Lattice Polygons and Surfaces with Torus Action

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Introduction

This thesis contributes to the classification of log del Pezzo surfaces with a torus action.

By *del Pezzo surface* we mean a normal projective algebraic surface X over an algebraically closed field of characteristic 0 that admits an ample anticanonical divisor $-\mathcal{K}_X$. The smooth del Pezzo surfaces can be classified by classical methods. They are known to be: the product $\mathbb{P}_1 \times \mathbb{P}_1$ of the projective line with itself, the projective plane $\mathbb{P}_2$ and the blow-ups of $\mathbb{P}_2$ in up to eight points in general position. For a smooth del Pezzo surface X, there is the relation

$$\mathcal{K}_X^2 + \rho(X) \ = \ 10$$

by Noether's formula. The self intersection number $\mathcal{K}_X^2$ is called the *degree* of X and $\rho(X)$ denotes the Picard number of X. The del Pezzo surfaces of degree at least 4 can be described as an intersection of quadrics in a projective space. Those of degree 3 are given as cubics in $\mathbb{P}_3$, those of degree 2 as quartics in $\mathbb{P}_{1,1,1,2}$ and those of degree 1 as sextics in $\mathbb{P}_{1,1,2,3}$.

Allowing for singularities on a del Pezzo surface X, a common measure for their mildness arises from looking at some resolution $\pi\colon X' \to X$ of singularities. The associated *ramification formula* is

$$\mathcal{K}_{X'} \ = \ \pi^*\mathcal{K}_X + \sum a(E)E.$$

Here, E runs through the exceptional prime divisors and the $a(E) \in \mathbb{Q}$ are the *discrepancies* of π. The surface X is called

- *log terminal*, if $a(E) > -1$ for each E,
- *ε-log terminal*, if $a(E) > -1 + \varepsilon$ for each E,
- *ε-log canonical*, if $a(E) \geq -1 + \varepsilon$ for each E,
- *terminal*, if it is 1-log terminal,
- *canonical*, if it is 1-log canonical.

This does not depend on the choice of π. A log terminal del Pezzo surface is also called a *log del Pezzo surface*. By [**38**, Prop. 3.6], log del Pezzo surfaces are necessarily rational. Alexeev showed that for given ε there are only finitely many families of ε-log terminal del Pezzo surfaces, see [**1**].

Another important invariant of a del Pezzo surface X is its *Gorenstein index*. By definition this is the smallest positive integer ι_X such that $\iota_X\mathcal{K}_X$ is a Cartier divisor. In this setting, the simplest class is given by the *Gorenstein del Pezzo surfaces* X, i.e. those with $\iota_X = 1$. Here, cones over elliptic curves provide the only non-rational examples, see [**32**, Thm. 2.2]. The Gorenstein del Pezzo surfaces X having only rational singularities are precisely the ones admitting at most ADE singularities, i.e. rational double points. This in turn is equivalent to X having at most canonical singularities. Their minimal resolutions are precisely the *weak del Pezzo surfaces*, i.e. the smooth rational surfaces with a big and nef anticanonical divisor. The weak del Pezzo surfaces turn out to be precisely the iterated blow-ups of the projective plane in up to eight points in almost general position. This finally leads to the classification of the Gorenstein rational del Pezzo surfaces, see [**17**, **18**, **32**].

Alexeev and Nikulin provided all possible intersection graphs of a certain resolution of singularities of log del Pezzo surfaces of Gorenstein index 2 in [**2**]. This classifies them up to equisingular deformation. The theory of $K3$ surfaces played a substantial role in their work. Independently and using a different approach, Nakayama also succeeded in classifying the log del Pezzo surfaces of Gorenstein index 2, see [**38**]. Nakayama's approach was adopted by Fujita and Yasutake in [**22**] to cover the case of Gorenstein index 3.

In this dissertation we focus on log del Pezzo surfaces X that come with an effective morphical action $\mathbb{T} \times X \to X$ of a non-trivial algebraic torus $\mathbb{T}$.

If $\mathbb{T} \cong \mathbb{K}^* \times \mathbb{K}^*$ we are in the setting of *toric surfaces*. These are particularly accessible via their combinatorial description in terms of fans. All toric surfaces have at most cyclic quotient singularities. In particular, they are all log terminal. Moreover, the Gorenstein index can be explicitly read off from the defining fan.

Toric (log) del Pezzo surfaces correspond to *LDP-polygons*, i.e. two-dimensional convex polytopes in $\mathbb{Q}^2$ having the origin in its interior and only primitive lattice points as vertices. This correspondence allowed explicit classifications up to Gorenstein index 17, see [**10**, **35**]. A toric del Pezzo surface X is ε-log canonical if and only if the associated LDP-polygon $\mathcal{P}_X$ satisfies

$$\varepsilon \mathcal{P}_X^\circ \cap \mathbb{Z}^2 \ = \ \{0\}.$$

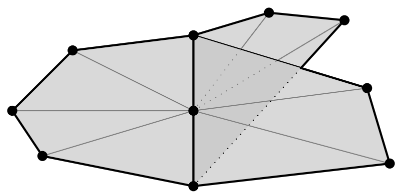
Using this criterion we develop methods for explicit classification and yield the following results.

Theorem 1. *We obtain the following statements on toric ε-log canonical del Pezzo surfaces.*

- $\varepsilon = 1$: *Up to isomorphy there are exactly* 16 *toric canonical del Pezzo surfaces. These are the well known Gorenstein toric del Pezzo surfaces. The maximum Picard number is* 4*, realized by exactly one surface.*
- $\varepsilon = \frac{1}{2}$: *Up to isomorphy there are exactly* 505 *toric* 1/2*-log canonical del Pezzo surfaces. The maximum Picard number is* 6*, realized by exactly one surface.*
- $\varepsilon = \frac{1}{3}$: *Up to isomorphy there are exactly* 48032 *toric* 1/3*-log canonical del Pezzo surfaces. The maximum Picard number is* 10*, realized by exactly one surface.*

The other possible case for the acting torus is $\mathbb{T} \cong \mathbb{K}^*$. This means we deal with so-called $\mathbb{K}^*$*-surfaces*. Similarly to toric surfaces, the $\mathbb{K}^*$-surfaces have been studied intensively for a long time, see for example [**19**–**21**, **41**–**44**]. It should be noted that all log terminal surface singularities are obtained as quotients of the affine plane $\mathbb{K}^2$ by finite subgroups of the general linear group $\mathrm{GL}_2(\mathbb{K})$ and thus come with a $\mathbb{K}^*$-action. This makes $\mathbb{K}^*$-surfaces particularly interesting for the general study of log del Pezzo surfaces. Moreover, different combinatorial approaches developed in [**3**, **5**, **27**, **30**] make $\mathbb{K}^*$-surfaces a very accessible class. This was used by Huggenberger in [**34**] to classify Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces and by Süß for the case of Picard number 1 and Gorenstein index at most 3, see [**45**].

To expand on these classifications, our combinatorial main tool is the *anticanonical complex*, first presented in [**6**]. It is a polytopal complex that generalizes the LDP-polygon from the toric case.



Anticanonical complex.

As with toric surfaces and their associated LDP-polygons, all geometric properties of a log del Pezzo $\mathbb{K}^*$-surface are encoded in its corresponding anticanonical complex. We exclusively use this language to obtain the following result.

Theorem 2. *There are exactly* 154161 *isomorphy classes of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Picard number* 1 *and Gorenstein index* $\iota \leq 200$.

A table with the specific numbers of isomorphy classes of given Gorenstein index is presented in Proposition 4.3.14.

We broaden our view to ε-log canonical del Pezzo $\mathbb{K}^*$-surfaces. In this context, there are characterizations analogous to the toric case. For a log del Pezzo $\mathbb{K}^*$-surface X and its anticanonical complex $\mathcal{A}_X$ we have:

- X is ε-log terminal if and only if 0 is the only lattice point in $\varepsilon\mathcal{A}_X$.
- X is ε-log canonical if and only if 0 is the only lattice point in $\varepsilon\mathcal{A}_X^\circ$.

In order to obtain explicit classifications for these classes of surfaces we need the notion of contractions and combinatorial minimality. A normal, complete surface X is *combinatorially minimal* if every contraction $X \to Y$ is an isomorphism. This can be expressed in terms of anticanonical complexes and is used to get the following.

Theorem 3. *We obtain the following statements on non-toric combinatorially minimal ε-log canonical del Pezzo $\mathbb{K}^*$-surfaces.*

$\varepsilon = 1$: *There are exactly* 13 *sporadic and* 2 *one-parameter families of non-toric combinatorially minimal canonical del Pezzo $\mathbb{K}^*$-surfaces.*

$\varepsilon = \frac{1}{2}$: *There are exactly* 62 *sporadic and* 5 *one-parameter families of non-toric combinatorially minimal* $1/2$*-log canonical del Pezzo $\mathbb{K}^*$-surfaces.*

$\varepsilon = \frac{1}{3}$: *There are exactly* 318 *sporadic and* 14 *one-parameter families of non-toric combinatorially minimal* $1/3$*-log canonical del Pezzo* $\mathbb{K}^*$*-surfaces.*

To make use of these results, we develop a process to systematically build up anticanonical complexes of ε-log canonical del Pezzo $\mathbb{K}^*$-surfaces from complexes corresponding to combinatorially minimal surfaces and LDP-polygons from the toric case. Algorithms have been implemented that yield these results:

Theorem 4. *We obtain the following statements on non-toric* ε*-log canonical del Pezzo* $\mathbb{K}^*$*-surfaces.*

$\varepsilon = 1$: *There are exactly* 30 *sporadic and* 4 *one-parameter families of canonical non-toric del Pezzo* $\mathbb{K}^*$*-surfaces. The maximal Picard number is* 4*, realized by* 1 *sporadic and* 1 *one-parameter family.*

$\varepsilon = \frac{1}{2}$: *There are exactly* 998 *sporadic,* 184 *one-parameter families,* 40 *two-parameter families,* 12 *three-parameter families,* 2 *four-parameter families and* 1 *five-parameter family of non-toric* $1/2$*-log canonical del Pezzo* $\mathbb{K}^*$*-surfaces. The maximal Picard number is* 8*, realized by the unique five-parameter family.*

$\varepsilon = \frac{1}{3}$: *There are exactly* 65022 *sporadic,* 12402 *one-parameter families,* 3190 *two-parameter families,* 917 *three-parameter families,* 254 *four-parameter families,* 64 *five-parameter families,* 14 *six-parameter families,* 6 *seven-parameter families,* 2 *eight-parameter families and* 1 *nine-parameter family of non-toric* $1/3$*-log canonical del Pezzo* $\mathbb{K}^*$*-surfaces. The maximal Picard number is* 12*, realized by the unique nine-parameter family.*

Since $1/k$-log canonical del Pezzo surfaces contain all log del Pezzo surfaces of Gorenstein index k, we can get the following by filtering the previous classifications.

Corollary 5. *We have the following statements on non-toric log del Pezzo* $\mathbb{K}^*$*-surfaces of Gorenstein index* ι*.*

$\iota = 1$: *There are exactly* 30 *sporadic and* 4 *one-parameter families of non-toric log del Pezzo* $\mathbb{K}^*$*-surfaces of Gorenstein index* 1*. The maximal Picard number is 4, realized by* 1 *sporadic and* 1 *one-parameter family.*

$\iota = 2$: *There are exactly* 53 *sporadic,* 17 *one-parameter families,* 7 *two-parameter families,* 3 *three-parameter families,* 1 *four-parameter family and* 1 *five-parameter family of non-toric log del Pezzo* $\mathbb{K}^*$*-surfaces of Gorenstein index* 2*. The maximal Picard number is* 8*, realized by the unique five-parameter family.*

$\iota = 3$: *There are exactly* 268 *sporadic,* 123 *one-parameter families,* 67 *two-parameter families,* 36 *three-parameter families,* 18 *four-parameter families,* 10 *five-parameter families,* 5 *six-parameter families,* 3 *seven-parameter families,* 1 *eight-parameter family and* 1 *nine-parameter family of non-toric log del Pezzo* $\mathbb{K}^*$*-surfaces of Gorenstein index* 3*. The maximal Picard number is* 12*, realized by the unique nine-parameter family.*

The defining data of all classified log del Pezzo $\mathbb{K}^*$-surfaces from Theorems 2 and 4 together with basic invariants such as Picard number, degree, Gorenstein index, number of singularities etc. will be made available in [**25**].

This dissertation is organized in the following way. The first chapter treats two-dimensional lattice polytopes, particulary those who do not contain k-fold lattice points, i.e. elements of $k\mathbb{Z}^n$. A description of a standard form for such lattice triangles is presented. Furthermore, the *Farey sequences* are used to classify these triangles. This view has found application in [**11**]. The second Chapter is dedicated to toric surfaces. We provide a quick reminder on toric varieties in general and present everything we need for the surface case, especially the methods necessary for the classifications mentioned above. Chapter 3 provides all the basic background on $\mathbb{K}^*$-surfaces in general and their combinatorial treatment. We show how to determine invariants like divisor class group, Cox Ring, Picard group, anticanonical divisor, singularities and the surface's intersection theory from its defining data. Moreover, we go into the details of the computation of resolutions of singularities and the surface's Gorenstein index. In the fourth Chapter we specify to (non-toric) log del Pezzo $\mathbb{K}^*$-surfaces and the anticanonical complex is introduced. We present algorithms to classify log del Pezzo $\mathbb{K}^*$-surfaces without quasismooth elliptic fixed points and ones of Picard number 1. The fifth Chapter treats $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces. First, we discuss contractions and combinatorial minimality. Then, using these concepts, the details of the classification for $k = 1, 2, 3$ are presented. Finally, Chapter 6 treats K-polystability and Ricci-flat Kähler cone metrics, especially in the case of $\mathbb{K}^*$-surfaces. We will algorithmically test the classified surfaces from previous chapters for these properties.

CHAPTER 1

Lattice polygons

1.1. k-empty lattice triangles

The major part of this Section contributes to [**11**] as Section 3.1. We give a description of a standard form of k-empty lattice triangles. Then, the Farey sequences are used to classify those. We set $\Delta := \{(x,y) \in \mathbb{Z}^2;\ 0 \le y < x\}$. Members of $k\mathbb{Z}^n$ for $k \in \mathbb{Z}_{\ge 1}$ are called *k-fold lattice points.*

Definition 1.1.1. Let $n, k \in \mathbb{Z}_{\ge 1}$ and consider a convex rational polytope $\mathcal{P} \subseteq \mathbb{Q}^n$. The set of vertices of $\mathcal{P}$ is denoted by $\mathcal{V}(\mathcal{P})$, the relative interior by $\mathcal{P}^\circ$ and the boundary by $\partial\mathcal{P}$. We call $\mathcal{P}$

(i) a *lattice polytope* if $\mathcal{V}(\mathcal{P}) \subseteq \mathbb{Z}^n$.
(ii) a *lattice polygon* if $\mathcal{P}$ is a lattice polytope and $n = 2$.
(iii) *k-empty* if $\mathcal{P} \cap k\mathbb{Z}^n \subseteq \mathcal{V}(\mathcal{P})$.
(iv) *almost k-empty* if $\mathcal{P} \cap k\mathbb{Z}^2 \subseteq \mathcal{V}(\mathcal{P}) \cup \{(0,0)\}$ and $(0,0) \in \mathcal{P}^\circ$.
(v) *k-hollow* if $\mathcal{P}^\circ \cap k\mathbb{Z}^2 = \emptyset$.
(vi) *almost k-hollow* if $\mathcal{P}^\circ \cap k\mathbb{Z}^2 = \{(0,0)\}$.

Definition 1.1.2. The group $\mathrm{Aff}_k^n(\mathbb{Z})$ of *k-affine unimodular transformations in $\mathbb{Q}^n$* is defined by

$$\mathrm{Aff}_k^n(\mathbb{Z}) \ := \ \{T\colon \mathbb{Q}^n \to \mathbb{Q}^n;\ T(v) = Av + w,\ A \in \mathrm{GL}_n(\mathbb{Z}), w \in k\mathbb{Z}^n\}\,.$$

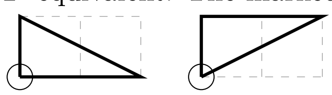
It naturally acts on the set of lattice polytopes in $\mathbb{Q}^n$. Lattice polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ are called *k-equivalent*, if $\mathcal{P}_2 \in \mathrm{Aff}_k(\mathbb{Z}) \cdot \mathcal{P}_1$. Additionally, we call 1-equivalent polytopes *lattice equivalent.*

Remark 1.1.3. Let $T \in \mathrm{Aff}_k^n(\mathbb{Z})$ and $\mathcal{P}$ be a lattice polytope. Then the following hold.

- $T(\mathbb{Z}^n) = \mathbb{Z}^n$.
- $T(k\mathbb{Z}^n) = k\mathbb{Z}^n$.
- $T(\mathcal{V}(\mathcal{P})) = \mathcal{V}(T(\mathcal{P}))$.
- $T(\partial\mathcal{P}) = \partial T(\mathcal{P})$.
- $T(\mathcal{P}^\circ) = T(\mathcal{P})^\circ$.
- $\mathrm{vol}(\mathcal{P}) = \mathrm{vol}(T(\mathcal{P}))$.

Therefore, the number of vertices, the number of (interior) lattice points and the number of (interior) k-fold lattice points are invariant under the action of $\mathrm{Aff}_k^n(\mathbb{Z})$.

Note that k-equivalence of lattice polytopes indeed depends on the specific value of k. Consider for example the following pair of lattice polygons. They are 1-equivalent but not 2- equivalent. The marked point is the origin.



Definition 1.1.4. Let $k \in \mathbb{Z}_{\geq 1}$ and $\mathcal{P}$ be a lattice polygon. We set

$$a_{\mathcal{P},k} \;:=\; \min\{\text{number of lattice points in the relative interior of } \mathcal{E}\} \;+\; 1,$$

where $\mathcal{E}$ runs through the edges of $\mathcal{P}$ which have a vertex in $k\mathbb{Z}^2$.

Definition 1.1.5 (Standard form of k-empty lattice triangles)**.** Let $k \in \mathbb{Z}_{\geq 1}$ and $\mathcal{S}$ be a k-empty lattice polygon with exactly three vertices such that one of them is in $k\mathbb{Z}^2$. We refer to $\mathcal{S}$ as in *standard form*, if the following conditions are satisfied.

(i) $\mathcal{S}$ has the vertices $(0,0),(0,a_{\mathcal{S},k})$ and (x,y) where $(x,y) \in \Delta$.
(ii) If $(x,y) \notin k\mathbb{Z}^2$ and $\gcd(x,y) = a_{\mathcal{S},k}$, then for each $z = 1,\dots,y-1$ we have $a_{\mathcal{S},k} \nmid z$ or $a_{\mathcal{S},k}x \nmid a_{\mathcal{S},k}^2 - zy$.
(iii) If $(x,y) \in k\mathbb{Z}^2$ and $\gcd(x,y) = a_{\mathcal{S},k}$, then for each $z = 1,\dots,y-1$ we have $a_{\mathcal{S},k} \nmid z$ or $a_{\mathcal{S},k}x \nmid a_{\mathcal{S},k}(z+y) - zy$.

If $\mathcal{S}$ is in standard form, we write $\mathcal{S} = \Delta(a_{\mathcal{S},k}, x, y)$. The simplex $\mathcal{S}$ is called *minimal* if $a_{\mathcal{S},k} = 1$.

Remark 1.1.6. The last two conditions of Definition 1.1.5 ensure that the second coordinate of the vertex $(x,y) \in \Delta$ is minimal. To illustrate this, consider the 2-equivalent 2-empty polytopes

$$\begin{aligned} \mathcal{P}_1 &= \operatorname{conv}((0,0),(0,1),(5,3)), \\ \mathcal{P}_2 &= \operatorname{conv}((0,0),(0,1),(5,2)). \end{aligned}$$

We can see that $\mathcal{P}_1$ does not fulfill Condition (ii) so it is not in standard form whereas $\mathcal{P}_2$ is. Therefore, these conditions make sure that the *right* value is chosen for the second component of the third vertex. This is relevant in case that there are several edges which attain the minimum of Definition 1.1.4.

Proposition 1.1.7. *Let $\mathcal{S}$ be a k-empty lattice triangle with a vertex $z \in k\mathbb{Z}^2$. Then there is a unique lattice triangle $\mathcal{S}'$ in standard form that is k-equivalent to $\mathcal{S}$.*

Proof. There are three cases depending on the number of vertices of $\mathcal{S}$ in $k\mathbb{Z}^2$.

Case 1. There is exactly one vertex $z \in k\mathbb{Z}^2$. Let v_1 and v_2 be the other two vertices and d_i the number of lattice points in the relative interior of the edge of $\mathcal{S}$ with vertices z and v_i. We have $a_{\mathcal{S},k} = \min\{d_1, d_2\} + 1$.

Case 1.1. $d_1 \neq d_2$. We can assume, without loss of generality, that $d_1 < d_2$. So we have $a_{\mathcal{S},k} = d_1 + 1$. Consider the k-affine unimodular transformation T_1 given by $T_1(v) = v - z$. The coordinates of $T_1(v_1)$ have the greatest common divisor $a_{\mathcal{S},k}$. Thus, there is a k-affine unimodular transformation T_2 that leaves the origin fixed and takes $T_1(v_1)$ to $(0, a_{\mathcal{S}})$. A third transformation T_3 sends $T_2(T_1(v_2))$ to a point $(x,y) \in \Delta$ without changing the coordinates of $T_2(T_1(v_1))$ and $T_2(T_1(z))$. The k-empty lattice triangle $\mathcal{S}' := T_3 \circ T_2 \circ T_1(S)$ satisfies the conditions of Definition 1.1.5. Note that $\gcd(x,y) \neq a_{\mathcal{S}}$ since $d_1 \neq d_2$. The representation is obviously unique in this case.

Case 1.2. $d_1 = d_2$. As before, let T_1 be given by $T_1(v) = v - z$. Then, let T_2 be the k-affine unimodular transformation that leaves the origin fixed and sends $T_1(v_1)$ to $(0, a_{\mathcal{S},k})$. We choose a transformation T_3 that takes $T_2(T_1(v_2))$ to a point $(x, y) \in \Delta$ without changing the coordinates of $(0, a_{\mathcal{S},k})$ and $(0, 0)$. Analogously, let T_2' be the k-affine unimodular transformation that leaves the origin fixed and sends $T_1(v_2)$ to $(0, a_{\mathcal{S},k})$. We choose a transformation T_3' that takes $T_2'(T_1(v_1))$ to a point $(x, y') \in \Delta$ without changing the coordinates of $(0, a_{\mathcal{S},k})$ and $(0, 0)$. Without loss of generality we have $y < y'$. Set $\mathcal{S}' := T_3 \circ T_2 \circ T_1(S)$.

The lattice triangle $\mathcal{S}'$ fulfills Condition (i) of Definition 1.1.5. Suppose that it does not satisfy Condition (ii). That means there is a z with $1 \leq z \leq y-1$ such that $a_{\mathcal{S},k}|z$ and $a_{\mathcal{S},k}x|a_{\mathcal{S},k}^2 - zy$. Consider the transformation T given by the matrix

$$A \; = \; \begin{bmatrix} -\frac{y}{a_{\mathcal{S},k}} & \frac{x}{a_{\mathcal{S},k}} \\ \frac{a_{\mathcal{S},k}^2 - zy}{a_{\mathcal{S},k}x} & \frac{z}{a_{\mathcal{S},k}} \end{bmatrix} \; \in \; \mathrm{GL}_2(\mathbb{Z})$$

and apply it to $\mathcal{S}'$. Since $(x, z) \in \Delta$ is a vertex of $T(\mathcal{S}')$, we have $z = y'$. So $y' = z < y < y'$ which is a contradiction. Thus $\mathcal{S}'$ is in standard form and by construction unique.

Case 2. There are exactly two vertices $z_1, z_2 \in k\mathbb{Z}^2$. Let v be the third vertex and d_i the number of lattice points in the relative interior of the edge of $\mathcal{S}$ with vertices z_i and v. As in Case 1 we have $a_{\mathcal{S},k} = \min\{d_1, d_2\} + 1$.

Case 2.1. $d_1 \neq d_2$. Without loss of generality $d_1 < d_2$. So we have $a_{\mathcal{S},k} = d_1 + 1$. Consider the k-affine unimodular transformation T_1 which is given by $T_1(v') = v' - z_1$. Let T_2 be such that $T_2(0) = 0$ and $T_2(v - z_1) = (0, a_{\mathcal{S},k})$. Then, choose a third transformation T_3 which fixes $(0, 0)$ and $(0, a_{\mathcal{S},k})$ and sends $T_2(T_1(z_2))$ to a point $(x, y) \in \Delta$. The lattice triangle $\mathcal{S}' := T_3 \circ T_2 \circ T_1(S)$ satisfies the conditions of Definition 1.1.5. As before, note that $\gcd(x, y) \neq a_{\mathcal{S},k}$ since $d_1 \neq d_2$. The representation is unique.

Case 2.2. $d_1 = d_2$. Let T_1 be given by $T_1(v') = v' - z_1$ and T_2 be the transformation fixing the origin and $T_2(T_1(v)) = (0, a_{\mathcal{S},k})$. Choose additionally T_3 such that the origin and $(0, a_{\mathcal{S},k})$ are fixed and $(x, y) := T_3(T_2(T_1(z_2))) \in \Delta$. Accordingly, let T_1' be defined by $T_1'(v') = v' - z_2$ and T_2' fixing $(0, 0)$ and $T_2'(T_1'(v)) = (0, a_{\mathcal{S},k})$. Finally, choose a transformation T_3' which fixes $(0, 0)$ and $(0, a_{\mathcal{S},k})$ and $(x, y') := T_3'(T_2'(T_1'(z_2))) \in \Delta$. Again, without loss of generality, we have $y < y'$. We set $\mathcal{S}' := T_3 \circ T_2 \circ T_1(S)$.

Assume that Condition (iii) of Definition 1.1.5 were not satisfied. Then there is a z with $1 \leq z \leq y - 1$ such that $a_{\mathcal{S},k}|z$ and $a_{\mathcal{S},k}x|a_{\mathcal{S},k}(z + y) - zy$. Consider the transformation T given by the matrix

$$A \; = \; \begin{bmatrix} 1 - \frac{y}{a_{\mathcal{S},k}} & \frac{x}{a_{\mathcal{S},k}} \\ \frac{a_{\mathcal{S},k}(z+y) - zy}{a_{\mathcal{S},k}x} & \frac{z}{a_{\mathcal{S},k}} - 1 \end{bmatrix} \; \in \; \mathrm{GL}_2(\mathbb{Z})$$

and apply it to the lattice triangle

$$\mathrm{conv}\left((0, 0), (x, y - a_{\mathcal{S},k})\right), (x, y))$$

which is k-equivalent to $\mathcal{S}'$. The lattice triangle $T(\mathcal{S}')$ has the vertex (x, z) and so we have $z = y'$. That means $y' = z < y < y'$ which is a contradiction. Therefore $\mathcal{S}'$ is in standard form and the uniqueness is clear by construction.

Case 3. There are three vertices $z_1, z_2, z_3 \in k\mathbb{Z}^2$. Since the number of lattice points in the relative interior of each edge of $\mathcal{S}$ is $k-1$, the only possible standard form is the lattice triangle $\operatorname{conv}((0,0),(0,k),(k,0))$. $\square$

Definition 1.1.8. Let $k \in \mathbb{Z}_{\geq 1}$. We define the *$k$-th Farey sequence* to be

$$F_k \ := \ \left(\frac{f_1}{f_2};\ 0 \leq f_1 < f_2 \leq k,\ \gcd(f_1, f_2) = 1\right).$$

Members of F_k are called *k-th Farey numbers.* Let $f = \frac{f_1}{f_2}$ be a k-th Farey number. We define the *k-th Farey strip* corresponding to f to be the convex set

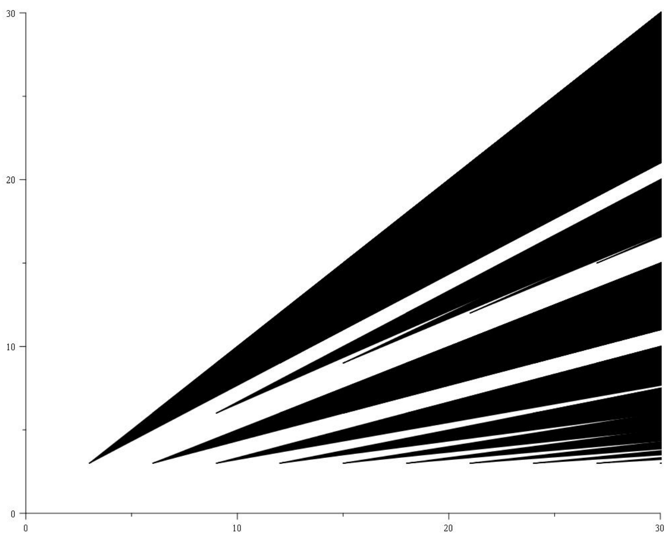
$$F_{k,f} \ := \ \begin{cases} \left\{(x,y)\,;\ 0 \ < \ \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -f_1 \\ f_2 \end{bmatrix} \ < \ k \right\}, & \text{if } f_2 = k, \\ \left\{(x,y)\,;\ 0 \ < \ \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -f_1 \\ f_2 \end{bmatrix} \ \leq \ k \right\}, & \text{if } f_2 \neq k. \end{cases}$$

Remark 1.1.9. By definition, the number of k-th Farey strips equals the length of the k-th Farey sequence. That is, there are $\varphi(1) + \cdots + \varphi(k)$ many k-th Farey strips where φ is the Euler totient function.

Definition 1.1.10. A *spike* attached to a k-th Farey strip $F_{k,f}$ is a 2-dimensional convex polytope $\mathcal{S}$ with exactly three rational vertices satisfying the following conditions.

(i) Two of the vertices of $\mathcal{S}$ are k-fold lattice points and lie the line $y = \frac{f_1}{f_2}x + \frac{k}{f_2}$.
(ii) One vertex of $\mathcal{S}$ is above the line $y = \frac{f_1}{f_2}x + \frac{k}{f_2}$.
(iii) If (x,y) is a lattice point in $\mathcal{S}$, then $\operatorname{conv}((0,0),(0,1),(x,y))$ is k-empty.

The following picture shows the k-th Farey strips and spikes attached to them for $k = 3$. There, we can see the four strips $F_{3,0}, F_{3,\frac{1}{3}}, F_{3,\frac{1}{2}}, F_{3,\frac{2}{3}}$ and the only occurring spikes attached to $F_{3,0}$ and $F_{3,\frac{1}{2}}$.



Proposition 1.1.11. *Let $k \in \mathbb{Z}_{\geq 1}$ and $f = \frac{f_1}{f_2}$ be a k-th Farey number. If the lattice triangle $\mathcal{S} = \operatorname{conv}((0,0),(0,1),(x,y))$ is contained in the k- th Farey strip $F_{k,f}$, it is k-empty.*

PROOF. Assume that $\mathcal{S}$ is not k-empty. Then we find $a, b \in \mathbb{Z}_{\geq 1}$ such that $(ka, kb) \in \mathcal{S} \setminus \mathcal{V}(\mathcal{P}) \subseteq F_{k,f}$. By definition we have $0 < f_2kb - f_1ka \leq k$ and therefore $0 < f_2b - f_1a \leq 1$. So $f_2b - f_1a = 1$. This means that (ka, kb) cannot lie in the interior of $F_{k,f}$. If $f_2 = k$, this is already a contradiction. If on the other hand $f_2 \neq k$, the point (ka, kb) must be a vertex of $\mathcal{S}$ which is again a contradiction. □

Definition 1.1.12. Let $k \in \mathbb{Z}_{\geq 1}$. A k-empty lattice triangle in standard form which is not contained in a k-th Farey strip is called *sporadic.*

Proposition 1.1.13. *Let $k \in \mathbb{Z}_{\geq 1}$. The number of minimal sporadic k-empty lattice triangles in standard form is finite. Explicitly, the first coordinate of the vertex in Δ is bounded by $(k^2-1)k-1$.*

PROOF. Let $(x, y) \in \mathbb{Z}^2$ such that $\operatorname{conv}((0,0),(0,1),(x,y))$ is a sporadic k-empty lattice triangle in standard form. Then there is a k-th Farey strip $F_{k,f}$ and there is a spike $\mathcal{S}$ attached to it, such that (x, y) is in the interior of it.

We have $f_2 \neq k$. Otherwise $\mathcal{S}$ is empty. For some $i > k - f_2$ the spike has the vertices

$$\left(ik, \frac{f_1}{f_2}ik + \frac{k}{f_2}\right), \ \left((i+f_2)k, \frac{f_1}{f_2}(i+f_2)k + \frac{k}{f_2}\right),$$
$$\left(\frac{(i+f_2)ik}{i-k+f_2}, \frac{\frac{f_1}{f_2}(i+f_2)ik + i\frac{k}{f_2}}{i-k+f_2}\right).$$

Its area is therefore given by

$$A(\mathcal{S}) \ := \ \frac{1}{2}\left(\frac{ik^2}{i-k+f_2} - k^2\right).$$

Let $I(\mathcal{S})$ be the number of interior integral points of $\mathcal{S}$ and $B(\mathcal{S})$ the number of integral points on the boundary of $\mathcal{S}$. By Pick's theorem we have

$$A(\mathcal{S}) = I(\mathcal{S}) + \frac{B(\mathcal{S})}{2} - 1.$$

Furthermore, it is $A(\mathcal{S}) < 1$ if and only if

$$i \ > \ \frac{k^3 - f_2k^2 + 2k - 2f_2}{2}.$$

Since $B(\mathcal{S}) \geq 1$, if $A(\mathcal{S}) < 1$ then $I(\mathcal{S}) < 1$. So x is bounded from above by $(k^2-1)k-1$. □

Corollary 1.1.14. *Let $k \in \mathbb{Z}_{\geq 1}$. The number of sporadic k-empty lattice triangles in standard form is finite.*

PROOF. Let $\mathcal{S} = \Delta(a, x, y)$ be a k-empty lattice triangle in standard form. Then there is a minimal k-empty lattice triangle $\mathcal{S}' = \Delta(1, x, y)$ contained in $\mathcal{S}$. □

Remark 1.1.15. The proof of Proposition 1.1.13 shows that we can list the sporadic minimal k-empty lattice triangles in standard form explicitly for a given $k \in \mathbb{Z}_{\geq 1}$. The following table lists the number of those simplices for low values of k.

k	1	2	3	4	5	6
#	0	2	7	32	96	279

Remark 1.1.16. It is possible to obtain the results of Propositions 1.1.11 and 1.1.13 in a different manner. We choose $d = 2$ and $s = k$ in Theorem 2.1 from [**39**] by Nill and Ziegler.

Remark 1.1.17. The terminology of Farey strips and spikes from this section can be extended to cover the more general case of k-hollow polygons. For this, we define the *k-th extended Farey strip* corresponding to a k-th Farey number $f = \frac{f_1}{f_2}$ to be

$$F_{k,f} \;:=\; \left\{(x,y)\,;\; 0 \;\leq\; \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -f_1 \\ f_2 \end{bmatrix} \;\leq\; k\right\}.$$

Then we get a natural analogue to Proposition 1.1.11. Using the same definition for spikes attached to Farey strips as before, we also get a finiteness statement analogous to Proposition 1.1.13.

1.2. Classification of 2-empty lattice polygons

In the following, we explicitly classify the 2-empty lattice polygons with a vertex $v \in 2\mathbb{Z}^2$ up to 2-affine unimodular transformation. First we need a definition and some lemmas.

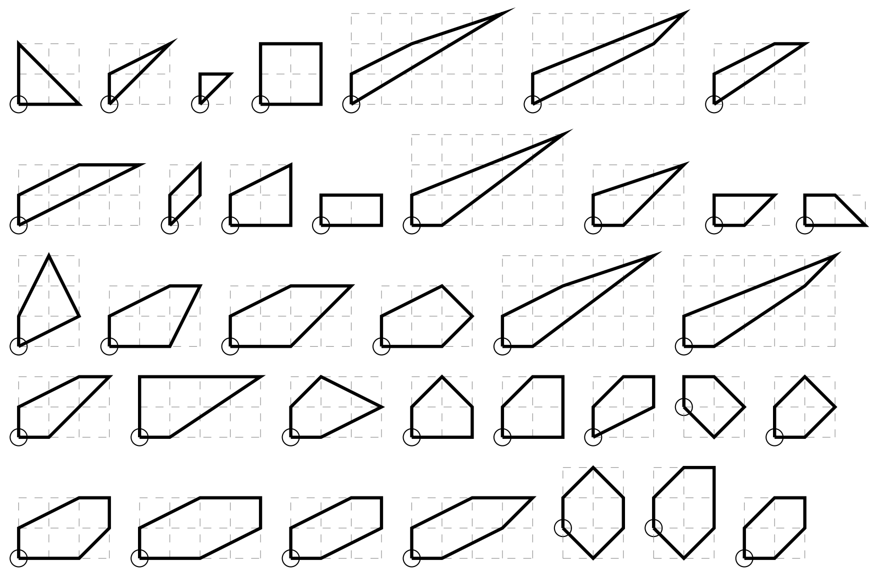


FIGURE 1. The 2-empty convex lattice polygons (with a vertex $v \in 2\mathbb{Z}^2$) up to equivalence with one representative per family. The marked point is the origin.

Definition 1.2.1. Let $C \subseteq \mathbb{R}^2$ be a convex set and $v \in \mathbb{R}^2$. The *shadow cast by C at v* is

$$S(C,v) \ := \ \{w \in \mathbb{R}^2 \ ; \ v \in \operatorname{conv}(C,w)\} \ \subseteq \ \mathbb{R}^2.$$

The *k-lattice shadow of C* is

$$S_k(C,\mathbb{Z}^2) \ := \ \bigcup_{v \in k\mathbb{Z}^2 \setminus C^\circ} S(C,v) \ \subseteq \ \mathbb{R}^2.$$

A k-fold lattice point $w \in S_k(C,\mathbb{Z}^2)$ is called a *vertex of $S_k(C,\mathbb{Z}^2)$* if

$$w \ \notin \ \bigcup_{\substack{v \in \mathbb{Z}^2 \setminus C^\circ \\ v \neq w}} S(C,v).$$

We denote the set of vertices of $S_k(C,\mathbb{Z}^2)$ by $\mathcal{V}(S_k(C,\mathbb{Z}^2))$.

Remark 1.2.2. Let $w_1,\dots,w_r,w_{r+1} \in \mathbb{Z}^2$ and $\mathcal{C} := \operatorname{conv}(w_1,\dots,w_r)$. Then $\operatorname{conv}(w_1,\dots,w_r,w_{r+1})$ is k-empty if and only if

$$w_{r+1} \notin S_k(C,\mathbb{Z}^2) \setminus \mathcal{V}(S_k(C,\mathbb{Z}^2)).$$

Also note that if $r = 2$ the shadow cast by C at $v \in k\mathbb{Z}^2$ is explicitly given by

$$S(C,v) \ = \ v \ + \ \operatorname{cone}(v - w_1, v - w_2) \ \subseteq \ \mathbb{R}^2.$$

Theorem 1.2.3. *Up to 2-affine unimodular transformation, 2-empty lattice polygons with a vertex in $2\mathbb{Z}^2$ are classified by the following list. It shows that the maximal number of vertices of such a polygon is 6. A polygon is represented by a matrix whose columns are its vertices.*

$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$		$\begin{bmatrix} 0 & 0 & x \\ 0 & 1 & 2 \end{bmatrix}$,	$x \geq 2$
$\begin{bmatrix} 0 & 0 & x \\ 0 & 1 & 1 \end{bmatrix}$,	$x \geq 1$		
$\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix}$		$\begin{bmatrix} 0 & 0 & 2 & 2x-1 \\ 0 & 1 & 2 & x \end{bmatrix}$,	$x \geq 3$
$\begin{bmatrix} 0 & 0 & 2x-1 & 4 \\ 0 & 1 & x & 2 \end{bmatrix}$,	$x \geq 3$	$\begin{bmatrix} 0 & 0 & x & x+1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$,	$x \geq 2$
$\begin{bmatrix} 0 & 0 & x & x+2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$,	$x \geq 2$ even	$\begin{bmatrix} 0 & 0 & x & y \\ 0 & 1 & 2 & 1 \end{bmatrix}$,	$2y > x \geq 1$
$\begin{bmatrix} 0 & 0 & x & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$,	$x \geq 2$	$\begin{bmatrix} 0 & 0 & x & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$,	$x \geq 2$
$\begin{bmatrix} 0 & 0 & 5 & 1 \\ 0 & 1 & 3 & 0 \end{bmatrix}$		$\begin{bmatrix} 0 & 0 & x & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}$,	$x \geq 3$
$\begin{bmatrix} 0 & 0 & x & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$,	$x \geq 2$	$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & x & 0 \end{bmatrix}$,	$x \geq 1$
$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & x & 1 \end{bmatrix}$,	$x \geq 3$		
$\begin{bmatrix} 0 & 0 & x & x+1 & 2 \\ 0 & 1 & 2 & 2 & 0 \end{bmatrix}$,	$x \geq 2$	$\begin{bmatrix} 0 & 0 & x & x+2 & 2 \\ 0 & 1 & 2 & 2 & 0 \end{bmatrix}$,	$x \geq 2$ even
$\begin{bmatrix} 0 & 0 & x & y & 2 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}$,	$2y-2 > x \geq 2$	$\begin{bmatrix} 0 & 0 & 2 & 5 & 1 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 5 & 4 & 1 \\ 0 & 1 & 3 & 2 & 0 \end{bmatrix}$,		$\begin{bmatrix} 0 & 0 & x & x+1 & 1 \\ 0 & 1 & 2 & 2 & 0 \end{bmatrix}$,	$x \geq 2$ even
$\begin{bmatrix} 0 & 0 & x & x+2 & 1 \\ 0 & 1 & 2 & 2 & 0 \end{bmatrix}$,	$x \geq 2$ even	$\begin{bmatrix} 0 & 0 & x & y & 1 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix}$,	$2y-1 > x \geq 1$, $x \geq 3$ or $y \geq 3$
$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & x & 1 & 0 \end{bmatrix}$,	$x \geq 2$	$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & x & 2 & 0 \end{bmatrix}$,	$x \geq 2$
$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & x & 2 & 1 \end{bmatrix}$,	$x \geq 2$	$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & x & 0 & y \end{bmatrix}$,	$x \geq 1,\ y \leq -1$
$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & x & 1 & y \end{bmatrix}$,	$x \geq 2,\ y \leq 0$		
$\begin{bmatrix} 0 & 0 & x & x+1 & y & 2 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$,	$2y-3 > x \geq 2$	$\begin{bmatrix} 0 & 0 & x & x+2 & y & 2 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$,	$2y-4 > x \geq 2$, x even
$\begin{bmatrix} 0 & 0 & x & x+1 & y & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$,	$2y-2 > x \geq 2$	$\begin{bmatrix} 0 & 0 & x & x+2 & y & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$,	$2y-3 > x \geq 2$, x even
$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & x & 1 & 0 & y \end{bmatrix}$,	$x \geq 2,\ y \leq -1$	$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & x & 2 & 0 & y \end{bmatrix}$,	$x \geq 2,\ y \leq -1$
$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & x & 2 & 1 & y \end{bmatrix}$,	$x \geq 2,\ y \leq 0$		

To prove the theorem we need some lemmas.

Lemma 1.2.4. *Let $\mathcal{P}$ be a 2-empty lattice polygon with a vertex that is a double lattice point. Then there exists a 2-affine unimodular transformation T, such that $T(\mathcal{P})$ satisfies one of the following.*

(i) *Each vertex is a double lattice point.*
(ii) *There is an edge given by the line segment from $(0,0)$ to $(0,1)$.*

Proof. By translating the polygon, we can assume that $(0,0)$ is a vertex of $\mathcal{P}$. Let $(x,y) \in \mathbb{Z}^2$ be the clockwise adjacent vertex to $(0,0)$ and $a = \gcd(x,y)$. Multiplication with a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ maps (x,y) to $(0,a)$. Because $\mathcal{P}$ is 2-empty, we have $a \leq 2$. If $a = 2$, we can repeat this process. Again, the clockwise adjacent vertex to $(0,a)$ is either double or primitive. By iteration, each vertex is a double lattice point or we find a pair of adjacent vertices, one of which can be assumed to be $(0,0)$ and the other one $(0,1)$. □

Lemma 1.2.5. *Let $\mathcal{P}$ be a 2-empty lattice polygon with adjacent vertices $(0,0)$ and $(0,a)$, where $a \in \{1,2\}$. Assume that $\mathcal{P}$ has no vertex on the line $x = 1$ and a vertex (x,y) with $x \geq 3$. Then there is a 2-affine unimodular* *transformation T such that $(0,0)$ and $(0,a)$ are fixed points and each vertex $(z,w) \neq (0,0),(0,a),(2,2)$ of $T(\mathcal{P})$ satisfies $0 \leq w < z$.*

PROOF. Let B be the line segment from $(0,0)$ to $(0,a)$ and $b \in \mathbb{Z}$. Consider the shadows $S(B,(2,2b))$. For each b, there cannot be a vertex of $\mathcal{P}$ in $S(B,(2,2b)) \setminus \{(2,2b)\}$. That is, the vertices have to be between those shadows. Therefore, the convexity and 2-emptiness of $\mathcal{P}$ yield a $b_0 \in \mathbb{Z}$ such that each vertex different from $(0,0),(0,a)$ is in

$$\begin{aligned} &\left\{(x,y)\ ;\ x \geq 3,\ b_0 \ \leq\ \frac{y}{x}\ \leq\ b_0 + 1 - \frac{a}{2} + \frac{a}{x}\ <\ b_0 + 1\right\} \\ \cup\ &\left\{(2,2b_0), (2,2b_0+1), (2,2b_0+2)\right\} \\ \subseteq\ &\left\{(x,y)\ ;\ x \geq 2,\ 0 \leq\ y - b_0 x\ <\ x\right\}\ \cup\ \left\{(2,2b_0+2)\right\}. \end{aligned}$$

Thus, after multiplying with a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ we have that each vertex $(z,w) \neq (0,0),(0,a),(2,2)$ satisfies $0 \leq w < z$. $\square$

PROOF OF THEOREM 1.2.3. Let $\mathcal{P}$ be a 2-empty lattice polygon with a vertex that is a double lattice point. By Lemma 1.2.4 there are two cases that need to be considered.

Case 1. Each vertex of $\mathcal{P}$ is a double lattice point. As seen in the proof of Lemma 1.2.4 we can assume that up to 2-affine unimodular transformation $(0,0)$ and $(0,2)$ are adjacent vertices of $\mathcal{P}$. Furthermore, by convexity we can assume that additional points are located in the right half space given by the vertical axis.

Case 1.1. There is a vertex (x,y) with $x \geq 3$. By Lemma 1.2.5 we can assume that each vertex $(z,w) \neq (0,0),(0,2),(2,2)$ of $\mathcal{P}$ satisfies $0 \leq w < z$. That means that each such vertex lies in the horizontal strip as illustrated in Figure 2. Since $x \geq 3$ that's not possible. Otherwise there would be an edge of $\mathcal{P}$ with $(2,2)$ or $(2,0)$ in its relative interior and thus violating 2-emptiness.

Case 1.2. There is no vertex (x,y) with $x \geq 3$. The first coordinate of each additional vertex has to be 2. The following possibilities remain.

$$\operatorname{conv}((0,0),(0,2),(2,0)), \qquad \operatorname{conv}((0,0),(0,2),(2,2),(2,0)).$$

Case 2. There is an edge of $\mathcal{P}$ given by the line segment B from $(0,0)$ to $(0,1)$. Again, without loss of generality, additional points are located in the right half space given by the vertical axis.

Case 2.1. There is no vertex on the line $x = 1$ and a vertex (x,y) with $x \geq 3$. By Lemma 1.2.5 each vertex $(z,w) \neq (0,0),(0,1),(2,2)$ satisfies $0 \leq w < z$. We find additional restrictions for the location of the vertices by considering the shadows $S(B,(2c,2))$, where $c \in \mathbb{Z}_{\geq 1}$. The values $c = 1$ and $c = 2$ give shadows that leave a strip in between with possible vertices. Its lattice points are given by

$$\{(2x,x), (2y-2,y), (2z-1,z);\ x,y,z \geq 3\}.$$

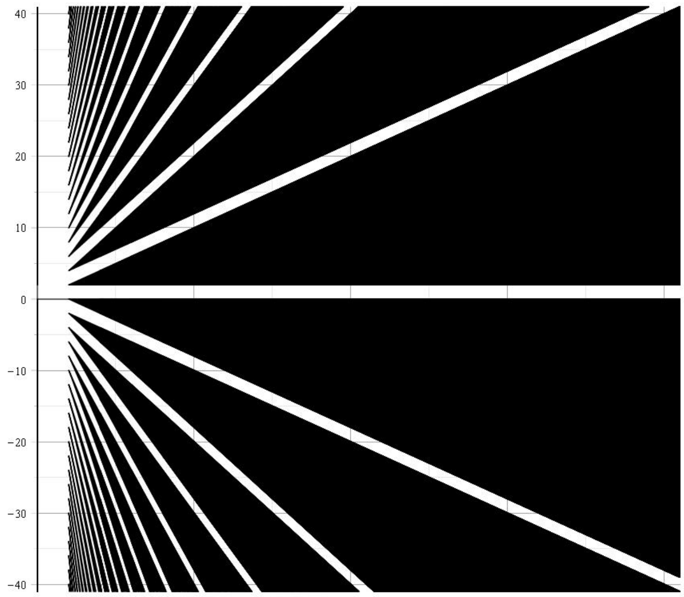


FIGURE 2. The shadows show forbidden areas for vertices of $\mathcal{P}$.

Suppose that one of the vertices of $\mathcal{P}$ is in this strip. Then $\mathcal{P}$ is of one of the following.

$$\begin{aligned}
&\operatorname{conv}\left((0,0),(0,1),(2z-1,z)\right), \ z \geq 3,\\
&\operatorname{conv}\left((0,0),(0,1),(2,2),(2z-1,z)\right), \ z \geq 3,\\
&\operatorname{conv}\left((0,0),(0,1),(2z-1,z),(4,2)\right), \ z \geq 3.
\end{aligned}$$

Now let $c \geq 2$. The shadows $S(B,(2c,2))$ and $S(B,(2(c+1),2))$ intersect in $(\frac{2c(c+1)}{c-1}, \frac{2c}{c-1})$. If $c > 3$, the second coordinate is strictly smaller than 3 and there can't be any lattice point between those shadows. So we only have to consider the cases $c = 2, 3$. Explicitly, one obtains the intersection points $(12,4)$ and $(12,3)$. The simplices forming the area between the shadows are given by $\operatorname{conv}((4,2),(6,2),(12,4)), \operatorname{conv}((6,2),(8,3),(12,3))$, respectively. We see that there can't be a vertex of $\mathcal{P}$ with second coordinate strictly bigger than 2 in these simplices because otherwise there would be a double lattice point in the relative interior of an edge of $\mathcal{P}$.

Consider the case that there are only vertices whose second coordinates are ≤ 2. One obtains the following possibilities for $\mathcal{P}$.

$$\begin{aligned}
&\operatorname{conv}\left((0,0),(0,1),(x,2)\right), \ x \geq 3,\\
&\operatorname{conv}\left((0,0),(0,1),(x,1)\right), \ x \geq 3,\\
&\operatorname{conv}\left((0,0),(0,1),(x,2),(x+1,2)\right), \ x \geq 2,\\
&\operatorname{conv}\left((0,0),(0,1),(x,2),(x+2,2)\right), \ x \geq 2, \ x \text{ even},\\
&\operatorname{conv}\left((0,0),(0,1),(x,2),(y,1)\right), \ 2y > x \geq 2 \text{ and } (x \geq 3 \text{ or } y \geq 3),\\
&\operatorname{conv}\left((0,0),(0,1),(x,2),(2,0)\right), \ x \geq 3,\\
&\operatorname{conv}\left((0,0),(0,1),(x,1),(2,0)\right), \ x \geq 3,
\end{aligned}$$

$$\operatorname{conv}\left((0,0),(0,1),(x,2),(x+1,2),(2,0)\right),\ x \geq 2,$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(x+2,2),(2,0)\right),\ x \geq 2,\ x \text{ even},$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(y,1),(2,0)\right),\ 2y-2 > x \geq 2,$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(x+1,2),(y,1),(2,0)\right),\ 2y-3 > x \geq 2,$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(x+2,2),(y,1),(2,0)\right),\ 2y-4 > x \geq 2,\ x \text{ even}.$$

Case 2.2. There is no vertex on the line $x = 1$ and no vertex (x, y) with $x \geq 3$. Then $\mathcal{P}$ has 3 or 4 vertices and is, up to 2-affine unimodular transformation, one of the following.

$$\operatorname{conv}\left((0,0),(0,1),(2,0)\right), \qquad \operatorname{conv}\left((0,0),(0,1),(2,1)\right),$$
$$\operatorname{conv}\left((0,0),(0,1),(2,1),(2,0)\right), \qquad \operatorname{conv}\left((0,0),(0,1),(2,2),(2,0)\right),$$
$$\operatorname{conv}\left((0,0),(0,1),(2,2),(2,1)\right).$$

Note that the first two polygons transform into each other by applying a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ and a translation by a lattice point. Thus they are 1-equivalent and can be identified as lattice polygons. They are, however, not 2-equivalent.

