimate the remaining terms we remember that for $X \in \mathcal{A}_t^c$ the probability for $X \in \mathcal{B}_t \cap \mathcal{C}_t$ decreases faster than any power of N. Further more right derivative of J_t^N with respect to t is given by

$$\begin{aligned}
&\partial_t^+ J_t^{N,\lambda}(X) \\
&\le \max\Big\{0, \frac{d}{dt}\Big(\sigma_{N,t}\Big(N^\alpha\sqrt{\ln(N)}\left|\Psi_{t,0}^{1,N}(X) - \Phi_{t,0}^{1,N}(X)\right|_\infty \\
&\qquad + N^\alpha\left|\Psi_{t,0}^{2,N}(X) - \Phi_{t,0}^{2,N}(X)\right|_\infty + N^{5\beta+\alpha-1}\Big)\Big)\Big\} \\
&\le \max\Big\{0, -\lambda\sqrt{\ln(N)}e^{\lambda\sqrt{\ln(N)}(T-t)}\Big(N^\alpha\sqrt{\ln(N)}|\,\Psi_{t,0}^{1,N}(X) - \Phi_{t,0}^{1,N}(X)|_\infty \\
&\qquad + |\Psi_{t,0}^{2,N}(X) - \Phi_{t,0}^{2,N}(X)|_\infty + N^{5\beta+\alpha-1}\Big)
\end{aligned}$$

$$+ e^{\lambda\sqrt{\ln(N)}(T-t)} N^{\alpha} \partial_t \Big(\sqrt{\ln(N)} | \Psi^{1,N}_{t,0}(X) - \Phi^{1,N}_{t,0}(X)|_\infty \\ + |\Psi^{2,N}_{t,0}(X) - \Phi^{2,N}_{t,0}(X)|_\infty \Big) \Big\}$$

For the derivative of the position coordinate and for the momentum coordinate we further estimate

$$\begin{aligned} \partial_t |\Psi^{1,N}_{t,0}(X) - \Phi^{1,N}_{t,0}(X)|_\infty &\leq |\partial_t (\Psi^{1,N}_{t,0}(X) - \Phi^{1,N}_{t,0}(X))|_\infty \\ &\leq |\Psi^{2,N}_{t,0}(X) - \Phi^{2,N}_{t,0}(X)|_\infty \\ &\leq \sup_{0\leq s\leq t} |\Psi^{2,N}_{s,0}(X) - \Phi^{2,N}_{s,0}(X)|_\infty \\ \partial_t |\Psi^{2,N}_{t,0}(X) - \Phi^{2,N}_{t,0}(X)|_\infty &\leq |\partial_t (\Psi^{2,N}_{t,0}(X) - \Phi^{2,N}_{t,0}(X))|_\infty \\ &\leq |F^N(\Psi^1_{t,0}(X)) - \overline{F}_t(\Phi^1_{t,0}(X))|_\infty . \end{aligned}$$

Secondly the total force is bounded $|F(X)|_\infty \leq N^{4\beta}$ and the mean-field force $\bar{F}$ is of order one. Since $X \in \mathcal{A}^c_t$ we get $N^\beta |\Psi^{2,N}_{t,0}(X) - \Phi^{2,N}_{t,0}(X)| \leq 1$ and hence $\sup\{|\partial^+_t J^N_t(X)| : X \in \mathcal{A}^c_t\} \leq C e^{\lambda\sqrt{\ln(N)}T} N^{4\beta}$ for some $C > 0$. According to Lemma 2.11 the probability for $X \in \mathcal{B}_t \cap \mathcal{C}_t$ decreases faster than any power of N. Hence, we can find for any $\gamma > 0$ a constant C_γ, such that

$$\begin{aligned} \partial^+_t \mathbb{E}(J^N_t \mid \mathcal{A}^c_t \setminus (\mathcal{B}^c_t \cap \mathcal{C}^c_t)) &\leq \sup\{|\partial^+_t J^N_t(X)| : X \in \mathcal{A}^c_t\} \, \mathbb{P}\big[(\mathcal{A}_t \cup \mathcal{B}_t)\big] \\ &\leq e^{\lambda\sqrt{\ln(N)}T} C_\gamma N^{-\gamma}. \end{aligned}$$

It remains to control $E_0(J^N_t \mid (\mathcal{A}_t \cup \mathcal{B}_t \cup \mathcal{C}_t)^c)$ which is defined on the most likely initial conditions. The relevant term can be estimated by

$$\begin{aligned} |F^N(\Psi^1_{t,0}(X)) - \overline{F}_t(\Phi^1_{t,0}(X))|_\infty &\leq |F^N(\Psi^1_{t,0}(X)) - F^N(\Phi^{1,N}_{t,0}(X))|_\infty \\ &\quad + |F^N(\Phi^{1,N}_{t,0}(X)) - \overline{F}_t(\Phi^{1,N}_{t,0}(X))|_\infty . \end{aligned}$$

Since $X \notin \mathcal{B}_t$, it follows for the second addend

$$|F^N(\Phi^1_{t,0}(X)) - \overline{F}(\Phi^1_{t,0}(X))|_\infty < N^{5\beta-1} \ln(N).$$

For the first addend we use the triangle inequality to get for any $1 \leq i \leq N$

$$\begin{aligned} \Big| \big(F^N(\Psi^1_{t,0}(X)) - F^N(\Phi^1_{t,0}(X))\big)_i \Big|_\infty &\leq \Big| \frac{1}{N} \sum_{j=1}^N f^N_\beta(\Psi^1_i - \Psi^1_j) - f^N_\beta(\Phi^1_i - \Phi^1_j) \Big|_\infty \\ &\leq \frac{1}{N} \sum_{j=1}^N \big| f^N_\beta(\Psi^1_i - \Psi^1_j) - f^N_\beta(\Phi^1_i - \Phi^1_j) \big|_\infty \end{aligned}$$

and application of a version of mean value theorem stated in Lemma 2.8 leads to

$$\begin{aligned} \big| f^N_\beta(\Psi^1_i - \Psi^1_j) - f^N_\beta(\Phi^1_i - \Phi^1_j) \big|_\infty &\leq g^\beta_N(\Phi^1_i - \Phi^1_j) |(\Psi^1_i - \Psi^1_j) - (\Phi^1_i - \Phi^1_j)|_\infty \\ &\leq 2\, g^\beta_N(\Phi^1_i - \Phi^1_j) |\Psi^1_{t,0} - \Phi^1_{t,0}|_\infty . \end{aligned}$$

Since $X \in \mathcal{A}_t^c$ we have, by the construction of $J_t^N(X)$, in particular for N large enough

$$\sup_{0\le s\le t} |\Psi_{s,0}^{1,N}(X) - \Phi_{s,0}^{1,N}(X)|_\infty < N^{-\alpha}.$$

Additionally $X \notin \mathcal{C}_t$ from which we can conclude that

$$|G(\Phi_{t,0}^N(X)) - \bar{G}(\Phi_{t,0}^N(X))|_\infty \le CN^{7\beta-1}\ln(N)$$

and in particular

$$\frac{1}{N}\sum_{j=1}^N g_N^\beta(\Phi_i^1 - \Phi_j^1) = \left(G^N(\Phi_{t,0}(X)\right)_i \qquad \le \|g_N^\beta * \tilde{k}_t^N(q)\|_\infty + N^{7\beta-1}\ln(N)$$
$$\le CN^{7\beta-1}\ln(N).$$

For the derivative of the momentum we can conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_t^2(X) - \Phi_{t,0}^2(X)|_\infty \le CN^{7\beta-1}\ln(N)\left|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)\right|_\infty + N^{5\beta-1}\ln(N).$$

We observe that for $X \in (A_t \cup B_t \cup C_t)^c$ and $\beta \le \frac{1}{7}$

$$\begin{aligned}
&\partial_t^+\left(\sqrt{\ln(N)}|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)|_\infty + |\Psi_{t,0}^2(X) - \Phi_{t,0}^2(X)|_\infty\right)\Big|_{\mathcal{A}_t\cap\mathcal{B}_t\cap\mathcal{C}_t}\\
&\le \sqrt{\ln(N)}\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)|_\infty + \frac{\mathrm{d}}{\mathrm{d}t}|\Psi_{t,0}^2(X) - \Phi_{t,0}^2(X)|_\infty\\
&\le \sqrt{\ln(N)}|\Psi_{t,0}^2(X) - \Phi_{t,0}^2(X)|_\infty\\
&\quad + C\ln(N)\left(N^{7\beta-1}|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)|_\infty + N^{5\beta-1}\right)\\
&\le C\sqrt{\ln(N)}\left(\sqrt{\ln(N)}|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)|_\infty + |\Psi_{t,0}^2(X) - \Phi_{t,0}^2(X)|_\infty\right) + N^{5\beta-1}.
\end{aligned}$$

In total the right derivative of J_t^N is given by

$$\begin{aligned}
\partial_t^+ J_t^N(X) \le \max\Bigg\{0, \frac{d}{dt}\bigg(\sigma_{N,t}\bigg(N^\alpha\bigg(&\sqrt{\ln(N)}|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)|_\infty\\
&+|\Psi_{t,0}^2(X) - \Phi_{t,0}^2(X)|_\infty + N^{5\beta-1}\bigg)\bigg)\bigg)\Bigg\}
\end{aligned}$$

with

$$\begin{aligned}
&\frac{d}{dt}\left(\sigma_{N,t}\left(N^\alpha\left(\sqrt{\ln(N)}|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)|_\infty + |\Psi_{t,0}^2(X) - \Phi_{t,0}^2(X)|_\infty\right) + N^{5\beta-1}\right)\right)\\
&\le -\lambda\sqrt{\ln(N)}e^{\lambda\sqrt{\ln(N)}(T-t)}\Big(N^\alpha\Big(\sqrt{\ln(N)}|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)|_\infty\\
&\qquad\qquad + |\Psi_{t,0}^2(X) - \Phi_{t,0}^2(X)|_\infty\Big) + N^{5\beta-1}\Big)\\
&+ e^{\lambda\sqrt{\ln(N)}(T-t)}N^\alpha\Big(C\sqrt{\log N}\Big(\sqrt{\ln(N)}|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)|_\infty\\
&\qquad\qquad + |\Psi_{t,0}^2(X) - \Phi_{t,0}^2(X)|_\infty\Big) + N^{5\beta-1}\Big)\\
&= \sqrt{\ln(N)}N^\alpha e^{\lambda\sqrt{\ln(N)}(T-t)}\Big[(C-\lambda)\Big(\sqrt{\ln(N)}|\Psi_{t,0}^1(X) - \Phi_{t,0}^1(X)|_\infty
\end{aligned}$$

$$+ |\Psi^2_{t,0}(X) - \Phi^2_{t,0}(X)|_\infty + \Big((\ln(N))^{-\frac{1}{2}} - \lambda\Big)\ N^{5\beta-1}\Big)\Big].$$

By choosing $\lambda = C$ this derivative is negative and thus we get

$$\partial_t^+ \mathbb{E}(J_t^N \mid \mathcal{A}_t^c \cap \mathcal{B}_t^c \cap \mathcal{C}_t^c) = 0.$$

Finally we can conclude for the right derivative of the expectation value of the auxiliary process

$$\partial_t^+ \mathbb{E}(J_t^N) \le e^{\lambda\sqrt{\ln(N)}T} C_\gamma N^{-\gamma}.$$

And thus by the linearity of the expectation value the following bound holds

$$\mathbb{E}_0(J_t^N) - \mathbb{E}_0(J_0^N) = \mathbb{E}_0\big(J_t^N - J_0^N\big) \le T e^{\lambda\sqrt{\ln(N)}T} C_\gamma N^{-\gamma},$$

uniformly in $t \in [0, T]$. The initial states where chosen such that

$$\Big(\sqrt{\ln(N)}|\Psi^1_{0,0}(X) - \Phi^1_{0,0}(X)|_\infty + |\Psi^2_{0,0}(X) - \Phi^2_{0,0}(X)|_\infty\Big) = 0$$

and thus at time $t = 0$ the auxiliary process $J_0^N(X) \equiv e^{\lambda\sqrt{\ln(N)}T} N^{5\beta+\alpha-1}$ which for N sufficient large and $0 < \alpha < \beta \le \frac{1}{7}$ is bounded by

$$e^{\lambda\sqrt{\ln(N)}T} N^{5\beta+\alpha-1} \le e^{\lambda\sqrt{\ln(N)}T} N^{5\beta+\alpha-1} \le \frac{1}{2}$$

as $e^{\sqrt{\ln(N)}}$ grows slower than than N^ε for all $\varepsilon > 0$. We observe that the random variable $J_t^N - J_0^N$ is certainly non-negative and it follows that

$$\mathbb{P}_0\Big[J_T^N(X) - J_0^N(X) \ge \frac{1}{2}\Big] \le 2T e^{\lambda\sqrt{\ln(N)}T} C_\gamma N^{-\gamma}.$$

In the case $J_t^N - J_0^N < \frac{1}{2}$ we have $J_t^N(X) < 1$ and thus we can conclude

$$\begin{aligned}
&\mathbb{P}_0\Big[\sup_{0\le s\le T}\Big\{\Big(\sqrt{\ln(N)}|\Psi^1_{t,0}(X) - \Phi^1_{t,0}(X)|_\infty + |\Psi^2_{t,0}(X) - \Phi^2_{t,0}(X)|_\infty\Big)\Big\} \ge N^{-\alpha}\Big]\\
&\le \mathbb{P}_0\Big[J_T^N(X) - J_0^N(X) \ge \frac{1}{2}\Big]\\
&\le 2T e^{\lambda\sqrt{\ln(N)}T} C_\gamma N^{-\gamma} \le 2T C_\gamma N^{1-\gamma-5\beta-\alpha}.
\end{aligned}$$

We can find for any given $\tilde{\gamma} > 0$ by choosing $\gamma := \tilde{\gamma} + 1 - 5\beta - \alpha$ a $C_{\tilde{\gamma}}$ such that

$$\begin{aligned}
&\mathbb{P}_0\Big[\sup_{0\le s\le T}\Big\{\Big(\sqrt{\ln(N)}|\Psi^1_{t,0}(X) - \Phi^1_{t,0}(X)|_\infty + |\Psi^2_{t,0}(X) - \Phi^2_{t,0}(X)|_\infty\Big)\Big\} \ge N^{-\alpha}\Big]\\
&\le T C_{\tilde{\gamma}} N^{-\tilde{\gamma}}.
\end{aligned}$$

This proves Lemma 2.5.

We are left to show that the we fully approximate the non regularized system and therefore we define

$$\Delta_N(t) := \sup_{x\in\mathbb{R}^6}\sup_{0\le s\le T} |\varphi^1_{s,0}(X) - \varphi^\infty_{s,0}(X)| \le 2N^{-\alpha},$$

for $f^\infty = \delta_0(x)$ which will conclude the proof of Theorem 2.2.

2.2 Proof of Theorem 2.2

Let $t \in [0,T]$ be such that still $\Delta_N(t) \leq N^{-\alpha}$, then it holds for $x \in \mathbb{R}^6$ and $N \in \mathbb{N}\backslash\{1\}$ that

$$\begin{aligned}
&\partial_t \sup_{x\in\mathbb{R}^6} |\varphi^{1,N}_{t,0}(x) - {}^1\varphi^{\infty}_{t,0}(x)| = \sup_{x\in\mathbb{R}^6} |\varphi^{2,N}_{t,0}(x) - \varphi^{2,\infty}_{t,0}(x)| \\
&\leq \sup_{q_0,p_0\in\mathbb{R}^3} |\widetilde{k}^N_t * f^N_\beta(q^N_t(q_0,p_0)) - \widetilde{k}^N_t * f^N_\beta(q^\infty_t(q_0,p_0))| \\
&+ \sup_{q_0,p_0\in\mathbb{R}^3} |\widetilde{k}^N_t * f^N_\beta(q^\infty_t(q_0,p_0)) - \widetilde{k}^N_t * f^\infty(q^\infty_t(q_0,p_0))| \\
&+ \sup_{q_0,p_0\in\mathbb{R}^3} |\widetilde{k}^N_t * f^\infty(q^\infty_t(q_0,p_0)) - \widetilde{k}^\infty_t * f^\infty(q^\infty_t(q_0,p_0))| \, . \\
&\leq \|\widetilde{k}^N_t * \nabla f^N_\beta\| \|q^N_t - q^\infty_t\|_\infty + \|\widetilde{k}^N_t * f^N_\beta - \widetilde{k}^N_t * f^\infty\|_\infty + \|\widetilde{k}^N_t * f^\infty - \widetilde{k}^\infty_t * f^\infty\|_\infty.
\end{aligned}$$

The first addend is bounded by $C\Delta_N(t)$ due to Lemma 2.9 and because of the restrictions on $\tilde{k}$

$$\begin{aligned}
\left|\widetilde{k}^N_t(q) - \widetilde{k}^\infty_t(q)\right| &\leq \int \left|k_0(q^N_t, p^N_t) - k_0(q^\infty_t, p^\infty_t)\right| d^3p_0 \\
&\leq \int \left(\sup_{h\in\mathbb{R}^3;|h|=1} \nabla k_0(q^\infty_t, p^\infty_t + h)\right) |x^N_t - x^\infty_t| d^3p_0 \, .
\end{aligned}$$

for a time t such that $\Delta_N(t) \leq 1$. The second one is also bounded due to Lemma 2.9 by

$$\|\widetilde{k}^N_t * f^N_\beta - \widetilde{k}^N_t * f^\infty\|_\infty \leq \|\widetilde{k}^N_t * (f^N_\beta - f^\infty)\|_\infty \leq \|\widetilde{k}^N_t * f^N_\beta\|_\infty \leq C\Delta_N(t)$$

So we get by Gronwalls lemma

$$\sup_{q_0,p_0\in\mathbb{R}^3} |x^N_s(q_0,p_0) - x^\infty_s(q_0,p_0)| \leq CN^{-\alpha} \, ,$$

which implies

$$\left\|\Phi^N_{s,0} - \Phi^\infty_{s,0}\right\|_\infty < CN^{-\alpha} \, .$$

That shows that the initial assumption $\Delta_N(t) \leq N^{-\alpha}$ stays true for times $t < T$ provided that $N \in \mathbb{N}$ is large enough.

2.3 Molecular chaos

This result implies molecular chaos in the sense of Corollary 2.15 which is stated below. Therefore we introduce the following notation of distance which we require to estimate the dissimilarity between the two probability measures.

Definition 2.14. *Let $\mathcal{P}(\mathbb{R}^n)$ be the set of probability measures on $\mathbb{R}^n$.For $\mu,\nu \in \mathcal{P}(\mathbb{R}^n)$, let $\Pi(\mu,\nu)$ be the set of all probability measures $\mathbb{R}^n \times \mathbb{R}^n$ with marginal μ and ν. Then, for $p \in [1,+\infty)$, the p'th Wasserstein distance on $\mathcal{P}(\mathbb{R}^n)$ is defined by*

$$W_p(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \Big(\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^p \, \mathrm{d}\pi(x,y) \Big)^{1/p}.$$

For $p = \infty$ the infinite Wasserstein distance is defined by

$$W_\infty(\mu,\nu) = \inf\{\pi - \operatorname{esssup} |x-y| \,:\, \pi \in \Pi(\mu,\nu)\}.$$

In particular this notion of distance implies weak convergence in $\mathcal{P}(\mathbb{R}^n)$. Theorem 2.2 implies molecular chaos in the following sense:

Corollary 2.15. *Let $K_0^N := \otimes^N k_0$ be the n-fold solution of the considered effective equation and $K_t^N := \Psi_{t,0}^N \# K_0$ the N-particle distribution at time $t \in [0,T]$ evolving with the microscopic flow (1.4). Then the i-particle marginal*

$${}^{(i)}K_t^N(x_1,...,x_i) := \int K_t^N(X)\,\mathrm{d}^6x_{i+1}...\mathrm{d}^6x_N$$

converges weakly to $\otimes^i k_t$ as $N \to \infty$ for all $k \in \mathbb{N}$, where k_t is the unique solution of the Vlasov-Dirac-Benney equation (1.6) with initial density $k^N|_{t=0} = k_0$. More precisely, under the assumptions of Theorem 2.2, we get a constant $C > 0$ such that for all $N \geq N_0$

$$W_1({}^{(i)}K_t^N, \otimes^i k_t) \leq i\, e^{TC\sqrt{\ln(N)}} N^{-\alpha}, \textit{ for all } \; 0 \leq t \leq T.$$

Proof. For a fixed time $0 \leq t \leq T$ and $\mathcal{A} \subset \mathbb{R}^{6N}$ defined in Definition 2.10 we have proven in Theorem 2.2, that $\mathbb{P}_0(\mathcal{A}) \leq TC_\gamma N^{-\gamma}$ for sufficiently large N. Using the notion of distance above an that all test-functions are Lipschitz with $\|h\|_{Lip} = 1$,

$$\begin{aligned} &W_1({}^{(i)}K_t^N, \otimes^i k_t) \\ &\quad= \sup_{\|h\|_{Lip}=1} \Big| \int \big(K_t^N(X) - \otimes^N k_t(X)\big) g(x_1,...,x_i) \mathrm{d}^6x_1 ... \, \mathrm{d}^6x_k ... \mathrm{d}^6x_N \Big| \\ &\quad= \sup_{\|h\|_{Lip}=1} \Big| \int \big(\Psi_{t,0}^N \# K_0^N(X) - \Phi_{t,0}^N \# K_0^N(X)\big) g(x_1,...,x_i)\, \mathrm{d}^{6N}X \Big| \\ &\quad= \sup_{\|h\|_{Lip}=1} \Big| \int K_0^N(X) \big(h(\pi_i \Psi_{t,0}^N(X)) - h(\pi_i \Phi_{t,0}^N(X))\big)\, \mathrm{d}^{6N}X \Big| \\ &\quad= \sup_{\|h\|_{Lip}=1} \Big| \int\limits_{\mathcal{A}} K_0^N(X) \big(h(\pi_i \Psi_{t,0}^N(X)) - h(\pi_i \Phi_{t,0}^N(X))\big)\, \mathrm{d}^{6N}X \Big| \\ &\quad+ \sup_{\|h\|_{Lip}=1} \Big| \int\limits_{\mathcal{A}^c} K_0^N(X) \big(h(\pi_i \Psi_{t,0}^N(X)) - h(\pi_i \Phi_{t,0}^N(X))\big)\, \mathrm{d}^{6N}X \Big| \end{aligned}$$

for the projection $\pi_i : \mathbb{R}^N \to \mathbb{R}^i, (x_1,...,x_N) \mapsto (x_1,...,x_i)$. The first addend is bounded by

$$\sup_{\|h\|_{Lip}=1} \Big| \int\limits_{\mathcal{A}} K_0^N(X)(h\big(\pi_i \Psi_{t,0}^N(X)) - h(\pi_i \Phi_{t,0}^N(X))\big)\, \mathrm{d}^{6N}X \Big|$$

$$\leq \mathbb{P}(\mathcal{A}^c)\|K_0^N\|_\infty |\Psi_{t,0}^N(X) - \Phi_{t,0}^N(X)|,$$

with $\|K_0^N\|_\infty = (\|k_0\|_\infty)^N$. By the initialization of the initial conditions we trivially have that $|\Psi_{0,0}^N(X) - \Phi_{0,0}^N(X)|_\infty = 0$ and by Newtons law

$$|\Psi_{t,0}^{2,N}(X) - \Phi_{t,0}^{2,N}(X)|_\infty \leq \int_0^t |F^N(\Psi_{s,0}^{1,N}(X)) - \overline{F}(\Phi_{s,0}^{1,N}(X))|_\infty \, \mathrm{d}s,$$

$$|\Psi_{t,0}^{1,N}(X) - \Phi_{t,0}^{1,N}(X)|_\infty \leq \int_0^t |\Psi_{s,0}^{2,N}(X) - \Psi_{s,0}^{2,N}(X)|_\infty \, \mathrm{d}s.$$

The mean-field force $\overline{F}$ is of order 1 and the microscopic force F^N is bounded by $N^{4\beta}$. Hence, there exists a constant $C > 0$ such that $|\Psi_{t,0}^{2,N}(X) - \Phi_{t,0}^{2,N}(C)|_\infty \leq TCN^{4\beta}$ and consequently to Newtons law $|\Psi_{t,0}^{1,N}(X) - \Phi_{t,0}^{1,N}(X)|_\infty \leq T^2CN^{4\beta}$ for all times $t \leq T$. Choosing $\gamma := 5\beta$ in Theorem 2.2 we thus get a constant C such that

$$\mathbb{P}(\mathcal{A}^c)\|K_0^N\|_\infty |\Psi_{t,0}^N(X) - \Phi_{t,0}^N(X)| \leq C \max\{T^2, T^3\} N^{-\beta},$$

for all times $0 \leq t \leq T$. On the other hand, for $X \in \mathcal{A}^c$, we have for any h with $\|h\|_{Lip} = 1$,

$$|h(\pi_i \Psi_{t,0}^N(X)) - h(\pi_i \Phi_{t,0}^N(X))| \leq |\Psi_{t,0}^N(X) - \Phi_{t,0}^N(X)|_\infty \leq N^{-\alpha}$$

for all $t \leq T$ and thus

$$\sup_{\|g\|_{Lip}=1} \Big| \int_{\mathcal{A}^c} K_0^N(X) \big(h(\pi_i \Psi_{t,0}^N(X)) - h(\pi_i \Phi_{t,0}^N(X)) \big) \, \mathrm{d}^{6N} X \Big| \leq N^{-\alpha}.$$

It follows that there exists a constant C such that

$$W_1 \left({}^{(i)}K_t^N, \otimes^i k_t^N \right) \leq C(1 + T^3) N^{-\alpha},$$

for all times $0 \leq t \leq T$ and due to the property that k^N approximate k (see [32, Prob. 9.1] the statement follows. □

Molecular chaos in the sense of Corollary 2.15 implies convergence in law of the empirical distribution to the solution of the Vlasov Dirac Benney equation k_t (see e.g. [29], [23], [42, Prop.2.2]). Finally one can derive the macroscopic mean-field equation (1.6) from the microscopic particle system 1.1. We define the empirical measure associated to the microscopic N-particle system and respectively to the macroscopic by

$$\mu_\Phi(t) := \frac{1}{N} \sum_{i=1}^N \delta(q - q_i^t)\delta(p - p_i^t), \quad \mu_\Psi(t) := \frac{1}{N} \sum_{i=1}^N \delta(q - \overline{q}_i^t)\delta(p - \overline{p}_i^t)$$

and will see that the empirical measure $\mu_\Phi(t)$ converges to the solution of the Vlasov-Dirac-Benney equation in W_p distance with high probability.

Theorem 2.16 (Particle approximation of the Vlasov-Dirac-Benney system)**.** *Let k_0 be a probability measure satisfying the assumptions of Theorem 2.2 and $\Psi_{t,s}$ be the N-particle flow defined in Definition 1.1. Then, the empirical density $\mu_{\Phi_0(t)}$ converges to the solution of the Vlasov-Dirac-Benney equation in the following sense: For any $T > 0$ there exists a constant C depending on the initial density k_0 and T such that for all $N \geq N_0$ and some $\eta, \iota > 0$*

$$\mathbb{P}\Big[\max_{t\in[0,T]} \in [0,T] : W_p(\mu_\Phi, k_t) > N^{-\eta}\Big] \leq Ce^{-CN^{1-\iota}},$$

where k is the unique solution of the Vlasov-Dirac-Benney system on $[0,T]$.

Proof. In order to prove this let us split $W_p(\mu_\Phi(t), k_t)$ into three parts

$$W_p(\mu_\Phi(t), k_t) \leq W_p(k_t, k_t^N) + W_p(k_t^N, \mu_\Psi(t)) + W_p(\mu_\Psi(t), \mu_\Phi(t)).$$

The convergence of the first addend is a deterministic result stated in [32, Prob. 9.1], the second addend is bounded in probability due to[32, Cor. 9.4] and the last term is bounded in probability due to Theorem 2.2. □

Chapter 3

On the mean-field limit for the Vlasov-Poisson system

The Poisson kernel has its origins in Newton's theory of gravity. In the field of astrophysics, it is employed to investigate the development and transformation of galaxies and galaxy clusters, particularly when relativistic effects can be disregarded on a large scale. In this context, individual particles represent stars or even more substantial structures.
When considering repulsive interactions, the Poisson kernel corresponds to the electrostatic forces acting between particles. This is commonly applied in plasma physics, often involving multiple species or components.
Additionally, the Poisson kernel finds application in first-order models, such as in the realm of chemotaxis, which involves the movement of bacteria or cells induced by a chemical gradient. In this scenario, the force experienced can be understood as the gradient of the concentration of a chemical substance produced by each particle. The singularity of the kernel creates challenges for both theoretical analysis and numerical simulations. To address this issue in numerical computations, a straightforward solution is to introduce regularization to the kernel. Consequently, instead of working with f^∞, the non-regularized singular force, the focus shifts to f^N, where the regularization depends on the parameter N.

In this Chapter we present a microscopic derivation of the Vlasov-Poisson system. Therefore we consider a system consisting of N interacting particles subject to Newtonian time evolution. Our system is distributed by a trajectory in phase space $\mathbb{R}^{6N}$ with $X = (Q, P) = (q_1, \ldots, q_N, p_1, \ldots, p_N) \in \mathbb{R}^{6N}$, where $(Q)_j = q_j \in \mathbb{R}^3$ denotes the one-particle position and and $(P)_j = p_j$ stands for its momentum. The evolution of the system is given by the coupled differential equations

$$i \in \{1, ..., N\}, \quad \begin{cases} \dot{q}_i = \frac{p_i}{m} \\ \dot{p}_i = \frac{1}{N} \sum_{j \neq i} f^N(q_i - q_j) \end{cases} \tag{3.1}$$

with particle mass $m > 0$, which will always be set equal 1 in our considerations. We consider a Coulomb force with a cut-off at $N^{-\beta}$ for $\beta \leq \frac{5}{12} - \sigma$ and arbitrary $\sigma > 0$. Remarkably this cut-off can be chosen distinctly smaller than the typical inter particle distance which is given by $N^{-\frac{1}{3}}$. The interaction force for this Chapter is defined by the following.

Definition 3.1. *For $N \in \mathbb{N} \cup \{\infty\}$ the interaction force is given by*

$$f^N : \mathbb{R}^3 \to \mathbb{R}^3, q \mapsto \begin{cases} aN^{3\beta}q & \text{if } |q| \leq N^{-\beta} \\ a\frac{q}{|q|^3} & \text{if } |q| > N^{-\beta} \end{cases}$$

for $0 < \beta \leq \frac{5}{12} - \sigma$, some positive σ and $a \in \mathbb{R}$.

Remark. *Note, that we do not have any further constraints on the choice of a. In particular we consider both, attractive and repulsive interactions. We will use the notation $F^N : \mathbb{R}^{6N} \to \mathbb{R}^{3N}$ the total force of the system. Thus the i'th component of F^N gives the force exhibited on a single coordinate j :*

$$(F^N(X))_j := \sum_{i \neq j} \frac{1}{N} f^N(q_i - q_j).$$

We also consider the system in the mean-field scaling. The prefactor $\frac{1}{N}$ constitutes such a scaling factor. This scaling factor is discussed in Section 1.4 and is the most common choice in this setting [31].

Note that the scaling we chose is such that, as $N \to \infty$ the dynamics of the cloud remains fixed (both in q and p) while the density grows linearly with N. Furthermore the scaling ensures that the interaction per particle remains of order 1.

We want to derive the Vlasov-Poisson equation from the microscopic Newtonian N-particle dynamics with an improved cut-off. According to the typical approach, we compare the microscopic N-particle time evolution $\Psi^N_{t,s}$ with an effective one-particle description given by the mean-field flow $(\varphi^N_{t,s})_{t,s\in\mathbb{R}} : \mathbb{R}^6 \to \mathbb{R}^6$ and prove convergence of $\Psi^N_{t,s}$ to the product of $\varphi^N_{t,s}$ in the limit $N \to \infty$ in a suitable sense. From this, weak convergence of the s-particle marginals of the N-particle system to the corresponding s-fold products of solutions of the Vlasov equation follows. This is usually referred to as propagation of molecular chaos [7, 32, 22].

3.0.1 Dynamics of the Newtonian and of the effective system

By introducing the N-particle Coulomb force with cut-off, defined in Definition 3.1, we can characterize the Newtonian flow $\Psi^N_{t,s\in\mathbb{R}} : \mathbb{R}^{6N} \to \mathbb{R}^{6N}$, defined in Definition 1.1, as a solution of the system of equations (3.1) since the vector field is Lipschitz for fixed N and thus we have global existence and uniqueness of solutions. $\Psi^N_{t,s}(X)$ indicates the position of the particles in phase space, the first component $\Psi^{1,N}_{t,s}(X)$ denotes the positions of the particles in physical space and the second component $\Psi^{2,N}_{t,s}(X)$ the respective velocities.

Looking for a macroscopic law of motion for the particle density leads us to a continuity equation. For $N \in \mathbb{N} \cup \{\infty\}$, and $k : \mathbb{R}^6 \to \mathbb{R}^+_0$ we consider the corresponding mean-field equation, namely the Vlasov-Poisson equation (1.5). For a fixed initial distribution $k_0 \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ with $k_0 \geq 0$ we denote by k^N_t the unique solution of the Vlasov-Poisson equation (1.5) with initial datum $k^N_t(0, \cdot, \cdot) = k_0$.
The global existence and uniqueness of solutions of the Vlasov-Poisson equation for suitable initial conditions is well understood, even for singular interactions (see Section 1.6.1). The characteristics of Vlasov equation similar to the system of

equations (3.1) are given by the solution of equation (1.9). This system is uniquely solvable on any interval $[0,T]$ and this provides us the flow $(\varphi^{\infty}_{s,t})_{s,t\in\mathbb{R}}$.

The one-particle flow $(\varphi_{s,t})_{s,t\in\mathbb{R}} = (\varphi^1_{\cdot,s}(x), \varphi^2_{\cdot,s}(x))$ solves the equations (1.9) where $\varphi_{s,s}(x) = x$ for any $x \in \mathbb{R}^6$ and $s \in \mathbb{R}$. With this construction we get a new trajectory which is influenced by the mean-field force and not by the pair interaction force like in the Newtonian system. Now we have two trajectories which we will compare and later show that they are close to each other. To this end, we consider the lift of $\varphi^N_{t,s}(\cdot)$ to the N-particle phase-space, which we denote by $\Phi^N_{t,s}$ (Definition 1.4). The lift of the mean field force to the N-particle phase-space $\bar{F} : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ is given by

$$(\overline{F}_t(X))_i := f^N * \tilde{k}^N_t[k_t](x_i)$$

for $X = (x_1, ..., x_N)$.

In contrast to Ψ^N, Φ^N conserves independence, which is crucial for the later proof. This is due to the fact that Φ^N consists of N copies of φ_t. Hence the particles are distributed i.i.d. with respect to the particle density k^N of the Vlasov-Dirac-Benney equation (1.6). The mean-field particles move independently, because we use the same force for every particle and thus we do not have pair interactions, which lead to correlations. In summary, for fixed k_0 and $N \in \mathbb{N}$, we consider for any initial configuration $X \in \mathbb{R}^{6N}$ two different time-evolutions: $\Psi^N_{t,0}(X)$, given by the microscopic equations and $\Phi^N_{t,0}(X)$, given by the time-dependent mean-field force generated by the force f^N_t. We are going to show that for typical X, the two time-evolutions are close in an appropriate sense. In other words, we have non-linear time-evolution in which $\varphi^N_{t,s}(\cdot\,; k_0)$ is the one-particle flow induced by the mean-field dynamics with initial density k_0, while, in turn, k_0 is transported with the mean-field flow $\varphi^N_{t,s}$.

3.0.2 Statement of the results

In the following section we show that the N-particle trajectory Ψ_t starting from Ψ_0 (i.i.d. with the common density k_0) remains close to the mean-field trajectory Φ_t with the same initial configuration $\Psi_0 = \Phi_0$ during any finite time $[0,T]$ and so the microscopic and the macroscopic approach describe the same system.

Theorem 3.2. *Let $T > 0$ and $k_0 \in L^1(\mathbb{R}^6)$ be a continuously differentiable probability density fulfilling $\sup_{N\in\mathbb{N}} \sup_{0\le s\le T} ||\tilde{k}^N_s||_\infty \le \infty$. Moreover, let $(\Phi^{\infty}_{t,s})_{t,s\in\mathbb{R}}$ be the related lifted effective flow defined in Definition 1.4 as well as $(\Psi^N_{t,s})_{t,s\in\mathbb{R}}$ the N-particle flow defined in Definition 1.1. If $\sigma > 0$ and $\beta = \frac{5}{12} - \sigma$, then for any $\gamma > 0$ there exists a $C_\gamma > 0$ such that for all $N \in \mathbb{N}$ it holds that*

$$\mathbb{P}\left(X \in \mathbb{R}^{6N} : \sup_{0\le s\le T} \left|\Psi^N_{s,0}(X) - \Phi^{\infty}_{s,0}(X)\right|_\infty > N^{-\frac{1}{6}} \right) \le C_\gamma N^{-\gamma}. \tag{3.2}$$

This Theorem implies Propagation of Chaos. The main difference to [7] and [32] is that in the current case we analyse the advantages of the second order nature of the equation to transfer more information from the mean-field system to the true particles as introduced in [22]. As long as the true and their related mean-field particles are close in phase space, the types of their collisions are expected to be similar.

Therefore we will divide the particles into sets, a 'good', a 'bad' and a 'superbad' set, depending on their mean-field particle partners. If for certain particles, pair collision are expected according to their auxiliary trajectories, then depending on the distance and their relative velocity, they will be labelled 'bad' or 'superbad'. As for such particles larger deviations are expected after the collisions, we will allow larger distances to their related mean-field particles. The greater the distance to their related mean-field particles is, the worse the label gets. An advantage is that the number of 'bad' or 'superbad' particles is typically much smaller than the total particle number N.