Case 2.3. There is a vertex $(1, d)$ with $d \in \mathbb{Z}$ and a vertex (x, y) with $x \geq 3$. Up to unimodular transformation we have $0 \leq y < x$. Considering the shadows $S(B, (2, 2b))$, where $b \in \mathbb{Z}$, we obtain $0 \leq d \leq 2$. If $d = 0$ there are the following possibilities for $\mathcal{P}$.

$$\operatorname{conv}\left((0,0),(0,1),(5,3),(1,0)\right),$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(1,0)\right),\ x \geq 3,$$
$$\operatorname{conv}\left((0,0),(0,1),(x,1),(1,0)\right),\ x \geq 3,$$
$$\operatorname{conv}\left((0,0),(0,1),(2,2),(5,3),(1,0)\right),$$
$$\operatorname{conv}\left((0,0),(0,1),(5,3),(4,2),(1,0)\right),$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(x+1,2),(1,0)\right),\ x \geq 2,$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(x+2,2),(1,0)\right),\ x \geq 2,\ x \text{ even},$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(y,1),(1,0)\right),\ 2y-1 > x \geq 1 \text{ and } (x \geq 3 \text{ or } y \geq 3),$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(x+1,2),(y,1),(1,0)\right),\ 2y-2 > x \geq 2,$$
$$\operatorname{conv}\left((0,0),(0,1),(x,2),(x+2,2),(y,1),(1,0)\right),\ 2y-3 > x \geq 2,\ x \text{ even}.$$

The convexity and 2-emptiness of $\mathcal{P}$ rule out the case $d = 1$. If $d = 2$, $\mathcal{P}$ is equal to

$$\operatorname{conv}\left((0,0),(0,1),(1,2),(x,1)\right),\ x \geq 3.$$

Case 2.3. There is a vertex $(1, d)$ with $d \in \mathbb{Z}$ and no vertex (x, y) with $x \geq 3$. If there are no more vertices, $\mathcal{P}$ is up to 2-affine unimodular equivalence of the shape

$$\operatorname{conv}\left((0,0),(0,1),(1,0)\right).$$

Suppose that there is no vertex on the line $x = 2$. Then $\mathcal{P}$ is 2-equivalent to

$$\operatorname{conv}\left((0,0),(0,1),(1,d),(1,0)\right),\ d \geq 1.$$

If on the other hand there is a vertex on the line $x = 2$, then $\mathcal{P}$ can be transformed into one of the following.

$$\begin{aligned}
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,0)\right), \ d \geq 1,\\
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,1)\right), \ d \geq 2,\\
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,1),(2,0)\right), \ d \geq 2,\\
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,2),(2,0)\right), \ d \geq 2,\\
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,2),(2,1)\right), \ d \geq 2,\\
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,0),(1,e)\right), \ d \geq 1, \ e < 0,\\
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,1),(1,e)\right), \ d \geq 2, \ e \leq 0,\\
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,1),(2,0),(1,e)\right), \ d \geq 2, \ e < 0,\\
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,2),(2,0),(1,e)\right), \ d \geq 2, \ e < 0,\\
&\operatorname{conv}\left((0,0),(0,1),(1,d),(2,2),(2,1),(1,e)\right), \ d \geq 2, \ e \leq 0.
\end{aligned}$$

Putting together the results from the different cases (and removing polygons such that there is only one representative per equivalence class) one obtains the list from the Theorem. $\square$

CHAPTER 2

Toric Surfaces

2.1. A quick reminder on toric geometry

We briefly gather general background from toric geometry that will be used in the subsequent sections. As detailed introductory texts, we refer to [**15**, **16**, **23**]. Toric geometry, initiated by Demazure's work [**17**] on the Cremona group, connects algebraic geometry with combinatorics and has rapidly become a rich and intensively studied interplay of these disciplines.

The objects on the side of algebraic geometry are the *standard n-torus*, i.e. the n-fold direct product $\mathbb{T}^n = \mathbb{K}^* \times \ldots \times \mathbb{K}^*$, and *toric varieties*, i.e. open embeddings $\mathbb{T}^n \subseteq Z$ into normal varieties Z such that the multiplication on $\mathbb{T}^n \times \mathbb{T}^n \to \mathbb{T}^n$ extends to a morphical action $\mathbb{T}^n \times Z \to Z$. A *toric morphism* between toric varieties $\mathbb{T}^m \subseteq Y$ and $\mathbb{T}^n \subseteq Z$ is a morphism $Y \to Z$ that restricts to a homomorphism $\mathbb{T}^m \to \mathbb{T}^n$ of tori. Usually we just write Z, Y, etc. for toric varieties and $\varphi\colon Y \to Z$, etc. for toric morphisms.

On the combinatorial side we use the following terminology. A *lattice fan* in $\mathbb{Z}^n$ is a finite collection Σ of pointed, convex, polyhedral cones in $\mathbb{Q}^n$ such that for any $\sigma \in \Sigma$ every face $\tau \preccurlyeq \sigma$ belongs to Σ and any two $\sigma, \sigma' \in \Sigma$ intersect in a common face. A *map of lattice fans* from a fan Σ in $\mathbb{Z}^n$ to a fan Δ in $\mathbb{Z}^m$ is a homomorphism $F\colon \mathbb{Z}^n \to \mathbb{Z}^m$ such that for every $\sigma \in \Sigma$ there is a $\tau \in \Delta$ with $F(\sigma) \subseteq \tau$. We also write F for the linear map $\mathbb{Q}^n \to \mathbb{Q}^m$ extending the homomorphism $\mathbb{Z}^n \to \mathbb{Z}^m$.

We present the basic construction of toric geometry, associating a toric variety with an arbitrary lattice fan. This construction is functorial and the fundamental theorem of toric geometry tells us that it even sets up a covariant equivalence between the category of lattice fans and the category of toric varieties.

Construction 2.1.1. Let Σ be a lattice fan in $\mathbb{Z}^n$. For a cone $\sigma \subseteq \mathbb{Q}^n$ of Σ denote the dual cone by $\sigma^\vee \subseteq \mathbb{Q}^n$ and consider the spectrum Z_σ of the monoid algebra $\mathbb{K}[M_\sigma]$ of the additive monoid $M_\sigma := \sigma^\vee \cap \mathbb{Z}^n$:

$$Z_\sigma \; = \; \operatorname{Spec} \mathbb{K}[M_\sigma], \qquad\qquad \mathbb{K}[M_\sigma] \; = \; \bigoplus_{u \in M_\sigma} \mathbb{K}\chi^u.$$

The acting torus $\mathbb{T}^n = \operatorname{Spec} \mathbb{K}[\mathbb{Z}^n]$ embeds via $\mathbb{K}[M_\sigma] \subseteq \mathbb{K}[\mathbb{Z}^n]$ canonically into Z_σ, turning it into an affine toric variety. Similarly, we have open embeddings $Z_\tau \subseteq Z_\sigma$ whenever $\tau \preceq \sigma$. This allows to glue together all the Z_σ to a variety Z:

$$Z \; = \; \bigcup_{\sigma \in \Sigma} Z_\sigma, \qquad \text{where } Z_\sigma \cap Z_{\sigma'} \; = \; Z_{\sigma \cap \sigma'} \; \subseteq \; Z.$$

The gluing respects the toric structures $\mathbb{T}^n \subseteq Z_\sigma$ such that we obtain a toric variety $\mathbb{T}^n \subseteq Z$, *the toric variety associated with the fan* Σ in $\mathbb{Z}^n$. Any $\chi^u \in \mathbb{K}[\mathbb{Z}^n]$ yields a rational function on Z satisfying

$$\chi^u(1,\ldots,1) \ = \ 1, \qquad\qquad \chi^u(t\cdot z) \ = \ t_1^{u_1}\cdots t_n^{u_n}\chi^u(z),$$

whenever defined at $z \in Z$. In particular, the restrictions $\chi^u\colon \mathbb{T}^n \to \mathbb{K}^*$ are precisely the characters of $\mathbb{T}^n \subseteq Z$. Given a map F of lattice fans Σ in $\mathbb{Z}^n$ and Δ in $\mathbb{Z}^m$, we obtain algebra homomorphisms

$$\mathbb{K}[M_\tau] \ \to \ \mathbb{K}[M_\sigma], \qquad \chi^u \mapsto \chi^{u\circ F},$$

whenever $\sigma \in \Sigma$ and $\tau \in \Delta$ satisfy $F(\sigma) \subseteq \tau$. Passing to the spectra, this defines morphisms $Z_\sigma \to Y_\tau$ which in turn glue together to a morphism $Z \to Y$ of the toric varieties associated with Σ and Δ.

The task of toric geometry is to link geometric properties on the one side to combinatorial ones on the other. We first indicate how to detect the orbit decomposition and the invariant divisors of a toric variety from its defining fan.

Summary 2.1.2. Let Σ be a lattice fan in $\mathbb{Z}^n$ and Z the associated toric variety. Every lattice vector $v \in \mathbb{Z}^n$ defines a *one-parameter subgroup*

$$\lambda_v\colon \mathbb{K}^* \ \to \ \mathbb{T}^n \ \subseteq Z, \qquad\qquad t \ \mapsto \ (t^{v_1},\ldots,t^{v_n}).$$

If v lies in the relative interior σ° of a cone $\sigma \in \Sigma$, then λ_v extends to a morphism $\bar\lambda_v\colon \mathbb{K} \to Z$. The associated *limit point* is

$$z_\sigma \ := \ \lim_{t\to 0} \lambda_v(t) \ := \ \bar\lambda_v(0) \ \in \ Z_\sigma \ \subseteq \ Z.$$

Here, z_σ does not depend on the particular choice of $v \in \sigma^\circ$. Note that the zero cone $\{0\} \in \Sigma$ yields the unit element $z_0 \in \mathbb{T}^n$. The limit points set up a bijection

$$\Sigma \ \to \ \{\mathbb{T}^n\text{-orbits of } Z\}, \qquad \sigma \ \mapsto \ \mathbb{T}^n\cdot z_\sigma.$$

The dimension of the orbit $\mathbb{T}^n\cdot z_\sigma$ is $n-\dim(\sigma)$. In particular, the *rays*, i.e. the one-dimensional cones $\varrho_1,\ldots,\varrho_r$ of Σ, define invariant prime divisors

$$D_i \ := \ \overline{\mathbb{T}^n\cdot z_{\varrho_i}} \ \subseteq \ Z.$$

We have $Z = \mathbb{T}^n \cup D_1 \cup \ldots \cup D_r$ and $D_i \cap D_j \neq \emptyset$ if and only if $\varrho_i, \varrho_j \subseteq \sigma$ for some cone $\sigma \in \Sigma$. Moreover, there is an isomorphism

$$\mathbb{Z}^r \ \to \ \mathrm{WDiv}(Z)^{\mathbb{T}^n}, \qquad a \ \mapsto \ a_1D_1+\ldots+a_rD_r.$$

In other words, the prime divisors $D_1,\ldots,D_r$ freely generate the group $\mathrm{WDiv}(Z)^{\mathbb{T}^n}$ of *invariant Weil divisors*. Finally, we obtain an invariant anti-canonical divisor

$$-\mathcal{K}_Z \ = \ D_1+\cdots+D_r \ \in \ \mathrm{WDiv}(Z)^{\mathbb{T}^n}.$$

Our next topic is the *divisor class group*. This is the group of Weil divisors modulo the subgroup of principal divisors. We obtain an explicit description in terms of the defining fan, which is also the key to other invariants. We briefly recall the terminology. For a point z of a normal variety Z, the *local class group* $\mathrm{Cl}(Z,z)$ is the group of Weil divisors modulo the subgroup of divisors that are principal near z. A *Cartier divisor* is a Weil divisor that is locally principal. The *Picard group* is the group of Cartier divisors modulo the subgroup of principal divisors. Moreover, in the rational vector space

$$\mathrm{Cl}_{\mathbb{Q}}(Z) \ = \ \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Cl}(Z)$$

associated with the divisor class group of a variety Z, there are the following important subsets. The *effective cone* $\mathrm{Eff}(Z)$ is generated by the classes of effective divisors. The *moving cone* $\mathrm{Mov}(Z)$ is generated by the classes of movable divisors, i.e. those with base locus of codimension one. The *semiample cone* $\mathrm{SAmple}(Z)$ is generated by the classes of semiample divisors and the *ample cone* $\mathrm{Ample}(Z)$ is generated by the classes of ample divisors. By a cone, we always mean a convex cone.

Summary 2.1.3. Consider a lattice fan Σ in $\mathbb{Z}^n$ and its rays $\varrho_1, \ldots, \varrho_r$ with corresponding unique primitive lattice vectors $v_i \in \varrho_i$. The *generator matrix of* Σ is

$$P \ := \ [v_1, \ldots, v_r].$$

Suppose that P is of rank n. We describe the divisor class group of the toric variety Z associated with Σ in terms of P. First recall the identification

$$\mathbb{Z}^r \ \to \ \mathrm{WDiv}(Z)^{\mathbb{T}^n}, \qquad a \ \mapsto \ D(a) := a_1 D_1 + \ldots + a_r D_r.$$

The divisor of a character function $\chi^u \in \Gamma(\mathbb{T}^n, \mathcal{O}) \subseteq \mathbb{K}(Z)$, where $u \in \mathbb{Z}^n$, is given via the transpose P^t of the generator matrix:

$$D(P^t(u)) \ = \ \langle u, v_1 \rangle D_1 + \ldots + \langle u, v_r \rangle D_r \ = \ \mathrm{div}(\chi^u).$$

The divisor class group $\mathrm{Cl}(Z)$ of Z turns out to be the group of invariant Weil divisors modulo the group of invariant principal divisors and thus we obtain

$$\mathrm{Cl}(Z) \ = \ K \ := \ \mathbb{Z}^r / \mathrm{im}(P^t).$$

Using the projection $Q \colon \mathbb{Z}^r \to K$, we have $[D_i] = Q(e_i)$ for the class of the invariant prime divisor $D_i \subseteq Z$. The local class group of $z_\sigma \in Z$ is given by

$$\mathrm{Cl}(Z, z_\sigma) \ = \ K / K_\sigma, \qquad\qquad K_\sigma \ := \ \langle Q(e_i); \ P(e_i) \notin \sigma \rangle.$$

Since Z is the union of the $\mathbb{T}^n$-orbits through the points z_σ, where $\sigma \in \Sigma$, this determines all local class groups. The Picard group of Z is given by

$$\mathrm{Pic}(Z) \ = \ \bigcap_{\sigma \in \Sigma} K_\sigma \ \subseteq \ \mathrm{Cl}(Z).$$

In the rational divisor class group $\mathrm{Cl}_{\mathbb{Q}}(Z) = K_{\mathbb{Q}}$, we identify the cones of effective and movable divisor classes as

$$\mathrm{Eff}(Z) \ = \ Q(\gamma), \qquad\qquad \mathrm{Mov}(Z) \ = \ \bigcap_{\substack{\gamma_0 \preccurlyeq \gamma \\ \text{facet}}} Q(\gamma_0),$$

where $\gamma = \mathbb{Q}^r_{\geq 0}$ denotes the positive orthant in $\mathbb{Q}^r$. The cones of semiample and ample divisor classes in $\mathrm{Cl}_{\mathbb{Q}}(Z)$ are given by

$$\mathrm{SAmple}(Z) \ = \ \bigcap_{\sigma \in \Sigma} \mathrm{cone}(Q(e_i); \ P(e_i) \notin \sigma), \qquad \mathrm{Ample}(Z) \ = \ \mathrm{SAmple}(Z)^\circ.$$

We want to look at smoothness and singularities. Recall that a point $z \in Z$ of a normal variety Z is called $\mathbb{Q}$-*factorial* if every Weil divisor admits a non-zero multiple that is principal near z. A normal variety is called $\mathbb{Q}$-*factorial* if each of its points is $\mathbb{Q}$-factorial.

Summary 2.1.4. Let Z be the toric variety arising from a lattice fan Σ in $\mathbb{Z}^n$. Consider the limit points $z_\sigma \in Z$, where $\sigma \in \Sigma$.

(i) The point $z_\sigma \in Z$ is $\mathbb{Q}$-factorial if and only if σ is *simplicial*, i.e. generated by part of a vector space basis of $\mathbb{Q}^n$.
(ii) The point $z_\sigma \in Z$ is smooth if and only if σ is *regular*, i.e. generated by part of a lattice basis of $\mathbb{Z}^n$.
(iii) The point $z_\sigma \in Z$ is $\mathbb{Q}$-factorial (smooth) if and only if all points of $\mathbb{T}^n \cdot z_\sigma$ are $\mathbb{Q}$-factorial (smooth).
(iv) The variety Z is $\mathbb{Q}$-factorial (smooth) if and only if all cones of the lattice fan Σ are simplicial (regular).

A lattice fan Σ in $\mathbb{Z}^n$ is called *complete* if its *support*, i.e. the union over all its cones, equals $\mathbb{Q}^n$. Moreover, Σ is called *polytopal* if it is spanned by an n-dimensional convex polytope $\mathcal{A} \subseteq \mathbb{Q}^n$ containing the origin in its interior. In this case the cones of Σ are precisely the cones over the faces of $\mathcal{A}$.

Summary 2.1.5. Let Z be the toric variety arising from a lattice fan Σ in $\mathbb{Z}^n$. Then Z is complete if and only Σ is complete and Z is projective if and only if Σ is polytopal. If the latter holds, then, for any $a \in \mathbb{Z}^r$ representing an ample divisor $D(a) = a_1D_1 + \cdots + a_rD_r$, the lattice fan Σ is spanned by the dual of the polytope

$$(P^t)^{-1}(B_a - a) \ \subseteq\ \mathbb{Q}^n, \qquad B_a \ :=\ Q^{-1}(Q(a)) \cap \gamma.$$

Here, as before, P is the generator matrix of Σ and $Q\colon \mathbb{Z}^r \to K = \mathbb{Z}^r/\mathrm{im}(P^t)$ the projection and $\gamma = \mathbb{Q}^r_{\geq 0}$ the positive orthant. In dimension one the only complete toric variety is the projective line. Every complete toric surface is projective. The first non-projective complete toric varieties occur in dimension three.

A *Fano variety* is a normal complete variety admitting an ample anticanonical divisor. Observe that by this definition, Fano varieties are projective and some non-zero multiple of the anticanonical divisor of a Fano variety is Cartier.

Summary 2.1.6. Let Σ be a complete lattice fan in $\mathbb{Z}^n$ and $\mathcal{A} \subseteq \mathbb{Q}^n$ the convex hull over the primitive generators $v_1, \ldots, v_r$ of Σ. The following statements are equivalent.

(i) The toric variety Z associated with Σ is a Fano variety.
(ii) The vector $Q(e_1) + \cdots + Q(e_r)$ lies in the ample cone of Z.
(iii) For every $\sigma \in \Sigma$ there is a $u \in \mathbb{Q}^r$ with $\langle u, v_i \rangle = 1$ whenever $v_i \in \sigma$.
(iv) The fan Σ is spanned by the polytope $\mathcal{A}$.

We will make use of *Cox's quotient presentation*, which generalizes the construction of the projective space $\mathbb{P}_n$ as the quotient of $\mathbb{K}^{n+1} \setminus \{0\}$ by $\mathbb{K}^*$ acting via scalar multiplication. See [**14**], also [**15**, Sec. 5] and [**4**, Sec. 2.1.3] for more details.

Construction 2.1.7. Consider a lattice fan Σ in $\mathbb{Z}^n$ with generator matrix P of rank n and let Z be the associated toric variety. The *Cox ring* of Z is given by

$$\mathcal{R}(Z) \ =\ \bigoplus_{D \in \mathrm{Cl}(Z)} \Gamma(Z, \mathcal{O}_D(Z)) \ \cong\ \bigoplus_{w \in K} \mathbb{K}[T_1, \ldots, T_r]_w \ =\ \mathbb{K}[T_1, \ldots, T_r].$$

The K-grading of the polynomial ring $\mathbb{K}[T_1, \ldots, T_r]$ is defined by $\deg(T_i) := [D_i] = Q(e_i)$. Consider the orthant $\gamma = \mathbb{Q}^r_{\geq 0}$, its fan of faces $\bar{\Sigma}$ and

$$\hat{\Sigma} \ := \ \{\tau \preccurlyeq \gamma;\ P(\tau) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

Then $\hat{\Sigma}$ is a subfan of the fan $\bar{\Sigma}$ and P sends cones from $\hat{\Sigma}$ into cones of Σ. For the associated toric varieties this leads to the picture

$$\begin{tikzcd}[column sep=small]
\mathbb{T}^r \arrow[d, "p", "/H"'] & \subseteq & \hat{Z} \arrow[d, "/\!\!/H", "p"'] & \subseteq & \bar{Z} & := & \mathbb{K}^r. \\
\mathbb{T}^n & \subseteq & Z & & & &
\end{tikzcd}$$

Here, $\hat{Z} \subseteq \bar{Z}$ is an open $\mathbb{T}^r$-invariant subvariety and p extends the homomorphism of tori having the rows of $P = (p_{ij})$ as its exponent vectors:

$$\mathbb{T}^r \ \to \ \mathbb{T}^n, \qquad t \ \mapsto \ (t^{P_{1*}}, \ldots, t^{P_{n*}}), \qquad t^{P_{i*}} \ := \ t_1^{p_{i1}} \cdots t_r^{p_{ir}}.$$

The morphism p is a so called *good quotient* for the action of the quasitorus $H = \ker(p) \subseteq \mathbb{T}^r$ on $\hat{Z}$. If Σ is simplicial, then each fiber of p is an H-orbit.

Remark 2.1.8. Let Z be a toric variety with quotient presentation $p\colon \hat{Z} \to Z$ as in Constructionn 2.1.7. Then every p-fiber contains a unique closed H-orbit. The presentation in *Cox coordinates* of a point $x \in Z$ is

$$x \ = \ [z_1, \ldots, z_r], \qquad \text{where } z = (z_1, \ldots, z_r) \in p^{-1}(x) \text{ with } H \cdot z \subseteq \hat{Z} \text{ closed.}$$

Thus, $[z]$ and $[z']$ represent the same point $x \in Z$ if and only if z and z' lie in the same closed H-orbit of $\hat{Z}$. For instance, the points $z_\sigma \in Z$, where $\sigma \in \Sigma$, are given in Cox coordinates as

$$z_\sigma \ = \ [\varepsilon_1, \ldots, \varepsilon_r], \qquad \varepsilon_i \ = \ \begin{cases} 0, & P(e_i) \in \sigma, \\ 1, & P(e_i) \notin \sigma. \end{cases}$$

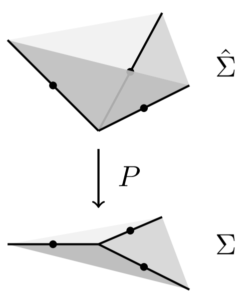
We conclude the section with an example from the surface case in order to see how its (well known) geometric features are obtained from its defining combinatorial data.

Example 2.1.9. Consider the weighted projective plane $\mathbb{P}_{1,2,3}$. As a toric variety it arises from the complete lattice fan Σ in $\mathbb{Z}^2$ with generator matrix

$$P \ = \ [v_1, v_2, v_3] \ = \ \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}.$$

In order to see that the toric variety Z associated with Σ indeed equals $\mathbb{P}_{1,2,3}$ we look at Cox's quotient construction.

$$\begin{tikzcd}
\mathbb{K}^3 \setminus \{0\} \ = \ \hat{Z} \arrow[d, "p", "/H"'] \\
\mathbb{P}_{1,2,3} \ = \ Z
\end{tikzcd}$$



Note that the homomorphism of tori defined by the generator matrix P and its kernel H are explicitly given by

$$p\colon \mathbb{T}^3 \ \to\ \mathbb{T}^2, \quad (t_1,t_2,t_3) \ \mapsto\ \left(\frac{t_1t_2}{t_3}, \frac{t_1^2}{t_2}\right), \qquad H \ = \ \{(t,t^2,t^3);\ t\in\mathbb{K}^*\}.$$

Thus, $H = \mathbb{K}^*$ acts with the weights 1, 2 and 3 in $\mathbb{K}^3$ and $Z = \mathbb{P}_{1,2,3}$ is the quotient of $\hat{Z} = \mathbb{K}^3 \setminus \{0\}$ by this action. Let us look at the geometry of $\mathbb{P}_{1,2,3}$. We have

$$\mathrm{Cl}(Z) \ = \ \mathbb{Z}^3/\mathrm{im}(P^t) \ = \ \mathbb{Z}$$

with the projection $Q\colon \mathbb{Z}^3 \to \mathbb{Z}$ sending (a_1,a_2,a_3) to $(a_1,2a_2,3a_3)$. This allows to recover the Picard group as

$$\mathrm{Pic}(Z) \ = \ \mathbb{Z}\cap 2\mathbb{Z}\cap 3\mathbb{Z} \ = \ 6\mathbb{Z} \ \subseteq \ \mathbb{Z} \ = \ \mathrm{Cl}(Z).$$

As Σ has three rays, there are three invariant prime divisors D_1, D_2 and D_3 on Z. In particular, the anticanonical class is given by

$$[-\mathcal{K}_Z] \ = \ [D_1]+[D_2]+[D_3] \ = \ 1+2+3 \ = \ 6 \ \in \ \mathbb{Z} \ = \ \mathrm{Cl}(Z).$$

Thus, $-\mathcal{K}_Z$ is an ample Cartier divisor generating $\mathrm{Pic}(Z)$. Note that Σ is spanned by $\mathcal{A} = \mathrm{conv}(v_1,v_2,v_3)$. There are two singularities, given in Cox-coordinates by

$$[0,1,0], \qquad\qquad [0,0,1].$$

2.2. Geometry of toric surfaces

In this Section, the specific case of two-dimensional toric geometry is treated, i.e. the case of toric surfaces. The combinatorial framework of lattice fans is much simpler in dimension two, which yields a particularly grateful example class. First we show how the general picture from the preceding Section boils down in the surface case.

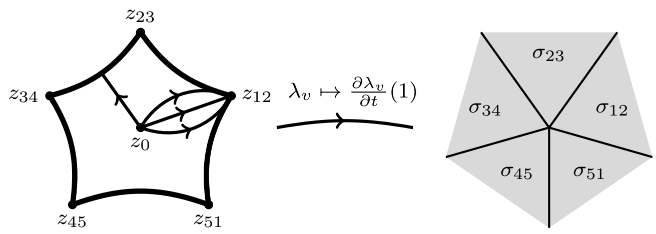
Summary 2.2.1. Consider the projective toric surface Z arising from a complete lattice fan Σ in $\mathbb{Z}^2$, the generator matrix $P = [v_1,\ldots,v_r]$ of Σ, the rays $\varrho_i = \mathrm{cone}(v_i) \in \Sigma$ and their limit points $z_i \in Z$. Then we have

$$Z \ = \ \mathbb{T}^2 \cup D_1 \cup \ldots \cup D_r, \qquad D_i \ := \ \overline{\mathbb{T}^2 \cdot z_i}.$$

Each of the orbit closures D_i is a smooth rational curve and $D_1\cup\ldots\cup D_r$ is a cycle in the sense that $D_i \cap D_j$ is nonempty if and only if the corresponding rays ϱ_i and ϱ_j are adjacent. If two rays ϱ_i and ϱ_j are adjacent, we have

$$D_i\cap D_j \ = \ \{z_{ij}\}, \qquad z_{ij} \ := \ z_{\sigma_{ij}}, \qquad \sigma_{ij} \ := \ \mathrm{cone}(v_i,v_j) \ \in \ \Sigma.$$

A one-parameter subgroup $\lambda_v\colon \mathbb{K}^* \to \mathbb{T}^2$, $t \mapsto (t^{v_1}, t^{v_2})$ approaches z_i or z_{ij} for $t \to 0$ if and only if $v = (v_1,v_2)$ lies in the relative interior of ϱ_i or σ_{ij}, respectively. Gathering derivatives of λ_v with common limit into cones we recover Σ from Z:



The orbits $\mathbb{T}^2 \cdot z_0$ and $\mathbb{T}^2 \cdot z_i$ only contain smooth points of Z. The z_{ij} are precisely the toric fixed points. Every toric fixed point z_{ij} is $\mathbb{Q}$-factorial. Moreover, z_{ij} is smooth if and only if $\det(v_i, v_j) = \pm 1$.

Being $\mathbb{Q}$-factorial, any complete toric surface comes with a well-defined intersection product. Here is how to compute the intersection numbers explicitly in terms of the defining fan.

Summary 2.2.2. Consider the toric surface Z arising from a complete lattice fan Σ in $\mathbb{Z}^2$ with generator matrix

$$P \;=\; [v_1, \dots, v_r].$$

For any two distinct generators v_i, v_j in positive orientation, the intersection number of the associated divisors D_i, D_j is given as

$$D_i \cdot D_j \;=\; \begin{cases} \det(v_i, v_j)^{-1}, & \text{if } \operatorname{cone}(v_i, v_j) \in \Sigma, \\ 0, & \text{else.} \end{cases}$$

Moreover, we can compute the self intersection number of a divisor D_j. Taking adjacent generators v_i, v_j, v_k in positive orientation, we have

$$D_j^2 \;=\; -\frac{\det(v_i, v_k)}{\det(v_i, v_j)\det(v_j, v_k)}.$$

Remark 2.2.3. Let Σ be a complete lattice fan in $\mathbb{Z}^2$ with generator matrix $P = [v_1, \dots, v_r]$, where the v_i are ordered counter-clockwise. We write $v_i = (l_i, d_i)$ with $l_i \neq 0$ for $i = 1, \dots, r$. Then

$$\mathcal{K}_Z^2 \;=\; \sum_{i=1}^{r} \left(2 - \frac{l_i}{l_{i+1}} - \frac{l_{i+1}}{l_i}\right) \frac{1}{\det(v_i, v_{i+1})}, \qquad v_{r+1} \;:=\; v_1$$

gives the self intersection number of any canonical divisor $\mathcal{K}_Z$ on the toric surface Z associated with Σ.

We take a closer look at the singularities of toric surfaces. As observed, these are necessarily fixed points. Thus, for the local study, we have to consider affine toric surfaces defined by two-dimensional lattice cones.

Summary 2.2.4. Consider the affine toric surface Z_σ associated with a two-dimensional lattice cone σ in $\mathbb{Z}^2$. After applying a suitable unimodular transformation, the generator matrix is of the shape

$$P \;=\; \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}, \qquad 0 \le a < b.$$

The cone σ is therefore generated by the columns of P. The point $z_\sigma \in Z_\sigma$ is singular if and only if $b > 1$ holds. Cox's quotient presentation from 2.1.7

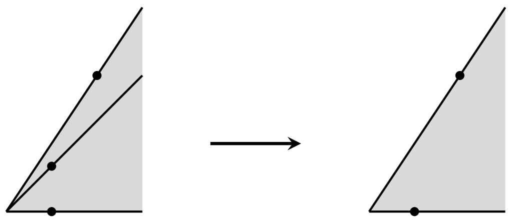
yields a morphism $p\colon \mathbb{K}^2 \to \mathbb{K}^2/H = Z_\sigma$, where

$$H \;=\; \ker(p) \;=\; \{(t^{-a}, t);\ t \in \Gamma_b\}, \qquad \Gamma_b \;:=\; \{t \in \mathbb{K}^*;\ t^b = 1\}.$$

Thus, Z_σ is the quotient of $\mathbb{K}^2$ by a diagonal action of a cyclic group of order b. In particular, for $b \ge 2$, we see that the point $z_\sigma = p(0,0)$ is a two-dimensional cyclic quotient singularity.

We discuss the resolution of toric surface singularities. Recall that the *Hilbert basis* $\mathcal{H}_\sigma$ of a pointed convex polyhedral cone $\sigma \subseteq \mathbb{Q}^n$ is the (finite) set of indecomposable elements of the additive monoid $\sigma \cap \mathbb{Z}^n$. Here, a non-zero element $v \in \sigma \cap \mathbb{Z}^n$ is *indecomposable* if $v = v' + v''$ with $v', v'' \in \sigma \cap \mathbb{Z}^n$ is only possible for $v = v'$ or $v = v''$.

Summary 2.2.5. Consider the affine toric surface Z_σ arising from a two-dimensional lattice cone σ in $\mathbb{Z}^2$. Subdividing σ by the members $v_1, \ldots, v_r$ of the Hilbert basis $\mathcal{H}_\sigma$ gives a lattice fan Σ in $\mathbb{Z}^2$ with support σ and generator matrix $P = [v_1, \ldots, v_r]$.



The toric surface Z associated with Σ is smooth and the canonical toric morphism $\pi\colon Z \to Z_\sigma$ is the minimal resolution of singularities. The exceptional curves of π are precisely the D_i given by the Hilbert basis members $v_i \in \sigma^\circ$.

We say that a normal variety X with canonical divisor $\mathcal{K}_X$ is $\mathbb{Q}$-*Gorenstein* if some non-zero integral multiple of $\mathcal{K}_X$ is Cartier. If this is fulfilled, the *Gorenstein index* of X is the smallest non-zero integer ι_X such that $\iota_X \mathcal{K}_X$ is Cartier. A variety is said to be *Gorenstein* if it is of Gorenstein index 1.

Summary 2.2.6. Consider a two-dimensional lattice cone σ in $\mathbb{Z}^2$ with primitive generators v_1 and v_2. Then there is a primitive $u_\sigma \in \mathbb{Z}^2$ and an $i_\sigma \in \mathbb{Z}_{>0}$ such that

$$\langle u_\sigma, v_1 \rangle \;=\; \iota_\sigma, \qquad\qquad \langle u_\sigma, v_2 \rangle \;=\; \iota_\sigma.$$

Moreover u_σ and ι_σ are uniquely determined by this property. For the associated affine toric surface Z_σ and its invariant canonical divisor, we have

$$-\iota_\sigma \mathcal{K}_{Z_\sigma} \;=\; \iota_\sigma D_1 + \iota_\sigma D_2 \;=\; \langle u_\sigma, v_1 \rangle D_1 + \langle u_\sigma, v_2 \rangle D_2 \;=\; \operatorname{div}(\chi^{u_\sigma}).$$

Thus, ι_σ is minimal with $\iota_\sigma \mathcal{K}_Z$ being Cartier and hence equals the Gorenstein index of Z_σ. For a toric surface Z arising from a complete lattice fan Σ in $\mathbb{Z}^2$, we obtain

$$\iota_Z \;=\; \operatorname{lcm}(\iota_\sigma;\ \sigma \in \Sigma,\ \dim(\sigma) = 2).$$

Remark 2.2.7. Consider two primitive vectors $v_1 = (a, c)$ and $v_2 = (b, d)$ in $\mathbb{Z}^2$ generating a two-dimensional cone $\sigma \subseteq \mathbb{Q}^2$. The linear form u_σ and the number ι_σ from Summary 2.2.6 are given as

$$u_\sigma = \frac{1}{\iota_\sigma}\left(\frac{c-d}{ad-bc}, \frac{b-a}{ad-bc}\right), \qquad \iota_\sigma = \frac{|ad-bc|}{\gcd(c-d,\, b-a)}.$$

In particular, this allows to compute the Gorenstein index ι_σ of the affine toric surface Z_σ in terms of the generator matrix $P = [v_1, v_2]$. Note that ι_σ equals the order of the class of $\mathcal{K}_{Z_\sigma}$ in the local class group $\mathrm{Cl}(Z_\sigma, z_\sigma)$.

Proposition 2.2.8. *In $\mathbb{Z}^2$, consider the vector $e := (1,0)$, for $a \in \mathbb{Z}_{\geq 1}$ the vectors $v_a := (1,a)$ and for $\iota, b \in \mathbb{Z}_{\geq 2}$ the vectors*

$$v_{\iota,b,\kappa} := \left(b,\, \iota\frac{b-1}{\kappa}\right), \ \kappa = 1,\dots,\iota-1,\ \kappa \mid b-1,\ \gcd(b, \iota\tfrac{b-1}{\kappa}) = \gcd(\kappa,\iota) = 1.$$

Set $\sigma_a := \mathrm{cone}(e, v_a) \subseteq \mathbb{Q}^2$ and $\sigma_{\iota,b,\kappa} := \mathrm{cone}(e, v_{\iota,b,\kappa}) \subseteq \mathbb{Q}^2$. Then the following statements hold.

(i) *Up to isomorphy the Gorenstein affine toric surfaces with fixed point are precisely the Z_{σ_a}.*
(ii) *Fix $\iota \in \mathbb{Z}_{\geq 2}$. Up to isomorphy the affine toric surfaces with fixed point being of Gorenstein index ι are precisely the $Z_{\sigma_{\iota,b,\kappa}}$.*

Proof. By Summary 2.2.6, a toric surface Z_σ is of Gorenstein index ι if there is a primitive $u \in \mathbb{Z}^2$ evaluating to ι on the primitive generators of σ. The linear forms

$$u_a := (1,0), \qquad u_{\iota,b,\kappa} := (\iota, -\kappa)$$

do so for Z_{σ_a} and $Z_{\sigma_{\iota,b,\kappa}}$. Conversely, given an affine toric surface Z_σ with fixed point of Gorenstein index ι, we may assume that

$$\sigma = \mathrm{cone}(v_1, v_2), \qquad v_1 = (1,0), \qquad v_2 = (a,b), \qquad 0 \leq a < b.$$

Then we directly see that the existence of a primitive $u \in \mathbb{Z}^2$ evaluating to ι on v_1 and v_2 forces $v_2 = v_a$ or $v_2 = v_{\iota,b,\kappa}$. $\square$

Example 2.2.9. We examine the minimal resolution of affine toric surfaces of small Gorenstein index. For $\iota = 1$, we have to look at

$$\sigma_a = \mathrm{cone}((1,0),\, (1,a)), \quad a = 1,2,3,\dots \qquad \mathcal{H}_{\sigma_a} = \{(1,j);\ j = 1,\dots,a\}.$$

Moreover for $\iota = 2$, by Proposition 2.2.8 we have to consider the cones $\sigma_{2,b,\kappa}$, where $b \geq 2$ must be odd and $\kappa = 1$. Thus, we end up with

$$\sigma_{2,b,1} = \mathrm{cone}((1,0),\, (b, 2b-2)), \quad b = 3,5,7,9,\dots$$

Besides the primitive generators, the Hilbert basis of $\sigma_{2,b,1}$ contains all interior lattice points of the line with slope 2 through $(1,1)$, i.e.

$$\mathcal{H}_{\sigma_{2,b,1}} = \{(1,0),\, (b, 2b-2)\} \cup \left\{(1+j, 1+2j);\ j = 0,\dots,\frac{b-3}{2}\right\}.$$

Using Summary 2.2.5 we can compute *resolution graphs*, the vertices of which represent the exceptional curves, labelled by their self intersection number. Two vertices are connected by an edge if and only if the corresponding curves intersect.

$$\begin{array}{lcr}
Z_{\sigma_a}\colon & (-2) - (-2) \cdots\cdots (-2) - (-2) & a = 2, 3, 4, \ldots, \\
Z_{\sigma_{2,b,1}}\colon & (-4) & b = 3, \\
Z_{\sigma_{2,b,1}}\colon & (-3) - (-3) & b = 5, \\
Z_{\sigma_{2,b,1}}\colon & (-3) - (-2) \cdots\cdots (-2) - (-3) & b = 7, 9, 11, \ldots.
\end{array}$$

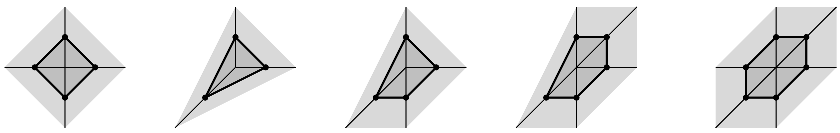
One calls the singularities of Z_{σ_a} of *type* A_n with $n = a-1$ and those of $Z_{\sigma_{2,b,1}}$ of *type* K_n with $n = (b-1)/2$. Note that n is the number of exceptional curves.

In the following, we dicuss toric del Pezzo surfaces. Recall that a *del Pezzo surface* is a two-dimensional Fano variety, i.e. a normal projective surface admitting an ample anticanonical divisor.

We give the combinatorial notions we need for the treatment of toric del Pezzo surfaces. By a *(lattice) polygon* we mean a convex polytope in $\mathbb{Q}^2$ (having only integral vertices). An *LDP-polygon* is a polygon in $\mathbb{Q}^2$ containing the origin $(0,0)$ as an interior point and having only primitive vectors from $\mathbb{Z}^2$ as vertices. We call two polygons *unimodularly equivalent* if they can be transformed into each other by a *unimodular matrix*, which is an integral 2×2 matrix with determinant ± 1.

Summary 2.2.10. Any LDP-polygon $\mathcal{A} \subseteq \mathbb{Q}^2$ spans a complete lattice fan $\Sigma_{\mathcal{A}}$ in $\mathbb{Z}^2$. The toric surface $Z_{\mathcal{A}}$ associated with $\Sigma_{\mathcal{A}}$ is a del Pezzo surface. The assignment $\mathcal{A} \mapsto Z_{\mathcal{A}}$ yields a bijection between the unimodular equivalence classes of LDP-polygons and the isomorphy classes of toric del Pezzo surfaces.

Example 2.2.11. Up to isomorphy, there are only five smooth toric del Pezzo surfaces. They are given by the following LDP-polygons.



The surfaces are $\mathbb{P}_1 \times \mathbb{P}_1$, the projective plane $\mathbb{P}_2$ and the blowing-up of $\mathbb{P}_2$ in up to three points in general position.

The situation gets much more lively if we look at singular del Pezzo surfaces. Besides the case of a given Gorenstein index, we will also consider singularities coming from the minimal model program. Let us briefly recall their definition. Given any $\mathbb{Q}$-Gorenstein variety X with canonical divisor $\mathcal{K}_X$, consider a resolution of singularities $\pi\colon X' \to X$. Then there are canonical divisors $\mathcal{K}_X$ on X and $\mathcal{K}'_X$ on X' such that the *ramification formula*

$$\mathcal{K}'_X \;=\; \pi^*\mathcal{K}_X + \sum a(E)E,$$

holds. Here, E runs through the exceptional prime divisors and the $a(E) \in \mathbb{Q}$ are the *discrepancies* of $\pi\colon X' \to X$. For $0 \le \varepsilon \le 1$, the singularities of X are called *ε-log terminal (ε-log canonical)* if $a(E) > \varepsilon - 1$ for each E ($a(E) \ge \varepsilon - 1$ for each E). For $\varepsilon = 0$, one speaks of log terminal (log canonical) singularities and for $\varepsilon = 1$ of terminal (canonical) ones.

Summary 2.2.12. Consider the toric del Pezzo surface $Z_{\mathcal{A}}$ defined by an LDP-polygon $\mathcal{A}$ and a toric resolution of singularities

$$\pi\colon Z \ \to \ Z_{\mathcal{A}}$$

given by a map of lattice fans Σ and $\Sigma_{\mathcal{A}}$. The discrepancy of an exceptional divisor $E_\varrho \subseteq Z$ corresponding to a ray $\varrho \in \Sigma$ is given by

$$a(E_\varrho) \ = \ \frac{\|v_\varrho\|}{\|v'_\varrho\|} - 1,$$

where $v_\varrho \in \varrho$ is the primitive lattice vector and $v'_\varrho \in \varrho$ denotes the intersection point of ϱ and the boundary $\partial\mathcal{A}$ of $\mathcal{A}$.

This shows in particular that toric del Pezzo surfaces are always log terminal, hence the "L" in LDP-polygon. Moreover, we directly obtain a simple characterization of ε-log canonicity for a toric del Pezzo surface via its defining polygon. Given $k \in \mathbb{Z}_{\geq 1}$, we call a polygon $\mathcal{A}$ *almost k-hollow* if the origin $(0,0)$ is the only point in $\mathcal{A}^\circ \cap k\mathbb{Z}^2$.

Proposition 2.2.13. *Let $\mathcal{A}$ be an LDP-polygon and consider the associated toric del Pezzo surface $Z_{\mathcal{A}}$. Then, for any $k \in \mathbb{Z}_{\geq 1}$, the following statements are equivalent.*

(i) *The polygon $\mathcal{A}$ is almost k-hollow.*
(ii) *The surface $Z_{\mathcal{A}}$ has only $1/k$-log canonical singularities.*

2.3. Classifying toric del Pezzo surfaces

We give a classification of toric $1/k$-log canonical del Pezzo surfaces. In combinatorial terms, this means to classify the almost k-hollow LDP-polygons. In Theorem 2.3.10 we gather basic features of the polygon classification, the full list will be available under [**25**]. The corresponding statements on the del Pezzo surfaces is given in Corollary 2.3.11.

Definition 2.3.1. Given a polygon $\mathcal{A}$ and a vector $v \in \mathbb{Q}^2$, we obtain new polygons $\mathcal{A}_v^+$ by *expanding* $\mathcal{A}$ at v and $\mathcal{A}_v^-$ by *collapsing* $\mathcal{A}$ at v:

$$\mathcal{A}_v^+ \ := \ \mathrm{conv}(\mathcal{A} \cup \{v\}), \qquad\qquad \mathcal{A}_v^- \ := \ \mathrm{conv}(\mathbb{Z}^2 \cap \mathcal{A} \setminus \{v\}).$$

A polygon $\mathcal{A}$ having the origin $(0,0)$ as an interior point is *minimal* if $(0,0) \notin (\mathcal{A}_v^-)^\circ$ holds for every vertex v of $\mathcal{A}$.

Example 2.3.2. Consider the canonical basis vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ in $\mathbb{Z}^2$. Given $k \in \mathbb{Z}_{\geq 1}$, we have minimal almost k-hollow lattice polygons

$$\Delta_\alpha := \mathrm{conv}(e_1,\, e_2,\, -\alpha e_1 - e_2),\ \alpha = 1, \ldots, 2k, \qquad \Box := \mathrm{conv}(\pm e_1,\, \pm e_2).$$

Proposition 2.3.3. *Let $k \in \mathbb{Z}_{\geq 1}$. Then, up to unimodular equivalence, $\Delta_1, \ldots, \Delta_{2k}$ and $\Box$ are the only minimal almost k-hollow lattice polygons.*

Lemma 2.3.4. *Let $\mathcal{A}$ be a minimal polygon. Then $\mathcal{A}$ has at most four vertices.*

PROOF. We number the vertices $v_1, \dots, v_r$ of $\mathcal{A}$ counter-clockwise. Assume $r \geq 5$. Then we have

$$\mathcal{A}^\circ \;=\; \operatorname{conv}(v_1, v_3, \dots, v_r)^\circ \cup \operatorname{conv}(v_1, \dots, v_{r-1})^\circ \;\subseteq\; (\mathcal{A}_{v_2}^-)^\circ \cup (\mathcal{A}_{v_r}^-)^\circ.$$

The origin lies in $\mathcal{A}^\circ$ and thus in $(\mathcal{A}_{v_2}^-)^\circ$ or in $(\mathcal{A}_{v_r}^-)^\circ$. This is a contradiction to the minimality of $\mathcal{A}$. □

PROOF OF PROPOSITION 2.3.3. Let $\mathcal{A}$ be a minimal almost k-hollow lattice polygon. By Lemma 2.3.4, we know that $\mathcal{A}$ has either four or three vertices.

We first treat the case, that $\mathcal{A}$ has four vertices. Say v_1, v_2, v_3, v_4, ordered counter-clockwise. We claim that the two diagonals of the quadrangle $\mathcal{A}$ intersect in the origin. Indeed, minimality of $\mathcal{A}$ yields

$$(0,0) \;\notin\; \operatorname{conv}(v_1, v_2, v_3)^\circ \;\subseteq\; (\mathcal{A}_{v_4}^-)^\circ, \quad (0,0) \;\notin\; \operatorname{conv}(v_1, v_3, v_4)^\circ \;\subseteq\; (\mathcal{A}_{v_2}^-)^\circ.$$

Since $(0,0) \in \mathcal{A}^\circ$, we have $(0,0) \in \operatorname{conv}(v_1, v_3)$. Analogously, $(0,0) \in \operatorname{conv}(v_2, v_4)$. Using minimality of $\mathcal{A}$ again, we see that each vertex $v_i \in \mathbb{Z}^2$ is primitive. Thus, after applying a suitable unimodular transformation of $\mathbb{Z}^2$, we can assume

$$v_1 = (1,0), \qquad v_2 = (a,b), \quad a, b \in \mathbb{Z},\ 0 \leq a < b,\ \gcd(a,b) = 1.$$

By primitivity, we also have $v_3 = -v_1$ and $v_4 = -v_2$. Finally, we claim $v_2 = (0,1)$. Otherwise $(1,1) \in \mathcal{A}$, which yields a contradiction to the minimality of $\mathcal{A}$, namely

$$(0,0) \;\in\; \operatorname{conv}(e_1,\, e_1 + e_2,\, -e_1, -e_1 - e_2)^\circ \;\subseteq\; (\mathcal{A}_{v_4}^-)^\circ.$$

Now assume that $\mathcal{A}$ has three vertices, say v_1, v_2, v_3, numbered counter-clockwise. As in the previous case, each of these vertices has to be primitive by minimality. Consider the case that $\mathcal{A}$ has an interior lattice point $v \neq (0,0)$. Then we can write

$$\mathcal{A} \;=\; \operatorname{conv}(v_1, v_2, v) \cup \operatorname{conv}(v_2, v_3, v) \cup \operatorname{conv}(v_3, v_1, v).$$

By minimality, $(0,0)$ lies on a line segment, say on $\operatorname{conv}(v, v_1)$. Then we can assume $v = -v_1$. Applying a suitable unimodular transformation gives

$$v_1 = (1,0), \qquad v_2 = (a,b), \quad a, b \in \mathbb{Z},\ 0 \leq a < b,\ \gcd(a,b) = 1.$$

We show that $\mathcal{A}$ equals Δ_α for some $1 \leq \alpha \leq 2k$. We claim $a = 0$. Otherwise, $(1,1) \in \mathcal{A}$ and, in contradiction to the minimality of $\mathcal{A}$ we obtain

$$(0,0) \;\in\; \operatorname{conv}(v_1,\, e_1 + e_2,\, -v_1, v_3)^\circ \;\subseteq\; (\mathcal{A}_{v_2}^-)^\circ.$$

Hence $a = 0$ and $v_2 = e_2$. Moreover, $v_3 = (c,d)$ with integers $c, d < 0$. Since $-v_1$ is an interior point of $\mathcal{A}$, the vertex v_3 lies above the line through v_2 and $-v_1$. In particular, $c \leq -3$. By minimality of $\mathcal{A}$, no point $(e,-1)$ with $e \in \mathbb{Z}_{\leq 0}$ lies in $\mathcal{A}^\circ$. This yields $d = -1$.

Now assume that the origin is the only interior lattice point of $\mathcal{A}$. As before, we adjust by means of a suitable unimodular transformation to

$$v_1 = (1,0), \qquad v_2 = (a,b), \quad a, b \in \mathbb{Z},\ 0 \leq a < b,\ \gcd(a,b) = 1.$$

Minimality implies $a \leq 1$. If $a = 0$, the only possibility is $v_2 = e_2$ and v_3 being one of $(-1,-1)$, $(-2,-1)$ or $(-1,-2)$. If $a = 1$, we have $v_2 = (2,1)$ and $v_3 = (-1,-1)$. For each of these constellations, $\mathcal{A}$ is equivalent to a Δ_α for $\alpha = 1, 2$. □

We are ready to provide effective bounds for almost k-hollow lattice polygons. Below, $D_r \subseteq \mathbb{R}^2$ denotes the disk of radius r centered around the origin $(0,0)$.