Additionally, by using the integral version of Gronwalls Lemma we will make full use of the second order nature of the dynamics. If two particle come exceptionally close to each other, one can expect a correspondingly large deviation of the true and mean-field trajectory. However, for the vast majority, these deviations are typically only of a very limited duration. In order not to overestimate the deviations between them, it makes sense to compare the dynamics on longer time periods.

The idea of dividing the particles into sets and using the integral version of Gronwalls Lemma were previously implemented in the work of Grass and Pickl [22] for two particle sets, a so called 'good' and a 'bad' one.

3.0.3 Heuristics for the particle groups

The technical implementation works by dividing the particles into subsets. The closer particles get to each other and the lower their relative speed, the worse they are in the sense that their interactions leads to comparably large deviations from their mean-field evolution. However, it will be shown that the number of bad particles compared to N is extremely small. The small number will be useful in the estimates. It helps to control the future effect of the other particles despite their comparably large deviation form the mean field particle. In a first step we will classify the particles according to their distance from one another and their relative velocities. Roughly one should think of

$$\begin{aligned}
M_0 &: = \{i \in \{1,\dots,N\} | \exists t \geq 0 : |\bar{q}_j - \bar{q}_k| \leq N^{-r_0} \text{ and } |\bar{p}_j - \bar{p}_k| \leq N^{-v_0}\} \\
M_1 &: = \{i \in \{1,\dots,N\} | \exists t \geq 0 : |\bar{q}_j - \bar{q}_k| < N^{-r_1} \text{ and } |\bar{p}_j - \bar{p}_k| \leq N^{-v_1}\} \setminus M_0 \\
&\vdots \\
M_l &: = \{i \in \{1,\dots,N\} | \exists t \geq 0 : |\bar{q}_j - \bar{q}_k| \leq N^{-r_l} \text{ and } |\bar{p}_j - \bar{p}_k| \leq N^{-v_l}\} \setminus \bigcup_{n=0}^{l-1} M_n.
\end{aligned}$$

for $0 \leq r_l \leq r_1 \leq r_0$ and $0 \leq v_l \leq v_1 \leq v_0$. It holds that $\{1 \dots N\} = \dot{\bigcup} M_n$. The particles contained in M_0 are the most problematic particles, the so-called 'superbad' particles. An adjusted definition to the precise technical needs will be defined in Section 3.1 by so called collision classes. As we are only interested to show the advantage of introducing more particle subsets we limit ourselves to three subsets.

Note that the definition of the sets M_l only refers to the mean-field dynamics Φ which conserves independence, not Ψ. This makes it easy to calculate a bound for the probability of X_i belonging to these sets. Standard law of large numbers arguments give that for all $\gamma \in \mathbb{N}$ there exists a C_γ such that $\mathbb{P}(|M_l| \geq N^{\delta_l}) \leq C_\gamma N^{-\gamma}$ for some $\delta_l > 0$.
Let us next calculate the probability for a hit. It should be given by the Bolzmanzylinder $\mathbb{P}(\text{hit}) = Cr^2 v_{rel}$ for the relative velocity v_{rel}. In our case v_{rel} is also probabilistic with $\mathbb{P}(v_{rel} \leq v_{cut}) \approx v_{cut}^3$. So we should get a probabilistic bound of the form

$$\mathbb{P}(v_{rel} \leq v_{cut} \text{ and hit}) \leq Cr^2 v_{cut}^4.$$

The probability of finding k particles inside the set M_l around a bad particle is thus bounded from above by the binomial probability mass function with parameter $p := \mathbb{P}(j \in M_l)$ at position k, i.e. for any natural number $0 \leq A \leq N$ and any $t_n \leq t \leq t_{n+1}$

$$\mathbb{P}\left(\text{card }(M_l \geq A)\right) \leq \sum_{j=A}^{N} \binom{N}{j} p^j (1-p)^{N-j}.$$

The mean of a binomially distributed random variables is given by Np and thus the standard deviation by $\sqrt{Np(1-p)} < \sqrt{Np}$. The probability to find more than $Np + a\sqrt{Np}$ particles in the set M_l is exponentially small in a, i.e. there is a sufficiently large N for any $\gamma > 0$ and any t with $t \in [t_n, t_{n+1}]$ such that

$$\mathbb{P}\left(\text{card }(M_l) \geq Np + a\sqrt{Np}\right) \leq a^{-\gamma} .$$

The binomial distribution can be seen as a normal distribution when N is sufficiently large because of the central limit theorem. Hence the probability of finding more than $2Np = Np + \sqrt{Np}\sqrt{Np}$ (i.e. $a = \sqrt{Np}$) particles in the set M_l is smaller than any polynomial in N, i.e. there is a C_γ for any $\gamma > 0$ and any t with $t_n \leq t \leq t_{n+1}$ such that

$$\mathbb{P}\left(\text{card }(M_l) \geq 2Np\right) \leq C_\gamma N^{-\gamma}.$$

This preliminary consideration leads us to assume that the number of particles in a bad subset can be estimated by $N^{2-2r_l-4v_l}$, which will also be proven later.

3.0.4 Preliminary studies

To implement this proposed strategy we collect and derive necessary results and properties. Constants appearing in this thesis will generically be denoted by C. More precisely we will not distinguish constants appearing in a sequence of estimates, i.e. in an inequality chain $a \leq Cb \leq Cd$, the constants C may differ.

The following Lemma constitutes the probability of a hit i.e. the probability of the different types of collisions.

Lemma 3.3. *Let $(\varphi_{t,s}^N)_{t,s\in\mathbb{R}}$ be the related effective flow for $\beta \geq 0$ then there is an $C > 0$ such that for $N^{-a_k}, N^{-b_k} > 0, N \in \mathbb{N}$ and $[t_1, t_2] \subset [0, T]$ it holds that*

$$\mathbb{P}\Big(X \in \mathbb{R}^6 : \big(\exists t \in [t_1, t_2] : |\varphi_{t,0}^1(X) - \varphi_{t,0}^1(Y)| \leq N^{-a_{k-1}}$$

$$\wedge |\varphi^2_{t,0}(X) - \varphi^2_{t,0}(Y)| \leq N^{-b_{k-1}})\Big)$$
$$\leq C((N^{-a_{k-1}})^2(N^{-b_{k-1}})^4(t_2 - t_1) + (N^{-a_{k-1}})^3 \max(N^{-a_{k-1}}, N^{-b_{k-1}})^3)$$

The proof of Lemma 3.3 can be found in [22, Lemma 2.1.4]. This Lemma constitutes a probability bound for amount of particles belonging to a certain particle group,i.e.

$$\mathbb{P}(Y \in \mathbb{R}^6 : Y \in M_l(X_k)) \leq C(N^{-a_{l-1}})^2((N^{-b_{l-1}})^4.$$

So far all N particles were taken into account as possible interaction partners for the considered particle X_i particle. This constitutes a worst case estimate. The possible types of collisions and, accordingly, the impact on the force term can differ. This will be taken into account later by defining collision classes.

We further introduce the underlying Gronwall Lemma, which takes into account the second order nature of the equation. The unlikely collisions are usually only of a limited duration. An integral Gronwall version can take that into account.

Lemma 3.4. *Let $u : [0, \infty) \to [0, \infty)$ be a continuous and monotonously increasing map as well as $l, f_1 : \mathbb{R} \to [0, \infty)$ and $f_2 : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ continuous maps such that for some $n \in \mathbb{N}$ and for all $t_1 > 0,\ x_1, x_2 \geq 0$*

(i) $x_1 < x_2 \Rightarrow f_2(t_1, x_1) \leq f_2(t_1, x_2)$

(ii) $\exists K_1, \delta > 0 : \sup_{\substack{x,y \in [f_1(0), f_1(0)+\delta] \\ s \in [0,\delta]}} |f_2(s, x) - f_2(s, y)| \leq K_1 |x - y|.$

(iii)
$$f_1(t_1) + \int_0^{t_1} ... \int_0^{t_n} f_2(s, u(s)) ds dt_n ... dt_2 < u(t_1) \ \wedge$$
$$f_1(t_1) + \int_0^{t_1} ... \int_0^{t_n} f_2(s, l(s)) ds dt_n ... dt_2 \geq l(t_1),$$

then it holds for all $t \geq 0$ that $l(t) \leq u(t)$.

The proof of Lemma 3.4 can be found in [22, Lemma 2.1.1]. We need the fluctuation and the rate of change in the proof and therefore we introduce a kind of first derivative of f given by

Definition 3.5. *For $N \in \mathbb{N} \cup \{\infty\}$ we define*

$$g^N : \mathbb{R}^3 \to \mathbb{R}^3, q \mapsto \begin{cases} 2N^{3\beta} & \text{if } |q| \leq 3N^{-\beta} \\ 54\frac{1}{|q|^3} & \text{if } |q| > 3N^{-\beta} \end{cases}$$

for $0 < \beta$.

Lemma 3.6. *a) For $a, b, c \in \mathbb{R}^3$ with $|a| \leq \min(|b|, |c|)$ the following relations hold*

$$|f^N(b) - f^N(c)| \leq g^N(a)|b - c|. \tag{3.3}$$

b) If $\|X_t - \overline{X}_t\|_\infty \le 2N^{-\beta}$, then it holds that

$$\left\|F^N(X_t) - F^N(\overline{X}_t)\right\|_\infty \le C\|G^N(\overline{X}_t)\|_\infty \|X_t - \overline{X}_t\|_\infty, \tag{3.4}$$

for some $C > 0$ independent of N.

Proof. a) For the case $|a| \le 3N^{-\beta}$ we have $\|\nabla f^N\|_\infty \le 2N^{3\beta}$ and thus $2N^{3\beta}$ constitutes a Lipschitz-constant for f^N.
For $|a| \ge 3N^{-\beta}$, we get by the mean value theorem and the fact, that $\nabla f^N(x)$ is decreasing

$$f^N(b) - f^N(c)|| \le \nabla f^N(a)||b-a| \le \left(\frac{C}{|a|}\right)^3 |b-c| \le C\frac{|b-c|}{|1|^3} \le Cg^N(a)|b-a|.$$

b) For any $x, \xi \in \mathbb{R}^3$ with $|\xi| < 2N^{-\beta}$, we have for $|x| < 3N^{-\beta}$

$$|f^N(x+\xi) - f^N(x)| \le 2N^{3\beta}|\xi| \le g^N(x)|\xi| \tag{3.5}$$

by applying estimate 3.3 and for choosing without loss of generality $a = b = x+\xi$ and $c = x$. For $|x| \ge 3N^{-\beta}$ we use the fact that in this case small changes in the argument of the function lead to small changes in the function values, i.e. for $\xi \le 2N^{-\beta}$ we have $g^N(x+\xi) \le Cg^N(x)$. Thus we have by estimate 3.3

$$|f^N(x+\xi) - f^N(x)| \le Cg^N(x+\xi)|\xi| \le Cg^N(x)|\xi|.$$

Applying claim (3.5) one has

$$\begin{aligned} |(F^N(X_t))_i - (F^N(\overline{X}_t))_i| &\le \frac{1}{N}\sum_{j\ne i}^N \left|f^N(x_i^t - x_j^t) - f^N(\overline{x}_i^t - \overline{x}_j^t)\right| \\ &\le \frac{C}{N}\sum_{j\ne i}^N g^N(\overline{x}_i^t - \overline{x}_j^t)\left|x_i^t - x_j^t - \overline{x}_i^t + \overline{x}_j^t\right| \\ &\le C(g^N(\overline{X}_t))_i \left|X_t - \overline{X}_t\right|_\infty, \end{aligned} \tag{3.6}$$

which leads to estimate (3.4).

□

Analogously to the total force of the system F^N, the total fluctuation of the system is given by $G^N : \mathbb{R}^{6N} \to \mathbb{R}^{3N}$, where the force exhibited on a single coordinate j is given by

$$(G^N(X))_j := \sum_{i\ne j} \frac{1}{N} g^N(q_i - q_j).$$

Since f and g are not differentiable, we now prove some estimates for differences of function values. Also an important fact of the system is that the distance between the mean-field particles stay of the same order over time. This is provided by the following Lemma.

Lemma 3.7. *Let $T > 0$ and k_0 be a probability density fulfilling the assumptions of Theorem 3.2 where $(\varphi^{N,c}_{t,s})_{t,s\in\mathbb{R}}$ shall be the related effective flow defined in Definition 1.4. Then there exist a $C_1, C_2 > 0$ such that for all configurations $X, Y \in \mathbb{R}^6$, $N \in \mathbb{N} \cup \{\infty\}$ and $t, t_0 \in [0, T]$ it holds that*

$$|\varphi^N_{t,t_0}(X) - \varphi^N_{t,t_0}(Y)| \leq |X - Y| e^{C_1|t-t_0|}$$

and

$$|f^N_c * \widetilde{k}^N_t(^1X) - f^N_c * \widetilde{k}^N_t(^1Y)| \leq C_2|^1X - {}^1Y|.$$

The proof of this Lemma can be found in [22] (Lemma 2.1.2). Last but not least we come to the most important corollary of this chapter. It provides suitable upper bounds for almost all integrals appearing in the proof of the main theorem.

Corollary 3.8. *Let k_0 be a probability density fulfilling the assumptions of Theorem 3.2 and $(\varphi^{N,c}_{t,s})_{t,s\in\mathbb{R}}$ be the related effective flow defined in Definition 1.4 as well as $(\Psi^{N,c}_{t,s})_{t,s\in\mathbb{R}}$ the N-particle flow defined in Definition 1.1. Let additionally for $N, n \in \mathbb{N}$, $1 < \lambda \leq 3$, $C_0 > 0$ and $c_N > 0$ $h_N : \mathbb{R}^3 \to \mathbb{R}^n$ be a continuous map fulfilling*

$$|h_N(q)| \leq \begin{cases} C_0 c_N^{-\lambda}, & |q| \leq c_N \\ \frac{C_0}{|q|^\lambda}, & |q| > c_N \end{cases}.$$

(i) Let for $Y, Z \in \mathbb{R}^6$ $t_{min} \in [0, T]$ be a point in time where

$$\min_{0\leq s\leq T} |\varphi^{1,N}_{s,0}(Z) - \varphi^{1,N}_{s,0}(Y)| = |\varphi^{1,N}_{t_{min},0}(Z) - \varphi^{1,N}_{t_{min},0}(Y)| =: \Delta r > 0 \wedge$$
$$|\varphi^{2,N}_{t_{min},0}(Z) - \varphi^{2,N}_{t_{min},0}(Y)| =: \Delta v > 0,$$

then there exists a $C_1 > 0$ (independent of $Y, Z \in \mathbb{R}^6$ and $N \in \mathbb{N}$) such that

$$\int_0^T |h_N(^1\varphi^N_{s,0}(Z) - {}^1\varphi^N_{s,0}(Y))| ds \leq C_1 \min\left(\frac{1}{\Delta r^\lambda}, \frac{1}{c_N^{\lambda-1}\Delta v}, \frac{1}{\Delta r^{\lambda-1}\Delta v}\right).$$

(ii) Let $T > 0$, $i, j \in \{1, ..., N\}$, $i \neq j$, $X \in \mathbb{R}^{6N}$ and $Y, Z \in \mathbb{R}^6$ be given such that for some $\delta > 0$

$$N^\delta |\varphi^{1,N}_{t_{min},0}(Y) - \varphi^{1,N}_{t_{min},0}(Z)| \leq |\varphi^{2,N}_{t_{min},0}(Y) - \varphi^{2,N}_{t_{min},0}(Z)| =: \Delta v$$

and

$$\sup_{0\leq s\leq T} |\varphi^N_{s,0}(Y) - [\Psi^N_{s,0}(X)]_i| \leq N^{-\delta}\Delta v \wedge \sup_{0\leq s\leq T} |\varphi^N_{s,0}(Z) - [\Psi^N_{s,0}(X)]_j| \leq N^{-\delta}\Delta v$$

where t_{min} shall fulfil the same conditions as in item (i). Then there exists a $N_0 \in \mathbb{N}$ and $C_2 > 0$ (independent of $X \in \mathbb{R}^{6N}$, $Y, Z \in \mathbb{R}^6$) such that for all $N \geq N_0$

$$\int_0^T |h_N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)| ds$$
$$\leq C_2 \min\left(\frac{1}{c_N^{\lambda-1}\Delta v}, \frac{1}{\min\limits_{0\leq s\leq T} |[\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j|^{\lambda-1}\Delta v}\right).$$

The proof of this Corollary can be found in [22](Corollary 2.1.1).

3.1 Proof of Theorem 3.2

This proof and the notation is based on the work of Pickl and Grass [22]. Some of their estimates can be directly implied in our situation. For simplification we consider three different subsets of particles depending on their distance and relative velocity to other particles. The first set $\mathcal{M}_s$ of the 'superbad' ones includes all particles $j \in \{1 \dots N\}$ for which there is a time $t \geq 0$ such that $|\bar{q}_j - \bar{q}_k| \leq N^{-s_r}$ and $|\bar{p}_j - \bar{p}_k| \leq N^{-s_v}$. They are expected to come very close to other particles with small relative velocity. The second set $\mathcal{M}_b$, containing the so called 'bad' particles, which come intermediate close with intermediate relative velocity, is defined by analogue conditions $|\bar{q}_j - \bar{q}_k| \leq N^{-b_r}$ and $|\bar{p}_j - \bar{p}_k| \leq N^{-b_v}$, excluding the particles already in $\mathcal{M}_s$. Finally the reaming unproblematic 'good' ones, which never come close to each other while having small relative velocity are contained in $\mathcal{M}_g = (M_b \cup M_s)^c$. An important point in the proof is that the better the particle is, we allow less distance to the mean-field particle. Furthermore it depends only on their corresponding mean-field particle whether a particle is considered good, bad or superbad . In the course of a simple notation we introduce collision classes, which turn out to be very important throughout the proof, as each collision class has a different impact on the force term. They are intended to cover all possible ways in which particles can interact and thus the particle subsets can be defined using this notation.

Definition 3.9. *For $r, R, v, V \in \mathbb{R}_0^+ \cup \{\infty\}, t_1, t_2 \in [0, T]$ and $Y \in \mathbb{R}^6$ the set $M^{N,(t_1,t_2)}_{(r,R),(v,V)}(Y) \subset \mathbb{R}^6$ is defined as follows:*

$$\begin{aligned} &Z \in M^{N,(t_1,t_2)}_{(r,R),(v,V)}(Y) \Leftrightarrow Z \neq Y \wedge \exists t \in [t_1, t_2] : \\ &r \leq \min_{t_1 \leq s \leq t_2} |\varphi^1_{s,0}(Z) - \varphi^1_{s,0}(Y)| = |\varphi^1_{t,0}(Z) - \varphi^1_{t,0}(Y)| \leq R \\ &\wedge v \leq |\varphi^2_{t,0}(X) - \varphi^2_{t,0}(Y)| \leq V. \end{aligned}$$

Here $(\varphi^N_{s,r})_{s,r\in\mathbb{R}}$ is the one particle mean-field flow, defined in Definition 1.3, related to the considered initial density k_0. In addition, we will use the following short notation:

$$\begin{aligned} M^{N,(t_1,t_2)}_{R;V}(Y) &:= M^{N,(t_1,t_2)}_{(0,R),(0,V)}(Y) \\ M^N_{(r,R),(v,V)}(Y) &:= M^{N,(0,T)}_{(r,R),(v,V)}(Y) \\ M^N_{R,V}(Y) &:= M^{N,(0,T)}_{(0,R),(0,V)}(Y). \end{aligned}$$

The set $G^N(Y) \subset \mathbb{R}^6$ of non-problematic particle interactions is defined by

$$G^N(Y) := (M^N_{6r_b,v_b} \cup M^N_{6r_s,v_s})^c = (M^N_{6r_b,v_b})^c, \tag{3.7}$$

for $r_b = N^{-\frac{7}{24}-\sigma}, v_b = N^{-\frac{1}{6}}, r_s = N^{-\frac{1}{3}-\sigma}$ and $v_s = N^{-\frac{5}{18}}$ by application of such collision classes. Next we split the particles in three subsets using the notation of the collision classes as mentioned before: A 'superbad' subset where super hard collisions are expected to happen, a 'bad' subset where hard collisions are expected and a subset of the remaining 'good' particles.

$$\mathcal{M}^N_g(X) := \{i \in \{1, \dots, N\} : \forall j \in \{1, \dots, N\} \setminus \{i\} : X_j \in G^N(X_i))\}$$

$$\mathcal{M}_s^N(X) := \{i \in \{1,\dots,N\} : \exists j \in \{1,\dots,N\} \setminus \{i\} : X_j \in M^N_{(0,r_s),(0,v_s)}(X_j)\}$$
$$\mathcal{M}_b^N(X) := \{1,\dots,N\} \setminus (\mathcal{M}_g^N(X) \cup \mathcal{M}_s^N(X)).$$

The labelling 'good', 'bad' or 'superbad' depends only on their corresponding mean-field particle, as the sets above are defined by application of the collision classes which themselves are defined by the mean-field flow.

Each of the three particle subsets has its own stopping time which is defined by

$$\tau_g^N := \sup\{t \in [0,T] : \max_{i\in\mathcal{M}_g^N} \sup_{0\le s\le t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \le \delta_g^N = N^{-\frac{5}{12}+\sigma}\}$$
$$\tau_b^N := \sup\{t \in [0,T] : \max_{i\in\mathcal{M}_b^N} \sup_{0\le s\le t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \le \delta_b^N = N^{-\frac{7}{24}-\sigma}\}$$
$$\tau_s^N := \sup\{t \in [0,T] : \max_{i\in\mathcal{M}_s^N} \sup_{0\le s\le t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \le \delta_{sb}^N = N^{-\frac{1}{6}-\sigma}\}.$$

The stopping time for the whole system is given by

$$\tau^N(X) := \min(\tau_g^N(X), \tau_b^N(X), \tau_s^N(X)), \tag{3.8}$$

where $\delta_g^N = N^{-\beta}$, $\delta_b^N = N^{-d_b}$ and $\delta_s^N = N^{-d_s}$.
We will see that configurations fulfilling $\tau^N(X) < T$ become sufficiently small in probability for large values of N and hence Theorem 3.2 follows.
The main part of the proof is based on the application of Gronwall's Lemma to show that $\sup_{0\le s\le t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)|_\infty$ stays typically small for large N.
Therefore we estimate the right derivative of $\sup_{0\le s\le t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)|$, which is given by

$$\begin{aligned}
&\frac{d}{dt_+} \sup_{0\le s\le t} \left| \left[\Psi^{1,N}_{s,0}(X)\right]_i - \varphi^{1,N}_{s,0}(X_i)\right| \\
&\le \left| \left[\Psi^{2,N}_{t,0}(X)\right]_i - \varphi^{2,N}_{t,0}(X_i)\right| \\
&\le \left| \int_0^t \frac{1}{N} \sum_{j\neq i} f^N\left(\left[\Psi^{1,N}_{s,0}(X)\right]_i - \left[\Psi^{1,N}_{s,0}(X)\right]_j \right) - f^N * \widetilde{k}^N_s(\varphi^{1,N}_{s,0}(X_i)) ds \right|.
\end{aligned}$$

For technical reasons we will distinguish between observing a 'good', 'bad' or 'superbad' particle for further estimation of this expression.

3.1.1 Controlling the deviations of good particles

In the first Section we focus on the case, that the considered particle X_i is 'good' and use a similar proof technique as presented in [7, 32, 22]. First we break down the equation in terms of interaction partners. They themselves can be 'superbad', 'bad' or 'good' relative to X_i. Of course the set of particles having a bad or superbad interaction is empty in this case as having an unpleasant collision is symmetrical and consequently the underlying term will vanish later, but still, it will be technically useful to split the equation in that way.

Let $i \in \mathcal{M}_g^N(X)$ and $0 \le t_1 \le t \le T$

$$\left| \int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j) - f^N * \widetilde{k}_s^N(\varphi_{s,0}^{1,N}(X_i)) ds \right| \tag{3.9}$$

$$\begin{aligned} &\le \left| \int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j) \mathbb{1}_{(G^N(X_i))^C}(X_j) ds \right| \\ &+ \left| \int_{t_1}^{t} \Big(\frac{1}{N} \sum_{j \neq i} f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j) \mathbb{1}_{G^N(X_i)}(X_j) \right. \\ &\left. - f^N * \widetilde{k}_s^N(\varphi_{s,0}^{1,N}(X_i)) \Big) ds \right| . \end{aligned} \tag{3.10}$$

Using triangle inequality in the last two lines of Equation 3.10 one gets that the previous Term 3.9 is bounded by

$$\left| \int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j) \mathbb{1}_{(G^N(X_i))^C}(X_j) ds \right| \tag{3.11}$$

$$\begin{aligned} &+ \left| \int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} \Big(f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j) \mathbb{1}_{G^N(X_i)}(X_j) \right. \\ &\left. - f^N(\varphi_{s,0}^{1,N}(X_i) - \varphi_{s,0}^{1,N}(X_j)) \mathbb{1}_{G^N(X_i)}(X_j) \Big) ds \right| \end{aligned} \tag{3.12}$$

$$\begin{aligned} &+ \left| \int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} f^N(\varphi_{s,0}^{1,N}(X_i) - \varphi_{s,0}^{1,N}(X_j)) \mathbb{1}_{G^N(X_i)}(X_j) ds \right. \\ &\left. - \int_{t_1}^{t} \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(X_i) - \varphi_{s,0}^{1,N}(Y)) \mathbb{1}_{G^N(X_i)}(Y) k_0(Y) d^6Y ds \right| \end{aligned} \tag{3.13}$$

$$\begin{aligned} &+ \left| \int_{t_1}^{t} \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(X_i) - \varphi_{s,0}^{1,N}(Y)) \mathbb{1}_{G^N(X_i)}(Y) k_0(Y) d^6Y ds \right. \\ &\left. - \int_{t_1}^{t} f^N * \widetilde{k}_s^N(\varphi_{s,0}^{1,N}(X_i)) ds \right| \end{aligned} \tag{3.14}$$

Estimate of Term 3.11 and Term 3.14

Recall that $i \in \mathcal{M}_g^N(X)$ and that the set $(G^N(X_i))^c = M_{6r_b,v_b}^N$ includes all particles which come close to X_i while having small relative velocity. Thus the characteristic function $\mathbb{1}_{(G^N(X_i))^c}(X_j) = 0$ for $i \in \mathcal{M}_g^N(X)$ and therefore Term 3.11 vanishes and we are left to estimate Term 3.14. For the Lebesgue measure preserving diffeomorphism the following holds

$$\begin{aligned} f^N * \widetilde{k}_s^N(\varphi_{s,0}^{1,N}(X_i)) &= \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(X_i) - {}^1Y) k_s^N(Y) d^6Y \\ &= \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(X_i) - \varphi_{s,0}^{1,N}(Y)) k_0(Y) d^6Y. \end{aligned}$$

So we get for Term 3.14

$$\begin{aligned}
&\Big|\int_{t_1}^{t}\int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(X_i)-\varphi^{1,N}_{s,0}(Y))k_0(Y)\mathbb{1}_{G^N(X_i)}(Y)d^6Y ds \\
&\qquad -\int_{t_1}^{t} f^N * \widetilde{k}^N_s(\varphi^{1,N}_{s,0}(X_i))ds\Big| \\
=&|\int_{t_1}^{t}\int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(X_i)-\varphi^{1,N}_{s,0}(Y))k_0(Y)(\mathbb{1}_{G^N(X_i)}(Y)-1)d^6Y ds| \\
\leq& T\|f^N\|_\infty \int_{\mathbb{R}^6} \mathbb{1}_{(G^N(X_i))^C}(Y)k_0(Y)d^6Y \\
\leq& TN^{2\beta}\mathbb{P}\big(Y\in\mathbb{R}^6 : Y\notin G^N(X_i)\big) \\
\leq& TN^{2\beta}\mathbb{P}\big(Y\in\mathbb{R}^6 : Y\in M^N_{r_b,v_b}(X_i)\big) \\
\leq& CTN^{2\beta}N^{-2b_r-4b_v}
\end{aligned}$$

This is small under a suitable choice of parameters.

Law of large numbers for Term 3.12 and Term 3.13

For the remaining Terms (3.12) and (3.13) we provide a version of law of large numbers which takes into account the different types of collision classes which could occur. Each collision type has a different impact on the force and a certain probability. For that reason it is useful for the estimates to distinguish between them.

Theorem 3.10. *Let $\delta, C_0 > 0$, $N\in\mathbb{N}$ and let $(X_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables $X_k : \Omega \to \mathbb{R}^6$ distributed with respect to a probability density $k\in\mathcal{L}^1(\mathbb{R}^6)$. Moreover, let $(M^N_i)_{i\in I}$ be a family of (possibly N-dependent) sets $M^N_i \subseteq \mathbb{R}^6$ fulfilling $\bigcup_{i\in I} M^N_i = \mathbb{R}^6$ where $|I| < C_0$ and $h_N := \mathbb{R}^6 \to \mathbb{R}$ are measurable functions which fulfil on the one hand $\|h_N\|_\infty \leq C_0 N^{1-\delta}$ and on the other hand*

$$\max_{i\in I}\int_{M^N_i} h_N(X)^2 k(X) d^6X \leq C_0 N^{1-\delta}.$$

Then for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ such that for all $N\in\mathbb{N}$

$$\mathbb{P}_t\Big[\Big|\frac{1}{N}\sum_{j\neq i}^{N} h_N(X_k) - \int_{\mathbb{R}^6} h_N(Z)k_t(Z)d^6Z\Big| \geq 1\Big] \leq \frac{C_\gamma}{N^\gamma}. \tag{3.15}$$

Proof. By Markov's inequality, we have for every $M\in\mathbb{N}$:

$$\mathbb{P}_t\Big[\Big|\frac{1}{N}\sum_{j\neq i}^{N} h_N(X_k) - \int_{\mathbb{R}^6} h_N(X)k_t(X)d^6X\Big]| \tag{3.16}$$

$$\leq \mathbb{E}\Big[N^{-2M}\Big|\frac{1}{N}\sum_{j=1}^{N} h_N(X_k) - \int_{\mathbb{R}^6} h_N(X)k_t(X)d^6X\Big|^{2M}\Big], \tag{3.17}$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to the N-fold product of k.

Let $\mathcal{M} := \{\gamma \in \mathbb{N}_0^N \mid |\gamma| = 2M\}$ be the set of multiindices $\gamma = (\gamma_1, ..., \gamma_N)$ with $\sum_{j=1}^{N} \gamma_j = 2M$. Let

$$G_\gamma(X) := \prod_{j=1}^{N} \big(h_N(X_j) - \int_{\mathbb{R}^6} h_N(X) k_t(X) d^6 X)^{\gamma_i}. \tag{3.18}$$

Then

$$\begin{aligned} & N^{-2M}\mathbb{E}\Big[\Big(\sum_{k=1}^{N} h_N(X_k) - \int_{\mathbb{R}^6} h_N(X) k_t(X) d^6 Z\Big)^{2M}\Big] \\ & \leq N^{-2M} \sum_{\gamma_1,...,\gamma_N \in \mathcal{M}} \mathbb{E}\Big[\Big(G_\gamma(X)\big)^{\gamma_k}\Big]. \end{aligned}$$

Note that $\mathbb{E}(G_\gamma) = 0$ whenever there is a $1 \leq j \leq N$ such that $\gamma_j = 1$. This can be seen by integrating the j'th variable first.

For the remaining terms, we have for any $1 \leq m \leq M$:

$$|\big(h_N(X_j) - \int_{\mathbb{R}^6} h_N(X) k_t(X) d^6 X\big)^{\gamma_i}| \leq 2^{\gamma_k} |h_N(X_j)|^{\gamma_k} + |\int_{\mathbb{R}^6} h_N(X) k_t(X) d^6 X|^{\gamma_k}.$$

As $||h_N|| \leq C_0 N^{1-\delta}$, it follows for $m \geq 2$

$$\begin{aligned} & \int_{\mathbb{R}^6} |h|^m(X) k_t(X) \, \mathrm{d}^6 X \leq C_0 \max_{i \in I} \int_{M_i^N} |h|^m(X) k_t(X) \, \mathrm{d}^6 X \\ \leq & C_0 ||h_N||_\infty^{m-2} \max_{i \in I} \int_{M_i^N} h_N(X)^2 k(X) \mathrm{d}^6 X \leq C_0 (C_0^{m-2} N^{(m-2)(1-\delta)})(C_0 N^{1-\delta}) \end{aligned}$$

Let $R := \sqrt{\int_{\mathbb{R}^6} h_N^2(X) \mathrm{d}^6 X}$, then it holds that

$$\begin{aligned} & \int_{\mathbb{R}^6} |h|(X) k_t(X) \, \mathrm{d}^6 X \leq \frac{1}{R} \underbrace{\int_{\mathbb{R}^6} h^2(X) k_t(X)}_{=R^2} + \underbrace{\int_{\mathbb{R}^6} |h(X)| \mathbb{1}_{[0,R]} k_t(X)}_{\leq R} \\ & \leq 2\big(C \max_{i \in I} \int_{M_i^N} h_N^2(X) k(X) \mathrm{d}^6 X\big)^{\frac{1}{2}} \leq C M^{\frac{1}{2}(1-\delta)}. \end{aligned}$$

Since the constraints on the maps h_N become more stringent with an increase in the chosen value of δ, we can restrict our consideration to specific values, such as the interval $(0, 1]$. If we additionally identify $|\gamma| := |\{i \in \{1, ..., N\} : \gamma_i \neq 0\}|$ and recall that only tuples matter where $\gamma_i \neq 1$ for all $i \in \{1, ..., N\}$ as well as $\sum_{i=1}^{N} \gamma_i = 2M$, then application of these estimates and relations above yield that for all other multiindices, we get

$$\mathbb{E}_t(G^\gamma) \leq \prod_{j=1: \gamma_i \geq 2}^{N} \big(C^{\gamma_i} N^{(\gamma_i - 2)(1-\delta)} N^{1-\delta}\big) \leq C^{2M} N^{2M(1\delta)} N^{|\gamma|(\delta-1)},$$

by using that the particles are statistically independent. Finally, we observe that for any $l \geq 1$, the number of multiindices $\gamma \in \mathcal{M}$ with $\#\gamma = l$ is bounded by

$$\sum_{\#\gamma=l} 1 \leq \binom{N}{l}(2M)^l \leq (2M)^{2M} N^l.$$

Thus

$$\begin{aligned}
&\frac{1}{N^{2M}} \sum_{\gamma \in \mathcal{M}} \mathbb{E}(G^\gamma) \\
&\leq \frac{N^{2M(1-\delta)}}{N^{2M}} \sum_{\gamma \in \mathcal{M}} C^M N^{|\gamma|(\delta-1)} \\
&\leq C^M N^{-2M(\delta} \sum_{k=1}^{M} N^k (2M)^M N^{k(\delta-1)} \\
&\leq (CM)^M N^{-\delta M},
\end{aligned}$$

where C is some constant depending on M. Choosing M arbitrary large proofs the Theorem. $\square$

Estimate of Term 3.13

It is left to show that the third Term 3.13, respectively

$$\left| \int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} f^N \left(\varphi_{s,0}^{1,N}(X_i) - \varphi_{s,0}^{1,N}(X_j) \right) \mathbb{1}_{G^N(X_i)}(X_j) \right.$$
$$\left. - \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(X_i) - \varphi_{s,0}^{1,N}(Y)) \mathbb{1}_{G^N(X_i)}(Y) k_0(Y) d^6Y ds \right|$$

stays small for typical initial data. Analogously to the function h_N from Theorem 3.10, we define for arbitrary $Y \in \mathbb{R}^6$ the function

$$h_{1,N}^t(y, \cdot) : \mathbb{R}^6 \to \mathbb{R}^3, Z \mapsto N^\alpha \int_0^t f^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(Z)) ds \mathbb{1}_{G^N(Y)}(Z), \tag{3.19}$$

with $0 < \alpha \leq \frac{5}{12}$ or more precisely $0 < \alpha = \beta + \sigma$. As $h_{1,N}^t(Y, \cdot)$ does not map to $\mathbb{R}$ as assumed in Theorem 3.10 it can still be applied on each component separately. If it holds for each component then it holds for the related vector valued map. The fact that the Theorem only makes statements for certain points in time will be generalized later.