Proposition 2.3.5. *Let $\mathcal{A}$ be an almost k-hollow lattice polygon. Then there is a unimodular transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that one of the following holds.*

(i) *We have $\Box \subseteq \varphi(\mathcal{A}) \subseteq D_r$ for $r = k^2/\sqrt{2}$.*
(ii) *We have $\Delta_\alpha \subseteq \varphi(\mathcal{A}) \subseteq D_r$ for $1 \le \alpha \le 2k$ and*

$$r = 2k^2\sqrt{\alpha^2+2\alpha+2}.$$

Lemma 2.3.6. *Let $\mathcal{A}$ be an almost k-hollow polygon and $r \in \mathbb{R}$ with $D_r \subseteq \mathcal{A}$ and let $v \in \mathbb{R}^2$. If the extension $\mathcal{A}_v^+$ is almost k-hollow, then $\|v\| \le k^2/2r$.*

Proof. First consider the case $k = 1$. Then the origin is the only interior lattice point of $\mathcal{A}$. Being contained in $\mathcal{A}_v^+$, the set $\mathrm{conv}(D_r \cup \{v\})$ has the origin as its only interior lattice point. Thus, also $\mathcal{B} := \mathrm{conv}(D_r \cup \{\pm v\})$ has the origin as its only interior lattice point. Since $\mathcal{B}$ is a centrally symmetric convex set, Minkowski's Theorem yields $\mathrm{vol}(\mathcal{B}) \le 4$. Moreover, we directly see $2r\|v\| \le \mathrm{vol}(\mathcal{B})$. This proves the assertion for $k = 1$. For the case of a general k, apply the previous consideration to $k^{-1}\mathcal{A}$ and $k^{-1}v$. $\Box$

Proof of Proposition 2.3.5. First observe that successively collapsing $\mathcal{A}$ at vertices, we arrive at a minimal almost k-hollow lattice polygon $\mathcal{A}'$. Proposition 2.3.3 provides a unimodular transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(\mathcal{A}')$ is one of the polygons mentioned in (i) and (ii). It remains to show that $\varphi(\mathcal{A})$ lies in the corresponding disks. For this, we have to bound the length of any given vertex $v \in \varphi(\mathcal{A})$ accordingly. Collapsing step by step suitable vertices turns $\varphi(\mathcal{A})$ into $\varphi(\mathcal{A}')_v^+$. Thus the assertion follows from Lemma 2.3.6 and the fact that we find a disk D_r of radius $r = 1/\sqrt{2}$ in $\Box$ and of radius $r = 1/\sqrt{\alpha^2+2\alpha+2}$ in Δ_α. $\Box$

Corollary 2.3.7. *Up to unimodular transformation every almost k-hollow lattice polygon is obtained by stepwise extending almost k-hollow lattice polygons inside D_R starting with $\Box$ and Δ_α for $R := R(k) := k^2\sqrt{4k^2+4k+2}$.*

In particular, this allows to construct all almost k-hollow lattice polygons up to unimodular equivalence. A naive way is to extend $\Box$ and Δ_α by lattice points from the box $\mathrm{conv}(\pm Re_1 + \pm Re_2)$ with $R = k^2\sqrt{4k^2+4k+2}$ and check in each step for almost k-hollowness. The following principle allows more target-oriented searching.

Remark 2.3.8. Consider a box $\mathcal{B} := \mathrm{conv}(\pm Re_1, \pm Re_2)$ in $\mathbb{R}^2$ and an almost k-hollow lattice polygon $\mathcal{A} \subseteq \mathcal{B}$. The *shadow* of $w \in k\mathbb{Z}^2$ with respect to $\mathcal{A}$ is

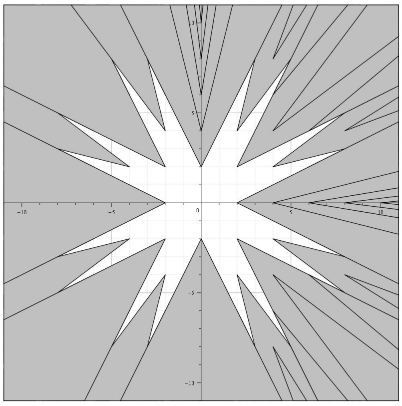
$$S(w,\mathcal{A}) \ := \ \mathrm{cone}(w-u;\ u \in \mathcal{A})^\circ + w \ \subseteq \ \mathbb{R}^2.$$

This is an open affine cone in $\mathbb{R}^2$. The lattice vectors $v \in \mathcal{B} \cap \mathbb{Z}^2$ such that $\mathcal{A}_v^+$ is almost k-hollow are all located in the star-shaped set

$$\Xi \ := \bigcap_{0 \ne w \in k\mathbb{Z}^2} \mathcal{B} \setminus S(w,\mathcal{A}) \ \subseteq \ \mathcal{B}.$$

Each $S(w, \mathcal{A})$ is described by two inequalities. This leads to an explicit description of the searching space Ξ and thus allows computational membership tests.

Example 2.3.9. Consider the case $k = 2$ and $\mathcal{A} = \square$. Then the searching space for possible $v \in \mathbb{Z}^2$ such that $\mathcal{A}_v^+$ is almost 2-hollow is the white area in the figure below.



With an ad hoc implementation using the bounds from Corollary 2.3.7 and Remark 2.3.8, we classified almost k-hollow lattice polygons computationally. Below we present some key data.

Theorem 2.3.10. *We obtain the following statements on almost k-hollow LDP-polygons.*

$k = 1$: *There are up to unimodular equivalence exactly* 16 *almost* 1*-hollow LDP-polygons. These are the well known reflexive polygons. The maximum number of vertices is* 6 *and the maximum volume is* $\frac{9}{2}$.

vertices	3	4	5	6
number	5	7	3	1
max.vol.	$\frac{9}{2}$	*4*	$\frac{7}{2}$	*3*

$k = 2$: *There are up to unimodular equivalence exactly* 505 *almost* 2*-hollow LDP-polygons. The maximum number of vertices is* 8*, realized by exactly one polygon, and the maximum volume is* 17.

vertices	3	4	5	6	7	8
number	42	181	202	74	5	1
max.vol.	16	16	17	$\frac{33}{2}$	14	14

$k = 3$: *There are up to unimodular equivalence exactly* 48032 *almost* 3*-hollow LDP-polygons. The maximum number of vertices is* 12*, realized by exactly one polygon, and the maximum volume is* 47.

vertices	3	4	5	6	7	8	9	10	11	12
number	355	3983	13454	17791	9653	2456	292	37	1	1
max.vol.	44	$\frac{91}{2}$	47	43	39	35	30	29	$\frac{47}{2}$	24

Here is the translation of Theorem 2.3.10 to the setting of toric del Pezzo surfaces. Recall that for log del Pezzo surfaces, the concepts of Gorenstein, ADE and canonical singularities coincide.

Corollary 2.3.11. *We obtain the following statements on toric ε-log canonical del Pezzo surfaces.*

$\varepsilon = 1$: *Up to isomorphy there are exactly* 16 *toric canonical del Pezzo surfaces. These are the well known toric Gorenstein del Pezzo surfaces. The maximum Picard number is* 4*, realized by exactly one surface.*

$\varepsilon = \frac{1}{2}$: *Up to isomorphy there are exactly* 505 *toric $\frac{1}{2}$-log canonical del Pezzo surfaces. The maximum Picard number is* 6*, realized by exactly one surface.*

$\varepsilon = \frac{1}{3}$: *Up to isomorphy there are exactly* 48032 *toric $\frac{1}{3}$-log canonical del Pezzo surfaces. The maximum Picard number is* 10*, realized by exactly one surface.*

CHAPTER 3

Basics on $\mathbb{K}^*$-surfaces

3.1. Rational T-varieties of complexity one

For the basics on toric varieties we refer the reader to Section 2.1 or to [**15**, **16**, **23**] for more in-depth treatments. Here, we only briefly recall Cox's quotient presentation of a toric variety from that area.

Construction 3.1.1. Let Z be the toric variety defined by a fan Σ in a lattice N such that the primitive generators $v_1, \ldots, v_r$ of the rays of Σ span the rational vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. We have a linear map

$$P\colon \mathbb{Z}^r \ \to \ N, \qquad e_i \ \mapsto \ v_i.$$

If $N = \mathbb{Z}^n$, we also speak of the generator matrix $P = [v_1, \ldots, v_r]$ of Σ. The divisor class group and the Cox ring of Z are

$$\mathrm{Cl}(Z) = K := \mathbb{Z}^r/\mathrm{im}(P^*), \qquad \mathcal{R}(Z) = \mathbb{K}[T_1, \ldots, T_r], \quad \deg(T_i) = Q(e_i),$$

where P^* denotes the dual map of P and $Q\colon \mathbb{Z}^r \to K$ the projection. Finally, we obtain a fan $\hat{\Sigma}$ in $\mathbb{Z}^r$ consisting of certain faces of the positive orthant, namely

$$\hat{\Sigma} \ := \ \{\delta_0 \preceq \mathbb{Q}^r_{\geq 0};\ P(\delta_0) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

The toric variety $\hat{Z}$ associated with $\hat{\Sigma}$ is an open toric subset in $\bar{Z} := \mathbb{K}^r$. As P is a map of the fans $\hat{\Sigma}$ and Σ, it defines a toric morphism $p\colon \hat{Z} \to Z$, the good quotient for the action of the quasitorus $H = \ker(p) \subseteq \mathbb{T}^r$ on $\hat{Z}$.

We briefly recall the Cox ring based approach to rational T-varieties X of complexity one as provided in the projective case by [**27**, **29**]; see also [**4**, Sec. 3.4]. We need the slightly more general version presented in [**30**, Constr. 1.6, Type 2]. This also includes affine X with only constant T-invariant functions.

Construction 3.1.2. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \ldots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \cdots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. The input data consists of matrices

$$A = [a_0, \ldots, a_r] \in \mathrm{Mat}(2, r{+}1; \mathbb{K}), \qquad P = \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix} \in \mathrm{Mat}(r{+}s, n{+}m; \mathbb{Z}),$$

where A has pairwise linearly independent columns and P is built from an $(s \times n)$-block d, an $(s \times m)$-block d' and an $(r \times n)$-block L of the form

$$L \ = \ \begin{bmatrix} -l_0 & l_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \ldots & l_r \end{bmatrix}, \qquad l_i \ = \ (l_{i1}, \ldots, l_{in_i}) \ \in \ \mathbb{Z}^{n_i}_{\geq 1}.$$

We require that the columns v_{ij}, v_k of P are pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a vector space. Consider the polynomial algebra

$$\mathbb{K}[T_{ij}, S_k] \ := \ \mathbb{K}[T_{ij}, S_k;\ 0 \le i \le r,\ 1 \le j \le n_i, 1 \le k \le m].$$

Denote by $\mathfrak{I}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \le i_1 < i_2 < i_3 \le r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I \ := \ g_{i_1,i_2,i_3} \ := \ \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}, \qquad T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(\mathrm{P}^*)$ and the projection $Q\colon \mathbb{Z}^{n+m} \to K$. We define a K-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ w_{ij} \ := \ Q(e_{ij}), \qquad \deg(S_k) \ := \ w_k \ := \ Q(e_k).$$

Then the trinomials g_I are K-homogeneous, all of the same degree. In particular, we obtain a K-graded factor algebra

$$R(A, P) \ := \ \mathbb{K}[T_{ij}, S_k] \ / \ \langle g_I;\ I \in \mathfrak{I} \rangle.$$

The ring $R(A, P)$ is a normal complete intersection ring and its ideal of relations is, for example, generated by $g_{i,i+1,i+2}$, where $i = 0, \ldots, r-2$. The varieties X with torus action of complexity one are constructed as quotients of $\operatorname{Spec} R(A, P)$ by the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$. Each of them comes embedded into a toric variety.

Construction 3.1.3. Situation as in Construction 3.1.2. Consider the common zero set of the defining relations of $R(A, P)$:

$$\bar{X} \ := \ V(g_I;\ I \in \mathfrak{I}) \ \subseteq \ \bar{Z} \ := \ \mathbb{K}^{n+m}.$$

Let Σ be any fan in the lattice $N = \mathbb{Z}^{r+s}$ having the columns of P as the primitive generators of its rays. Construction 3.1.1 leads to a commutative diagram

$$\begin{tikzcd} \bar{X} \arrow[r, phantom, "\subseteq"] & \bar{Z} \\ \hat{X} \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d, "p"', "/\!\!/ H"] & \hat{Z} \arrow[u, phantom, "\cup"] \arrow[d, "p", "/\!\!/ H"'] \\ X \arrow[r] & Z \end{tikzcd}$$

with a variety $X = X(A, P, \Sigma)$ embedded into the toric variety Z associated with Σ. Dimension, divisor class group and Cox ring of X are given by

$$\dim(X) = s + 1, \qquad \mathrm{Cl}(X) \ \cong \ K, \qquad \mathcal{R}(X) \ \cong \ R(A, P).$$

The subtorus $T \subseteq \mathbb{T}^{r+s}$ of the acting torus of Z associated with the sublattice $\mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$ leaves X invariant and the induced T-action on X is of complexity one.

Remark 3.1.4. In Construction 3.1.3, the group $H \cong \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)]$ is the *characteristic quasitorus* and $\bar{X} \cong \operatorname{Spec} \mathcal{R}(X)$ is the *total coordinate space* of X. Moreover, $p\colon \widehat{X} \to X$ is the *characteristic space* over X.

Remark 3.1.5. As in the toric case, Construction 3.1.3 yields *Cox coordinates* for the points of $X = X(A, P, \Sigma)$. Every $x \in X \subseteq Z$ can be written as $x = p(z)$, where $z \in \hat{X} \subseteq \hat{Z}$ is a point with closed H-orbit in $\hat{X}$ and this presentation is unique up to multiplication by elements of H.

The results of [**30**] tell us in particular the following; see also [**28**] for a generalization to higher complexity.

Theorem 3.1.6. *Every normal semiprojective rational variety with a torus action of complexity one having only constant invariant functions is equivariantly isomorphic to some $X(A, P, \Sigma)$.*

Proposition 3.1.7. *Consider a projective variety X defined by (A, P, Σ) and an ample divisor class $[D] \in \mathrm{Cl}(X)$ given by*

$$\mathrm{Cl}(X) \ni [D] = \sum_{i,j} \alpha_{ij} w_{ij} + \sum_k \alpha_k w_k \in K.$$

Then the corresponding affine cone X' over X is given by the data (A, P', Σ') with the stack matrix

$$P' = \begin{bmatrix} P \\ \alpha \end{bmatrix}, \qquad \alpha = (\alpha_{ij}, \alpha_k)$$

obtained by adding a row to P listing the coefficients α_{ij}, α_k and the fan of faces Σ' of the cone generated by the columns of P'.

3.2. Rational $\mathbb{K}^*$-surfaces

A $\mathbb{K}^*$*-surface* is an irreducible, normal surface X endowed with an effective morphical action $\mathbb{K}^* \times X \to X$ of the multiplicative group $\mathbb{K}^*$. In this section, we recall the basic background and present our working environment for rational $\mathbb{K}^*$-surfaces, the Cox ring based approach from [**27, 29, 30**]. We begin with a brief reminder on the raw geometric picture of projective $\mathbb{K}^*$-surfaces, the major part of which has been drawn in the work of Orlik and Wagreich, see [**43**].

Summary 3.2.1. Let X be a projective $\mathbb{K}^*$-surface. We discuss the basic geometric properties of the $\mathbb{K}^*$-action. The possible isotropy groups $\mathbb{K}^*_x$, where $x \in X$, are $\mathbb{K}^*$ itself and the subgroups of order $l \in \mathbb{Z}_{\geq 1}$ consisting of the l-th roots of unity. Thus, the non-trivial orbits are locally closed curves of the form

$$\mathbb{K}^* \cdot x \cong \mathbb{K}^*/\mathbb{K}^*_x \cong \mathbb{K}^*.$$

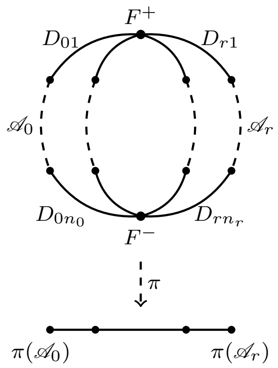
There are three types of fixed points. A fixed point $x \in X$ is *elliptic (hyperbolic, parabolic)* if it lies in the closure of infinitely many (precisely two, precisely one) non-trivial $\mathbb{K}^*$-orbit(s). Elliptic and hyperbolic fixed points are isolated, whereas the parabolic fixed points form a curve in X. The *limit points* x_0 and x_∞ of an orbit $\mathbb{K}^* \cdot x$ are obtained by extending the orbit map $t \to t \cdot x$ to a morphism $\varphi_x \colon \mathbb{P}_1 \to X$ and setting

$$x_0 := \lim_{t \to 0} t \cdot x := \varphi_x(0), \qquad x_\infty := \lim_{t \to \infty} t \cdot x := \varphi_x(\infty).$$

These limit points x_0, x_∞ are fixed points and together with $\mathbb{K}^* \cdot x$ they form the closure of the orbit $\mathbb{K}^* \cdot x$. Every projective $\mathbb{K}^*$-surface X has a *source* and a *sink*, i.e. irreducible components $F^+, F^- \subseteq X$ of the fixed point set admitting open $\mathbb{K}^*$-invariant neighborhoods $U^+, U^- \subseteq X$ such that

$$x_0 \ \in \ F^+ \text{ for all } x \in U^+, \qquad\qquad x_\infty \in \ F^- \text{ for all } x \in U^-.$$

The source and sink each consist of either a single elliptic fixed point or it is a smooth irreducible curve of parabolic fixed points. Apart from the source and the sink, we find at most hyperbolic fixed points. The raw geometric picture of a projective $\mathbb{K}^*$-surface X is as follows.



The general orbit $\mathbb{K}^* \cdot x \subseteq X$ has trivial isotropy group and connects the source and the sink in the sense that its closure contains one fixed point from F^+ and one from F^-. Besides the general orbits, there are special non-trivial orbits. Their closures are rational curves $D_{ij} \subseteq X$ forming the *arms*

$$\mathscr{A}_i \ := \ D_{i1} \cup \cdots \cup D_{in_i} \ \subseteq \ X, \qquad i = 0, \ldots, r.$$

The intersections $F^+ \cap D_{i1}$ and $D_{in_i} \cap F^-$ each consist of a fixed point and any two subsequent D_{ij}, D_{ij+1} intersect in a hyperbolic fixed point. Finally, the field of invariant rational functions $\mathbb{L} \subseteq \mathbb{K}(X)$ yields a projective curve C and a surjective rational map

$$\pi \colon X \dashrightarrow C.$$

It is defined everywhere except at possible elliptic fixed points. The $\mathbb{K}^*$-surface X is rational if and only if $C = \mathbb{P}_1$ holds. We will also call $\pi \colon X \dashrightarrow C$ the (rational) quotient of X. The critical fibers of π are up to elliptic fixed points precisely the arms containing two or more non-trivial orbits or an orbit with non-trivial finite isotropy group.

Our working environment for a sufficently explicit treatment is the Cox ring based approach to rational normal varieties with torus action of complexity one from [**27**, **29**, **30**]; see also [**28**] for torus actions of higher complexity. The following is a play back of [**30**, Constr. 1.1, 1.4 and 1.6] adapted to the surface case. The procedure starts with a pair (A, P) of matrices as defining data and puts out a *semiprojective* rational $\mathbb{K}^*$-surface $X(A, P)$ having only constant invariant functions and coming embedded into a toric variety Z. Here, semiprojective means projective over an affine variety. This comprises for instance the projective and the affine varieties.

Construction 3.2.2. Fix integers $r, n_0, \ldots, n_r \geq 1$ and $0 \leq m \leq 2$. We construct a rational semiprojective $\mathbb{K}^*$-surface X coming from a $2 \times (r+1)$

matrix A over $\mathbb{K}$ and an integral $(r+1) \times (n+m)$ matrix P, both given as a list of columns

$$A = [a_0, \dots, a_r], \qquad P = [v_{01}, \dots, v_{0n_0}, \dots, v_{r1}, \dots, v_{rn_r}, v_1, \dots, v_m].$$

Here $r+1 < n+m$ for $n = n_0 + \dots + n_r$. The columns of A are pairwise linearly independent and those of P generate $\mathbb{Q}^{r+1}$ as a vector space. Moreover, the columns v_{ij} of P are pairwise distinct and of the form

$$v_{0j} = (-l_{0j}, \dots, -l_{0j}, d_{0j}), \quad v_{ij} = (0, \dots, 0, l_{ij}, 0 \dots, 0, d_{ij}), \quad i = 1, \dots, r,$$

where l_{ij} sits at the i-th place for $i = 1, \dots, r$. We always have $\gcd(l_{ij}, d_{ij}) = 1$. The columns v_k are pairwise distinct as well and of the form

$$v^+ := (0, \dots, 0, 1), \qquad v^- := (0, \dots, 0, -1).$$

The idea is to let X come embedded into a toric variety Z. The corresponding defining fan Σ can be written down comfortably if P is *slope-ordered*, i.e.

$$m_{i1} > \dots > m_{in_i}, \quad m_{ij} := \frac{d_{ij}}{l_{ij}}, \quad i = 0, \dots, r, \ j = 1, \dots, n_i.$$

This can always be achieved by suitable numbering. In this setting, Σ has the columns of P as its rays and there are the maximal cones

$$\tau_{ij} := \operatorname{cone}(v_{ij}, v_{ij+1}) \in \Sigma, \qquad i = 0, \dots, r, \ j = 1, \dots, n_i - 1.$$

Depending on the existence of v^+ and v^- and the values of $\mathcal{m}^+ := m_{01} + \dots + m_{r1}$ and $\mathcal{m}^- := m_{0n_0} + \dots + m_{rn_r}$, we complement the collection of maximal cones by

$$\begin{aligned}
&\tau_i^+ := \operatorname{cone}(v^+, v_{i1}) \text{ for } i = 0, \dots, r \text{ if } P \text{ has a column } v^+,\\
&\tau_i^- := \operatorname{cone}(v^-, v_{in_i}) \text{ for } i = 0, \dots, r \text{ if } P \text{ has a column } v^-,\\
&\sigma^+ := \operatorname{cone}(v_{01}, \dots, v_{r1}) \text{ if } \mathcal{m}^+ > 0 \text{ and there is no } v^+,\\
&\sigma^- := \operatorname{cone}(v_{0n_0}, \dots, v_{rn_r}) \text{ if } \mathcal{m}^- < 0 \text{ and there is no } v^-.
\end{aligned}$$

The toric variety Z associated with Σ will be the ambient variety for X. Consider $\mathbb{K}^{n+m}$ with the coordinate functions T_{ij}, S_k and for $\iota = 0, \dots, r-2$ the trinomials

$$g_\iota := \det \begin{bmatrix} T_\iota^{l_\iota} & T_{\iota+1}^{l_{\iota+1}} & T_{\iota+2}^{l_{\iota+2}} \\ a_\iota & a_{\iota+1} & a_{\iota+2} \end{bmatrix} \in \mathbb{K}[T_{ij}, S_k],$$

where $T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$. Then, with $\bar{X} := V(g_0, \dots, g_{r-2}) \subseteq \mathbb{K}^{m+n} =: \bar{Z}$, we have a commutative diagram

$$\begin{array}{ccc}
\bar{X} & \subseteq & \bar{Z} \\
\cup\!| & & \cup\!| \\
\hat{X} & \longrightarrow & \hat{Z} \\
{\scriptstyle /H}\downarrow{\scriptstyle p} & & {\scriptstyle p}\downarrow{\scriptstyle /H} \\
X & \longrightarrow & Z.
\end{array}$$

Here $p\colon \hat{Z} \to Z$ denotes Cox's quotient presentation from 2.1.7. We set $\hat{X} := \bar{X} \cap \hat{Z}$ and $X := p(\hat{X})$. Moreover, we look at the acting torus $\mathbb{T}^{r+1}$ of Z and the homomorphism

$$\mathbb{K}^* \to \mathbb{T}^{r+1}, \qquad t \mapsto = (1, \dots, 1, t).$$

This defines a $\mathbb{K}^*$-action on Z which leaves $X \subseteq Z$ invariant. Altogether, we end up with a rational semiprojective $\mathbb{K}^*$-surface X with $\Gamma(X, \mathcal{O})^{\mathbb{K}^*} = \mathbb{K}$ coming embedded into a toric variety Z. Furthermore,

- X is projective if and only if the columns of P generate $\mathbb{Q}^{r+1}$ as a cone,
- X is affine if and only if $n_0 = \cdots = n_r = 1$ and P has neither v^+ nor v^-.

Theorem 3.2.3. *See* [**29, 30**]. *Every rational, semiprojective $\mathbb{K}^*$-surface having only constant invariant global functions is equivariantly isomorphic to a $\mathbb{K}^*$-surface arising from Construction 3.2.2.*

Remark 3.2.4. For $r = 1$, Construction 3.2.2 precisely gives the semiprojective toric surfaces with a $\mathbb{K}^*$-action. To be more specific, given $r = 1$, the following holds for $X = X(A, P)$.

(i) The fan Σ in $\mathbb{Z}^2$ has convex support and its one-dimensional cones are the rays given by the columns of P.
(ii) The surface X coincides with the toric surface Z and $\mathbb{K}^*$ acts via the one-parameter subgroup $\mathbb{K}^* \to \mathbb{T}^2$, $t \mapsto (1, t)$.

Let us see how to extract the general raw geometric picture in the case of a rational projective $\mathbb{K}^*$-surface $X = X(A, P)$ from its defining matrices A and P.

Summary 3.2.5. Consider a projective $\mathbb{K}^*$-surface $X = X(A, P)$. Let $D_Z^{ij} \subseteq Z$ denote the toric prime divisor corresponding to the ray generated by the column v_{ij} of P. Then we have $r + 1$ arms in X:

$$\mathscr{A}_i = D_X^{i1} \cup \cdots \cup D_X^{in_i}, \qquad D_X^{ij} = D_Z^{ij} \cap X.$$

The toric orbit of Z corresponding to a cone τ_{ij} cuts out the hyperbolic fixed point in $D_X^{ij} \cap D_X^{ij+1}$. According to the possible constellations of v^+, v^-, σ^+ and σ^- source and sink look as follows.

(e-e) The fan Σ has the cones σ^+ and σ^-. Then the associated toric orbits are elliptic fixed points $x^+, x^- \in X$ forming source and sink.
(e-p) The fan Σ has σ^+ as a cone and P has v^- as a column. The toric orbit given by σ^+ is an elliptic fixed point $x^+ \in X$ forming the source. The toric divisor D_Z^- given by the ray through v^- cuts out a curve D_X^- of parabolic fixed points forming the sink.
(p-e) The matrix P has v^+ as a column and Σ has σ^- as a cone. The toric divisor D_Z^+ given by the ray through v^+ cuts out a curve D_X^+ of parabolic fixed points forming the source. The toric orbit given by σ^- is an elliptic fixed point $x^- \in X$ forming the sink.
(p-p) The matrix P has v^+ and v^- as columns. The toric divisors D_Z^+, D_Z^- given by the rays through v^+, v^- cut out curves D_X^+, D_X^- of parabolic fixed points forming source and sink.

The entry l_{ij} of P equals the order of the isotropy group $\mathbb{K}_x^*$ of a general $x \in D_X^{ij}$ and the associated d_{ij} yields the weight of the tangent representation of $\mathbb{K}_x^*$. Moreover, we retrieve the quotient map $\pi\colon X \dashrightarrow \mathbb{P}_1$ from the commutative diagram

$$\begin{tikzcd} X \arrow[r, phantom, "\subseteq"] \arrow[d, dashed, "/\mathbb{K}^*"', "\pi"] & Z \arrow[d, dashed, "\pi"', "/\mathbb{K}^*"] \\ \mathbb{P}_1 \arrow[r, phantom, "\subseteq"] & \mathbb{P}_r. \end{tikzcd}$$

In the lower row, the homogeneous coordinates $[a_{i1}, a_{i2}]$ of the value $\pi(\mathscr{A}_i) \in \mathbb{P}_1$ of the i-th arm are given by the i-th column of the defining matrix A. Additionally, $\mathbb{P}_1 \subseteq \mathbb{P}_r$ is cut out by the linear forms

$$h_\iota \;:=\; \det \begin{bmatrix} U_\iota & U_{\iota+1} & U_{\iota+2} \\ a_\iota & a_{\iota+1} & a_{\iota+2} \end{bmatrix} \;\in\; \mathbb{K}[U_0, \ldots, U_r], \qquad \iota = 0, \ldots, r-2.$$

Remark 3.2.6 (Cox coordinates)**.** Each $x \in X$ is of the form $x = p(z)$ with a point

$$z \;=\; (z_{ij}, z_k) \;\in\; \hat{X} \;=\; \bar{X} \cap \hat{Z} \;\subseteq\; \bar{Z} \;=\; \mathbb{K}^{n+m}.$$

Here, $z = (z_{ij}, z_k)$ is unique up to multiplication by the quasitorus H. We call $x = [z_{ij}, z^{\pm}]$ a presentation in *Cox coordinates*. For any $x = [z_{ij}, z^{\pm}] \in X$, we have

$$\begin{aligned} x \text{ general} &\iff \text{all } z_{ij}, z_k \neq 0, \quad x \in D^+ \\ &\iff \text{only } z^+ = 0, \quad x \in D^- \\ &\iff \text{only } z^- = 0, \\ x \in D_{ij} &\iff \text{only } z_{ij} = 0, \quad x \in D_{ij} \cap D_{ij+1} \\ &\iff \text{only } z_{ij} = z_{ij+1} = 0, \\ x = x^+ &\iff \text{only } z_{01} = \cdots = z_{r1} = 0, \quad x = x^- \\ &\iff \text{only } z_{0n_0} = \cdots = z_{rn_r} = 0. \end{aligned}$$

By “general” we mean it is not contained in any arm of X. This collects all possibilities of vanishing and non-vanishing of Cox coordinates of the points of X.

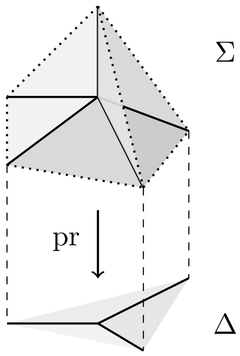
Example 3.2.7. Given formatting data $r = 2$, $n_0 = n_1 = 2, n_2 = 1$ and $m = 1$ consider the following defining matrices A and P.

$$A \;=\; \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \; P \;=\; [v_{01}, v_{02}, v_{11}, v_{21}, v^+] \;=\; \begin{bmatrix} -3 & -5 & 2 & 0 & 0 \\ -3 & -5 & 0 & 2 & 0 \\ -4 & -8 & 1 & 1 & 1 \end{bmatrix}.$$

For $i = 0, 1, 2$, the projection $\mathrm{pr}\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ onto the first two coordinates sends the columns v_{ij} of the matrix P into the rays $\varrho_i \subseteq \mathbb{Q}^2$ spanned by the vectors

$$(-1, -1), \qquad (1, 0), \qquad (0, 1).$$

These are the primitive generators of the fan Δ of the projective plane $\mathbb{P}_2$. For the fan Σ we obtain the picture



In terms of Cox coordinates, the surface $X = X(A, P)$ sitting in the toric variety Z defined by Σ is given as

$$\bar{X} \ = \ V(T_{01}^3 T_{02}^5 + T_{11}^2 + T_{21}^2) \ \subseteq \ \bar{Z} \ = \ \mathbb{K}^5.$$

The rational toric morphism $\pi\colon Z \dashrightarrow \mathbb{P}_2$ given by $\mathrm{pr}\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ is defined everywhere except at the point $x^- \in X \subseteq Z$. Restricting to X gives the map

$$\pi\colon X \ \dashrightarrow \ \mathbb{P}_1 \ = \ V(S_0 + S_1 + S_2) \ \subseteq \ \mathbb{P}_2,$$

which is the rational quotient. The source $D^+ \subseteq X$ maps onto $\mathbb{P}_1$ and the critical values of π are cut out by $S_i = 0$ for $i = 0, 1, 2$.

We conclude the section by discussing how to decide if two pairs of defining matrices give rise to isomorphic $\mathbb{K}^*$-surfaces.

Definition 3.2.8. Let P a defining matrix as in Construction 3.2.2.

(i) We call P *irredundant* if $l_{i1} n_i > 1$ for $i = 0, \ldots, r$. It is *redundant* if it is not irredundant.
(ii) A column v_{i1} of P is called *erasable* if $i > 0, n_i = 1, l_{i1} = 1$ and $d_{i1} = 0$.

Remark 3.2.9. Consider a $\mathbb{K}^*$-surface $X = X(A, P)$ with an irredundant P. Then we have the following.

(i) The surface X is isomorphic to a toric surface if and only if $r = 1$ holds.
(ii) The arms of X coincide with the critical fibers of the map $\pi\colon X \dashrightarrow \mathbb{P}_1$.

Now we introduce operations on a pair of defining matrices (A, P) that do not change the isomorphy type of the resulting $\mathbb{K}^*$-surface $X(A, P)$. As the i-th block $v_{i1}, \ldots, v_{in_i}$ of columns of P reflects the i-th arm of X, we also refer to $v_{i1}, \ldots, v_{in_i}$ as the i-th *arm of* P.

Definition 3.2.10. Consider pair (A, P) of defining matrices as in Construction 3.2.2 The *admissible operations* on (A, P) are the following ones.

(i) Add a multiple of one of the upper r rows to the last row of P.
(ii) Multiply the last row of P by -1.
(iii) Swap two columns v_{ij_1} and v_{ij_2} inside the i-th arm of P.
(iv) Swap the i-th and j-th column of A, the i-th and j-th arm of P and rearrange the shape of P by elementary operations on the first r rows.

(v) Swap two columns v_{k_1} and v_{k_2} of the d'-block.
(vi) *Erase* an erasable column v_{i1} by removing the i-th row and the $i1$-th column from P and the i-th column of A.
(vii) Transform A into BAD with $B \in \mathrm{GL}_2(\mathbb{K})$ and diagonal $D \in \mathrm{GL}_{r+1}(\mathbb{K})$.

We say that two pairs (A, P) and (A', P') of defining matrices are *equivalent* if we can transform one of them into the other via admissible operations. Also note that P is redundant if and only if there is a series of admissible operations of type (i) and (iv) on P such that the resulting matrix has an erasable column.

Recall that a *morphism* of $\mathbb{K}^*$-surfaces X and X' is a pair (φ, ψ) with a morphism $\varphi\colon X \to X'$ of varieties and a homomorphism $\psi\colon \mathbb{K}^* \to \mathbb{K}^*$ of algebraic groups such that we always have $\varphi(t \cdot x) = \psi(t) \cdot \varphi(x)$. So, in this setting, an equivariant morphism is a morphism (φ, ψ) with ψ being the identity.

Example 3.2.11. Given a rational projective $\mathbb{K}^*$-surface X, consider the automorphism $\jmath(t) = t^{-1}$ of $\mathbb{K}^*$. Then $(\mathrm{id}_X, \jmath)$ is a non-equivariant isomorphism $X \to X$ of $\mathbb{K}^*$-surfaces, swapping the source and the sink. For $X = X(A, P)$ the isomorphism $(\mathrm{id}_X, \jmath)$ is given by multiplying the last row of P by -1.

Proposition 3.2.12. *Consider defining pairs (A, P) and (A', P') of non-toric projective $\mathbb{K}^*$-surfaces X and X'. Then the following statements are equivalent.*
(i) *(A, P) and (A', P') are equivalent.*
(ii) *X and X' are isomorphic as $\mathbb{K}^*$-surfaces.*
(iii) *X and X' are isomorphic as surfaces.*

Note that the assumption of X and X' being non-toric is essential. For instance, on the projective plane $\mathbb{P}_2$ the $\mathbb{K}^*$-actions $[z_0, tz_1, t^2z_2]$ and $[z_0, z_1, tz_2]$ give rise to non-isomorphic $\mathbb{K}^*$-surfaces.

Remark 3.2.13. As observed in Remark 3.2.5, the defining matrix A of the $\mathbb{K}^*$-surface $X(A, P)$ is directly related to the quotient map $\pi\colon X \dashrightarrow \mathbb{P}_1$. By suitable scaling of the variables in the defining equations from Construction 3.2.2, we arrive at the simpler equations

$$\tilde{g}_\iota \;:=\; \lambda_\iota T_\iota^{l_\iota} + T_{\iota+1}^{l_{\iota+1}} + T_{\iota+2}^{l_{\iota+2}}, \qquad \iota = 0, \ldots, r-2,$$

where $\lambda_0, \ldots, \lambda_r \in \mathbb{K}^*$ with $\lambda_0 = 1$ are pairwise distinct. These equations define up to equivariant isomorphisms the same $\mathbb{K}^*$-surfaces but leave the framework of Construction 3.2.2. We see that for fixed P the surfaces $X(A, P)$ come in an $(r-2)$-dimensional family.

3.3. Geometry of rational $\mathbb{K}^*$-surfaces

We show how to read off basic geometric properties of rational $\mathbb{K}^*$-surfaces from their defining data. For a given $X = X(A, P)$ we explicitly determine divisor class group, Cox ring, Picard group, cones of effective, movable, semiample and ample divisor classes, canonical divisor, singularities and its intersection theory.

The key observation is that we directly get from Construction 3.2.2 of the $\mathbb{K}^*$-surface $X = X(A,P)$ its divisor class group $\mathrm{Cl}(X)$ and Cox ring

$$\mathcal{R}(X) \ = \ \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

We refer to [**4**, Sec. 1.1.4] for the precise definition of the Cox ring and to [**4**, Sec. 3.4.3] for details of the following.

Summary 3.3.1. Consider a $\mathbb{K}^*$-surface $X = X(A,P)$ and the toric embedding $X \subseteq Z$ as in Construction 3.2.2. Recall that the columns v_{ij} and $v^{\pm}$ of P give prime divisors

$$D_X^{ij} \ = \ X \cap D_Z^{ij} \ \subseteq \ X, \qquad\qquad D_X^{\pm} \ = \ X \cap D_Z^{\pm} \ \subseteq \ X.$$

These are obtained by intersecting with the corresponding toric prime divisors of Z. Moreover, every character function $\chi^u \in \mathbb{K}(Z)$ restricts to a rational function on X with divisor

$$\mathrm{div}(\chi^u) \ = \ \sum_{i,j} \langle u, v_{ij} \rangle D_X^{ij} + \sum_{+,-} \langle u, v^{\pm} \rangle D_X^{\pm}.$$

The terms of the second sum are understood to equal zero if there is no v^+ or v^-. The subvariety $X \subseteq Z$ inherits its divisor class group from Z:

$$\mathrm{Cl}(X) \ = \ \mathrm{Cl}(Z) \ = \ K \ := \ \mathbb{Z}^{n+m} / \mathrm{im}(P^t).$$

Let $Q\colon \mathbb{Z}^{n+m} \to K$ be the projection and let e_{ij}, $e^{\pm}$ be the canonical basis vectors of $\mathbb{Z}^{n+m}$, indexed in accordance with the variables T_{ij} and $S^{\pm}$. Then

$$\deg(T_{ij}) := w_{ij} := Q(e_{ij}) = [D_X^{ij}], \qquad \deg(S^{\pm}) := w^{\pm} := Q(e^{\pm}) = [D_X^{\pm}]$$

defines a K-grading on the polynomial ring $\mathbb{K}[T_{ij}, S^{\pm}]$ such that the trinomials $g_0, \ldots, g_{r-2}$ are homogeneous. We have isomorphisms

$$\mathcal{R}(X) \ \cong \ \mathbb{K}[T_{ij}, S_k]/\langle g_0, \ldots, g_{r-2} \rangle \ \cong \ \mathcal{R}(Z)/\langle g_0, \ldots, g_{r-2} \rangle$$

of K-graded $\mathbb{K}$-algebras. In particular, the Cox ring of X is a factor algebra of the Cox ring of its ambient toric variety Z.

In particular, this will allow to apply the whole machinery around Cox rings from [**4**]. Our first closer look focuses on data in the divisor class group; see Summary 2.1.3 for corresponding statements in the toric case.

Summary 3.3.2. Let $X = X(A,P)$ be a $\mathbb{K}^*$-surface and $X \subseteq Z$ the associated toric embedding. We take a look at various data in the divisor class groups

$$\mathrm{Cl}(X) \ = \ K \ = \ \mathrm{Cl}(Z).$$

For $\sigma \in \Sigma$ define a subgroup of K by $K_\sigma := \langle w_{ij}, w^{\pm};\ v_{ij}, v^{\pm} \not\in \sigma \rangle$. Then the *local class group* of $x \in X \cap \mathbb{T}^n \cdot z_\sigma$ is given by

$$\mathrm{Cl}(X, x) \ = \ K/K_\sigma \ = \ \mathrm{Cl}(Z, z_\sigma).$$

As a consequence, X also shares its *Picard group* with the ambient toric variety Z. More precisely, we obtain

$$\mathrm{Pic}(X) \ = \ \bigcap_{\sigma \in \Sigma} K_\sigma \ = \ \mathrm{Pic}(Z).$$

The *monoid of effective divisor classes* is generated by the classes w_{ij} of D_X^{ij} and $w^{\pm}$ of $D_X^{\pm}$. Thus, in $K_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} K$, the effective cone of X is

$$\mathrm{Eff}(X) \;=\; \mathrm{cone}(w_{ij}, w^{\pm}) \;=\; \mathrm{Eff}(Z),$$

For $\sigma \in \Sigma$ set $\sigma^* := \mathrm{cone}(w_{ij}, w^{\pm};\; v_{ij}, v^{\pm} \not\in \sigma)$. Then, for the *cones of movable and semiample divisor classes* of X, we have

$$\mathrm{Mov}(Z) = \bigcap_{\substack{\gamma_0 \preccurlyeq \gamma \\ \text{facet}}} Q(\gamma_0) = \mathrm{Mov}(X) = \mathrm{SAmple}(X) = \bigcap_{\sigma \in \Sigma} \sigma^* = \mathrm{SAmple}(Z).$$

Furthermore, the *ample cones* are given by

$$\mathrm{Ample}(X) \;=\; \mathrm{SAmple}(X)^{\circ} \;=\; \mathrm{SAmple}(Z)^{\circ} \;=\; \mathrm{Ample}(Z).$$

For the common degree $\mu \in \mathrm{Cl}(X)$ of the defining polynomials $g_0, \dots, g_{r-2}$ of $X \subseteq Z$, we have

$$\mu \;=\; \sum_{j=0}^{n_i} l_{ij} w_{ij} \;\in\; \bigcap_{i=0}^{r} \mathrm{cone}(w_{i1}, \dots, w_{in_i}) \;\subseteq\; \mathrm{SAmple}(X).$$

Finally, for each $i = 0, \dots, r$ we obtain an anticanonical divisor of X by the following adjunction formula.

$$-\mathcal{K}_X^i \;=\; \sum D_X^{ij} + \sum D_X^k - (r-1) \sum_{j=1}^{n_i} l_{ij} D_X^{ij}.$$

In particular, X is a del Pezzo surface if and only if the corresponding divisor class $-w_X = -w_Z - (r-1)\mu$ lies in the ample cone, which can be checked explicitly.

Proposition 3.3.3. *Every $\mathbb{K}^*$-surface $X = X(A, P)$ is $\mathbb{Q}$-factorial.*

Proof. We have to show that every Weil divisor has a non-zero Cartier multiple. For this consider the associated toric embedding $X \subseteq Z$. The defining fan of Z is simplicial and hence Z is $\mathbb{Q}$-factorial by Summary 2.1.4. Thus $\mathrm{Pic}(Z)$ is of finite index in $\mathrm{Cl}(X)$. Summary 3.3.2 shows that $\mathrm{Pic}(X)$ is of finite index in $\mathrm{Cl}(Z)$. Consequently X is $\mathbb{Q}$-factorial. ☐

Let us emphasize that the descriptions of the Picard group and the cones of (semi-)ample divisor classes essentially depend on the fact that we work with the toric embedding $X \subseteq Z$ provided by Construction 3.2.2. In contrast, the descriptions of the cone of movable divisor classes and the anticanonical divisors are more robust and allow going over to certain completions of Z as presented below.

Summary 3.3.4. Consider a projective $\mathbb{K}^*$-surface $X = X(A, P)$ and the toric embedding $X \subseteq Z$ as provided by Construction 3.2.2. In general, Z is not complete and we have several choices of possible toric completions $Z \subseteq Z'$. For instance, every divisor class $w \in \mathrm{Mov}(Z)^{\circ}$ yields a fan

$$\Sigma(w) \;:=\; \{P(\gamma_0^*);\; \gamma_0 \preccurlyeq \gamma,\; w \in Q(\gamma_0)^{\circ}\}.$$

Any such fan $\Sigma(w)$ is polytopal, has the same generator matrix as Σ and contains all cones of Σ. The associated open embeddings $Z \subseteq Z(w)$ are precisely the projective toric completions such that $Z(w)$ has the same Cox ring as Z. For a precise picture, one associates with $w \in \mathrm{Eff}(Z) = Q(\gamma)$ the polyhedral cone

$$\lambda(w) \ := \bigcap_{\substack{\gamma_0 \preccurlyeq \gamma,\\ w \in Q(\gamma_0)}} Q(\gamma_0).$$

These $\lambda(w)$ form a fan supported on $Q(\gamma) = \mathrm{Eff}(Z)$, the so-called *secondary fan*. For $w, w' \in \mathrm{Mov}(Z)^\circ$ we have $\lambda(w) \preccurlyeq \lambda(w')$ if and only if $\Sigma(w')$ refines $\Sigma(w)$. In particular, all $w' \in \lambda(w)^\circ$ share the same $\Sigma(w)$. A fan $\Sigma(w)$ is simplicial if and only if $\lambda(w)$ is full-dimensional.

We come to the intersection theory of rational $\mathbb{K}^*$-surfaces $X = X(A, P)$. As just noted, X is $\mathbb{Q}$-factorial and thus has indeed a well-defined intersection product. The aim is to compute intersection numbers in terms of the defining matrix P. As a preparation, we assign the following numbers to P, which in fact turn out to be ubiquitous in all of the subsequent considerations.

Definition 3.3.5. With any slope-ordered defining matrix P in the sense of Construction 3.2.2, we associate the numbers

$$l^+ := l_{01} \cdots l_{r1}, \quad \mathscr{m}^+ := m_{01} + \cdots + m_{r1}, \quad \ell^+ := \frac{1}{l_{01}} + \cdots + \frac{1}{l_{r1}} - r + 1,$$

$$l^- := l_{0n_0} \cdots l_{rn_r}, \quad \mathscr{m}^- := m_{0n_0} + \cdots + m_{rn_r}, \quad \ell^- := \frac{1}{l_{0n_0}} + \cdots + \frac{1}{l_{rn_r}} - r + 1.$$

Remark 3.3.6. Let P be a defining matrix as in Construction 3.2.2. Then we always have $\ell^+ \leq 2$ and $\ell^- \leq 2$. Moreover,

$$\begin{array}{rclcrcl} \mathscr{m}^+ & = & \dfrac{1}{l^+}\det(\sigma^+), & \qquad & \det(\sigma^+) & := & (-1)^r \det(v_{01}, \ldots, v_{r1}), \\ \mathscr{m}^- & = & \dfrac{1}{l^-}\det(\sigma^-), & \qquad & \det(\sigma^-) & := & (-1)^r \det(v_{0n_0}, \ldots, v_{rn_r}). \end{array}$$

If $X = X(A, P)$ has an elliptic fixed point $x^+ \in X$, then $\mathscr{m}^+ > 0$ and if there is $x^- \in X$, then $\mathscr{m}^- < 0$.

We first explain the basic principles of intersecting the relevant invariant curves on $X = X(A, P)$. Then we list all their intersection numbers.

Summary 3.3.7. Consider a $\mathbb{K}^*$-surface $X = X(A, P)$ with slope-ordered P. The fact that X comes as a complete intersection in a $\mathbb{Q}$-factorial projective ambient toric variety Z allows to perform intersections of curves via intersecting suitable ambient toric divisors. For $i = 1, \ldots, r$, consider the divisors

$$D_Z^i \ := \ \sum_{j=1}^{n_i} l_{ij} D_Z^{ij} \ \in \ \mathrm{WDiv}(Z).$$

The divisor class of D_Z^i equals the common degree the defining relations of $X \subseteq Z$. Thus, by general intersection theory, the intersection number of any two of the D_X^{ij} and $D_X^{\pm}$ equals the intersection of the corresponding toric prime divisors with $r-1$ of the D_Z^i. For instance, for D_X^{01} and D_X^{11} we can write

$$D_X^{01} \cdot D_X^{11} \ = \ D_Z^{01} \cdot D_Z^{11} \cdot D_Z^2 \cdots D_Z^r.$$

The toric intersection number expands into intersection numbers of pairwise distinct toric prime divisors. These, if non-zero, are given by toric intersection theory as one over the determinant of the involved primitive generators. If D_X^{01} and D_X^{11} intersect precisely in $x^+ \in X$, we obtain

$$D_X^{01} \cdot D_X^{11} \;=\; D_Z^{01} \cdot D_Z^{11} \cdot D_Z^{21} \cdots D_Z^{r1} \;=\; \frac{l_{21} \cdots l_{r1}}{\det(\sigma^+)} \;=\; \frac{1}{l_{01}l_{11}} \frac{1}{m^+}.$$

Note that any two of the D_Z^i are linearly equivalent and that $D_Z^{\pm}$ is linearly equivalent to $\mp \sum E_Z^i$ with $E_Z^i := \sum d_{ij} D_Z^{ij}$. This enables us to represent any intersection between D_X^{ij} and $D_X^{\pm}$ as a linear combination of intersection numbers of pairwise distinct toric prime divisors and hence to proceed as above.

Summary 3.3.8. Let $X = X(A, P)$ be a projective $\mathbb{K}^*$-surface with slope-ordered P. For $i = 0, \ldots, r$, set

$$\begin{aligned}
m_{i0} \quad &:= \quad \begin{cases} 0, & \text{if there is } D_X^+ \subseteq X, \\ -\frac{1}{m^+}, & \text{if there is } x^+ \in X, \end{cases} \\
m_{ij} \quad &:= \quad \tfrac{1}{m_{ij} - m_{ij+1}}, \qquad j = 1, \ldots, n_i - 1, \\
m_{in_i} \quad &:= \quad \begin{cases} 0, & \text{if there is } D_X^- \subseteq X, \\ \frac{1}{m^-}, & \text{if there is } x^- \in X. \end{cases}
\end{aligned}$$

Then the non-zero intersection numbers of (possible) curves $D_X^{\pm}$ and the curves D_X^{ij} in X are given by

$$D_X^+ \cdot D_X^{i1} \;=\; \frac{1}{l_{i1}}, \qquad D_X^{ij} \cdot D_X^{ij+1} \;=\; \frac{1}{l_{ij} l_{ij+1}} m_{ij}, \qquad D_X^{in_i} \cdot D_X^- \;=\; \frac{1}{l_{in_i}}.$$

Moreover, in case there is an elliptic fixed point $x^+ \in X$ or $x^- \in X$, for $i \neq k$ we have the following intersections

$$\begin{aligned}
D_X^{i1} \cdot D_X^{k1} \quad &= \quad -\frac{1}{l_{i1} l_{k1}} (m_{i0} + m_{in_i}), \qquad && n_i n_k = 1, \\
D_X^{i1} \cdot D_X^{k1} \quad &= \quad -\frac{1}{l_{i1} l_{k1}} m_{i0}, && n_i n_k > 1, \\
D_X^{in_i} \cdot D_X^{kn_k} \quad &= \quad -\frac{1}{l_{in_i} l_{kn_k}} m_{in_i}, && n_i n_k > 1.
\end{aligned}$$

Lastly, the self intersection numbers of the (possible) curves $D_X^{\pm}$ and the curves D_X^{ij} in X are given by

$$D_X^+ \cdot D_X^+ \;=\; -m^+, \qquad D_X^- \cdot D_X^- \;=\; m^-, \qquad D_X^{ij} \cdot D_X^{ij} \;=\; -\frac{1}{l_{ij}^2} (m_{ij-1} + m_{ij}).$$

As the D_X^{ij} and $D_X^{\pm}$ generate the divisor class group of $X = X(A, P)$, the above computations determine the entire intersection theory of X. In particular, we can directly compute intersections of the anticanonical divisor with the relevant curves and, subsequently, the anticanonical self intersection.

Proposition 3.3.9. *Consider a projective $\mathbb{K}^*$-surface $X = X(A,P)$ with slope-ordered P. For $i = 0,\dots,r$, set*

$$\begin{aligned}
\ell_{i0} &:= \begin{cases} \infty, & \text{if there is } D_X^+ \subseteq X, \\ -\ell^+, & \text{if there is } x^+ \in X, \end{cases} \\
\ell_{ij} &:= \tfrac{1}{l_{ij}} - \tfrac{1}{l_{ij+1}}, \quad j = 1,\dots,n_i - 1, \\
\ell_{in_i} &:= \begin{cases} -\infty, & \text{if there is } D_X^- \subseteq X, \\ \ell^-, & \text{if there is } x^- \in X. \end{cases}
\end{aligned}$$

Setting $\infty \cdot 0 = 1$ and $-\infty \cdot 0 = -1$, we can write the intersections of an anticanonical divisor $-\mathcal{K}_X$ with the D_X^{ij} and $D_X^{\pm}$ as

$$\begin{aligned}
-\mathcal{K}_X \cdot D_X^+ &= \ell^+ - m^+, \\
-\mathcal{K}_X \cdot D_X^{ij} &= \tfrac{1}{l_{ij}}\left(\ell_{ij-1} m_{ij-1} - \ell_{ij} m_{ij}\right), \\
-\mathcal{K}_X \cdot D_X^- &= \ell^- + m^-.
\end{aligned}$$

Proof. This is an explicit computation. Take the anticanonical divisor $\mathcal{K}_X^0$ from Summary 3.3.2 and then apply Summary 3.3.8. □

Proposition 3.3.10. *Consider a projective $\mathbb{K}^*$-surface $X = X(A,P)$ with slope-ordered P. To any arm $\mathscr{A}_i$, where $i = 0,\dots,r$, we attach the number*

$$\alpha_i := \sum_{j=1}^{n_i-1} \frac{\lambda_{ij}}{\Delta_{ij}}, \qquad \Delta_{ij} := l_{ij+1} d_{ij} - l_{ij} d_{ij+1}, \qquad \lambda_{ij} := 2 - \frac{l_{ij+1}}{l_{ij}} - \frac{l_{ij}}{l_{ij+1}},$$

where $\alpha_i = 0$ if $n_i = 1$. Then, according to the possible constellations of source and sink, the anticanonical self intersection number of X is given by

$$\begin{aligned}
&\text{(e-e)} \quad \mathcal{K}_X^2 = \frac{(\ell^+)^2}{m^+} + \alpha_0 + \dots + \alpha_r - \frac{(\ell^-)^2}{m^-}, \\
&\text{(e-p)} \quad \mathcal{K}_X^2 = \frac{(\ell^+)^2}{m^+} + \alpha_0 + \dots + \alpha_r + 2\ell^- + m^-, \\
&\text{(p-e)} \quad \mathcal{K}_X^2 = 2\ell^+ - m^+ + \alpha_0 + \dots + \alpha_r - \frac{(\ell^-)^2}{m^-}, \\
&\text{(p-p)} \quad \mathcal{K}_X^2 = 2\ell^+ - m^+ + \alpha_0 + \dots + \alpha_r + 2\ell^- + m^-.
\end{aligned}$$

Proof. Take again the anticanonical divisor $\mathcal{K}_X^0$ from Summary 3.3.2 and use Proposition 3.3.9. □

Remark 3.3.11. In the case $r = 1$ with two elliptic fixed points, Proposition 3.3.10 exactly reproduces Remark 2.2.3. Note that on every complete toric surface, we find a $\mathbb{K}^*$-action with two elliptic fixed points.

3.4. Singularities

We can now take a close look at the singularities of a $\mathbb{K}^*$-surface $X(A,P)$. First, we explain how to detect them in terms of the defining matrix P. Then we present the canonical resolution of singularities and give a detailed discussion of the local Gorenstein index and log terminality.

A point $x \in X$ is called *quasismooth* if $x = p(\hat{x})$ holds with a smooth point $\hat{x} \in \hat{X}$. A point $x \in X$ is called *factorial* if its local divisor class group is trivial. The latter holds if and only if the local ring $\mathcal{O}_{X,x}$ admits unique factorization.

Summary 3.4.1. Let $X = X(A,P)$ be a $\mathbb{K}^*$-surface $X = X(A,P)$ where P is slope-ordered. Consider the toric embedding $X \subseteq Z$ from Construction 3.2.2. First we note the following.

- As X is normal, its singularities are necessarily fixed points.
- The surface X inherits $\mathbb{Q}$-factoriality from Z; see Proposition 3.3.3.

Parabolic fixed points are always quasismooth and any parabolic fixed point not contained in an arm of X is even smooth. For every point

$$x_i^+ \ \in \ D_X^+ \cap \mathscr{A}_i, \qquad\qquad x_i^- \ \in \ D_X^- \cap \mathscr{A}_i$$

we find a $\mathbb{K}^*$-invariant open neighborhood $U_i^{\pm} \subseteq X$ isomorphic to a $\mathbb{K}^*$-invariant open subset of the toric surface with generator matrix

$$P_i^+ \ = \ \begin{bmatrix} 0 & l_{i1} \\ 1 & d_{i1} \end{bmatrix}, \qquad\qquad P_i^- \ = \ \begin{bmatrix} l_{in_i} & 0 \\ d_{in_i} & -1 \end{bmatrix}.$$

In particular, x_i^+ (x_i^-) is smooth if and only if $l_{i1} = 1$ ($l_{in_i} = 1$). We examine the hyperbolic fixed points. The intersection point

$$x_{ij} \ \in \ D_X^{ij} \cap D_X^{ij+1}$$

is always quasismooth and it admits a $\mathbb{K}^*$-invariant open neighborhood isomorphic to a $\mathbb{K}^*$-invariant open subset of the toric surface with generator matrix

$$P_{ij} \ = \ \begin{bmatrix} l_{ij} & l_{ij+1} \\ d_{ij} & d_{ij+1} \end{bmatrix}.$$

In particular x_{ij} is smooth if and only if $\det(P_{ij}) = -1$. For the possible elliptic fixed points, we have the affine $\mathbb{K}^*$-invariant open neighborhoods

$$x^+ \in U^+ = \{x \in X;\ x_0 = x^+\} \subseteq X, \quad x^- \in U^- = \{x \in X;\ x_\infty = x^-\} \subseteq X.$$

As $\mathbb{K}^*$-surfaces they are given by $U^+ = X(A, P^+)$ and $U^- = X(A, P^-)$ with the defining matrices

$$P^+ \ = \ [v_{01}, \ldots, v_{r1}], \qquad\qquad P^- \ = \ [v_{0n_0}, \ldots, v_{rn_r}].$$

The local class group of an elliptic fixed point $x^{\pm}$ equals the divisor class group of $U^{\pm}$. Moreover, x^+ (x^-) is

- quasismooth if and only if $l_{i1} = 1$ ($l_{in_i} = 1$) for at least $r-1$ of $i = 0, \ldots, r$,
- factorial if and only if $\det(P^+) = l^+ m^+ = 1$ ($\det(P^-) = l^- m^- = -1$),
- smooth if and only if it is factorial and quasismooth.

Note that $x^{\pm} \in X$ is factorial if and only if the Cox ring of the affine $\mathbb{K}^*$-surface $U^{\pm}$ coincides with its total coordinate ring.

We present the resolution of singularities of a $\mathbb{K}^*$-surface $X = X(A,P)$ in terms of the defining matrix P. Note that the procedure yields in particular the star-shaped resolution graphs observed in [**41**, **42**, **44**].

Summary 3.4.2. See [**4**, Constr. 5.4.3.2]. The *canonical resolution* of singularities $X'' \to X$ of $X = X(A, P)$ is obtained by the following two-step procedure.

(i) Enlarge P to a matrix P' by adding the columns v^+ and v^- if not already present. Then we have the surface $X' := X(A, P')$ and a canonical morphism $X' \to X$.

(ii) Let P'' be the defining matrix having the primitive generators of the regular subdivision Σ'' of Σ' as its columns. Then $X'' := X(A, P'')$ is smooth and there is a canonical morphism $X'' \to X'$.

Both fans Σ' and Σ'' have the *tropical variety* of $X \subseteq Z$ as their support. With the canonical basis vectors $e_1, \ldots, e_{r+1} \in \mathbb{Z}^{r+1}$ and $e_0 := -e_1 - \cdots - e_r$, it is given by

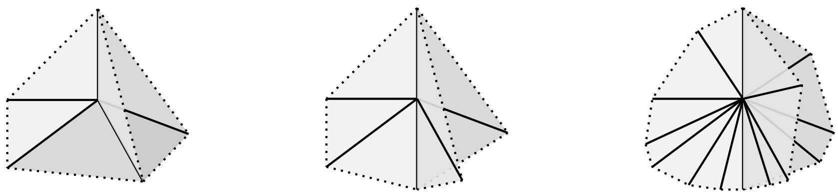
$$\operatorname{trop}(X) \ = \ \lambda_0 \cup \cdots \cup \lambda_r \ \subseteq \ \mathbb{Q}^{r+1}, \qquad \lambda_i \ := \ \operatorname{cone}(e_i, \pm e_{r+1}).$$

Contracting all (-1)-curves inside the smooth locus that lie over singularities of X gives $X'' \to \tilde{X} \to X$, where $X(A, \tilde{P}) = \tilde{X} \to X$ is the *minimal resolution* of X.

Example 3.4.3. Consider again the $\mathbb{K}^*$-surface $X = X(A, P)$ from Example 3.2.7, given by the defining data

$$A \ = \ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad\qquad P \ = \ \begin{bmatrix} -3 & -5 & 2 & 0 & 0 \\ -3 & -5 & 0 & 2 & 0 \\ -4 & -8 & 1 & 1 & 1 \end{bmatrix}.$$

The four hyperbolic fixed points and the elliptic fixed point are singular. The two resolution steps from Summary 3.4.2 schematically look as follows.



Σ $\qquad\qquad$ Σ' $\qquad\qquad$ Σ''

Explicitly, the defining matrix $\tilde{P}$ of the minimal resolution $\tilde{X} = X(A, \tilde{P})$ is given by

$$\tilde{P} \ = \ \begin{bmatrix} -1 & -3 & -2 & -5 & -3 & -1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -3 & -2 & -5 & -3 & -1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ -1 & -4 & -3 & -8 & -5 & -2 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

Remark 3.4.4. Consider $X = X(A, P)$ and the associated surface X' from Step (i) of Summary 3.4.2. Then we have

$$m^+ \ = \ -D^+_{X'} \cdot D^+_{X'}, \qquad\qquad \ell^+ \ = \ D^+_{X'} \cdot D^+_{X'} - \mathcal{K}^0_{X'} \cdot D^+_{X'},$$
$$m^- \ = \ D^-_{X'} \cdot D^-_{X'}, \qquad\qquad \ell^- \ = \ D^-_{X'} \cdot D^-_{X'} - \mathcal{K}^0_{X'} \cdot D^-_{X'}.$$

Construction 3.4.5. Consider a rational projective $\mathbb{K}^*$-surface X, the quotient $\pi \colon X \dashrightarrow \mathbb{P}_1$, its domain of definition $U \subseteq X$ and the closure of the graph

$$X' \ := \ \overline{\Gamma}_\pi \ \subseteq \ X \times \mathbb{P}_1, \qquad \Gamma_\pi \ = \ \{(x, \pi(x));\ x \in U\} \ \subseteq \ X \times \mathbb{P}_1.$$

Then X' comes with a $\mathbb{K}^*$-action given by $t * (x, z) = (t \cdot x, z)$ and the projection yields an equivariant birational morphism

$$X' \ \to \ X, \qquad (x, \pi(x)) \ \mapsto \ x.$$

Furthermore, for every equivariant morphism $\varphi\colon X_1 \to X_2$ of $\mathbb{K}^*$-surfaces with rational quotients $\pi_i\colon X_i \dashrightarrow \mathbb{P}_1$, we have an induced equivariant morphism

$$\varphi'\colon X_1' \ \to \ X_2', \qquad (x, \pi_1(x)) \ \mapsto \ (\varphi(x), \pi_2(x)).$$

Proposition 3.4.6. *Consider $X = X(A, P)$. Then the morphism $X' \to X$ from Construction 3.4.5 equals the one presented in Summary 3.4.2 (i).*

Corollary 3.4.7. *Let the $\mathbb{K}^*$-surfaces X_1 and X_2 arise from defining data (A_1, P_1) and (A_2, P_2). If there is an equivariant isomorphism $X_1 \cong X_2$, then we have*

$$m_1^+ = m_2^+, \qquad \ell_1^+ = \ell_2^+, \qquad m_1^- = m_2^-, \qquad \ell_1^- = \ell_2^-.$$

In particular, for every rational projective $\mathbb{K}^*$-surface X, we can choose any isomorphism $X \cong X(A, P)$ and obtain well defined numbers

$$m_X^+ := m^+, \qquad \ell_X^+ := \ell^+, \qquad m_X^- := m^-, \qquad \ell_X^- := \ell^-.$$

Recall that a normal variety X is $\mathbb{Q}$-*Gorenstein* if some non-zero multiple of a canonical divisor $\mathcal{K}_X$ is Cartier. Moreover, in this case, the *Gorenstein index* of X is the smallest non-zero integer ι_X such that $\iota_X \mathcal{K}_X$ is Cartier and the *local Gorenstein index* of $x \in X$ is the smallest non-zero integer ι_x such that $\iota_x \mathcal{K}_X$ is principal in some neighborhood of x. Note that ι_X is the least common multiple of all ι_x, where $x \in X$.

Proposition 3.4.8. *Consider a $\mathbb{K}^*$-surface $X = X(A, P)$ with slope-ordered P and the (possible) parabolic and hyperbolic fixed points of X.*

(i) *The local Gorenstein index of a parabolic fixed point $x_i^+ \in D_X^+ \cap \mathscr{A}_i$ is given by*

$$\iota_i^+ \ := \ \iota(x_i^+) \ = \ \frac{l_{i1}}{\gcd(d_{i1} - 1, l_{i1})}.$$

(ii) *The local Gorenstein index of a parabolic fixed point $x_i^- \in D_X^- \cap \mathscr{A}_i$ is given by*

$$\iota_i^- \ := \ \iota(x_i^-) \ = \ \frac{l_{in_i}}{\gcd(d_{in_i} + 1, l_{in_i})}.$$

(iii) *The local Gorenstein index of a hyperbolic fixed point $x_{ij} \in D_X^{ij} \cap D_X^{ij+1}$ is given by*

$$\iota_{ij} \ := \ \iota(x_{ij}) \ = \ \frac{l_{ij+1} d_{ij} - l_{ij} d_{ij+1}}{\gcd(l_{ij} - l_{ij+1}, d_{ij} - d_{ij+1})}.$$

Proof. A neighborhood of x_i^+, x_i^-, x_{ij} is isomorphic to a neighborhood of the toric fixed point of the affine toric surface with generator matrix

$$P_i^+ \ = \ \begin{bmatrix} 0 & l_{i1} \\ 1 & d_{i1} \end{bmatrix}, \qquad P_i^- \ = \ \begin{bmatrix} l_{in_i} & 0 \\ d_{in_i} & -1 \end{bmatrix}, \qquad P_{ij} \ = \ \begin{bmatrix} l_{ij} & l_{ij+1} \\ d_{ij} & d_{ij+1} \end{bmatrix},$$

respectively. Consequently, Remark 2.2.7 provides the desired formulae for the local Gorenstein indices. $\square$

Proposition 3.4.9. *Consider a $\mathbb{K}^*$-surface $X = X(A,P)$ with slope-ordered P. We define linear forms $u^\pm \in \mathbb{Q}^{r+1}$ by*

$$u^+ \;:=\; \frac{1}{l^+m^+}\left(u_1^+,\dots,u_r^+,l^+\ell^+\right), \qquad u^- \;:=\; \frac{1}{l^-m^-}\left(u_1^-,\dots,u_r^-,l^-\ell^-\right),$$

where $u_i^\pm \in \mathbb{Z}$ is given by

$$u_i^+ \;=\; (r-1)d_{i1}\frac{l^+}{l_{i1}} + \sum_{j\neq i}(d_{i1}-d_{j1})\frac{l^+}{l_{i1}l_{j1}}, \qquad i = 1,\dots,r,$$

$$u_i^- \;=\; (r-1)d_{in_i}\frac{l^-}{l_{in_i}} + \sum_{j\neq i}(d_{in_i}-d_{jn_j})\frac{l^-}{l_{in_i}l_{jn_j}}, \qquad i = 1,\dots,r.$$

Then the linear forms $u^\pm \in \mathbb{Q}^{r+1}$ are uniquely determined by satisfying the properties

$$\langle u^+, v_{01}\rangle = 1-(r-1)l_{01}, \qquad \langle u^+, v_{i1}\rangle = 1,\; i = 1,\dots,r,$$

$$\langle u^-, v_{0n_0}\rangle = 1-(r-1)l_{0n_0}, \qquad \langle u^-, v_{in_i}\rangle = 1,\; i = 1,\dots,r.$$

If there is an elliptic fixed point $x^\pm \in X$, then the following holds for the local Gorenstein index.

(i) *$\iota(x^+)$ is the unique positive integer with $\iota(x^+)u^+ \in \mathbb{Z}^{r+1}$ being primitive and it is explicitly given by*

$$\iota^+ \;:=\; \iota(x^+) \;=\; \frac{l^+m^+}{\gcd(u_1^+,\dots,u_r^+,l^+\ell^+)}.$$

(ii) *$\iota(x^-)$ is the unique positive integer with $\iota(x^-)u^- \in \mathbb{Z}^{r+1}$ being primitive and it is explicitly given by*

$$\iota^- \;:=\; \iota(x^-) \;=\; -\frac{l^-m^-}{\gcd(u_1^-,\dots,u_r^-,l^-\ell^-)}.$$

Proof. The characterizing properties of the linear forms $u^\pm$ are a result of direct computation. Near $x^\pm$, the characterizing properties of $u^\pm$ yield

$$l^+m^+\mathcal{K}_X^0 \;=\; \mathrm{div}(\chi^{l^+m^+u^+}), \qquad l^-m^-\mathcal{K}_X^0 \;=\; \mathrm{div}(\chi^{l^-m^-u^-}).$$

Note that $l^\pm m^\pm u^\pm$ is an integral vector and thus $l^\pm m^\pm$ is a multiple of the local Gorenstein index $\iota(x^\pm)$. The remaining assertions follow. $\square$

We characterize log terminality of rational $\mathbb{K}^*$-surfaces. Recall that for any variety X with a $\mathbb{Q}$-Cartier canonical divisor $\mathcal{K}_X$ one considers a resolution of singularities $\pi\colon X \to X'$ and the associated *ramification formula*

$$\mathcal{K}'_X \;=\; \pi^*\mathcal{K}_X + \sum a(E)E.$$

Here, E runs through the exceptional prime divisors and the $a(E) \in \mathbb{Q}$ are the *discrepancies* of $\pi\colon X' \to X$. The singularities of X are called *log terminal* if $a(E) > -1$ for each E.

Proposition 3.4.10. *Consider a $\mathbb{K}^*$-surface $X = X(A,P)$ and its canonical resolution $\pi\colon X'' \to X$.*

(i) *If there is an elliptic fixed point $x^+ \in X$, then the discrepancy of $D^+_{X''}$ is given by*

$$a(D^+_{X''}) \;=\; \frac{\ell^+}{m^+} - 1.$$

(ii) *If there is an elliptic fixed point $x^- \in X$, then the discrepancy of $D^-_{X''}$ is given by*

$$a(D^-_{X''}) \;=\; -\frac{\ell^-}{m^-} - 1.$$

(iii) *If $\ell^\pm > 0$ holds for $x^\pm \in X$, then every exceptional divisor $D \subseteq \pi^{-1}(x^\pm)$ has discrepancy strictly greater than -1.*

(iv) *Every exceptional divisor $D \subseteq X'' \backslash \pi^{-1}(x^\pm)$ has discrepancy strictly greater than -1.*

In particular, all points $x \in X$ distinct from $x^\pm$ are log terminal and $x^\pm \in X$ is log terminal if and only if $\ell^\pm > 0$.

Proof. For (i), we compute the discrepancy on the affine open subset $U^+ \subseteq X$ containing all orbits that have $x^+ \in X$ in their closure. That means we are in the case $P = [v_{01}, \dots, v_{r1}]$. Let $u \in \mathbb{Q}^{r+1}$ represent $\mathcal{K}^0_X$. Then

$$\langle u, v_{01} \rangle \;=\; -1 + (r-1)l_{01}, \qquad \langle u, v_{i1} \rangle \;=\; -1, \quad i = 1, \dots, r.$$

Additionally, with the Gorenstein index $\iota = \iota_X$, we have $\pi^*(\iota \mathcal{K}^0_X) = \mathrm{div}(\chi^{\iota u})$. Plugging this into the ramification formula we get

$$-D^+_{X''} \;=\; \langle u, v^+ \rangle D^+_{X''} + a(D^+_{X''}) D^+_{X''} \;=\; -\frac{\ell^+}{m^+} D^+_{X''} + a(D^+_{X''}) D^+_{X''}.$$

The evaluation of u at v^+ is a direct computation. We conclude that the discrepancy $a(D^+_{X''})$ is as claimed in the assertion. The case of an elliptic fixed point $x^- \in X$ is treated analogously.

We show (iii) and (iv). First look at the case that D comes from a column v of P'' with $v \in \mathrm{cone}(v^+, v_{i1})$ for $i = 1, \dots, r$. Then $v = bv^+ + cv_{i1}$ with positive $b, c \in \mathbb{Q}$. Using the linear form u from above, we compute

$$\begin{aligned} -D &= \langle u, v \rangle D + a(D) = \langle u, bv^+ + cv_{i1} \rangle D + a(D) \\ &= -\left(b\frac{\ell^+}{m^+} + c \right) D + a(D)D. \end{aligned}$$

Since $\ell^+ > 0$, we can conclude $a(D) > -1$. If $v \in \mathrm{cone}(v^+, v_{01})$ the same reasoning works with the canonical divisor $\mathcal{K}^1_X$. Moreover, the arguments adapt to the cases $v \in \sigma^-$ and $v \in \tau_{ij}$. ☐

Remark 3.4.11. A tuple $(q_0, \dots, q_r)$ of positive integers is called *platonic* if

$$q_0^{-1} + \dots + q_r^{-1} \;>\; r - 1.$$

If $q_0 \geq \dots \geq q_r$ holds, platonicity is equivalent to $q_3 = \dots = q_r = 1$ and (q_0, q_1, q_2) being one of

$$(q_0, q_1, 1), \quad (q_0, 2, 2), \quad (5, 3, 2), \quad (4, 3, 2), \quad (3, 3, 2).$$

Consider $X = X(A,P)$ with slope-ordered P and look at the tuples of exponents associated with possible elliptic fixed points x^+ and x^-:

$$(l_{01},\ldots,l_{r1}), \qquad\qquad (l_{0n_0},\ldots,l_{rn_r}).$$

We have $\ell^\pm > 0$, i.e. X is log terminal, if and and only if the corresponding tuple is platonic.

Log terminal surface singularities have been studied intensively, see [**12**] for a classical reference. They are known to be precisely the quotient singularities $\mathbb{K}^2/G$, where $G \subseteq \mathrm{GL}(2,\mathbb{K})$ is a finite subgroup. Any such affine surface $\mathbb{K}^2/G$ comes with the $\mathbb{K}^*$-action induced by scalar multiplication, which allows an easy treatment in terms of the defining matrices.

Summary 3.4.12. Consider a rational affine $\mathbb{K}^*$-surface X with an elliptic fixed point $x^\pm \in X$. Then there is a rational function $f \in \mathbb{K}(X)$ with

$$\mathrm{div}(f) \ = \ \iota_X \mathcal{K}_X,$$

where $\iota_X = \iota(x^\pm)$ is the Gorenstein index of X. The *canonical multiplicity* ζ_X *of* X is the weight of f with respect to the $\mathbb{K}^*$-action, hence

$$f(t\cdot x) \ = \ t^\zeta f(x),$$

whenever f is defined at x. If $X = X(A,P)$, using the linear form $u^\pm$ from Proposition 3.4.9, we can write

$$\iota_X \mathcal{K}_X \ = \ \mathrm{div}(\chi^{\iota_X u^\pm}).$$

Keeping in mind that $\mathbb{K}^*$ acts on $X \subseteq Z$ as the subgroup of the acting torus $\mathbb{T}^{r+1}$ given by $t \mapsto (1,\ldots,1,t)$, we see

$$\zeta_X \ = \ \iota_X \langle u^\pm, e_{r+1}\rangle \ = \ \iota_X \frac{\ell^\pm}{m^\pm}.$$

Now we are able to describe the log terminal surface singularities in terms of defining matrices. The case of a toric singularity is settled by Proposition 2.2.8. So we only have to examine the non-toric ones.

Proposition 3.4.13. *Let X be a non-toric log terminal affine $\mathbb{K}^*$-surface with an elliptic fixed point $x \in X$. Then $X \cong X(A,P)$ with P being*

Type $D_n^{\zeta,\iota}$:

$$\begin{bmatrix} -l_0 & 2 & 0 \\ -l_0 & 0 & 2 \\ \frac{\pm\iota-\zeta l_0}{\zeta} & 1 & 1 \end{bmatrix}, \qquad {\scriptstyle \zeta=1,2,} \quad {\scriptstyle \gcd(\iota,\zeta l_0)=\zeta.}$$

Type $E_6^{\zeta,\iota}$:

$$\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ \frac{\pm\iota-5\zeta}{2\zeta} & 1 & 1 \end{bmatrix}, \qquad {\scriptstyle \zeta=1,3,} \quad {\scriptstyle \gcd(\zeta\pm\iota,6\zeta)=2\zeta.}$$

Type E_7^{ι}:

$$\begin{bmatrix} -4 & 3 & 0 \\ -4 & 0 & 2 \\ \frac{\pm\iota-10}{3} & 1 & 1 \end{bmatrix}, \qquad {\scriptstyle \zeta=1,} \quad {\scriptstyle \gcd(2\pm\iota,12)=3.}$$

Type E_8^{ι}:

$$\begin{bmatrix} -5 & 3 & 0 \\ -5 & 0 & 2 \\ \frac{\pm\iota-25}{6} & 1 & 1 \end{bmatrix}, \qquad {\scriptstyle \zeta=1,} \quad {\scriptstyle \gcd(5\pm\iota,30)=6.}$$

In all cases, ι denotes the Gorenstein index and ζ the canonical multiplicity of X. Moreover, a "$+\iota$" in the defining matrix gives $x = x^+$ and a "$-\iota$" gives $x = x^-$.

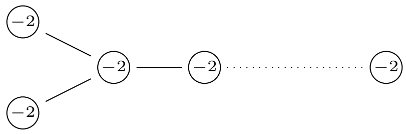
Proof. We may assume $X = X(A, P)$. Then, as X is affine, $n_0 = \cdots = n_r = 1$ holds. Since X is non-toric and log terminal, we have $l_{i1} > 1$ exactly three times. Thus, we arrive at defining 3×3 matrices P of the shape

$$\begin{bmatrix} -l_0 & 2 & 0 \\ -l_0 & 0 & 2 \\ d_0 & 1 & 1 \end{bmatrix}, \quad l_0 \geq 2, \qquad\qquad \begin{bmatrix} -l_0 & 3 & 0 \\ -l_0 & 0 & 2 \\ d_0 & 1 & 1 \end{bmatrix}, \quad l_0 = 3, 4, 5$$

by applying suitable admissible operations and removing erasable columns. Now compute the linear forms $u^{\pm}$ from Proposition 3.4.9 for these matrices. Then $\iota_X u^{\pm}$ being primitive, ι_X dividing $\det(P)$ and $\zeta_X m^{\pm} = \iota_X \ell^{\pm}$ lead to the assertion. $\square$

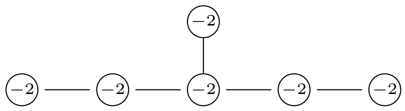
Example 3.4.14. Look again at the log terminal elliptic fixed points of given Gorenstein index ι from Proposition 3.4.13. Resolving the singularity according to Summary 3.4.2 and computing the self intersection numbers with the aid of Summary 3.3.8, we get for $\iota = 1$ the classical resolution graphs of type D_n, E_6, E_7 and E_8:

$D_n^{1,1}$:

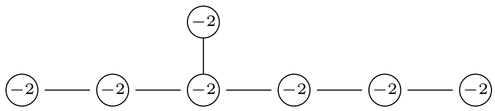


$n \in \mathbb{Z}_{\geq 1}$

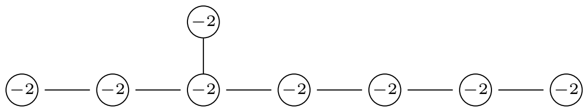
$E_6^{1,1}$:



$E_7^{1,1}$:

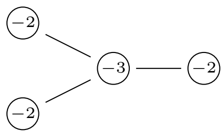


$E_8^{1,1}$:

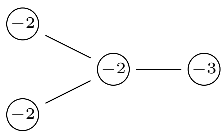


For Gorenstein index $\iota = 2$, none of the types D and E can occur. In Gorenstein indices $\iota = 3, 4$, the simplest examples are:

$D_4^{1,3}$: $\quad P = \begin{bmatrix} -2 & 2 & 0 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$



$D_4^{2,4}$: $\quad P = \begin{bmatrix} -3 & 2 & 0 \\ -3 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$



CHAPTER 4

Log del Pezzo $\mathbb{K}^*$-surfaces

4.1. Log del Pezzo $\mathbb{K}^*$-surfaces

We discuss the impact of the del Pezzo property and log terminality on rational projective $\mathbb{K}^*$-surfaces. Recall that a del Pezzo surface is a two-dimensional Fano variety and a Fano variety is a normal projective variety X admitting an ample anticanonical divisor $-\mathcal{K}_X$. The following observation links del Pezzo $\mathbb{K}^*$-surfaces to toric Fano varieties.

Proposition 4.1.1. *Consider $X = X(A, P)$ and the toric embedding $X \subseteq Z$ from Construction 3.2.2. If X is a del Pezzo surface the following holds.*

(i) *The anticanonical class $w = -w_Z \in K$ of Z yields a Fano toric completion $X \subseteq Z \subseteq Z(w)$.*
(ii) *The defining fan $\Sigma(w)$ of $Z(w)$ from (i) is spanned by the convex hull $\mathcal{A} := \mathrm{conv}(v_1, \dots, v_r)$ over the columns of P.*

Proof. By Summary 3.3.2 the relation degree μ lies in

$$\mathrm{Mov}(Z) = \mathrm{SAmple}(X).$$

The anticanonical divisor classes $-w_X$ of X and $-w_Z$ of Z are related via

$$-w_X \;=\; -w_Z - (r-1)\mu.$$

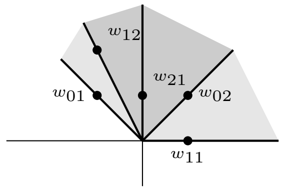
Since $-w_X$ is ample, we conclude that $-w_Z$ lies in the interior of the moving cone $\mathrm{Mov}(Z)$. Thus, Summary 3.3.4 gives the desired completion. $\square$

Given $X = X(A, P)$, the Fano toric completion $Z \subseteq Z(w)$ from Proposition 4.1.1 is uniquely determined by the Cox ring of Z and hence by P. Although there always exist $\mathbb{Q}$-factorial toric completions $Z \subseteq Z'$, the Fano toric completion $Z \subseteq Z(w)$ need not be $\mathbb{Q}$-factorial in general. We present a concrete example.

Example 4.1.2. Consider the projective $\mathbb{K}^*$-surface $X = X(A, P)$ and its toric embedding $X \subseteq Z$ for the defining matrix

$$P \;=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix}.$$

Then X is a Gorenstein del Pezzo surface and in $\mathrm{Cl}(X) = \mathbb{Z}^2 = \mathrm{Cl}(Z)$ the degrees $w_{ij} = \deg(T_{ij})$ are located as follows.



The anticanonical divisor classes of X and Z are $-w_X = (0,3)$ and $-w_Z = (0,5)$. With $w := -w_z$ the maximal cones of the fan $\Sigma(w)$ are given by

$$\begin{array}{ccc} & \mathrm{cone}(v_{01}, v_{02}, v_{21}), & \\ \mathrm{cone}(v_{01}, v_{11}, v_{21}), & \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{12}), & \mathrm{cone}(v_{02}, v_{12}, v_{21}), \\ & \mathrm{cone}(v_{11}, v_{12}, v_{21}). & \end{array}$$

In particular, the Fano toric completion $Z \subseteq Z(w)$ is not $\mathbb{Q}$-factorial. Refining the fan $\Sigma(w)$ by inserting into $\mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{12})$ one of the diagonals

$$\mathrm{cone}(v_{01}, v_{12}), \qquad \mathrm{cone}(v_{11}, v_{02})$$

yields the two $\mathbb{Q}$-factorial toric completions $Z \subseteq Z'$ and $Z \subseteq Z''$ given by the members $\mathrm{cone}(w_{02}, w_{21})$ and $\mathrm{cone}(w_{21}, w_{12})$ of the secondary fan.

We take another look at the del Pezzo property of a given $X = X(A, P)$. So far Summary 3.3.2 tells us how to check explicitly whether or not the class of an anticanonical divisor is ample. The following alternative relies on Kleiman's criterion for ampleness. It states that a divisor is ample if and only if it has positive intersection with any effective curve.

Proposition 4.1.3. *A projective $\mathbb{K}^*$-surface $X = X(A, P)$ is a del Pezzo surface if and only if the following inequalities are satisfied.*

- $\ell^+ > m^+$ *if there is* $D_X^+ \subseteq X$,
- $\ell_{ij-1} m_{ij-1} > \ell_{ij} m_{ij}$, $i = 0, \ldots, r$, $j = 1, \ldots, n_i$,
- $\ell^- > -m^-$ *if there is* $D_X^- \subseteq X$.

Proof. By Kleiman's criterion, X is del Pezzo if and only $-\mathcal{K}_X \cdot D > 0$ for all effective curves D on X. The latter is true if and only if $-\mathcal{K}_X$ has positive intersection with all D_X^{ij} and $D_X^{\pm}$. Due to Proposition 3.3.9, the latter is equivalent to the inequalities of the assertion. $\square$

We will use this characterization to obtain basic geometric properties. The first one is that a rational del Pezzo $\mathbb{K}^*$-surface admits at most one parabolic fixed point curve. As a preparation we need the subsequent series of general estimates.

Proposition 4.1.4. *Let P be a defining matrix as in Construction 3.2.2. Assume that P is slope-ordered, irredundant and $r \geq 2$. Then we have the following estimates.*

(i) $\lceil m_{ij} \rceil - \frac{l_{ij}-1}{l_{ij}} \ \leq \ m_{ij} \ \leq \ \lfloor m_{ij} \rfloor + \frac{l_{ij}-1}{l_{ij}}$.
(ii) $\lceil m_{i1} \rceil - \lfloor m_{in_i} \rfloor \ \geq \ 1$.
(iii) $r + 1 \ \leq \ (m^+ - \ell^+) - (m^- + \ell^-) + 4$.
(iv) $m^+ \geq \ell^+$ *or* $m^- \leq -\ell^-$.

Proof. The first assertion is obvious. For the second one note that $m_{i1} \geq m_{in_i}$ holds due to slope-orderedness. Furthermore

$$\lceil m_{i1} \rceil = \lfloor m_{in_i} \rfloor \implies m_{i1} = m_{in_i} \in \mathbb{Z} \implies n_i = 1, \ l_{i1} = 1.$$

Thus, $\lceil m_{i1} \rceil = \lfloor m_{in_i} \rfloor$ cannot happen since P is irredundant with $r \geq 2$. The third assertion is now deduced from the first two:

$$\begin{aligned} r+1 &\leq \sum_{i=0}^{r} \lceil m_{i1} \rceil - \lfloor m_{in_i} \rfloor \\ &\leq \sum_{i=0}^{r} m_{i1} + \frac{l_{i1}-1}{l_{i1}} - \sum_{i=0}^{r} m_{in_i} - \frac{l_{in_i}-1}{l_{in_i}} \\ &= (m^+ - \ell^+) - (m^- + \ell^-) + 4. \end{aligned}$$

For the fourth assertion, assume $m^+ < \ell^+$ and $m^- > -\ell^-$. Then (iii) gives the estimate

$$r+1 \;\leq\; (m^+ - \ell^+) - (m^- + \ell^-) + 4 \;<\; 4.$$

So $r+1 = 3$. Consequently, using (ii), we obtain

$$3 \;\leq\; \sum_{i=0}^{2} \lceil m_{i1} \rceil - \lfloor m_{in_i} \rfloor \;=\; \sum_{i=0}^{2} \lceil m_{i1} \rceil - \sum_{i=0}^{2} \lfloor m_{in_i} \rfloor.$$

This contradicts the subsequent two estimates, showing that the right hand side equals at most two.

$$\begin{aligned} 2 \;>\; 2 + m^+ - \ell^+ &= \sum_{i=0}^{2} m_{i1} + \frac{l_{i1}-1}{l_{i1}} &&\geq \sum_{i=0}^{2} \lceil m_{i1} \rceil, \\ 2 \;>\; 2 - m^- - \ell^- &= -\sum_{i=0}^{2} m_{in_i} - \frac{l_{in_i}-1}{l_{in_i}} &&\geq -\sum_{i=0}^{2} \lfloor m_{in_i} \rfloor. \end{aligned}$$

□

Proposition 4.1.5. *Let X be a non-toric rational del Pezzo $\mathbb{K}^*$-surface. Then X admits at most one parabolic fixed point curve.*

Proof. We may assume $X = X(A, P)$ with slope-ordered, irredundant P and $r \geq 2$. Suppose that there are D_X^+ and D_X^-. Then Proposition 4.1.3 yields $\ell^+ > m^+$ and $\ell^- > -m^-$. This contradicts Proposition 4.1.4 (iv). □

Let us explore the impact of the del Pezzo property on the possible singularities of a rational projective $\mathbb{K}^*$-surface. Recall that at most elliptic fixed points can be non log terminal and thus the number of non log terminal singularities of any rational $\mathbb{K}^*$-surface is bounded by two.

Proposition 4.1.6. *Let X be a rational del Pezzo $\mathbb{K}^*$-surface. Then X can have at most one non log terminal singularity.*

Proof. It suffices to treat the case that X is non-toric and comes with $x^+ \in X$ and $x^- \in X$. Moreover, we may assume $X = X(A, P)$ with slope-ordered, irredundant P and $r \geq 2$. For any i, Proposition 4.1.3 yields

$$\frac{\ell^+}{m^+} \;=\; \ell_{i0} m_{i_0} \;>\; \ldots \;>\; \ell_{in_i} m_{in_i} \;=\; \frac{\ell^-}{m^-}.$$

Now, if $\ell^+ > 0$ then at most x^- can be non log terminal and we are done. If $\ell^+ \leq 0$, then $m^- < 0$ and the above (strict) inequalities imply $\ell^- > 0$. Thus, at most x^+ can be non log terminal. □

The following example gives an infinite series of rational del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index two, each coming with a non log terminal singularity. It also shows that the class of del Pezzo surfaces of fixed Gorenstein index is not finite.

Example 4.1.7. Consider the projective $\mathbb{K}^*$-surfaces $X_l = X(A, P_l)$ given by the defining data

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad P = \begin{bmatrix} -l & 2 & 0 & 0 \\ -l & 0 & 6 & 1 \\ -\frac{l+1}{2} & 1 & 1 & 0 \end{bmatrix}, \quad 5 \le l \in 2\mathbb{Z}+1.$$

Then each X_l is a del Pezzo surface having x^+ as its only singularity. The corresponding discrepancy is -2, so x^+ is not log terminal. The Gorenstein index of X_l equals 2.

We introduce the main tool of our classification, the anticanonical complex from [**6**, **33**] in the setting of log del Pezzo $\mathbb{K}^*$-surfaces.

Definition 4.1.8. We call the defining matrix P of $X = X(A, P)$ an *LDP-matrix* if X is a del Pezzo surface with at most log terminal singularities.

Construction 4.1.9. Consider an LDP-matrix P. Define vectors $\tilde{v}^+ := \mathscr{d}^+ v^+$ and $\tilde{v}^- := \mathscr{d}^- v^-$ in $\mathbb{R}^{r+1}$ by

$$\mathscr{d}^+ := \frac{\mathscr{m}^+}{\ell^+}, \qquad \mathscr{d}^- := \frac{\mathscr{m}^-}{\ell^-}.$$

We associate with P the two-dimensional simplicial complex $\mathcal{A}_P$ in $\mathbb{R}^{r+1}$ having as its cells

$$\kappa_{ij} \ := \ \mathrm{conv}(0, v_{ij}, v_{ij+1}), \qquad i = 0, \ldots, r, \ j = 1, \ldots, n_i - 1$$

and, according to the possible cases (e-e), (e-p) and (p-e), for $i = 0, \ldots, r$ the simplices

$$\begin{array}{lll} \text{(p-e)} & \kappa_i^+ := \mathrm{conv}(0, v^+, v_{i1}), & \kappa_i^- := \mathrm{conv}(0, \tilde{v}^-, v_{in_i}), \\[1em] \text{(e-p)} & \kappa_i^+ := \mathrm{conv}(0, \tilde{v}^+, v_{i1}), & \kappa_i^- := \mathrm{conv}(0, v^-, v_{in_i}), \\[1em] \text{(e-e)} & \kappa_i^+ := \mathrm{conv}(0, \tilde{v}^+, v_{i1}), & \kappa_i^- := \mathrm{conv}(0, \tilde{v}^-, v_{in_i}). \end{array}$$

Observe that the support of the simplicial complex $\mathcal{A}_P$ is a subset of the tropical variety $\mathrm{trop}(X) \subseteq \mathbb{Q}^{r+1}$.

Remark 4.1.10. Corollary 3.4.7 ensures that for any log del Pezzo $\mathbb{K}^*$-surface $X = X(A, P)$ we can set in accordance with Construction 4.1.9:

$$\mathscr{d}_X^+ \ := \ \frac{\mathscr{m}_X^+}{\ell_X^+}, \qquad \mathscr{d}_X^- \ := \ \frac{\mathscr{m}_X^-}{\ell_X^-}.$$

Moreover, using the properties of the surface X' from Remark 3.4.5, we have the following descriptions in terms of intersection numbers.

$$\mathscr{d}_X^+ \ = \ -\frac{D_{X'}^+ \cdot D_{X'}^+}{D_{X'}^+ \cdot (D_{X'}^+ - \mathcal{K}_{X'}^0)}, \qquad \mathscr{d}_X^- \ = \ -\frac{D_{X'}^- \cdot D_{X'}^-}{D_{X'}^- \cdot (D_{X'}^- + \mathcal{K}_{X'}^0)}.$$

Remark 4.1.11. Consider a del Pezzo surface $X = X(A, P)$ with $r = 1$. Then X is toric and thus log terminal. The support of the complex $\mathcal{A}_P$ equals the LDP-polygon of X.

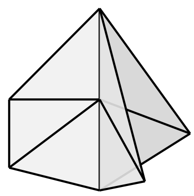
In the following we will see that the complex $\mathcal{A}_P$ of a $\mathbb{K}^*$-surface $X(A, P)$ naturally generalizes the LDP-polygon of a toric del Pezzo surface arising from a fan.

Definition 4.1.12. Let P be an LDP-matrix and $X = X(A, P)$ the corresponding $\mathbb{K}^*$-surface. Consider the simplicial complex $\mathcal{A}_P$ associated with P.

(i) The *interior* of $\mathcal{A}_P$ is the relative interior $\mathcal{A}_P^\circ$ of its support with respect to $\mathrm{trop}(X)$.
(ii) The *outer vertices* of $\mathcal{A}_P$ are the vertices of the complex $\mathcal{A}_P$ apart from the origin.
(iii) Given $k \in \mathbb{Z}_{\geq 1}$, we say that the complex $\mathcal{A}_P$ is *almost k-hollow* if $\mathcal{A}_P^\circ \cap k\mathbb{Z}^{r+1} = \{0\}$.

Example 4.1.13. Consider again the $\mathbb{K}^*$-surface $X = X(A, P)$ from Example 3.4.3. The defining matrix P and the complex $\mathcal{A}_P$ are given as

$$P = \begin{bmatrix} -3 & -5 & 2 & 0 & 0 \\ -3 & -5 & 0 & 2 & 0 \\ -4 & -8 & 1 & 1 & 1 \end{bmatrix}$$



$\mathcal{A}_P$

The outer vertices of $\mathcal{A}_P$ are the columns $v_{01}, v_{02}, v_{11}, v_{21}, v^+$ of P and $\tilde{v}^- = (0, 0, 3)$.

In [**6**] the *anticanonical complex* was introduced. For the case of rational Fano varieties with a torus action of complexity one, it was established as a tool for the treatment of log terminal singularities; see also [**31, 33**] for further results.

Theorem 4.1.14. *Consider an LDP-matrix P, its associated simplicial complex $\mathcal{A}_P$ and the projective $\mathbb{K}^*$-surface $X = X(A, P)$.*

(i) *According to the possible constellations of source and sink, the outer vertices of the complex $\mathcal{A}_P$ are given by*

(e-e) v_{ij} *for* $i = 0, \dots, r, j = 1, \dots, n_i$ *and* $\tilde{v}^+$, $\tilde{v}^-$,
(e-p) v_{ij} *for* $i = 0, \dots, r, j = 1, \dots, n_i$ *and* $\tilde{v}^+$, v^-,
(p-e) v_{ij} *for* $i = 0, \dots r$, $j = 1, \dots, n_i$ *and* v^+, $\tilde{v}^-$.

(ii) *The simplicial complex $\mathcal{A}_P$ equals the anticanonical complex of the log del Pezzo $\mathbb{K}^*$-surface $X = X(A, P)$.*
(iii) *For all $i = 0, \dots, r$, the intersection $\mathcal{A}_P \cap \lambda_i$ with the i-th arm $\lambda_i \subseteq \mathrm{trop}(X)$ is a convex polygon. Regarding possible outer vertices $\tilde{v}^\pm$ we have*

$$\tilde{v}^+ \notin \mathrm{conv}(0, v_{01}, \dots, v_{r1}), \qquad \tilde{v}^- \notin \mathrm{conv}(0, v_{0n_0}, \dots, v_{rn_r}).$$

(iv) *The discrepancy along an exceptional divisor $E_\varrho \subseteq X''$ of the canonical resolution $X'' \to X$ is given by*

$$a(E_\varrho) \;=\; \frac{\|v_\varrho\|}{\|\tilde{v}_\varrho\|} - 1,$$

where $v_\varrho \in \varrho$ is the primitive vector and $\tilde{v}_\varrho \in \varrho$ is the intersection point of ϱ and the boundary $\partial\mathcal{A}_P$.

(v) *The surface X has at most $1/k$-log canonical singularities if and only if the anticanonical complex $\mathcal{A}_X$ is almost k-hollow.*

Proof. Assertions (i), (ii) and (iv), (v) are covered by [**6**, Thm. 1.4, Prop. 2.3, Prop. 3.7 and Cor. 4.10]. We show (iii). The intersection points of the boundary of $\mathcal{A}_P$ and the facet $\mathrm{conv}(v_{01}, \ldots, v_{r1})$ with the ray through v^+ are given by

$$\tilde{v}^+ \;=\; \frac{m^+}{\ell^+} v^+, \qquad \frac{m^+}{\ell^+ + r - 1} v^+.$$

Since X is log terminal we have $\ell^+ > 0$. Together with $m^+ > 0$, this gives the assertion in case of the existence of an elliptic fixed point x^+. The case of an elliptic fixed point x^- is analogous. □

Proposition 4.1.15. *For any non-toric rational $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surface X, the number $r+1$ of critical values of $\pi\colon X \dashrightarrow \mathbb{P}_1$ is bounded as follows.*

(i) *If X has two elliptic fixed points, then $r + 1 \le 4k$ holds.*
(ii) *If X has a parabolic fixed point curve, then $r + 1 \le 2k + 1$ holds.*

Proof. We may assume $X = X(A, P)$ with slope-ordered and irredundant P and $r \ge 2$. According to Theorem 4.1.14, we have $d^+ \le k$ and $d^- \ge -k$. This gives

$$m^+ \;\le\; k\ell^+, \qquad m^- \;\ge\; -k\ell^-.$$

Assume that X has two elliptic fixed points. Combining the above estimates with Lemma 4.1.4 (iii) and using $\ell^+ + \ell^- \le 4$ as well as $k \ge 1$ leads to

$$r + 1 \;\le\; (m^+ - \ell^+) - (m^- + \ell^-) + 4 \;\le\; (k-1)(\ell^+ + \ell^-) + 4 \;\le\; 4k.$$

Assume that X has a parabolic fixed point curve, say D_X^+. Then $-\mathcal{K}_X \cdot D_X^+ > 0$ implies $m^+ < \ell^+$. Similarly as before we conclude

$$r + 1 \;\le\; (m^+ - \ell^+) - (m^- + \ell^-) + 4 \;<\; (k-1)\ell^- + 4 \;\le\; 2k + 2.$$

□

We discuss bounds for the number of singularities of X in terms of the Picard number $\rho(X)$. First note that for a projective toric surface X the number of singularities is at most the number of its fixed points and is therefore bounded by $\rho(X) + 2$. For non-toric rational projective $\mathbb{K}^*$-surfaces, we obtain bounds in the case that there is a log terminal elliptic fixed point.

Proposition 4.1.16. *Let $X = X(A, P)$ be a non-toric $\mathbb{K}^*$-surface with irredundant P. If X has a log terminal elliptic fixed point, then the number $r+1$ of arms of X is bounded by*

$$r + 1 \;\le\; \rho(X) + 3 - m.$$

Moreover, according to the possible constellations of source and sink in X, the number $s(X)$ of singularities of X is bounded as follows:

(e-e): $s(X) \le \rho(X) + 2$, (e-p), (p-e): $s(X) \le r + 1 + \rho(X) \le 2\rho(X) + 2$.

Proof. Since X is $\mathbb{Q}$-factorial, $\rho(X)$ coincides with the rank of the divisor class group $\mathrm{Cl}(X)$ and hence equals $n + m - r - 1$; see Summary 3.3.1. By assumption, we have a log terminal elliptic fixed point. Using Remark 3.4.11 and the irredundancy of P we see that there are at least $r - 2$ arms having length greater or equal than 2. Therefore

$$\rho(X) \;=\; m + \sum_{i=0}^{r}(n_i - 1) \;\ge\; m + r + 1 - 3.$$

This proves the first statement. To estimate the number of singularities we use that each one is a fixed point. Note that in any case the number of hyperbolic fixed points of $X = X(A,P)$ is given by

$$(n_0 - 1) + \cdots + (n_r - 1).$$

Thus, in the case (e-e), we have $\rho(X) + 2$ fixed points in total. For the cases (e-p) and (p-e) recall from Summary 3.4.1 that there are at most $r+1$ singular parabolic fixed points. Moreover, we have $\rho(X) - 1$ hyperbolic fixed points. □

Remark 4.1.17. There are no bounds on the number of singularities in terms of the Picard number if there are two parabolix fixed point curves. Note that this case leaves the class of rational del Pezzo $\mathbb{K}^*$-surfaces. For an example, let $m = 2$ and $n_0 = \cdots = n_r = 1$ with

$$l_{01} = \cdots = l_{r1} = 2, \qquad d_{01} = -1, \qquad d_{11} = \cdots = d_{r1} = 1.$$

This gives a defining matrix P and thus an $X = X(A,P)$. We have $\rho(X) = 2$ and in each arm of X there are two singular parabolic fixed points. Thus X has $2r + 2$ singularities in total.

For a log del Pezzo surface of Picard number one, results of Keel/McKernan [**36**] and Belousov [**7**] tell us that the number of singularities is sharply bounded by four. In presence of a $\mathbb{K}^*$-action, we can extend this statement to higher Picard numbers.

Corollary 4.1.18. *Let X be a log del Pezzo $\mathbb{K}^*$-surface. Then X has at most $2\rho(X) + 2$ singularities.*

Proof. If X is a toric surface the assertion is clear as mentioned before. For a non-toric X Corollary 4.1.5 ensures the existence of an elliptic fixed point and thus Proposition 4.1.16 applies. □

4.2. Log del Pezzo $\mathbb{K}^*$-surfaces without quasismooth elliptic fixed points

In the following we present an algorithm to effectively classify non-toric log del Pezzo $\mathbb{K}^*$-surfaces $X = X(A,P)$ without quasismooth elliptic fixed points of specified Gorenstein index. This is done by exclusively working with defining P-matrices. First, we look at the case of X having two elliptic fixed points.

Definition 4.2.1. Let $X = X(A, P)$ be a projective $\mathbb{K}^*$-surface. We use the following terminology.

(i) P is *irredundant* if $n_i l_{i1} > 1$ for $i = 0, \ldots, r$.
(ii) P is *slope-ordered* if $m_{i1} > \cdots > m_{in_i}$ for $i = 0, \ldots, r$.
(iii) P is *adapted to the source* if $0 \le d_{i1} < l_{i1}$ for $i = 1, \ldots, r$ and $d_1 = 1$ if $m \ge 1$.

If additionally $n_0 \ge \cdots \ge n_r$ we call the defining matrix P *adjusted.*

Proposition 4.2.2. *Consider a non-toric rational log terminal projective $\mathbb{K}^*$-surface X with two elliptic fixed points. Then $X \cong X(A, P)$ where P is irredundant, slope-ordered, adapted to the source and of type (e-e). Moreover, P satisifes the following: Removing all columns v_{ij} with $1 < j < n_i$ and collapsing all arms with $l_{i1} = l_{in_i} = 1$ we arrive at one of the following matrices P'.*

(i) $r' - 1 = 1$.
 (a) $\bar{n} = (2, 1, 1)$,
$$P' = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{21} \end{bmatrix}.$$
 (b) $\bar{n} = (2, 2, 1)$,
$$P' = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} \end{bmatrix}.$$
 (c) $\bar{n} = (2, 2, 2)$,
$$P' = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} & l_{22} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{22} \end{bmatrix}.$$
(ii) $r' - 1 = 2$.
 (a) $\bar{n} = (2, 1, 1, 1)$,
$$P' = \begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & 0 & l_{21} & 0 \\ -1 & -1 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & d_{11} & d_{21} & d_{31} \end{bmatrix}.$$
 (b) $\bar{n} = (2, 2, 1, 1)$,
$$P' = \begin{bmatrix} -1 & -l_{02} & l_{11} & 1 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{31} \end{bmatrix}.$$
 (c) $\bar{n} = (2, 2, 2, 1)$,
$$P' = \begin{bmatrix} -1 & -l_{02} & l_{11} & 1 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & l_{22} & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{22} & d_{31} \end{bmatrix}.$$
 (d) $\bar{n} = (2, 2, 2, 2)$,
$$P' = \begin{bmatrix} -1 & -l_{02} & l_{11} & 1 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & l_{22} & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & l_{32} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} \end{bmatrix}.$$
(iii) $r' - 1 = 3$.

(a) $\bar{n} = (2,2,2,2,1)$,

$$P' = \begin{bmatrix} -1 & -l_{02} & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & 1 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} \\ d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} & d_{41} \end{bmatrix}.$$

(b) $\bar{n} = (2,2,2,2,2)$,

$$P' = \begin{bmatrix} -1 & -l_{02} & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & 1 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & l_{42} \\ d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} & d_{41} & d_{42} \end{bmatrix}.$$

(iv) $r' - 1 = 4$.