We are left to check if the assumptions of Theorem 3.10 on the force term are fulfilled. Therefore we abbreviate $\tilde{r} := \max(r, N^{-\beta})$ for $r \geq 0$ and we obtain by Corollary 3.8 and Lemma 3.3 for $0 \leq v \leq V$, $0 \leq r \leq R$ and $\lambda = 2$ that

$$\int_{M^N_{(r,R),(v,V)}(Y)} \left(\int_0^t |f^N(\varphi_{s,0}^{1,N}(Z) - \varphi_{s,0}^{1,N}(Y))| ds \right)^2 k_0(Z) d^6 Z$$

$$\begin{aligned}
&\leq C\big(\min(\frac{1}{\widetilde{r}^{\lambda}},\frac{1}{\widetilde{r}^{\lambda-1}v})\big)^2\int_{M^N_{(r,R),(v,V)}(Y)}k_0(Z)d^6Z\\
&\leq C\min\big(\frac{1}{\widetilde{r}^{2\lambda}},\frac{1}{\widetilde{r}^{2(\lambda-1)}v^2}\big)\min\big(1,R^2,R^2V^4+R^3\max(V^3,R^3)\big)\\
&\leq C\min\big(\frac{1}{\widetilde{r}^{2(\lambda-1)}v^2},\frac{R^2}{\widetilde{r}^{2(\lambda-1)}v^2},\frac{R^2V^4}{\widetilde{r}^{2(\lambda-1)}\max(\widetilde{r},v)^2}+\frac{R^6}{\widetilde{r}^{2\lambda}}\big)\\
&\leq C\min\big(\frac{1}{\widetilde{r}^{2}v^2},\frac{R^2}{\widetilde{r}^{2}v^2},\frac{R^2V^4}{\widetilde{r}^{2}\max(\widetilde{r},v)^2}+\frac{R^6}{\widetilde{r}^{4}}\big).
\end{aligned}\tag{3.20}$$

Let us define a suitable cover of $\mathbb{R}^6$, i.e. the collision classes, in order to apply Theorem 3.10. The classes are chosen finer as the collision strength becomes larger. If the particles keep distance of order 1 no splitting will be necessary. Let therefore be $k,l\in\mathbb{Z}, N\in\mathbb{N}\setminus\{1\},\delta>0$ and $0\leq r,v\leq 1$ and the family of sets given by

$$\begin{aligned}
&(i)\ M^N_{(0,r)(0,v)}(Y) && (ii)\ M^N_{(0,r)(N^{l\delta}v,N^{N(l+1)\delta}v)}(Y)\\
&(iii)\ M^N_{(0,r)(1,\infty)}(Y) && (iv)\ M^N_{(N^{k\delta}r,N^{N(k+1)\delta}r)(0,v)}(Y)\\
&(v)\ M^N_{(N^{k\delta}r,N^{N(k+1)\delta}r)(N^{l\delta}v,N^{N(l+1)\delta}v)}(Y) && (vi)\ M^N_{(N^{k\delta}r,N^{(k+1)\delta}r)(1,\infty)}(Y)\\
&(vii)\ M^N_{(N^{-\delta}r,\infty)(0,\infty)}(Y),
\end{aligned}\tag{3.21}$$

for $0\leq k\leq\lfloor\frac{\ln(\frac{1}{r})}{\delta\ln(N)}\rfloor, 0\leq l\leq\lfloor\frac{\ln(\frac{1}{v})}{\delta\ln(N)}\rfloor$. In this case we choose $r=v=N^{-\beta}$ and the number of sets belonging to this list is some integer I_δ independent of N.

We will apply 3.20 for each collision class family and get the bounds

$$\begin{aligned}
&(i)\ \frac{(N^{-\beta})^6}{(N^{-\beta})^4}=N^{-2\beta} && (ii)\ \frac{N^{-2\beta}N^{4[(k+1)\delta-\beta]}}{N^{-2\beta}N^{2(k\delta-\beta)}}=N^{-2\beta+2k\delta+4\delta}\\
&(iii)\ \frac{(N^{-\beta})^2}{(N^{-\beta})^2}=1 && (iv)\ \frac{N^{6(k\delta-\beta)}}{N^{4(k\delta-\beta)}}=N^{-2\beta+2k\delta+6\delta}\\
&(v)\ \frac{N^{2(k\delta+\delta-\beta)}N^{4(l\delta+\delta-\beta)}}{N^{2(k\delta-\beta)}N^{2(l\delta-\beta)}}+\frac{N^{6(k\delta-\beta)}}{N^{4(k\delta-\beta)}}=N^{-2\beta+2l\delta+6\delta}+N^{-2\beta+2k\delta+6\delta}\\
&(vi)\ \frac{(N^{k\delta}N^{-\beta})^2}{(N^{k\delta-\beta})^2}=N^{2\delta} && (vii)\ \frac{1}{(N^{-\delta})^4}=N^{4\delta}
\end{aligned}$$

for $0\leq k,l\leq\lfloor\frac{\beta}{\delta}\rfloor$. All these terms are bounded by $N^{6\delta}$.

For a law of large numbers argument we need

$$\|h_{1,N}\|_\infty\leq C_0N^{1-\delta}\ \text{ and }\ \max_{i\in I}\int_{M_i^N}h_{1,N}(X)^2k(X)d^6X\leq C_0N^{1-\delta}.$$

Due to the estimates for each collision class it follows for all $i\in I$

$$\int_{M^N_{(r_i,R_i),(v_i,V_i)}(Y)}h^t_{1,N}(Y,Z)^2k_0(Z)d^6Z\leq CN^{2\alpha}N^{6\delta}\leq C(N^\alpha)^2\big(CN^{6\delta}\big)\leq CN^{2(3\delta+\alpha)}.$$

For $\delta>0$ small enough and due to the fact that $\alpha=\beta+\sigma$ it follows that $6\delta+2\alpha<1$ and the first assumption of Theorem 3.10 is fulfilled as $\beta<\frac{1}{2}-3\delta$.

It holds due to Corollary 3.8 that for a point in time t_{min}, where the mean-field particles are close

$$\begin{aligned}
&\int_0^t |f^N(\varphi^{1,N}_{s,0}(Y) - \varphi^{1,N}_{s,0}(Z))| \mathbb{1}_{G^N(Z)}(Y) ds \\
&\leq \min\Big(\frac{Ct}{|\varphi^{1,N}_{t_{min},0}(Y) - \varphi^{1,N}_{t_{min},0}(Z)|^2}, \frac{CN^\beta}{|\varphi^{2,N}_{t_{min},0}(Y) - \varphi^{2,N}_{t_{min},0}(Z)|}, \\
&\frac{C}{|\varphi^{1,N}_{t_{min},0}(Y) - \varphi^{1,N}_{t_{min},0}(Z)| \cdot |\varphi^{2,N}_{t_{min},0}(Y) - \varphi^{2,N}_{t_{min},0}(Z)|} \Big) \mathbb{1}_{G^N(Z)}(Y). \qquad (3.22)
\end{aligned}$$

This is where we break down the time integral into several parts. If v is large, the assumptions of Theorem 3.10 are fulfilled directly. If v is small we made use of the fact that the collision time is not very large. Remember the definition of the 'good' set

$$G^N(Z) := \Big((M^N_{r_b,v_b}(Z) \setminus M^N_{r_s,v_s}(Z)) \cup M^N_{r_s,v_s}(Z) \Big)^C.$$

For $x_{min} := |\varphi^{1,N}_{t_{min},0}(Y) - \varphi^{1,N}_{t_{min},0}(Z)|$ and $v_{min} := |\varphi^{2,N}_{t_{min},0}(Y) - \varphi^{2,N}_{t_{min},0}(Z)|$ the following implication holds due to the definition of $G^N(Z)$

$$x_{min} \leq N^{-r_s} \Rightarrow v_{min} \geq N^{-v_b} \qquad (3.23)$$

$$N^{-s_r} \leq x_{min} \leq N^{-b_r} \Rightarrow v_{min} \geq N^{-b_v} \qquad (3.24)$$

$$N^{-b_r} \leq x_{min} \Rightarrow v_{min} \in \mathbb{R}^+ \qquad (3.25)$$

and thus the term is bounded in the first case (3.23) by

$$CN^{\beta+b_v}.$$

for the second case (3.24), the term is bounded by

$$\min(CN^{\beta+b_v}, CN^{s_r+b_v}).$$

And for the last case (3.25) we get a bound of

$$CtN^{2b_r}.$$

As $\alpha = \beta + \sigma$ from Theorem 3.10 the term is bounded by

$$CtN^{2b_r} + CN^{\beta+b_v}.$$

The second upper bound controls the cases where $x_{min} \leq 6N^{-b_r}$. This yields for small enough $\sigma > 0$ and $\beta + \alpha + b_v < 1$ that

$$||h^t_{1,N}(Y, \cdot)||_\infty \leq N^\alpha C(N^{2b_r} + N^{\beta+b_v}) \leq CN^{1-\sigma}.$$

We now apply our estimate on $h^t_{1,N}(y)$ defined in (3.19) to control Term 3.13. Therefore we introduce the set $\mathcal{B}^{N,\sigma}_{1,i} \subset \mathbb{R}^{6N}, i \in \{1, \dots, N\}$:

$$\begin{aligned} &X \in \mathcal{B}^{N,\sigma}_{1,i} \subseteq \mathbb{R}^{6N} \\ \Leftrightarrow &\exists t_1, t_2 \in [0,T] : \\ &\Big| \frac{1}{N} \sum_{j \neq i} \int_{t_1}^{t_2} f^N(\varphi^{1,N}_{s,0}(X_i) - \varphi^{1,N}_{s,0}(X_j)) \mathbb{1}_{G^N(X_i)}(X_j) ds \\ &- \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f^N(\varphi^N_{s,0}(X_i) - \varphi^N_{s,0}(Y)) \mathbb{1}_{G^N(X_i)}(Y) ds k_0(Y) d^6Y \Big| > N^{-\alpha} = N^{-\beta-\sigma}. \end{aligned} \tag{3.26}$$

The law of large numbers makes only statements for certain points in time. We can overcome this problem, because it further tells us that at one considered moment large fluctuations are extremely unlikely. On very short time intervals fluctuations cannot change significantly since the force is bounded due to the cut off by $N^{2\beta}$. By the definition of the set $\mathcal{B}^{N,\sigma}_{1,i}$ and by the fact that any continuous map $a : \mathbb{R} \to \mathbb{R}^m$ fulfills

$$\begin{aligned} &\Big| \int_{t_1}^{t_2} a(s) ds \Big| = \Big| \int_0^{t_2} a(s) ds - \int_0^{t_1} a(s) ds \Big| \\ \leq &\Big| \int_0^{\lfloor \frac{t_2}{\delta_N} \rfloor \delta_N} a(s) ds \Big| + \int_{\lfloor \frac{t_2}{\delta_N} \rfloor \delta_N}^{t_2} |a(s)| ds + \Big| \int_0^{\lfloor \frac{t_1}{\delta_N} \rfloor \delta_N} a(s) ds \Big| + \int_{\lfloor \frac{t_1}{\delta_N} \rfloor \delta_N}^{t_1} |a(s)| ds \\ \leq &2 \max_{k \in \{0, \dots, \lfloor \frac{T}{\delta_N} \rfloor\}} \Big(\Big| \int_0^{k\delta_N} a(s) ds \Big| + \int_{k\delta_N}^{(k+1)\delta_N} |a(s)| ds \Big), \end{aligned}$$

for $m \in \mathbb{N}, t_1, t_2 \in [0,T]$ it follows for $\delta_N > 0$ that

$$\begin{aligned} &X \in \mathcal{B}^{N,\sigma}_{1,i} \\ \Rightarrow &\exists k \in \{0, ..., \lfloor \frac{T}{\delta_N} \rfloor\} : \\ &\Big(\Big| \int_0^{k\delta_N} \Big(\frac{1}{N} \sum_{j \neq i} f^N(\varphi^{1,N}_{s,0}(X_i) - \varphi^{1,N}_{s,0}(X_j)) \mathbb{1}_{G^N(X_i)}(X_j) \\ &- \int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(X_i) - \varphi^{1,N}_{s,0}(Y)) \mathbb{1}_{G^N(X_i)}(Y) k_0(Y) d^6Y \Big) ds \Big| \geq \frac{N^{-\frac{5}{12}}}{4} \Big) \vee \\ &\Big(\int_{k\delta_N}^{(k+1)\delta_N} \Big(\Big| \frac{1}{N} \sum_{j \neq i} f^N(\varphi^{1,N}_{s,0}(X_i) - \varphi^{1,N}_{s,0}(X_j)) \mathbb{1}_{G^N(X_i)}(Y) \Big| \\ &+ \Big| \int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(X_i) - \varphi^{1,N}_{s,0}(Y)) \mathbb{1}_{G^N(X_i)}(Y) k_0(Y) d^6Y \Big| \Big) ds \geq \frac{N^{-\frac{5}{12}}}{4} \Big) \end{aligned}$$

If we choose $\delta_N := \frac{N^{-\alpha}}{8||f_N||_\infty} \leq CN^{-\alpha-\beta\lambda} = N^{-\alpha-2\beta} = N^{-3\beta-\sigma}$ the second constraint of the assumption is true. For the current estimate we assumed that all particles form a single cluster because it is sufficient for our estimates. We could choose δ_N of much larger order.

According to the previous reasoning for at least one $k \in \{0, \dots, \lfloor \frac{T}{\delta_N} \rfloor\}$ the event related to the first constraint must occur if $X \in \mathcal{B}_{1,i}^{N,\sigma}$, but the law of large numbers tells us that for any of these events and any $\gamma > 0$ there exists a $C_\gamma > 0$ such that its probability is smaller than $C_\gamma N^{-\gamma}$ since $h_{1,N}^t(Y, \cdot)$ fulfils the assumptions of Theorem 3.10.

As $\beta = \frac{5}{12} - \sigma$ and $\alpha = \beta + \sigma$ the number of such events is bounded by

$$\lfloor \frac{T}{\delta_N} \rfloor + 1 \leq CN^{\alpha+2\beta} \leq CN^{\sigma+3\beta} = CN^{\frac{5}{4}+\sigma}$$

and thus it holds for all $N \in \mathbb{N}$ that

$$\begin{aligned}
\mathbb{P}(\exists i \in \{1, \dots, N\} : X \in \mathcal{B}_{1,i}^{N\sigma}) &\leq N\mathbb{P}(X \in \mathcal{B}_{1,i}^{N\sigma})) \\
&\leq N \left(CN^{\frac{5}{4}} (C_{\gamma+\frac{9}{4}} N^{-(\gamma+\frac{9}{4})}) \right) \\
&\leq C_\gamma N^{-\gamma}.
\end{aligned}$$

For typical initial data and large enough $N \in \mathbb{N}$ Term 3.13 stays smaller than $N^{-\frac{5}{12}+\sigma}$.

Estimate of Term 3.12

Let us estimate Term 3.12, i.e. the difference of the real force acting on the real particles and the real force acting on the mean-field particles

$$\left| \int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} \left(f^N \left([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j \right) \right.\right. \\
\left.\left. - f^N(\varphi_{s,0}^{1,N}(X_i) - \varphi_{s,0}^{1,N}(X_j)) \right) \mathbb{1}_{G^N(X_i)}(X_j) ds \right| .$$

We abbreviate the following notation for the allowed difference between mean-field particle and the real one, depending on the subset membership. We allow less control if the particle is bad but have strict requirements if the particle is good. $\Delta_g^N(t, X)$ describes the largest spatial deviation of the 'good' particles, $\Delta_b^N(t, X)$ the corresponding value for the 'bad' ones and $\Delta_{sb}^N(t, X)$ the corresponding value for the 'superbad' ones. The worse the subset (in the sense of 'bad' or 'superbad'), the more deviation is allowed.

$$\begin{aligned}
\Delta_g^N(t, X) &:= \max_{j \in \mathcal{M}_g^N(X)} \sup_{0 \leq s \leq t} \left| \left[\Psi_{s,0}^{1,N}(X) \right]_j - \varphi_{s,0}^{1,N}(X_j) \right| = N^{-\frac{5}{12}+\sigma} \\
\Delta_b^N(t, X) &:= \max_{j \in \mathcal{M}_b^N(X)} \sup_{0 \leq s \leq t} \left| \left[\Psi_{s,0}^{1,N}(X) \right]_j - \varphi_{s,0}^{1,N}(X_j) \right| = N^{-\frac{7}{24}-\sigma} \\
\Delta_{sb}^N(t, X) &:= \max_{j \in \mathcal{M}_{sb}^N(X)} \sup_{0 \leq s \leq t} \left| \left[\Psi_{s,0}^{1,N}(X) \right]_j - \varphi_{s,0}^{1,N}(X_j) \right| = N^{-\frac{1}{6}-\sigma}.
\end{aligned}$$

We further introduce a subset of the good particles

$$\widetilde{G}^N(\cdot) := G^N(\cdot) \cap \left(M_{3N^{-\frac{1}{2}+\sigma},\infty}^N(\cdot) \right)^C$$

which helps us to shorten the upcoming estimates. By definition of $\widetilde{G}^N(\cdot)$ (applied for the first inequality) and the stopping time $\tau^N(X)$

$$\tau_g^N := \sup\left\{t \in [0,T] : \max_{i\in\mathcal{M}_g^N} \sup_{0\le s\le t} \left|\left[\Psi_{s,0}^N(X)\right]_i - \varphi_{s,0}^N(X_i)\right| \le \delta_g^N\right\}$$
$$\tau_b^N := \sup\left\{t \in [0,T] : \max_{i\in\mathcal{M}_b^N} \sup_{0\le s\le t} \left|\left[\Psi_{s,0}^N(X)\right]_i - \varphi_{s,0}^N(X_i)\right| \le \delta_b^N\right\}$$
$$\tau_{sb}^N := \sup\left\{t \in [0,T] : \max_{i\in\mathcal{M}_{sb}^N} \sup_{0\le s\le t} \left|\left[\Psi_{s,0}^N(X)\right]_i - \varphi_{s,0}^N(X_i)\right| \le \delta_{sb}^N\right\}$$

as well as $\tau^N(X) := \min(\tau_g^N(X), \tau_b^N(X), \tau_{sb}^N(X))$ with $\delta_g^N = N^{-\beta} = N^{-\frac{5}{12}+\sigma}$, $\delta_b^N = N^{-d_b} = N^{-\frac{7}{24}-\sigma}$ and $\delta_{sb}^N = N^{-d_{sb}} = N^{-\frac{1}{6}-\sigma}$ it holds for $X_j \in \widetilde{G}^N(X_i)$ and times $s \in [0, \tau^N(X)]$ that

$$\max\left(2N^{-\beta}, \frac{2}{3}|\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i)|\right) \ge \max\left(2N^{-\beta}, 2N^{-\frac{1}{2}+\sigma}\right) \ge 2\Delta_g^N(t,X).$$

In the next step we subdivide the sum according to whether the particle interacting with i is itself 'superbad', 'bad' or 'good'. Furthermore, the map g^N was defined such that $|f^N(q+\delta) - f^N(q)| \le g^N(q)|\delta|$ for $q, \delta \in \mathbb{R}^3$ where $\max\left(2N^{-\beta}, \frac{2}{3}|q|\right) \ge |\delta|$, see Definition 3.5. Thus the subsequent estimates are fulfilled for all times $0 \le t_1 \le t \le \tau^N(X)$.

$$\begin{aligned}
&\Big|\int_{t_1}^t \Big(\frac{1}{N}\sum_{j\ne i}\Big(f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i) \\
&- f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\Big)\mathbb{1}_{G^N(X_i)}(X_j)\Big)ds\Big| &(3.27)\\
\le &\int_0^t \Big(\frac{1}{N}\sum_{\substack{j\ne i\\ j\in\mathcal{M}_{sb}^N(X)}}\Big(\big|f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i) \\
&- f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\big|\Big)\mathbb{1}_{G^N(X_i)}(X_j)\Big)ds &(3.28)\\
+&\int_0^t \Big(\frac{1}{N}\sum_{\substack{j\ne i\\ j\in\mathcal{M}_{b}^N(X)}}\Big(\big|f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i) \\
&- f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\big|\Big)\mathbb{1}_{G^N(X_i)}(X_j)\Big)ds &(3.29)\\
&+\int_0^t \Big(\frac{1}{N}\sum_{\substack{j\ne i\\ j\in\mathcal{M}_{g}^N(X)}}\Big(\big|f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i)\big| \\
&+\big|f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\big|\Big)\mathbb{1}_{G^N(X_i)\cap M^N_{3N^{-\frac{1}{2}+\sigma},\infty}(X_i)}(X_j)\Big)ds &(3.30)\\
&+\int_0^t \frac{2}{N}\sum_{\substack{j\ne i\\ j\in\mathcal{M}_{g}^N(X)}} g^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\Delta_g^N(s,X)\mathbb{1}_{\widetilde{G}^N(X_i)}(X_j)ds. &(3.31)
\end{aligned}$$

For the last term we applied the previous considerations and to estimate this one we define a set

$$\begin{aligned}
&X \in \mathcal{B}_{2,i}^{N,\sigma} \subseteq \mathbb{R}^{6N}\\
\Leftrightarrow &\exists t_1, t_2 \in [0,T]:\\
&\Big|\frac{1}{N}\sum_{j\neq i}\int_{t_1}^{t_2} g^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\mathbb{1}_{\widetilde{G}^N(X_i)}(X_j)ds \qquad (3.32)\\
&-\int_{\mathbb{R}^6}\int_{t_1}^{t_2} g^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(X_i))\mathbb{1}_{\widetilde{G}^N(X_i)}(Y)dsk_0(Y)d^6Y\Big| > 1
\end{aligned}$$

For $Y, Z \in \mathbb{R}^6$ it holds by definition of $\widetilde{G}^N(\cdot)$ and the definition of g^N (see 3.5) that

$$\begin{aligned}
&\int_0^t g^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(Z))\mathbb{1}_{\widetilde{G}^N(Z)}(Y)ds\\
\leq &CN^{\beta}\int_0^t |f^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(Z))|\mathbb{1}_{G^N(Z)}(Y)ds \qquad (3.33)\\
\leq &CN^{\frac{5}{12}-\sigma}\int_0^t |f^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(Z))|\mathbb{1}_{G^N(Z)}(Y)ds. \qquad (3.34)
\end{aligned}$$

Analogously to the previous section, Term 3.34 fulfils the assumptions of Theorem 3.10. Following the same reasoning for the map $h_N^t(Y,\cdot)$ one can show that for an arbitrary $\gamma > 0$ there exists a $C_\gamma > 0$ such that for all $N \in \mathbb{N}$

$$\mathbb{P}\big(\exists i \in \{1, ..., N\} : X \in \mathcal{B}_{2,i}^{N,\sigma}\big) \leq C_\gamma N^{-\gamma}. \qquad (3.35)$$

It remains to determine an upper bound for the terms (3.30), (3.28) and (3.29). We start with the last two terms, which describe the interaction of a good particle with a superbad particle respectively bad one. We show that the 'superbad' and 'bad' particles do typically not infect the 'good' ones which corresponds to deriving a suitable bound for Term (3.28) and (3.29). Since the allowed maximal value for for the largest deviation of a 'bad' or 'superbad' particle $\Delta_b^N(t,X)$ and $\Delta_s^N(t,X)$ is distinctly larger than the corresponding value for the good particle $\Delta_g^N(t,X)$, problems could arise if the number of 'bad' or 'superbad' particles coming close to a 'good' one exceeds a certain value. But we can show that the probability of such events is sufficiently small for large N.

Analogously to the previous section we introduce $h_{2,N}^t(Y,\cdot)$ according to Theorem 3.10 with

$$h_{2,N}^t(y,\cdot) : \mathbb{R}^6 \to \mathbb{R}^3, Z \mapsto N^{\alpha}\int_0^t f^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(Z))ds\mathbb{1}_{G^N(Y)}(Z). \qquad (3.36)$$

Let us also implement a family of 'collision classes' $\big(M_{(r_i,R_i),(v_i,V_i)}^N(Y)\big)_{i\in I_\delta}$ which covers $\mathbb{R}^6$ and check if $h_{2,N}^t(Y,\cdot)$ in combination with this cover fulfils the assumptions of Theorem 3.10 to derive an upper bound for the terms (3.28) and (3.29). Similar to the list stated in (3.21) we define $\big(M_{(r_i,R_i),(v_i,V_i)}^N(Y)\big)_{i\in I_\delta}$ for the parameters $r := r_b = 6N^{-\frac{7}{24}-\sigma}$ and $v := 6v_b = 6N^{-\frac{1}{6}}$ for Term (3.29) and for the parameters

$r := r_s = 6N^{-\frac{1}{3}-\sigma}$ and $v := 6v_s = 6N^{-\frac{5}{18}}$ for Term (3.28) (instead of $r = v := N^{-c}$ and $\delta := \sigma$ like in (3.21)). Thus we define for $i \in \{1, ..., N\}$ the sets $\mathcal{B}^{N,\sigma}_{3_b,i}, \mathcal{B}^{N,\sigma}_{3_s,i} \subseteq \mathbb{R}^{6N}$ as follows

$$\begin{aligned}
&X \in \mathcal{B}^{N,\sigma}_{3_b,i} \subseteq \mathbb{R}^{6N} \\
\Leftrightarrow &\exists l \in I_\sigma : \Big(R_l \neq \infty \wedge \\
&\sum_{j \in \mathcal{M}^N_b(X)} \mathbf{1}_{M^N_{(r_l,R_l),(v_l,V_l)}(X_i)}(X_j) \geq N^{\sigma\frac{3}{4}} \lceil N^{\frac{3}{4}} R_l^2 \min\big(\max(V_l, R_l), 1\big)^4 \rceil\Big) \vee \\
&\sum_{j \in \mathcal{M}^N_b(X)} 1 = |\mathcal{M}^N_b(X)| \geq N^2 v_b^4 r_b^2 \geq N^{\frac{3}{4}(1+\sigma)}.
\end{aligned} \tag{3.37}$$

Respectively for Term (3.28)

$$\begin{aligned}
&X \in \mathcal{B}^{N,\sigma}_{3_{sb},i} \subseteq \mathbb{R}^{6N} \\
\Leftrightarrow &\exists l \in I_\sigma : \Big(R_l \neq \infty \wedge \\
&\sum_{j \in \mathcal{M}^N_{sb}(X)} \mathbf{1}_{M^N_{(r_l,R_l),(v_l,V_l)}(X_i)}(X_j) \geq N^{\sigma(\frac{2}{9})} \lceil N^{\frac{2}{9}} R_l^2 \min\big(\max(V_l, R_l), 1\big)^4 \rceil\Big) \vee \\
&\sum_{j \in \mathcal{M}^N_{sb}(X)} 1 = |\mathcal{M}^N_{sb}(X)| \geq N^2 v_{sb}^4 r_{sb}^2 \geq N^{\frac{2}{9}(1+\sigma)}.
\end{aligned} \tag{3.38}$$

The last line in each case gives an estimate of the absolute number of bad or superbad particles and the line above an estimate of how many bad or superbad particles come close to a good one given a certain inter-particle distance and velocity. We now derive an upper bound for Term (3.28) and (3.29) under the condition that $X \in \big(\mathcal{B}^{N,\sigma}_{3_{sb},i}\big)^C$ respectively $X \in \big(\mathcal{B}^{N,\sigma}_{3_b,i}\big)^C$ and prove later that $\mathbb{P}\big(X \in \mathcal{B}^{N,\sigma}_{3_{sb},i}\big)$ and $\mathbb{P}\big(X \in \mathcal{B}^{N,\sigma}_{3_b,i}\big)$ get small as N increases.
To this end, we abbreviate for $0 \leq r \leq R$ and $0 \leq v \leq V$

$$\widetilde{M}^N_{(r,R),(v,V)}(X_i) := G^N(X_i) \cap M^N_{(r,R),(v,V)}(X_i)$$

to distinguish between the collision classes. As mentioned before, for Term (3.28) we only consider values of r and R that satisfy the constraint

$$\big(r = 0 \wedge R = 6\delta^N_{sb} = 6N^{-\delta_s}\big) \vee \big(r \geq 6\delta^N_{sb} \wedge R = N^\sigma r\big), \tag{3.39}$$

respectively for Term (3.29)

$$\big(r = 0 \wedge R = 6\delta^N_b = 6N^{-\delta_b}\big) \vee \big(r \geq 6\delta^N_b \wedge R = N^\sigma r\big). \tag{3.40}$$

We will see in Section 3.1.1 that those are the worst case options for the estimates. Recall that

$$\sup_{0 \leq s \leq t} |\Psi^N_{s,0}(X) - \Phi^N_{s,0}(X)|_\infty \leq N^{-s_\delta} = \delta^N_{sb} = N^{-\frac{1}{6}-\sigma}$$

and

$$\sup_{0 \leq s \leq t} |\Psi^N_{s,0}(X) - \Phi^N_{s,0}(X)|_\infty \leq N^{-b_\delta} = \delta^N_b = N^{-\frac{7}{24}-\sigma}$$

depending on which of the two term we devote ourselves to and for times before the stopping time is 'triggered'. Thus, we obtain for $0 \leq t \leq \tau^N(X)$ depending on the choice of r that Term (3.28) can be estimated by

$$\begin{aligned}
&\int_0^t \frac{1}{N} \sum_{\substack{j \neq i \\ j \in \mathcal{M}_{sb}^N(X)}} \Big(\big| f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i) \\
&- f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i)) \big| \Big) \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j) ds \\
\leq &\int_0^t \frac{1}{N} \sum_{\substack{j \neq i \\ j \in \mathcal{M}_{sb}^N(X)}} \Big(\big| f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i) \big| \\
&+ \big| f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i)) \big| \Big) \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j) ds \mathbf{1}_{[0,6\delta_{sb}^N]}(r) \\
&+ \frac{2}{N} \Delta_{sb}^N(t,X) \sup_{Y \in \widetilde{M}^N_{(r,R),(v,V)}(X_i)} \int_0^t g^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(X_i)) ds \\
&\cdot \sum_{\substack{j \neq i \\ j \in \mathcal{M}_b^N(X)}} \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j) \mathbf{1}_{[6\delta_{sb}^N, \infty)}(r).
\end{aligned}$$

Analogously Term (3.29) can be estimated by

$$\begin{aligned}
&\int_0^t \frac{1}{N} \sum_{\substack{j \neq i \\ j \in \mathcal{M}_b^N(X)}} \Big(\big| f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i) \\
&- f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i)) \big| \Big) \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j) ds \\
\leq &\int_0^t \frac{1}{N} \sum_{\substack{j \neq i \\ j \in \mathcal{M}_b^N(X)}} \Big(\big| f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i) \big| \\
&+ \big| f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i)) \big| \Big) \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j) ds \mathbf{1}_{[0,6\delta_b^N]}(r) \\
&+ \frac{2}{N} \Delta_b^N(t,X) \sup_{Y \in \widetilde{M}^N_{(r,R),(v,V)}(X_i)} \int_0^t g^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(X_i)) ds \\
&\cdot \sum_{\substack{j \neq i \\ j \in \mathcal{M}_b^N(X)}} \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j) \mathbf{1}_{[6\delta_b^N, \infty)}(r)
\end{aligned}$$

where we utilzied that $|f^N(q+\delta) - f^N(q)| \leq g^N(q)|\delta|$ for $q, \delta \in \mathbb{R}^3$ provided that $\max\left(2N^{-c}, \frac{2}{3}|q|\right) \geq |\delta|$.

Application of Corollary 3.8 yields that the previous terms are bounded by

$$(3.28) \leq \frac{C}{N} \frac{1}{N^{-\beta} v} \sum_{\substack{j \neq i \\ j \in \mathcal{M}_{sb}^N(X)}} \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j) \mathbf{1}_{[0,6\delta_{sb}^N]}(r)$$

$$+\frac{C}{N}\frac{\Delta_b^N(t,X)}{r^2\max(r,v)}\sum_{\substack{j\neq i\\ j\in\mathcal{M}_{sb}^N(X)}}\mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)\mathbf{1}_{[6\delta_{sb}^N,\infty)}(r). \quad (3.41)$$

and

$$(3.29)\leq\frac{C}{N}\frac{1}{N^{-\beta}v}\sum_{\substack{j\neq i\\ j\in\mathcal{M}_b^N(X)}}\mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)\mathbf{1}_{[0,6\delta_b^N]}(r)$$
$$+\frac{C}{N}\frac{\Delta_b^N(t,X)}{r^2\max(r,v)}\sum_{\substack{j\neq i\\ j\in\mathcal{M}_b^N(X)}}\mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)\mathbf{1}_{[6\delta_b^N,\infty)}(r). \quad (3.42)$$

Estimate of Term 3.29 (i good j bad)

Remark that the assumptions of the Corollary 3.8 are indeed fulfilled in the current situation since according to the constraints on the possible parameters (see (3.40)) $r\in[0,6\delta_b^N]$ implies $R=\delta_b^N$ and $r=0$. Considering the definition of the set of 'good' particles $G^N(X_i)$ it follows that

$$\widetilde{M}^N_{(0,6\delta_b^N),(v,V)}(X_i)=M^N_{(0,6\delta_b^N),(v,V)}(X_i)\cap G^N(X_i)\subseteq\left(M^N_{6\delta_b^N,N^{-\frac{1}{6}}}(X_i)\right)^C$$

which in turn provides

$$X_j\in\widetilde{M}^N_{(0,6\delta_b^N),(v,V)}(X_i)$$
$$\Rightarrow|\varphi^{2,N}_{t_{min},0}(X_j)-\varphi^{2,N}_{t_{min},0}(X_i)|\geq N^{-\frac{1}{6}} \quad (3.43)$$

where t_{min} shall denote a point in time where $|\varphi^{1,N}_{\cdot,0}(X_j)-\varphi^{1,N}_{\cdot,0}(X_i)|$ takes its minimum on $[0,T]$.
Now we want to derive an upper bound for Term 3.42 under the condition that

$$\sum_{j\in\mathcal{M}_b^N(X)}\mathbf{1}_{M^N_{(r,R),(v,V)}(X_i)}(X_j)\leq N^{\frac{3\sigma}{4}}\left\lceil N^{\frac{3}{4}}R^2\min\left(\max(V,R),1\right)^4\right\rceil.$$

We will deal with the addends related to $\mathbf{1}_{[0,6\delta_b^N]}(r)$ and $\mathbf{1}_{[6\delta_b^N,\infty)}(r)$ separately. Regarding the first addend, we already discussed that $r=0$ and $R=6\delta_b^N$ due to condition (3.40). We obtain

$$\frac{C}{N}\frac{1}{N^{-\beta}\Delta v}\sum_{\substack{j\neq i\\ j\in\mathcal{M}_b^N(X)}}\mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)\mathbf{1}_{[0,6\delta_b^N]}(r) \quad (3.44)$$
$$\leq\frac{N^{\frac{3\sigma}{4}}}{N^{\beta+1}\max(v,\delta_b)}+\frac{C|\mathcal{M}_b|R^2\min(V,1)^4}{N^{-\beta-1}\max(v,\delta_b)} \quad (3.45)$$
$$\leq\frac{N^{\frac{3\sigma}{4}}}{N^{\beta+1}\max(v,\delta_b)}+\frac{CR^2\min(V,1)^4N^{\frac{3}{4}(1+\sigma)}}{N^{-\beta+1}\max(N^{-b_\delta},v)} \quad (3.46)$$

$$\leq CN^{-3\sigma-\frac{5}{12}} + CN^{-\frac{5}{12}-3\sigma}\frac{\min(V,1)^4}{\max(N^{-b_\delta},v)} \tag{3.47}$$

for $R = \delta_b^N = N^{-b_\delta} = N^{-\frac{7}{24}-\sigma}$ since we only have to consider values with $v > N^{-b_v} = N^{-\frac{1}{6}}$, see (3.43).

For the allowed deviation $\Delta_b^N(t,X) \leq N^{-b_\delta} = \delta_b^N = N^{-\frac{7}{24}-\sigma}$ and $R = N^\sigma r$ for $r \geq 6\delta_b^N$ (see (3.40)) it follows for the second term of (3.42) that

$$\begin{aligned}
&\frac{C}{N}\frac{\Delta_b^N(t,X)}{r^2\max(r,v)}\sum_{\substack{j\neq i\\ j\in\mathcal{M}_b^N(X)}} \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)\\
\leq&\frac{C}{N}\Big(\frac{N^{-2b_r-4b_v+2+\sigma}R^2\min\big(\max(V,R),1\big)^4}{r^2\max(r,v)} + \frac{N^{\frac{3\sigma}{4}}}{r^2\max(r,v)}\Big)N^{-b_\delta}\\
\leq&C\Big(\frac{\min\big(\max(V,R),1\big)^4}{\max(r,v)}N^{-\frac{13}{24}} + N^{-\frac{41}{48}+3\sigma}\Big) \leq CN^{-\frac{13}{24}} \leq CN^{-\frac{5}{12}}.
\end{aligned} \tag{3.48}$$

In total we got an upper bound for Term (3.29). All sets belonging to the family $\big(M^N_{(r_i,R_i),(v_i,V_i)}(Y)\big)_{i\in I_\sigma}$ are contained in a 'collision class' which takes one of the subsequent forms for suitable parameter $r,v\in[0,1]$

(i) $M^N_{(0,6\delta_b^N),(0,6\delta_b^N)}(Y)$

(ii) $M^N_{(0,6\delta_b^N),(v,N^\sigma v)}(Y)$

(iii) $M^N_{(0,6\delta_b^N),(1,\infty)}(Y)$

(iv) $M^N_{(r,N^\sigma r),(0,6\delta_b^N)}(Y)$

(v) $M^N_{(r,N^\sigma r),(v,N^\sigma v)}(Y)$

(vi) $M^N_{(r,N^\sigma r),(1,\infty)}(Y)$,

except for $M^N_{(N^{-\sigma},\infty),(0,\infty)}(Y)$, which will be considered separately. Recall that the number of 'collision classes' belonging to the cover $|I_\sigma|$ is independent of N, analogously to Section 3.1.1. By comparing the possible values of r, R, v, and V with the estimates (3.47) and (3.48), it is evident that if $X\in\big(\mathcal{B}_{3,i}^{N,\sigma}\big)^C$ and $\sigma>0$ is chosen sufficiently small for the relevant terms, a set of type (ii), (iv), or (v) with $v=N^{-\sigma}$ or $r=N^{-\sigma}$ results in the 'worst-case scenario.' Consequently, the overall expression for Term (3.29) can be bounded as follows:

$$CN^{-\frac{5}{12}}. \tag{3.49}$$

The class where the previous general considerations can not be applied, $M^N_{(N^{-\sigma},\infty),(0,\infty)}(Y)$, the following holds:

$$\begin{aligned}
&\int_0^t \frac{1}{N}\sum_{\substack{j\neq i\\ j\in\mathcal{M}_b^N(X)}}\Big(\big|f^N([\Psi^{1,N}_{s,0}(X)]_j - [\Psi^{1,N}_{s,0}(X)]_i)\\
&- f^N(\varphi^{1,N}_{s,0}(X_j) - \varphi^{1,N}_{s,0}(X_i))\big|\Big)\mathbf{1}_{M^N_{(N^{-\sigma},\infty),(0,\infty)}(X_i)}(X_j)ds\\
\leq&\frac{2}{N}\sup_{Y\in M^N_{(N^{-\sigma},\infty),(0,\infty)}(X_i)}\int_0^t g^N(\varphi^{1,N}_{s,0}(Y)-\varphi^{1,N}_{s,0}(X_i))ds\sum_{\substack{j\neq i\\ j\in\mathcal{M}_b^N(X)}}\underbrace{\Delta_b^N(t,X)}_{\leq N^{-b_\delta}}
\end{aligned}$$

$$\leq \frac{2}{N}\big(T\frac{C}{(N^{-\sigma})^3}\big)N^{-b_\delta} \underbrace{|\mathcal{M}_b^N(X)|}_{\leq N^{2-2b_r-4b_v(1+\sigma)}}$$
$$\leq CN^{1-2b_r-4b_v-b_\delta+c\sigma}$$
$$\leq CN^{-\frac{13}{24}+\frac{3}{4}\sigma}$$

for $X \in \big(\mathcal{B}_{3,i}^{N,\sigma}\big)^C$ and $t \leq \tau^N(X)$.