(a) $\bar{n} = (2,2,2,2,2,2)$,

$$P' = \begin{bmatrix} -1 & -l_{02} & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 1 & l_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & 1 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{51} & 1 \\ d_{01} & d_{02} & 0 & d_{12} & 0 & d_{22} & d_{31} & d_{32} & d_{41} & d_{42} & d_{51} & d_{52} \end{bmatrix}.$$

PROOF. By log terminality, at most 3 entries of the tuples $(l_{01}, \dots, l_{r1})$, $(l_{0n_0}, \dots, l_{rn_r})$ are different from 1. Hence, there are at most 6 arms that satisfy $l_{i1} \neq 1$ or $l_{in_i} \neq 1$. Going through the different possible constellations and arm lengths (and swapping arms if necessary) we arrive at the shapes from above. $\square$

Remark 4.2.3. Note that the columns of the matrices P' from Proposition 4.2.2 do not necessarily generate $\mathbb{Q}^{r'+1}$ as a cone. Therefore, they are not necessarily P-matrices in the sense of Construction 3.2.2. Nonetheless, we can look at the numbers $m^+_{P'}$, $m^-_{P'}$, $\ell^+_{P'}$, $\ell^-_{P'}$, defined in the same way as before. We have

$$m^+_{P'} = m^+_P > 0, \qquad \ell^+_{P'} = \ell^+_P > 0, \qquad \ell^-_{P'} = \ell^-_P > 0,$$

but not necessarily $m^-_{P'} < 0$.

Lemma 4.2.4. *Consider a log terminal projective $\mathbb{K}^*$-surface $X = X(A, P)$ of Gorenstein index ι with adjusted P of type (e-e) and the corresponding matrix P' from Proposition 4.2.2. The following assertions hold.*

(i) *Since P is slope-ordered and adapted to the source, its entries satisfy $d_{r'+1,n_{r'+1}}, \dots, d_{rn_r} < 0$. The condition $d^-_P \geq -\iota$ therefore gives*

$$m^-_{P'} \geq m^-_P \geq -\iota \cdot \ell^-_P = -\iota \cdot \ell^-_{P'}.$$

This means we also have $d^-_{P'} \geq -\iota$. Furthermore, the above inequality yields the following estimate for the d_{in_i}, where $0 \leq i \leq r'$.

$$\begin{aligned} d_{in_i} &\geq l_{in_i} \cdot \left(-\iota \cdot \ell^- - \sum_{j \neq i} \frac{d_{jn_j}}{l_{jn_j}} \right), \\ &\geq l_{in_i} \cdot \left(-\iota \cdot \ell^- - \sum_{j \neq i} \frac{d_{j1}}{l_{j1}} \right). \end{aligned}$$

(ii) *Since P is slope-ordered and $\cancel{d}_P^- \geq -\iota$, for each $i = r'+1, \ldots, r$, we have*

$$\begin{aligned} 0 \ > \ d_{in_i} \ &\geq \ d_{r'+1,n_{r'+1}} + \cdots + d_{rn_r} \\ &\geq \ -\iota \cdot \ell^- - \frac{d_{0n_0}}{l_{0n_0}} - \cdots - \frac{d_{r'n_{r'}}}{l_{r'n_{r'}}} \\ &> \ -\iota \cdot \ell^- - \frac{d_{01}}{l_{01}} - \cdots - \frac{d_{r'1}}{l_{r'1}}. \end{aligned}$$

Hence, each entry of P is bounded as long as the entries of P' are.

Proposition 4.2.5. *Consider case (i)(a) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type E_6 and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $\begin{bmatrix} -3 & -l_{02} & 3 & 0 \\ -3 & -l_{02} & 0 & 2 \\ \frac{\iota^+-5}{2} & d_{02} & 1 & 1 \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-1, \quad 2 \leq l_{02} \leq 5, \quad l_{02} \cdot \frac{\iota-5}{6} - \iota \leq d_{02} < l_{02} \cdot \frac{\iota^+-5}{6}.$

(ii) $\begin{bmatrix} -3 & -l_{02} & 3 & 0 \\ -3 & -l_{02} & 0 & 2 \\ \frac{\iota^+-7}{2} & d_{02} & 2 & 1 \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-5, \quad 2 \leq l_{02} \leq 5, \quad l_{02} \cdot \frac{\iota-7}{6} - \iota \leq d_{02} < l_{02} \cdot \frac{\iota^+-7}{6}.$

(iii) $\begin{bmatrix} -3 & 3 & l_{12} & 0 \\ -3 & 0 & 0 & 2 \\ \frac{\iota^+-5}{2} & 1 & d_{12} & 1 \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-1, \quad 2 \leq l_{12} \leq 5, \quad l_{12} \cdot \frac{\iota+2-\iota^+}{6} - \iota \leq d_{12} < \frac{l_{12}}{3}.$

(iv) $\begin{bmatrix} -3 & 3 & l_{12} & 0 \\ -3 & 0 & 0 & 2 \\ \frac{\iota^+-7}{2} & 2 & d_{12} & 1 \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-5, \quad 2 \leq l_{12} \leq 5, \quad l_{12} \cdot \frac{\iota+4-\iota^+}{6} - \iota \leq d_{12} < \frac{2 \cdot l_{12}}{3}.$

(v) $\begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 2 \\ \frac{\iota^+-5}{2} & 1 & 1 & d_{22} \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-1, \quad \frac{2\iota+3-\iota^+}{3} - \iota \leq d_{22} \leq 0.$

(vi) $\begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 2 \\ \frac{\iota^+-7}{2} & 2 & 1 & d_{22} \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-5, \quad \frac{2\iota+3-\iota^+}{3} - \iota \leq d_{22} \leq 0.$

(vii) $\begin{bmatrix} -3 & -l_{02} & 3 & 0 \\ -3 & -l_{02} & 0 & 2 \\ \frac{\iota^+-15}{6} & d_{02} & 1 & 1 \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-3, \quad 2 \leq l_{02} \leq 5, \quad l_{02} \cdot \frac{\iota-5}{6} - \iota \leq d_{02} < l_{02} \cdot \frac{\iota^+-15}{18}.$

(viii) $\begin{bmatrix} -3 & -l_{02} & 3 & 0 \\ -3 & -l_{02} & 0 & 2 \\ \frac{\iota^+-21}{6} & d_{02} & 2 & 1 \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-3, \quad 2 \leq l_{02} \leq 5, \quad l_{02} \cdot \frac{\iota-7}{6} - \iota \leq d_{02} < l_{02} \cdot \frac{\iota^+-21}{18}.$

(ix) $\begin{bmatrix} -3 & 3 & l_{12} & 0 \\ -3 & 0 & 0 & 2 \\ \frac{\iota^+-15}{6} & 1 & d_{12} & 1 \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-3, \quad 2 \leq l_{12} \leq 5, \quad l_{12} \cdot \frac{3\iota+6-\iota^+}{18} - \iota \leq d_{12} < \frac{l_{12}}{3}.$

(x) $\begin{bmatrix} -3 & 3 & l_{12} & 0 \\ -3 & 0 & 0 & 2 \\ \frac{\iota^+-21}{6} & 2 & d_{12} & 1 \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-3, \quad 2 \leq l_{12} \leq 5, \quad l_{12} \cdot \frac{3\iota+12-\iota^+}{18} - \iota \leq d_{12} < \frac{2 \cdot l_{12}}{3}.$

(xi) $\begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 2 \\ \frac{\iota^+-15}{6} & 1 & 1 & d_{22} \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-3, \quad \frac{6\iota+9-\iota^+}{9} - \iota \leq d_{22} \leq 0.$

(xii) $\begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 2 \\ \frac{\iota^+-21}{6} & 2 & 1 & d_{22} \end{bmatrix}$, $\quad \iota^+|\iota, \quad 6|\iota^+-3, \quad \frac{6\iota+9-\iota^+}{9} - \iota \leq d_{22} \leq 0.$

Proposition 4.2.6. *Consider case (i)(a) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type E_7 and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $$\begin{bmatrix} -4 & -l_{02} & 3 & 0 \\ -4 & -l_{02} & 0 & 2 \\ \frac{\iota^+-10}{3} & d_{02} & 1 & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 1, \\ 2 \le l_{02} \le 5, \\ l_{02} \cdot \frac{\iota-5}{6} - \iota \le d_{02} < l_{02} \cdot \frac{\iota^+-10}{12}. \end{gathered}$$

(ii) $$\begin{bmatrix} -4 & -l_{02} & 3 & 0 \\ -4 & -l_{02} & 0 & 2 \\ \frac{\iota^+-14}{3} & d_{02} & 2 & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 5, \\ 2 \le l_{02} \le 5, \\ l_{02} \cdot \frac{\iota-7}{6} - \iota \le d_{02} < l_{02} \cdot \frac{\iota^+-14}{12}. \end{gathered}$$

(iii) $$\begin{bmatrix} -4 & 3 & l_{12} & 0 \\ -4 & 0 & 0 & 2 \\ \frac{\iota^+-10}{3} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 1, \\ 2 \le l_{12} \le 3, \\ l_{12} \cdot \frac{3\iota+4-\iota^+}{12} - \iota \le d_{12} \le 0. \end{gathered}$$

(iv) $$\begin{bmatrix} -4 & 3 & l_{12} & 0 \\ -4 & 0 & 0 & 2 \\ \frac{\iota^+-14}{3} & 2 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 5, \\ 2 \le l_{12} \le 3, \\ l_{12} \cdot \frac{3\iota+8-\iota^+}{12} - \iota \le d_{12} < \frac{2 \cdot l_{12}}{3}. \end{gathered}$$

(v) $$\begin{bmatrix} -4 & 3 & 0 & 0 \\ -4 & 0 & 2 & 2 \\ \frac{\iota^+-10}{3} & 1 & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 1, \\ \frac{5\iota+6-\iota^+}{6} - \iota \le d_{22} \le 0. \end{gathered}$$

(vi) $$\begin{bmatrix} -4 & 3 & 0 & 0 \\ -4 & 0 & 2 & 2 \\ \frac{\iota^+-14}{3} & 2 & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 5, \\ \frac{5\iota+6-\iota^+}{6} - \iota \le d_{22} \le 0. \end{gathered}$$

Proposition 4.2.7. *Consider case (i)(a) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type E_8 and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $$\begin{bmatrix} -5 & -l_{02} & 3 & 0 \\ -5 & -l_{02} & 0 & 2 \\ \frac{\iota^+-25}{6} & d_{02} & 1 & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 1, \\ 2 \le l_{02} \le 5, \\ l_{02} \cdot \frac{\iota-5}{6} - \iota \le d_{02} < l_{02} \cdot \frac{\iota^+-25}{30}. \end{gathered}$$

(ii) $$\begin{bmatrix} -5 & -l_{02} & 3 & 0 \\ -5 & -l_{02} & 0 & 2 \\ \frac{\iota^+-35}{6} & d_{02} & 2 & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 5, \\ 2 \le l_{02} \le 5, \\ l_{02} \cdot \frac{\iota-7}{6} - \iota \le d_{02} < l_{02} \cdot \frac{\iota^+-35}{30}. \end{gathered}$$

(iii) $$\begin{bmatrix} -5 & 3 & l_{12} & 0 \\ -5 & 0 & 0 & 2 \\ \frac{\iota^+-25}{6} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 1, \\ 2 \le l_{12} \le 3, \\ l_{12} \cdot \frac{9\iota+10-\iota^+}{30} - \iota \le d_{12} \le 0. \end{gathered}$$

(iv) $$\begin{bmatrix} -5 & 3 & l_{12} & 0 \\ -5 & 0 & 0 & 2 \\ \frac{\iota^+-35}{6} & 2 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 5, \\ 2 \le l_{12} \le 3, \\ l_{12} \cdot \frac{9\iota+20-\iota^+}{30} - \iota \le d_{12} < \frac{2 \cdot l_{12}}{3}. \end{gathered}$$

(v) $$\begin{bmatrix} -5 & 3 & 0 & 0 \\ -5 & 0 & 2 & 2 \\ \frac{\iota^+-25}{6} & 1 & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 1, \\ \frac{14\iota+15-\iota^+}{15} - \iota \le d_{22} \le 0. \end{gathered}$$

(vi) $$\begin{bmatrix} -5 & 3 & 0 & 0 \\ -5 & 0 & 2 & 2 \\ \frac{\iota^+-35}{6} & 2 & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+ \mid \iota, \quad 6 \mid \iota^+ - 5, \\ \frac{14\iota+15-\iota^+}{15} - \iota \le d_{22} \le 0. \end{gathered}$$

Proof of Propositions 4.2.7, 4.2.6, 4.2.5. The shape of the columns v_{01}, v_{11}, v_{21}, is given in Proposition 3.4.13. The divisibility claims come from the Gorenstein conditions in 3.4.9. Log terminality gives the bounds on l_{i2}. Slope-orderedness and $\mathscr{d}^-_{P'} \ge -\iota$ lead to the bounds for d_{i2}. $\square$

Proposition 4.2.8. *Consider case (i)(a) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type D_n and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & 0 \\ -l_{01} & -l_{02} & 0 & 2 \\ \iota^+ - l_{01} & d_{02} & 1 & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+ - 1, \\ 2 \le l_{01}, l_{02} \le 2\iota^2, \\ -l_{02} - \iota \le d_{02} < l_{02} \cdot \tfrac{\iota^+ - l_{01}}{l_{01}}. \end{gathered}$$

(ii) $$\begin{bmatrix} -l_{01} & 2 & 1 & 0 \\ -l_{01} & 0 & 0 & 2 \\ \iota^+ - l_{01} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+ - 1, \\ 2 \le l_{01} \le \tfrac{2\iota + 2\iota^+}{1 - \iota - 2d_{12}}, \\ \tfrac{1-\iota}{2} - \tfrac{\iota + \iota^+}{l_{01}} \le d_{12} < \tfrac{1-\iota}{2}. \end{gathered}$$

(iii) $$\begin{bmatrix} -2 & 2 & l_{12} & 0 \\ -2 & 0 & 0 & 2 \\ \iota^+ - 2 & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+ - 1, \\ 2 \le l_{12} \le 2\iota^2, \\ l_{12} \cdot \tfrac{1 - \iota^+}{2} - \iota \le d_{12} < \tfrac{l_{12}}{2}. \end{gathered}$$

(iv) $$\begin{bmatrix} -l_{01} & 2 & 2 & 0 \\ -l_{01} & 0 & 0 & 2 \\ \iota^+ - l_{01} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+ - 1, \\ 2 \le l_{01} \le 4\iota^2, \\ \tfrac{l_{01} - 2\iota - 2\iota^+}{l_{01}} \le d_{12} \le 0. \end{gathered}$$

(v) $$\begin{bmatrix} -l_{01} & 2 & l_{12} & 0 \\ -l_{01} & 0 & 0 & 2 \\ \iota^+ - l_{01} & 1 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+ - 1, \\ (l_{01}, l_{12}) \in \{(z,3), (3,z) \mid z = 3,4,5\}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{01}} + \tfrac{1}{l_{12}} - \tfrac{1}{2} \right) - \tfrac{\iota^+ - l_{01}}{l_{01}} - \tfrac{1}{2} \right) \le d_{12} < \tfrac{l_{12}}{2}. \end{gathered}$$

(vi) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & 0 \\ -l_{01} & -l_{02} & 0 & 2 \\ \tfrac{\iota^+ - 2l_{01}}{2} & d_{02} & 1 & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+, \\ 2 \le l_{01}, l_{02} \le 2\iota^2, \\ -l_{02} - \iota \le d_{02} < l_{02} \cdot \tfrac{\iota^+ - 2l_{01}}{2l_{01}}. \end{gathered}$$

(vii) $$\begin{bmatrix} -l_{01} & 2 & 1 & 0 \\ -l_{01} & 0 & 0 & 2 \\ \tfrac{\iota^+ - 2l_{01}}{2} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+, \\ 2 \le l_{01} \le \tfrac{2\iota + \iota^+}{1 - \iota - 2d_{12}}, \\ \tfrac{1-\iota}{2} - \tfrac{2\iota + \iota^+}{2l_{01}} \le d_{12} < \tfrac{1-\iota}{2}. \end{gathered}$$

(viii) $$\begin{bmatrix} -2 & 2 & l_{12} & 0 \\ -2 & 0 & 0 & 2 \\ \tfrac{\iota^+ - 4}{2} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+, \\ 2 \le l_{12} \le 6\iota^2, \\ l_{12} \cdot \tfrac{2 - \iota^+}{4} - \iota \le d_{12} < \tfrac{l_{12}}{2}. \end{gathered}$$

(ix) $$\begin{bmatrix} -l_{01} & 2 & 2 & 0 \\ -l_{01} & 0 & 0 & 2 \\ \tfrac{\iota^+ - 2l_{01}}{2} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+ \\ 2 \le l_{01} \le 4\iota^2, \\ \tfrac{l_{01} - 2\iota - \iota^+}{l_{01}} \le d_{12} \le 0. \end{gathered}$$

(x) $$\begin{bmatrix} -l_{01} & 2 & l_{12} & 0 \\ -l_{01} & 0 & 0 & 2 \\ \tfrac{\iota^+ - 2l_{01}}{2} & 1 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered} \iota^+ | \iota, \quad 2 | \iota^+, \\ (l_{01}, l_{12}) \in \{(z,3), (3,z) \mid z = 3,4,5\}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{01}} + \tfrac{1}{l_{12}} - \tfrac{1}{2} \right) - \tfrac{\iota^+ - 2l_{01}}{2l_{01}} - \tfrac{1}{2} \right) \le d_{12} < \tfrac{l_{12}}{2}. \end{gathered}$$

PROOF. As before, the shape of the columns v_{01}, v_{11}, v_{21} is given in Proposition 3.4.13. The divisibility claims come from the Gorenstein conditions, see Proposition 3.4.9. Now note that $\not{d}^+ = \iota^+$ in each case. Furthermore, in cases (i) and (vi) we have $\not{d}^-_{P'} = l_{02} + d_{02}$. Since $\not{d}^-_P - \not{d}^-_{P'} \in \mathbb{Z}$, we also have $\not{d}^-_P \in \mathbb{Z}$. Therefore, by Minkowski's Theorem, we get the bounds for l_{01} and l_{02}. The bounds for d_{02} in these cases are due to slope-orderedness and the condition $\not{d}^-_{P'} \ge -\iota$.

Cases (ii) to (v) go through the possible platonic tuples corresponding to the exponents associated with the elliptic fixed point $x^- \in X$. The respective bounds for l_{01} and l_{12} either come from considering $\not{d}_P^-$ and using Minkowki's Theorem or using the condition $\not{d}_{P'}^- \geq -\iota$. The cases (vii) to (x) are treated analogously. $\square$

We will only list the results for the rest of the formats from case (i) in Proposition 4.2.2. The proofs follow the same structure as before.

Proposition 4.2.9. *Consider case (i)(b) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type E_6 and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i)
$$\begin{bmatrix} -3 & -l_{02} & 3 & l_{12} & 0 \\ -3 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-5}{2} & d_{02} & 1 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{02}, l_{12}) \in \{(z,3),(3,z) \mid z = 3,4,5\}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{5}{6}\right) < d_{02} < l_{02} \cdot \tfrac{\iota^+-5}{6}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{2}\right) \leq d_{12} < \tfrac{l_{12}}{3}. \end{gathered}$$

(ii)
$$\begin{bmatrix} -3 & -2 & 3 & l_{12} & 0 \\ -3 & -2 & 0 & 0 & 2 \\ \frac{\iota^+-5}{2} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \leq l_{12} \leq 4\iota^2, \\ -\tfrac{2\iota}{l_{12}} - 1 \leq d_{02} < \tfrac{\iota^+-5}{3}, \\ -\iota - \tfrac{l_{12}(d_{02}+1)}{2} \leq d_{12} < \tfrac{l_{12}}{3}. \end{gathered}$$

(iii)
$$\begin{bmatrix} -3 & -l_{02} & 3 & 2 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-5}{2} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \leq l_{02} \leq 4\iota^2, \\ -\iota - \tfrac{l_{02}}{2} \leq d_{02} < l_{02} \cdot \tfrac{\iota^+-5}{6}, \\ -\iota - d_{02} - \tfrac{l_{02}}{2} \leq d_{12} \leq 0. \end{gathered}$$

(iv)
$$\begin{bmatrix} -3 & -l_{02} & 3 & l_{12} & 0 \\ -3 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-7}{2} & d_{02} & 2 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{02}, l_{12}) \in \{(z,3),(3,z) \mid z = 3,4,5\}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{7}{6}\right) < d_{02} < l_{02} \cdot \tfrac{\iota^+-7}{6} \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{2}\right) \leq d_{12} < \tfrac{2l_{12}}{3}. \end{gathered}$$

(v)
$$\begin{bmatrix} -3 & -2 & 3 & l_{12} & 0 \\ -3 & -2 & 0 & 0 & 2 \\ \frac{\iota^+-7}{2} & d_{02} & 2 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \leq l_{12} \leq 4\iota^2, \\ -\tfrac{2\iota}{l_{12}} - 1 \leq d_{02} < \tfrac{\iota^+-7}{3}, \\ -\iota - \tfrac{l_{12}(d_{02}+1)}{2} \leq d_{12} < \tfrac{2l_{12}}{3}. \end{gathered}$$

(vi)
$$\begin{bmatrix} -3 & -l_{02} & 3 & 2 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-7}{2} & d_{02} & 2 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \leq l_{02} \leq 4\iota^2, \\ -\iota - \tfrac{l_{02}}{2} \leq d_{02} < l_{02} \cdot \tfrac{\iota^+-7}{6}, \\ -\iota - d_{02} - \tfrac{l_{02}}{2} \leq d_{12} \leq 1. \end{gathered}$$

(vii)
$$\begin{bmatrix} -3 & -l_{02} & 3 & 0 & 0 \\ -3 & -l_{02} & 0 & 2 & l_{22} \\ \frac{\iota^+-5}{2} & d_{02} & 1 & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{02}, l_{22}) \in \{(z,2),(2,z) \mid z = 2,3,4,5\}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{5}{6}\right) < d_{02} < l_{02} \cdot \tfrac{\iota^+-5}{6}, \\ l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{3}\right) \leq d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(viii) $$\begin{bmatrix} -3 & -l_{02} & 3 & 0 & 0 \\ -3 & -l_{02} & 0 & 2 & l_{22} \\ \frac{\iota^+-7}{2} & d_{02} & 2 & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{02}, l_{22}) \in \{(z,2),(2,z) \mid z = 2,3,4,5\}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{7}{6}\right) < d_{02} < l_{02} \cdot \tfrac{\iota^+-7}{6}, \\ l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{2}{3}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(ix) $$\begin{bmatrix} -3 & 3 & l_{12} & 0 & 0 \\ -3 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-5}{2} & 1 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{12}, l_{22}) \in \{(z,2),(2,z) \mid z = 2,3,4,5\}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{\iota^+-5}{6} - \tfrac{l_{22}}{2}\right) \le d_{12} < \tfrac{l_{12}}{3}, \\ l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{\iota^+-5}{6} - \tfrac{d_{12}}{l_{12}}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(x) $$\begin{bmatrix} -3 & 3 & l_{12} & 0 & 0 \\ -3 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-7}{2} & 2 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{12}, l_{22}) \in \{(z,2),(2,z) \mid z = 2,3,4,5\}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{\iota^+-7}{6} - \tfrac{l_{22}}{2}\right) \le d_{12} < \tfrac{2l_{12}}{3}, \\ l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{\iota^+-7}{6} - \tfrac{d_{12}}{l_{12}}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(xi) $$\begin{bmatrix} -3 & -l_{02} & 3 & l_{12} & 0 \\ -3 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-15}{6} & d_{02} & 1 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ (l_{02}, l_{12}) \in \{(z,3),(3,z) \mid z = 3,4,5\}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{5}{6}\right) < d_{02} < l_{02} \cdot \tfrac{\iota^+-15}{18}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{2}\right) \le d_{12} < \tfrac{l_{12}}{3}. \end{gathered}$$

(xii) $$\begin{bmatrix} -3 & -2 & 3 & l_{12} & 0 \\ -3 & -2 & 0 & 0 & 2 \\ \frac{\iota^+-15}{6} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \le l_{12} \le 6\iota^2, \\ -\tfrac{2\iota}{l_{12}} - 1 \le d_{02} < \tfrac{\iota^+-15}{9}, \\ -\iota - \tfrac{l_{12}(d_{02}+1)}{2} \le d_{12} < \tfrac{l_{12}}{3}. \end{gathered}$$

(xiii) $$\begin{bmatrix} -3 & -l_{02} & 3 & 2 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-15}{6} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \le l_{02} \le 6\iota^2, \\ -\iota - \tfrac{l_{02}}{2} \le d_{02} < l_{02} \cdot \tfrac{\iota^+-15}{18}, \\ -\iota - d_{02} - \tfrac{l_{02}}{2} \le d_{12} \le 0. \end{gathered}$$

(xiv) $$\begin{bmatrix} -3 & -l_{02} & 3 & l_{12} & 0 \\ -3 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-21}{6} & d_{02} & 2 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ (l_{02}, l_{12}) \in \{(z,3),(3,z) \mid z = 3,4,5\}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{7}{6}\right) < d_{02} < l_{02} \cdot \tfrac{\iota^+-21}{18}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{2}\right) \le d_{12} < \tfrac{2l_{12}}{3}. \end{gathered}$$

(xv) $$\begin{bmatrix} -3 & -2 & 3 & l_{12} & 0 \\ -3 & -2 & 0 & 0 & 2 \\ \frac{\iota^+-21}{6} & d_{02} & 2 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \le l_{12} \le 6\iota^2, \\ -\tfrac{2\iota}{l_{12}} - 1 \le d_{02} < \tfrac{\iota^+-21}{9}, \\ -\iota - \tfrac{l_{12}(d_{02}+1)}{2} \le d_{12} < \tfrac{2l_{12}}{3}. \end{gathered}$$

(xvi) $\begin{bmatrix} -3 & -l_{02} & 3 & 2 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-21}{6} & d_{02} & 2 & d_{12} & 1 \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \le l_{02} \le 6\iota^2, \\ -\iota - \tfrac{l_{02}}{2} \le d_{02} < l_{02} \cdot \tfrac{\iota^+-21}{18}, \\ -\iota - d_{02} - \tfrac{l_{02}}{2} \le d_{12} \le 1. \end{gathered}$$

(xvii) $\begin{bmatrix} -3 & -l_{02} & 3 & 0 & 0 \\ -3 & -l_{02} & 0 & 2 & l_{22} \\ \frac{\iota^+-15}{6} & d_{02} & 1 & 1 & d_{22} \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ (l_{02}, l_{22}) \in \{(z,2), (2,z) \mid z = 2,3,4,5\}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{5}{6}\right) < d_{02} < l_{02} \cdot \tfrac{\iota^+-15}{18}, \\ l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{3}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(xviii) $\begin{bmatrix} -3 & -l_{02} & 3 & 0 & 0 \\ -3 & -l_{02} & 0 & 2 & l_{22} \\ \frac{\iota^+-21}{6} & d_{02} & 2 & 1 & d_{22} \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ (l_{02}, l_{22}) \in \{(z,2), (2,z) \mid z = 2,3,4,5\}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{7}{6}\right) < d_{02} < l_{02} \cdot \tfrac{\iota^+-21}{18}, \\ l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{2}{3}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(xix) $\begin{bmatrix} -3 & 3 & l_{12} & 0 & 0 \\ -3 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-15}{6} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ (l_{12}, l_{22}) \in \{(z,2), (2,z) \mid z = 2,3,4,5\}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{\iota^+-15}{18} - \tfrac{l_{22}}{2}\right) \le d_{12} < \tfrac{l_{12}}{3}, \\ l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{\iota^+-15}{18} - \tfrac{d_{12}}{l_{12}}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(xx) $\begin{bmatrix} -3 & 3 & l_{12} & 0 & 0 \\ -3 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-21}{6} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ (l_{12}, l_{22}) \in \{(z,2), (2,z) \mid z = 2,3,4,5\}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{\iota^+-21}{18} - \tfrac{l_{22}}{2}\right) \le d_{12} < \tfrac{2l_{12}}{3}, \\ l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{\iota^+-21}{18} - \tfrac{d_{12}}{l_{12}}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

Proposition 4.2.10. *Consider case (i)(b) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type E_7 and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $\begin{bmatrix} -4 & -l_{02} & 3 & l_{12} & 0 \\ -4 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-10}{3} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{02}, l_{12}) \in \{(z,3), (3,z) \mid z = 3,4,5\}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{5}{6}\right) < d_{02} < l_{02} \cdot \tfrac{\iota^+-10}{12}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{2}\right) \le d_{12} < \tfrac{l_{12}}{3}. \end{gathered}$$

(ii) $\begin{bmatrix} -4 & -2 & 3 & l_{12} & 0 \\ -4 & -2 & 0 & 0 & 2 \\ \frac{\iota^+-10}{3} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \le l_{12} \le 4\iota^2, \\ -\tfrac{2\iota}{l_{12}} - 1 \le d_{02} < \tfrac{\iota^+-10}{6}, \\ -\iota - \tfrac{l_{12}(d_{02}+1)}{2} \le d_{12} < \tfrac{l_{12}}{3}. \end{gathered}$$

(iii) $$\begin{bmatrix} -4 & -l_{02} & 3 & 2 & 0 \\ -4 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-10}{3} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \le l_{02} \le 4\iota^2, \\ -\iota - \tfrac{l_{02}}{2} \le d_{02} < l_{02}\cdot \tfrac{\iota^+-10}{12}, \\ -\iota - d_{02} - \tfrac{l_{02}}{2} \le d_{12} \le 0. \end{gathered}$$

(iv) $$\begin{bmatrix} -4 & -l_{02} & 3 & l_{12} & 0 \\ -4 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-14}{3} & d_{02} & 2 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{02}, l_{12}) \in \{(z,3), (3,z) \mid z = 3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{7}{6}\right) < d_{02} < l_{02}\cdot\tfrac{\iota^+-14}{12}, \\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} - \tfrac{1}{2}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{2}\right) \le d_{12} < \tfrac{2l_{12}}{3}. \end{gathered}$$

(v) $$\begin{bmatrix} -4 & -2 & 3 & l_{12} & 0 \\ -4 & -2 & 0 & 0 & 2 \\ \frac{\iota^+-14}{3} & d_{02} & 2 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \le l_{12} \le 4\iota^2, \\ -\tfrac{2\iota}{l_{12}} - 1 \le d_{02} < \tfrac{\iota^+-14}{6}, \\ -\iota - \tfrac{l_{12}(d_{02}+1)}{2} \le d_{12} < \tfrac{2l_{12}}{3}. \end{gathered}$$

(vi) $$\begin{bmatrix} -4 & -l_{02} & 3 & 2 & 0 \\ -4 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-14}{3} & d_{02} & 2 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \le l_{02} \le 4\iota^2, \\ -\iota - \tfrac{l_{02}}{2} \le d_{02} < l_{02}\cdot \tfrac{\iota^+-14}{12}, \\ -\iota - d_{02} - \tfrac{l_{02}}{2} \le d_{12} \le 1. \end{gathered}$$

(vii) $$\begin{bmatrix} -4 & -l_{02} & 3 & 0 & 0 \\ -4 & -l_{02} & 0 & 2 & l_{22} \\ \frac{\iota^+-10}{3} & d_{02} & 1 & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{02}, l_{22}) \in \{(z,2), (2,z) \mid z = 2,3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{5}{6}\right) < d_{02} < l_{02}\cdot\tfrac{\iota^+-10}{12}, \\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{3}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(viii) $$\begin{bmatrix} -4 & -l_{02} & 3 & 0 & 0 \\ -4 & -l_{02} & 0 & 2 & l_{22} \\ \frac{\iota^+-14}{3} & d_{02} & 2 & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{02}, l_{22}) \in \{(z,2), (2,z) \mid z = 2,3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{7}{6}\right) < d_{02} < l_{02}\cdot\tfrac{\iota^+-14}{12}, \\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} - \tfrac{2}{3}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{2}{3}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(ix) $$\begin{bmatrix} -4 & 3 & l_{12} & 0 & 0 \\ -4 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-10}{3} & 1 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{12}, l_{22}) \in \{(3,2), (2,2), (2,3)\}, \\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{3}{4}\right) - \tfrac{\iota^+-10}{12} - \tfrac{l_{22}}{2}\right) \le d_{12} \le 0, \\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{3}{4}\right) - \tfrac{\iota^+-10}{12} - \tfrac{d_{12}}{l_{12}}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(x) $$\begin{bmatrix} -4 & 3 & l_{12} & 0 & 0 \\ -4 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-14}{3} & 2 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{12}, l_{22}) \in \{(3,2), (2,2), (2,3)\}, \\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{3}{4}\right) - \tfrac{\iota^+-14}{12} - \tfrac{l_{22}}{2}\right) \le d_{12} < \tfrac{2l_{12}}{3}, \\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - \tfrac{3}{4}\right) - \tfrac{\iota^+-14}{12} - \tfrac{d_{12}}{l_{12}}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

Proposition 4.2.11. *Consider case (i)(b) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type E_8 and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i)
$$\begin{bmatrix} -5 & -l_{02} & 3 & l_{12} & 0 \\ -5 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-25}{6} & d_{02} & 1 & d_{12} & 1 \end{bmatrix},$$
$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{02}, l_{12}) \in \{(z,3),(3,z) \mid z=3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{12}}-\tfrac{1}{2}\right)-\tfrac{5}{6}\right) < d_{02} < l_{02}\cdot\tfrac{\iota^+-25}{30}, \\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{12}}-\tfrac{1}{2}\right)-\tfrac{d_{02}}{l_{02}}-\tfrac{1}{2}\right) \le d_{12} < \tfrac{l_{12}}{3}. \end{gathered}$$

(ii)
$$\begin{bmatrix} -5 & -2 & 3 & l_{12} & 0 \\ -5 & -2 & 0 & 0 & 2 \\ \frac{\iota^+-25}{6} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \le l_{12} \le 4\iota^2, \\ -\tfrac{2\iota}{l_{12}}-1 \le d_{02} < \tfrac{\iota^+-25}{15}, \\ -\iota-\tfrac{l_{12}(d_{02}+1)}{2} \le d_{12} < \tfrac{l_{12}}{3}. \end{gathered}$$

(iii)
$$\begin{bmatrix} -5 & -l_{02} & 3 & 2 & 0 \\ -5 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-25}{6} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \le l_{02} \le 4\iota^2, \\ -\iota-\tfrac{l_{02}}{2} \le d_{02} < l_{02}\cdot\tfrac{\iota^+-25}{30}, \\ -\iota-d_{02}-\tfrac{l_{02}}{2} \le d_{12} \le 0. \end{gathered}$$

(iv)
$$\begin{bmatrix} -5 & -l_{02} & 3 & l_{12} & 0 \\ -5 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-35}{6} & d_{02} & 2 & d_{12} & 1 \end{bmatrix},$$
$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{02}, l_{12}) \in \{(z,3),(3,z) \mid z=3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{12}}-\tfrac{1}{2}\right)-\tfrac{7}{6}\right) < d_{02} < l_{02}\cdot\tfrac{\iota^+-35}{30} \\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{12}}-\tfrac{1}{2}\right)-\tfrac{d_{02}}{l_{02}}-\tfrac{1}{2}\right) \le d_{12} < \tfrac{2l_{12}}{3}. \end{gathered}$$

(v)
$$\begin{bmatrix} -5 & -2 & 3 & l_{12} & 0 \\ -5 & -2 & 0 & 0 & 2 \\ \frac{\iota^+-35}{6} & d_{02} & 2 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \le l_{12} \le 4\iota^2, \\ -\tfrac{2\iota}{l_{12}}-1 \le d_{02} < \tfrac{\iota^+-35}{15}, \\ -\iota-\tfrac{l_{12}(d_{02}+1)}{2} \le d_{12} < \tfrac{2l_{12}}{3}. \end{gathered}$$

(vi)
$$\begin{bmatrix} -5 & -l_{02} & 3 & 2 & 0 \\ -5 & -l_{02} & 0 & 0 & 2 \\ \frac{\iota^+-35}{6} & d_{02} & 2 & d_{12} & 1 \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \le l_{02} \le 4\iota^2, \\ -\iota-\tfrac{l_{02}}{2} \le d_{02} < l_{02}\cdot\tfrac{\iota^+-35}{30}, \\ -\iota-d_{02}-\tfrac{l_{02}}{2} \le d_{12} \le 1. \end{gathered}$$

(vii)
$$\begin{bmatrix} -5 & -l_{02} & 3 & 0 & 0 \\ -5 & -l_{02} & 0 & 2 & l_{22} \\ \frac{\iota^+-25}{6} & d_{02} & 1 & 1 & d_{22} \end{bmatrix},$$
$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{02}, l_{22}) \in \{(z,2),(2,z) \mid z=2,3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{22}}-\tfrac{2}{3}\right)-\tfrac{5}{6}\right) < d_{02} < l_{02}\cdot\tfrac{\iota^+-25}{30}, \\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{22}}-\tfrac{2}{3}\right)-\tfrac{d_{02}}{l_{02}}-\tfrac{1}{3}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(viii)
$$\begin{bmatrix} -5 & -l_{02} & 3 & 0 & 0 \\ -5 & -l_{02} & 0 & 2 & l_{22} \\ \frac{\iota^+-35}{6} & d_{02} & 2 & 1 & d_{22} \end{bmatrix},$$
$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{02}, l_{22}) \in \{(z,2),(2,z) \mid z=2,3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{22}}-\tfrac{2}{3}\right)-\tfrac{7}{6}\right) < d_{02} < l_{02}\cdot\tfrac{\iota^+-35}{30}, \\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{22}}-\tfrac{2}{3}\right)-\tfrac{d_{02}}{l_{02}}-\tfrac{2}{3}\right) \le d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(ix)
$$\begin{bmatrix} -5 & 3 & l_{12} & 0 & 0 \\ -5 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-25}{6} & 1 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 6|\iota^+-1,\\ (l_{12},l_{22}) \in \{(3,2),(2,2),(2,3)\},\\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{12}}+\tfrac{1}{l_{22}}-\tfrac{4}{5}\right)-\tfrac{\iota^+-25}{30}-\tfrac{l_{22}}{2}\right)\le d_{12}\le 0,\\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{12}}+\tfrac{1}{l_{22}}-\tfrac{4}{5}\right)-\tfrac{\iota^+-25}{30}-\tfrac{d_{12}}{l_{12}}\right)\le d_{22}<\tfrac{l_{22}}{2}.\end{gathered}$$

(x) $$\begin{bmatrix} -5 & 3 & l_{12} & 0 & 0 \\ -5 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-35}{6} & 2 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 6|\iota^+-5,\\ (l_{12},l_{22}) \in \{(3,2),(2,2),(2,3)\},\\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{12}}+\tfrac{1}{l_{22}}-\tfrac{4}{5}\right)-\tfrac{\iota^+-35}{30}-\tfrac{l_{22}}{2}\right)\le d_{12}<\tfrac{2l_{12}}{3},\\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{12}}+\tfrac{1}{l_{22}}-\tfrac{4}{5}\right)-\tfrac{\iota^+-35}{30}-\tfrac{d_{12}}{l_{12}}\right)\le d_{22}<\tfrac{l_{22}}{2}.\end{gathered}$$

Proposition 4.2.12. *Consider case (i)(b) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type D_n and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & l_{12} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 2 \\ \iota^+-l_{01} & d_{02} & 1 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 2|\iota^+-1,\\ (l_{02},l_{12}) \in \{(z,3),(3,z) \mid z=3,4,5\},\\ 1\le l_{01}\le (2l_{02}+2l_{12}-l_{02}l_{12}+1)\,\iota^2,\\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{12}}-\tfrac{1}{2}\right)-\tfrac{5}{6}\right)<d_{02}<l_{02}\cdot\tfrac{\iota^+-l_{01}}{l_{01}},\\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{12}}-\tfrac{1}{2}\right)-\tfrac{d_{02}}{l_{02}}-\tfrac{1}{2}\right)\le d_{12}<\tfrac{l_{12}}{2}.\end{gathered}$$

(ii) $$\begin{bmatrix} -l_{01} & -2 & 2 & l_{12} & 0 \\ -l_{01} & -2 & 0 & 0 & 2 \\ \iota^+-l_{01} & d_{02} & 1 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 2|\iota^+-1,\\ 2\le l_{01},l_{12}\le 4\iota^2,\\ -\tfrac{2\iota}{l_{12}}-1\le d_{02}<2\cdot\tfrac{\iota^+-l_{01}}{l_{01}},\\ -\iota-\tfrac{l_{12}(d_{02}+1)}{2}\le d_{12}<\tfrac{l_{12}}{2}.\end{gathered}$$

(iii) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & 2 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 2 \\ \iota^+-l_{01} & d_{02} & 1 & d_{12} & 1 \end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 2|\iota^+-1,\\ 2\le l_{01},l_{02}\le 4\iota^2,\\ -\iota-\tfrac{l_{02}}{2}\le d_{02}<l_{02}\cdot\tfrac{\iota^+-l_{01}}{l_{01}},\\ -\iota-d_{02}-\tfrac{l_{02}}{2}\le d_{12}\le 0.\end{gathered}$$

(iv) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 2 & l_{22} \\ \iota^+-l_{01} & d_{02} & 1 & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 2|\iota^+-1,\\ (l_{02},l_{22}) \in \{(z,3),(3,z) \mid z=3,4,5\},\\ 1\le l_{01}\le (2l_{02}+2l_{22}-l_{02}l_{22}+1)\,\iota^2,\\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{22}}-\tfrac{1}{2}\right)-1\right)<d_{02}<l_{02}\cdot\tfrac{\iota^+-l_{01}}{l_{01}},\\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{22}}-\tfrac{1}{2}\right)-\tfrac{d_{02}}{l_{02}}-\tfrac{1}{2}\right)\le d_{22}<\tfrac{l_{22}}{2}.\end{gathered}$$

(v) $$\begin{bmatrix} -l_{01} & -2 & 2 & 0 & 0 \\ -l_{01} & -2 & 0 & 2 & l_{22} \\ \iota^+-l_{01} & d_{02} & 1 & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 2|\iota^+-1,\\ 2\le l_{01},l_{22}\le 4\iota^2,\\ \tfrac{-2\iota-2}{l_{22}}<d_{02}<2\cdot\tfrac{\iota^+-l_{01}}{l_{01}},\\ -\iota-l_{22}\cdot\tfrac{d_{02}+1}{2}\le d_{22}<\tfrac{l_{22}}{2}.\end{gathered}$$

(vi) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 2 & 2 \\ \iota^+-l_{01} & d_{02} & 1 & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 2|\iota^+-1,\\ 2\le l_{01},l_{02}\le 4\iota^2,\\ -\iota-\tfrac{l_{02}}{2}\le d_{02}<l_{02}\cdot\tfrac{\iota^+-l_{01}}{l_{01}},\\ \tfrac{-2\iota-2d_{02}}{l_{02}}-1\le d_{22}\le 0.\end{gathered}$$

(vii) $$\begin{bmatrix} -l_{01} & 2 & l_{12} & 0 & 0 \\ -l_{01} & 0 & 0 & 2 & l_{22} \\ \iota^+-l_{01} & 1 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 2|\iota^+-1,\\ (l_{01},l_{12},l_{22}) \in \{(z,3,2),(3,z,2),(2,z,3) \mid z=3,4,5\},\\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{01}}+\tfrac{1}{l_{12}}+\tfrac{1}{l_{22}}-1\right)-\tfrac{\iota^+-l_{01}}{l_{01}}-\tfrac{1}{2}\right) < d_{12} < \tfrac{l_{12}}{2},\\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{01}}+\tfrac{1}{l_{12}}+\tfrac{1}{l_{22}}-1\right)-\tfrac{\iota^+-l_{01}}{l_{01}}-\tfrac{d_{12}}{l_{12}}\right) \le d_{22} < \tfrac{l_{22}}{2}.\end{gathered}$$

(viii) $$\begin{bmatrix} -2 & 2 & 2 & 0 & 0\\ -2 & 0 & 0 & 2 & l_{22}\\ \iota^+-2 & 1 & d_{12} & 1 & d_{22}\end{bmatrix}, \qquad \begin{gathered}\iota^+|\iota, \quad 2|\iota^+-1,\\ 2\le l_{22}\le 4\iota^2,\\ \tfrac{-2\iota+l_{22}}{l_{22}}-\iota^+ \le d_{12}\le 0,\\ -\iota-l_{22}\cdot\tfrac{\iota^+-2+d_{12}}{2} \le d_{22} < \tfrac{l_{22}}{2}.\end{gathered}$$

(ix) $$\begin{bmatrix} -l_{01} & 2 & 2 & 0 & 0\\ -l_{01} & 0 & 0 & 2 & 2\\ \iota^+-l_{01} & 1 & d_{12} & 1 & d_{22}\end{bmatrix}, \qquad \begin{gathered}\iota^+|\iota, \quad 2|\iota^+-1,\\ 2\le l_{01}\le 4\iota^2,\\ \tfrac{-2\iota-2\iota^++2l_{01}}{l_{01}}-1 \le d_{12}\le 0,\\ \tfrac{-2\iota-2\iota^++2l_{01}}{l_{01}}-d_{12} \le d_{22} \le 0.\end{gathered}$$

(x) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & l_{12} & 0\\ -l_{01} & -l_{02} & 0 & 0 & 2\\ \tfrac{\iota^+-2l_{01}}{2} & d_{02} & 1 & d_{12} & 1\end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 2|\iota^+,\\ (l_{02},l_{12}) \in \{(z,3),(3,z) \mid z=3,4,5\},\\ 2\le l_{01}\le (2l_{02}+2l_{12}-l_{02}l_{12}+2)\,\iota^2,\\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{12}}-\tfrac{1}{2}\right)-\tfrac{5}{6}\right) < d_{02} < l_{02}\cdot\tfrac{\iota^+-2l_{01}}{2l_{01}},\\ l_{12}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{12}}-\tfrac{1}{2}\right)-\tfrac{d_{02}}{l_{02}}-\tfrac{1}{2}\right) \le d_{12} < \tfrac{l_{12}}{2}.\end{gathered}$$

(xi) $$\begin{bmatrix} -l_{01} & -2 & 2 & l_{12} & 0\\ -l_{01} & -2 & 0 & 0 & 2\\ \tfrac{\iota^+-2l_{01}}{2} & d_{02} & 1 & d_{12} & 1\end{bmatrix}, \qquad \begin{gathered}\iota^+|\iota, \quad 2|\iota^+,\\ 2\le l_{01},l_{12}\le 4\iota^2,\\ -\tfrac{2\iota}{l_{12}}-1 \le d_{02} < 2\cdot\tfrac{\iota^+-2l_{01}}{2l_{01}},\\ -\iota-\tfrac{l_{12}(d_{02}+1)}{2} \le d_{12} < \tfrac{l_{12}}{2}.\end{gathered}$$

(xii) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & 2 & 0\\ -l_{01} & -l_{02} & 0 & 0 & 2\\ \tfrac{\iota^+-2l_{01}}{2} & d_{02} & 1 & d_{12} & 1\end{bmatrix}, \qquad \begin{gathered}\iota^+|\iota, \quad 2|\iota^+,\\ 2\le l_{01},l_{02}\le 4\iota^2,\\ -\iota-\tfrac{l_{02}}{2} \le d_{02} < l_{02}\cdot\tfrac{\iota^+-2l_{01}}{2l_{01}},\\ -\iota-d_{02}-\tfrac{l_{02}}{2} \le d_{12} \le 0.\end{gathered}$$

(xiii) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & 0 & 0\\ -l_{01} & -l_{02} & 0 & 2 & l_{22}\\ \tfrac{\iota^+-2l_{01}}{2} & d_{02} & 1 & 1 & d_{22}\end{bmatrix},$$

$$\begin{gathered}\iota^+|\iota, \quad 2|\iota^+,\\ (l_{02},l_{22}) \in \{(z,3),(3,z) \mid z=3,4,5\},\\ 2\le l_{01}\le (2l_{02}+2l_{22}-l_{02}l_{22}+2)\,\iota^2,\\ l_{02}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{22}}-\tfrac{1}{2}\right)-1\right) < d_{02} < l_{02}\cdot\tfrac{\iota^+-2l_{01}}{2l_{01}},\\ l_{22}\cdot\left(-\iota\cdot\left(\tfrac{1}{l_{02}}+\tfrac{1}{l_{22}}-\tfrac{1}{2}\right)-\tfrac{d_{02}}{l_{02}}-\tfrac{1}{2}\right) \le d_{22} < \tfrac{l_{22}}{2}.\end{gathered}$$

(xiv) $$\begin{bmatrix} -l_{01} & -2 & 2 & 0 & 0\\ -l_{01} & -2 & 0 & 2 & l_{22}\\ \tfrac{\iota^+-2l_{01}}{2} & d_{02} & 1 & 1 & d_{22}\end{bmatrix}, \qquad \begin{gathered}\iota^+|\iota, \quad 2|\iota^+,\\ 2\le l_{01},l_{22}\le 4\iota^2,\\ \tfrac{-2\iota-2}{l_{22}} < d_{02} < \tfrac{\iota^+-2l_{01}}{l_{01}},\\ -\iota-l_{22}\cdot\tfrac{d_{02}+1}{2} \le d_{22} < \tfrac{l_{22}}{2}.\end{gathered}$$

(xv) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & 0 & 0\\ -l_{01} & -l_{02} & 0 & 2 & 2\\ \tfrac{\iota^+-2l_{01}}{2} & d_{02} & 1 & 1 & d_{22}\end{bmatrix}, \qquad \begin{gathered}\iota^+|\iota, \quad 2|\iota^+,\\ 2\le l_{01},l_{02}\le 4\iota^2,\\ -\iota-\tfrac{l_{02}}{2} \le d_{02} < l_{02}\cdot\tfrac{\iota^+-2l_{01}}{2l_{01}},\\ \tfrac{-2\iota-2d_{02}}{l_{02}}-1 \le d_{22} \le 0.\end{gathered}$$

(xvi) $$\begin{bmatrix} -l_{01} & 2 & l_{12} & 0 & 0\\ -l_{01} & 0 & 0 & 2 & l_{22}\\ \tfrac{\iota^+-2l_{01}}{2} & 1 & d_{12} & 1 & d_{22}\end{bmatrix},$$

$$
\begin{gathered}
\iota^+|\iota, \quad 2|\iota^+, \\
(l_{01}, l_{12}, l_{22}) \in \{(z,3,2), (3,z,2), (2,z,3) \mid z = 3,4,5\}, \\
l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{01}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{\iota^+ - 2l_{01}}{2l_{01}} - \tfrac{1}{2}\right) < d_{12} < \tfrac{l_{12}}{2}, \\
l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{01}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{\iota^+ - 2l_{01}}{2l_{01}} - \tfrac{d_{12}}{l_{12}}\right) \leq d_{22} < \tfrac{l_{22}}{2}.
\end{gathered}
$$

(xvii) $$\begin{bmatrix} -2 & 2 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-4}{2} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 2|\iota^+, \\ 2 \leq l_{22} \leq 4\iota^2, \\ \tfrac{-4\iota - l_{22}}{2l_{22}} - \iota^+ \leq d_{12} \leq 0, \\ -\iota - l_{22} \cdot \tfrac{\iota^+ - 4 + 2d_{12}}{4} \leq d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(xviii) $$\begin{bmatrix} -l_{01} & 2 & 2 & 0 & 0 \\ -l_{01} & 0 & 0 & 2 & 2 \\ \frac{\iota^+-2l_{01}}{2} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 2|\iota^+, \\ 2 \leq l_{01} \leq 4\iota^2, \\ \tfrac{-2\iota - \iota^+ + 2l_{01}}{l_{01}} - 1 \leq d_{12} \leq 0, \\ \tfrac{-2\iota - \iota^+ + 2l_{01}}{l_{01}} - d_{12} \leq d_{22} \leq 0. \end{gathered}$$

Proposition 4.2.13. *Consider case (i)(c) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type E_6 and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $$\begin{bmatrix} -3 & -l_{02} & 3 & l_{12} & 0 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-5}{2} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$
\begin{gathered}
\iota^+|\iota, \quad 6|\iota^+ - 1, \\
(l_{02}, l_{12}, l_{22}) \in \{(z,3,2), (z,2,3), (3,z,2), (3,2,z), (2,z,3), (2,3,z) \mid z = 3,4,5\}, \\
l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{5}{6}\right) \leq d_{02} < l_{02} \cdot \tfrac{\iota^+ - 5}{6}, \\
l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{2}\right) \leq d_{12} < \tfrac{l_{12}}{3}, \\
l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{12}}{l_{12}}\right) \leq d_{22} < \tfrac{l_{22}}{2}.
\end{gathered}
$$