Estimate of Term 3.28 (i good j superbad)

The estimates on Term 3.29 are quite similar to the previous one, except that now $j \in \mathcal{M}_{sb}^N(X)$.

We get for times $0 \leq t \leq \tau^N(X)$ the following r-depending estimate

$$\int_0^t \frac{1}{N} \sum_{\substack{j\neq i \\ j\in\mathcal{M}_{sb}^N(X)}} \Big(\big|f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i)$$

$$- f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\big|\Big)\mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)ds \tag{3.50}$$

$$\leq \frac{C}{N}\frac{1}{N^{-\beta}\Delta v} \sum_{\substack{j\neq i \\ j\in\mathcal{M}_s^N(X)}} \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)\mathbf{1}_{[0,6\delta_s^N]}(r) \tag{3.51}$$

$$+ \frac{C\Delta_{sb}^N(t,X)}{N\max(r,v)r^2} \sum_{\substack{j\neq i \\ j\in\mathcal{M}_{sb}^N(X)}} \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)\mathbf{1}_{[6\delta_s^N,\infty]}(r). \tag{3.52}$$

For the first summand, $r_s := N^{-\frac{1}{3}-\sigma}, v_s := N^{-\frac{5}{18}}$ and $\delta_s = N^{-\frac{1}{6}}$ and in view of the definition of $G^N(X_i)$ it follows that

$$\widetilde{M}^N_{(0,6\delta_s^N),(v,V)}(X_i) = M^N_{(0,6\delta_s^N),(v,V)}(X_i) \cap G^N(X_i) \subseteq \big(M^N_{6\delta_s^N,N^{-\frac{1}{6}}}(X_i)\big)^C,$$

were $N^{-\frac{1}{6}}$ is the velocity cut off of the bad particles not the superbad ones. This provides us the necessary implication

$$X_j \in \widetilde{M}^N_{(0,6\delta_b^N),(v,V)}(X_i)$$
$$\Rightarrow |\varphi^{2,N}_{t_{min},0}(X_j) - \varphi^{2,N}_{t_{min},0}(X_i)| \geq N^{-\frac{1}{6}} \tag{3.53}$$

where t_{min} shall denote a point in time where $|\varphi_{\cdot,0}^{1,N}(X_j) - \varphi_{\cdot,0}^{1,N}(X_i)|$ takes its minimum on $[0,T]$.

We derive an upper bound for Term 3.29 under the condition that

$$\sum_{j\in\mathcal{M}s^N(X)} \mathbf{1}_{M^N_{(r,R),(v,V)}(X_i)}(X_j) \leq N^{\frac{2\sigma}{9}}\big\lceil N^{\frac{2}{9}}R^2\min\big(\max(V,R),1\big)^4\big\rceil.$$

For the first summand we have for $R = \delta_s$

$$\frac{C}{N}\frac{1}{N^{-\beta}\Delta v} \sum_{\substack{j\neq i \\ j\in\mathcal{M}_{sb}^N(X)}} \mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)\mathbf{1}_{[0,6\delta_{sb}^N]}(r)$$

$$
\begin{aligned}
&\leq \frac{C}{N}\frac{R^2\min(V,1)^4|\mathcal{M}_{sb}|}{N^{-\beta+1}\max(v,N^{-\frac{1}{6}})}+\frac{CN^{\frac{2}{9}\sigma}}{N^{-\beta}\max(v,N^{-\frac{1}{6}})}\\
&\leq \frac{CR^2\min(V,1)^4|\mathcal{M}_{sb}|}{\max(N^{-\frac{1}{6}},v)N^{-\beta+1}}+\frac{CN^{\frac{2}{9}\sigma}}{N^{-\beta+1}\max(N^{-\frac{1}{6}},v)}\\
&\leq CN^{1+\beta-2s_r-4s_v-2s_\delta}\frac{\min(V,1)^4}{\max(N^{-\frac{1}{6}},v)}+N^{\frac{2}{9}\sigma+\frac{5}{12}-\sigma+\frac{1}{6}-1} \qquad (3.54)\\
&\leq CN^{-\frac{19}{12}}+CN^{-\frac{5}{12}} \qquad (3.55)
\end{aligned}
$$

Taking additionally into account that $\Delta_{sb}^N(t,X)\leq N^{s_\delta}=\delta_{sb}^N$ as well as $R=N^\sigma r$ for $r\geq 6\delta_{sb}^N$ (see (3.39)) it follows for the second term of (3.28) that

$$
\begin{aligned}
&\frac{C}{N}\frac{\Delta_{sb}^N(t,X)}{r^2\max(r,v)}\sum_{\substack{j\neq i\\ j\in\mathcal{M}_b^N(X)}}\mathbf{1}_{\widetilde{M}^N_{(r,R),(v,V)}(X_i)}(X_j)\\
\leq&\frac{C}{N}\Big(\frac{N^{-2s_r-4s_v+2+\sigma}R^2\min\big(\max(V,R),1\big)^4}{r^2\max(r,v)}+\frac{N^{\frac{2\sigma}{3}}}{r^2\max(r,v)}\Big)N^{-s_\delta}\\
\leq&C\Big(\frac{\min\big(\max(V,R),1\big)^4}{\max(r,v)}N^{1-2s_r-4s_v-s_\delta}+N^{c\sigma-1-s_\delta+b_v}\Big)\\
\leq&CN^{-\frac{17}{18}}+CN^{-1}. \qquad (3.56)
\end{aligned}
$$

The sum of Terms (3.55) and (3.56) forms an upper bound for Term (3.28) under the current assumption. All sets which belong to the family $\big(M^N_{(r_i,R_i),(v_i,V_i)}(Y)\big)_{i\in I_\sigma}$ are contained in a 'collision class' which takes one of the subsequent forms for suitable parameter $r,v\in[0,1]$

(i) $M^N_{(0,6\delta_s^N),(0,6\delta_s^N)}(Y)$

(ii) $M^N_{(0,6\delta_s^N),(v,N^\sigma v)}(Y)$

(iii) $M^N_{(0,6\delta_s^N),(1,\infty)}(Y)$

(iv) $M^N_{(r,N^\sigma r),(0,6\delta_s^N)}(Y)$

(v) $M^N_{(r,N^\sigma r),(v,N^\sigma v)}(Y)$

(vi) $M^N_{(r,N^\sigma r),(1,\infty)}(Y)$,

except for $M^N_{(N^{-\sigma},\infty),(0,\infty)}(Y)$, which will be discussed separately like in the previous section. A set of kind (ii), (iv) or (v) with $v=N^{-\sigma}$ or $r=N^{-\sigma}$ yields the 'worst case option' and thus in total Term (3.28) is bounded by

$$CN^{-\frac{5}{12}}\quad\text{if}\quad X\in\big(\mathcal{B}_{3,i}^{N,\sigma}\big)^C. \qquad (3.57)$$

For the last class $M^N_{(N^{-\sigma},\infty),(0,\infty)}(Y)$ the following holds

$$
\begin{aligned}
&\int_0^t\frac{1}{N}\sum_{\substack{j\neq i\\ j\in\mathcal{M}_{sb}^N(X)}}\Big(\big|f^N([\Psi_{s,0}^{1,N}(X)]_j-[\Psi_{s,0}^{1,N}(X)]_i)\\
&-f^N(\varphi_{s,0}^{1,N}(X_j)-\varphi_{s,0}^{1,N}(X_i))\big|\Big)\mathbf{1}_{M^N_{(N^{-\sigma},\infty),(0,\infty)}(X_i)}(X_j)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{2}{N} \sup_{Y \in M^N_{(N^{-\sigma},\infty),(0,\infty)}(X_i)} \int_0^t g^N(\varphi^{1,N}_{s,0}(Y) - \varphi^{1,N}_{s,0}(X_i))ds \sum_{\substack{j \neq i \\ j \in \mathcal{M}^N_{sb}(X)}} \underbrace{\Delta^N_s(t,X)}_{\leq N^{-s\delta}} \\
&\leq \frac{2}{N}\left(T\frac{C}{(N^{-\sigma})^3}\right)N^{-s_\delta} \underbrace{|\mathcal{M}^N_{sb}(X)|}_{\leq N^{\frac{2}{9}(1+\sigma)}} \\
&\leq C N^{-1+\frac{2}{9}-s_\delta+c\sigma} \\
&\leq C N^{-\frac{19}{18}+C\sigma},
\end{aligned}
\tag{3.58}
$$

for $X \in \left(\mathcal{B}^{N,\sigma}_{3sb,i}\right)^C$ and $t \leq \tau^N(X)$. This is distinctly smaller than necessary for small enough $\sigma > 0$ and concludes the estimates for Term (3.28).

Unlikely sets $\mathcal{B}^{N,\sigma}_{3b,i}$ and $\mathcal{B}^{N,\sigma}_{3s,i}$

It only remains to show that the probability related to the sets $\mathcal{B}^{N,\sigma}_{3b,i}$ and $\mathcal{B}^{N,\sigma}_{3s,i}$ is indeed small enough, i.e. that for any $\gamma > 0$ there exists a C_γ such that

$$
\begin{aligned}
\mathbb{P}\Big(&\sum_{j \in \mathcal{M}^N_b(X)} \mathbf{1}_{M^N_{R,V}(X_i)}(X_j) \geq N^{\frac{3\sigma}{4}} \lceil N^{\frac{3}{4}} R^2 \min\left(\max(R,V),1\right)^4 \rceil \\
&\vee |\mathcal{M}^N_b(X)| > N^{\frac{3}{4}(1+\sigma)}\Big) \leq C_\gamma N^{-\gamma}
\end{aligned}
$$

and analogously that for any $\eta > 0$ there exists a C_η such that

$$
\begin{aligned}
\mathbb{P}\Big(&\sum_{j \in \mathcal{M}^N_b(X)} \mathbf{1}_{M^N_{R,V}(X_i)}(X_j) \geq N^{\frac{2\sigma}{9}} \lceil N^{\frac{2}{2}} R^2 \min\left(\max(R,V),1\right)^4 \rceil \\
&\vee |\mathcal{M}^N_b(X)| > N^{\frac{2}{9}(1+\sigma)}\Big) \leq C_\eta N^{-\eta}
\end{aligned}
$$

The proof follows the same pattern as in [22] and is similar in both cases ('bad' and 'superbad'), so we confine ourselves to the proof in the bad particles case. For clarity, we define

$$
M := \lceil N^{\frac{3}{4}\sigma} \lceil N^{\frac{3}{4}} R^2 \min\left(\max(V,R),1\right)^4 \rceil \rceil.
$$

Recall that $j \in \mathcal{M}^N_b(X)$ implies that there is at least on $X_k \in \left(G^N(X_j)\right)^C$ for some $k \in \{1,...,N\} \setminus \{j\}$. We will see that for $R, V > 0$

$$
\sum_{j \in \mathcal{M}^N_b(X)} \mathbf{1}_{M^N_{R,V}(X_i)}(X_j) \geq M \tag{3.59}
$$

either implies that there exists a $j \in \{1,...,N\}$ such that

$$
\sum_{k=1}^{N} \mathbf{1}_{(G^N(X_j))^C}(X_k) \geq \lceil \frac{N^{\frac{\sigma}{4}}}{2} \rceil) \tag{3.60}
$$

or there exists a set $\mathcal{S} \subseteq \{1, ..., N\}^2 \setminus \bigcup_{n=1}^{N} \{(n, n)\}$ with the following properties

$$\begin{aligned}
&\text{(i)} \quad |\mathcal{S}| = \lceil \frac{N^{-\frac{\sigma}{4}} M}{2} \rceil \\
&\text{(ii)} \quad \forall (j,k) \in \mathcal{S} : X_j \in (G^N(X_k))^C \cap M^N_{R,V}(X_i) \\
&\text{(iii)} \ (j_1, k_1), (j_2, k_2) \in \mathcal{S} \Rightarrow \{j_1, k_1\} \cap \{j_2, k_2\} = \emptyset. \qquad (3.61)
\end{aligned}$$

In the proof of this implication we will name the event $X_m \in M^N_{R,V}(X_n)$ by the phrase 'collision between particles m, n' and the phrase 'hard collision between particles m, n' will be applied synonymously to the event $X_m \in (G(X_n))^C$. Note that if assumption (3.60) is not fulfilled, it implies that a given 'bad' particle can have at least $\lceil \frac{N^{\frac{\sigma}{4}}}{2} \rceil$ 'hard collisions' with different particles. Such a 'bad' particle can, 'infect' not more than $\lceil \frac{N^{\frac{\sigma}{4}}}{2} \rceil$ other particles, causing them to be included in the set $\mathcal{M}^N_b(X)$.

For the following considerations we stick to this case and we will see that under this constraint the relation (3.59), i.e.

$$\sum_{j \in \mathcal{M}^N_b(X)} \mathbf{1}_{M^N_{R,V}(X_i)}(X_j) \geq M$$

implies that the event related to (3.61) is fulfilled.

In this case there is a set $\mathcal{C}_0 \subseteq \mathcal{M}^N_b(X)$ of 'bad' particles which have 'collisions' with the particle i. By assumption (3.59) we have $|\mathcal{C}_0| \geq M$ and as the event related to (3.60) does not occur, there are at most $\lfloor \frac{N^{\frac{\sigma}{4}}}{2} \rfloor$ particles having a 'hard collision' with particle i. We construct a new set $\mathcal{C}_1 \subseteq \mathcal{C}_0$ by 'detaching' all of these at most $\lfloor \frac{N^{\frac{\sigma}{4}}}{2} \rfloor$ particles, which are possibly contained in $\mathcal{C}_0$, and it obviously holds that

$$|\mathcal{C}_1| \geq M - \lfloor \frac{N^{\frac{\sigma}{4}}}{2} \rfloor \geq 1,$$

for N large enough. Similarly we take one of these remaining 'bad' particles j_1 out of $\mathcal{C}_1$ and since $j_1 \in \mathcal{C}_1 \subseteq \mathcal{C}_0 \subseteq \mathcal{M}^N_b(X)$, there must be at least one further particle having a 'hard collision' with j_1. By construction of $\mathcal{C}_1$ this can not be i, so lets call it k_1. This gets us our first tuple (j_1, k_1) which fulfils condition (ii) of the set $\mathcal{S}$ appearing in (3.61). In a next step we 'detach' j_1 and k_1 and all of their at most $2\lfloor \frac{N^{\frac{\sigma}{4}}}{2} \rfloor - 2$ remaining 'hard collision partners' from $\mathcal{C}_1$ to obtain a new set $\mathcal{C}_2 \subseteq \mathcal{C}_1$. This gives us an iteration process (provided that $\mathcal{C}_2 \neq \emptyset$) by choosing the next particle j_2 out of $\mathcal{C}_2$ and afterwards an arbitrary one of its 'hard collision partners' k_2. Then the next round can start after having removed j_2 and k_2 as well as their remaining 'hard collision partners' from $\mathcal{C}_2$ to obtain $\mathcal{C}_3 \subseteq \mathcal{C}_2$. By construction after each round of this process at most $2\lfloor \frac{N^{\frac{\sigma}{4}}}{2} \rfloor$ 'particle labels' are removed from the set $\mathcal{C}_k$ to obtain $\mathcal{C}_{k+1}$. Considering that $M \geq N^{\frac{3\sigma}{4}}$, we can reiterate this procedure at least

$$\lceil \frac{M - \lfloor \frac{N^{\frac{\sigma}{4}}}{2} \rfloor}{N^{\frac{\sigma}{4}}} \rceil \geq \lceil \frac{N^{-\frac{\sigma}{4}} M}{2} \rceil$$

times. The removal of the 'hard collision partners' of the occurring tuples after each round ensures that condition (iii) is fulfilled and thus this provides us a set $\mathcal{S}$ consisting of tuples (j_i, k_i) like claimed in (3.61).

Due to this considerations we can determine an upper bound for the probability $\mathbb{P}(X \in \mathcal{B}^{N,\sigma}_{3b,i})$. Starting with assumption (3.61) we abbreviate

$$M_1 := \lceil \frac{N^{-\frac{\sigma}{4}} M}{2} \rceil \text{ with } M = \lceil N^{\frac{3}{4}\sigma} \lceil N^{\frac{3}{4}} R^2 \min\big(\max(V,R),1\big)^4 \rceil \rceil.$$

There are less than $\binom{N^2}{K}$ different possibilities to choose K 'disjoint' (condition (iii) of (3.61) is fulfilled) pairs (j,k) belonging to $\{1,...,N\}^2 \setminus \bigcup_{n=1}^N \{(n,n)\}$. Application of this, Lemma 3.3 and $\sup_{Y \in \mathbb{R}^6} \mathbb{P}\big(X_1 \in (G^N(Y))^C\big) \leq CN^{-\frac{11}{6}-2\sigma}$ yields that the probability of the existence of a set S satisfying the three conditions in 3.61 is small for large N, i.e.

$$\begin{aligned}
&\mathbb{P}\Big(\exists \mathcal{S} \subseteq \{1,...,N\}^2 \setminus \bigcup_{n=1}^N \{(n,n)\} : |\mathcal{S}| = M_1 \wedge \\
&\qquad \big(\forall (j,k) \in \mathcal{S} : X_j \in (G^N(X_k))^C \cap M^N_{R,V}(X_i)\big) \wedge \\
&\qquad \big((j_1,k_1),(j_2,k_2) \in \mathcal{S} \Rightarrow \{j_1,k_1\} \cap \{j_2,k_2\} = \emptyset\big)\Big) \\
\leq& \binom{N^2}{M_1} \mathbb{P}\Big(\forall (j,k) \in \{(2,3),(4,5),...,(2M_1,2M_1+1)\} : \\
&\qquad\qquad X_j \in (G^N(X_k))^C \cap M^N_{R,V}(X_1)\Big) \\
\leq& \frac{N^{2M_1}}{M_1!} \Big(\sup_{Y \in \mathbb{R}^6} \mathbb{P}\big(X \in (G^N(Y))^C\big) \sup_{Z \in \mathbb{R}^6} \mathbb{P}\big(X \in M^N_{R,V}(Z)\big)\Big)^{M_1} \\
\leq& C^{M_1} \frac{N^{2M_1}}{M_1^{M_1}} \big(N^{-\frac{11}{6}-2\sigma}\big)^{M_1} \Big(R^2 \min\big(\max(V,R),1\big)^4\Big)^{M_1} \\
\leq& (CN^{-\frac{5\sigma}{4}})^{\frac{N^{\frac{\sigma}{4}}}{2}},
\end{aligned} \tag{3.62}$$

since $M_1 \geq \frac{N^{\frac{\sigma}{4}}}{2}$ for

$$M_1 = \lceil \frac{N^{-\frac{\sigma}{4}}}{2} M \rceil \text{ with } M = \lceil N^{\frac{3}{4}\sigma} \lceil N^{\frac{3}{4}} R^2 \min\big(\max(V,R),1\big)^4 \rceil \rceil.$$

For any class which appears in $\big(M^N_{(r_i,R_i),(v_i,V_i)}(Y)\big)_{i \in I_\delta}$ where $R_l \neq \infty$ this probability decays distinctly faster than necessary.

To prove that $\sum_{k \in \mathcal{M}^N_b(X)} 1 \leq N^{\frac{3}{4}(1+\sigma)}$ we can also apply the considerations from above by setting the collision class parameters R, V to infinity and thus we obtain the event $\mathbf{1}_{M^N_{\infty,\infty}(X_i)}(X_j) = 1$. In the case $M_1 := \lceil \frac{N^{\frac{3}{4}+\frac{\sigma}{4}}}{2} \rceil$ and $\mathbb{P}\big(X_1 \in M^N_{R,V}(Y)\big) = 1$. Applying the above procedure, we get

$$\mathbb{P}\big(\sum_{k \in \mathcal{M}^N_b(X)} 1 \leq N^{\frac{3}{4}(1+\sigma)}\big) \leq CN^{-\sigma N^{\frac{3}{4}}}$$

which is small enough. Now, let's proceed with the considerations regarding assumption (3.60). Therefore we abbreviate $M_2 := \lceil \frac{N^{\frac{\sigma}{4}}}{2} \rceil$ and estimate

$$\mathbb{P}\Big(X \in \mathbb{R}^{6N} : \big(\exists j \in \{1,...,N\} : \sum_{k \neq j} \mathbf{1}_{(G^N(X_j))^C}(X_k) \geq M_2\big)\Big)$$

$$\begin{aligned}
&\le N\mathbb{P}\Big(X\in\mathbb{R}^{6N}:\sum_{k=2}^{N}\mathbf{1}_{(G^N(X_1))^C}(X_k)\ge M_2\Big)\\
&\le N\binom{N}{M_2}\sup_{Y\in\mathbb{R}^6}\mathbb{P}\big(Z\in\mathbb{R}^6: Z\in(G^N(Y))^C\big)^{M_2}\\
&\le N\frac{N^{M_2}}{M_2!}\big(CN^{-\frac{11}{6}-2\sigma}\big)^{M_2}\\
&\le CN^{-\frac{1}{4}\lceil\frac{N^{\frac{\sigma}{4}}}{2}\rceil},
\end{aligned}\tag{3.63}$$

which decreases fast enough as N increases. In total we obtain as desired

$$\begin{aligned}
&\mathbb{P}\big(X\in\mathcal{B}^{N,\sigma}_{3b,i}\big)\\
\le&|I_\sigma|\sup_{R,V>0}\mathbb{P}\Big(\sum_{j\in\mathcal{M}^N_b(X)}\mathbf{1}_{M^N_{R,V}(X_i)}(X_j)\ge N^{\frac{3\sigma}{4}}\big\lceil N^{\frac{3}{4}}R^2\min\big(\max(R,V),1\big)^4\big\rceil\Big)\\
&+\mathbb{P}\Big(\sum_{k\in\mathcal{M}^N_b(X)}1\ge N^{\frac{3}{4}(1+\sigma)}\Big)\\
\le&(CN^{-\frac{5\sigma}{4}})^{\frac{N^{\frac{\sigma}{4}}}{2}}
\end{aligned}\tag{3.64}$$

Similarly we can show that the probability related to the set $\mathcal{B}^{N,\sigma}_{3s,i}$ in the superbad particle case is indeed small enough. A similar estimate holds for the superbad particles

$$\begin{aligned}
&\mathbb{P}\big(X\in\mathcal{B}^{N,\sigma}_{3s,i}\big)\\
\le&|I_\sigma|\sup_{R,V>0}\mathbb{P}\Big(\sum_{j\in\mathcal{M}^N_b(X)}\mathbf{1}_{M^N_{R,V}(X_i)}(X_j)\ge N^{\frac{2\sigma}{9}}\big\lceil N^{\frac{2}{9}}R^2\min\big(\max(R,V),1\big)^4\big\rceil\Big)\\
&+\mathbb{P}\Big(\sum_{k\in\mathcal{M}^N_b(X)}1\ge N^{\frac{2}{9}(1+\sigma)}\Big)\\
\le&(CN^{-\frac{16\sigma}{9}})^{\frac{N^{\frac{\sigma}{9}}}{2}}
\end{aligned}\tag{3.65}$$

Estimate of Term 3.30 (i good j good)

Now we are left with the last Term 3.30 which measures the fluctuation between two good particles. To estimate the term we identify

$$v^N_{min}=N^{b_v}=N^{-\frac{1}{6}},$$

since $i,j\in\mathcal{M}^N_g(X)$. To estimate the term we apply Corollary 3.8 and subdivide the term depending on the relative velocity of the particles so that the first term deals with collisions where the relative velocity is below order $N^{-\frac{1}{9}+3\sigma}$ and the second deals with the rest. The choice of the value is more or less random as long as the equations stay small. Corollary 3.8 (ii) is applicable since the relative velocity values for the considered ‘collision classes’ are of distinctly larger order than the deviation between corresponding particle trajectories of the microscopic and the auxiliary system. Note that $G^N(X_i) \subseteq M(^N_{6\delta_b^N, v_{min}^N}(X_i))^c$ where $\delta_b^N = N^{-\frac{7}{24}-\sigma}$ and

$$\max_{i \in \mathcal{M}_g^N(X)} \sup_{0 \le s \le \tau^N(X)} |[\Psi_{s,0}^N(X)]_i - \varphi_{s,0}^N(X_i)| \le N^{-\frac{5}{12}+\sigma} = N^{-\frac{1}{4}+\sigma} v_{min}^N.$$

Thus Term 3.30 is bounded by

$$\begin{aligned}
&\int_0^t \Big(\frac{1}{N} \sum_{\substack{j \neq i \\ j \in \mathcal{M}_g^N(X)}} \Big(\big| f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i) \big| \\
&+ \big| f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i)) \big| \Big) \mathbf{1}_{G^N(X_i) \cap M^N_{3N^{-\frac{1}{2}+\sigma}, \infty}(X_i)}(X_j) \Big) ds \\
\le & \frac{C}{N} \frac{1}{N^{-\beta} v_{min}^N} \sum_{j \neq i} \mathbf{1}_{G^N(X_i) \cap M^N_{3N^{-\frac{1}{2}+\sigma}, N^{-\frac{1}{9}+3\sigma}}(X_i)}(X_j) \\
&+ \frac{C}{N} \frac{1}{N^{-\beta} N^{-\frac{1}{9}+3\sigma}} \sum_{j \neq i} \mathbf{1}_{G^N(X_i) \cap M^N_{3N^{-\frac{1}{2}+\sigma}, \infty}(X_i)}(X_j).
\end{aligned} \tag{3.66}$$

This stays sufficiently small since the concerned sets are very unlikely. To prove this we define

$$\begin{aligned}
&X \in \mathcal{B}_{4,i}^{N,\sigma} \subseteq \mathbb{R}^{6N} \\
\Leftrightarrow & \sum_{j \neq i} \mathbf{1}_{M^N_{6N^{-\frac{1}{2}+\sigma}, N^{-\frac{1}{9}+3\sigma}}(X_i)}(X_j) \ge N^{\frac{\sigma}{2}} \wedge \\
& \sum_{j \neq i} \mathbf{1}_{M^N_{6N^{-\frac{1}{2}+\sigma}, \infty}(X_i)}(X_j) \ge N^{3\sigma}
\end{aligned}. \tag{3.67}$$

Set $M_1 := \lceil N^{\frac{\sigma}{2}} \rceil$ and $M_2 := \lceil N^{3\sigma} \rceil$. By the same proof as applied in (3.63) and application of Lemma 3.3 we can estimate the probability

$$\begin{aligned}
&\mathbb{P}\big(X \in \mathcal{B}_{4,i}^{N,\sigma}\big) \\
\le & \frac{N^{M_1}}{M_1!} \sup_{Y \in \mathbb{R}^6} \mathbb{P}\big(X_i \in M^N_{6N^{-\frac{1}{2}+\sigma}, N^{-\frac{1}{9}+3\sigma}}(Y)\big)^{M_1} \\
&+ \frac{N^{M_2}}{M_2!} \sup_{Y \in \mathbb{R}^6} \mathbb{P}\big(X_i \in M^N_{6N^{-\frac{1}{2}+\sigma}, \infty}(Y)\big)^{M_2} \\
\le & \frac{(CN)^{M_1}}{M_1!} \big(N^{2(-\frac{1}{2}+\sigma)}\big)^{M_1} \big(N^{4(-\frac{1}{9}+3\sigma)}\big)^{M_1} + C^{M_2} \frac{N^{M_2}}{(N^{3\sigma})^{M_2}} \big(N^{2(-\frac{1}{2}+\sigma)}\big)^{M_2} \\
\le & C\big(N^{-\frac{4}{9}+14\sigma}\big)^{N^{\frac{\sigma}{2}}} + \big(CN^{-\sigma}\big)^{N^{3\sigma}},
\end{aligned} \tag{3.68}$$

which for $\sigma > 0$ small enough decreases fast enough.

Due to our estimates it holds for $X \in (\mathcal{B}_{4,i}^{N,\sigma})^C$ that Term (3.66), and thereby Term (3.30), is bounded by

$$\frac{C}{N} \frac{1}{N^{-\beta} v_b^N} N^{\frac{\sigma}{2}} + \frac{C}{N} \frac{1}{N^{-\beta} N^{-\frac{1}{9}+3\sigma}} N^{3\sigma}$$

$$\leq CN^{-\frac{5}{12}-\frac{\sigma}{2}} + CN^{-\frac{17}{36}} \leq CN^{-\frac{5}{12}} \tag{3.69}$$

Due to the previous probability estimates on the unlikely sets it easily follows that for small enough $\sigma > 0$ and an arbitrary $\gamma > 0$ there is a constant $C > 0$ such that

$$\mathbb{P}(\bigcup_{j\in\{1,2,3,4\}} \bigcup_{i=1}^{N} \mathcal{B}_{j,i}^{N,\sigma}) \leq CN^{-\gamma}.$$

Conclusion for case 1 (labelled particle X_i is good)

For $i \in \mathcal{M}_g^N(X)$ we determined an upper bound for the term

$$|\int_{t_1}^{t} \frac{1}{N} \sum_{j\neq i} f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j) - f^N * \widetilde{k}_s^N(\varphi_{s,0}^{1,N}(X_i))ds|$$

which is given by the sum of bounds of the four Terms (3.11), (3.12), (3.13) and (3.14). We restrict ourselves to the configurations $X \in \left(\bigcup_{j\in\{1,2,3,4\}} \bigcup_{i=1}^{N} \mathcal{B}_{j,i}^{N,\sigma}\right)^C$ and all upper bounds hold for any times $t_1, t \in [0, \tau^N(X)]$. For a suitable constant $C > 0$, $CN^{-\frac{5}{12}}$ dominates all of these upper bounds except for Term (3.31). But for our underlying configurations it holds for any $i \in \{1, ..., N\}$ and times $t_1, t \in [0, T]$ that

$$\begin{aligned}
&|\int_{t_1}^{t} \Big(\frac{1}{N} \sum_{j=1}^{N} g^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\mathbf{1}_{G^N(X_i)}(X_j) \\
&- \int_{\mathbb{R}^6} g^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(X_i))\mathbf{1}_{G^N(X_i)}(Y)k_0(Y)d^6Y\Big)ds\Big| \leq 1.
\end{aligned}$$

By the definition of $\mathcal{B}_{2,i}^{N,\sigma}$ and thus for $N > 1$ and $t_1 \leq t$ we receive

$$\begin{aligned}
&\int_{t_1}^{t} \frac{1}{N} \sum_{j=1}^{N} g^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\mathbf{1}_{G^N(X_i)}(X_j)ds \\
\leq& 1 + \int_{t_1}^{t} \int_{\mathbb{R}^6} g^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(X_i))\mathbf{1}_{G^N(X_i)}(Y)k_0(Y)d^6Y ds \\
\leq& 1 + C\ln(N)(t - t_1).
\end{aligned} \tag{3.70}$$

We used the fact that for $N > 1$

$$\begin{aligned}
&\sup_{t_1\leq s\leq t} \int_{\mathbb{R}^6} g^N(\varphi_{s,0}^{1,N}(Y) - \varphi_{s,0}^{1,N}(X_i))\mathbf{1}_{G^N(X_i)}(Y)k_0(Y)d^6Y \\
\leq& C \sup_{t_1\leq s\leq t} \int_{\mathbb{R}^3} \min\big(N^{3\beta}, \frac{1}{|Y - \varphi_{s,0}^{1,N}(X_i)|^3}\big)\widetilde{k}_s^N(Y)d^3Y \\
\leq& C\ln(N)
\end{aligned}$$

holds. This leads us in particular for times $t_1 \leq t$ to

$$\Delta_g^N(t, X) \leq \Delta_g^N(t_1, X) + \int_{t_1}^{t} \delta_g^N(s, X)ds,$$

with the common abbreviations

$$\begin{aligned}\delta_g^N(t,X) &:= \max_{i\in\mathcal{M}_g^N(X)} |[\Psi_{t,0}^{2,N}(X)]_i - \varphi_{t,0}^{2,N}(X_i)|,\\ \Delta_g^N(t,X) &:= \max_{i\in\mathcal{M}_g^N(X)} \sup_{0\le s\le t} |[\Psi_{s,0}^{1,N}(X)]_i - \varphi_{s,0}^{1,N}(X_i)|.\end{aligned} \tag{3.71}$$

By choosing the subsequent sequence of time steps $t^* := t_{n+1} - t_n = \frac{C}{\sqrt{\ln(N)}}$ for some constant $C > 0$ with

$$\begin{aligned}&t_n = n\frac{C}{\sqrt{\ln(N)}} \text{ for } n \in \{0, ..., \lceil \frac{\sqrt{\ln(N)}}{C}\tau^N(X)\rceil - 1\},\\ &t_{\lceil \frac{\sqrt{\ln(N)}}{C}\tau^N(X)\rceil} = \tau^N(X)\end{aligned}$$

the previous relation implies that for $t_n \le t \le \tau^N(X)$

$$\Delta_g^N(t,X) \le \sum_{k=1}^{n} \sup_{0\le s\le t_k} \delta_g^N(s,X)t^* + \int_{t_n}^{t} \delta_g^N(s,X)ds. \tag{3.72}$$

It follows that for any 'good' particle $i \in \mathcal{M}_g^N(X)$, the considered configurations and for all times $t \in [t_n, t_{n+1}]$, where $n \in \{0, ..., \lceil \frac{\sqrt{\ln(N)}}{C}\tau^N(X)\rceil - 1\}$ the following inequality holds

$$\begin{aligned}&\delta_g^N(t,X)\\ \le&\delta_g^N(t_n,X) + \max_{i\in\mathcal{M}_g^N(X)} \Big| \int_{t_n}^{t} \Big(\frac{1}{N}\sum_{j\neq i} f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j)\\ &- f^N * \widetilde{k}_s^N(\varphi_{s,0}^{1,N}(X_i))\Big)ds\Big|\\ \le&\max_{i\in\{1,...,N\}} \int_{t_n}^{t} \frac{2}{N}\sum_{j=1}^{N} g^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(X_i))\mathbf{1}_{G^N(X_i)}(X_j)\underbrace{\Delta_g^N(s,X)}_{\le\Delta_g^N(t,X)}ds\\ &+ \delta_g^N(t_n,X) + CN^{-\frac{5}{12}}\\ \le&\big(1 + C\ln(N)\underbrace{(t-t_n)}_{\le t_{n+1}-t_n=t^*}\big)\Big(\sum_{k=1}^{n}\sup_{0\le s\le t_k}\delta_g^N(s,X)t^* + \int_{t_n}^{t}\delta_g^N(r,X)dr\Big)\\ &+ \delta_g^N(t_n,X) + CN^{-\frac{5}{12}}\\ \le&\big(1 + C\ln(N)t^*\big)\int_{t_n}^{t}\delta_g^N(r,X)dr\\ &+ \big(2 + C\ln(N)(t^*)^2\big)\sum_{k=1}^{n}\sup_{0\le s\le t_k}\delta_g^N(s,X) + CN^{-\frac{5}{12}}.\end{aligned} \tag{3.73}$$

Application of Gronwall's Lemma implies that for all times $t \in [t_n, t_{n+1}]$ it holds that

$$\delta_g^N(t,X)$$

$$\leq\Big(\big(2+C\ln(N)(t^*)^2\big)\sum_{k=1}^{n}\sup_{0\leq s\leq t_k}\delta_g^N(s,X)+CN^{-\frac{5}{12}}\Big)e^{t^*+C\ln(N)(t^*)^2}. \tag{3.74}$$

Especially for $t\in[0,t_n]$, we can exchange the left-hand side by its supremum over $[0,t_{n+1}]$. For $t^*=\frac{C_1}{\sqrt{\ln(N)}}$ with $C_1:=\min\big(\frac{1}{\sqrt{C}},1\big)$ the previous relation implies

$$\sup_{0\leq s\leq t_{n+1}}\delta_g^N(s,X)\leq 3e^2\sum_{k=1}^{n}\sup_{0\leq s\leq t_k}\delta_g^N(s,X)+Ce^2N^{-\frac{5}{12}}. \tag{3.75}$$

Due to this relation it follows for $n\in\{1,...,\lceil\frac{\sqrt{\ln(N)}}{C_1}\tau^N(X)\rceil\}$ that

$$\sup_{0\leq s\leq t_n}\delta_g^N(s,X)\leq Ce^2N^{-\frac{5}{12}}(3e^2+1)^{n-1}. \tag{3.76}$$

For $n=1$ the relation is obvious due to (3.75) and if it holds for $k\in\{1,...,n\}$, $n\in\mathbb{N}$, where we fix the constant C for these estimates, then we obtain that

$$\begin{aligned}&\sup_{0\leq s\leq t_{n+1}}\delta_g^N(s,X)\\ \leq&3e^2\sum_{k=1}^{n}\underbrace{\sup_{0\leq s\leq t_k}\delta_g^N(s,X)}_{\leq Ce^2N^{-\frac{5}{12}}(3e^2+1)^{k-1}}+Ce^2N^{-\frac{5}{12}}\\ \leq&3e^2\big(Ce^2N^{-\frac{5}{12}}\frac{(3e^2+1)^n-1}{(3e^2+1)-1}\big)+Ce^2N^{-\frac{5}{12}}\\ =&Ce^2N^{-\frac{5}{12}}(3e^2+1)^n.\end{aligned}$$

This confirms the claim and it follows that

$$\begin{aligned}\sup_{0\leq s\leq\tau^N(X)}\delta_g^N(s,X)\leq&Ce^2N^{-\frac{5}{12}}(3e^2+1)^{\lceil\frac{\sqrt{\ln(N)}}{C_1}\tau^N(X)\rceil-1}\\ \leq&Ce^2N^{-\frac{5}{12}}N^{\frac{\ln(3e^2+1)}{\ln(N)}\frac{\sqrt{\ln(N)}}{C_1}T}\\ \leq&CN^{-\frac{5}{12}+\frac{\sigma}{2}},\end{aligned} \tag{3.77}$$

for N large enough. The received upper bound for the velocity deviation implies that

$$\max_{i\in\mathcal{M}_g^N(X)}\sup_{0\leq s\leq\tau^N(X)}|[\Psi_{s,0}^N(X)]_i-\varphi_{s,0}^N(X_i)|\leq CN^{-\frac{5}{12}+\frac{\sigma}{2}}, \tag{3.78}$$

which is smaller than necessary since $CN^{-\frac{5}{12}+\frac{\sigma}{2}}<N^{-\frac{5}{12}+\sigma}$ for $\sigma>0$ and N large enough.