(ii) $$\begin{bmatrix} -3 & -2 & 3 & 2 & 0 & 0 \\ -3 & -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-5}{2} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+ - 1, \\ 2 \leq l_{22} \leq 4\iota^2, \\ \tfrac{-6\iota - 5l_{22}}{3l_{22}} \leq d_{02} < \tfrac{\iota^+ - 5}{3}, \\ \tfrac{-2\iota - d_{02}l_{22} - l_{22}}{l_{22}} \leq d_{12} \leq 0, \\ -\iota - l_{22} \cdot \tfrac{d_{02} + d_{12}}{2} \leq d_{22} < \tfrac{l_{22}}{2}. \end{gathered}$$

(iii) $$\begin{bmatrix} -3 & -2 & 3 & l_{12} & 0 & 0 \\ -3 & -2 & 0 & 0 & 2 & 2 \\ \frac{\iota^+-5}{2} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+ - 1, \\ 2 \leq l_{12} \leq 4\iota^2, \\ \tfrac{-6\iota - 5l_{12}}{3l_{12}} \leq d_{02} < \tfrac{\iota^+ - 5}{3}, \\ -\iota - l_{12} \cdot \tfrac{d_{02} + 1}{2} \leq d_{12} < \tfrac{l_{12}}{3}, \\ \tfrac{-2\iota - d_{02}l_{12} - 2d_{12}}{l_{12}} \leq d_{22} \leq 0. \end{gathered}$$

(iv) $$\begin{bmatrix} -3 & -l_{02} & 3 & 2 & 0 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 & 2 \\ \frac{\iota^+-5}{2} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+ - 1, \\ 2 \leq l_{02} \leq 4\iota^2, \\ -\iota - \tfrac{5l_{02}}{6} \leq d_{02} < l_{02} \cdot \tfrac{\iota^+ - 5}{6}, \\ \tfrac{-2\iota - 2d_{02} - l_{02}}{l_{02}} \leq d_{12} \leq 0, \\ \tfrac{-2\iota - 2d_{02} - d_{12}l_{02}}{l_{02}} \leq d_{22} \leq 0. \end{gathered}$$

(v) $$\begin{bmatrix} -3 & -l_{02} & 3 & l_{12} & 0 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-7}{2} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$
\begin{gathered}
\iota^+|\iota, \quad 6|\iota^+ - 5, \\
(l_{02}, l_{12}, l_{22}) \in \{(z,3,2), (z,2,3), (3,z,2), (3,2,z), (2,z,3), (2,3,z) \mid z = 3,4,5\}, \\
l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{5}{6}\right) \leq d_{02} < l_{02} \cdot \tfrac{\iota^+ - 7}{6}, \\
l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{1}{2}\right) \leq d_{12} < \tfrac{2l_{12}}{3}, \\
l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{12}}{l_{12}}\right) \leq d_{22} < \tfrac{l_{22}}{2}.
\end{gathered}
$$

(vi) $$\begin{bmatrix} -3 & -2 & 3 & 2 & 0 & 0 \\ -3 & -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-7}{2} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \leq l_{22} \leq 4\iota^2, \\ \frac{-6\iota-5l_{22}}{3l_{22}} \leq d_{02} < \frac{\iota^+-7}{3}, \\ \frac{-2\iota-d_{02}l_{22}-l_{22}}{l_{22}} \leq d_{12} \leq 1, \\ -\iota-l_{22}\cdot\frac{d_{02}+d_{12}}{2} \leq d_{22} < \frac{l_{22}}{2}. \end{gathered}$$

(vii) $$\begin{bmatrix} -3 & -2 & 3 & l_{12} & 0 & 0 \\ -3 & -2 & 0 & 0 & 2 & 2 \\ \frac{\iota^+-7}{2} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \leq l_{12} \leq 4\iota^2, \\ \frac{-6\iota-5l_{12}}{3l_{12}} \leq d_{02} < \frac{\iota^+-7}{3}, \\ -\iota-l_{12}\cdot\frac{d_{02}+1}{2} \leq d_{12} < \frac{2l_{12}}{3}, \\ \frac{-2\iota-d_{02}l_{12}-2d_{12}}{l_{12}} \leq d_{22} \leq 0. \end{gathered}$$

(viii) $$\begin{bmatrix} -3 & -l_{02} & 3 & 2 & 0 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 & 2 \\ \frac{\iota^+-7}{2} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \leq l_{02} \leq 4\iota^2, \\ -\iota-\frac{5l_{02}}{6} \leq d_{02} < l_{02}\cdot\frac{\iota^+-7}{6}, \\ \frac{-2\iota-2d_{02}-l_{02}}{l_{02}} \leq d_{12} \leq 1, \\ \frac{-2\iota-2d_{02}-d_{12}l_{02}}{l_{02}} \leq d_{22} \leq 0. \end{gathered}$$

(ix) $$\begin{bmatrix} -3 & -l_{02} & 3 & l_{12} & 0 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-15}{6} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ (l_{02},l_{12},l_{22}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z=3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{5}{6}\right) \leq d_{02} < l_{02}\cdot\frac{\iota^+-15}{18}, \\ l_{12}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{1}{2}\right) \leq d_{12} < \frac{l_{12}}{3}, \\ l_{22}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{d_{12}}{l_{12}}\right) \leq d_{22} < \frac{l_{22}}{2}. \end{gathered}$$

(x) $$\begin{bmatrix} -3 & -2 & 3 & 2 & 0 & 0 \\ -3 & -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-15}{6} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \leq l_{22} \leq 6\iota^2, \\ \frac{-6\iota-5l_{22}}{3l_{22}} \leq d_{02} < \frac{\iota^+-15}{9}, \\ \frac{-2\iota-d_{02}l_{22}-l_{22}}{l_{22}} \leq d_{12} \leq 0, \\ -\iota-l_{22}\cdot\frac{d_{02}+d_{12}}{2} \leq d_{22} < \frac{l_{22}}{2}. \end{gathered}$$

(xi) $$\begin{bmatrix} -3 & -2 & 3 & l_{12} & 0 & 0 \\ -3 & -2 & 0 & 0 & 2 & 2 \\ \frac{\iota^+-15}{6} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \leq l_{12} \leq 6\iota^2, \\ \frac{-6\iota-5l_{12}}{3l_{12}} \leq d_{02} < \frac{\iota^+-15}{9}, \\ -\iota-l_{12}\cdot\frac{d_{02}+1}{2} \leq d_{12} < \frac{l_{12}}{3}, \\ \frac{-2\iota-d_{02}l_{12}-2d_{12}}{l_{12}} \leq d_{22} \leq 0. \end{gathered}$$

(xii) $$\begin{bmatrix} -3 & -l_{02} & 3 & 2 & 0 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 & 2 \\ \frac{\iota^+-15}{6} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \leq l_{02} \leq 6\iota^2, \\ -\iota-\frac{5l_{02}}{6} \leq d_{02} < l_{02}\cdot\frac{\iota^+-15}{18}, \\ \frac{-2\iota-2d_{02}-l_{02}}{l_{02}} \leq d_{12} \leq 0, \\ \frac{-2\iota-2d_{02}-d_{12}l_{02}}{l_{02}} \leq d_{22} \leq 0. \end{gathered}$$

$$\begin{bmatrix} -3 & -l_{02} & 3 & l_{12} & 0 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-21}{6} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix},$$

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ (l_{02},l_{12},l_{22}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z=3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{5}{6}\right) \leq d_{02} < l_{02}\cdot\frac{\iota^+-21}{18}, \\ l_{12}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{1}{2}\right) \leq d_{12} < \frac{2l_{12}}{3}, \\ l_{22}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{d_{12}}{l_{12}}\right) \leq d_{22} < \frac{l_{22}}{2}. \end{gathered}$$

(xiv) $$\begin{bmatrix} -3 & -2 & 3 & 2 & 0 & 0 \\ -3 & -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-21}{6} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \leq l_{22} \leq 6\iota^2, \\ \frac{-6\iota-5l_{22}}{3l_{22}} \leq d_{02} < \frac{\iota^+-21}{9}, \\ \frac{-2\iota-d_{02}l_{22}-l_{22}}{l_{22}} \leq d_{12} \leq 1, \\ -\iota-l_{22}\cdot\frac{d_{02}+d_{12}}{2} \leq d_{22} < \frac{l_{22}}{2}. \end{gathered}$$

(xv) $\begin{bmatrix} -3 & -2 & 3 & l_{12} & 0 & 0 \\ -3 & -2 & 0 & 0 & 2 & 2 \\ \frac{\iota^+-21}{6} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{array}{c} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \leq l_{12} \leq 6\iota^2, \\ \frac{-6\iota-5l_{12}}{3l_{12}} \leq d_{02} < \frac{\iota^+-21}{9}, \\ -\iota - l_{12}\cdot\frac{d_{02}+1}{2} \leq d_{12} < \frac{2l_{12}}{3}, \\ \frac{-2\iota-d_{02}l_{12}-2d_{12}}{l_{12}} \leq d_{22} \leq 0. \end{array}$

(xvi) $\begin{bmatrix} -3 & -l_{02} & 3 & 2 & 0 & 0 \\ -3 & -l_{02} & 0 & 0 & 2 & 2 \\ \frac{\iota^+-21}{6} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{array}{c} \iota^+|\iota, \quad 6|\iota^+-3, \\ 2 \leq l_{02} \leq 6\iota^2, \\ -\iota - \frac{5l_{02}}{6} \leq d_{02} < l_{02}\cdot\frac{\iota^+-21}{18}, \\ \frac{-2\iota-2d_{02}-l_{02}}{l_{02}} \leq d_{12} \leq 1, \\ \frac{-2\iota-2d_{02}-d_{12}l_{02}}{l_{02}} \leq d_{22} \leq 0. \end{array}$

Proposition 4.2.14. *Consider case (i)(c) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type E_7 and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $\begin{bmatrix} -4 & -l_{02} & 3 & l_{12} & 0 & 0 \\ -4 & -l_{02} & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-10}{3} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$,

$$\begin{array}{c} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{02},l_{12},l_{22}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z=3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{5}{6}\right) \leq d_{02} < l_{02}\cdot\frac{\iota^+-10}{12}, \\ l_{12}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{1}{2}\right) \leq d_{12} < \frac{l_{12}}{3}, \\ l_{22}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{d_{12}}{l_{12}}\right) \leq d_{22} < \frac{l_{22}}{2}. \end{array}$$

(ii) $\begin{bmatrix} -4 & -2 & 3 & 2 & 0 & 0 \\ -4 & -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-10}{3} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{array}{c} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \leq l_{22} \leq 4\iota^2, \\ \frac{-6\iota-5l_{22}}{3l_{22}} \leq d_{02} < \frac{\iota^+-10}{6}, \\ \frac{-2\iota-d_{02}l_{22}-l_{22}}{l_{22}} \leq d_{12} \leq 0, \\ -\iota - l_{22}\cdot\frac{d_{02}+d_{12}}{2} \leq d_{22} < \frac{l_{22}}{2}. \end{array}$

(iii) $\begin{bmatrix} -4 & -2 & 3 & l_{12} & 0 & 0 \\ -4 & -2 & 0 & 0 & 2 & 2 \\ \frac{\iota^+-10}{3} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{array}{c} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \leq l_{12} \leq 4\iota^2, \\ \frac{-6\iota-5l_{12}}{3l_{12}} \leq d_{02} < \frac{\iota^+-10}{6}, \\ -\iota - l_{12}\cdot\frac{d_{02}+1}{2} \leq d_{12} < \frac{l_{12}}{3}, \\ \frac{-2\iota-d_{02}l_{12}-2d_{12}}{l_{12}} \leq d_{22} \leq 0. \end{array}$

(iv) $\begin{bmatrix} -4 & -l_{02} & 3 & 2 & 0 & 0 \\ -4 & -l_{02} & 0 & 0 & 2 & 2 \\ \frac{\iota^+-10}{3} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{array}{c} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \leq l_{02} \leq 4\iota^2, \\ -\iota - \frac{5l_{02}}{6} \leq d_{02} < l_{02}\cdot\frac{\iota^+-10}{12}, \\ \frac{-2\iota-2d_{02}-l_{02}}{l_{02}} \leq d_{12} \leq 0, \\ \frac{-2\iota-2d_{02}-d_{12}l_{02}}{l_{02}} \leq d_{22} \leq 0. \end{array}$

(v) $\begin{bmatrix} -4 & -l_{02} & 3 & l_{12} & 0 & 0 \\ -4 & -l_{02} & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-14}{3} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$,

$$\begin{array}{c} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{02},l_{12},l_{22}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z=3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{5}{6}\right) \leq d_{02} < l_{02}\cdot\frac{\iota^+-14}{12}, \\ l_{12}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{1}{2}\right) \leq d_{12} < \frac{2l_{12}}{3}, \\ l_{22}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{d_{12}}{l_{12}}\right) \leq d_{22} < \frac{l_{22}}{2}. \end{array}$$

(vi) $\begin{bmatrix} -4 & -2 & 3 & 2 & 0 & 0 \\ -4 & -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-14}{3} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{array}{c} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \leq l_{22} \leq 4\iota^2, \\ \frac{-6\iota-5l_{22}}{3l_{22}} \leq d_{02} < \frac{\iota^+-14}{6}, \\ \frac{-2\iota-d_{02}l_{22}-l_{22}}{l_{22}} \leq d_{12} \leq 1, \\ -\iota - l_{22}\cdot\frac{d_{02}+d_{12}}{2} \leq d_{22} < \frac{l_{22}}{2}. \end{array}$

(vii) $\begin{bmatrix} -4 & -2 & 3 & l_{12} & 0 & 0 \\ -4 & -2 & 0 & 0 & 2 & 2 \\ \frac{\iota^+-14}{3} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \leq l_{12} \leq 4\iota^2, \\ \frac{-6\iota-5l_{12}}{3l_{12}} \leq d_{02} < \frac{\iota^+-14}{6}, \\ -\iota - l_{12}\cdot\frac{d_{02}+1}{2} \leq d_{12} < \frac{2l_{12}}{3}, \\ \frac{-2\iota-d_{02}l_{12}-2d_{12}}{l_{12}} \leq d_{22} \leq 0. \end{gathered}$

(viii) $\begin{bmatrix} -4 & -l_{02} & 3 & 2 & 0 & 0 \\ -4 & -l_{02} & 0 & 0 & 2 & 2 \\ \frac{\iota^+-14}{3} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \leq l_{02} \leq 4\iota^2, \\ -\iota - \frac{5l_{02}}{6} \leq d_{02} < l_{02}\cdot\frac{\iota^+-14}{12}, \\ \frac{-2\iota-2d_{02}-l_{02}}{l_{02}} \leq d_{12} \leq 1, \\ \frac{-2\iota-2d_{02}-d_{12}l_{02}}{l_{02}} \leq d_{22} \leq 0. \end{gathered}$

Proposition 4.2.15. *Consider case (i)(c) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type E_8 and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $\begin{bmatrix} -5 & -l_{02} & 3 & l_{12} & 0 & 0 \\ -5 & -l_{02} & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-25}{6} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ (l_{02},l_{12},l_{22}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z=3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{5}{6}\right) \leq d_{02} < l_{02}\cdot\frac{\iota^+-25}{30}, \\ l_{12}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{1}{2}\right) \leq d_{12} < \frac{l_{12}}{3}, \\ l_{22}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{d_{12}}{l_{12}}\right) \leq d_{22} < \frac{l_{22}}{2}. \end{gathered}$$

(ii) $\begin{bmatrix} -5 & -2 & 3 & 2 & 0 & 0 \\ -5 & -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-25}{6} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \leq l_{22} \leq 4\iota^2, \\ \frac{-6\iota-5l_{22}}{3l_{22}} \leq d_{02} < \frac{\iota^+-25}{15}, \\ \frac{-2\iota-d_{02}l_{22}-l_{22}}{l_{22}} \leq d_{12} \leq 0, \\ -\iota - l_{22}\cdot\frac{d_{02}+d_{12}}{2} \leq d_{22} < \frac{l_{22}}{2}. \end{gathered}$

(iii) $\begin{bmatrix} -5 & -2 & 3 & l_{12} & 0 & 0 \\ -5 & -2 & 0 & 0 & 2 & 2 \\ \frac{\iota^+-25}{6} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \leq l_{12} \leq 4\iota^2, \\ \frac{-6\iota-5l_{12}}{3l_{12}} \leq d_{02} < \frac{\iota^+-25}{15}, \\ -\iota - l_{12}\cdot\frac{d_{02}+1}{2} \leq d_{12} < \frac{l_{12}}{3}, \\ \frac{-2\iota-d_{02}l_{12}-2d_{12}}{l_{12}} \leq d_{22} \leq 0. \end{gathered}$

(iv) $\begin{bmatrix} -5 & -l_{02} & 3 & 2 & 0 & 0 \\ -5 & -l_{02} & 0 & 0 & 2 & 2 \\ \frac{\iota^+-25}{6} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-1, \\ 2 \leq l_{02} \leq 4\iota^2, \\ -\iota - \frac{5l_{02}}{6} \leq d_{02} < l_{02}\cdot\frac{\iota^+-25}{30}, \\ \frac{-2\iota-2d_{02}-l_{02}}{l_{02}} \leq d_{12} \leq 0, \\ \frac{-2\iota-2d_{02}-d_{12}l_{02}}{l_{02}} \leq d_{22} \leq 0. \end{gathered}$

(v) $\begin{bmatrix} -5 & -l_{02} & 3 & l_{12} & 0 & 0 \\ -5 & -l_{02} & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-35}{6} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ (l_{02},l_{12},l_{22}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z=3,4,5\}, \\ l_{02}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{7}{6}\right) \leq d_{02} < l_{02}\cdot\frac{\iota^+-35}{30}, \\ l_{12}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{1}{2}\right) \leq d_{12} < \frac{2l_{12}}{3}, \\ l_{22}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{d_{12}}{l_{12}}\right) \leq d_{22} < \frac{l_{22}}{2}. \end{gathered}$$

(vi) $\begin{bmatrix} -5 & -2 & 3 & 2 & 0 & 0 \\ -5 & -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-35}{6} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \leq l_{22} \leq 4\iota^2, \\ \frac{-6\iota-5l_{22}}{3l_{22}} \leq d_{02} < \frac{\iota^+-35}{15}, \\ \frac{-2\iota-d_{02}l_{22}-l_{22}}{l_{22}} \leq d_{12} \leq 1, \\ -\iota - l_{22}\cdot\frac{d_{02}+d_{12}}{2} \leq d_{22} < \frac{l_{22}}{2}. \end{gathered}$

(vii) $\begin{bmatrix} -5 & -2 & 3 & l_{12} & 0 & 0 \\ -5 & -2 & 0 & 0 & 2 & 2 \\ \frac{\iota^+-35}{6} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\quad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \le l_{12} \le 4\iota^2, \\ \frac{-6\iota-5l_{12}}{3l_{12}} \le d_{02} < \frac{\iota^+-35}{15}, \\ -\iota - l_{12}\cdot\frac{d_{02}+1}{2} \le d_{12} < \frac{2l_{12}}{3}, \\ \frac{-2\iota-d_{02}l_{12}-2d_{12}}{l_{12}} \le d_{22} \le 0. \end{gathered}$

(viii) $\begin{bmatrix} -5 & -l_{02} & 3 & 2 & 0 & 0 \\ -5 & -l_{02} & 0 & 0 & 2 & 2 \\ \frac{\iota^+-35}{6} & d_{02} & 2 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\quad \begin{gathered} \iota^+|\iota, \quad 6|\iota^+-5, \\ 2 \le l_{02} \le 4\iota^2, \\ -\iota-\frac{5l_{02}}{6} \le d_{02} < l_{02}\cdot\frac{\iota^+-35}{30}, \\ \frac{-2\iota-2d_{02}-l_{02}}{l_{02}} \le d_{12} \le 1, \\ \frac{-2\iota-2d_{02}-d_{12}l_{02}}{l_{02}} \le d_{22} \le 0. \end{gathered}$

Proposition 4.2.16. *Consider case (i)(c) from Proposition 4.2.2 with X having Gorenstein index ι. Suppose $x^+ \in X$ is of type D_n and $x^- \in X$ not quasismooth. Then, up to admissible operations, P' is one of the following.*

(i) $\begin{bmatrix} -l_{01} & -l_{02} & 2 & l_{12} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 2 & l_{22} \\ \iota^+-l_{01} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 2|\iota^+-1, \\ (l_{02},l_{12},l_{22}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z=3,4,5\}, \\ 2 \le l_{01} \le (l_{02}l_{12}+l_{02}l_{22}+l_{12}l_{22}-l_{02}l_{12}l_{22}+1)\,\iota^2 \\ l_{02}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-1\right) \le d_{02} < l_{02}\cdot\frac{\iota^+-l_{01}}{l_{01}}, \\ l_{12}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{1}{2}\right) \le d_{12} < \frac{l_{12}}{2}, \\ l_{22}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{d_{12}}{l_{12}}\right) \le d_{22} < \frac{l_{22}}{2}. \end{gathered}$$

(ii) $\begin{bmatrix} -l_{01} & -2 & 2 & 2 & 0 & 0 \\ -l_{01} & -2 & 0 & 0 & 2 & l_{22} \\ \iota^+-l_{01} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\quad \begin{gathered} \iota^+|\iota, \quad 2|\iota^+-1, \\ 2 \le l_{01}, l_{22} \le 4\iota^2, \\ \frac{-2\iota-2l_{22}}{l_{22}} \le d_{02} < \frac{2\iota^+-2l_{01}}{l_{01}}, \\ \frac{-2\iota-d_{02}l_{22}-l_{22}}{l_{22}} \le d_{12} \le 0, \\ -\iota-l_{22}\cdot\frac{d_{02}+d_{12}}{2} \le d_{22} < \frac{l_{22}}{2}. \end{gathered}$

(iii) $\begin{bmatrix} -l_{01} & -2 & 2 & l_{12} & 0 & 0 \\ -l_{01} & -2 & 0 & 0 & 2 & 2 \\ \iota^+-l_{01} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\quad \begin{gathered} \iota^+|\iota, \quad 2|\iota^+-1, \\ 2 \le l_{01}, l_{12} \le 4\iota^2, \\ \frac{-2\iota-2l_{12}}{l_{12}} \le d_{02} < \frac{2\iota^+-2l_{01}}{l_{01}}, \\ -\iota-l_{12}\cdot\frac{d_{02}+1}{2} \le d_{12} < \frac{l_{12}}{2}, \\ \frac{-2\iota-d_{02}l_{12}-2d_{12}}{l_{12}} \le d_{22} \le 0. \end{gathered}$

(iv) $\begin{bmatrix} -l_{01} & -l_{02} & 2 & 2 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 2 & 2 \\ \iota^+-l_{01} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\quad \begin{gathered} \iota^+|\iota, \quad 2|\iota^+-1, \\ 2 \le l_{01}, l_{02} \le 4\iota^2, \\ -\iota-l_{02} \le d_{02} < l_{02}\cdot\frac{\iota^+-l_{01}}{l_{01}}, \\ \frac{-2\iota-2d_{02}-l_{02}}{l_{02}} \le d_{12} \le 0, \\ \frac{-2\iota-2d_{02}-d_{12}l_{02}}{l_{02}} \le d_{22} \le 0. \end{gathered}$

(v) $\begin{bmatrix} -l_{01} & -l_{02} & 2 & l_{12} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-2l_{01}}{2} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$,

$$\begin{gathered} \iota^+|\iota, \quad 2|\iota^+, \\ (l_{02},l_{12},l_{22}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z=3,4,5\}, \\ 2 \le l_{01} \le (l_{02}l_{12}+l_{02}l_{22}+l_{12}l_{22}-l_{02}l_{12}l_{22}+2)\,\iota^2 \\ l_{02}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-1\right) \le d_{02} < l_{02}\cdot\frac{\iota^+-2l_{01}}{2l_{01}}, \\ l_{12}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{1}{2}\right) \le d_{12} < \frac{l_{12}}{2}, \\ l_{22}\cdot\left(-\iota\cdot\left(\frac{1}{l_{02}}+\frac{1}{l_{12}}+\frac{1}{l_{22}}-1\right)-\frac{d_{02}}{l_{02}}-\frac{d_{12}}{l_{12}}\right) \le d_{22} < \frac{l_{22}}{2}. \end{gathered}$$

(vi) $\begin{bmatrix} -l_{01} & -2 & 2 & 2 & 0 & 0 \\ -l_{01} & -2 & 0 & 0 & 2 & l_{22} \\ \frac{\iota^+-2l_{01}}{2} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}$, $\quad \begin{gathered} \iota^+|\iota, \quad 2|\iota^+, \\ 2 \le l_{01}, l_{22} \le 6\iota^2, \\ \frac{-2\iota-2l_{22}}{l_{22}} \le d_{02} < \frac{\iota^+-2l_{01}}{l_{01}}, \\ \frac{-2\iota-d_{02}l_{22}-l_{22}}{l_{22}} \le d_{12} \le 0, \\ -\iota-l_{22}\cdot\frac{d_{02}+d_{12}}{2} \le d_{22} < \frac{l_{22}}{2}. \end{gathered}$

(vii) $$\begin{bmatrix} -l_{01} & -2 & 2 & l_{12} & 0 & 0 \\ -l_{01} & -2 & 0 & 0 & 2 & 2 \\ \frac{\iota^+ - 2l_{01}}{2} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 2|\iota^+, \\ 2 \le l_{01}, l_{12} \le 6\iota^2, \\ \frac{-2\iota - 2l_{12}}{l_{12}} \le d_{02} < \frac{\iota^+ - 2l_{01}}{l_{01}}, \\ -\iota - l_{12} \cdot \frac{d_{02}+1}{2} \le d_{12} < \frac{l_{12}}{2}, \\ \frac{-2\iota - d_{02}l_{12} - 2d_{12}}{l_{12}} \le d_{22} \le 0. \end{gathered}$$

(viii) $$\begin{bmatrix} -l_{01} & -l_{02} & 2 & 2 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 2 & 2 \\ \frac{\iota^+ - 2l_{01}}{2} & d_{02} & 1 & d_{12} & 1 & d_{22} \end{bmatrix}, \qquad \begin{gathered} \iota^+|\iota, \quad 2|\iota^+, \\ 2 \le l_{01}, l_{02} \le 6\iota^2, \\ -\iota - l_{02} \le d_{02} < l_{02} \cdot \frac{\iota^+ - 2l_{01}}{2l_{01}}, \\ \frac{-2\iota - 2d_{02} - l_{02}}{l_{02}} \le d_{12} \le 0, \\ \frac{-2\iota - 2d_{02} - d_{12}l_{02}}{l_{02}} \le d_{22} \le 0. \end{gathered}$$

Proposition 4.2.17. *Consider case (ii)(a) from Proposition 4.2.2 with X having Gorenstein index ι. Then neither $x^+ \in X$ nor $x^- \in X$ are quasismooth and P' is, up to admissible operations, one of the following.*

(i) $$\begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & 0 & 3 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ d_{01} & d_{02} & d_{11} & d_{21} & 1 \end{bmatrix},$$

$$\begin{gathered} 3 \le l_{11} \le 5, \ 1 \le d_{11} \le l_{11} - 1 \\ 1 \le d_{21} \le 2, \\ -\frac{d_{11}}{l_{11}} - \frac{d_{21}}{3} - \frac{1}{2} < d_{01} \le \iota \cdot \left(\frac{1}{l_{11}} + \frac{5}{6}\right) - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{3} - \frac{1}{2}, \\ -\iota \cdot \left(\frac{1}{l_{11}} + \frac{5}{6}\right) - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{3} - \frac{1}{2} \le d_{02} \le d_{01} - 1. \end{gathered}$$

(ii) $$\begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ d_{01} & d_{02} & d_{11} & 1 & 1 \end{bmatrix}, \qquad \begin{gathered} 2 \le l_{11} \le 2\iota^2, \\ 1 \le d_{11} \le l_{11} - 1, \\ \frac{-d_{11} - l_{11}}{l_{11}} < d_{01} \le \frac{\iota - d_{11} - l_{11}}{l_{11}}, \\ \frac{-\iota - d_{11} - l_{11}}{l_{11}} \le d_{02} \le d_{01} - 1. \end{gathered}$$

Proof. The bounds in the first case are obtained from log terminality, slope-orderedness and the conditions

$$0 < \not{d}_P^+ = \not{d}_{P'}^+ \le \iota, \quad \not{d}_P^- \ge -\iota.$$

In the second case we can use Minkowski's Theorem since $\not{d}_P^- - \not{d}_{P'}^- \in \mathbb{Z}$ and

$$\begin{aligned} \not{d}_P^+ &= d_{01}l_{11} + d_{11} + l_{11}, \\ \not{d}_{P'}^- &= d_{02}l_{11} + d_{11} + l_{11} \in \mathbb{Z}. \end{aligned}$$

□

Proposition 4.2.18. *Consider case (ii)(b) from Proposition 4.2.2 with X having Gorenstein index ι. Then neither $x^+ \in X$ nor $x^- \in X$ are quasismooth P' is, up to admissible operations, one of the following.*

(i) $$\begin{bmatrix} -1 & -l_{02} & l_{11} & 1 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{31} \end{bmatrix},$$

$$\begin{gathered} (l_{11}, l_{21}, l_{31}) \in \{(z,3,2), (3,z,2), (2,z,3) \mid z = 3,4,5\}, \\ 2 \le l_{02} \le 5, \\ -\frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} < d_{01} \le \iota \cdot \left(\frac{1}{l_{11}} + \frac{1}{l_{21}} + \frac{1}{l_{31}} - 1\right) - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}}, \\ l_{02} \cdot \left(-\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{21}} + \frac{1}{l_{31}} - 1\right) - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}}\right) \le d_{02} < d_{01}l_{02}, \\ -\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{21}} + \frac{1}{l_{31}} - 1\right) - \frac{d_{02}}{l_{02}} - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} \le d_{12} \le 0. \end{gathered}$$

(ii) $$\begin{bmatrix} -1 & -l_{02} & l_{11} & 1 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 2 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 2 \\ d_{01} & d_{02} & d_{11} & d_{12} & 1 & 1 \end{bmatrix},$$

$$\begin{gathered} 2 \leq l_{02} \leq 2\iota^2, \\ 3 \leq l_{11} \leq 2\iota^2, \\ 1 \leq d_{11} \leq l_{11} - 1, \\ \tfrac{-d_{11}-l_{11}}{l_{11}} < d_{01} \leq \tfrac{\iota - d_{11} - l_{11}}{l_{11}}, \\ -\iota - l_{02} \leq d_{02} < d_{01} l_{02}, \\ \tfrac{-\iota - l_{02} - d_{02}}{l_{02}} \leq d_{12} \leq 0. \end{gathered}$$

(iii) $$\begin{bmatrix} -1 & -l_{02} & 2 & 1 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 2 \\ d_{01} & d_{02} & 1 & d_{12} & d_{21} & 1 \end{bmatrix},$$

$$\begin{gathered} (l_{02}, l_{21}) \in \{(z,3), (3,z) \mid z = 3,4,5\}, \\ 1 \leq d_{21} \leq l_{21} - 1, \\ \tfrac{-d_{21}-l_{21}}{l_{21}} < d_{01} \leq \tfrac{\iota - d_{21} - l_{21}}{l_{21}}, \\ l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{21}} - \tfrac{1}{2}\right) - \tfrac{d_{21}}{l_{21}} - \tfrac{1}{2}\right) \leq d_{02} < d_{01} l_{02}, \\ -\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{21}} - \tfrac{1}{2}\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{21}}{l_{21}} - \tfrac{1}{2} \leq d_{12} \leq 0. \end{gathered}$$

(iv) $$\begin{bmatrix} -1 & -2 & 2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & l_{21} & 0 \\ -1 & -2 & 0 & 0 & 0 & 2 \\ d_{01} & d_{02} & 1 & d_{12} & d_{21} & 1 \end{bmatrix}, \qquad \begin{gathered} 2 \leq l_{21} \leq 4\iota^2, \\ 1 \leq d_{21} \leq l_{11} - 1, \\ \tfrac{-d_{21}-l_{21}}{l_{21}} < d_{01} \leq \tfrac{\iota - d_{21} - l_{21}}{l_{21}}, \\ \tfrac{-2\iota - 2d_{21}}{l_{21}} - 1 \leq d_{02} < 2d_{01}, \\ \tfrac{-\iota - d_{21}}{l_{21}} - \tfrac{d_{02}+1}{2} \leq d_{12} \leq 0. \end{gathered}$$

(v) $$\begin{bmatrix} -1 & -l_{02} & 2 & 1 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 2 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 2 \\ d_{01} & d_{02} & 1 & d_{12} & 1 & 1 \end{bmatrix}, \qquad \begin{gathered} 2 \leq l_{02} \leq 2\iota^2 \\ -1 \leq d_{01} \leq \tfrac{\iota - 3}{2}, \\ -\iota - l_{02} \leq d_{02} < d_{01} l_{02}, \\ \tfrac{-\iota - l_{02} - d_{02}}{l_{02}} \leq d_{12} \leq 0. \end{gathered}$$

Proof. First note that we have $l_{02}, l_{11} > 1$, otherwise we would be in case (i)(a). The first case goes through the possibilities if the exponents associated with x^+ are of the form $(z, 3, 2)$ with $3 \leq z \leq 5$. As before, the bounds for $\not{d}_P^+$ and $\not{d}_{P'}^-$ and slope-orderedness are used to obtain the bounds for the d_{ij}.

In the second case the exponents associated with the elliptic fixed point x^+ are $(2, 2, y)$ and $l_{21} = l_{31} = 2$. Then $\not{d}_P^+, \not{d}_{P'}^- \in \mathbb{Z}$ so we can use Minkowski's Theorem for the bounds on l_{02} and l_{11}.

The third, fourth and fifth case treats the exponents associated with x^+ being of type $(2, 2, y)$ and $l_{11} = l_{31} = 2$. Using the fact that $(l_{02}, l_{21}, 2)$ is platonic, we go through the different constellations and use the bounds on $\not{d}_P^+$ and $\not{d}_{P'}^-$ and then Minkowski's Theorem to obtain the claimed bounds. □

As before, we will only list the results for the rest of the cases as the proofs proceed in a completely analogous manner as previously.

Proposition 4.2.19. *Consider case (ii)(c) from Proposition 4.2.2 with X having Gorenstein index ι. Then neither $x^+ \in X$ nor $x^- \in X$ are quasi-smooth and P' is, up to admissible operations, one of the following.*

(i) $$\begin{bmatrix} -1 & -l_{02} & l_{11} & 1 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & l_{22} & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{22} & d_{31} \end{bmatrix},$$

$$
\begin{gathered}
(l_{11}, l_{21}, l_{31}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z = 3,4,5\}, \\
2 \le l_{02}, l_{22} \le 5, \\
1 \le d_{i1} \le l_{i1} - 1, \ i > 0, \\
-\frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} < d_{01} \le \iota \cdot \left(\frac{1}{l_{11}} + \frac{1}{l_{21}} + \frac{1}{l_{31}} - 1\right) - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}}, \\
l_{02} \cdot \left(-\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{22}} + \frac{1}{l_{31}} - 1\right) - \frac{d_{31}}{l_{31}} - 1\right) < d_{02} < d_{01} l_{02}, \\
-\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{22}} + \frac{1}{l_{31}} - 1\right) - \frac{d_{02}}{l_{02}} - \frac{d_{31}}{l_{31}} - 1 < d_{12} \le 0, \\
l_{22} \cdot \left(-\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{22}} + \frac{1}{l_{31}} - 1\right) - \frac{d_{02}}{l_{02}} - d_{12} - \frac{d_{31}}{l_{31}}\right) \le d_{22} \le 0.
\end{gathered}
$$

(ii) $$\begin{bmatrix} -1 & -2 & l_{11} & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & l_{21} & l_{22} & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 2 \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{22} & 1 \end{bmatrix},$$

$$
\begin{gathered}
(l_{11}, l_{21}) \in \{(z,3),(3,z), \mid z = 3,4,5\}, \\
2 \le l_{22} \le (2l_{11} + 2l_{21} - l_{11}l_{21} + 2)\iota^2, \\
1 \le d_{i1} \le l_{i1} - 1, \ i > 0, \\
-\frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} - \frac{1}{2} < d_{01} \le \iota \cdot \left(\frac{1}{l_{11}} + \frac{1}{l_{21}} - \frac{1}{2}\right) - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} - \frac{1}{2}, \\
-\frac{2\iota}{l_{22}} - 3 < d_{02} < 2d_{01}, \\
-\frac{\iota}{l_{22}} - \frac{d_{02}+1}{2} - 1 < d_{12} \le 0, \\
-\frac{d_{02}+1}{2} - d_{12} - \iota \le d_{22} \le 0.
\end{gathered}
$$

(iii) $$\begin{bmatrix} -1 & -l_{02} & l_{11} & 1 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & 2 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & 2 \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{22} & 1 \end{bmatrix},$$

$$
\begin{gathered}
(l_{11}, l_{21}) \in \{(z,3),(3,z), \mid z = 3,4,5\}, \\
2 \le l_{02} \le (2l_{11} + 2l_{21} - l_{11}l_{21} + 2)\iota^2, \\
1 \le d_{i1} \le l_{i1} - 1, \ i > 0, \\
-\frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} - \frac{1}{2} < d_{01} \le \iota \cdot \left(\frac{1}{l_{11}} + \frac{1}{l_{21}} - \frac{1}{2}\right) - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} - \frac{1}{2}, \\
-\iota - \frac{3l_{02}}{2} < d_{02} < d_{01} l_{02}, \\
-\frac{\iota + d_{02}}{l_{02}} - \frac{1}{2} \le d_{12} \le 0, \\
-\frac{2\iota + 2d_{02}}{l_{02}} - 2d_{12} - 1 \le d_{22} \le 0.
\end{gathered}
$$

(iv) $$\begin{bmatrix} -1 & -2 & 2 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 2 & 2 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & 1 & d_{12} & 1 & d_{22} & d_{31} \end{bmatrix},$$

$$
\begin{gathered}
2 \le l_{31} \le 4\iota^2, \\
1 \le d_{31} \le l_{31} - 1, \\
-\frac{d_{31}}{l_{31}} - 1 < d_{01} \le \frac{\iota + d_{31}}{l_{31}} - 1, \\
-\frac{2\iota + 2d_{31}}{l_{31}} - 2 < d_{02} < 2d_{01}, \\
-\frac{\iota + d_{31}}{l_{31}} - \frac{d_{02}}{2} - 1 < d_{12} \le 0, \\
-\frac{2\iota + 2d_{31}}{l_{31}} - d_{02} - 2d_{12} \le d_{22} \le 0.
\end{gathered}
$$

Proposition 4.2.20. *Consider case (ii)(d) from Proposition 4.2.2 with X having Gorenstein index ι. Then neither $x^+ \in X$ nor $x^- \in X$ are quasismooth and P' is, up to admissible operations, one of the following.*

(i) $$\begin{bmatrix} -1 & -l_{02} & l_{11} & 1 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & l_{22} & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & l_{32} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} \end{bmatrix},$$

$$
\begin{gathered}
(l_{11}, l_{21}, l_{31}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) |\ z=3,4,5\}, \\
(l_{02}, l_{22}, l_{32}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) |\ z=3,4,5\}, \\
1 \le d_{i1} \le l_{i1}-1,\ i>0, \\
-\tfrac{d_{11}}{l_{11}} - \tfrac{d_{21}}{l_{21}} - \tfrac{d_{31}}{l_{31}} < d_{01} \le \iota \cdot \left(\tfrac{1}{l_{11}} + \tfrac{1}{l_{21}} + \tfrac{1}{l_{31}} - 1\right) - \tfrac{d_{11}}{l_{11}} - \tfrac{d_{21}}{l_{21}} - \tfrac{d_{31}}{l_{31}}, \\
l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} + \tfrac{1}{l_{32}} - 1\right) - 2\right) < d_{02} < d_{01} l_{02}, \\
-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} + \tfrac{1}{l_{32}} - 1\right) - \tfrac{d_{02}}{l_{02}} - 2 < d_{12} \le 0, \\
l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} + \tfrac{1}{l_{32}} - 1\right) - \tfrac{d_{02}}{l_{02}} - d_{12} - 1\right) < d_{22} \le 0, \\
l_{32} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{22}} + \tfrac{1}{l_{32}} - 1\right) - \tfrac{d_{02}}{l_{02}} - d_{12} - \tfrac{d_{22}}{l_{22}}\right) < d_{32} \le 0.
\end{gathered}
$$

(ii)
$$
\begin{bmatrix}
-1 & -2 & l_{11} & 1 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & l_{21} & 2 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & l_{31} & l_{32} \\
d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{22} & d_{31} & d_{32}
\end{bmatrix},
$$

$$
\begin{gathered}
(l_{11}, l_{21}, l_{31}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) |\ z=3,4,5\}, \\
2 \le l_{32} \le (l_{11}l_{21} + l_{11}l_{31} + l_{21}l_{31} - l_{11}l_{21}l_{31} + 2)\iota^2, \\
1 \le d_{i1} \le l_{i1}-1,\ i>0, \\
-\tfrac{d_{11}}{l_{11}} - \tfrac{d_{21}}{l_{21}} - \tfrac{d_{31}}{l_{31}} < d_{01} \le \iota \cdot \left(\tfrac{1}{l_{11}} + \tfrac{1}{l_{21}} + \tfrac{1}{l_{31}} - 1\right) - \tfrac{d_{11}}{l_{11}} - \tfrac{d_{21}}{l_{21}} - \tfrac{d_{31}}{l_{31}}, \\
-\tfrac{2\iota}{l_{32}} - 4 < d_{02} < 2d_{01}, \\
-\tfrac{\iota}{l_{32}} - \tfrac{d_{02}}{2} - 2 < d_{12} \le 0, \\
-\tfrac{2\iota}{l_{32}} - \tfrac{d_{02}}{2} - 2d_{12} - 2 < d_{22} \le 0, \\
-\iota - \tfrac{l_{32}(d_{02} - 2d_{12} - d_{22})}{2} < d_{32} \le 0.
\end{gathered}
$$

(iii)
$$
\begin{bmatrix}
-1 & -l_{02} & l_{11} & 1 & 0 & 0 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & l_{21} & 2 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & 2 \\
d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{22} & d_{31} & d_{32}
\end{bmatrix},
$$

$$
\begin{gathered}
(l_{11}, l_{21}, l_{31}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) |\ z=3,4,5\}, \\
2 \le l_{02} \le (l_{11}l_{21} + l_{11}l_{31} + l_{21}l_{31} - l_{11}l_{21}l_{31} + 2)\iota^2, \\
1 \le d_{i1} \le l_{i1}-1,\ i>0, \\
-\tfrac{d_{11}}{l_{11}} - \tfrac{d_{21}}{l_{21}} - \tfrac{d_{31}}{l_{31}} < d_{01} \le \iota \cdot \left(\tfrac{1}{l_{11}} + \tfrac{1}{l_{21}} + \tfrac{1}{l_{31}} - 1\right) - \tfrac{d_{11}}{l_{11}} - \tfrac{d_{21}}{l_{21}} - \tfrac{d_{31}}{l_{31}}, \\
l_{02} \cdot (-\iota - 2l_{02}) < d_{02} < d_{01} l_{02}, \\
-\tfrac{\iota + d_{02}}{l_{02}} - 2 < d_{12} \le 0, \\
-\tfrac{2\iota + 2d_{02}}{l_{02}} - 2d_{12} - 2 < d_{22} \le 0, \\
-\tfrac{2\iota + 2d_{02}}{l_{02}} - 2d_{12} - d_{22} < d_{32} \le 0.
\end{gathered}
$$

(iv)
$$
\begin{bmatrix}
-1 & -2 & 2 & 1 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 2 & 2 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & l_{31} & l_{32} \\
d_{01} & d_{02} & 1 & d_{12} & 1 & d_{22} & d_{31} & d_{32}
\end{bmatrix},
$$

$$
\begin{gathered}
2 \le l_{31}, l_{32} \le 4\iota^2, \\
1 \le d_{31} \le l_{31} - 1, \\
-\tfrac{d_{31}}{l_{31}} - 1 < d_{01} \le \tfrac{\iota - d_{31}}{l_{31}} - 1, \\
-\tfrac{2\iota}{l_{32}} - 4 < d_{02} < 2d_{01}, \\
-\tfrac{\iota}{l_{32}} - \tfrac{d_{02}}{2} - 2 < d_{12} \le 0, \\
-\tfrac{2\iota}{l_{32}} - d_{02} - 2d_{12} - 2 < d_{22} \le 0, \\
-\iota - \tfrac{l_{32}(d_{02} - 2d_{12} + d_{22})}{2} < d_{32} \le 0.
\end{gathered}
$$

(v)
$$
\begin{bmatrix}
-1 & -l_{02} & 2 & 1 & 0 & 0 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & 2 & 2 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & 2 \\
d_{01} & d_{02} & 1 & d_{12} & 1 & d_{22} & d_{31} & d_{32}
\end{bmatrix},
$$

$$
\begin{gathered}
2 \leq l_{02}, l_{31} \leq 4\iota^2, \\
1 \leq d_{31} \leq l_{31} - 1, \\
-\tfrac{d_{31}}{l_{31}} - 1 < d_{01} \leq \tfrac{\iota - d_{31}}{l_{31}} - 1, \\
-\iota - 2l_{02} < d_{02} < d_{01} l_{02}, \\
-\tfrac{\iota + d_{02}}{l_{02}} - 2 < d_{12} \leq 0, \\
-\tfrac{2\iota + 2d_{02}}{l_{02}} - 2d_{12} - 2 < d_{22} \leq 0, \\
\tfrac{2\iota + 2d_{02}}{l_{02}} - 2d_{12} - d_{22} < d_{32} \leq 0.
\end{gathered}
$$

(vi)
$$
\begin{bmatrix}
-1 & -2 & l_{11} & 1 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 2 & 2 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & 2 & l_{32} \\
d_{01} & d_{02} & d_{11} & d_{12} & 1 & d_{22} & 1 & d_{32}
\end{bmatrix},
$$

$$
\begin{gathered}
2 \leq l_{11}, l_{32} \leq 4\iota^2, \\
1 \leq d_{11} \leq l_{11} - 1, \\
-\tfrac{d_{11}}{l_{11}} - 1 < d_{01} \leq \tfrac{\iota - d_{11}}{l_{11}} - 1, \\
-\tfrac{\iota l_{02}}{l_{32}} - 2 < d_{02} < 2d_{01}, \\
-\tfrac{\iota}{l_{32}} - \tfrac{d_{02}}{2} - 2 < d_{12} \leq 0, \\
-\tfrac{2\iota}{l_{32}} - d_{02} - 2d_{12} - 2 < d_{22} \leq 0, \\
-\iota - \tfrac{l_{32}(d_{02} + 2d_{12} + d_{22})}{2} < d_{32} \leq 0.
\end{gathered}
$$

(vii)
$$
\begin{bmatrix}
-1 & -l_{02} & l_{11} & 1 & 0 & 0 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & 2 & 2 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & 0 & 0 & 2 & 2 \\
d_{01} & d_{02} & d_{11} & d_{12} & 1 & d_{22} & 1 & d_{32}
\end{bmatrix},
$$

$$
\begin{gathered}
2 \leq l_{02}, l_{11} \leq 4\iota^2, \\
1 \leq d_{11} \leq l_{11} - 1, \\
-\tfrac{d_{11}}{l_{11}} - 1 < d_{01} \leq \tfrac{\iota}{l_{11}} - \tfrac{d_{31}}{l_{31}} - 1, \\
-\iota - 2l_{02} < d_{02} < d_{01} l_{02}, \\
-\tfrac{\iota + d_{02}}{l_{02}} - 2 < d_{12} \leq 0, \\
-\tfrac{2\iota + 2d_{02}}{l_{02}} - 2d_{12} - 2 < d_{22} \leq 0, \\
-\tfrac{2\iota + 2d_{02}}{l_{02}} - 2d_{12} - d_{22} < d_{32} \leq 0.
\end{gathered}
$$

Proposition 4.2.21. *Consider case (iii)(a) from Proposition 4.2.2 with X having Gorenstein index ι. Then neither $x^+ \in X$ nor $x^- \in X$ are quasismooth and P' is, up to admissible operations, one of the following.*

(i)
$$
\begin{bmatrix}
-1 & -l_{02} & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & l_{21} & 1 & 0 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 \\
-1 & -l_{02} & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} \\
d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} & d_{41}
\end{bmatrix},
$$

$$
\begin{gathered}
(l_{21}, l_{31}, l_{41}) \in \{(z,3,2), (z,2,3), (3,z,2), (3,2,z), (2,z,3), (2,3,z) |\ z = 3,4,5\}, \\
2 \leq l_{02}, l_{12} \leq 5, \\
1 \leq d_{i1} \leq l_{i1} - 1,\ i = 2,3,4, \\
-\tfrac{d_{21}}{l_{21}} - \tfrac{d_{31}}{l_{31}} - \tfrac{d_{41}}{l_{41}} < d_{01} \leq \iota \cdot \left(\tfrac{1}{l_{21}} + \tfrac{1}{l_{31}} + \tfrac{1}{l_{41}} - 1\right) - \tfrac{d_{21}}{l_{21}} - \tfrac{d_{31}}{l_{31}} - \tfrac{d_{41}}{l_{41}}, \\
l_{02} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{41}} - 1\right) - \tfrac{d_{41}}{l_{41}} - 1\right) < d_{02} < d_{01} l_{02}, \\
l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{41}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{41}}{l_{41}}\right) < d_{12} \leq -1, \\
-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{41}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{12}}{l_{12}} - \tfrac{d_{41}}{l_{41}} < d_{22} \leq 0, \\
-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{41}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{12}}{l_{12}} - d_{22} - \tfrac{d_{41}}{l_{41}} < d_{32} \leq 0.
\end{gathered}
$$

(ii)
$$
\begin{bmatrix}
-1 & -2 & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & l_{21} & 1 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} & 1
\end{bmatrix},
$$

$$
\begin{gathered}
(l_{21}, l_{31}) \in \{(z,3),(3,z) |\ z = 3,4,5\}, \\
2 \le l_{12} \le (2l_{21} + 2l_{31} - l_{21}l_{31} + 2)\iota^2, \\
1 \le d_{i1} \le l_{i1} - 1,\ i = 2,3, \\
-\frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} - \frac{1}{2} < d_{01} \le \iota \cdot \left(\frac{1}{l_{21}} + \frac{1}{l_{31}} - \frac{1}{2}\right) - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} - \frac{1}{2}, \\
-\frac{2\iota}{l_{12}} - 3 < d_{02} < 2d_{01}, \\
-\iota - \frac{l_{12}(d_{02}+1)}{2} < d_{12} \le -1, \\
-\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02}+1}{2} < d_{22} \le 0, \\
-\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02}+1}{2} - d_{22} < d_{32} \le 0.
\end{gathered}
$$

(iii)
$$
\begin{bmatrix}
-1 & -2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} \\
d_{01} & d_{02} & 0 & d_{12} & 1 & d_{22} & 1 & d_{32} & d_{41}
\end{bmatrix},
$$

$$
\begin{gathered}
2 \le l_{41} \le 4\iota^2, \\
1 \le d_{41} \le l_{41} - 1, \\
\frac{d_{41}}{l_{41}} - 1 < d_{01} \le \frac{\iota - d_{41}}{l_{41}} - \frac{d_{41}}{l_{41}} - 1, \\
-\frac{2\iota + 2d_{41}}{l_{41}} - 2 < d_{02} < 2d_{01}, \\
-\frac{2\iota + 2d_{41}}{l_{41}} - d_{02} < d_{12} \le -1, \\
-\frac{\iota + d_{41}}{l_{41}} - \frac{d_{02}+d_{12}}{2} < d_{22} \le 0, \\
-\frac{\iota + d_{41}}{l_{41}} - \frac{d_{02}+d_{12}}{2} - d_{22} < d_{32} \le 0.
\end{gathered}
$$

(iv)
$$
\begin{bmatrix}
-1 & -2 & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
d_{01} & d_{02} & 0 & d_{12} & 1 & d_{22} & 1 & d_{32} & 1
\end{bmatrix},
$$

$$
\begin{gathered}
2 \le l_{12} \le 4\iota^2, \\
-\frac{3}{2} < d_{01} \le \frac{\iota - 3}{2}, \\
-\frac{2\iota}{l_{12}} - 3 < d_{02} < 2d_{01}, \\
-\iota - \frac{l_{12}(d_{02}+1)}{2} < d_{12} \le -1, \\
-\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02}+1}{2} < d_{22} \le 0, \\
-\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02}+1}{2} - d_{22} < d_{32} \le 0.
\end{gathered}
$$

(v)
$$
\begin{bmatrix}
-1 & -2 & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
d_{01} & d_{02} & 0 & d_{12} & 1 & d_{22} & d_{31} & d_{32} & 1
\end{bmatrix},
$$

$$
\begin{gathered}
2 \le l_{12}, l_{31} \le 4\iota^2, \\
1 \le d_{31} \le l_{31} - 1, \\
-\frac{d_{31}}{l_{31}} - 1 < d_{01} \le \frac{\iota - d_{31}}{l_{31}} - 1, \\
-\frac{2\iota}{l_{12}} - 3 < d_{02} < 2d_{01}, \\
-\iota - \frac{l_{12}(d_{02}+1)}{2} < d_{12} \le -1, \\
-\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02}+1}{2} < d_{22} \le 0, \\
-\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02}+1}{2} - d_{22} < d_{32} \le 0.
\end{gathered}
$$

(vi)
$$
\begin{bmatrix}
-1 & -l_{02} & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
-1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 \\
-1 & -l_{02} & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
d_{01} & d_{02} & 0 & d_{12} & 1 & d_{22} & d_{31} & d_{32} & 1
\end{bmatrix},
$$

$$
\begin{gathered}
2 \le l_{02}, l_{31} \le 4\iota^2,\\
1 \le d_{31} \le l_{31}-1,\\
-\frac{d_{31}}{l_{31}} - 1 < d_{01} \le \frac{\iota - d_{31}}{l_{31}} - 1,\\
-\iota - \frac{3l_{02}}{2} < d_{02} < d_{01}l_{02},\\
-\frac{2\iota+2d_{02}}{l_{02}} - 1 < d_{12} \le -1,\\
-\frac{\iota+d_{02}}{l_{02}} - \frac{d_{12}+1}{2} < d_{22} \le 0,\\
-\frac{\iota+d_{02}}{l_{02}} - \frac{d_{12}+1}{2} - d_{22} < d_{32} \le 0.
\end{gathered}
$$

Proposition 4.2.22. *Consider case (iii)(b) from Proposition 4.2.2 with X having Gorenstein index ι. Then neither $x^+ \in X$ nor $x^- \in X$ are quasismooth and P' is, up to admissible operations, one of the following.*

(i)
$$
\begin{bmatrix}
-1 & -l_{02} & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & -l_{02} & 0 & 0 & l_{21} & 1 & 0 & 0 & 0 & 0\\
-1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 & 0\\
-1 & -l_{02} & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & l_{42}\\
d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} & d_{41} & d_{42}
\end{bmatrix},
$$

$$
\begin{gathered}
(l_{21}, l_{31}, l_{41}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid\ z = 3,4,5\},\\
(l_{02}, l_{12}, l_{42}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid\ z = 3,4,5\},\\
1 \le d_{i1} \le l_{i1}-1,\ \ i = 2,3,4,\\
-\frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} - \frac{d_{41}}{l_{41}} < d_{01} \le \iota \cdot \left(\frac{1}{l_{21}} + \frac{1}{l_{31}} + \frac{1}{l_{41}} - 1\right) - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} - \frac{d_{41}}{l_{41}},\\
l_{02} \cdot \left(-\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{12}} + \frac{1}{l_{42}} - 1\right) - 2\right) < d_{02} < d_{01}l_{02},\\
l_{12} \cdot \left(-\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{12}} + \frac{1}{l_{42}} - 1\right) - \frac{d_{02}}{l_{02}} - 1\right) < d_{12} \le -1,\\
-\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{12}} + \frac{1}{l_{42}} - 1\right) - \frac{d_{02}}{l_{02}} - \frac{d_{12}}{l_{12}} - 1 < d_{22} \le 0,\\
-\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{12}} + \frac{1}{l_{42}} - 1\right) - \frac{d_{02}}{l_{02}} - \frac{d_{12}}{l_{12}} - d_{22} - 1 < d_{32} \le 0,\\
l_{42} \cdot \left(-\iota \cdot \left(\frac{1}{l_{02}} + \frac{1}{l_{12}} + \frac{1}{l_{42}} - 1\right) - \frac{d_{02}}{l_{02}} - \frac{d_{12}}{l_{12}} - d_{22} - d_{32}\right) < d_{42} \le 0.
\end{gathered}
$$

(ii)
$$
\begin{bmatrix}
-1 & -2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & -2 & 0 & 0 & l_{21} & 1 & 0 & 0 & 0 & 0\\
-1 & -2 & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 & 0\\
-1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & l_{42}\\
d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} & d_{41} & d_{42}
\end{bmatrix},
$$

$$
\begin{gathered}
(l_{21}, l_{31}, l_{41}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid\ z = 3,4,5\},\\
2 \le l_{42} \le (l_{21}l_{31} + l_{21}l_{41} + l_{31}l_{41} - l_{21}l_{31}l_{41} + 2)\iota^2,\\
1 \le d_{i1} \le l_{i1}-1,\ \ i = 2,3,4,\\
-\frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} - \frac{d_{41}}{l_{41}} < d_{01} \le \iota \cdot \left(\frac{1}{l_{21}} + \frac{1}{l_{31}} + \frac{1}{l_{41}} - 1\right) - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} - \frac{d_{41}}{l_{41}},\\
-\frac{2\iota+2d_{42}}{l_{42}} - 2 < d_{02} < 2d_{01},\\
-\frac{2\iota+2d_{42}}{l_{42}} - d_{02} < d_{12} \le -1,\\
-\frac{\iota+d_{42}}{l_{42}} - \frac{d_{02}+d_{12}}{2} < d_{22} \le 0,\\
-\frac{\iota+d_{42}}{l_{42}} - \frac{d_{02}+d_{12}}{2} - d_{22} < d_{32} \le 0,\\
-\iota - \frac{l_{42}(d_{02}+d_{12}+2d_{22}+2d_{32})}{2} < d_{42} \le 0.
\end{gathered}
$$

(iii)
$$
\begin{bmatrix}
-1 & -2 & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & -2 & 0 & 0 & l_{21} & 1 & 0 & 0 & 0 & 0\\
-1 & -2 & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 & 0\\
-1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & 2\\
d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} & d_{41} & d_{42}
\end{bmatrix},
$$

$$
\begin{gathered}
(l_{21}, l_{31}, l_{41}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid\ z = 3,4,5\},\\
2 \le l_{12} \le (l_{21}l_{31} + l_{21}l_{41} + l_{31}l_{41} - l_{21}l_{31}l_{41} + 2)\iota^2,\\
1 \le d_{i1} \le l_{i1}-1,\ \ i = 2,3,4,\\
-\frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} - \frac{d_{41}}{l_{41}} < d_{01} \le \iota \cdot \left(\frac{1}{l_{21}} + \frac{1}{l_{31}} + \frac{1}{l_{41}} - 1\right) - \frac{d_{21}}{l_{21}} - \frac{d_{31}}{l_{31}} - \frac{d_{41}}{l_{41}},\\
-\frac{2\iota}{l_{12}} - d_{42} - 2 < d_{02} < 2d_{01},\\
-\iota - \frac{l_{12}(d_{02}+d_{42})}{2} < d_{12} \le -1,\\
-\frac{\iota+d_{12}}{l_{12}} - \frac{d_{02}+d_{42}}{2} < d_{22} \le 0,\\
-\frac{\iota+d_{12}}{l_{12}} - \frac{d_{02}+d_{42}}{2} - d_{22} < d_{32} \le 0,\\
-\frac{2\iota+2d_{12}}{l_{12}} - d_{02} - 2d_{22} - 2d_{32} < d_{42} \le 0.
\end{gathered}
$$

(iv)
$$\begin{bmatrix} -1 & -2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & l_{42} \\ d_{01} & d_{02} & 0 & d_{12} & 1 & d_{22} & 1 & d_{32} & d_{41} & d_{42} \end{bmatrix},$$

$$\begin{gathered} 2 \leq l_{41}, l_{42} \leq 4\iota^2, \\ 1 \leq d_{41} \leq l_{41} - 1, \\ -\frac{d_{41}}{l_{41}} - 1 < d_{01} \leq \frac{\iota - d_{41}}{l_{41}} - 1, \\ 2 \cdot \left(-\iota \cdot \left(\frac{1}{l_{42}}\right) - \frac{d_{42}}{l_{42}} - 1\right) < d_{02} < 2d_{01}, \\ -\frac{2\iota + d_{42}}{l_{42}} - \frac{d_{02}}{2} < d_{12} \leq -1, \\ -\frac{\iota + d_{42}}{l_{42}} - \frac{d_{02} + d_{12}}{2} < d_{22} \leq 0, \\ -\frac{\iota + d_{42}}{l_{42}} - \frac{d_{02} + d_{12}}{2} - d_{22} < d_{32} \leq 0, \\ -\iota - \frac{l_{42}(d_{02} + d_{12} + 2d_{22} + 2d_{32})}{2} < d_{42} \leq 0. \end{gathered}$$

(v)
$$\begin{bmatrix} -1 & -2 & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & 2 \\ d_{01} & d_{02} & 0 & d_{12} & 1 & d_{22} & 1 & d_{32} & d_{41} & d_{42} \end{bmatrix},$$

$$\begin{gathered} 2 \leq l_{12}, l_{41} \leq 4\iota^2, \\ 1 \leq d_{41} \leq l_{41} - 1, \\ -\frac{d_{41}}{l_{41}} - 1 < d_{01} \leq \frac{\iota - d_{41}}{l_{41}} - 1, \\ -\frac{2\iota}{l_{12}} - d_{42} - 2 < d_{02} < 2d_{01}, \\ -\iota - \frac{l_{12}(d_{02} + d_{42})}{2} < d_{12} \leq -1, \\ -\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02} + d_{42}}{2} < d_{22} \leq 0, \\ -\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02} + d_{42}}{2} - d_{22} < d_{32} \leq 0, \\ -\frac{2\iota + 2d_{12}}{l_{12}} - d_{02} - 2d_{22} - 2d_{32} < d_{42} \leq 0. \end{gathered}$$

(vi)
$$\begin{bmatrix} -1 & -2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & l_{42} \\ d_{01} & d_{02} & 0 & d_{12} & 1 & d_{22} & d_{31} & d_{32} & 1 & d_{42} \end{bmatrix},$$

$$\begin{gathered} 2 \leq l_{31}, l_{42} \leq 4\iota^2, \\ 1 \leq d_{31} \leq l_{31} - 1, \\ -\frac{d_{31}}{l_{31}} - 1 < d_{01} \leq \frac{\iota - d_{31}}{l_{31}} - 1, \\ -\frac{2\iota + 2d_{42}}{l_{42}} - 2 < d_{02} < 2d_{01}, \\ -\frac{2\iota + 2d_{42}}{l_{42}} - d_{02} < d_{12} \leq -1, \\ -\frac{\iota + d_{42}}{l_{42}} - \frac{d_{02} + d_{12}}{2} < d_{22} \leq 0, \\ -\frac{\iota + d_{42}}{l_{42}} - \frac{d_{02} + d_{12}}{2} - d_{22} < d_{32} \leq 0, \\ -\iota - \frac{l_{42}(d_{02} + d_{12} + 2d_{22} + 2d_{32})}{2} < d_{42} \leq 0. \end{gathered}$$

(vii)
$$\begin{bmatrix} -1 & -2 & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ d_{01} & d_{02} & 0 & d_{12} & 1 & d_{22} & d_{31} & d_{32} & 1 & d_{42} \end{bmatrix},$$

$$\begin{gathered} 2 \leq l_{12}, l_{31} \leq 4\iota^2, \\ 1 \leq d_{31} \leq l_{31} - 1, \\ -\frac{d_{31}}{l_{31}} - 1 < d_{01} \leq \frac{\iota - d_{31}}{l_{31}} - 1, \\ -\frac{2\iota}{l_{12}} - d_{42} - 2 < d_{02} < 2d_{01}, \\ -\iota - \frac{l_{12}(d_{02} + d_{42})}{2} < d_{12} \leq -1, \\ -\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02} + d_{42}}{2} < d_{22} \leq 0, \\ -\frac{\iota + d_{12}}{l_{12}} - \frac{d_{02} + d_{42}}{2} - d_{22} < d_{32} \leq 0, \\ -\frac{2\iota + 2d_{12}}{l_{12}} - d_{02} - 2d_{22} - 2d_{32} < d_{42} \leq 0. \end{gathered}$$

Proposition 4.2.23. *Consider case (iv)(a) from Proposition 4.2.2 with X having Gorenstein index ι. Then neither $x^+ \in X$ nor $x^- \in X$ are quasismooth and P' is, up to admissible operations, one of the following.*

(i)
$$\begin{bmatrix} -1 & -l_{02} & 1 & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 1 & l_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 & 0 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & 1 & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{51} & 1 \\ d_{01} & d_{02} & 0 & d_{12} & 0 & d_{22} & d_{31} & d_{32} & d_{41} & d_{42} & d_{51} & d_{52} \end{bmatrix},$$

$$\begin{gathered} (l_{31}, l_{41}, l_{51}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z = 3,4,5\}, \\ (l_{31}, l_{41}, l_{51}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z = 3,4,5\}, \\ 1 \le d_{i1} \le l_{i1} - 1, \ i = 3,4,5, \\ -\tfrac{d_{31}}{l_{31}} - \tfrac{d_{41}}{l_{41}} - \tfrac{d_{51}}{l_{51}} < d_{01} \le \iota \cdot \left(\tfrac{1}{l_{31}} + \tfrac{1}{l_{41}} + \tfrac{1}{l_{51}} - 1\right) - \tfrac{d_{31}}{l_{31}} - \tfrac{d_{41}}{l_{41}} - \tfrac{d_{51}}{l_{51}}, \\ -l_{02}\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) < d_{02} < d_{01} l_{02}, \\ l_{12} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{d_{02}}{l_{02}}\right) < d_{12} \le -1, \\ l_{22} \cdot \left(-\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{12}}{l_{12}}\right) < d_{22} \le -1, \\ -\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{12}}{l_{12}} - \tfrac{d_{22}}{l_{22}} < d_{32} \le 0, \\ -\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{12}}{l_{12}} - \tfrac{d_{22}}{l_{22}} - d_{32} < d_{42} \le 0, \\ -\iota \cdot \left(\tfrac{1}{l_{02}} + \tfrac{1}{l_{12}} + \tfrac{1}{l_{22}} - 1\right) - \tfrac{d_{02}}{l_{02}} - \tfrac{d_{12}}{l_{12}} - \tfrac{d_{22}}{l_{22}} - d_{32} - d_{42} < d_{52} \le 0. \end{gathered}$$

(ii)
$$\begin{bmatrix} -1 & -2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & l_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & l_{31} & 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{51} & 1 \\ d_{01} & d_{02} & 0 & d_{12} & 0 & d_{22} & d_{31} & d_{32} & d_{41} & d_{42} & d_{51} & d_{52} \end{bmatrix},$$

$$\begin{gathered} (l_{31}, l_{41}, l_{51}) \in \{(z,3,2),(z,2,3),(3,z,2),(3,2,z),(2,z,3),(2,3,z) \mid z = 3,4,5\}, \\ 2 \le l_{22} \le (l_{31}l_{41} + l_{31}l_{51} + l_{41}l_{51} - l_{31}l_{41}l_{51} + 2)\iota^2, \\ 1 \le d_{i1} \le l_{i1} - 1, \ i = 3,4,5, \\ -\tfrac{d_{31}}{l_{31}} - \tfrac{d_{41}}{l_{41}} - \tfrac{d_{51}}{l_{51}} < d_{01} \le \iota \cdot \left(\tfrac{1}{l_{31}} + \tfrac{1}{l_{41}} + \tfrac{1}{l_{51}} - 1\right) - \tfrac{d_{31}}{l_{31}} - \tfrac{d_{41}}{l_{41}} - \tfrac{d_{51}}{l_{51}}, \\ -\tfrac{2\iota}{l_{22}} < d_{02} < 2d_{01}, \\ -\tfrac{2\iota}{l_{22}} - d_{02} < d_{12} \le -1, \\ -\iota - \tfrac{l_{22}(d_{02}+d_{12})}{2} < d_{22} \le -1, \\ -\tfrac{\iota+d_{22}}{l_{22}} - \tfrac{d_{02}+d_{12}}{2} < d_{32} \le 0, \\ -\tfrac{\iota+d_{22}}{l_{22}} - \tfrac{d_{02}+d_{12}}{2} - d_{32} < d_{42} \le 0, \\ -\tfrac{\iota+d_{22}}{l_{22}} - \tfrac{d_{02}+d_{12}}{2} - d_{32} - d_{42} < d_{52} \le 0. \end{gathered}$$

(iii)
$$\begin{bmatrix} -1 & -2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & l_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{51} & 1 \\ d_{01} & d_{02} & 0 & d_{12} & 0 & d_{22} & 1 & d_{32} & 1 & d_{42} & d_{51} & d_{52} \end{bmatrix},$$

$$\begin{gathered} 2 \le l_{22}, l_{51} \le 4\iota^2, \\ 1 \le d_{51} \le l_{51} - 1, \\ -\tfrac{d_{51}}{l_{51}} - 1 < d_{01} \le \tfrac{\iota - d_{51}}{l_{51}} - 1, \\ -\tfrac{2\iota}{l_{22}} < d_{02} < 2d_{01}, \\ -\tfrac{2\iota}{l_{22}} - d_{02} < d_{12} \le -1, \\ -\iota - \tfrac{l_{22}(d_{02}+d_{12})}{2} < d_{22} \le -1, \\ -\tfrac{\iota+d_{22}}{l_{22}} - \tfrac{d_{02}+d_{12}}{2} < d_{32} \le 0, \\ -\tfrac{\iota+d_{22}}{l_{22}} - \tfrac{d_{02}+d_{12}}{2} - d_{32} < d_{42} \le 0, \\ -\tfrac{\iota+d_{22}}{l_{22}} - \tfrac{d_{02}+d_{12}}{2} - d_{32} - d_{42} < d_{52} \le 0. \end{gathered}$$

Now we will take a look at the less complicated case of X having a parabolic fixed point curve. The following Proposition is obvious by Remark 3.4.11.

Proposition 4.2.24. *Consider a non-toric rational log terminal projective $\mathbb{K}^*$-surface X with a parabolic fixed point curve. Then $X \cong X(A,P)$ with adjusted P of type (e-p). Removing all columns v_{ij} with $j > 1$ and collapsing all arms with $l_{i1} = 1$ we arrive at one of the following matrices P'.*

$$P' = \begin{bmatrix} -l_{01} & l_{11} & 0 \\ d_{01} & d_{11} & -1 \end{bmatrix}, \qquad P' = \begin{bmatrix} -l_{01} & l_{11} & 0 & 0 \\ -l_{01} & 0 & l_{21} & 0 \\ d_{01} & d_{11} & d_{21} & -1 \end{bmatrix}.$$

If $x^+ \in X$ is not quasismooth we are in the second case. If additionally X has Gorenstein index ι, then P' is, up to admissible operations, one of the following.

(i) $$\begin{bmatrix} -l_{01} & 2 & 0 & 0 \\ -l_{01} & 0 & 2 & 0 \\ d_{01} & 1 & 1 & -1 \end{bmatrix}, \qquad \begin{array}{c} 2 \le l_{01} \le 2\iota^2, \\ 1 - l_{01} \le d_{01} \le \iota - l_{01}. \end{array}$$

(ii) $$\begin{bmatrix} -z & 3 & 0 & 0 \\ -z & 0 & 2 & 0 \\ d_{01} & d_{11} & 1 & -1 \end{bmatrix}, \qquad \begin{array}{c} z = 3,4,5, \ d_{11} = 1,2, \\ 1 - \frac{2d_{11}}{3} < d_{01} \le \iota \cdot \left(\frac{2}{z} + \frac{5}{3}\right) - \frac{2d_{11}}{3} - 1. \end{array}$$

For classifications we have to be able to reduce lists of P-matrices to lists of pairwise non-equivalent ones. This is achieved by using a normal form defined as follows.

Construction 4.2.25. Consider an adjusted P and denote by $0 \le \iota_l \le r$ the positions where the numbers n_i decrease. That means that, setting $\kappa_0 = 0$ and $\kappa_l = \iota_{l-1} + 1$ for $l \ge 1$, we have

$$n_0 = n_{\kappa_0} = \ldots = n_{\iota_0} \ > \ n_{\kappa_1} = \ldots = n_{\iota_1} \ > \ \ldots \ > \ n_{\kappa_q} = \ldots = n_{\iota_q} = n_r.$$

With this notation, let us list the slopes m_{ij} of P according to the scheme

$$\begin{array}{lclclcl} \mu_{01} & := & (m_{\kappa_0 1}, \ldots, m_{\iota_0 1}) & \ldots & \mu_{q1} & := & (m_{\kappa_q 1}, \ldots, m_{\iota_q 1}) \\ & \ldots & & & & \ldots & \\ \mu_{0n_{\kappa_0}} & := & (m_{\kappa_0 n_{\kappa_0}}, \ldots, m_{\iota_0 n_{\kappa_0}}) & \ldots & \mu_{qn_{\kappa_q}} & := & (m_{\kappa_q n_{\kappa_q}}, \ldots, m_{\iota_q n_{\kappa_q}}). \end{array}$$

Observe that the numbers $\kappa_0, \ldots, \kappa_q$ only depend on the equivalence class of P. The *slope vector* of an adjusted P is the concatenated vector

$$S(P) \ := \ (s(P,0), \ldots, s(P,q)), \qquad s(P,i) \ := \ (\mu_{i1}, \ldots, \mu_{in_{\kappa_i}}).$$

Definition 4.2.26. A defining matrix P is in *normal form* if it is adjusted and its slope vector $S(P)$ is lexicographically maximal among the slope vectors $S(P')$ of all adjusted P' equivalent to P.

Proposition 4.2.27. *Each pair (A,P) of defining matrices is equivalent to some pair $(\tilde{A}, \tilde{P})$ of defining matrices, where $\tilde{P}$ is in normal form and uniquely determined by P.*

Proof. Let P be a defining matrix. Using admissible operations of type (iv) in Definition 3.2.10 we can order the arms of P according to length. Admissible operations of type (iii) make sure that each arm is slope-ordered. Moreover, admissible operations of type (i) lead to P being adapted to the source. If $m \ge 1$ we can use admissible operations of type (ii) and (v) to obtain $d_1 = 1$ if necessary. This shows that P can be transformed into an adjusted matrix via admissible operations. Furthermore, choosing a lexicographically maximal slope vector among the adjusted matrices equivalent to P gives a matrix in normal form.

To show that it is unique, consider matrices P_1 and P_2 that are in normal form and equivalent to P. Note that the number of arms of P_1 and P_2 and their respective lengths are equal. Since the matrices are in normal form they are both adjusted and their slope vectors coincide. Therefore $m^1_{ij} = m^2_{ij}$ for all $0 \leq i \leq r$ and $1 \leq j \leq n_i$. The columns of P_1 and P_2 are primitive, hence this equality of fractions implies equality of the integers $l^1_{ij} = l^2_{ij}$ and $d^1_{ij} = d^2_{ij}$. So $P_1 = P_2$. $\square$

The computation of the normal form can be directly implemented. In order to obtain all non-toric log del Pezzo $\mathbb{K}^*$-surfaces without quasismooth elliptic fixed points of Gorenstein index ι, we have to build up the classified matrices P'. This is done by using the following terminology.

Construction 4.2.28. Consider defining data (A, P) of a projective $\mathbb{K}^*$-surface as in Construction 3.2.2.

(i) A *redundant extension* of (A, P) is the defining data (A', P'), where

$$A' = [A, a_{r+1}], \qquad P' = \begin{bmatrix} L & 0 & 0 \\ 0 & 1 & 0 \\ d & 0 & d' \end{bmatrix}.$$

We will also call any defining data equivalent to (A', P') a redundant extension of (A, P).

(ii) A *proper extension* of (A, P) is the defining data (A', P'), where $A = A'$ and P' arises from P via inserting either a column v_{in_i+1} into the i-th arm or inserting a column of type $v^{\pm}$ at the end of P.

Remark 4.2.29. Consider a projective $\mathbb{K}^*$-surface X arising from defining data (A, P).

(i) Every redundant extension (A', P') of (A, P) defines a $\mathbb{K}^*$-surface X' isomorphic to X.

(ii) For every proper extension (A, P') of (A, P), the associated $\mathbb{K}^*$-surface X' comes with a non-trivial contraction $X' \to X$.

Remark 4.2.30. As mentioned, the matrices P' from Proposition 4.2.2 don't necessarily fulfill $m^-_{P'} < 0$. By abuse of notation we will still talk about redundant and proper extensions in this case. Then, every log del Pezzo $\mathbb{K}^*$-surface $X = X(A, P)$ can be obtained by using a series of redundant and proper extensions on a matrix from Proposition 4.2.2 or 4.2.24.

We are now ready to formulate two algorithms that put together a way to explicitly classifiy all non-toric log del Pezzo $\mathbb{K}^*$-surfaces of fixed Gorenstein index ι and without quasismooth elliptic fixed points.

Algorithm 4.2.31. Let $\iota \in \mathbb{Z}_{\geq 1}$. The input set $\mathfrak{S}_0$ consists of all matrices P' from Propositions 4.2.5 to 4.2.23 having primitive columns. Set $\mathfrak{S} := \mathfrak{S}_1 := \emptyset$. For each element of $\mathfrak{S}_0$ do the following steps.

- For each i with $n_i = 2$ test the polygon

$$\operatorname{conv}((0, \iota), (0, 0), (l_{i1}, d_{i1}), (l_{i2}, d_{i2}))$$

for k-hollowness. If all tests are positive, proceed to the next step.

- For each $0 \le s \le 4\iota - r' - 1$ do s redundant and subsequent proper extensions with $l_{i2} = 1$ in the corresponding arm and using the bounds from Lemma 4.2.4 for d_{i2}. If the resulting matrix satisfies $m^- < 0$, ι^- divides ι and for each i the polygon
$$\operatorname{conv}\Big((0, d^+), (0, d^-), (l_{i1}, d_{i1}), \ldots, (l_{in_i}, d_{in_i})\Big)$$
is k-hollow, add it to $\mathfrak{S}_1$.
- For each element P' of $\mathfrak{S}_1$ do:
  - For each arm with $n_i \ge 2$ successively add columns between the second to last and last one starting from the left and continuing descendingly by slopes. In each step, check if the "upper" new hyperbolic fixed point's Gorenstein index divides ι, see Proposition 3.4.9. Bounds for this process are given by Minkowski's Theorem and convexity of the arms. If the above is fulfilled and additionally the Gorenstein index of the "lowest" hyperbolic fixed point in that arm divides ι, add the corresponding matrix to $\mathfrak{S}$.
  - Repeat the previous step with each matrix of $\mathfrak{S}$ that was added there.
- Test each element of $\mathfrak{S}$ for the del Pezzo property using Proposition 4.1.3 and delete those from the list who fail.
- Bring the matrices of $\mathfrak{S}$ into normal form and, in this way, remove all entries of $\mathfrak{S}$ that define the same surface.

The output set $\mathfrak{S}$ consists of defining P-matrices of type (e-e) that deliver all non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index ι with two non-quasismooth elliptic fixed points.

Algorithm 4.2.32. Let $\iota \in \mathbb{Z}_{\ge 1}$. The input set $\mathfrak{S}_0$ consists of all matrices P' from (i) and (ii) in Proposition 4.2.24 having primitive columns.

- For each $0 \le s \le 4\iota - r' - 1$ do s redundant and subsequent proper extensions with $l_{i2} = 1$ in the corresponding arm and using the bounds from Lemma 4.2.4 for d_{i2}. If all ι_i^- in the resulting matrix divide ι and for each i the polygon
$$\operatorname{conv}\Big((0, d^+), (0, -1), (l_{i1}, d_{i1}), \ldots, (l_{in_i}, d_{in_i})\Big)$$
is k-hollow, add it to $\mathfrak{S}_1$.
- For each element P' of $\mathfrak{S}_1$ do:
  - For each arm successively add columns at the end (while maintaining slope-orderedness). In each step, check if the "upper" new hyperbolic fixed point's Gorenstein index divides ι, see Proposition 3.4.9. Bounds for this process are given by Minkowski's Theorem and convexity of the arms. If the above is fulfilled and additionally the Gorenstein index of the "lowest" hyperbolic fixed point in that arm divides ι, add the corresponding matrix to $\mathfrak{S}$.
  - Repeat the previous step with each matrix of $\mathfrak{S}$ that was just added.
- Test each element of $\mathfrak{S}$ for the del Pezzo property using Proposition 4.1.3 and delete those from the list who fail.

- Bring the matrices of $\mathfrak{S}$ into normal form and, in this way, remove all entries of $\mathfrak{S}$ that define the same surface.

The output set $\mathfrak{S}$ consists of defining P-matrices of type (e-p) that deliver all non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index ι with a non-quasismooth elliptic fixed point and a parabolic fixed point.

4.3. Log del Pezzo $\mathbb{K}^*$-surfaces of Picard number 1

We consider non-toric log del Pezzo $\mathbb{K}^*$-surfaces X of Picard number 1. We present a classification strategy for given Gorenstein index and provide results for Gorenstein indices up to 200. These have been published in [**24**].

A first step is to take a closer look at the quasismooth elliptic fixed points of given local Gorenstein index. These are toric singularities and thus Proposition 2.2.8 gives a complete picture in terms of two-dimensional lattice cones. The following description fits directly into the setting of $\mathbb{K}^*$-surfaces $X(A,P)$ and will be used frequently throughout the subsequent classifications.

Proposition 4.3.1. *Let X be a quasismooth affine $\mathbb{K}^*$-surface of Gorenstein index ι having an elliptic fixed point $x \in X$. Then $X \cong X(A,P)$ with P of the form*

$$\begin{bmatrix} -1 & l_1 & 0 \\ -1 & 0 & ab - l_1 \\ 0 & \frac{\iota - cl_1}{b} & ac + \frac{\iota - cl_1}{b} \end{bmatrix}, \qquad \begin{array}{c} 1 \le a, \quad 1 \le b, \\ l_1{+}1 \le ab \le 2l_1, \\ -b{+}\frac{\iota+b}{l_1} \le c \le \frac{\iota}{l_1}, \end{array}$$

$$\begin{bmatrix} -1 & l_1 & 0 \\ -1 & 0 & ab - l_1 \\ 0 & \frac{\iota - cl_1}{b} & ac + \frac{\iota - cl_1}{b} \end{bmatrix}, \qquad \begin{array}{c} -1 \ge a, \quad -1 \ge b, \\ l_1{+}1 \le ab \le 2l_1, \\ \frac{\iota}{l_1} \le c \le -b{+}\frac{\iota+b}{l_1} \end{array}$$

with primitive vectors of $\mathbb{Z}^3$ as columns and $\gcd(b,c) = 1$*. Moreover, for $a \ge 1$, we have $x = x^+ \in X$ and for $a \le -1$, we have $x = x^- \in X$.*

Proof. We may assume $X = X(A,P)$. As X is affine, $n_0 = \ldots = n_r = 1$ holds and quasismoothness allows $l_{i1} > 1$ at most twice. Thus, we have

$$P = \begin{bmatrix} -1 & l_1 & 0 \\ -1 & 0 & l_2 \\ 0 & d_1 & d_2 \end{bmatrix}, \qquad \begin{array}{c} 1 \le l_2 \le l_1, \\ 0 \le d_1 < l_1 \end{array}$$

by applying suitable admissible operations and removing redundant columns. Then the linear form u from Proposition 3.4.9 is given by

$$u = \left(\frac{d_2 - d_1}{l_2 d_1 + l_1 d_2}, \frac{d_1 - d_2}{l_2 d_1 + l_1 d_2}, \frac{l_1 + l_2}{l_2 d_1 + l_1 d_2} \right), \qquad l_2 d_1 + l_1 d_2 = \det(P).$$

We have $\det(P) = a\iota$ and $\iota u \in \mathbb{Z}^3$ being primitive gives $\iota u_3 = b$ and $\iota u_1 = c$ with suitable integers a, b, c. This allows us to express l_2, d_1, d_2 as claimed. □

Using the description of the defining matrix P for the affine case, we can systematically build up the defining matrices P in the projective case. Here is the result for Picard number 1.

Proposition 4.3.2. *Let X be a non-toric rational quasismooth projective $\mathbb{K}^*$-surface with $\rho(X) = 1$. Then X is log del Pezzo and we have $X \cong X(A,P)$ with*

$$P \;=\; \left[\begin{array}{cccc} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & ab-l_1 \\ 0 & \frac{q\iota^- - a\iota^+}{l_1(ab-l_1)} & \frac{\iota^+ - cl_1}{b} & ac + \frac{\iota^+ - cl_1}{b} \end{array}\right],$$

where ι^+, ι^- are the local Gorenstein indices of $x^+, x^- \in X$. The columns of P are primitive vectors of $\mathbb{Z}^3$, we have $\gcd(b,c) = 1$ and $q|ab$. Moreover,

$$1 \;\le\; a, \quad 1 \;\le\; b, \quad l_1 + 2 \;\le\; ab \;\le\; 2l_1, \quad -b + \frac{\iota + b}{l_1} \;\le\; c \;\le\; b + \frac{\iota - b}{l_1}.$$

Finally, in this setting, the (unique) hyperbolic fixed point $[0,0,1,1]$ is Gorenstein and the Gorenstein index of X is $\iota_X = \operatorname{lcm}(\iota^+, \iota^-)$.

Proof. We may assume that P is irredundant and slope-ordered with $n_0 \ge \ldots \ge n_r$. We show that the number m of parabolic fixed points curves is zero, meaning that there are $x^+, x^- \in X$, and that $r = 2$ holds. Recall that $r-1$ is the number of defining equations and that we have

$$n_0 + \ldots + n_r + m - (r-1) \;=\; \dim(X) + \rho(X) \;=\; 3.$$

This immediately excludes $m = 2$. Moreover, $m = 1$ would force $n_0 = \ldots = n_r = 1$. By quasismoothness, $l_{i1} \ne 1$ holds at most twice, hence irredundance of P implies $r = 1$. This is a contradiction to X being non-toric. Thus, $m = 0$. So we have to consider the case that $n_0 = 2$ and $n_1 = \ldots = n_r = 1$. Quasismoothness and irredundance give $r = 2$. Proposition 4.3.1 allows to write

$$P \;=\; \left[\begin{array}{cccc} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & ab-l_1 \\ 0 & d_{02} & \frac{\iota^+ - cl_1}{b} & ac + \frac{\iota^+ - cl_1}{b} \end{array}\right],$$

where ι^+ is the local Gorenstein index of the elliptic fixed point x^+ sitting in the affine open subest $X^+ \subseteq X$ defined by $\sigma^+ = \operatorname{cone}(v_{01}, v_{11}, v_{21})$. Now consider $X^- \subseteq X$ defined by $\sigma^- = \operatorname{cone}(v_{02}, v_{11}, v_{21})$. Then

$$\det(v_{02}, v_{11}, v_{21}) = q\iota^-$$

holds with local Gorenstein index ι^- of $x^- \in X^-$ and an integer $q \in \mathbb{Z}$. This allows us to express the entry d_{02} as in the assertion.

Now the displayed bounds on the entries stem from Proposition 4.3.1. In order to see $\gcd(b,c) = 1$ and $q \mid ab$, compute the linear form u^- associated with x^- as in Proposition 3.4.9. The fact that $\iota^- u^-$ is a primitive vector in $\mathbb{Z}^3$ gives the desired properties.

Finally, X is del Pezzo because $-\mathcal{K}_X^0 = D_X^{11} + D_X^{21}$ is obviously ample. Moreover, the (unique) hyperbolic fixed point is of local Gorenstein index 1 according to Proposition 3.4.8. $\square$

Proposition 4.3.2 bounds in particular all entries of P in terms of the Gorenstein index ι of X and the entry l_1. Thus, the task is to find a sufficiently tight bound for l_1. The key is the following relation to the collection of possible partitions of the unit fraction $1/\iota$ into sums of four unit fractions.

Proposition 4.3.3. *Consider $X = X(A,P)$ with a defining matrix P as in Proposition 4.3.2. Then X is of Gorenstein index $\iota = \mathrm{lcm}(\iota^+, \iota^-)$ and there are integers $1 \le a_1 \le a_2$ such that with l_1 and $l_2 := ab - l_1$ we have*

$$\frac{1}{\iota} \;=\; \frac{1}{a_1 l_1} + \frac{1}{a_2 l_1} + \frac{1}{a_1 l_2} + \frac{1}{a_2 l_2}.$$

Proof. Let $(w_{01}, w_{02}, w_{11}, w_{21})$ be the $\mathbb{Z}$-parts of the classes of the divisors D_X^{ij}. Then $w_{11} + w_{21}$ is the $\mathbb{Z}$-part of the class of $-\mathcal{K}_X^0$. Moreover we have the relations

$$a_1 w_{01} = \iota w_{11} + \iota w_{21}, \qquad a_2 w_{02} = \iota w_{11} + \iota w_{21}$$

reflecting the fact that $-\mathcal{K}_X^0$ is Cartier near x^-, x^+ and hence on the affine open subsets X^+, X^- defined by σ^+, σ^-. Thus, we have

$$\begin{pmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ -a_1 & 0 & \iota & \iota \\ 0 & -a_2 & \iota & \iota \end{pmatrix} \cdot \begin{pmatrix} w_{01} \\ w_{02} \\ w_{11} \\ w_{21} \end{pmatrix} \;=\; 0,$$

where the first two rows of the matrix stem from P. Now the identity displayed in the assertion just means that the above matrix has vanishing determinant. $\square$

Remark 4.3.4. Given any rational number $0 < p < 1$ and positive integer k, the number of possible partitions of p into a sum of k unit fractions is finite. The partitions fitting into the particular shape of Proposition 4.3.3 form a considerably smaller collection than the general ones. Also the fact that any three entries of the weight vector $(w_{01}, w_{02}, w_{11}, w_{21})$ from the proof of Proposition 4.3.3 are coprime helps to reduce the amount of partitions that need to be considered.

The next steps are to consider rational projective $\mathbb{K}^*$-surfaces X with a quasismooth elliptic fixed point $x^+ \in X$ and a non-quasismooth but log terminal elliptic fixed point $x^- \in X$. We begin with x^- being of type D_n.

Proposition 4.3.5. *Let X be a non-toric rational projective $\mathbb{K}^*$-surface of Picard number 1 with $x^+ \in X$ quasismooth and $x^- \in X$ of type D_n. Then X is log del Pezzo and we have $X \cong X(A,P)$, specified as follows.*

(i) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -\frac{b\iota^- + 4c\iota_h}{a\iota^+} & 2 & 0 \\ -1 & -\frac{b\iota^- + 4c\iota_h}{a\iota^+} & 0 & 2 \\ 0 & -c\iota_h & \frac{a}{2}\iota^+ - 1 & 1 \end{bmatrix}, \qquad \begin{array}{c} a = 1,2,4, \;\; b = -2,-4, \\ -\frac{b}{4}\frac{\iota^-}{\iota_h} < c \le a\iota^+ - b\iota^- \end{array}$$

with $\iota^+ u^+, \iota^- u^-$ and $\iota_h u_h$ given by

$$\left(\tfrac{4 - a\iota^+}{2a}, \tfrac{a\iota^+ - 4}{2a}, \tfrac{4}{a}\right), \quad \left(\tfrac{b\iota^- - 2a\iota^+ + 4}{2b}, \tfrac{b\iota^- - 4}{2b}, \tfrac{4}{b}\right), \quad \left(\iota_h, \tfrac{-a\iota^+ + b\iota^- + 4c\iota_h}{ac\iota^+}\right).$$

(ii) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -2 & \pm\frac{b\iota^- - 2a\iota^+}{2\iota_h} & 0 \\ -1 & -2 & 0 & 2 \\ 0 & \pm\iota_h & \frac{a}{2}\iota^+ \pm \frac{2a\iota^+ - b\iota^-}{4\iota_h} & 1 \end{bmatrix}, \qquad \begin{array}{c} b = -2,-4, \\ 1 < a \le 2\iota_h - \frac{b}{2}\iota^- \end{array}$$

with $\iota^+ u^+, \iota^- u^-$ and $\iota_h u_h$ given by

$$\left(\pm\frac{(4-2a\iota^+)\iota_h-2a\iota^++b\iota^-}{4a\iota_h}, \mp\frac{(4-2a\iota^+)\iota_h-2a\iota^++b\iota^-}{4a\iota_h}, \frac{\pm(b\iota^--2a\iota^+)+4\iota_h}{2a\iota_h}\right),$$

$$\left(\frac{\pm 2\iota_h+2}{b}, \frac{\iota^--4}{b}, \frac{4}{b}\right), \qquad\qquad (-\iota_h, -1).$$

Here, ι^+, ι^- and ι_h are the local Gorenstein indices of x^+, x^- and the hyperbolic fixed point. The vectors $\iota^+ u^+, \iota^- u^- \in \mathbb{Z}^3$ and $\iota_h u_h \in \mathbb{Z}^2$ are primitive and we have

$$\iota_X \;=\; \operatorname{lcm}(\iota^+, \iota^-, \iota_h).$$

Proof. We may assume that P is irredundant. Proposition 3.4.13 and $\rho(X) = 1$ ensure that we can bring the defining matrix P via suitable admissible operations into one of the shapes

$$\begin{bmatrix} -1 & -l_{02} & 2 & 0 \\ -1 & -l_{02} & 0 & 2 \\ 0 & d_{02} & d_{11} & 1 \end{bmatrix}, \qquad\qquad \begin{bmatrix} -1 & -2 & l_{11} & 0 \\ -1 & -2 & 0 & 2 \\ 0 & d_{02} & d_{11} & 1 \end{bmatrix}.$$

We show that $X = X(A,P)$ is del Pezzo. According to the two possible shapes, K-homogeneity of the defining relation of $X \subseteq Z$ gives

$$w_{21} \;=\; w_{11}, \qquad\qquad w_{21} \;=\; \frac{1}{2}w_{01} + w_{02}.$$

Thus, again according to the two shapes, we see that the anticanonical class $-w_X = w_{01}+w_{02}+w_{11}+w_{21}-2w_{21}$ in $K_{\mathbb{Q}} \cong \mathbb{Q}$ is positive and hence ample:

$$-w_X \;=\; w_{01}+w_{02} \;>\; 0, \qquad -w_X \;=\; \frac{1}{2}w_{01}+w_{02} \;>\; 0.$$

Propositions 3.4.8 and 3.4.9 allow us to express l_{02}, d_{11}, d_{02} in the first shape and l_{11}, d_{11}, d_{02} in the second one in terms of integers a, b, c by resolving the equations

$$\det(v_{01}, v_{11}, v_{21}) = a\iota^+, \quad \det(v_{02}, v_{11}, v_{21}) = b\iota^-, \quad \det\begin{bmatrix} -1 & -l_{02} \\ 0 & d_{02} \end{bmatrix} = c\iota_h.$$

Then $m^+ > 0$ implies $a > 0$ and $m^- < 0$ implies $b < 0$. The linear forms u^+, u^-, u_h as above locally represent $-\mathcal{K}_X$. Their primitivity and the fact that x^- has canonical multiplicity 1 or 2 yield the claimed conditions on a, b, c. $\square$

Using the same pattern of arguments, we treat the cases of a quasismooth elliptic fixed point x^+ coming together with an elliptic fixed point x^- of type E_6, E_7 or E_8. We restrict ourselves to listing the results.

Proposition 4.3.6. *Let X be a non-toric rational projective $\mathbb{K}^*$-surface of Picard number one with $x^+ \in X$ quasismooth and $x^- \in X$ of type E_6. Then X is log del Pezzo and we have $X \cong X(A,P)$, specified as follows.*

(i) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ 0 & \frac{b\iota^- - 3a\iota^+}{6} & \frac{a\iota^+ - 3}{2} & 1 \end{bmatrix}, \qquad \begin{array}{c} 6c\iota_h \;=\; 3a\iota^+ - b\iota^-, \\ c = 1,2, \;\; a = 1,5, \\ b = -1,-3 \end{array}$$

with ι^+u^+, ι^-u^- *and* $\iota_h u_h$ *given by*

$$\left(\frac{5-a\iota^+}{2a}, \frac{a\iota^+-5}{2a}, \frac{5}{a}\right), \quad \left(\frac{2b\iota^- -3a\iota^+ +9}{6b}, \frac{b\iota^- -3}{2b}, \frac{3}{b}\right), \quad \left(\frac{b\iota^- -3a\iota^+}{6c}, \frac{2}{c}\right).$$

(ii) *The defining matrix* P *is of the shape*

$$\begin{bmatrix} -1 & -2 & 3 & 0 \\ -1 & -2 & 0 & 3 \\ 0 & \frac{b\iota^- -2a\iota^+}{9} & \frac{a\iota^+ -3}{3} & 1 \end{bmatrix}, \qquad \begin{gathered} 2a\iota^+ - b\iota^- = 9\iota_h, \\ a = 1,2,3,6, \\ b = -1,-3 \end{gathered}$$

with ι^+u^+, ι^-u^- *and* $\iota_h u_h$ *given by*

$$\left(\frac{6-a\iota^+}{3a}, \frac{a\iota^+-6}{3a}, \frac{6}{a}\right), \quad \left(\frac{b\iota^- -a\iota^+ +3}{3b}, \frac{b\iota^- -3}{3b}, \frac{3}{b}\right), \quad \left(\frac{b\iota^- -2a\iota^+}{9}, 1\right).$$

(iii) *The defining matrix* P *is of the shape*

$$\begin{bmatrix} -1 & -2 & 3 & 0 \\ -1 & -2 & 0 & 3 \\ 0 & \frac{b\iota^- -2a\iota^+}{9} & \frac{a\iota^+ -6}{3} & 2 \end{bmatrix}, \qquad \begin{gathered} 2a\iota^+ - b\iota^- = 9\iota_h, \\ a = 1,2,3,6, \\ b = -1,-3 \end{gathered}$$

with ι^+u^+, ι^-u^- *and* $\iota_h u_h$ *given by*

$$\left(\frac{12-a\iota^+}{3a}, \frac{a\iota^+-12}{3a}, \frac{6}{a}\right), \quad \left(\frac{b\iota^- -a\iota^+ +6}{3b}, \frac{b\iota^- -6}{3b}, \frac{3}{b}\right), \quad \left(\frac{b\iota^- -2a\iota^+}{9}, 1\right).$$

Proposition 4.3.7. *Let* X *be a non-toric rational projective* $\mathbb{K}^*$*-surface of Picard number* 1 *with* $x^+ \in X$ *quasismooth and* $x^- \in X$ *of type* E_7*. Then* X *is log del Pezzo and we have* $X \cong X(A,P)$*, specified as follows.*

(i) *The defining matrix* P *is of the shape*

$$\begin{bmatrix} -1 & -4 & 3 & 0 \\ -1 & -4 & 0 & 2 \\ 0 & -\frac{2a\iota^+ +\iota^-}{3} & \frac{a\iota^+ -3}{2} & 1 \end{bmatrix}, \qquad \begin{gathered} 2a\iota^+ + 2\iota^- = 3c\iota_h, \\ a = 1,5, \\ c = 1,3 \end{gathered}$$

with ι^+u^+, ι^-u^- *and* $\iota_h u_h$ *given by*

$$\left(\frac{5-a\iota^+}{2a}, \frac{a\iota^+-5}{2a}, \frac{5}{a}\right), \quad \left(\frac{a\iota^+ +2\iota^- -3}{6}, \frac{\iota^- +1}{2}, -1\right), \quad \left(-\frac{2a\iota^+ +\iota^-}{3c}, \frac{3}{c}\right).$$

(ii) *The defining matrix* P *is of the shape*

$$\begin{bmatrix} -1 & -3 & 4 & 0 \\ -1 & -3 & 0 & 2 \\ 0 & -\frac{3a\iota^+ +2\iota^-}{8} & \frac{a\iota^+ -4}{2} & 1 \end{bmatrix}, \qquad \begin{gathered} 3a\iota^+ + 2\iota^- = 8c\iota_h, \\ a = 1,2,3,6, \\ c = 1,2. \end{gathered}$$

with ι^+u^+, ι^-u^- *and* $\iota_h u_h$ *given by*

$$\left(\frac{6-a\iota^+}{2a}, \frac{a\iota^+-6}{2a}, \frac{6}{a}\right), \quad \left(\frac{a\iota^+ +2\iota^- -4}{8}, \frac{\iota^- +1}{2}, -1\right), \quad \left(-\frac{3a\iota^+ +2\iota^-}{8c}, \frac{2}{c}\right).$$

(iii) *The defining matrix* P *is of the shape*

$$\begin{bmatrix} -1 & -2 & 4 & 0 \\ -1 & -2 & 0 & 3 \\ 0 & -\frac{a\iota^+ +\iota^-}{6} & \frac{a\iota^+ -4}{3} & 1 \end{bmatrix}, \qquad \begin{gathered} a\iota^+ + \iota^- = 6\iota_h, \\ a = 1,7 \end{gathered}$$

with ι^+u^+, ι^-u^- *and* $\iota_h u_h$ *given by*

$$\left(\frac{7-a\iota^+}{3a}, \frac{a\iota^+-7}{3a}, \frac{7}{a}\right), \quad \left(\frac{a\iota^+ +3\iota^- -4}{12}, \frac{\iota^- +1}{3}, -1\right), \quad \left(-\frac{a\iota^+ +\iota^-}{6}, 1\right).$$

(iv) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -2 & 4 & 0 \\ -1 & -2 & 0 & 3 \\ 0 & -\frac{a\iota^+ + \iota^-}{6} & \frac{a\iota^+ - 8}{3} & 2 \end{bmatrix}, \qquad \begin{array}{c} a\iota^+ + \iota^- = 6\iota_h, \\ a = 1, 7 \end{array}$$

with $\iota^+ u^+, \iota^- u^-$ and $\iota_h u_h$ given by

$$\left(\frac{14 - a\iota^+}{3a}, \frac{a\iota^+ - 14}{3a}, \frac{7}{a}\right), \quad \left(\frac{a\iota^+ + 3\iota^- - 8}{12}, \frac{\iota^- + 2}{3}, -1\right), \quad \left(-\frac{a\iota^+ + \iota^-}{6}, 1\right).$$

Proposition 4.3.8. *Let X be a non-toric rational projective $\mathbb{K}^*$-surface of Picard number one with $x^+ \in X$ quasismooth and $x^- \in X$ of type E_8. Then X is log del Pezzo and we have $X \cong X(A, P)$, specified as follows.*

(i) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -5 & 3 & 0 \\ -1 & -5 & 0 & 2 \\ 0 & -\frac{5a\iota^+ + \iota^-}{6} & \frac{a\iota^+ - 3}{2} & 1 \end{bmatrix}, \qquad \begin{array}{c} 5a\iota^+ + \iota^- = 6c\iota_h, \\ a = 1, 5, \\ c = 1, 2, 4 \end{array}$$

with $\iota^+ u^+, \iota^- u^-$ and $\iota_h u_h$ given by

$$\left(\frac{5 - a\iota^+}{2a}, \frac{a\iota^+ - 5}{2a}, \frac{5}{a}\right), \quad \left(\frac{a\iota^+ + 2\iota^- - 5}{10}, \frac{\iota^- + 1}{2}, -1\right), \quad \left(-\frac{5a\iota^+ + \iota^-}{6c}, \frac{4}{c}\right).$$

(ii) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -3 & 5 & 0 \\ -1 & -3 & 0 & 2 \\ 0 & -\frac{3a\iota^+ + 2\iota^-}{10} & \frac{a\iota^+ - 5}{2} & 1 \end{bmatrix}, \qquad \begin{array}{c} 3a\iota^+ + \iota^- = 10c\iota_h, \\ a = 1, 7, \\ c = 1, 2 \end{array}$$

with $\iota^+ u^+, \iota^- u^-$ and $\iota_h u_h$ given by

$$\left(\frac{7 - a\iota^+}{2a}, \frac{a\iota^+ - 7}{2a}, \frac{7}{a}\right), \quad \left(\frac{a\iota^+ + 2\iota^- - 4}{8}, \frac{\iota^- + 1}{2}, -1\right), \quad \left(-\frac{3a\iota^+ + 2\iota^-}{10c}, \frac{2}{c}\right).$$

(iii) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -2 & 5 & 0 \\ -1 & -2 & 0 & 3 \\ 0 & -\frac{2a\iota^+ + \iota^-}{15} & \frac{a\iota^+ - 5}{3} & 1 \end{bmatrix}, \qquad \begin{array}{c} 2a\iota^+ + \iota^- = 15\iota_h, \\ a = 1, 2, 4, 8 \end{array}$$

with $\iota^+ u^+, \iota^- u^-$ and $\iota_h u_h$ given by

$$\left(\frac{8 - a\iota^+}{3a}, \frac{a\iota^+ - 8}{3a}, \frac{8}{a}\right), \quad \left(\frac{a\iota^+ + 3\iota^- - 5}{15}, \frac{\iota^- + 1}{3}, -1\right), \quad \left(-\frac{2a\iota^+ + \iota^-}{15}, 1\right).$$

(iv) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -2 & 5 & 0 \\ -1 & -2 & 0 & 3 \\ 0 & -\frac{2a\iota^+ + \iota^-}{15} & \frac{a\iota^+ - 10}{3} & 2 \end{bmatrix}, \qquad \begin{array}{c} 2a\iota^+ + \iota^- = 15\iota_h, \\ a = 1, 2, 4, 8 \end{array}$$

with $\iota^+ u^+, \iota^- u^-$ and $\iota_h u_h$ given by

$$\left(\frac{16 - a\iota^+}{3a}, \frac{a\iota^+ - 16}{3a}, \frac{8}{a}\right), \quad \left(\frac{a\iota^+ + 3\iota^- - 10}{15}, \frac{\iota^- + 2}{3}, -1\right), \quad \left(-\frac{2a\iota^+ + \iota^-}{15}, 1\right).$$

Now we treat the cases of non-quasismooth log terminal elliptic fixed points x^+ and x^-, i.e. of type D_n, E_6, E_7 or E_8. Again, the arguments being analogous, we restrict ourselves to giving the results.