3.1.2 Controlling the deviation of the bad and superbad particles

Most estimates for the second part can be applied analogously, except that we allow more distance of the observed ‘bad’ or ‘superbad’ particle to its mean-field partner, since $\delta_s = N^{-\frac{1}{6}-\sigma} > N^{-\frac{5}{12}+\sigma}$ and $\delta_b = N^{-\frac{7}{24}-\sigma} > N^{-\frac{5}{12}+\sigma}$. For the 'good' particle this distance is of the same order as the cut-off radius. The vast majority of particles is typically 'good', so we have control over the 'collision partners' in most cases. By the definition of the distance, the considered 'bad' or 'superbad' particle is inside a ball of radius $N^{-\frac{1}{6}-\sigma}$ or respectively $N^{-\frac{7}{24}-\sigma}$ around its related mean-field particle. To circumvent this problem, we define a cloud of auxiliary 'mean-field particles' around the 'bad' or 'superbad' particle, like proposed in [22]. 'Hard' or 'Superhard' collisions might cause that the observed particle departs too far from its initially corresponding mean-field particle, that propagates homogeneously in time. Phillip was able to show that for any point in time, we can find an auxiliary particle around the 'bad' or 'superbad' particle with a disance small than the cut-off. By exchanging the these particles, we can copy the estimates from Section 3.1.1.

To ensure that we can apply Theorem 3.10, we have to introduce a 'cloud' of auxiliary particles instead of a single one when needed, because the introduced auxiliary particle would depend on the whole configuration and thus be correlated with the remaining particles and we would loose the big advantage of the 'mean-field particle'. If we propagate the whole 'cloud' from the beginning at the time of a 'hard collision' for a certain particle the initial positions of the related auxiliary particles are chosen independently of the remaining configuration. We will show that all of the auxiliary particles which belong to the small 'cloud' fulfill corresponding demands with high probability like in the previous situation, where we could show for typical initial data that the related mean-field particles fulfill properties which made it possible to prove that the effective and the microscopic dynamics are usually close. In the upcoming part we will end up in a very similar situation as in Section 3.1.1 and we will benefit from the proof techniques of the previous chapter.

3.1.3 Controlling the deviation of the superbad particles

To create the particle cloud we first define

$$Q_N := \{-\lceil N^{\frac{1}{4}} \rceil, ..., -1, 0, 1, ..., \lceil N^{\frac{1}{4}} \rceil\}^6 \tag{3.79}$$

and for $(k_1, ..., k_6) \in Q_N$ the positions or the initial data of the auxiliary particles $X^i_{k_1,...,k_6} := X_i + \sum_{j=1}^6 k_j N^{-\frac{5}{12}+\frac{\sigma}{2}} e_j$, where e_j, $j \in \{1, ..., 6\}$ is the j-th basis vector of $\mathbb{R}^6$. According to Lemma 3.7, which ensures that the distance between mean-field particles stays of the same order, and $\delta_s = N^{-\frac{1}{6}-\sigma}$ for $t \le \tau$, it holds for arbitrary $t_1 \in [0, \tau^N(X)]$ and large enough N that

$$|\varphi^N_{0,t_1}([\Psi^N_{t_1,0}(X)]_i) - X_i| \le C|[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(X_i)| < CN^{-\frac{1}{6}-\sigma} \le N^{-\frac{1}{6}}. \tag{3.80}$$

It is always possible to find a tuple $(k_1, ..., k_6) \in Q_N$ for N large enough such that

$$|\varphi^N_{0,t_1}([\Psi^N_{t_1,0}(X)]_i) - X^i_{k_1,...,k_6}| \le \frac{\sqrt{6}}{2} N^{-\frac{5}{12}+\frac{\sigma}{2}} \tag{3.81}$$

since (3.80) is of smaller order with respect to N than the diameter of the auxiliary 'particle cloud' around X_i. Lemma 3.7 implies in turn that

$$|[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(X^i_{k_1,...,k_6})| \le CN^{-\frac{5}{12}+\frac{\sigma}{2}}. \tag{3.82}$$

If we choose $N \in \mathbb{N}$ large enough such that $CN^{-\frac{5}{12}+\frac{\sigma}{2}} < \frac{1}{2}N^{-\frac{5}{12}+\sigma}$ and $\sigma > 0$ sufficiently small, then there exists a further point in time $t_2 \in (t_1, T]$ such that not only

$$\sup_{s\in[t_1,t_2]} |[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(X^i_{k_1,\dots,k_6})| \leq N^{-\frac{5}{12}+\sigma}$$

holds, but also the following bound for the velocity deviation

$$\sup_{s\in[t_1,t_2]} |[\Psi^{2,N}_{s,0}(X)]_i - {}^2\varphi^{N}_{s,0}(X^i_{k_1,\dots,k_6})| \leq N^{-\frac{1}{6}-\sigma}.$$

Now we have a sufficiently good approximation for the trajectory of real particle, given by the trajectory of the auxiliary particle with initial datum $X^i_{k_1,\dots,k_6}$ for this time span. We apply this to prove that

$$\begin{aligned} &\sup_{t_1\leq s\leq t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \\ &\leq \sup_{t_1\leq s\leq t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X^i_{k_1,\dots,k_6})| + \sup_{t_1\leq s\leq t} |\varphi^N_{s,0}(X^i_{k_1,\dots,k_6}) - \varphi^N_{s,0}(X_i)| \end{aligned} \tag{3.83}$$

grows slow enough on this interval. The considerations for the first term is mostly analogous to the estimates of case 1, see Section 3.1.1, because the spatial distance between the considered auxiliary particle and the 'real' particle is bounded by $N^{-\frac{5}{12}+\sigma}$ like the largest allowed deviation for a 'good' particle.

From now on we will assume that for an arbitrary point in time $t_1 \in [0, \tau^N(X))$ and $X \in \mathbb{R}^{6N}$ the initial position of the auxiliary particle $X^i_{k_1,\dots,k_6}$ and $t_2 \in (t_1, \tau^N(X)]$ are chosen such that the previously introduced demands are fulfilled on $[t_1, t_2]$. Following the notation of [22] we abbreviate $\widetilde{X}_i := X^i_{k_1,\dots,k_6}$ but remind that t_2 and the choice of $(k_1, \dots, k_6) \in Q_N$ depends on i, t_1 and X.
Controlling the growth of the second term is a simple application of Lemma 3.7. It follows for arbitrary $t \in [t_1, t_2]$ that

$$\begin{aligned} &|\varphi^N_{t,0}(\widetilde{X}_i) - \varphi^N_{t,0}(X_i)| \\ \leq &e^{C(t-t_1)} |\varphi^N_{t_1,0}(\widetilde{X}_i) - \varphi^N_{t_1,0}(X_i)| \\ \leq &e^{C(t-t_1)} \big(|\varphi^N_{t_1,0}(X_i) - [\Psi^N_{t_1,0}(X)]_i| + |[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(\widetilde{X}_i)| \big) \\ \leq &e^{C(t-t_1)} \big(|\varphi^N_{t_1,0}(X_i) - [\Psi^N_{t_1,0}(X)]_i| + N^{-\frac{5}{12}+\sigma} \big), \end{aligned} \tag{3.84}$$

where we applied bound (3.80) according to the choice of $\widetilde{X}_i$. This concludes the estimates for this term and we will return to it at the end of this subsection after estimating Term (3.88), Term (3.89) and Term (3.87).
For the second term we first remark that

$$\begin{aligned} &|[{}^2\Psi^N_{t,0}(X)]_i - {}^2\varphi^N_{t,0}(\widetilde{X}_i)| \\ \leq &|[{}^2\Psi^N_{t_1,0}(X)]_i - {}^2\varphi^N_{t_1,0}(\widetilde{X}_i)| \\ &+ \Big| \int_{t_1}^t \Big(\frac{1}{N} \sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j) - f^N * \widetilde{k}_s(\varphi^{1,N}_{s,0}(\widetilde{X}_i)) \Big) ds \Big|. \end{aligned} \tag{3.85}$$

To derive an upper bound for the force term, note that the same structure as in the previous case. Thus we can again apply multiple times triangle inequality and obtain essentially the four terms of case 1, see Section 3.1.1,

$$|\int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j) - f^N * \widetilde{k}_s(\varphi_{s,0}^{1,N}(\widetilde{X}_i))ds| \tag{3.86}$$

$$\leq |\int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j)\mathbb{1}_{(G^N(\widetilde{X}_i))^C}(X_j)ds| \tag{3.87}$$

$$+ |\int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} \Big(f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j)\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)$$
$$- f^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - \varphi_{s,0}^{1,N}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big) ds| \tag{3.88}$$

$$+ |\int_{t_1}^{t} \frac{1}{N} \sum_{j \neq i} f^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - \varphi_{s,0}^{1,N}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)ds$$
$$- \int_{t_1}^{t} \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - \varphi_{s,0}^{1,N}(Y))\mathbb{1}_{G^N(\widetilde{X}_i)}(Y)k_0(Y)d^6Y ds| \tag{3.89}$$

$$+ |\int_{t_1}^{t} \Big(\int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - \varphi_{s,0}^{1,N}(Y))\mathbb{1}_{G^N(\widetilde{X}_i)}(Y)k_0(Y)d^6Y$$
$$- f^N * \widetilde{k}_s^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i))\Big) ds|. \tag{3.90}$$

Estimate of Term 3.90

An upper bound for Term (3.90) can be derived analogously to the estimates of Term 3.14 and thus is also given by $CN^{-\frac{5}{12}}$ as

$$\begin{aligned} &f^N * \widetilde{k}_s^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i)) \\ &= \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - {}^1Y)k_s^N(Y)d^6Y \\ &= \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - \varphi_{s,0}^{1,N}(Y))k_0(Y)d^6Y, \end{aligned}$$

which yields

$$\begin{aligned} &|\int_{t_1}^{t} \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - \varphi_{s,0}^{1,N}(Y))k_0(Y)\mathbb{1}_{G^N(\widetilde{X}_i)}(Y)d^6Y ds \\ &- \int_{t_1}^{t} f^N * \widetilde{k}_s^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i))ds| \\ =&|\int_{t_1}^{t} \int_{\mathbb{R}^6} f^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - \varphi_{s,0}^{1,N}(Y))k_0(Y)(\mathbb{1}_{G^N(\widetilde{X}_i)}(Y) - 1)d^6Y ds| \\ \leq& T\|f^N\|_\infty \int_{\mathbb{R}^6} \mathbb{1}_{(G^N(\widetilde{X}_i))^C}(Y)k_0(Y)d^6Y \\ \leq& CTN^{2\beta}(N^{-2b_r-4b_v} + N^{-2s_r-4s_v}) \\ \leq& CTN^{\frac{15}{18}-2b_r-4b_v} \end{aligned}$$

$$\leq CN^{-\frac{5}{12}-4\sigma}$$

Estimate of Term 3.88 and Term 3.89

For the Terms (3.88) and (3.89) we will utilize Theorem 3.10. Since according to the choice of t_1, t_2 and $\widetilde{X}_i$ it holds that $\sup_{t_1\leq s\leq t_2} |[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| \leq N^{-\frac{5}{12}+\sigma}$, it follows by estimating with the map g^N that

$$\begin{aligned}
&\Big|\int_{t_1}^{t} \frac{1}{N}\sum_{j\neq i}\Big(f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\\
&- f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big)ds\Big| \qquad (3.91)\\
\leq&\Big|\int_{t_1}^{t} \frac{1}{N}\sum_{j\in\mathcal{M}^N_b(X)\setminus\{i\}}\Big(f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\\
&- f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big)ds\Big|\\
&+\int_{t_1}^{t} \frac{1}{N}\sum_{j\in\mathcal{M}^N_g(X)\setminus\{i\}}\Big(g^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\\
&\cdot\big(|[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| + |[\Psi^{1,N}_{s,0}(X)]_j - \varphi^{1,N}_{s,0}(X_j)|\big)\Big)ds. \qquad (3.92)
\end{aligned}$$

All of these terms have basically the same structure as in case 1, see Section 3.1.1, and the upper bound of the deviation of the true and the auxiliary dynamic is the same as the allowed deviations of 'good' particles and so we only have to make minor modifications to the definitions of the unlikely sets $\mathcal{B}^{N,\sigma}_{i,j}$. We define for $(k_1,...,k_6)\in Q_N$

$$\begin{aligned}
&X\in\mathcal{B}^{N,\sigma}_{1,i,(k_1,..,k_6)}\subseteq\mathbb{R}^{6N}\\
\Leftrightarrow&\exists t_1', t_2'\in[0,T]:\\
&\Big|\int_{t_1'}^{t_2'}\Big(\frac{1}{N}\sum_{j\neq i} f^N(\varphi^{1,N}_{s,0}(X^i_{k_1,\dots,k_6}) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(X^i_{k_1,\dots k_6})}(X_j)\\
&-\int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(X^i_{k_1,\dots,k_6}) - \varphi^{1,N}_{s,0}(Y))\\
&\qquad\cdot\mathbb{1}_{G^N(X^i_{k_1,\dots k_6})}(Y)k_0(Y)d^6Y\Big)ds\Big| > N^{-\frac{5}{12}}\vee \qquad (3.93)\\
&\Big|\int_{t_1'}^{t_2'}\Big(\frac{1}{N}\sum_{j\neq i} g^N(\varphi^{1,N}_{s,0}(X^i_{k_1,\dots,k_6}) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(X^i_{k_1,\dots k_6})}(X_j)\\
&-\int_{\mathbb{R}^6} g^N(\varphi^{1,N}_{s,0}(X^i_{k_1,\dots,k_6}) - \varphi^{1,N}_{s,0}(Y))\\
&\qquad\cdot\mathbb{1}_{G^N(X^i_{k_1,\dots k_6})}(Y)k_0(Y)d^6Y\Big)ds\Big| > 1 \qquad (3.94)
\end{aligned}$$

Hence, statement (3.93) has the same structure as $\mathcal{B}^{N,\sigma}_{1,i}$ but note that in this case X_i is replaced by the initial data of another auxiliary particle $X^i_{k_1,\dots,k_6} :=$

$X_i + \sum_{j=1}^{6} k_j N^{-\frac{5}{12}+\frac{\sigma}{2}} e_j$. For statement (3.94) a corresponding relationship holds, however with respect to $\mathcal{B}_{2,i}^{N,\sigma}$. It follows analogous, to the reasoning applied for the sets $\mathcal{B}_{j,i}^{N,\sigma}$, $j \in \{1,2\}$ that for any $\gamma > 0$ there exists a $C_\gamma > 0$ such that for all $N \in \mathbb{N}$

$$\mathbb{P}\big(X \in \mathcal{B}_{1,i,(k_1,\dots,k_6)}^{N,\sigma}\big) \le C_\gamma N^{-\gamma}.$$

By restricting the initial data to this set we can estimate Term (3.89) and the second term of (3.92). We are left with the considerations for the first term of (3.92) and Term (3.87). In the proof of case 1, see Section 3.1.1 the set $\mathcal{B}_{3,i}^{N,\sigma}$ was introduced to deal with the corresponding term of (3.92). Since the situation is basically the same we just have to modify the definition such that it applies for $X^i_{k_1,\dots,k_6}$ and for $(k_1,\dots,k_6) \in Q_N$:

$$\begin{aligned}
&X \in \mathcal{B}_{2,i,(k_1,\dots,k_6)}^{N,\sigma} \subseteq \mathbb{R}^{6N} \\
\Leftrightarrow &\exists l \in I_\sigma : \Big(R_l \neq \infty \wedge \\
&\sum_{j \in \mathcal{M}_s^N(X)\setminus\{i\}} \mathbb{1}_{M^N_{(r_l,R_l),(v_l,V_l)}(X^i_{k_1,\dots,k_6})}(X_j) \ge N^{\frac{2\sigma}{9}} \lceil N^{\frac{2}{9}} R_l^2 \min\big(\max(V_l,R_l),1\big)^4 \rceil\Big) \vee \\
&\sum_{j \in \mathcal{M}_s^N(X)\setminus\{i\}} 1 \ge N^{\frac{2}{9}(1+\sigma)}
\end{aligned} \tag{3.95}$$

For $X \in \big(\mathcal{B}_{2,i,(k_1,\dots,k_6)}^{N,\sigma}\big)^C$ and $t \in [t_1,t_2]$ the term

$$\begin{aligned}
&\Big|\int_{t_1}^{t} \frac{1}{N} \sum_{j \in \mathcal{M}_s^N(X)\setminus\{i\}} \Big(f^N([\Psi^{1,N}_{s,0}(X)]_j - [\Psi^{1,N}_{s,0}(X)]_i)\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) \\
&\quad - f^N(\varphi^{1,N}_{s,0}(X_j) - \varphi^{1,N}_{s,0}(\widetilde{X}_i))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big) ds\Big|
\end{aligned} \tag{3.96}$$

can be estimated similar to case 1, see Section 3.1.1. For this purpose, one has to take into account the choice of the interval $[t_1,t_2]$, because for this time span it holds that

$$\begin{aligned}
&\sup_{t \in [t_1,t_2]} |[\Psi^{1,N}_{s,0}(X)]_j - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| \le N^{-\frac{5}{12}+\sigma} \wedge \\
&\sup_{t \in [t_1,t_2]} |[\Psi^{2,N}_{s,0}(X)]_j - {}^2\varphi^{N}_{s,0}(\widetilde{X}_i)| \le N^{-\frac{1}{6}-\sigma}.
\end{aligned}$$

The estimates from case 1, see Section 3.1.1, can be copied to the current situation and hence the previously derived upper bound $CN^{-\frac{5}{12}}$ can be applied.
This concludes the considerations for Term (3.88). Due to the definition of the set (3.94) and the subsequent reasoning it holds for configurations $X \in \big(\mathcal{B}_{1,i,(k_1,\dots,k_6)}^{N,\sigma} \cup \mathcal{B}_{2,i,(k_1,\dots,k_6)}^{N,\sigma}\big)^C$ and $t \in [t_1,t_2]$ that

$$\Big|\int_{t_1}^{t} \frac{1}{N} \sum_{j \in \mathcal{M}_b^N(X)\setminus\{i\}} \Big(f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)$$

$$
\begin{aligned}
&- f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big)ds\Big| \\
&+ \int_{t_1}^{t} \frac{1}{N} \sum_{j\in\mathcal{M}^N_g(X)\setminus\{i\}} \Big(g^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) \\
&\cdot \big(|[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| + |[\Psi^{1,N}_{s,0}(X)]_j - \varphi^{1,N}_{s,0}(X_j)|\big)\Big)ds \\
\le& CN^{-\frac{5}{12}} \\
&+ \Big(1 + \int_{t_1}^{t}\int_{\mathbb{R}^6} g^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(Y))k_0(Y)d^6Y ds\Big) \\
&\cdot \sup_{s\in[t_1,t]} \Big(|[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| + \max_{j\in\mathcal{M}^N_g(X)} |[\Psi^{1,N}_{s,0}(X)]_j - \varphi^{1,N}_{s,0}(X_j)|\Big) \\
\le& CN^{-\frac{5}{12}} + C\big(1 + (t-t_1)\ln(N)\big)N^{-\frac{5}{12}}.
\end{aligned} \tag{3.97}
$$

The derivation of the upper bound for the first term was already discussed previously. For the upper bound of the second term we remind that $0 \le g^N(q) \le C\min(N^{3\beta}, \frac{1}{|q|^3})$ which leads to the factor $C\ln(N)$ after the integration. Further for $s \in [t_1, t]$ since $t \in [t_1, t_2] \subseteq [t_1, \tau^N(X)]$ it holds that

$$
|[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| + \max_{j\in\mathcal{M}^N_g(X)} |[\Psi^{1,N}_{s,0}(X)]_j - \varphi^{1,N}_{s,0}(X_j)| \le 2N^{-\frac{5}{12}},
$$

by the constraints on t_2 and the definition of the stopping time, see 3.8).

Estimate of Term 3.87

In contrast to case 1, see Section 3.1.1, the last remaining Term (3.87) has impact on the prove. It takes into account the impact of the 'superhard' collisions with 'superbad' or 'hard' with 'bad' collision partners and is given by

$$
\left|\int_{t_1}^{t} \frac{1}{N}\sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{(G^N(\widetilde{X}_i))^C}(X_j)ds\right|.
$$

The non-negligibility of this term is the first significant modification in contrast to the considerations for the 'good' particles in case 1, see Section 3.1.1. For this reason we introduce a set of inappropriate initial data for $(k_1, \ldots, k_6) \in Q_N$ and $i \in \{1, \ldots, N\}$

$$
X \in \mathcal{B}^{N,\sigma}_{3,i,(k_1,\ldots,k_6)} \subseteq \mathbb{R}^{6N} \Leftrightarrow \sum_{j\neq i} \mathbb{1}_{M^N_{6N^{-\frac{1}{3}-\sigma}, N^{-\frac{5}{18}}}(X^i_{k_1,\ldots,k_6})}(X_j) \ge N^{\frac{\sigma}{2}} \tag{3.98}
$$

It measures the amount of particles coming very close to the auxiliary particle cloud. For configurations $X \notin \mathcal{B}^{N,\sigma}_{3,i,(k_1,\ldots,k_6)}$ it holds that this last remaining term is bounded by

$$
CN^{\frac{\sigma}{2}-1}\|f^N\|_\infty|t-t_1| \le CN^{\frac{\sigma}{2}-1}\big(N^{\frac{5}{12}-\sigma}\big)^2|t-t_1|
$$

$$\leq CN^{-\frac{1}{6}-\frac{3\sigma}{2}}|t-t_1|. \tag{3.99}$$

As $\mathbb{P}\big(Y \in \mathbb{R}^6 : Y \notin G^N(X_i)\big) \leq CN^{-\frac{5}{4}-2\sigma}$ it follows that

$$\mathbb{P}\big(X \in \mathcal{B}^{N,\sigma}_{3,i,(k_1,\dots,k_6)}\big) \;\leq \binom{N}{\lceil N^{\frac{\sigma}{2}}\rceil}\big(CN^{-\frac{5}{4}-2\sigma}\big)^{\lceil N^{\frac{\sigma}{2}}\rceil} \leq CN^{-\frac{1}{4}\lceil N^{\frac{\sigma}{2}}\rceil}. \tag{3.100}$$

Conclusion case 2 (labelled particle X_i is superbad)

All applied estimates work for arbitrary t_1, t_2 fulfilling the initially introduced demands

$$X \in \Big(\bigcup_{j\in\{1,2,3\}}\bigcup_{i=1}^{N}\bigcup_{(k_1,\dots,k_6)\in Q_N}\mathcal{B}^{N,\sigma}_{j,i,(k_1,\dots,k_6)}\Big)^C.$$

From now on we restrict ourselves to these good configurations. We already discussed that for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ such that $\mathbb{P}\big(X \in \mathcal{B}^{N,\sigma}_{1,i,(k_1,\dots,k_6)}\big) \leq C_\gamma N^{-\gamma}$ and according to the proof of the first case it holds that $\mathbb{P}\big(X \in \mathcal{B}^{N,\sigma}_{2,i,(k_1,\dots,k_6)}\big) \leq (CN^{-\frac{16\sigma}{9}})^{\frac{N^{\frac{\sigma}{9}}}{2}}$. Since $|Q_N| \leq (3\lceil N^{\frac{1}{4}}\rceil)^6 \leq CN^{\frac{3}{2}}$ (see (3.79)), it is possible to choose the constant $C_\gamma > 0$ such that

$$\mathbb{P}\Big(\bigcup_{j\in\{1,2,3\}}\bigcup_{i=1}^{N}\bigcup_{(k_1,\dots,k_6)\in Q_N}\mathcal{B}^{N,\sigma}_{j,i,(k_1,\dots,k_6)}\Big) \leq C_\gamma N^{-\gamma}$$

holds for a given $\gamma > 0$ and all $N \in \mathbb{N}$ and for all configurations

$$X \in \Big(\bigcup_{j\in\{1,2,3\}}\bigcup_{i=1}^{N}\bigcup_{(k_1,\dots,k_6)\in Q_N}\mathcal{B}^{N,\sigma}_{j,i,(k_1,\dots,k_6)}\Big)^C$$

all derived upper bounds are fulfilled for arbitrary 'triples' t_1, t_2 and $\widetilde{X}_i$ provided they are chosen according to the introduced constraints on them. We obtain that Term (3.88) is bounded by $C(1+(t-t_1)\ln(N))N^{-\frac{5}{12}}$, see (3.124). The upper bound for Term (3.89) and Term (3.90) is given by $N^{-\frac{5}{12}}$. The upper bound for Term (3.87) is given by $CN^{-\frac{1}{6}-\frac{3\sigma}{2}}(t-t_1)$. It follows for $t \in [t_1, t_2]$ and for small enough $\sigma > 0$ that the Term (3.86) is bounded by

$$C\big(N^{-\frac{1}{6}-\frac{3\sigma}{2}}(t-t_1) + N^{-\frac{5}{12}}\big).$$

With $|[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(\widetilde{X}_i)| \leq \frac{N^{-\frac{5}{12}+\sigma}}{2}$ we obtain that for any $i \in \{1,\dots,N\}$ and for all times $t \in [t_1, t_2]$ the following inequality holds

$$\begin{aligned}
&|[\Psi^{2,N}_{t,0}(X)]_i - \varphi^{2,N}_{t,0}(\widetilde{X}_i)| \\
\leq&|[\Psi^{2,N}_{t_1,0}(X)]_i - \varphi^{2,N}_{t_1,0}(\widetilde{X}_i)| \\
&+ \Big|\int_{t_1}^{t}\Big(\frac{1}{N}\sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j) - f^N * \widetilde{k}^N_s(\varphi^{1,N}_{s,0}(\widetilde{X}_i))\Big)ds\Big|
\end{aligned}$$

$$\leq |[\Psi^{2,N}_{t_1,0}(X)]_i - \varphi^{2,N}_{t_1,0}(\widetilde{X}_i)| + C\big(N^{-\frac{1}{6}-\frac{3\sigma}{2}}(t-t_1) + N^{-\frac{5}{12}}\big) \tag{3.101}$$

$$\leq \frac{N^{-\frac{5}{12}+\sigma}}{2} + C\big(N^{-\frac{1}{6}-\frac{3\sigma}{2}}(t-t_1) + N^{-\frac{5}{12}}\big). \tag{3.102}$$

Now it is straightforward to find an upper bound for the spatial deviation for $t \in [t_1, t_2]$:

$$\begin{aligned}
&|[\Psi^{1,N}_{t,0}(X)]_i - \varphi^{1,N}_{t,0}(\widetilde{X}_i)| \\
\leq &|[\Psi^{1,N}_{t,0}(X)]_i - \varphi^{1,N}_{t,0}(\widetilde{X}_i)| + \int_{t_1}^{t} |[{}^2\Psi^N_{s,0}(X)]_i - {}^2\varphi^N_{s,0}(\widetilde{X}_i)| ds \\
\leq &\frac{N^{-\frac{5}{12}+\sigma}}{2} + C\big(N^{-\frac{1}{6}-\frac{3\sigma}{2}}(t-t_1)^2 + N^{-\frac{5}{12}}(t-t_1)\big).
\end{aligned} \tag{3.103}$$

The time t_1 denotes an arbitrary moment in $[0, \tau^N(X))$ before the stopping time is triggered. At this point in time we argued that it is always possible to find an auxiliary particle of the introduced 'auxiliary cloud' which is closer in phase space to the observed 'real' particle than $\frac{N^{-\beta}}{2} = \frac{N^{-\frac{5}{12}+\sigma}}{2}$. At time $t_2 \in (t_1, \tau^N(X)]$ the distance in (physical) space between this auxiliary particle and the 'real' one still fulfils

$$\sup_{t_1 \leq t \leq t_2} |[\Psi^{1,N}_{t,0}(X)]_i - \varphi^{1,N}_{t,0}(\widetilde{X}_i)| \leq N^{-\frac{5}{12}+\sigma},$$

while for the velocity deviation the much larger upper bound

$$\sup_{t_1 \leq t \leq t_2} |[\Psi^{2,N}_{t,0}(X)]_i - \varphi^{2,N}_{t,0}(\widetilde{X}_i)| \leq N^{-\frac{1}{6}-\sigma}$$

was allowed. After that point in time maybe a new auxiliary particle of the 'auxiliary cloud' which is closer to the observed 'real' particle must be chosen for further estimates. The possible length of such an interval $[t_1, t_2]$, where the same auxiliary particle can be applied can be derived by (3.102) and (3.103).

However, for large enough $N \in \mathbb{N}$ and $\sigma > 0$ small enough the subsequent implication holds

$$t - t_1 \leq N^{-\frac{1}{8}} \Rightarrow \begin{cases} \frac{N^{-\frac{5}{12}+\sigma}}{2} + C\big(N^{-\frac{1}{6}-\frac{3\sigma}{2}}(t-t_1)^2 + N^{-\frac{5}{12}}(t-t_1)\big) \leq N^{-\frac{5}{12}+\sigma} \\ \frac{N^{-\frac{5}{12}+\sigma}}{2} + C\big(N^{-\frac{1}{6}-\frac{3\sigma}{2}}(t-t_1) + N^{-\frac{5}{12}}\big) \leq CN^{-\frac{7}{24}-\frac{3\sigma}{2}} \leq N^{-\frac{1}{6}-\sigma} \end{cases}$$

and thus, according to relations (3.102) and (3.103), the point in time $t_2 := t_1 + N^{-\frac{1}{8}}$ is a possible option such that the constraints on t_2 are fulfilled. Hence, bound (3.102) and (3.103) yield for t_2 and small enough $\sigma > 0$ that

$$\sup_{t_1 \leq s \leq t_2} |[\Psi^N_{t,0}(X)]_i - \varphi^N_{t,0}(\widetilde{X}_i)| \leq CN^{-\frac{1}{6}-\frac{3\sigma}{2}}(t_2 - t_1) = CN^{-\frac{7}{24}-\frac{3\sigma}{2}}.$$

Considering estimate (3.84) we obtain for $t \in [t_1, t_1 + N^{-\frac{1}{8}}]$, the considered configurations, large enough N and sufficiently small $\sigma > 0$ that Term (3.83) is bounded by

$$\sup_{t_1 \leq s \leq t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)|$$

$$\leq \sup_{t_1 \leq s \leq t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X^i_{k_1,\dots,k_6})| + \sup_{t_1 \leq s \leq t} |\varphi^N_{s,0}(X^i_{k_1,\dots,k_6}) - \varphi^N_{s,0}(X_i)|$$
$$\leq CN^{-\frac{7}{24}-\frac{3\sigma}{2}} + e^{C(t-t_1)}\big|[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(X_i)\big|. \tag{3.104}$$

The first point in time $t_1 \in [0, \tau^N(X))$ was chosen arbitrarily and based on that we define a sequence of time steps

$$t_n := nN^{-\frac{1}{8}} \text{ for } n \in \{0, \dots, \lceil \tau^N(X)N^{\frac{1}{8}} \rceil - 1\} \text{ and } t_{\lceil \tau^N(X)N^{\frac{1}{8}} \rceil} := \tau^N(X)$$

and thereby receive a corresponding sequence of inequalities

$$\sup_{t_n \leq s \leq t_{n+1}} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \leq CN^{-\frac{7}{24}-\frac{3\sigma}{2}} + e^{CN^{-\frac{1}{8}}}\big|[\Psi^N_{t_n,0}(X)]_i - \varphi^N_{t_n,0}(X_i)\big|.$$

Inductively we derive that

$$\sup_{0 \leq s \leq t_n} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \leq CN^{-\frac{7}{24}-\frac{3\sigma}{2}} \sum_{k=0}^{n-1} e^{2CN^{-\frac{1}{8}}k}.$$

An upper bound for the possible values of n is given by $\lceil TN^{\frac{1}{8}} \rceil$ and this yields that

$$\sup_{0 \leq s \leq \tau^N(X)} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \leq CN^{-\frac{1}{6}-\frac{3}{2}\sigma}.$$

For sufficiently large N this value stays smaller than the allowed distance between the mean-field and the real trajectory $N^{-\frac{1}{6}-\sigma}$, which shows that also the 'superbad' particles do typically not 'trigger' the stopping time for the relevant N and σ.

3.1.4 Controlling the deviation of the bad particles

Now we are left with the last set, the set of bad particles. This intermediate set was defined as

$$\mathcal{M}^N_b(X) := \{1, \dots, N\} : \exists j \in \{1, \dots, N\} \setminus \{i\} : X_j \in (M^N_{(r_b,v_b)}(X_j) \setminus M^N_{(r_s,v_s)}(X_j)).$$

The advantage of this set is that it contains less particles than the amount of good particles, but more than amount of superbad ones. For particles in this set we allow intermediate deviation to their mean-field partners as bad events, i.e. particles coming close to each other, still occur. We would also like to use the estimates of case 1, see Section 3.1.1, and therefore we introduce the particle cloud which provides us the auxiliary particles like in case 2. This time Q_N is given by

$$Q_N := \{-\lceil N^{\frac{1}{8}} \rceil, \dots, -1, 0, 1, \dots, \lceil N^{\frac{1}{8}} \rceil\}^6 \tag{3.105}$$

for $(k_1, \dots, k_6) \in Q_N$ the positions $X^i_{k_1,\dots,k_6} := X_i + \sum_{j=1}^6 k_j N^{-\frac{5}{12}+\frac{\sigma}{2}} e_j$. Let us apply Lemma 3.7 and the condition on the distance between the corresponding 'real' and mean-field particle before the stopping time is 'triggered'. This gets us for the point in time $t_1 \in [0, \tau^N(X)]$ and large enough N that

$$|\varphi^N_{0,t_1}([\Psi^N_{t_1,0}(X)]_i) - X_i| \leq C|[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(X_i)| < CN^{-\frac{7}{24}-\sigma}.$$

By construction, this distance is of smaller order with respect to N than the diameter of the auxiliary 'particle cloud' around X_i and if N is sufficiently large it is always possible to find a tuple $(k_1, \dots, k_6) \in Q_N$ such that

$$|\varphi^N_{0,t_1}([\Psi^N_{t_1,0}(X)]_i) - X^i_{k_1,\dots,k_6}| \leq \frac{\sqrt{6}}{2} N^{-\frac{5}{12}-\frac{\sigma}{2}}. \tag{3.106}$$

Lemma 3.7 implies in turn that

$$|[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(X^i_{k_1,\dots,k_6})| \leq CN^{-\frac{5}{12}-\frac{\sigma}{2}}. \tag{3.107}$$

For $CN^{-\frac{5}{12}-\frac{\sigma}{2}} < \frac{1}{2}N^{-\frac{5}{12}}$ with $N \in \mathbb{N}$ large enough, there exists a further point in time $t_2 \in (t_1, T]$ such that

$$\sup_{s\in[t_1,t_2]} |[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(X^i_{k_1,\dots,k_6})| \leq N^{-\frac{5}{12}}$$

and the bound for the velocity deviation

$$\sup_{s\in[t_1,t_2]} |[\Psi^{2,N}_{s,0}(X)]_i - {}^2\varphi^N_{s,0}(X^i_{k_1,\dots,k_6})| \leq N^{-\frac{1}{6}}$$

holds for $\sigma > 0$ sufficiently small. Like in the previous cases we have to show that $\sup_{t_1\leq s\leq t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)|$ grows slow enough on this time interval. Since this variable is bounded by

$$\sup_{t_1\leq s\leq t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X^i_{k_1,\dots,k_6})| + \sup_{t_1\leq s\leq t} |\varphi^N_{s,0}(X^i_{k_1,\dots,k_6}) - \varphi^N_{s,0}(X_i)| \tag{3.108}$$

and estimate the growth of these deviations instead.

The considerations for the first term is mostly analogous to the estimates of case 1 or case 2, see Section 3.1.1 and 3.1.3. By construction, the spatial distance between the considered auxiliary particle and the 'real' particle is bounded from above by $N^{-\frac{5}{12}+\sigma}$.

We use the abbreviation $\widetilde{X}_i := X^i_{k_1,\dots,k_6}$ and assume for the rest of the proof that for an arbitrary point in time $t_1 \in [0, \tau^N(X))$ and $X \in \mathbb{R}^{6N}$ the initial position of the auxiliary particle $X^i_{k_1,\dots,k_6}$ and $t_2 \in (t_1, \tau^N(X)]$ are chosen such that the previously introduced demands are fulfilled on $[t_1, t_2]$.

The second term has the same structure like 3.84 in case 2 an can be controlled by application of Lemma 3.7. It follows for arbitrary $t \in [t_1, t_2]$ that

$$\begin{aligned}
&|\varphi^N_{t,0}(\widetilde{X}_i) - \varphi^N_{t,0}(X_i)| \\
\leq& e^{C(t-t_1)}|\varphi^N_{t_1,0}(\widetilde{X}_i) - \varphi^N_{t_1,0}(X_i)| \\
\leq& e^{C(t-t_1)}\big(|\varphi^N_{t_1,0}(X_i) - [\Psi^N_{t_1,0}(X)]_i| + |[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(\widetilde{X}_i)|\big) \\
\leq& e^{C(t-t_1)}\big(|\varphi^N_{t_1,0}(X_i) - [\Psi^N_{t_1,0}(X)]_i| + N^{-\frac{5}{12}}\big),
\end{aligned} \tag{3.109}$$

where we regarded the allowed upper bound for $\big|[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(\widetilde{X}_i)\big|$ according to the choice of $\widetilde{X}_i$. We will return to this term later.

For the second term we remark that

$$
\begin{aligned}
&|[\Psi^{2,N}_{t,0}(X)]_i - {}^2\varphi^N_{t,0}(\widetilde{X}_i)| \\
\leq &|[\Psi^{2,N}_{t_1,0}(X)]_i - \varphi^{2,N}_{t_1,0}(\widetilde{X}_i)| \\
&+ |\int_{t_1}^t \Big(\frac{1}{N}\sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j) - f^N * \widetilde{k}_s(\varphi^{1,N}_{s,0}(\widetilde{X}_i))\Big) ds|. \quad (3.110)
\end{aligned}
$$

The second summand can be estimated by multiple applications of the triangle inequality and we essentially obtain the four terms of case 1 or 2, Section 3.1.1 and 3.1.3.

$$
\begin{aligned}
&|\int_{t_1}^t \frac{1}{N}\sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j) - f^N * \widetilde{k}_s(\varphi^{1,N}_{s,0}(\widetilde{X}_i)) ds| \quad (3.111)\\
\leq &|\int_{t_1}^t \frac{1}{N}\sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j) \mathbb{1}_{(G^N(\widetilde{X}_i))^C}(X_j) ds| \quad (3.112)\\
&+ |\int_{t_1}^t \frac{1}{N}\sum_{j\neq i} \Big(f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j) \mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) \\
&- f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j)) \mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big) ds| \quad (3.113)\\
&+ |\int_{t_1}^t \frac{1}{N}\sum_{j\neq i} f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j)) \mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) ds \\
&- \int_{t_1}^t \int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(Y)) \mathbb{1}_{G^N(\widetilde{X}_i)}(Y) k_0(Y) d^6Y ds| \quad (3.114)\\
&+ |\int_{t_1}^t \Big(\int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(Y)) \mathbb{1}_{G^N(\widetilde{X}_i)}(Y) k_0(Y) d^6Y \\
&- f^N * \widetilde{k}^N_s(\varphi^{1,N}_{s,0}(\widetilde{X}_i))\Big) ds| \quad (3.115)
\end{aligned}
$$

Estimate of Term 3.115

A suitable upper bound for Term (3.115) can be derived analogously to the previous two cases and thus is given by $CN^{-\frac{5}{12}}$.

$$
\begin{aligned}
&|\int_{t_1}^t \int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(Y)) k_0(Y) \mathbb{1}_{G^N(\widetilde{X}_i)}(Y) d^6Y ds \\
&- \int_{t_1}^t \int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(Y)) k_0(Y) d^6Y ds| \\
=&|\int_{t_1}^t \int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(Y)) k_0(Y) (\mathbb{1}_{G^N(\widetilde{X}_i)}(Y) - 1) d^6Y ds| \\
\leq &T\|f^N\|_\infty \int_{\mathbb{R}^6} \mathbb{1}_{(G^N(\widetilde{X}_i))^C}(Y) k_0(Y) d^6Y \\
\leq &TN^{2\beta}\mathbb{P}\big(Y \in \mathbb{R}^6 : Y \notin G^N(\widetilde{X}_i)\big) \leq TN^{2\beta}\mathbb{P}\big(Y \in \mathbb{R}^6 : Y \in M^N_{6N^{-b_r}, N^{-b_v}}(\widetilde{X}_i)\big) \\
\leq &TN^{2\beta}C(N^{-b_r})^2(N^{-b_v})^4
\end{aligned}
$$

$$\leq TN^{\frac{5}{12}-2\sigma} \tag{3.116}$$

Estimate of Term 3.113 and Term 3.114

Let us focus on the two Terms (3.113) and (3.114), as both can be estimated by Theorem 3.10. Since according to the choice of t_1, t_2 and $\widetilde{X}_i$ it holds that $\sup_{t_1 \leq s \leq t_2} |[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| \leq N^{-\frac{5}{12}+\sigma}$. It follows by estimating with the map g^N that

$$\begin{aligned}
&\Big|\int_{t_1}^{t} \frac{1}{N}\sum_{j\neq i}\Big(f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) \\
&- f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big)ds\Big| \qquad (3.117)\\
\leq&\Big|\int_{t_1}^{t} \frac{1}{N}\sum_{j\in\mathcal{M}^N_b(X)\setminus\{i\}}\Big(f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) \\
&- f^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big)ds\Big| \\
&+\int_{t_1}^{t} \frac{1}{N}\sum_{j\in\mathcal{M}^N_g(X)\setminus\{i\}}\Big(g^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) \\
&\cdot\big(|[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| + |[\Psi^{1,N}_{s,0}(X)]_j - \varphi^{1,N}_{s,0}(X_j)|\big)\Big)ds. \qquad (3.118)
\end{aligned}$$

All these terms have basically the same structure as in case 1 or 2, see Section 3.1.1 and 3.1.3. We just have to amend the definitions of the sets $\mathcal{B}^{N,\sigma}_{i,j}$ from the previous to the current situation. We define for $(k_1, \ldots, k_6) \in Q_N$

$$\begin{aligned}
&X \in \mathcal{B}^{N,\sigma}_{1,i,(k_1,\ldots,k_6)} \subseteq \mathbb{R}^{6N} \\
\Leftrightarrow&\exists t_1', t_2' \in [0,T]: \\
&\Big|\int_{t_1'}^{t_2'}\Big(\frac{1}{N}\sum_{j\neq i} f^N(\varphi^{1,N}_{s,0}(X^i_{k_1,\ldots,k_6}) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(X^i_{k_1,\ldots,k_6})}(X_j) \\
&- \int_{\mathbb{R}^6} f^N(\varphi^{1,N}_{s,0}(X^i_{k_1,\ldots,k_6}) - \varphi^{1,N}_{s,0}(Y)) \\
&\qquad\cdot \mathbb{1}_{G^N(X^i_{k_1,\ldots,k_6})}(Y)k_0(Y)d^6Y\Big)ds\Big| > N^{-\beta+\sigma} \vee \qquad (3.119)\\
&\Big|\int_{t_1'}^{t_2'}\Big(\frac{1}{N}\sum_{j\neq i} g^N(\varphi^{1,N}_{s,0}(X^i_{k_1,\ldots,k_6}) - \varphi^{1,N}_{s,0}(X_j))\mathbb{1}_{G^N(X^i_{k_1,\ldots k_6})}(X_j) \\
&- \int_{\mathbb{R}^6} g^N(\varphi^{1,N}_{s,0}(X^i_{k_1,\ldots,k_6}) - \varphi^{1,N}_{s,0}(Y)) \\
&\qquad\cdot \mathbb{1}_{G^N(X^i_{k_1,\ldots,k_6})}(Y)k_0(Y)d^6Y\Big)ds\Big| > 1. \qquad (3.120)
\end{aligned}$$

For the second statement (3.120) we proceed similarly. It follows analogous to the reasoning applied for the sets $\mathcal{B}^{N,\sigma}_{j,i}$, $j \in \{1,2\}$ that for any $\gamma > 0$ there exists a $C_\gamma > 0$ such that for all $N \in \mathbb{N}$

$$\mathbb{P}\big(X \in \mathcal{B}^{N,\sigma}_{1,i,(k_1,\ldots,k_6)}\big) \leq C_\gamma N^{-\gamma}.$$

Like in case 1 or 2, see Section 3.1.1 and 3.1.3, restricting the initial data to this set is already enough to handle Term (3.114) and the second term of (3.118). Thus, we continue with the first term of (3.118) and finally deal with Term (3.115). Therefore we modify the definition of the set $\mathcal{B}_{3,i}^{N,\sigma}$ such that it applies for $X_{k_1,\dots,k_6}^i$ for $(k_1,\dots,k_6) \in Q_N$

$$
\begin{aligned}
&X \in \mathcal{B}_{2,i,(k_1,\dots,k_6)}^{N,\sigma} \subseteq \mathbb{R}^{6N} \\
\Leftrightarrow &\exists l \in I_\sigma : \Big(R_l \neq \infty \wedge \\
&\sum_{j \in \mathcal{M}_b^N(X)\setminus\{i\}} \mathbb{1}_{M^N_{(r_l,R_l),(v_l,V_l)}(X^i_{k_1,\dots,k_6})}(X_j) \geq N^{(2-2b_v-4b_r)\sigma} \qquad (3.121)\\
&\qquad\qquad \lceil N^{2-2b_v-4b_r} R_l^2 \min\big(\max(V_l,R_l),1\big)^4 \rceil\Big) \vee \\
&\sum_{j \in \mathcal{M}_b^N(X)\setminus\{i\}} 1 \geq N^{\frac{3}{4}(1+\sigma)} \qquad (3.122)
\end{aligned}
$$

For $X \in \big(\mathcal{B}_{2,i,(k_1,\dots,k_6)}^{N,\sigma}\big)^C$ and $t \in [t_1,t_2]$ the term

$$
\begin{aligned}
&\Big|\int_{t_1}^t \frac{1}{N} \sum_{j \in \mathcal{M}_b^N(X)\setminus\{i\}} \Big(f^N([\Psi_{s,0}^{1,N}(X)]_j - [\Psi_{s,0}^{1,N}(X)]_i)\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) \\
&- f^N(\varphi_{s,0}^{1,N}(X_j) - \varphi_{s,0}^{1,N}(\widetilde{X}_i))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big) ds \Big| \qquad (3.123)
\end{aligned}
$$

can be handled by the same estimates as in case 1, see Section 3.1.1. For this purpose, one has to take into account the choice of the interval $[t_1,t_2]$ because for this time span it holds that

$$
\begin{aligned}
&\sup_{t \in [t_1,t_2]} |[\Psi_{s,0}^{1,N}(X)]_j - \varphi_{s,0}^{1,N}(\widetilde{X}_i)| \leq N^{-\frac{5}{12}} \wedge \\
&\sup_{t \in [t_1,t_2]} |[\Psi_{s,0}^{2,N}(X)]_j - \varphi_{s,0}^{2,N}(\widetilde{X}_i)| \leq N^{-\frac{1}{6}}.
\end{aligned}
$$

The estimates can be copied form the previous cases and hence also the previously derived upper bound $CN^{-\frac{5}{12}}$ can be applied.

This concludes the considerations for Term (3.113) and Term (3.117). Due to Definition (3.120) and the subsequent reasoning it holds for configurations $X \in \big(\mathcal{B}_{1,i,(k_1,\dots,k_6)}^{N,\sigma} \cup \mathcal{B}_{2,i,(k_1,\dots,k_6)}^{N,\sigma}\big)^C$ and $t \in [t_1,t_2]$ that

$$
\begin{aligned}
&\Big|\int_{t_1}^t \frac{1}{N} \sum_{j \in \mathcal{M}_b^N(X)\setminus\{i\}} \Big(f^N([\Psi_{s,0}^{1,N}(X)]_i - [\Psi_{s,0}^{1,N}(X)]_j)\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) \\
&- f^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - \varphi_{s,0}^{1,N}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j)\Big) ds \Big| \\
&+ \int_{t_1}^t \frac{1}{N} \sum_{j \in \mathcal{M}_g^N(X)\setminus\{i\}} \Big(g^N(\varphi_{s,0}^{1,N}(\widetilde{X}_i) - \varphi_{s,0}^{1,N}(X_j))\mathbb{1}_{G^N(\widetilde{X}_i)}(X_j) \\
&\cdot \big(|[\Psi_{s,0}^{1,N}(X)]_i - \varphi_{s,0}^{1,N}(\widetilde{X}_i)| + |[\Psi_{s,0}^{1,N}(X)]_j - \varphi_{s,0}^{1,N}(X_j)|\big)\Big) ds
\end{aligned}
$$

$$\begin{aligned}
\leq & CN^{-\frac{5}{12}} \\
& + \Big(1 + \int_{t_1}^{t} \int_{\mathbb{R}^6} g^N(\varphi^{1,N}_{s,0}(\widetilde{X}_i) - \varphi^{1,N}_{s,0}(Y))k_0(Y)d^6Y ds\Big) \\
& \cdot \sup_{s\in[t_1,t]} \Big(|[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| + \max_{j\in\mathcal{M}^N_g(X)} |[\Psi^{1,N}_{s,0}(X)]_j - \varphi^{1,N}_{s,0}(X_j)|\Big) \\
\leq & CN^{-\frac{5}{12}+\sigma} + C\big(1+(t-t_1)\ln(N)\big)N^{-\frac{5}{12}}
\end{aligned} \tag{3.124}$$

The upper bound for the first summand was already discussed in the previously part. For the second summand we regarded that $0 \leq g^N(q) \leq C\min(N^{-\frac{5}{12}}, \frac{1}{|q|^3})$. This leads to the factor $C\ln(N)$ after the integration. Further it holds for $s \in [t_1, t]$ due to $t \in [t_1, t_2] \subseteq [t_1, \tau^N(X)]$, that

$$|[\Psi^{1,N}_{s,0}(X)]_i - \varphi^{1,N}_{s,0}(\widetilde{X}_i)| + \max_{j\in\mathcal{M}^N_g(X)} |[\Psi^{1,N}_{s,0}(X)]_j - \varphi^{1,N}_{s,0}(X_j)| \leq 2N^{-\frac{5}{12}}$$

by the constraints on t_2 and the definition of the stopping time.

Estimate of Term 3.112

We finally arrived at the last remaining Term (3.112)

$$|\int_{t_1}^{t} \frac{1}{N} \sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{(G^N(\widetilde{X}_i))^C}(X_j)ds|$$

Remember that $i \in \mathcal{M}^N_b(X)$. This term takes into account the impact of the 'hard' collisions which were excluded for the 'good' particles. But 'superhard' collisions are excluded again like in case 1, see Section 3.1.1, because the considered particle X_i is 'bad'. That simplifies the situation for us to

$$\begin{aligned}
&|\int_{t_1}^{t} \frac{1}{N} \sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{(G^N(\widetilde{X}_i))^C\setminus M_{sb}(\widetilde{X}_i)}(X_j)ds| = \\
&|\int_{t_1}^{t} \frac{1}{N} \sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{M_b(\widetilde{X}_i)}(X_j)ds|.
\end{aligned}$$

Fortunately, the estimates for this remaining term are straightforward and a simple application of Corollary 3.8 but first we need to define a set of inappropriate initial data for $(k_1, ..., k_6) \in Q_N$ and $i \in \{1, ..., N\}$:

$$\begin{aligned}
& X \in \mathcal{B}^{N,\sigma}_{3,i,(k_1,...,k_6)} \subseteq \mathbb{R}^{6N} \\
\Leftrightarrow & \sum_{j\neq i} \mathbb{1}_{M^N_{6N^{-\frac{7}{24}-\sigma}, N^{-\frac{1}{6}}}(X^i_{k_1,...,k_6})}(X_j) \geq N^{\frac{3\sigma}{4}}
\end{aligned} \tag{3.125}$$

It follows for configurations $X \notin \mathcal{B}^{N,\sigma}_{3,i,(k_1,...,k_6)}$ that

$$|\int_{t_1}^{t} \frac{1}{N} \sum_{j\neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j)\mathbb{1}_{M_b(\widetilde{X}_i)}(X_j)ds|$$

$$\leq \frac{N^{\frac{3\sigma}{4}}}{N} C \min\Big(\frac{1}{N^{-\beta}\Delta v}, \frac{1}{\min\limits_{0\leq s\leq T} |[{}^1\Psi^{N,\beta}_{s,0}(X)]_i - [{}^1\Psi^{N,\beta}_{s,0}(X)]_j|\Delta v}\Big)$$

$$\leq \frac{N^{\frac{3\sigma}{4}}}{N} C \min\Big(\frac{1}{N^{-\beta}N^{-\frac{1.5}{9}}}, \frac{1}{N^{-\frac{7}{24}-\sigma}N^{-\frac{1.5}{9}}}\Big) \leq CN^{-\frac{7}{8}+\frac{3\sigma}{4}}.$$

This last remaining term is bounded by

$$CN^{-\frac{7}{8}+\frac{3\sigma}{4}} \tag{3.126}$$

Moreover, by taking into account that $\mathbb{P}\big(Y \in \mathbb{R}^6 : Y \notin G^N(X_i)\big) \leq CN^{-\frac{5}{4}-2\sigma}$ it follows that

$$\mathbb{P}\big(X \in \mathcal{B}^{N,\sigma}_{3,i,(k_1,\dots,k_6)}\big) \leq \binom{N}{\lceil N^{\frac{3\sigma}{4}}\rceil}\big(CN^{-\frac{5}{4}-2\sigma}\big)^{\lceil N^{\frac{3\sigma}{4}}\rceil} \leq CN^{-\frac{1}{4}\lceil N^{\frac{3\sigma}{4}}\rceil} \tag{3.127}$$

which obviously drops sufficiently fast.

Conclusion case 3 (labelled particle X_i is bad)

Analogously to case 2, see Section 3.1.3 we have to merge all upper bounds. All applied estimates work for arbitrary t_1, t_2 fulfilling the initially in

$$X \in \Big(\bigcup_{j\in\{1,2,3\}}\bigcup_{i=1}^{N}\bigcup_{(k_1,\dots,k_6)\in Q_N} \mathcal{B}^{N,\sigma}_{j,i,(k_1,\dots,k_6)}\Big)^C.$$

We already discussed that for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ such that $\mathbb{P}\big(X \in \mathcal{B}^{N,\sigma}_{1,i,(k_1,\dots,k_6)}\big) \leq C_\gamma N^{-\gamma}$ and according to the proof of the first case it holds that $\mathbb{P}\big(X \in \mathcal{B}^{N,\sigma}_{2,i,(k_1,\dots,k_6)}\big) \leq (CN^{-\frac{7\sigma}{3}})^{\frac{N^{\frac{\sigma}{3}}}{2}}$, see (3.64). Since $|Q_N| \leq (3\lceil N^{\frac{1}{8}}\rceil)^6 \leq CN$, see (3.105), it is possible to choose the constant $C_\gamma > 0$ such that

$$\mathbb{P}\Big(\bigcup_{j\in\{1,2,3\}}\bigcup_{i=1}^{N}\bigcup_{(k_1,\dots,k_6)\in Q_N} \mathcal{B}^{N,\sigma}_{j,i,(k_1,\dots,k_6)}\Big) \leq C_\gamma N^{-\gamma}$$

holds for a given $\gamma > 0$ and all $N \in \mathbb{N}$. For arbitrary 'triples' t_1, t_2 and $\widetilde{X}_i$ all derived upper bounds are fulfilled for configurations

$$X \in \Big(\bigcup_{j\in\{1,2,3\}}\bigcup_{i=1}^{N}\bigcup_{(k_1,\dots,k_6)\in Q_N} \mathcal{B}^{N,\sigma}_{j,i,(k_1,\dots,k_6)}\Big)^C,$$

provided they are chosen according to the introduced constraints. We obtain that (3.113) is bounded by $C(1 + (t - t_1)\ln(N))N^{-\frac{5}{12}+\sigma}$, the bound for Term (3.114) is $N^{-\frac{5}{12}+\sigma}$ by definition, the bound for (3.115) is $CN^{-\frac{5}{12}}$, as derived in case 1, see Section 3.1.1 and $CN^{-\frac{7}{8}+\frac{3\sigma}{4}}$ constitutes an upper bound for (3.112). Hence the force term can be estimated by

$$CN^{-\frac{5}{12}+\sigma}.$$

With $|[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(\widetilde{X}_i)| \leq \frac{N^{-\frac{5}{12}+\sigma}}{2}$ for $t \in [t_1, t_2]$ and $\sigma > 0$ for small enough. We obtain that for any $i \in \{1, \dots, N\}$ and for all times $t \in [t_1, t_2]$ the following holds

$$\begin{aligned}
&|[\Psi^{2,N}_{t,0}(X)]_i - \varphi^{2,N}_{t,0}(\widetilde{X}_i)| \\
\leq &|[\Psi^{2,N}_{t_1,0}(X)]_i - \varphi^{2,N}_{t_1,0}(\widetilde{X}_i)| \\
&+ \left| \int_{t_1}^t \Big(\frac{1}{N} \sum_{j \neq i} f^N([\Psi^{1,N}_{s,0}(X)]_i - [\Psi^{1,N}_{s,0}(X)]_j) - f^N * \widetilde{k}^N_s(\varphi^{1,N}_{s,0}(\widetilde{X}_i)) \Big) ds \right| \\
\leq &\left| [\Psi^{2,N}_{t_1,0}(X)]_i - \varphi^{2,N}_{t_1,0}(\widetilde{X}_i) \right| + CN^{-\frac{5}{12}} \qquad (3.128) \\
\leq &\frac{N^{-\frac{5}{12}+\sigma}}{2} + CN^{-\frac{5}{12}} \qquad (3.129)
\end{aligned}$$

Now it is straightforward to find an upper bound for the spatial deviation for $t \in [t_1, t_2]$:

$$\begin{aligned}
&|[\Psi^{1,N}_{t,0}(X)]_i - \varphi^{1,N}_{t,0}(\widetilde{X}_i)| \\
\leq &|[\Psi^{1,N}_{t,0}(X)]_i - \varphi^{1,N}_{t,0}(\widetilde{X}_i)| + \int_{t_1}^t |[^2\Psi^N_{s,0}(X)]_i - {}^2\varphi^N_{s,0}(\widetilde{X}_i)| ds \\
\leq &\frac{N^{-\frac{5}{12}+\sigma}}{2} + C\big(N^{-\frac{5}{12}-\frac{\sigma}{2}}(t - t_1)\big) \qquad (3.130)
\end{aligned}$$

It is always possible to find an auxiliary particle of the introduced ‘cloud’ which is closer in phase space to the observed ‘real’ particle due to previous considerations. After the time t_2 it may be necessary for further estimates to choose a new auxiliary particle of the ‘cloud’ which is closer to the observed ‘real’ particle. For large enough $N \in \mathbb{N}$ and small enough $\sigma, \delta > 0$ the subsequent implication holds

$$t - t_1 \leq N^{-\delta} \Rightarrow \begin{cases} \frac{N^{-\frac{5}{12}}}{2} + C\big(N^{-\frac{5}{12}-\frac{\sigma}{2}}(t - t_1)\big) \leq N^{-\frac{5}{12}} \\ \frac{N^{-\frac{5}{12}}}{2} + C\big(N^{-\frac{5}{12}-\frac{1\sigma}{2}}(t - t_1)\big) \leq CN^{-\frac{5}{12}} \end{cases}$$

and thus according to relations (3.129) and (3.130), $t_2 := t_1 + N^{-\delta}$ is a possible option such that the constraints on t_2 are fulfilled. Hence, relation (3.129) and (3.130) yield for this choice of t_2 and small enough $\sigma > 0$ that

$$\sup_{t_1 \leq s \leq t_2} |[\Psi^N_{t,0}(X)]_i - \varphi^N_{t,0}(\widetilde{X}_i)| \leq CN^{-\frac{5}{12}-\frac{3\sigma}{2}}(t_2 - t_1) = CN^{-\frac{5}{12}-\delta-\frac{3\sigma}{2}}.$$

For Term (3.108) and by additionally considering estimate (3.109), we obtain for $t \in [t_1, t_1 + N^{-\delta}]$, the considered configurations, large enough N and $\sigma > 0$ small enough that

$$\begin{aligned}
&\sup_{t_1 \leq s \leq t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \\
\leq &\sup_{t_1 \leq s \leq t} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X^i_{k_1,\dots,k_6})| + \sup_{t_1 \leq s \leq t} |\varphi^N_{s,0}(X^i_{k_1,\dots,k_6}) - \varphi^N_{s,0}(X_i)| \\
\leq &CN^{-\frac{5}{12}-\delta-\frac{3\sigma}{2}} + e^{C(t-t_1)}\big|[\Psi^N_{t_1,0}(X)]_i - \varphi^N_{t_1,0}(X_i)\big|. \qquad (3.131)
\end{aligned}$$

Since the point in time $t_1 \in [0, \tau^N(X))$ before the stopping time was triggered was chosen arbitrarily, we can define a sequence of time steps

$$t_n := nN^{-\delta} \text{ for } n \in \{0, ..., \lceil \tau^N(X)N^\delta \rceil - 1\} \text{ and } t_{\lceil \tau^N(X)N^\delta \rceil} := \tau^N(X).$$

Thus we receive a corresponding sequence of inequalities

$$\begin{aligned} &\sup_{t_n \leq s \leq t_{n+1}} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \\ \leq& CN^{-\frac{5}{12}-\delta-\frac{3\sigma}{2}} + e^{CN^{-\delta}} \big|[\Psi^N_{t_n,0}(X)]_i - \varphi^N_{t_n,0}(X_i)\big|. \end{aligned}$$

Inductively we derive that

$$\sup_{0 \leq s \leq t_n} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \leq CN^{-\frac{5}{12}-\delta-\frac{3\sigma}{2}} \sum_{k=0}^{n-1} e^{2CN^{-\delta}k}.$$

An upper bound for the possible values of n is given by $\lceil TN^\delta \rceil$ and this yields that

$$\sup_{0 \leq s \leq \tau^N(X)} |[\Psi^N_{s,0}(X)]_i - \varphi^N_{s,0}(X_i)| \leq CN^{-\frac{5}{12}-\delta-\frac{3}{2}\sigma}.$$

For sufficient large N this value stays smaller than the allowed distance between the mean-field and the real trajectory $N^{-\frac{7}{24}-\sigma}$, which shows that also the 'bad' particles do typically not 'trigger' the stopping time for the relevant N and σ.
This finally completes the main part of the proof.

We conclude the proof of Theorem 3.2 by showing that for $N > 1$

$$\sup_{x \in \mathbb{R}^6} \sup_{0 \leq s \leq T} |\varphi^{1,N}_{s,0}(x) - \varphi^{1,\infty}_{s,0}(x)| \leq e^{C\sqrt{\ln(N)}} N^{-2\beta} \tag{3.132}$$

which is smaller than necessary.

3.2 Molecular of chaos

As mentioned in Section 1.5 and analogously to Chapter 2, we finally prove Theorem 3.2 by showing that

$$\Delta_N(t) := \sup_{x \in \mathbb{R}^6} \sup_{0 \leq s \leq T} |\varphi^{1,N}_{s,0}(x) - \varphi^{1,\infty}_{s,0}(x)| \leq e^{C\sqrt{\ln(N)}} N^{-2\beta} \tag{3.133}$$

holds for N large enough. Note, that this bound is much smaller than necessary. Therefore let $t \in [0, T]$ be such that $\Delta_N(t) \leq N^{-\frac{5}{12}+\sigma}$. It holds for $x \in \mathbb{R}^6$ and $N \in \mathbb{N} \setminus \{1\}$ that

$$\begin{aligned} &|\varphi^{2,N}_{t,0}(x) - \varphi^{2,\infty}_{t,0}(x)| \\ \leq& \Big| \int_0^t \int_{\mathbb{R}^6} \big(f^N(\varphi^{1,N}_{s,0}(x) - \varphi^{1,N}_{s,0}(y)) - f^\infty(\varphi^{1,\infty}_{s,0}(x) - \varphi^{1,\infty}_{s,0}(y))\big) k_0(y) d^6y ds \Big| \\ \leq& \Big| \int_0^t \int_{\mathbb{R}^6} \big(f^N(\varphi^{1,N}_{s,0}(x) - \varphi^{1,N}_{s,0}(y)) - f^N(\varphi^{1,\infty}_{s,0}(x) - \varphi^{1,\infty}_{s,0}(y))\big) k_0(y) d^6y ds \Big| \end{aligned}$$

$$
\begin{aligned}
&+\Big|\int_0^t\int_{\mathbb{R}^6}\big(f^N(\varphi^{1,\infty}_{s,0}(x)-\varphi^{1,\infty}_{s,0}(y))-f^\infty(\varphi^{1,\infty}_{s,0}(x)-\varphi^{1,\infty}_{s,0}(y))\big)k_0(y)d^6yds\Big|\\
\leq&\int_0^t 2\Delta_N(s)\int_{\mathbb{R}^6}g^N(\varphi^{1,N}_{s,0}(x)-\varphi^{1,N}_{s,0}(y))k_0(y)d^6yds\\
&+\Big|\int_0^t\int_{\mathbb{R}^6}\big(f^N(\varphi^{1,\infty}_{s,0}(x)-{}^1y)-f^\infty(\varphi^{1,\infty}_{s,0}(x)-{}^1y)\big)k^\infty_s(y)d^6yds\Big|\\
\leq&C\ln(N)\int_0^t\Delta(s)ds+\Big|\int_0^t\int_{\mathbb{R}^6}\frac{{}^1y}{|{}^1y|^3}\mathbb{1}_{(0,N^{-\beta}]}(|{}^1y|)k^\infty_s(y+\varphi^\infty_{s,0}(x))d^6yds\Big|\\
&+\Big|\int_0^t\int_{\mathbb{R}^6}{}^1yN^{3\beta}\mathbb{1}_{(0,N^{-\beta}]}(|{}^1y|)k^\infty_s(y+\varphi^\infty_{s,0}(x))d^6yds\Big|.
\end{aligned}
$$

In the second step we applied the assumption $\Delta_N(t)\leq N^{-\beta}$. Remember $g^N(q)$ is bounded by $C\min\big(N^{3\beta},\frac{1}{|q|^3}\big)$ for all $q\in\mathbb{R}^3$. The last two terms are quite similar. Let us consider the first term and let us use the notation $x=({}^1x,{}^2x)\in\mathbb{R}^6$. Due to the slowly varying mass or charge density, cancellations arise such that this term keeps small enough, i.e.

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}^6}\frac{{}^1y}{|{}^1y|^3}\mathbb{1}_{(0,N^{-\beta}]}(|{}^1y|)k^\infty_s(y+\varphi^\infty_{s,0}(x))d^6yds\Big|\\
=&\Big|\int_0^t\int_{\mathbb{R}^6}\frac{{}^1y}{|{}^1y|^3}\mathbb{1}_{(0,N^{-\beta}]}(|{}^1y|)\Big(\big(k^\infty_s(y+\varphi^\infty_{s,0}(x))-k^\infty_s((0,{}^2y)+\varphi^\infty_{s,0}(x))\big)\\
&+k^\infty_s((0,{}^2y)+\varphi^\infty_{s,0}(x))\Big)d^6yds\Big|\\
\leq&\int_0^t\int_{\mathbb{R}^6}\frac{1}{|{}^1y|^2}\mathbb{1}_{(0,N^{-\beta}]}(|{}^1y|)\Big(\big|k^\infty_s(y+\varphi^\infty_{s,0}(x))-k^\infty_s((0,{}^2y)+\varphi^\infty_{s,0}(x))\big|\Big)d^6yds.
\end{aligned}
\tag{3.134}
$$

Note that due to symmetry

$$
\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}^6}\frac{{}^1y}{|{}^1y|^3}\mathbb{1}_{(0,N^{-\beta}]}(|{}^1y|)k^\infty_s((0,{}^2y)+\varphi^\infty_{s,0}(x))d^6yds\Big|\\
=&\Big|\int_0^t\widetilde{k}^\infty_s(\varphi^{1,\infty}_{s,0}(x))\int_{\mathbb{R}^3}\frac{q}{|q|^3}\mathbb{1}_{(0,N^{-\beta}]}(|q|)d^3qds\Big|=0.
\end{aligned}
$$

Remember that the initial density fulfills $|\nabla k_0(x)|\leq\frac{C}{(1+|x|)^{3+\delta}}$. It follows, that for arbitrary $z\in\mathbb{R}^6$ and $s\in[0,T]$

$$
\begin{aligned}
&\big|k^\infty_s(y+z)-k^\infty_s((0,{}^2y)+z)\big|\mathbb{1}_{(0,N^{-\beta}]}(|{}^1y|)\\
=&\big|k_0(\varphi^\infty_{0,s}(y+z))-k_0(\varphi^\infty_{0,s}((0,{}^2y)+z))\big|\mathbb{1}_{(0,N^{-\beta}]}(|{}^1y|)\\
\leq&\sup_{z'\in\overline{\varphi^\infty_{0,s}(y+z)\varphi^\infty_{0,s}((0,{}^2y)+z)}}|\nabla k_0(z')|\\
&\cdot\mathbb{1}_{(0,N^{-\beta}]}(|{}^1y|)\Big(\big|\varphi^\infty_{0,s}(y+z)-\varphi^\infty_{0,s}((0,{}^2y)+z)\big|\Big)\\
\leq&\sup_{z'\in\overline{\varphi^\infty_{0,s}(y+z)\varphi^\infty_{0,s}((0,{}^2y)+z)}}\frac{C}{(1+|z'|)^{3+\delta}}
\end{aligned}
$$

$$\cdot \mathbb{1}_{(0,N^{-\beta}]}(|^1y|)\Big(C\big|(y+z)-\big((0,{}^2y)+z\big)\big|\Big)$$
$$\leq \sup_{y'\in\mathbb{R}^3:|y'|\leq N^{-\beta}} \sup_{z'\in\overline{\varphi^{\infty}_{0,s}((y',{}^2y)+z)\varphi^{\infty}_{0,s}((0,{}^2y)+z)}} \frac{CN^{-\beta}}{(1+|z'|)^{3+\delta}} \tag{3.135}$$

where $\overline{xy} := \{(1-\eta)x+\eta y \in \mathbb{R}^6 : \eta \in [0,1]\}$ for $x, y \in \mathbb{R}^6$ and Lemma 3.7 was applied in the second last step. Note that by choosing a sufficiently large value for $|^2y|$, as it appears in this expression, then all configurations within the set, over which the supremum is taken, exhibit velocities of this magnitude due to the bounded mean-field force. Consequently, Term (3.135) diminishes as $|^2y|$ increases, following a decay pattern of $\frac{CN^{-\beta}}{(1+|^2y|)^{3+\delta}}$. Now we can estimate Term (3.134). For arbitrary $z \in \mathbb{R}^6$, in particular $z := \varphi^{\infty}_{s,0}(x)$, we get that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^6} \frac{^1y}{|^1y|^3}\mathbb{1}_{(0,N^{-\beta}]}(|^1y|)k^{\infty}_s(y+z)d^6y\Big| \\
\leq &\int_{\mathbb{R}^3} \frac{1}{|^1y|^2}\mathbb{1}_{(0,N^{-\beta}]}(|^1y|)d^3(^1y) \\
&\cdot \int_{\mathbb{R}^3} \sup_{y'\in\mathbb{R}^3:|y'|\leq N^{-\beta}} \sup_{z'\in\overline{\varphi^{\infty}_{0,s}((y',{}^2y)+z)\varphi^{\infty}_{0,s}((0,{}^2y)+z)}} \frac{CN^{-\beta}}{(1+|z'|)^{3+\delta}} d^3(^2y) \\
\leq &CN^{-2\beta}.
\end{aligned}$$

So for any $x \in \mathbb{R}^6$ it follows that

$$\begin{aligned}
&\sup_{0\leq s\leq t} |\varphi^{1,N}_{s,0}(x)-\varphi^{1,\infty}_{s,0}(x)| \\
\leq &\int_0^t |\varphi^{2,N}_{s,0}(x)-\varphi^{2,\infty}_{s,0}(x)|ds \\
\leq &C\ln(N)\int_0^t\int_0^s \Delta_N(r)drds + CN^{-2\beta}t.
\end{aligned} \tag{3.136}$$

By means of this inequality, one derives by Gronwall Lemma 3.4 that

$$\Delta_N(t) = \sup_{x\in\mathbb{R}^6}\sup_{0\leq s\leq t} |\varphi^{1,N}_{s,0}(x)-\varphi^{1,\infty}_{s,0}(x)| \leq CN^{-2\beta}te^{\sqrt{C\ln(N)}t}$$

which shows that the initial assumption $\Delta_N(t) \leq N^{-\beta} = N^{-\frac{5}{12}+\sigma}$ stays true for arbitrarily large times t provided that $N \in \mathbb{N}$ is large enough.

Applying the stated bound to the relation

$$\Big|\varphi^{2,N}_{t,0}(x)-\varphi^{2,\infty}_{t,0}(x)\Big| \leq C\ln(N)\int_0^t \Delta_N(s)ds + CN^{-2\beta},$$

yields the asserted result

$$\sup_{x\in\mathbb{R}^6}\sup_{0\leq s\leq T} |\varphi^{N}_{s,0}(x)-\varphi^{\infty}_{s,0}(x)| \leq e^{C\sqrt{\ln(N)}}N^{-2\beta} \tag{3.137}$$

for sufficiently large N. This completes the proof of Theorem 3.2.

Chapter 4

New Notion of distance

In the following, we want to examine what happens when two particles collide. Within this section, we provide a preview of a pioneering technique, poised to yield substantial enhancements in the realm of the full Coulomb case.

Effectively managing high singularities necessitates a precise estimation of probabilities associated with exceedingly rare events like particles coming extremely close to each other. However, relying solely on the information that the true and the mean-field trajectories exhibit a certain distance offers only a rudimentary approximation.

The ability to estimate the impact of variations in the initial trajectory on subsequent changes is paramount for achieving superior results. In essence, the transition from convergence in probability to convergence in a distributional sense becomes imperative.

The first step is to require a consistency argument for particle evolution. Minor deviations in the initial configuration should not lead to substantial deviations at a later point in time. We still consider a system consisting of N interacting particles subject to Newtonian time evolution. By a probabilistic mean-field approach we will show, that a small displacement of a particle at the beginning entails a small effect for the dynamics of the whole system, i.e. the distance between the true dynamics and the disturbed dynamic will be small at later times. Therefore, we are able to show, that the deviation remains in the order of magnitude of the shift. For the remaining particles, which were not disturbed at the beginning, we are able to show an even stronger result. The deviation of the not disturbed particles from the true and disturbed system decreases as N increases.

4.1 Introduction to the basic objectives

Our system is distributed as a trajectory in phase space $\mathbb{R}^{6N}$ and the dynamic is given by the respective Newtonian flow $\Psi^N_{t,s} : \mathbb{R}^{6N} \to \mathbb{R}^{6N}$, which was introduced in 1.4. The core of the idea shall be presented for a Coulomb-like model force with a parameter λ to weaken the singularity and a cut-off at $N^{-\beta}$ for $(\lambda + 1)\beta < 1$.

Definition 4.1. *For $N \in \mathbb{N} \cup \{\infty\}$ the interaction force is given by*

$$f^N_\beta : \mathbb{R}^3 \to \mathbb{R}^3, q \mapsto \begin{cases} aN^{(\lambda+1)\beta} q & \text{if } |q| \leq N^{-\beta} \\ a\frac{q}{|q|^{\lambda+1}} & \text{if } |q| > N^{-\beta} \end{cases}$$

for $\frac{3}{2} < \lambda \leq 2, 0 < \beta \leq \frac{1}{3}$*, such that* $(\lambda+1)\beta < 1$ *and* $a \in \{-1, 1\}$*. The total force of the system is given by* $F : \mathbb{R}^{6N} \to \mathbb{R}^{3N}$*, where* $(F(X))_j := \sum_{i \neq j} \frac{1}{N} f_\beta^N(q_i - q_j)$ *is the force exhibited on a single coordinate* j*.*

In our pursuit of approximating a kinetic equation of the Vlasov type, we opt to examine the system in the mean-field scaling. As discusseed in Section 1.5, this choice ensures that the total mass of the system stays of order 1. In order to establish the fluctuation and the changing rate of force, we will introduce a kind of „first and second derivative“of f denoted as g and h respectively, which are required for the proof. Both fulfil a mean value theorem.

Definition 4.2. *a) For* $N \in \mathbb{N} \cup \{\infty\}$ *a function to control* $|f_\beta^N(q) - f_\beta^N(q+\xi)|$ *is given by*

$$g_\beta^N : \mathbb{R}^3 \to \mathbb{R}, q \mapsto \begin{cases} \lambda N^{(\lambda+1)\beta} & \text{if } |q| \leq 3N^{-\beta} \\ \lambda 3^{\lambda+1} \frac{1}{|q|^{\lambda+1}} & \text{if } |q| > 3N^{-\beta} \end{cases}$$

for $\frac{3}{2} < \lambda \leq 2, 0 < \beta \leq \frac{1}{3}$ *such that* $(\lambda+1)\beta < 1$*.*
And G *is given by* $G : \mathbb{R}^{6N} \to \mathbb{R}^N$*, where* $(G(X))_j := \sum_{i \neq j} \frac{1}{N} g_\beta^N(q_i - q_j)$*.*

b) For $N \in \mathbb{N} \cup \{\infty\}$ *a function to control* $|g_\beta^N(q) - g_\beta^N(q+\xi)|$ *is given by*

$$h_\beta^N : \mathbb{R}^3 \to \mathbb{R}, q \mapsto \begin{cases} CN^{(\lambda+2)\beta} & \text{if } |q| \leq 3N^{-\beta} \\ C \frac{1}{|q|^{\lambda+2}} & \text{if } |q| > 3N^{-\beta} \end{cases}$$

for $\frac{3}{2} \leq \lambda \leq 2, 0 < \beta \leq \frac{1}{3}$ *such that* $(\lambda+1)\beta < 1$*. Analogously* H *is given by* $H : \mathbb{R}^{6N} \to \mathbb{R}^N$*, where* $(H(X))_j := \sum_{i \neq j} \frac{1}{N} h_\beta^N(q_i - q_j)$*.*

In the mean-field scaling, the equations of motion for the regularized N-particle system are given by the Newtonian equations of motion, as introduced in the system of equations (1.2). Since the vector field is Lipschitz for fixed β and N, we have global existence and uniqueness of solutions and hence a N-particle flow, which we denote by

$$\Psi_{t,s}^N(X) = \left(\Psi_{t,s}^{1,N}(X), \Psi_{t,s}^{2,N}(X)\right) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N},$$

as introduced in Definition 1.1. For the sake of readability, from now on we will omit the index N. The proof of the main result is based on mean-field arguments for the Vlasov Poisson equation 1.5 and it applies the results of [7, 32]. As introduced in 1.4, we have a mean-field flow and the lift of the mean-field force to the N-particle space is given by

$$(\overline{F}_t(X))_i := f_\beta^N * \tilde{k}_t^N(x_i), \quad X = (x_1, ..., x_N). \tag{4.1}$$

Analogously we denote $\overline{G} : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ and $\overline{H} : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$, the lifts of the derivatives of the mean-field force defined in 4.2 and 4.2 to the N-particle phase-space, i.e.

$$(\overline{G}_t(X))_i := g_\beta^N * \tilde{k}_t^N(x_i), \quad X = (x_1, ..., x_N), \tag{4.2}$$

$$(\overline{H}_t(X))_i := h_\beta^N * \tilde{k}_t^N(x_i), \quad X = (x_1, ..., x_N). \tag{4.3}$$

Our objective is to show that a small displacement of a particle at the beginning entails a small effect for the dynamics of the whole system at later times.

Theorem 4.3. *Let $t > 0$ and consider the N-particle Newtonian flows $\Psi_t(X)$ and $\Psi_t^\delta(X^\delta)$ for $X^\delta = X + (\delta, 0, \ldots, 0)$ with $\delta \geq 0$ given by Definition 1.1 and the parameters of the interaction force chosen such that $(\lambda + 1)\beta < 1$, then there is a constant such that for the first component*

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} \left| \left[\Psi_t^\delta \left(X^\delta \right) - \Psi_t \left(X \right) \right]_1 \right| > \delta \right) \xrightarrow{N \to \infty} 0.$$

In the case $(\lambda + 1)\beta < 1$ there is a zero sequence $(a_N(t))_{N \in \mathbb{N}}$ such that for all components $j \neq 1$

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} \sup_{j \neq 1} \left| \left[\Psi_t^\delta \left(X^\delta \right) - \Psi_t \left(X \right) \right]_j \right| > a_N(t)\delta \right) \xrightarrow{N \to \infty} 0.$$

For the parameters chosen such that $(\lambda + 1)\beta = 1$, we have that

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} \sup_{j \neq 1} \left| \left[\Psi_t^\delta \left(X^\delta \right) - \Psi_t \left(X \right) \right]_j \right| > C \ln(N)\delta \right) \xrightarrow{N \to \infty} 0.$$

4.2 Preliminary studies

In this section we prove several lemmata that will be essential in the proof of Theorem 4.3. The constants, which are independent of N will generically be denoted by C. The constants appearing in a sequence of estimates may differ. For reasons of clarity, we will usually forgo indexing $f_\beta^N, g_\beta^N, h_\beta^N$ and use the short notation f^N, g^N, h^N.

4.2.1 Estimates on f, g and h

Since f and g are generally not differentiable, we prove estimates for differences of their function values.

Lemma 4.4. *a) May $\|X_t - \overline{X}_t\|_\infty \leq 2N^{-\beta}$, then it holds that*

$$\left\| F^N(X_t) - F^N(\overline{X}_t) \right\|_\infty \leq C \|G^N(\overline{X}_t)\|_\infty \|X_t - \overline{X}_t\|_\infty,$$

for some $C > 0$ independent of N.

b) May $||X_t - \bar{X}_t||_\infty \leq 2N^{-\beta}$, then it holds that

$$\left\| G(X_t) - G(\bar{X}_t) \right\|_\infty \leq C \left\| H(\bar{X}_t) \right\|_\infty ||X_t - \bar{X}_t||_\infty$$

for some C independent of N.

c) For $a, b, c \in \mathbb{R}^3$ with $|a| \leq \min(|b|, |c|)$ the following relations hold

$$|f^N(b) - f^N(c)| \leq g^N(a)|b - c|$$
$$|g^N(b) - g^N(c)| \leq h^N(a)|b - c|.$$

Proof. a) See 3.6

b) For any $\xi \in \mathbb{R}^3$ with $|\xi| < 2N^{-\beta}$, we claim that

$$|g^N(x+\xi) - g^N(x)| \leq Ch^N(x)|\xi|, \tag{4.4}$$

where $h^N(x)$ is defined in 4.2. For $|x| < 3N^{-\beta}$ the estimate holds due to the fact that $||\nabla g^N||_\infty \leq N^{(\lambda+2)\beta}$. For $|x| \geq 3N^{-\beta}$, there exists $\tau \in [0,1]$ such that

$$|g^N(x+\xi) - g^N(x)| \leq |\nabla g^N(x+\tau\xi)||\xi|,$$

where

$$|\nabla g^N(x+\tau\xi)| \leq C|x+\tau\xi|^{-(\lambda+2)}.$$

The right hand side of the above expression takes its largest value when $\tau = 1$ and

$$|x+\tau\xi|^{-(\lambda+2)} \leq |x(1-\frac{|\xi|}{|x|})|^{-(\lambda+2)}.$$

Since $|\xi| < 2N^{-\beta}$ and $|x| \geq 3N^{-\beta}$, it follows that $\frac{|\xi|}{|x|} < \frac{2}{3}$. Therefore, we get

$$|g^N(x+\xi) - g^N(x)| \leq C\left(\frac{3}{|x|}\right)^{(\lambda+2)}|\xi| \leq C\frac{|\xi|}{|x|^{(\lambda+2)}}.$$

Applying claim (4.4) one has

$$\begin{aligned}|(G^N(X_t))_i - (G^N(\bar{X}_t))_i| &\leq \frac{1}{N}\sum_{j\neq i}^{N}|g^N(x_i^t - x_j^t) - g^N(\overline{x}_i^t - \overline{x}_j^t)| \\ &\leq \frac{C}{N}\sum_{j\neq i}^{N} h^N(\overline{x}_i^t - \overline{x}_j^t)|x_i^t - x_j^t - \overline{x}_i^t + \overline{x}_j^t| \\ &\leq C(H^N(\overline{X}_t))_i\|X_t - \bar{X}_t\|_\infty,\end{aligned}$$

which leads to the desired estimate of Lemma 4.4 b).

c) In case $|a| \leq 3N^{-\beta}$ the term $g^N(a) = \lambda N^{(\lambda+1)\beta}$ constitutes a Lipschitz-constant for f^N. In case $|a| \geq 3N^{-\beta}$ we get by the mean value theorem and the fact, that $\nabla f^N(x)$ is decreasing

$$|f^N(b) - f^N(c)|| \leq \nabla f^N(a)||b-a| \leq \left(\frac{C}{|a|}\right)^3|b-c| \leq C\frac{|b-c|}{|1|^3} \leq Cg^N(a)|b-a|.$$

In case $|a| \leq 6N^{-\beta}$ the term $h^N(a) = \lambda N^{(\lambda+2)\beta}$ constitutes a Lipschitz-constant for g^N. In case $|a| \geq 3N^{-\beta}$ we get by the mean value theorem and the fact, that $\nabla g^N(x)$ is decreasing

$$|g^N(b) - g^N(c)|| \leq \nabla h^N(a)||b-a| \leq \left(\frac{C}{|a|}\right)^{\lambda+2}|b-c| \leq C\frac{|b-c|}{|a|^{\lambda+2}} \leq Cg^N(a)|b-a|.$$

□

The forces considered here become singular in the limit $N \to \infty$ and hence do not satisfy a uniform Lipschitz bound. Nevertheless, for the mean-field force $f_\beta^N * \tilde{k}_t^N$, the global Lipschitz constant $\|f^N * \tilde{k}_t^N\|_L$ diverges only logarithmically in the full Coulomb-case with $\lambda = 2$ as the cut-off is lifted with increasing N. This statement will be part of the next lemma, which will also contain important estimates for the following proofs.

Lemma 4.5. *Let $0 < \beta \leq \frac{1}{3}$, $\frac{3}{2} < \lambda \leq 2$ and assume that $g : \mathbb{R}^3 \to \mathbb{R}$ satisfies*

$$|g(q)| \leq c \cdot \min\{N^{(\lambda+1)\beta}, |q|^{-\lambda-1}\} \tag{4.5}$$

for some $c > 0$, then there exists a constant $C > 0$ such that the following estimates hold.

a)

$$\|g * \tilde{k}_t(x)\|_\infty \leq C \left(\|\tilde{k}\|_1 + \|\tilde{k}_t\|_\infty\right) \textit{ for } \lambda < 2,$$

respectively

$$\|g * \tilde{k}_t(x)\|_\infty \leq C \ln(N) \left(\|\tilde{k}\|_1 + \|\tilde{k}_t\|_\infty\right) \textit{ for } \lambda = 2.$$

b)

$$\|(g^N)^2 * \tilde{k}_t(x)\|_\infty \leq C N^{(2\lambda-1)\beta} \left(\|\tilde{k}\|_1 + \|\tilde{k}_t\|_\infty\right) \textit{ for } \lambda < 2,$$

respectively

$$\|(g^N)^2 * \tilde{k}_t(x)\|_\infty \leq C N^{3\beta} \left(\|\tilde{k}\|_1 + \|\tilde{k}_t\|_\infty\right) \textit{ for } \lambda = 2.$$

Proof. a) Let's begin by estimating

$$\|g * \tilde{k}_t(x)\|_\infty = \Big\| \int g(x-y)\tilde{k}_t(y)\, \mathrm{d}^3 y \Big\|_\infty \tag{4.6}$$

$$\leq \Big\| \int_{|x-y|<3N^{-\beta}} g(x-y)\tilde{k}_t(y)\, \mathrm{d}^3 y \Big\|_\infty \tag{4.7}$$

$$+ \Big\| \int_{3N^{-\beta}<|x-y|<1} g(x-y)\tilde{k}_t(y)\, \mathrm{d}^3 y \Big\|_\infty \tag{4.8}$$

$$+ \Big\| \int_{|x-y|>1} g(x-y)\tilde{k}_t(y)\, \mathrm{d}^3 y \Big\|_\infty . \tag{4.9}$$

We can bound the first Term (4.7) as follows:

$$\Big\| \int_{|x-y|<3N^{-\beta}} g(x-y)\tilde{k}_t(y)\, \mathrm{d}^3 y \Big\|_\infty \leq \|\tilde{k}_t\|_\infty \|g^N\|_\infty |\mathcal{B}(3N^{-\beta})|$$

$$\leq \frac{4}{3}\pi(3N^{-\beta})^3 N^{(\lambda+1)\beta}\, \|\tilde{k}_t\|_\infty \leq C\|\tilde{k}_t\|_\infty,$$

where $\mathcal{B}(r)$ represents a ball with radius r in $\mathbb{R}^3$. For the second Term (4.8), we have:

$$\begin{aligned}\Big\| \int\limits_{3N^{-\beta}<|x-y|<1} g(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y\Big|_\infty &\leq \|\tilde{k}_t\|_\infty \int\limits_{3N^{-\beta}<|y|<1} \frac{c}{|y|^{\lambda+1}}\,\mathrm{d}^3y \\ &\leq 4\pi C\,\|\tilde{k}_t\|_\infty N^{(\lambda-2)\beta} \leq C\|\tilde{k}_t\|_\infty.\end{aligned}$$

In case $\lambda = 2$, the second term can be estimated as:

$$\begin{aligned}\Big\| \int\limits_{3N^{-\beta}<|x-y|<1} g(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y\Big|_\infty &\leq \|\tilde{k}_t\|_\infty \int\limits_{3N^{-\beta}<|y|<1} \frac{c}{|y|^{3}}\,\mathrm{d}^3y \\ &\leq C\|\tilde{k}_t\|_\infty \ln(N).\end{aligned}$$

Finally, the last Term (4.9) is bounded by

$$\Big\| \int\limits_{|x-y|>1} g(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y\Big\|_\infty \leq c\,\|\tilde{k}_t\|_1.$$

In total, we obtain

$$\begin{aligned}\|g*\tilde{k}_t(x)\|_\infty =& \Big\|\int g(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y\Big\|_\infty \\ \leq& C\ln(N)(\|\tilde{k}_t\|_\infty + \|\tilde{k}_t\|_1)\end{aligned}$$

b) Similarly, we estimate

$$\|g^2*\tilde{k}_t(x)\|_\infty = \Big\|\int g^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y\Big\|_\infty \tag{4.10}$$

$$\leq \Big\| \int\limits_{|x-y|<3N^{-\beta}} g^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y\Big\|_\infty \tag{4.11}$$

$$+ \Big\| \int\limits_{3N^{-\beta}<|x-y|<1} g^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y\Big\|_\infty \tag{4.12}$$

$$+ \Big\| \int\limits_{|x-y|>1} g^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y\Big\|_\infty. \tag{4.13}$$

The first Term (4.11) is bounded by

$$\Big\| \int\limits_{|x-y|<3N^{-\beta}} g^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y\Big\|_\infty$$

$$\leq \|\tilde{k}_t\|_\infty \|(g^N)^2\|_\infty |\mathcal{B}(3N^{-\beta})| \leq \frac{4}{3}\pi(3N^{-\beta})^3 N^{2(\lambda+1)\beta} \|\tilde{k}_t\|_\infty$$
$$\leq CN^{2\lambda\beta+2\beta-3\beta}\|\tilde{k}_t\|_\infty \leq CN^{2\lambda\beta-\beta}\|\tilde{k}_t\|_\infty,$$

where $\mathcal{B}(r)$ is the ball with radius r in $\mathbb{R}^3$. The second Term (4.12) can be estimated with

$$\Big\| \int\limits_{3N^{-\beta}<|x-y|<1} g^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq \|\tilde{k}_t\|_\infty \int\limits_{3N^{-\beta}<|y|<1} \frac{c}{|y|^{2\lambda+2}}\,\mathrm{d}^3y$$
$$\leq 4\pi C\,\|\tilde{k}_t\|_\infty N^{(2\lambda+2-3)\beta}$$
$$\leq CN^{2\lambda\beta-\beta}\|\tilde{k}_t|_\infty.$$

Finally, the last Term (4.13) is bounded by

$$\Big\| \int\limits_{|x-y|>1} g^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq c\,\|\tilde{k}_t|_1.$$

In total, we have

$$\|g^2 * \tilde{k}_t(x)\|_\infty = \Big\| \int g^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq CN^{(2\lambda-1)\beta}(\|\tilde{k}_t|_\infty + \|\tilde{k}_t|_1)$$

□

Analogously, the estimates for the expectation value and the variance of H defined in Definition 4.2 are valid.

Lemma 4.6. *Let* $0 < \beta \leq \frac{1}{3}, \lambda \leq 2$ *and assume that* $h : \mathbb{R}^3 \to \mathbb{R}$ *satisfies*

$$|h(q)| \leq C \cdot \min\{N^{(\lambda+2)\beta}, |q|^{-\lambda-2)}\}$$

for some $c > 0$. *Then there exists a constant* $C_l > 0$ *such that*

$$\|h * \tilde{k}_t(x)\|_\infty \leq CN^{\lambda\beta-\beta}\left(\|\tilde{k}\|_1 + \|\tilde{k}_t\|_\infty\right).$$

and

$$\|h^2 * \tilde{k}_t(x)\|_\infty \leq CN^{2\lambda\beta+\beta}\left(\|\tilde{k}\|_1 + \|\tilde{k}_t\|_\infty\right).$$

Proof. Similarily to the previous Lemma 4.5, we estimate the integral by splitting it into three parts

$$\|h * \tilde{k}_t(x)\|_\infty = \Big\| \int h(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \tag{4.14}$$
$$\leq \Big\| \int\limits_{|x-y|<N^{-\beta}} h(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \tag{4.15}$$

$$+\Big\| \int\limits_{N^{-\beta}<|x-y|<1} h(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \tag{4.16}$$

$$+\Big\| \int\limits_{|x-y|>1} h(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty . \tag{4.17}$$

The first Term (4.15) is bounded by

$$\Big\| \int\limits_{|x-y|<N^{-\beta}} h(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq \|\tilde{k}_t\|_\infty N^{(\lambda+2)\beta}|\mathcal{B}(N^{-\beta})| \leq \frac{4}{3}\pi \,\|\tilde{k}_t\|_\infty N^{\lambda\beta-\beta},$$

where $\mathcal{B}(r)$ is the ball with radius r in $\mathbb{R}^3$. The last Term (4.17) is bounded by

$$\Big\| \int\limits_{|x-y|>1} h(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq C\,\|\tilde{k}_t\|_1.$$

Finally, the second Term (4.16) yields

$$\begin{aligned}\Big\| \int\limits_{N^{-\beta}<|x-y|<1} h(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty &\leq \|\tilde{k}_t\|_\infty \int\limits_{N^{-\beta}<|y|<1} \frac{c}{|y|^{\lambda+2}}\,\mathrm{d}^3y \\ &\leq 4\pi c\beta\,\|\tilde{k}_t\|_\infty N^{(\lambda+2)\beta-3\beta} \\ &\leq C\,\|\tilde{k}_t\|_\infty N^{\lambda\beta-\beta}.\end{aligned}$$

We estimate, analogously to the previous lemma,

$$\|h^2 * \tilde{k}_t(x)\|_\infty = \Big\| \int h^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \tag{4.18}$$

$$\leq \Big\| \int\limits_{|x-y|<N^{-\beta}} h^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \tag{4.19}$$

$$+\Big\| \int\limits_{N^{-\beta}<|x-y|<1} h^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \tag{4.20}$$

$$+\Big\| \int\limits_{|x-y|>1} h^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty . \tag{4.21}$$

The first Term (4.19) is bounded by

$$\Big\| \int\limits_{|x-y|<N^{-\beta}} h^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq \|\tilde{k}_t\|_\infty N^{(2\lambda+4)\beta}|\mathcal{B}(N^{-\beta})| \leq \frac{4}{3}\pi \,\|\tilde{k}_t\|_\infty N^{2\lambda\beta+\beta},$$

where $\mathcal{B}(r)$ is the ball with radius r in $\mathbb{R}^3$. The last Term (4.21) is bounded by

$$\Big\| \int\limits_{|x-y|>1} h^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq c\,\|\tilde{k}_t\|_1.$$

Finally, the second Term (4.20) yields

$$\Big\| \int\limits_{N^{-\beta}<|x-y|<1} h^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq \|\tilde{k}_t\|_\infty \int\limits_{N^{-\beta}<|y|<1} \frac{c}{|y|^{2\lambda+4}}\,\mathrm{d}^3y$$
$$\leq 4\pi c\beta\, \|\tilde{k}_t\|_\infty N^{(2\lambda+4)\beta-3\beta}$$
$$\leq C\, \|\tilde{k}_t\|_\infty N^{2\lambda\beta+\beta}.$$

In total we get for the variance

$$\|h^2 * \tilde{k}_t(x)\|_\infty = \Big\| \int h^2(x-y)\tilde{k}_t(y)\,\mathrm{d}^3y \Big\|_\infty$$
$$\leq CN^{2\lambda\beta+\beta}(\|\tilde{k}_t\|_\infty + \|\tilde{k}_t\|_1)$$

□

4.2.2 Law of large numbers for G and H

We want to show two probability bounds for G and H, i.e. we will show, that the random variables and their expectation value will not deviate much. Therefore we apply the concentration inequality defined in Lemma 2.11 on

$$Z_j = \left(g^N(\bar{q}_1^t - \bar{q}_j^t) - \bar{G}(\bar{Q})_j\right)$$

in the first part of the proof and on

$$Z_j = \left(h^N(\bar{q}_1^t - \bar{q}_j^t) - \bar{H}(\bar{Q})_j\right)$$

in the second part. Therefore we prove the following version of the law of large numbers.

Lemma 4.7. *At any fixed time $t \in [0,T]$, suppose that $\bar{X}_t$ satisfies the mean-field dynamics, G^N and $\bar{G}^N$ are defined in Definition (4.22) and (4.2), H^N and $\overline{H}^N$ are introduced in Definition (4.23) and (4.3). For any $\alpha > 0$ and $(\lambda+1)\beta < 1$, there is a constant $C > 0$ such that*

$$\mathbb{P}\left(\|G^N(\bar{X}_t) - \overline{G}^N(\bar{X}_t)\|_\infty \geq CN^{(\lambda+1)\beta-1}\ln(N)\right) \leq N^{-\alpha} \tag{4.22}$$

and

$$\mathbb{P}\left(N^{-\beta}\|H^N(\bar{X}_t) - \overline{H}^N(\bar{X}_t)\|_\infty \geq CN^{(\lambda+1)\beta-1}\ln(N)\right) \leq N^{-\alpha}. \tag{4.23}$$

Proof. We apply Lemma 2.11 to $Z_j := g^N(\bar{q}_1^t - \bar{q}_j^t) - \int_{\mathbb{R}^3} g^N(\bar{q}_1^t - q)\tilde{k}^N(q,t)dq$ and due to the exchangeability of the particles, we can estimate

$$(G^N(\bar{X}_t))_1 - (\bar{G}^N(\bar{X}_t))_1 = \frac{1}{N}\sum_{j=2}^{N} g^N(\bar{q}_1^t - \bar{q}_j^t) - \int_{\mathbb{R}^3} g^N(\bar{q}_1^t - q)\tilde{k}^N(q,t)d^3q$$

$$= \frac{1}{N} \sum_{j=2}^{N} Z_j.$$

The mean-field particles $\overline{q}_1^t$ and $\overline{q}_j^t$ are independent for $j \neq 1$ and $g^N(0) = 0$. If we denote $\mathbb{E}'[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\overline{q}_1^t]$ and consider $\overline{q}_1^t$ as given, then by

$$\mathbb{E}'\left[g^N(\overline{q}_1^t - \overline{q}_j^t)\right] = \int_{\mathbb{R}^6} g^N(\overline{q}_1^t - q) k^N(q,p,t) d^3q d^3p = \int_{\mathbb{R}^3} g^N(\overline{q}_1^t - q) \tilde{k}^N(q,t) d^3q.$$

it follows, that $\mathbb{E}'[Z_j] = 0$. So the first assumption on the expectation value of Lemma 2.11 is fulfilled. We also get a bound for the variance

$$\mathbb{E}'\left[|Z_j|^2\right] = \mathbb{E}'\left[\left|g^N(\overline{q}_1^t - \overline{q}_j^t) - \int_{\mathbb{R}^3} g^N(\overline{q}_1^t - q)\tilde{k}^N(q,t) d^d q\right|^2\right]$$

by applying the previous Lemma 4.5. It follows in the case $\lambda = 2$, that

$$\mathbb{E}'\left[g^N(\overline{q}_1^t - \overline{q}_j^t)\right] = \int_{\mathbb{R}^3} g^N(\overline{q}_1^t - q)\tilde{k}^N(q,t) dq \le C\ln(N)(\|\tilde{k}^N\|_1 + \|\tilde{k}^N\|_\infty) \le C\ln(N),$$

and

$$\mathbb{E}'\left[g^N(\overline{q}_1^t - \overline{q}_j^t)^2\right] = \int_{\mathbb{R}^3} g^N(\overline{q}_1^t - q)^2 \tilde{k}^N(q,t) dq \le C N^{3\beta}.$$

similarly

$$\mathbb{E}'\left[g^N(\overline{q}_1^t - \overline{q}_j^t)^2\right] = \int_{\mathbb{R}^3} g^N(\overline{q}_1^t - q)^2 \tilde{k}^N(q,t) dq \le C N^{(\lambda+1)\beta}.$$

for $\lambda < 2$. So one can apply Lemma 2.11 with the increasing sequence $r(N) = CN^{(2(\lambda+1)\beta-1}$ and it follows that $|Z_j| \le CN^{(\lambda+1)\beta} \le C\sqrt{Ng(N)}$. By using Lemma 2.11, we get the bounds in probability

$$\mathbb{P}\left(\left|(G^N(\bar{X}_t))_1 - (\bar{G}^N(\bar{X}_t))_1\right| \ge CN^{(\lambda+1)\beta-1}\ln(N)\right) \le N^{-\alpha}. \tag{4.24}$$

and so for the indices $i = 2, \cdots, N$ we have

$$\mathbb{P}\left(\|G^N(\bar{X}_t) - \bar{G}^N(\bar{X}_t)\|_\infty \ge CN^{(\lambda+1)\beta-1}\ln(N)\right) \le N^{1-\alpha}.$$

To prove (4.23) we estimate, like above,

$$\begin{aligned}
&(H^N(\bar{X}_t))_1 - (\overline{H}^N(\bar{X}_t))_1 \\
&= \frac{1}{N}\sum_{j=2}^{N} h^N(\overline{q}_1^t - \overline{q}_j^t) - \int_{\mathbb{R}^3} h^N(\overline{q}_1^t - q)\tilde{k}^N(q,t) d^3q = \frac{1}{N}\sum_{j=2}^{N} Z_j,
\end{aligned}$$

with

$$Z_j = h^N(\overline{q}_1^t - \overline{q}_j^t) - \int_{\mathbb{R}^3} h^N(\overline{q}_1^t - q)\tilde{k}^N(q,t) d^3q.$$

It is easy to show that $\mathbb{E}'[Z_j] = 0$ as well in the case of h^N.

$$\mathbb{E}'\left[h^N(\overline{q}_1^t - \overline{q}_j^t)\right] = \int_{\mathbb{R}^6} h^N(\overline{q}_1^t - q)k^N(q,p,t)d^3q d^3p = \int_{\mathbb{R}^3} h^N(\overline{q}_1^t - q)\tilde{k}^N(q,t)d^3q.$$

To use Lemma 2.11, we also need a bound for the expectation value and the variance. We estimate by applying lemma 4.6

$$\mathbb{E}'\left[h^N(\overline{q}_1^t - \overline{q}_j^t)\right] = \int_{\mathbb{R}^3} h^N(\overline{q}_1^t - q)\tilde{k}^N(q,t)d^3q \leq CN^{\lambda\beta-\beta}(\|\tilde{k}\|_1 + \|\tilde{k}\|_\infty) \leq CN^{\lambda\beta-\beta},$$

and

$$\begin{aligned}\mathbb{E}'\left[h^N(\overline{q}_1^t - \overline{x}_j^t)^2\right] &= \int_{\mathbb{R}^3} h^N(\overline{q}_1^t - q)^2\tilde{k}^N(q,t)dq \\ &\leq CN^{(2\lambda+1)\beta}(\|\tilde{k}\|_1 + \|\tilde{k}\|_\infty) \\ &\leq CN^{2\lambda\beta+\beta}.\end{aligned}$$

Hence one has for the variance

$$\mathbb{E}'\left[|Z_j|^2\right] \leq CN^{2\lambda\beta+\beta}.$$

So the hypotheses of Lemma 2.11 are satisfied with $r(N) = CN^{2\lambda\beta+4\beta-1}$. In addition, it follows that $N^{-\beta}|Z_j| \leq CN^{(\lambda+2)\beta-\beta} \leq C\sqrt{N \cdot r(N)}N^{-\beta}$. Hence we get the probability bound

$$\mathbb{P}\left(\left|N^{-\beta}(H^N(\overline{X}_t))_1 - (\bar{H}^N(\overline{X}_t))_1\right| \geq CN^{\lambda\beta+\beta-1}\ln(N)\right) \leq N^{-\alpha},$$

by Lemma 2.11, which leads to

$$\mathbb{P}\left(\|N^{-\beta}(H^N(\overline{X}_t) - \overline{H}^N(\overline{X}_t))\|_\infty \geq CN^{\lambda\beta+\beta-1}\ln(N)\right) \leq N^{1-\alpha}.$$

□

As a direct consequence of Lemma 4.7 we have the following statements.

Corollary 4.8. *Let $G, H : \mathbb{R}^3 \to \mathbb{R}$ be functions which fulfill the assumptions in Lemma 4.4 for some constant C and let the parameters be chosen such that $(\lambda+1)\beta < 1$. Then there exists a zero sequence $(a_N)_{N\in\mathbb{N}}$ such that*

$$\mathbb{P}(||G(X) - \bar{G}(X)||_\infty > a_N) \xrightarrow{N\to\infty} 0$$

and a zero sequence $(b_N)_{N\in\mathbb{N}}$ such that

$$\mathbb{P}(N^{-\beta}||H(X) - \bar{H}_s(X)||_\infty > b_N) \xrightarrow{N\to\infty} 0.$$

Due to the proofs presented in [7, 32] the following holds.

Lemma 4.9. *For the parameters chosen such that $(\lambda+1)\beta < 1$, let $\Psi_{t,s}$ be the N-particle flow defined in Definition 1.1 with cut-off width $N^{-\beta}$ and let $\Phi_{t,s}$ be the N-particle mean-field flow (1.4) induced by f^N as defined in (1.4). Then, for any $T > 0$, there is a constant C_δ such that for any $\delta > 0$*

$$\mathbb{P}\Big[\exists t \in [0,T] : ||\Psi_{t,0}^N(Z) - \Phi_{t,0}^N(X)||_\infty \geq N^{-\beta}\Big] \leq \frac{TC_\delta}{N^\delta}, \tag{4.25}$$

where $|\cdot|_\infty$ denotes the maximum-norm on $\mathbb{R}^{6N}$.

4.3 Effects of the Disturbance for the whole System

In this chapter, we examine the impact of a singular perturbation in the position on the system's evolution. The initial conditions of the disturbed system X^δ is given by $X_1^\delta(0) = X_1(0) + \delta$ and $X_k^\delta(0) = X_k(0)$ for all $k \neq 1$. Our primary interest lies in investigating the distance between X and X^δ after a time t. We are interested in how a single disturbance in the position affects the evolution of the whole system. Therefore, in Chapter 4.3.1, we consider the effect of the disturbance on the first coordinate, which is the one that was disturbed at time $t = 0$. In Chapter 4.3.2, we will compute the impact of this disturbance on the remaining coordinates.
Since we consider a second order system X including a position and a momentum part

$$\left|\Psi_t(X) - \Psi_t^\delta(X^\delta)\right|_* = a\left|\Psi_t^1(X) - \Psi_t^{1,\delta}(X^\delta)\right|_\infty + b\left|\Psi_t^2(X) - \Psi_t^{2,\delta}(X^\delta)\right|_\infty$$

. The derivative of the difference is given by

$$\frac{d\left|\Psi_t(X) - \Psi_t^\delta(X^\delta)\right|_*}{dt} \leq a\left|\Psi_t^1(X) - \Psi_t^{1,\delta}(X^\delta)\right|_\infty + b\left|F(\Psi_t^1(X)) - F(\Psi_t^{1,\delta}(X^\delta))\right|_\infty .$$

In the case of $\lambda = 2$ we have to weight the norm by setting $a = \sqrt{\ln(N)}, b = 1$ while in the case of $\lambda < 2$ we can set $a = b = 1$. The first term is already a sufficient bound in view of Gronwall's Lemma. To estimate this equation for proving Theorem 4.3 we split the norm into two parts

$$\left|\Psi_t(X) - \Psi_t^\delta(X^\delta)\right|_\infty = \left|\left[\Psi_t(X) - \Psi_t^\delta(X^\delta)\right]_1\right|_\infty + \left|\left[\Psi_t(X) - \Psi_t^\delta(X^\delta)\right]_{\setminus 1}\right|_\infty ,$$

where $[\cdot]_1$ stands for the first component of the vector and $[\cdot]_{\setminus 1}$ for all other components and use a Gronwall type argument for $\left|\Psi_t(X) - \Psi_t^\delta(X^\delta)\right|_\infty$.

4.3.1 Impact of the disturbance on the first coordinate

At first, we will estimate the difference in the first coordinate. Note that the first coordinate of the second component of $\Psi_t(X) - \Psi_t^\delta(X^\delta)$, namely the distance of the momentum, is determined by

$$\begin{aligned}&\left[\Psi_t^2(X) - \Psi_t^{2,\delta}\left(X^\delta\right)\right]_1 \\ &= \int_0^t ds \frac{1}{N}\sum_{k=2}^N \left(f\left(\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^1(X)\right]_k\right) - f\left(\left[\Psi_t^{1,\delta}\left(X^\delta\right)\right]_1 - \left[\Psi_t^{1,\delta}\left(X^\delta\right)\right]_k\right)\right).\end{aligned}$$

As the growth stems from fluctuations in the forces, we will estimate the difference in the forces exhibited in the first coordinate, where $\left[\Psi_t^{1,\delta}(X^\delta)\right]$ denotes the disturbed position developing in time.

$$\frac{d}{dt}\left|\left[\Psi_t^2(X) - \Psi_t^{2,\delta}(X^\delta)\right]_1\right|_\infty$$

$$
\begin{aligned}
&\leq \left| \frac{1}{N} \sum_{k=2}^{N} \left(f\left(\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) - f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - \left[\Psi_t^{1,\delta}(X^\delta)\right]_k \right) \right) \right|_\infty \\
&\leq \left| \frac{1}{N} \sum_{k=2}^{N} \left(f\left(\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) - f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) \right) \right. \\
&\qquad \left. + f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) - f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - \left[\Psi_t^{1,\delta}(X^\delta)\right]_k \right) \right|_\infty \\
&\leq \left| \frac{1}{N} \sum_{k=2}^{N} \left(f\left(\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) - f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) \right) \right|_\infty \qquad (4.26) \\
&+ \left| \frac{1}{N} \sum_{k=2}^{N} f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) - f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - \left[\Psi_t^{1,\delta}(X^\delta)\right]_k \right) \right|_\infty \qquad (4.27)
\end{aligned}
$$

Both Terms (4.26) and (4.27) can be estimated by stochastic arguments using the mean-field-force $\bar{g}_t^N$ defined by $\bar{g}_t^N(q) = g^N * \tilde{k}_t^N(q)$, where $\tilde{k}_t^N : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}_0^+$ is the spatial density from the Vlasov equation and $\bar{X} := (\bar{Q}, \bar{P})$ as defined in (4.2). Heuristically spoken, we replace the force by its expectation value. We can show that for typical initial conditions the disturbance in one coordinate does not effect the system a lot, but in the case of clustering things can go wrong.
For f and g defined in Definition 4.1 and 4.2 we can show with the equalities from Lemma 4.4 and by setting $|[\Psi_t(X)]_1 - [\Psi_t(X)]_k| \leq \left| \left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - [\Psi_t(X)]_k \right|$ without loss of generality, that

$$
\begin{aligned}
(4.26) &= \left| \frac{1}{N} \sum_{k=2}^{N} f\left(\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) - f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) \right|_\infty \\
&\leq \frac{1}{N} \sum_{k=2}^{N} C g\left(\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) \left| \left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^{1,\delta}(X^\delta)\right]_1 \right|_\infty \\
&\leq C \left(G\left(\Psi_t^1(X)\right)_1 \cdot \left| \left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^{1,\delta}(X^\delta)\right]_1 \right|_\infty \right) \\
&\leq C \left(\frac{1}{N} \sum_{k=2}^{N} g\left(\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^1(X)\right]_k \right) + \bar{G}\left(\Phi_t(X)\right)_1 - \bar{G}\left(\Phi_t(X)\right)_1 \right. \\
&\qquad \left. + \frac{1}{N} \sum_{k=2}^{N} g\left(\varphi_t^1(X_1) - \varphi_t^1(X_k) \right) - \frac{1}{N} \sum_{k=2}^{N} g\left(\varphi_t^1(X_1) - \varphi_t^1(X_k) \right) \right) \\
&\qquad \cdot \left| \left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^{1,\delta}(X^\delta)\right]_k \right|_\infty \\
&\quad \left(\bar{G}\left(\Phi_t(X)\right)_1 + \frac{1}{N} \sum_{k=2}^{N} h\left(\varphi_t^1(X_1) - \varphi_t^1(X_k) \right) \right. \\
&\qquad \cdot \left| \left[-\Psi_t^1(X)\right]_k + \varphi_t^1(X_1) + \left[\Psi_t^1(X)\right]_1 - \varphi_t^1(X_1) \right|_\infty \\
&\qquad \left. + \frac{1}{N} \sum_{k=2}^{N} g\left(\varphi_t^1(X_1) - \varphi_t^1(X_k) \right) - \bar{G}\left(\Phi_t^1(X)\right)_1 \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\left|\left[\Psi_t^1(X)\right]_1-\left[\Psi_t^{1,\delta}(X^\delta)\right]_1\right| \\
&\qquad\cdot\Bigg(\bar G\left(\Phi_t^1(X)\right)_1 && (4.28)\\
&\qquad+CH^N\left(\Phi_t^1(X)\right)_1\cdot\left|\Psi_t^1(X)-\Phi_t^1(X)\right|_\infty && (4.29)\\
&\qquad+\frac{1}{N}\sum_{k=2}^N g\left(\varphi_t^1(X_1)-\varphi_t^1(X_k)\right)-\bar G\left(\Phi_t(X)\right)_1\Bigg) && (4.30)
\end{aligned}
$$

The second summand (4.27) is bounded by similar arguments, because we can show with the equalities from Lemma 4.4, for f and g defined in (4.1) and (4.2) and by setting $\left|\left[\Psi_t^\delta(X^\delta)\right]_1-\left[\Psi_t^\delta(X^\delta)\right]_k|_\infty\leq|\left[\Psi_t^{1,\delta}(X^\delta)\right]_1-\left[\Psi_t(X)\right]_k\right|_\infty$ without loss of generality, that

$$
\begin{aligned}
(4.27)&=\left|\frac{1}{N}\sum_{k=2}^N f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1-\left[\Psi_t^1(X)\right]_k\right)-f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1-\left[\Psi_t^{1,\delta}(X^\delta)\right]_k\right)\right|_\infty\\
&\leq\frac{C}{N}\sum_{k=2}^N g\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1-\left[\Psi_t^{1,\delta}(X^\delta)\right]_k\right)\cdot\left|\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]-\left[\Psi_t^1(X)\right]\right)_{\setminus 1}\right|_\infty\\
&\leq C\left|\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]-\left[\Psi_t^1(X)\right]\right)_{\setminus 1}\right|_\infty\cdot\Big|G(\Psi^{1,\delta}(X^\delta))_1-\bar G(\Phi^{1,\delta}(X^\delta))_1\\
&\qquad\qquad+\bar G(\Phi^{1,\delta}(X^\delta))_1-G(\Phi^{1,\delta}(X^\delta))_1+G(\Phi^{1,\delta}(X^\delta))_1\Big|_\infty\\
&\leq C\Bigg(\frac{1}{N}\sum_{k=2}^N g\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_1-\left[\Psi_t^{1,\delta}(X^\delta)\right]_k\right)+\bar G\left(\Phi_t^{1,\delta}(X)\right)_1-\bar G\left(\Phi_t^{1,\delta}(X)\right)_1\\
&\qquad+\frac{1}{N}\sum_{k=2}^N g\left(\varphi_t^{1,\delta}(X_1^\delta)-\varphi_t^{1,\delta}(X_k^\delta)\right)-\frac{1}{N}\sum_{k=2}^N g\left(\varphi_t^{1,\delta}(X_1^\delta)-\varphi_t^{1,\delta}(X_k^\delta)\right)\Bigg)\\
&\qquad\cdot\left|\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]-\left[\Psi_t^1(X)\right]\right)_{\setminus 1}\right|_\infty\\
&\leq C\cdot\left|\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]-\left[\Psi_t^1(X)\right]\right)_{\setminus 1}\right|_\infty\\
&\qquad\cdot\Bigg(\bar G\left(\Phi_t^\delta(X)\right)_1+\frac{1}{N}\sum_{k=2}^N h\left(\varphi_t^{1,\delta}(X_1^\delta)-\varphi_t^{1,\delta}(X_k^\delta)\right)\\
&\qquad\qquad\cdot\left|\left[-\Psi_t^{1,\delta}(X^\delta)\right]_k+\varphi_t^{1,\delta}(X_1^\delta)+\left[\Psi_t^{1,\delta}(X^\delta)\right]_1-\varphi_t^{1,\delta}(X_1^\delta)\right|_\infty\\
&\qquad+\frac{1}{N}\sum_{k=2}^N g\left(\varphi_t^{1,\delta}(X_1^\delta)-\varphi_t^{1,\delta}(X_k^\delta)\right)-\bar G\left(\Phi_t^{1,\delta}(X^\delta)\right)_1\Bigg)\\
&\leq C\left|\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]-\left[\Psi_t^1(X)\right]\right)_{\setminus 1}\right|_\infty\cdot\\
&\qquad\Bigg(\bar G\left(\Phi_t^{1,\delta}(X^\delta)\right)_1 && (4.31)
\end{aligned}
$$

$$+ CH^N \left(\Phi_t^{1,\delta}(X^\delta)\right)_1 \cdot |\Psi_t^{1,\delta}(X^\delta) - \Phi_t^{1,\delta}(X^\delta)|_\infty \tag{4.32}$$

$$+ \frac{1}{N}\sum_{k=2}^{N} g\left(\varphi_t^{1,\delta}(X_1^\delta) - \varphi_t^{1,\delta}(X_k^\delta)\right) - \bar{G}\left(\Phi_t^\delta(X^\delta)\right)_1 \Bigg) \tag{4.33}$$

To shorten the notation we will partition the phase space into subsets. In Lemma 4.7 and 4.9 it is shown that these configurations of the system are the most likely ones.

Definition 4.10. *For any $t \in \mathbb{R}$ the sets $\mathcal{A}_t, \mathcal{A}_t^\delta, \mathcal{B}_t, \mathcal{B}_t^\delta, \mathcal{C}_t, \mathcal{C}_t^\delta$ are given by*

$$\begin{aligned}
\mathcal{A}_t &= \left\{|\Psi_{t,0}(X) - \Phi_{t,0}(X)|_\infty \le N^{-\beta}\right\}\\
\mathcal{A}_t^\delta &= \left\{|\Psi_{t,0}^\delta(X^\delta) - \Phi_{t,0}^\delta(X^\delta)|_\infty \le N^{-\beta}\right\}\\
\mathcal{B}_t &= \left\{\left|G^N(\bar{X}_t) - \overline{G}^N(\bar{X}_t)\right|_\infty \le C\ln(N)N^{(\lambda+1)\beta-1}\right\}\\
\mathcal{B}_t^\delta &= \left\{\left|G^N(\bar{X}\delta_t) - \overline{G}^N(\bar{X^\delta}_t)\right|_\infty \le C\ln(N)N^{(\lambda+1)\beta-1}\right\}\\
\mathcal{C}_t &= \left\{(N^{-\beta}\left|H^N(\bar{X}_t) - \overline{H}^N(\bar{X}_t)\right|_\infty \le CN^{(\lambda+1)\beta}\ln(N)\right\}\\
\mathcal{C}_t^\delta &= \left\{(N^{-\beta}\left|H^N(\bar{X^\delta}_t) - \overline{H}^N(\bar{X^\delta}_t)\right|_\infty \le CN^{(\lambda+1)\beta}\ln(N)\right\}.
\end{aligned}$$

Under the event $\mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t$ we can estimate Term (4.28) by Lemma 4.4. For the second Term (4.29) it follows by 2.11 that $H^N(\bar{Q})_1 \le ||h * \tilde{k}_t||_\infty + N^{(\lambda+2)\beta}\ln(N)$ and we can estimate

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{k=2}^{N} h\left(\varphi_t^1(X_1) - \varphi_t^1(X_k)\right) - \left[\Psi_t^1(X)\right]_k + \varphi_t^1(X_k) + \left[\Psi_t^1(X)\right]_1 - \varphi_t^1(X_1)\right|_\infty\\
&\le CH^N\left(\varphi_t^1(X_1)\right)\cdot\left|\Psi_t^1(X) - \Phi_t^1(X)\right|_\infty\\
&\le \left(\|h * \tilde{k}_t\|_\infty + N^{(\lambda+2)\beta-1}\ln(N)\right)\left|\Psi_t^1(X) - \Phi_t^1(X)\right|_\infty\\
&\le \left(C\cdot N^{\lambda\beta+2\beta-1}(\|\tilde{k}\|_1 + \|\tilde{k}_t\|_\infty) + N^{(\lambda+1)\beta-1}\ln(N)\right)\left|\Psi_t^1(X) - \Phi_t^1(X)\right|_\infty.
\end{aligned}$$

For the last Term (4.30) we apply 2.11, i.e. that $\mathbb{P}(|\frac{1}{N}\sum_{k=2}^N g(\bar{q}_1 - \bar{q}_k) - \bar{G}(\bar{Q})_1|_\infty > \ln(N)N^{(\lambda+1)\beta}) \le C_\varepsilon N^{-\varepsilon}$ for some $\varepsilon > 0$.
In total we get a probabilistic bound for (4.26) and for the difference in the first component that

$$\begin{aligned}
\frac{d}{dt}\left[\Psi_t^2(X) - \Psi_t^{2,\delta}(X^\delta)\right]_1 &\le C\ln(N)N^{(\lambda+1)\beta-1}\left|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^{1,\delta}(X^\delta)\right]_1\right|_\infty\\
&\quad + CN^{(\lambda+1)\beta-1}\left|\left(\left[\Psi_t^1(X)\right] - \left[\Psi_t^{1,\delta}(X^\delta)\right]\right)_{\setminus 1}\right|_\infty.
\end{aligned}$$

Under the event $\mathcal{A}_t^\delta \cap \mathcal{B}_t^\delta \cap \mathcal{C}_t^\delta$ we can estimate Term (4.31) by Lemma 4.4. It is either of order one, or in case $\lambda = 2$ of order $\ln(N)$. For the second Term (4.32) it follows by 2.11 that $H^N(\Phi_t^{1,\delta})_1 \le |h * \tilde{k}_t| + N^{(\lambda+2)\beta}\ln(N)$ and we can estimate

$$\left|\frac{1}{N}\sum_{k=2}^{N} h\left(\varphi_t^{1,\delta}(X_1^\delta) - \varphi_t^{1,\delta}(X_k^\delta)\right) - \left[\Psi_t^{1,\delta}(X^\delta)\right]_k + \varphi_t^{1,\delta}(X_k^\delta) + \left[\Psi_t^{1,\delta}(X^\delta)\right]_1 - \varphi_t^{1,\delta}(X_1^\delta)\right|_\infty$$

$$\leq CH^N\left(\varphi_t^{1,\delta}(X_1^\delta)\right)\cdot\left|\Psi_t^{1,\delta}(X^\delta)-\Phi_t^{1,\delta}(X^\delta)\right|_\infty$$
$$\leq\left(||h*\tilde{k}_t||_\infty+N^{(\lambda+2)\beta-1}\ln(N)\right)\left|\Psi_t^{1,\delta}(X^\delta)-\Phi_t^{1,\delta}(X^\delta)\right|_\infty$$
$$\leq\left(C\cdot N^{\lambda\beta+2\beta-1}(\left|\tilde{k}\right|_1+\left|\tilde{k}_t\right|_\infty)+N^{(\lambda+1)\beta-1}\ln(N)\right)\left|\Psi_t^{1,\delta}(X^\delta)-\Phi_t^{1,\delta}(X^\delta)\right|_\infty.$$

In the last Term (4.33) we applied 2.11 again.
In total we get a probabilistic bound for (4.27) and for the difference in the first component that

$$\begin{aligned}\frac{d}{dt}\left[\Psi_t^2(X)-\Psi_t^{2,\delta}(X^\delta)\right]_1 &\leq C\ln(N)N^{(\lambda+1)\beta-1}\left|\left[\Psi_t^1(X)\right]_1-\left[\Psi_t^{1,\delta}(X^\delta)\right]_1\right|_\infty\\ &\quad+C\ln(N)N^{(\lambda+1)\beta-1}\left|\left(\left[\Psi_t^1(X)\right]-\left[\Psi_t^{1,\delta}(X^\delta)\right]\right)_{\backslash 1}\right|_\infty\\ &\leq C\ln(N)N^{(\lambda+1)\beta-1}\left[\Psi_t^1(X)-\Psi_t^{1,\delta}(X^\delta)\right].\end{aligned}$$

4.3.2 Impact of the disturbance on $1\neq j-$th component

Now we give an estimate for the influence of the disturbance for the other $N-1$ components of the vector $X-\bar{X}$.

$$\begin{aligned}\frac{d}{dt}\left|\left[\Psi_t^2(X)-\Psi_t^{2,\delta}(X^\delta)\right]_{\backslash 1}\right|_\infty=\sum_{j=2}^N\Bigg|\frac{1}{N}\sum_{k\neq j}&\Big(f(\left[\Psi_t^1(X)\right]_j-\left[\Psi_t^1(X)\right]_k)\\ &-f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_j-\left[\Psi_t^{1,\delta}(X^\delta)\right]_k\right)\Big)\Bigg|_\infty\end{aligned}$$

We will have a look at the difference in the j-th coordinate with $j\neq 1$ and as the difference in the trajectories stems from the difference in the respective force we argue analogously to the previous section. To estimate the sum we split it into two terms, one for $k=1$ and one for the rest, as the first coordinate is the one that got disturbed. Let $j\neq 1$ and let us assume $\left|\left[\Psi_t^1(X)\right]_j-\left[\Psi_t^1(X)\right]_k\right|_\infty\leq\left|\left[\Psi_t^\delta(X^\delta)\right]_j-\left[\Psi_t^\delta(X^\delta)\right]_k\right|$ without loss of generality.

$$\begin{aligned}&\sum_{j=2}^N\frac{1}{N}\left|\sum_{k\neq j}\left(f\left(\left[\Psi_t^1(X)\right]_j-\left[\Psi_t^1(X)\right]_k\right)-f\left(\left[\Psi_t^\delta(X^\delta)\right]_j-\left[\Psi_t^\delta(X^\delta)\right]_k\right)\right)\right|_\infty\\ &=\sum_{j=2}^N\frac{1}{N}\Bigg|\left(\sum_{k\neq j}\left(f\left(\left[\Psi_t^1(X)\right]_j-\left[\Psi_t^1(X)\right]_k\right)-f\left(\left[\Psi_t^\delta(X^\delta)\right]_j-\left[\Psi_t^\delta(X^\delta)\right]_k\right)\right)\right)\\ &\qquad+\left(f\left(\left[\Psi_t^1(X)\right]_j-\left[\Psi_t^1(X)\right]_1\right)-f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_j-\left[\Psi_t^{1,\delta}(X^\delta)\right]_1\right)\right)\Bigg|_\infty\\ &\leq\sum_{j=2}^N\frac{1}{N}\Bigg(\sum_{k\neq j,k\neq 1}\left(g\left(\left[\Psi_t^1(X)\right]_j-\left[\Psi_t^1(X)\right]_k\right)\left(\left|\left[\Psi_t^1(X)\right]_j-\left[\Psi_t^{1,\delta}(X^\delta)\right]_j\right|_\infty\right.\right.\\ &\qquad\left.\left.+\left|\left[\Psi_t^1(X)\right]_k-\left[\Psi_t^{1,\delta}(X^\delta)\right]_k\right|\right)\right) \qquad (4.34)\end{aligned}$$

$$+\, g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_1\right)$$
$$\cdot\left(\left|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^{1,\delta}(X^\delta)\right]_j\right| + \left|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^{1,\delta}(X^\delta)\right]_1\right|\right)\right) \quad (4.35)$$

For the first summand (4.34) we get

$$\begin{aligned}
&\frac{1}{N}\sum_{j=2}^{N}\sum_{k\neq j} g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right)\\
&\quad\cdot\left(\left|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\right|_\infty + \left|\left[\Psi_t^1(X)\right]_k - \left[\Psi_t^\delta(X^\delta)\right]_k\right|_\infty\right)\\
&\leq \frac{1}{N}\sum_{j=2}^{N}\sum_{k\neq j} g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right)\left|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\right|_\infty\\
&\quad+\frac{1}{N}\sum_{j=2}^{N}\sum_{k\neq j} g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right)\left|\left[\Psi_t^1(X)\right]_k - \left[\Psi_t^\delta(X^\delta)\right]_k\right|_\infty\\
&\leq \frac{1}{N}\sum_{j=2}^{N}\left|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\right|_\infty\sum_{k\neq j} g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right)\\
&\quad+\frac{1}{N}\sum_{k=2}^{N}\left|\left[\Psi_t^1(X)\right]_k - \left[\Psi_t^\delta(X^\delta)\right]_k\right|_\infty\sum_{j\neq k} g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right)\\
&\leq \frac{2}{N-1}\sum_{j=2}^{N}\left|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\right|_\infty\sum_{k\neq j} g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right),
\end{aligned}$$

We can switch the sums because they are finite and, like g, symmetric in k an j. The inner sum can be estimated as follows

$$\begin{aligned}
&\frac{1}{N}\sum_{k\neq j} g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right)\\
&\leq C\Big(\bar{G}\left(\Phi_t^1(X)\right)_j\\
&\quad+\frac{1}{N}\sum_{k=2}^{N} h\left(\left[\Psi_t^1(X)\right]_j - \varphi_t^1(X_k)\right)\left|-\left[\Psi_t^1(X)\right]_k + \varphi_t^1(X_k) + \left[\Psi_t^1(X)\right]_j - \varphi_t^1(X_j)\right|_\infty\\
&\quad+\frac{1}{N}\sum_{k=2}^{N} g\left(\varphi_t^1(X_j) - \varphi_t^1(X_k)\right) - \bar{G}\left(\Phi_t^1(X)\right)_j\Big)\\
&\leq C\ln(N)N^{(\lambda+1)\beta-1}.
\end{aligned}$$

For the second summand (4.35) we get

$$\frac{1}{N}\sum_{j=2}^{N}(g(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_1)$$

$$
\begin{aligned}
&\cdot \Big(|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j| + \Big|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_1\Big|_\infty \Big) \\
\leq\ & \frac{1}{N}\sum_{j=2}^N N^{\lambda\beta+\beta} |\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j|_\infty \\
&+ \frac{1}{N}\sum_{j=2}^N g(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_1) \cdot |\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^\delta(X^\delta)\right]_1|_\infty \\
\leq\ & N^{\lambda\beta+\beta-1}\sum_{j=2}^N \Big|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\Big|_\infty \\
&+ C\Big(\frac{1}{N}\sum_{j=2}^N g\left(\left[\Psi_t^\delta(X^\delta)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_1\right) + \bar{G}(\Phi_t^1(X))_1 - \bar{G}(\Phi_t^1(X))_1 \\
&+ \frac{1}{N}\sum_{j=2}^N g(\varphi_t^1(X_1) - \varphi_t^1(X_k)) - \frac{1}{N}\sum_{j=2}^N g(\varphi_t^1(X_1) - \varphi_t^1(X_k))\Big) \\
&\qquad \cdot \Big|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^\delta(X^\delta)\right]_1\Big|_\infty \\
\leq\ & N^{\lambda\beta+\beta-1}\sum_{j=2}^N \Big|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\Big|_\infty \\
&+ C\Big(\bar{G}(\Phi_t^1(X_j))_1 && (4.36) \\
&\quad + CH^N(\Phi_t^1(X))_1 \cdot |\Phi_t^1 - \Phi_t^1|_\infty) && (4.37) \\
&\quad + \frac{1}{N}\sum_{j=2}^N g(\varphi_t^1(X_1) - \varphi_t^1(X_j)) - \bar{G}(\Phi_t^1(X))_1\Big) && (4.38) \\
&\qquad \cdot \Big|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^\delta(X^\delta)\right]_1\Big|_\infty .
\end{aligned}
$$

Under the event $\mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t$, by estimating the terms analogous to the previous section we get that

$$
\begin{aligned}
&\frac{d}{dt}\Big|\left[\Psi_t^2(X) - \Psi_t^{2,\delta}(X^\delta)\right]_{\setminus 1}\Big|_\infty \\
&\leq \frac{C}{N}\ln(N) N^{(\lambda+1)\beta}\sum_{j=2}^N \Big|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\Big|_\infty \\
&\quad + CN^{\lambda\beta+\beta-1}\sum_{j=2}^N \Big|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\Big|_\infty \\
&\quad + CN^{(\lambda+1)\beta-1}\ln(N)\Big|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^\delta(X^\delta)\right]_1\Big|_\infty \\
&\leq C\ln(N) N^{(\lambda+1)\beta-1}\Big|\left[\Psi_t^1(X) - \Psi_t^{1,\delta}(X^\delta)\right]_{\setminus 1}\Big|_\infty \\
&\quad + C\ln(N) N^{(\lambda+1)\beta-1}\sum_{j=2}^N \Big|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^\delta(X^\delta)\right]_1\Big|_\infty .
\end{aligned}
$$

4.4 Definition of distance

To provide the probabilistic bound of Theorem 4.3 we introduce a suitable notion of distance, which is defined similarly to Definition 1.4. This time it measures the distance between $\Psi_t(X) - \Psi_t^\delta(X^\delta)$

Definition 4.11. *Let $\Psi_{s,0}(X)$ be the microscopic flow and the disturbed microscopic flow $\Psi_{s,0}^\delta(X^\delta)$ defined in Definition 1.1. We denote the projection onto the spatial or respectively the momentum coordinates by $\Psi_{s,0}^1(X) = (q_i(t))_{1\le i\le N}$ and $\Psi_{s,0}^2(X) = (p_i(t))_{1\le i\le N}$. For $T>0$ and without loss of generality $N>1$ the auxiliary process is defined by*

$$J_t^N(X) := \min\left\{1, \sup_{0\le s\le t}\left\{\frac{1}{\delta}\cdot\left(\sqrt{\ln(N)}\left|\Psi_{t,0}^1(X) - \Psi_{t,0}^{1,\delta}(X^\delta)\right|_\infty \right.\right.\right. \\ \left.\left.\left. + \left|\Psi_{t,0}^2(X) - \Psi_{t,0}^{2,\delta}(X^\delta)\right|_\infty\right)\right\}\right\}$$

for $0 \le t \le T$.

The factor $\sqrt{\ln(N)}$ is only necessary in the case $\beta = \frac{1}{3}$. Remember from Chapter 1.5 that

$$\mathbb{P}\left(\sup_{0\le s\le t}\left|\left[\Psi_t(X) - \Psi_t^{1,\delta}(X^\delta)\right]\right|_\infty \ge \delta\right) = \mathbb{P}(J_t^N = 1) \le \mathbb{E}(J_t^N).$$

Proof of Theorem 4.3. Let $\mathcal{L}_t = \mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t \cap \mathcal{A}_t^\delta \cap \mathcal{B}_t^\delta \cap \mathcal{C}_t^\delta$ and consider the expectation $\mathbb{E}(J_t^N)$ which we split as follows

$$\mathbb{E}(J_t^N) = \mathbb{E}\left(J_t^N \mid |J_t^N(X)|_\infty = 1\right) + \mathbb{E}\left(J_t^N \mid |J_t^N(X)|_\infty < 1 \ \wedge \ X \notin \mathcal{L}_t\right) \\ + \mathbb{E}(J_t^N \mid X \in \mathcal{L}_t).$$

For $|J_t^N(X)|_\infty = 1$, we have $\frac{\mathrm{d}}{\mathrm{d}t} J_t^N(X) = 0$, since $J_t^N(Z)$ is already maximal and thus also

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_t(J_t^N \mid |J_t^N(X)|_\infty = 1) = 0. \tag{4.39}$$

Furthermore, according to Lemma 4.7, for any $\kappa > 0$ we can find the constant C_κ such that

$$\frac{d}{dt}\mathbb{E}\left(J_t^N \mid |J_t^N(X)|_\infty < 1 \ \wedge \ X \notin \mathcal{L}_t\right) \le e^{\lambda\sqrt{\log(N)}T}\frac{C_\kappa}{N^\kappa}.$$

The last summand $\mathbb{E}(J_t^N \mid X \in \mathcal{L}_t)$ is bounded due to Chapter 4.3.1. Under the event $\mathcal{L}_t$ it holds that

$$\frac{d}{dt}\left|\left[\Psi_t^2(X) - \Psi_t^{2,\delta}(X^\delta)\right]_1\right|_\infty \le C\ln(N)N^{(\lambda+1)\beta-1}\left|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^\delta(X^\delta)\right]_1\right|_\infty \\ + CN^{(\lambda+1)\beta-1}|(\Psi_t^1(X) - \Psi_t^{;\delta}(X^\delta))_{\backslash 1}|_\infty$$

$$\frac{d}{dt}\left|\left[\Psi_t^2(X) - \Psi_t^{2,\delta}(X^\delta)\right]_{\backslash 1}\right|_\infty \le \frac{C}{N}\ln(N)N^{(\lambda+1)\beta-1}\sum_{j=2}^N\left|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\right|_\infty$$

$$
\begin{aligned}
&+ CN^{\lambda\beta+\beta-1}\sum_{j=2}^{N}\left|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\right|_\infty \\
&+ CN^{(\lambda+1)\beta-1}\ln(N)\left|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^\delta(X^\delta)\right]_1\right|_\infty \\
\leq\, & C\ln(N)N^{(\lambda+1)\beta-1}|(\Psi_t^1(X) - \Psi_t^{;\delta}(X^\delta))_{\backslash 1}|_\infty \\
&+ C\ln(N)N^{(\lambda+1)\beta-1}\sum_{j=2}^{N}\left|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^\delta(X^\delta)\right]_1\right|_\infty
\end{aligned}
$$

For the whole vector we conclude that

$$
\frac{d}{dt}\left|\left[\Psi_t^2(X) - \Psi_t^{2,\delta}(X^\delta)\right]\right|_\infty \leq CN^{(\lambda+1)\beta-1}\ln(N)\left|\left[\Psi_t^1(X)\right]_1 - \left[\Psi_t^\delta(X^\delta)\right]_1\right|_\infty
$$

We can consequently estimate

$$
\frac{d}{dt}\mathbb{E}\left(J_t^N\right) \leq CN^{(\lambda+1)\beta}\ln(N)\mathbb{E}\left(J_t^N\right).
$$

Gronwall's inequality and the definition of J_t^N implies that the magnitude of the difference is bounded by δ with high probability, given that the initial difference is only δ. Therefore there exists a η for every C_η such that

$$
\mathbb{P}\left(\sup_{0\leq s\leq t}\left|\left[\Psi_t(X) - \Psi_t^{1,\delta}(X^\delta)\right]\right|_\infty \geq \delta\right) \leq C_\eta N^\eta
$$

As a direct result we have

$$
\mathbb{P}(\sup_{0\leq s\leq t}\left|\left[\Psi_t(X) - \Psi_t^{1,\delta}(X^\delta)\right]_1\right|_\infty > \delta) \xrightarrow{N\to\infty} 0,
$$

for the parameters chosen such that $(\lambda+1)\beta < 1$. □

This result tells us that after some time the deviation of these two systems is only of size of the initial disturbance δ.

4.4.1 Effect of the disturbed particles on not disturbed ones

In the preceding Chapter we observed that the perturbation on the first particle remains at a magnitude of δ over time. Furthermore, we can demonstrate that the influence on the remaining particles is even more negligible.

Lemma 4.12. *For parameters chosen such that $(\lambda+1)\beta < 1$ there is a zero sequence $(a_N(t))_{N\in\mathbb{N}}$ such that*

$$
\mathbb{P}(\sup_{0\leq s\leq t}\sup_{j\neq 1}\left|[\Psi_t(X)]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\right| > a_N(t)\delta) \xrightarrow{N\to\infty} 0.
$$

For the parameters chosen such that $(\lambda+1)\beta = 1$ we have that

$$
\mathbb{P}(\sup_{0\leq s\leq t}\sup_{j\neq 1}\left|[\Psi_t(X)]_j - \left[\Psi_t^\delta(X^\delta)\right]_j\right| > C\ln(N)\delta) \xrightarrow{N\to\infty} 0.
$$

Proof. Under the event of $\mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t$ Theorem 4.3 holds and therefore

$$
\begin{aligned}
&\left|\left[\Psi_t^2(X) - \Psi_t^{2,\delta}(X^\delta)\right]_{\setminus 1}\right|_\infty \\
&\leq \frac{d}{dt}\max_j \left|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^{1,\delta}(X^\delta)\right]_j\right|_\infty \\
&\leq \frac{1}{N}\sum_{k\neq j}\left(f\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right) - f\left(\left[\Psi_t^{1,\delta}(X^\delta)\right]_j - \left[\Psi_t^{1,\delta}(X^\delta)\right]_k\right)\right) \\
&\leq \frac{1}{N}\sum_{k\neq j} g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right) \\
&\qquad \cdot \left(\underbrace{\left|\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^{1,\delta}(X^\delta)\right]_j\right|_\infty + \left|\left[\Psi_t^1(X)\right]_k - \left[\Psi_t^{1,\delta}(X^\delta)\right]_k\right|_\infty}_{\leq 2\delta \text{ by Theorem 4.3}}\right) \\
&\leq \frac{1}{N}\sum_{k\neq j} g\left(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k\right)\cdot C\delta \\
&\leq C\delta\Big(\frac{1}{N}\sum_{k=2}^N g(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k) + \bar{G}(\Phi_t^1(X))_j - \bar{G}(\Phi_t^1(X))_j \\
&\qquad + \frac{1}{N}\sum_{k=2}^N g(\varphi_t^1(X_j) - \varphi_t^1(X_k)) - \frac{1}{N}\sum_{k=2}^N g(\varphi_t^1(X_j) - \varphi_t^1(X_k))\Big) \\
&\leq C\delta\Big(\bar{G}(\Phi_t^1(X)_j + \frac{1}{N}\sum_{k=2}^N g(\left[\Psi_t^1(X)\right]_j - \left[\Psi_t^1(X)\right]_k) \\
&\qquad - \frac{1}{N}\sum_{k=2}^N g(\varphi_t^1(X_j) - \varphi_t^1(X_k)) + \frac{1}{N}\sum_{k=2}^N g(\varphi_t^1(X_j) - \varphi_t^1(X_k)) - \bar{G}(\Phi_t^1(X)_j\Big) \\
&\leq \big(\bar{G}(\Phi_t^1(X)_j \\
&\qquad + \frac{1}{N}\sum_{k=2}^N h\left(\varphi_t^1(X_j) - \varphi_t^1(X_k)\right)\left|-\left[\Psi_t^1(X)\right]_k + \varphi_t^1(X_k) + \left[\Psi_t^1(X)\right]_j - \varphi_t^1(X_j)\right|_\infty \\
&\qquad + \frac{1}{N}\sum_{k=2}^N g(\varphi_t^1(X_j) - \varphi_t^1(X_k)) - \bar{G}(\Phi_t^1(X)_j\Bigg)\cdot C\delta \\
&\leq C\delta\Big(\underbrace{\bar{G}(\Phi_t^1(X)_j}_{(4.36)} + \underbrace{CH^N(\Phi_t^1(X))\cdot\left|\left[\Psi_t^1(X)\right] - (\Phi_t^1(X)_j\right|_\infty}_{(4.37)} \\
&\qquad \underbrace{+ \frac{1}{N}\sum_{k=2}^N g(\varphi_t^1(X_j) - \varphi_t^1(X_k)) - \bar{G}(\Phi_t^1(X)_j\Big)}_{(4.38)} \\
&\leq C\delta\ln(N)N^{(\lambda+1)\beta-1}
\end{aligned}
$$

So there is a zero sequence $(a_n)_{n\in N}$ and ε for every C_ε such that

$$
\begin{aligned}
&\mathbb{P}(\sup_{0\le s\le t}\sup_{j\neq 1}|\left[\Psi_t^1(X)-\Psi_t^{1,\delta}(X^\delta)\right]_j|>a_N(t)\delta)\\
&\le \mathbb{P}(\left|G^N(\bar{X}_t)-\overline{G}^N(\bar{X}_t)\right|_\infty \ge C\ln(N)N^{(\lambda+1)\beta-1})\\
&\quad+\mathbb{P}((N^{-\beta}|H^N(\bar{X}_t)-\overline{H}^N(\bar{X}_t)|_\infty \ge CN^{(\lambda+1)\beta}\ln(N))\\
&\quad+\mathbb{P}(|\Psi_{t,0}(Z)-\Phi_{t,0}(X)|_\infty \ge N^{-\beta})\\
&\quad+\mathbb{P}_0(\sup_{0\le s\le t}|\left[\Psi_t^1(X)-\Psi_t^{1,\delta}(X^\delta)\right]|_\infty>\delta)\le C_\varepsilon N^{-\varepsilon}
\end{aligned}
$$

This proves the lemma. □

This shows, that a small displacement in the initial condition leads to small effects on the time evolution of the system. The error term stays in size of the displacement and this result provides the basis for introducing convergence in distribution in later projects.

Chapter 5

Conclusion

This thesis has made significant strides in mathematically deriving the Vlasov-Poisson equation concerning the cut-off size and provides a rigorous mathematical derivation of the Vlasov-Dirac-Benney equation in the large N limit for N interacting particles. In the first segment, a probabilistic proof of the mean-field limit and chaos propagation of an N-particle system with pair potentials of the form

$$\phi_N^\beta = N^{3\beta-1}\phi(N^\beta x) \text{ for } \beta \in \left[0, \frac{1}{7}\right]$$

and $\phi \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ was established. Under specific assumptions about the initial density k_0, it was demonstrated that the characteristics of the Vlasov-Dirac-Benney equation offer a reliable approximation of the N-particle trajectories, provided their initial positions are independently and identically distributed with respect to density k_0.

The second part presented a probabilistic proof of the mean-field limit and chaos propagation of an N-particle system in three dimensions for a Coulomb force

$$f^N(q) = \pm\frac{q}{|q|^3} \text{ with a cutoff } |q| > N^{-\frac{5}{12}+\sigma},$$

where $\sigma > 0$ can be arbitrarily small. Notably the cut-off diameter was of a smaller order compared to the average distance between particles and their nearest neighbors. Nevertheless the results we obtained are just one step towards a conclusive derivation and leave room for improvements. One likes to further reduce the size of the cut-off or, ideally, dispense with the microscopic regularization altogether. But we can see that adding a additional particle group improves the cut off size. This can be utilized to further develop the optimal cut-off size, but it will result in significantly more estimates. Due to Law of Large numbers this strategy has its limit at a cut-off size of $N^{-\frac{1}{2}}$, because we can not expect better control than $N^{-\frac{1}{2}}$ fluctuations around the expectation. To further improve the cut-off size or give a derivation for Vlasov-Poisson equation without regularization we believe one has to change the notion of distance. The third section provided an outlook on a novel technique that could lead to significant improvements for the full Coulomb case. This required redefining the concept of convergence to a convergence in the distributional sense. Using a probabilistic mean-field approach, it was demonstrated that a small displacement of a particle at time zero results in a small effect on the dynamics of the entire system. Thus the deviation remains in the order of magnitude of the shift. Furthermore an even stronger result was showcased for the remaining particles that were initially undisturbed. The observation was made that the deviation of the undisturbed particles from the true and disturbed system decreases as N increases.

Overall, this work has not only made substantial progress in mathematically deriving Vlasov-like equations for highly singular interactions, but has also revealed new avenues for improving these models. The probabilistic methods and approximations presented here offer promising ideas for studying complex systems with a large particle limit and highly singular interaction. However there is still room for further research, particularly regarding the complete Coulomb interaction without regularization and new proof techniques within this context.

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Eidesstattliche Erklärung

Hiermit versichere ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt worden ist.

....

Manuela P. Feistl

München, den 23. Januar 2024