Proposition 4.3.9. *Let X be a non-toric rational projective $\mathbb{K}^*$-surface of Picard number 1 with $x^+ \in X$ and $x^- \in X$ of type D_n. Then X is log del Pezzo and we have $X \cong X(A, P)$, specified as follows.*

(i) *The defining matrix P is of the shape*

$$\begin{bmatrix} -2 & -2 & \frac{a\iota^+ - b\iota^-}{2c} & 0 \\ -2 & -2 & 0 & 2 \\ -1 & -c-1 & \frac{a\iota^+}{4} & 1 \end{bmatrix}, \qquad \begin{array}{c} a = 2,4,\ b = -2,-4, \\ a\iota^+ - b\iota^- > 0, \\ 1 \le c \le \frac{a\iota^+ - b\iota^-}{2} \end{array}$$

with ι^+u^+, ι^-u^- and $\iota_h u_h$ given by

$$\left(0, \frac{a\iota^+ - 4}{2a}, \frac{4}{a}\right), \quad \left(-\frac{2c}{b}, \frac{b\iota^- - 4}{2b}, \frac{4}{b}\right), \quad (-1, 0).$$

(ii) *The defining matrix P is of the shape*

$$\begin{bmatrix} -l_{01} & -\frac{b\iota^- l_{01} + 4c\iota_h}{a\iota^+} & 2 & 0 \\ -l_{01} & -\frac{b\iota^- l_{01} + 4c\iota_h}{a\iota^+} & 0 & 2 \\ \frac{a\iota^+}{4} - l_{01} & \frac{b\iota^-}{4} - \frac{b\iota^- l_{01} + 4c\iota_h}{a\iota^+} & 1 & 1 \end{bmatrix}, \qquad \begin{array}{c} a = 2,4, \\ b = -2,-4, \\ a\iota^+ - b\iota^- > 0, \\ 1 \le c \le \frac{a\iota^+ - b\iota^-}{4}, \\ 2 \le l_{01} < \frac{a\iota^+ - 4c\iota_h}{b\iota^-} \end{array}$$

with ι^+u^+, ι^-u^- and $\iota_h u_h$ given by

$$\left(\frac{a\iota^+ - 4}{2a}, \frac{a\iota^+ - 4}{2a}, \frac{4}{a}\right), \qquad \left(\frac{b\iota^- - 4}{2b}, \frac{b\iota^- - 4}{2b}, \frac{4}{b}\right),$$
$$\left(\frac{a^2(\iota^+)^2 - ab\iota^+\iota^- - 4a\iota^+ l_{01} + 4b\iota^- l_{01} + 16c\iota_h}{4ac\iota^+}, -\frac{a\iota^+ l_{01} - b\iota^- l_{01} - 4c\iota_h}{ac\iota^+}\right).$$

(iii) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & -\frac{a\iota^+ + b\iota^-}{4l_{11}} & \frac{a\iota^+}{4} - l_{11} & 1 & 1 \end{bmatrix}, \qquad \begin{array}{c} a, b = 2,4, \\ l_{11} \mid \frac{a\iota^+ + b\iota^-}{4} \end{array}$$

with ι^+u^+, ι^-u^- and $\iota_h u_h$ given by

$$\left(\frac{4}{a}, \frac{a\iota^+ - 4}{2a}, \frac{a\iota^+ - 4}{2a}, \frac{4}{a}\right), \left(\frac{a\iota^+ + b\iota^- - 4l_{11}}{l_{11}b}, \frac{b\iota^- + 4}{2b}, \frac{b\iota^- + 4}{2b}, -\frac{4}{b}\right), (-1, 0).$$

Here, ι^+, ι^- and ι_h are the local Gorenstein indices of x^+, x^- and the hyperbolic fixed point. The vectors $\iota^+u^+, \iota^-u^- \in \mathbb{Z}^{r+1}$ and $\iota_h u_h \in \mathbb{Z}^2$ are primitive and we have

$$\iota_X \ = \ \operatorname{lcm}(\iota^+, \iota^-, \iota_h).$$

Proposition 4.3.10. *Let X be a non-toric rational projective $\mathbb{K}^*$-surface of Picard number 1 with $x^+ \in X$ of type D_n and $x^- \in X$ of type E_6, E_7 or E_8. Then X is log del Pezzo and we have $X \cong X(A, P)$, specified as follows.*

(i) *The defining matrix P is of the shape*

$$\begin{bmatrix} -2 & -l_{02} & 3 & 0 \\ -2 & -l_{02} & 0 & 2 \\ -1 & -\frac{c\iota_h + l_{02}}{2} & d_{11} & 1 \end{bmatrix}, \qquad \begin{array}{c} l_{02} = 3,4,5, \\ d_{11} = \frac{\iota^+}{2}, \\ c \mid l_{02} - 2 \end{array}$$

with ι^+u^+, ι^-u^- and $\iota_h u_h$ given by

$$\left(0, \frac{\iota^+(d_{11}-1)}{2d_{11}}, \frac{\iota^+}{d_{11}}\right),$$
$$\left(\frac{\iota^-(c\iota_h-d_{11}l_{02}+2d_{11})}{3c\iota_h-2d_{11}l_{02}}, \frac{\iota^-(3c\iota_h-2d_{11}l_{02}-l_{02}+6)}{6c\iota_h-4d_{11}l_{02}}, \frac{\iota^-(l_{02}-6)}{3c\iota_h-2d_{11}l_{02}}\right),$$
$$\left(\frac{2-c\iota_h-l_{02}}{2c}, \frac{l_{02}-2}{c}\right).$$

(ii) *The defining matrix P is of the shape*

$$\begin{bmatrix} -2 & -3 & l_{11} & 0 \\ -2 & -3 & 0 & 2 \\ -1 & -\frac{\iota_h+3}{2} & \frac{a\iota^+}{4} & 1 \end{bmatrix}, \qquad \begin{array}{c} a = 2, 4, \\ l_{11} = 3, 4, 5 \end{array}$$

with ι^+u^+, ι^-u^- and $\iota_h u_h$ given by

$$\left(0, \frac{a\iota^+-4}{2a}, \frac{4}{a}\right),$$
$$\left(\frac{\iota^-(a\iota^+-4\iota_h)}{6a\iota^+-4\iota_h l_{11}}, \frac{\iota^-(3a\iota^+-2\iota_h l_{11}+2l_{11}-12)}{6a\iota^+-4\iota_h l_{11}}, \frac{2\iota^-(6-l_{11})}{3a\iota^+-2\iota_h l_{11}}\right),$$
$$\left(-\frac{\iota_h+1}{2}, 1\right).$$

Here, ι^+, ι^- and ι_h are the local Gorenstein indices of x^+, x^- and the hyperbolic fixed point. The vectors $\iota^+u^+, \iota^-u^- \in \mathbb{Z}^3$ and $\iota_h u_h \in \mathbb{Z}^2$ are primitive and we have

$$\iota_X = \operatorname{lcm}(\iota^+, \iota^-, \iota_h).$$

Proposition 4.3.11. *Let X be a non-toric rational projective $\mathbb{K}^*$-surface of Picard number 1 with $x^+ \in X$ and $x^- \in X$ of type E_6, E_7 or E_8. Then X is log del Pezzo and we have $X \cong X(A,P)$, specified as follows.*

(i) *The defining matrix P is of the shape*

$$\begin{bmatrix} -2 & -2 & l_{11} & 0 \\ -2 & -2 & 0 & 3 \\ -1 & -c-1 & \frac{a\iota^+-2l_{11}d_{21}+3l_{11}}{6} & d_{21} \end{bmatrix}, \qquad \begin{array}{c} l_{11} = 3, 4, 5, \\ a > 0, \ a \mid 6 - l_{11}, \\ \frac{a\iota^+}{3l_{11}} < c < \frac{a\iota^+ + \iota^-(6-l_{11})}{3}, \\ \frac{a\iota^++3l_{11}-3c-3}{2l_{11}-2} < d_{21} < \frac{a\iota^++3l_{11}-3}{2l_{11}-2} \end{array}$$

with ι^+u^+, ι^-u^- and $\iota_h u_h$ given by

$$\left(\frac{a\iota^+-2d_{21}l_{11}+12d_{21}+3l_{11}-18}{6a}, \frac{a\iota^++d_{21}l_{11}-6d_{21}}{3a}, \frac{6-l_{11}}{a}\right),$$
$$\left(\frac{\iota^-(a\iota_+-2d_{21}l_{11}-18c+12d_{21}+3l_{11}-18)}{6a\iota^+-18cl_{11}}, \frac{\iota^-(a\iota^+-3cl_{11}+d_{21}l_{11}-6d_{21})}{3a\iota^+-9cl_{11}}, \frac{\iota^-(6-l_{11})}{a\iota^+-3cl_{11}}\right),$$
$$(-1, 0).$$

(ii) *The defining matrix P is of the shape*

$$\begin{bmatrix} -3 & -3 & l_{11} & 0 \\ -3 & -3 & 0 & 2 \\ d_{01} & \frac{-a\iota^++b\iota^-+2d_{01}l_{11}}{2l_{11}} & \frac{a\iota^+-l_{11}(2d_{01}+3)}{6} & 1 \end{bmatrix}, \qquad \begin{array}{c} l_{11} = 4, 5, \\ d_{01} = -1, -2, \\ a > 0, \ b < 0, \\ a, b \mid 6 - l_{11} \end{array}$$

with ι^+u^+, ι^-u^- and $\iota_h u_h$ given by

$$\left(\frac{a\iota^+-2d_{01}l_{11}+12d_{01}-3l_{11}+18}{6a}, \frac{a\iota^++l_{11}-6}{2a}, \frac{6-l_{11}}{a}\right),$$
$$\left(\frac{a\iota^+l_{11}-2d_{01}l_{11}^2-6a\iota^++6b\iota^-+12d_{01}l_{11}-3l_{11}^2+18l_{11}}{6bl_{11}}, \frac{b\iota^-+l_{11}-6}{2b}, \frac{6-l_{11}}{b}\right),$$
$$\left(-\frac{\iota_h}{3}, 0\right).$$

(iii) *The defining matrix P is of the shape*

$$\begin{bmatrix} -l_{01} & -l_{02} & 3 & 0 \\ -l_{01} & -l_{02} & 0 & 2 \\ \frac{a\iota^+-l_{01}(2d_{11}+3)}{6} & \frac{b\iota^--l_{02}(2d_{11}+3)}{6} & d_{11} & 1 \end{bmatrix}, \qquad \begin{array}{c} l_{01}, l_{02} = 3, 4, 5, \\ d_{11} = 1, 2, \\ a > 0, \ b < 0, \\ a \mid 6 - l_{01}, \ b \mid 6 - l_{02} \end{array}$$

with ι^+u^+, ι^-u^- and $\iota_h u_h$ given by

$$\left(\frac{a\iota^++d_{11}l_{01}-6d_{11}}{3a}, \frac{a\iota^++l_{01}-6}{2a}, \frac{6-l_{01}}{a}\right),$$
$$\left(\frac{b\iota^-+d_{11}l_{02}-6d_{11}}{3b}, \frac{b\iota^-+l_{02}-6}{2b}, \frac{6-l_{02}}{b}\right),$$
$$\left(-\frac{\iota_h(a\iota^+-b\iota^--2d_{11}l_{01}+2d_{11}l_{02}-3l_{01}+3l_{02})}{a\iota^+l_{02}-b\iota^-l_{01}}, \frac{6\iota_h(l_{02}-l_{01})}{a\iota^+l_{02}-b\iota^-l_{01}}\right).$$

(iv) *The defining matrix P is of the shape*

$$\begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & 0 & 3 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & -\frac{a\iota^++b\iota^-}{6l_{11}} & \frac{a\iota^+-2l_{11}d_{21}-3l_{11}}{6} & d_{21} & 1 \end{bmatrix}, \qquad \begin{array}{c} l_{11} = 3,4,5, \\ 1 \le d_{11} \le l_{11}-1, \\ d_{21} = 1,2, \\ a,b > 0,\ a,b | 6-l_{11}, \\ l_{11} | \frac{a\iota^++b\iota^-}{4} \end{array}$$

with ι^+u^+, ι^-u^- and $\iota_h u_h$ given by

$$\left(\frac{a\iota^+-2d_{21}l_{11}+12d_{21}-3l_{11}+18}{6a}, \frac{a\iota^++d_{21}l_{11}-6d_{21}}{3a}, \frac{a\iota^++l_{11}-6}{2a}, \frac{6-l_{11}}{a}\right),$$
$$\left(\frac{6a\iota^++6b\iota^-+2d_{21}l_{11}^2+3l_{11}^2-a\iota^+l_{11}-12d_{21}l_{11}-18l_{11}}{6l_{11}b}, \frac{b\iota^--d_{21}l_{11}+6d_{21}}{3b}, \frac{b\iota^--l_{11}+6}{2b}, \frac{l_{11}-6}{b}\right),$$
$$(-1,0).$$

Here, ι^+, ι^- and ι_h are the local Gorenstein indices of x^+, x^- and the hyperbolic fixed point. The vectors $\iota^+u^+, \iota^-u^- \in \mathbb{Z}^{r+1}$ and $\iota_h u_h \in \mathbb{Z}^2$ are primitive and we have

$$\iota_X \ = \ \mathrm{lcm}(\iota^+, \iota^-, \iota_h).$$

Lastly, we have to consider the case of the existence of a parabolic fixed point curve.

Proposition 4.3.12. *Let X be a non-toric rational projective $\mathbb{K}^*$-surface of Picard number 1 with $x^+ \in X$ of type D_n, E_6, E_7 or E_8 and a parabolic fixed point curve D_X^-. Then X is log del Pezzo and we have $X \cong X(A,P)$, specified as follows.*

(i) *The defining matrix P is of the shape*

$$\begin{bmatrix} -l_{01} & 2 & 0 & 0 \\ -l_{01} & 0 & 2 & 0 \\ \frac{a\iota^+}{4} - l_{01} & 1 & 1 & -1 \end{bmatrix}, \qquad \begin{array}{c} a = 2,4, \\ l_{01} | \frac{\iota_0^-(a\iota^++4)}{4}, \end{array}$$

with ι^+u^+, $\iota_0^-u_0^-$, $\iota_1^-u_1^-$ and $\iota_2^-u_2^-$ given by

$$\left(\frac{a\iota^+-4}{2a}, \frac{a\iota^+-4}{2a}, \frac{4}{a}\right), \qquad \left(\frac{\iota_0^-(4l_{01}-a\iota^+-4)}{4l_{01}}, -\iota_0^-\right),$$
$$(1,-1), \qquad (1,-1).$$

(ii) *The defining matrix P is of the shape*

$$\begin{bmatrix} -l_{01} & 3 & 0 & 0 \\ -l_{01} & 0 & 2 & 0 \\ \frac{a\iota^+-l_{01}(2d_{11}+3)}{6} & d_{11} & 1 & -1 \end{bmatrix}, \qquad \begin{array}{c} l_{01} = 3,4,5, \\ a > 0,\ a | 6-l_{01}, \\ d_{11} = 1,2 \end{array}$$

with ι^+u^+, $\iota_0^-u_0^-$, $\iota_1^-u_1^-$ and $\iota_2^-u_2^-$ given by

$$\left(\frac{a\iota^++d_{11}l_{01}-6d_{11}}{3a}, \frac{a\iota^++l_{01}-6}{2a}, \frac{6-l_{01}}{a}\right),$$
$$\left(\frac{\iota_0^-(2d_{11}l_{01}+3l_{01}-a\iota^+-6)}{6l_{01}}, -\iota_0^-\right),$$
$$\left(\frac{\iota_1^-(d_{11}+1)}{3}, -\iota_1^-\right), \qquad (1,-1).$$

Here, ι^+, ι_0^-, ι_1^- and ι_2^- are the local Gorenstein indices of x^+ and the parabolic fixed points $x_i^- \in \mathscr{A}_i$. The vectors $\iota^+ u^+ \in \mathbb{Z}^3$ and $\iota_i^- u_i^- \in \mathbb{Z}^2$ are primitive and we have

$$\iota_X \ = \ \mathrm{lcm}(\iota^+, \iota_0^-, \iota_1^-, \iota_2^-).$$

Remark 4.3.13. Note that the preceding considerations show that rational projective log terminal $\mathbb{K}^*$-surfaces of Picard number 1 are always del Pezzo.

Implementing Remark 4.3.4 and the bounds from Propositions 4.3.2, 4.3.5, 4.3.6, 4.3.7, 4.3.8, 4.3.9, 4.3.10, 4.3.11, 4.3.12 we can classify. Using the normal form from Definition 4.2.26 to filter out equivalent matrices, we obtain the following results.

Theorem 4.3.14. *There are exactly* 154161 *isomorphy classes of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Picard number* 1 *and Gorenstein index $\iota \leq 200$. The numbers $\gamma(\iota)$ of isomorphy classes corresponding to ι are given in the table on the next page.*

ι	$\gamma(\iota)$	ι	$\gamma(\iota)$	ι	$\gamma(\iota)$	ι	$\gamma(\iota)$
1	13	51	570	101	831	151	1047
2	10	52	354	102	535	152	931
3	36	53	532	103	880	153	1464
4	25	54	334	104	786	154	963
5	80	55	776	105	1378	155	1693
6	37	56	427	106	449	156	1084
7	100	57	516	107	1006	157	1002
8	56	58	328	108	693	158	709
9	109	59	846	109	844	159	1538
10	71	60	493	110	748	160	1062
11	176	61	459	111	988	161	1626
12	85	62	349	112	758	162	694
13	158	63	730	113	866	163	1107
14	105	64	364	114	530	164	1016
15	200	65	845	115	1250	165	1773
16	102	66	366	116	743	166	808
17	226	67	570	117	1115	167	1789
18	102	68	449	118	713	168	1185
19	241	69	770	119	1919	169	1171
20	178	70	556	120	914	170	922
21	253	71	797	121	838	171	1520
22	150	72	464	122	450	172	857
23	312	73	531	123	1021	173	1240
24	176	74	365	124	708	174	878
25	269	75	811	125	1531	175	2021
26	149	76	494	126	731	176	1159
27	336	77	1046	127	841	177	1402
28	224	78	482	128	706	178	833
29	395	79	734	129	1141	179	2095
30	192	80	592	130	750	180	1302
31	309	81	683	131	1220	181	1015
32	216	82	410	132	978	182	971
33	381	83	993	133	1337	183	1260
34	207	84	640	134	619	184	974
35	592	85	881	135	1525	185	1611
36	230	86	383	136	823	186	848
37	336	87	899	137	1032	187	1808
38	239	88	613	138	695	188	1214
39	497	89	998	139	1251	189	2054
40	312	90	537	140	1242	190	1162
41	481	91	952	141	1011	191	1462
42	266	92	584	142	749	192	1047
43	405	93	750	143	1853	193	1145
44	348	94	549	144	836	194	805
45	526	95	1229	145	1371	195	2294
46	270	96	596	146	553	196	1092
47	549	97	716	147	1340	197	1538
48	317	98	522	148	787	198	1017
49	497	99	1105	149	1249	199	1387
50	277	100	599	150	802	200	1206

CHAPTER 5

$1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces

5.1. Contractions and combinatorial minimality

We discuss contractions and combinatorial minimality of $\mathbb{K}^*$-surfaces. In particular, Proposition 5.1.14 provides an explicit description of contractions in terms of defining matrices, Proposition 5.1.16 tells about the effect of a contraction on the anticanonical complex and Proposition 5.1.21 presents geometric properties of combinatorially minimal $\mathbb{K}^*$-surfaces.

Definition 5.1.1. A *contraction* is a proper, birational morphism $\psi\colon X \to Y$ of normal varieties such that $\psi\colon \psi^{-1}(V) \to V$ is an isomorphism for some open subset $V \subseteq Y$ with complement of codimension at least two in Y.

We gather basic general properties of contractions. Recall that for any proper morphism $\psi\colon X \to Y$ of normal varieties, we have the push forward homomorphisms

$$\psi_*\colon \mathrm{WDiv}(X) \ \to \ \mathrm{WDiv}(Y), \qquad\qquad \psi_*\colon \mathrm{Cl}(X) \ \to \ \mathrm{Cl}(Y),$$

defined by sending a prime divisor $D \subseteq X$ to $\psi(D) \subseteq Y$ if $\psi(D)$ is a prime divisor in Y and to zero else.

Proposition 5.1.2. *Let $\psi\colon X \to Y$ be a contraction and consider the associated push forward homomorphisms on the Weil divisors and the divisor classes.*

(i) *If $\mathcal{K}_X$ is a canonical divisor on X, then its push forward $\psi_*(\mathcal{K}_X)$ is a canonical divisor on Y.*

(ii) *The homomorphisms $\psi_*\colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(Y)$ and $\psi_*\colon \mathrm{Cl}X \to \mathrm{Cl}(Y)$ are both surjective.*

(iii) *For the cones of effective and movable divisor classes in $\mathrm{Cl}_{\mathbb{Q}}(X)$ and $\mathrm{Cl}_{\mathbb{Q}}(Y)$ we have*

$$\psi_*(\mathrm{Eff}(X)) = \mathrm{Eff}(Y), \qquad \psi_*(\mathrm{Mov}(X)) = \mathrm{Mov}(Y).$$

Proof. Take any open subset $V \subseteq Y$ with complement of codimension at least two in Y such that $\psi\colon \psi^{-1}(V) \to V$ is an isomorphism. Then canonical (principal, effective, movable) divisors D on X restrict to canonical (principal, effective, movable) divisors on $\psi^{-1}(V) \cong V$ and thus yield canonical (principal, effective, movable) divisors $\psi_*(D)$ on Y. $\square$

If a contraction $\psi\colon X \to Y$ maps a prime divisor $E \subseteq X$ onto a subset of codimension at least two in Y, then we say ψ *contracts* E and call E an *exceptional divisor* of ψ. Moreover, a prime divisor on a normal variety X is called *contractible* if it gets contracted by some contraction $X \to Y$.

Remark 5.1.3. For any contraction $\psi\colon X \to Y$, there is a finite (possibly empty) collection $E_1, \ldots, E_q \subseteq X$ of exceptional divisors.

Remark 5.1.4. Consider normal complete varieties X, Y and let $\psi\colon X \to Y$ be a contraction with the exceptional divisors $E_1, \ldots, E_q$.

(i) If X comes with a morphical action of a connected algebraic group G, then $E_1, \ldots, E_q \subseteq X$ are invariant and Y admits a morphical G-action making $\psi\colon X \to Y$ equivariant, see [**9**, Prop. I.1].

(ii) Assume that X has finitely generated Cox ring $\mathcal{R}(X)$ and let $f_i \in \mathcal{R}(X)$ represent the canonical section of E_i. Then we have an isomorpism

$$\mathcal{R}(X)/\langle 1 - f_i;\ i = 1, \ldots, l\rangle \ \cong\ \mathcal{R}(Y)$$

induced by sending homogeneous elements $f \in \mathcal{R}(X)$ of degree $[D]$ to homogeneous elements $\psi_* f \in \mathcal{R}(Y)$ of degree $[\psi_* D]$, see [**4**, Prop. 4.1.3.1].

We focus on the surface case. We also perform some basic general observations before entering the setting of surfaces with $\mathbb{K}^*$-action.

Remark 5.1.5. Let $\psi\colon X \to Y$ be a contraction of surfaces. Then the exceptional divisors $E_1, \ldots, E_q \subseteq X$ map to points $y_1, \ldots, y_q \in Y$ and $X \setminus (E_1 \cup \ldots \cup E_q)$ maps isomorphically onto $Y \setminus \{y_1, \ldots, y_q\}$.

Proposition 5.1.6. *Let $\psi\colon X \to Y$ be a contraction of surfaces, where X has finitely generated Cox ring. Then Y has finitely generated Cox ring and we have*

$$\psi_*(\mathrm{SAmple}(X)) = \mathrm{SAmple}(Y), \qquad \psi_*(\mathrm{Ample}(X)) = \mathrm{Ample}(Y)$$

for the cones of semiample and ample divisor classes. Moreover, if X is a del Pezzo surface, then Y is a del Pezzo surface.

Proof. According to Remark 5.1.4 (ii), Y also has a finitely generated Cox ring. Thus, the movable and semiample cones coincide in $\mathrm{Cl}_{\mathbb{Q}}(X)$ and as well in $\mathrm{Cl}_{\mathbb{Q}}(Y)$, see [**4**, Thm. 4.3.3.5]. Thus, the first displayed equation follows from Proposition 5.1.2 (iii). Moreover, the respective ample cones are the relative interiors of the semiample cones, see [**4**, Prop. 3.3.2.9]. Hence, the second displayed equation follows from the first one and the fact that any linear map sends the interior of a cone onto the interior of the image cone. For the supplement, we use Proposition 5.1.2 (i) to see that Y has an ample anticanonical divisor. □

We enter the setting of $\mathbb{K}^*$-surfaces. First we note an immediate consequence of Remark 5.1.4 and Propositions 5.1.6.

Corollary 5.1.7. *Any contraction of a toric del Pezzo surface is a toric del Pezzo surface and any contraction of a del Pezzo $\mathbb{K}^*$-surface is a del Pezzo $\mathbb{K}^*$-surface.*

Proposition 5.1.8. *For a contraction $X \to Y$ of $\mathbb{K}^*$-surfaces, every exceptional divisor $E \subseteq X$ is either a parabolic fixed point curve or it is an orbit closure containing a hyperbolic fixed point.*

Proof. Otherwise, being invariant, E is an orbit closure containing a point from the source and a point from the sink. Hence, source and sink of Y would intersect in the image point of E, which is impossible. □

Now we study contractions of $\mathbb{K}^*$-surfaces in terms of defining data. Observe that the special case of defining matrices P with $r = 1$ provides a full treatment of contractions of toric surfaces, see also Remark 3.2.4.

Definition 5.1.9. Let P be a defining matrix as in Construction 3.2.2. We call a column of P *contractible* if it lies in the cone generated by the remaining ones.

Remark 5.1.10. Consider a slope-ordered defining matrix P. Then, a column v is contractible if and only if the matrix arising from P by deleting v is a defining matrix as well. That means, v is contractible if and only if one of the following conditions is satisfied.

(i) $n_i \geq 2, v = v_{i1}$ and $m^+ - m_{i1} + m_{i2} > 0$.
(ii) $v = v_{ij}$ and $1 < j < n_i$.
(iii) $n_i \geq 2, v = v_{in_i}$ and $m^- - m_{in_i} + m_{in_i-1} < 0$.
(iv) $v = v^+$ and $m^+ > 0$.
(v) $v = v^-$ and $m^- > 0$.

Note that we have

$$\frac{m^+}{m_{i1} - m_{i2}} \;>\; 1, \qquad \frac{m^-}{m_{in_i} - m_{i,n_i-1}} \;>\; 1$$

in case (i) and (iii) respectively.

Construction 5.1.11. Consider a projective $\mathbb{K}^*$-surface $X_1 = X(A, P_1)$ and assume that P_1 has a contractible column v. Then, erasing v from P_1 yields a defining matrix P_2 of a projective $\mathbb{K}^*$-surface $X_2 = X(A, P_2)$. Moreover, we obtain a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "\psi_v"'] & Z_1 \arrow[d, "\psi_v"] \\ X_2 \arrow[r] & Z_2 \end{tikzcd}$$

involving the $\mathbb{K}^*$-surfaces X_i and their ambient toric varieties Z_i. The downwards maps contract the prime divisors $D_{X_1} \subseteq X_1$ and $D_{Z_1} \subseteq Z_1$ corresponding to the contracted column v of P_1. In particular, the downward maps are non-trivial contractions.

Remark 5.1.12. Consider a $\mathbb{K}^*$-surface $X = X(A, P)$, where the matrix P is slope-ordered.

(i) Assume that there is a curve $D_X^+ \subseteq X$. Then for every $i = 0, \ldots, r$ with $n_i \geq 2$, each column v_{ij} with $j = 1, \ldots, n_{i-1}$ is contractible.
(ii) Assume that there is a curve $D_X^- \subseteq X$. Then for every $i = 0, \ldots, r$ with $n_i \geq 2$, each column v_{ij} with $j = 2, \ldots, n_i$ is contractible.
(iii) Assume that X has two elliptic fixed points. Then for every $i = 0, \ldots, r$ with $n_i \geq 3$, each column v_{ij} with $j = 2, \ldots, n_{i-1}$ is contractible.

In particular, we obtain a contraction $X \to X'$ onto a $\mathbb{K}^*$-surface X' given by defining data (A, P') such that

(iv) in the case that X has of two elliptic fixed points, we have $n_i' \leq 2$ for $i = 0, \ldots, r$,

(v) in the case that X has a parabolic fixed point curve, we have $n_i' = 1$ for $i = 0, \ldots, r$.

Proposition 5.1.13. *Let $X = X(A, P)$ be projective, v a column of P and $D \subseteq X$ the corresponding prime divisor. Then the following statements are equivalent.*

(i) *The column v is contractible.*
(ii) *The curve $D \subseteq X$ is contractible.*
(iii) *We have $D^2 < 0$.*
(iv) *The divisor D is not movable.*

Proof. If (i) holds, then we contract D by means of Construction 5.1.11. The implications from (ii) to (iii) and from (iii) to (iv) are standard surface geometry. If D is not movable, then, in $\mathrm{Cl}_{\mathbb{Q}}(X)$, the ray through $[D]$ intersects the cone generated by the remaining w_{ij} and $w^{\pm}$ in the origin. By [**4**, Lemma 2.2.3.2], the column v lies in the interior of the cone generated by the remaining columns of P and thus is contractible. $\square$

Proposition 5.1.14. *Every contraction $X \to Y$ of rational projective $\mathbb{K}^*$-surfaces decomposes as $X \cong X_0 \to \ldots \to X_q \cong Y$ with $X_i \to X_{i+1}$ as in Construction 5.1.11.*

Proof. We may assume $X = X_0$ with X_0 arising from the defining data (A, P_0). As observed in Proposition 5.1.8, the exceptional divisors

$$E_1, \ldots, E_q \subseteq X$$

are taken from the $D_X^{\pm}$ and the D_X^{ij} that contain a hyperbolic fixed point. In particular, E_1 correponds to a column of P_0 and Proposition 5.1.13 provides $\psi\colon X_0 \to X_1$, as in Construction 5.1.11, contracting E_1. Now observe that none of $\psi_*(E_2), \ldots, \psi_*(E_q)$ are movable and hence by Proposition 5.1.13, they are all contractible. Iterating this consideration, we arrive at a sequence $X_0 \to \ldots \to X_q$, contracting $E_1, \ldots, E_q$ stepwise. The remaining task is to show that $X \to Y$ factors via an isomorphism through $X \to X_q$. By construction, we obtain such a factorization apart from the respective image points of $E_1, \ldots, E_q$ in X_q and Y. Being an isomorphism up to codimension two, $X_q \dashrightarrow Y$ lifts to the total coordinate spaces and descends again to an isomorphism of the surfaces X_q and Y. $\square$

We show that via contractions one does not leave the class of log del Pezzo $\mathbb{K}^*$-surfaces. Moreover, we study their effect on the invariants $đ^{\pm}$ defined in Construction 4.1.9 and Remark 4.1.10.

Proposition 5.1.15. *Let X be a log del Pezzo $\mathbb{K}^*$-surface and $X \to Y$ a contraction of surfaces. Then Y is a log del Pezzo $\mathbb{K}^*$-surface and, according to the constellations of source and sink, we have:*

(i) *For $x^+ \in X$ and $y^+ \in Y$, we have $đ_X^+ \geq đ_Y^+$.*
(ii) *For $D_X^+ \subseteq X$ and $y^+ \in Y$, we have $1 > đ_X^+ \geq đ_Y^+$.*
(iii) *For $x^- \in X$ and $y^- \in Y$, we have $đ_Y^- \geq đ_X^-$.*
(iv) *For $D_X^- \subseteq X$ and $y^- \in Y$, we have $đ_Y^- \geq đ_X^- > -1$.*

Proof. Remark 5.1.4 and Proposition 5.1.6 tell us that Y is a del Pezzo $\mathbb{K}^*$-surface such that $X \to Y$ is equivariant. By Proposition 5.1.14 it suffices to prove the assertion for X and Y arising from defining data (A, P_X) and (A, P_Y) with P_X, P_Y slope-ordered and P_Y obtained by erasing a column v from P_X. The task is to show that elliptic fixed points of Y are at most log terminal singularities.

Consider the case that we have elliptic fixed points $x^+ \in X$ and $y^+ \in Y$. Then only for $v = v_{i1}$ there is something to show. As v is erasable, we have $n_i \geq 2$. The tuples of exponents associated with x^+ and y^+ differ only in the i-th place and are given as

$$(l_{01}, \ldots, l_{i1}, \ldots, l_{r1}), \qquad\qquad (l_{01}, \ldots, l_{i2}, \ldots, l_{r1}).$$

For the first one, we have $\ell_X^+ > 0$ by log terminality. For the second one, we have to show this property. For $l_{i2} \leq l_{i1}$ this is obvious. So let $l_{i2} > l_{i1}$. First observe

$$m_Y^+ \;=\; m_X^+ + m_{i2} - m_{i1}, \qquad\qquad \ell_Y^+ \;=\; \ell_X^+ + \frac{1}{l_{i2}} - \frac{1}{l_{i1}}.$$

Due to slope-orderedness of P_X and $m_Y^+ > 0$, we obtain $0 < m_{i1} - m_{i2} < m_X^+$. Since X is del Pezzo, we have the positive intersection number

$$0 \;<\; -\mathcal{K}_X^0 \cdot D_X^{i1} \;=\; \frac{(m_{i1} - m_{i2})\ell_X^+ - \left(\frac{1}{l_{i1}} - \frac{1}{l_{i2}}\right) m_X^+}{l_{i1}(m_{i1} - m_{i2})},$$

where we use Summary 3.3.8 for the computation. Now $\ell_Y^+ > 0$ is an immediate consequence of the estimates

$$\frac{1}{l_{i1}} - \frac{1}{l_{i2}} \;<\; \left(\frac{1}{l_{i1}} - \frac{1}{l_{i2}}\right) \frac{m_X^+}{m_{i1} - m_{i2}} \;<\; \ell_X^+ \;=\; \ell_Y^+ + \frac{1}{l_{i1}} - \frac{1}{l_{i2}}.$$

Knowing that Y is a log del Pezzo surface, the number d_Y^+ is defined and we compare it with d_X^+. Consider the difference

$$d_X^+ - d_Y^+ \;=\; \frac{m_X^+}{\ell_X^+} - \frac{m_Y^+}{\ell_Y^+} \;=\; \frac{\ell_Y^+ m_X^+ - \ell_X^+ m_Y^+}{\ell_X^+ \ell_Y^+}.$$

For $d_X^+ \geq d_Y^+$ this fraction has to be non-negative. The denominator is obviously positive and for the enumerator we compute

$$\ell_Y^+ m_X^+ - \ell_X^+ m_Y^+ \;=\; (m_{i1} - m_{i2})\ell_X^+ - \left(\frac{1}{l_{i1}} - \frac{1}{l_{i2}}\right) m_X^+$$

which is also positive, as seen in the above computation of $-\mathcal{K}_X^0 \cdot D_X^{i1}$. Thus, we verified $d_X^+ \geq d_Y^+$. The case of elliptic fixed points $x^- \in X$ and $y^- \in Y$ is transformed in to the present one via swapping the action and needs no extra treatment.

Now assume that we have a parabolic fixed point curve $D_X^+ \subseteq X$ and an elliptic fixed point $y^+ \in Y$. Then $v = v_+$ holds and we obtain

$$0 \;<\; -(D_X^+)^2 \;=\; m_X^+, \qquad 0 \;<\; -\mathcal{K}_X^0 \cdot D_X^+ \;=\; -m_X^+ + \ell_X^+,$$

as D_X^+ is contracted and X del Pezzo. Thus, $\ell_Y^+ = \ell_X^+ > m_X^+ > 0$ and $d_Y^+ = d_X^+ < 1$. Again, for the case $D_X^- \subseteq X$ and $y^- \in Y$, we just swap the action. $\square$

Proposition 5.1.16. *Consider the defining matrices P_1 and P_2 of log del Pezzo $\mathbb{K}^*$-surfaces X_1 and X_2, where P_2 arises from P_1 by removing a column.*

(i) *We have $\mathcal{A}_{P_2} \subseteq \mathcal{A}_{P_1}$ for the associated anticanonical complexes.*
(ii) *If X_1 is $1/k$-log canonical, then X_2 is $1/k$-log canonical.*

Proof. The first assertion is a consequence of Proposition 5.1.15 and the description of the anticanonical complex provided by Proposition 4.1.14 (i) and (ii). The second assertion follows from the first one and Proposition 4.1.14 (v). □

The notion of combinatorial minimality was introduced in [**26**]. The following version is adapted to the setting of rational projective $\mathbb{K}^*$-surfaces.

Definition 5.1.17. We call a normal, complete surface X *combinatorially minimal* if every contraction $X \to Y$ is an isomorphism.

Example 5.1.18. Up to isomorphy, the combinatorially minimal toric surfaces arise from fans with a generator matrix of the form $[v_1, v_2, v_3]$ or $[v_1, v_2, -v_1, -v_2]$.

Remark 5.1.19. Consider a projective $\mathbb{K}^*$-surface $X = X(A, P)$. Then the following statements are equivalent.

(i) The surface X is combinatorially minimal.
(ii) The matrix P has no contractible column.
(iii) We have $\mathrm{Eff}(X) = \mathrm{Mov}(X)$.
(iv) Each of the D_X^{ij}, D_X^+, D_X^- has non-negative self intersection.

Remark 5.1.20. For every rational, projective $\mathbb{K}^*$-surface X, there is a sequence $X \cong X_0 \to \ldots \to X_q$ with contractions as in Construction 5.1.11 such that X_q is combinatorially minimal. The length of such a sequence is bounded by the Picard number, i.e. we have $q < \rho(X)$.

Proposition 5.1.21. *Let $X = X(A, P)$ be non-toric, projective, and combinatorially minimal. Then the following statements hold.*

(i) *We have $1 \le \rho(X) \le 2$ for the Picard number $\rho(X)$.*
(ii) *If X has two elliptic fixed points, then $1 \le n_i \le 2$ for $i = 0, \ldots, r$ and $n_i = 2$ happens at most twice.*
(iii) *If X has a parabolic fixed point curve, then $n_i = 1$ for $i = 0, \ldots, r$.*

Proof. Due to Remark 5.1.12 combinatorial minimality implies $n_i \le 2$ for $i = 0, \ldots, r$ in case of two elliptic fixed points and $n_i = 1$ for $i = 0, \ldots, r$ if X admits a parabolic fixed point curve. In particular, the third assertion holds.

We prove (i). Remark 5.1.19 gives $Q(\gamma) = \mathrm{Eff}(X) = \mathrm{Mov}(X)$. Thus, on each extremal ray of the effective cone, we find at least two of the generator degrees

$$w_{ij} \;=\; \deg(T_{ij}), \qquad w_k \;=\; \deg(S_k).$$

Assume $\rho(X) \ge 3$. Then $\dim(\mathrm{Eff}(X)) \ge 3$ and we find an extremal ray $\varrho \preccurlyeq \mathrm{Eff}(X)$ hosting two generator degrees $w_{i_1 j_1}$ and $w_{i_2 j_2}$. We claim $i_1 \ne i_2$.

Otherwise $n_i \le 2$ for $i := i_1 = i_2$ implies that T_{ij_1} and T_{ij_2} are the variables of a monomial h of a defining relation g_ι. Consequently,

$$\mu \;=\; \deg(g_\iota) \;=\; \deg(h) \;=\; \deg\left(T_{ij_1}^{l_{ij_1}} T_{ij_2}^{l_{ij_2}}\right) \;=\; l_{ij_1}w_{ij_1} + l_{ij_2}w_{ij_2} \;\in\; \varrho$$

with the exponents l_{ij_1} and l_{ij_2} of h. But then all generator degrees except the w_k are located on ϱ. This is a contradiction to $m \le 2$ and $\mathrm{Eff}(X)$ being of dimension at least three. Thus, $i_1 \ne i_2$ and we find monomials

$$T_{i_1j_1}^{l_{i_1j_1}} T_{i_3j_3}^{l_{i_3j_3}}, \qquad T_{i_2j_2}^{l_{i_2j_2}} T_{i_4j_4}^{l_{i_4j_4}}$$

in the defining relations. Observe that $w_{i_1j_1}$, $w_{i_3j_3}$ as well as $w_{i_2j_2}$, $w_{i_4j_4}$ generate two-dimensional cones, both containing $\mu \in \mathrm{Mov}(X)^\circ$ in their relative interior. Hence the same holds for $\eta = \mathrm{cone}(w_{i_1j_1}, w_{i_4j_4})$. Moreover, the point $z \in \bar{X}$ with

$$z_{i_1j_1} \;=\; z_{i_4j_4} \;=\; 1$$

and all other coordinates equal to zero belongs to $\hat{X}$, see [**4**, Constr. 3.3.1.3]. This contradicts Remark 2.1.8. Thus, we verified the first assertion.

For the second assertion, note that we have $\dim(X) + \rho(X) \le 4$ and $m = 0$. Thus, the claim follows from

$$n \;=\; n_0 + \ldots + n_r \;=\; \dim(X) + \rho(X) + r - 1 \;\le\; r + 3.$$

□

5.2. The combinatorially minimal case

In this section, we classify the non-toric combinatorially minimal $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces. The classification process runs entirely in terms of the defining matrices P from Construction 3.2.2. Theorem 5.2.1 provides bounds for the entries of the defining matrices for arbitrary k and in Theorem 5.2.5 the concrete classification for $k = 1, 2, 3$ is presented.
We denote by $c(k)$ the maximum volume of almost k-hollow lattice simplices. Due to Corollary 2.3.7 we always have

$$c(k) \;\le\; \pi R(k)^2 \;=\; 2\pi k^4(2k^2 + 2k + 1).$$

Theorem 5.2.1. *Every non-toric combinatorially minimal $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surface is isomorphic to an $X(A, P)$ with P from the following list.*

(i) $\begin{bmatrix} -l_{01} & l_{11} & 0 & 0 \\ -l_{01} & 0 & l_{21} & 0 \\ d_{01} & d_{11} & d_{21} & 1 \end{bmatrix}$,

$2 \le l_{01} \le \max(2k^2, 5)$, $2 \le l_{11} \le \max(4k^2, 5)$, $2 \le l_{21} \le 5$, $-k - 2l_{01} + 1 \le d_{01} \le -1$, $1 \le d_{11} \le l_{11} - 1$, $1 \le d_{21} \le l_{21} - 1$.

(ii) $\begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{21} \end{bmatrix}$,

$1 \le l_{01} \le 2k^2$, $1 \le l_{02} \le 2k^2$, $2 \le l_{11} \le \max(6, 4k^2, c(k))$, $2 \le l_{21} \le \max(6, 4k^2, c(k))$, $-2l_{01} + 1 \le d_{01} \le k - 1$, $-2l_{02} - k + 1 \le d_{02} \le -1$, $1 \le d_{11} \le l_{11} - 1$, $1 \le d_{21} \le l_{21}$.

(iii) $$\begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & 0 & l_{21} & 0 \\ -1 & -1 & 0 & 0 & 2 \\ d_{01} & d_{02} & d_{11} & d_{21} & 1 \end{bmatrix},$$

$$2 \le l_{11} \le \max(2k^2, 5), \qquad -2 \le d_{01} < \tfrac{k}{2}, \qquad 1 \le d_{11} \le l_{11} - 1,$$
$$2 \le l_{21} \le 3, \qquad -\tfrac{k}{2} - 3 < d_{02} \le -1, \qquad 1 \le d_{21} \le l_{21} - 1.$$

(iv) $$\begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} \end{bmatrix},$$

$$1 \le l_{01} \le k^3 + 3k^2, \qquad -2(k+3)k^2 \le d_{01} \le k^4,$$
$$1 \le l_{02} \le 6k^2, \qquad -25k^5 - 15k^2 \le d_{02} \le 5k^6 + 15k^3,$$
$$1 \le l_{11} \le 6k^2, \qquad 0 \le d_{11} < l_{11},$$
$$1 \le l_{12} \le 6k^2, \qquad -5k^6 - 3k \le d_{12} \le 125k^7 + 75k^4,$$
$$2 \le l_{21} \le 6k^2, \qquad 1 \le d_{21} \le l_{21} - 1.$$

(v) $$\begin{bmatrix} -1 & -l_{02} & l_{11} & l_{12} & 0 & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} & 0 \\ -1 & -l_{02} & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{31} \end{bmatrix},$$

$$-2 \le d_{01} < k,$$
$$1 \le l_{02} \le 5, \qquad -k - 3 \le d_{02} \le k - 2,$$
$$1 \le l_{11} \le 2k^2, \qquad 0 \le d_{11} \le l_{11} - 1,$$
$$1 \le l_{12} \le \max(2k^2, 5), \qquad -20k^3 \le d_{12} \le l_{12} - 1,$$
$$2 \le l_{21} \le \max(2k^2, 5), \qquad 1 \le d_{21} \le l_{21} - 1,$$
$$2 \le l_{31} \le l_{21}, \qquad 1 \le d_{31} \le l_{31} - 1.$$

Lemma 5.2.2. *Fix $k \in \mathbb{Z}_{\ge 1}$. Let $a, b, l, d \in \mathbb{Z}_{\ge 1}$ and consider the polygon*

$$\mathcal{B} \ := \ \mathrm{conv}((0, a/b),\ (l, d),\ (0, -a/b)) \ \subseteq \ \mathbb{R}^2.$$

If $\mathcal{B}$ is k-hollow, then $l \le 2bk^2/a$.

Proof. The set $\mathcal{C} := k^{-1}(-\mathcal{B} \cup \mathcal{B})$ is centrally symmetric and convex having the origin as its only interior lattice point. The volume of $\mathcal{C}$ equals $2alb^{-1}k^{-2}$ and is due to Minkowski's Theorem bounded by 4. $\square$

Remark 5.2.3. We will use Lemma 5.2.2 to bound the entries l_{ij} in P-matrices. The polygon $\mathcal{B}$ takes the role of (part of) an arm of the corresponding anticanonical complex $\mathcal{A}_P$ and a/b is a lower bound for $\min(d^+, -d^-)$.

Lemma 5.2.4. *Let P be a defining matrix for a non-toric combinatorially minimal log del Pezzo $\mathbb{K}^*$-surface of format $(n_0, \ldots, n_r; m)$. The following assertions hold.*

(i) *If P has the format $(2,2,1;0)$ the following equations hold.*
   (a) $m_{01} + m_{12} + m_{21} \ = \ 0,$
   (b) $m_{02} + m_{11} + m_{21} \ = \ 0,$
   (c) $m_{01} - m_{02} \ = \ m_{11} - m_{12}.$

(ii) *If P has the format $(2,2,1,1;0)$ the following equations hold.*
   (a) $m_{01} + m_{12} + m_{21} + m_{31} \ = \ 0,$
   (b) $m_{02} + m_{11} + m_{21} + m_{31} \ = \ 0,$
   (c) $m_{01} - m_{02} \ = \ m_{11} - m_{12}.$

(iii) *If P has the format $(2,2,1,1,1;0)$ the following equations hold.*
   (a) $m_{01} + m_{12} + m_{21} + m_{31} + m_{41} \ = \ 0,$
   (b) $m_{02} + m_{11} + m_{21} + m_{31} + m_{41} \ = \ 0,$
   (c) $m_{01} - m_{02} \ = \ m_{11} - m_{12}.$

Proof. Exemplarily, we prove (i). The proofs of the other two assertions are analogous. Since $X = X(A,P)$ is combinatorially minimal, no column of P is contractible. Applying Remark 5.1.10 (i) and (iii) to the first four columns gives the following inequalities.

$$
\begin{aligned}
m_{02} + m_{11} + m_{21} &\leq 0,\\
m_{01} + m_{12} + m_{21} &\geq 0,\\
m_{01} + m_{12} + m_{21} &\leq 0,\\
m_{02} + m_{11} + m_{21} &\geq 0.
\end{aligned}
$$

These yield (a) and (b). Equality (c) is a direct consequence. □

Proof of Theorem 5.2.1. Let X be a non-toric combinatorially minimal $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surface. We may assume $X = X(A,P)$, where P is adjusted, see Definition 4.2.1. By Proposition 5.1.21, combinatorial minimality forces the format $(n_0,\ldots,n_r;m)$ to be one of

$$(1,\ldots,1;1), \qquad (2,1,\ldots,1;0), \qquad (2,2,1,\ldots,1;0).$$

Moreover, X is log terminal and thus Summary 3.4.11 leaves the following six possibilities for $(n_0,\ldots,n_r;m)$, listed according to Case (i) to (vi):

$$(1,1,1;1),\ (2,1,1;0),\ (2,1,1,1;0),\ (2,2,1;0),\ (2,2,1,1;0),\ (2,2,1,1,1;0).$$

Case (i). The matrix P has the shape

$$\begin{bmatrix} -l_{01} & l_{11} & 0 & 0\\ -l_{01} & 0 & l_{21} & 0\\ d_{01} & d_{11} & d_{21} & 1 \end{bmatrix}.$$

Since P is adjusted, we have $l_{i1} \geq 2$ for $i = 0,1,2$ and $1 \leq d_{i1} < l_{i1}$ for $i = 1,2$. Without loss of generality we can assume $l_{11} \geq l_{21}$. Consider the vertex $\tilde{v}^- = \mathscr{d}^- v^-$ of the anticanonical complex $\mathcal{A}_P$, where

$$\mathscr{d}^- \;=\; \frac{\mathscr{m}^-}{\ell^-} \;=\; \frac{m_{01}+m_{11}+m_{21}}{l_{01}^{-1}+l_{11}^{-1}+l_{21}^{-1}-1}.$$

Note that according to Remark 3.4.11 the denominator is positive because X is log terminal. As the numerator $\mathscr{m}^-$ is negative and $d_{11}, d_{21} \geq 1$ we have $d_{01} < 0$. On the other hand X is $1/k$-log canonical and therefore $\mathscr{d}^- \geq -k$. This gives

$$
\begin{aligned}
d_{01} &\geq l_{01}\left(-k\cdot\left(\frac{1}{l_{01}}+\frac{1}{l_{11}}+\frac{1}{l_{21}}-1\right) - m_{11} - m_{21}\right)\\
&> -kl_{01}\left(\frac{1}{l_{11}}+\frac{1}{l_{21}}\right) - k + kl_{01} - 2l_{01}\\
&\geq -kl_{01} - k + kl_{01} - 2l_{01}\\
&= -k - 2l_{01}.
\end{aligned}
$$

That means we have

$$-k-2l_{01} \;<\; d_{01} \;<\; 0.$$

We go through the possible constellations of the platonic triple (l_{01}, l_{11}, l_{21}).

Case (i).1. At least two of the coordinates of (l_{01}, l_{11}, l_{21}) *are different from* 2. Then $l_{01}, l_{11}, l_{21} \leq 5$ and every entry of P is therefore bounded.

Case (i).2. Two of the coordinates of (l_{01}, l_{11}, l_{21}) *are equal to* 2. Then $l_{11} = l_{21} = 2$ or $l_{01} = l_{21} = 2$. In the first case we have $\not{d}^- = d_{01} + l_{01} \in \mathbb{Z}$ and in the second $\not{d}^- = \frac{d_{01}l_{11}+2d_{11}+l_{11}}{2} \in \frac{1}{2}\mathbb{Z}$. Lemma 5.2.2 gives the bounds $l_{01} \le 2k^2, l_{11} \le 4k^2$ respectively. Hence, every entry of P is bounded.

Case (ii). The matrix P has the shape

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{21} \end{bmatrix}.$$

As P is adjusted, we have $1 \le d_{i1} < l_{i1}$ for $i = 1, 2$. Consider the vertices $\tilde{v}^+ = \not{d}^+ v^+$ and $\tilde{v}^- = \not{d}^- v^-$ of the anticanonical complex $\mathcal{A}_P$, where

$$\not{d}^+ \;=\; \frac{m_{01} + m_{11} + m_{21}}{l_{01}^{-1} + l_{11}^{-1} + l_{21}^{-1} - 1}, \qquad \not{d}^- \;=\; \frac{m_{02} + m_{11} + m_{21}}{l_{02}^{-1} + l_{11}^{-1} + l_{21}^{-1} - 1}.$$

Again, according to Remark 3.4.11, log terminality of X ensures that the denominators are both positive. Using $l_{11}, l_{21} \ge 2$, we obtain

$$0 \;<\; l_{01}^{-1} + l_{11}^{-1} + l_{21}^{-1} - 1 \;\le\; l_{01}^{-1}, \qquad 0 \;<\; l_{02}^{-1} + l_{11}^{-1} + l_{21}^{-1} - 1 \;\le\; l_{02}^{-1}.$$

Observe that the expressions $\not{d}^+$, m_{01} and $\not{d}^-$, m_{02} are strictly increasing in d_{01} and d_{02}, respectively. Moreover, for any $\alpha \in \mathbb{Q}$, we have

$$\not{d}^+ = \alpha \iff m_{01} = \left(l_{01}^{-1} + l_{11}^{-1} + l_{21}^{-1} - 1\right)\alpha - m_{11} - m_{21},$$

$$\not{d}^- = \alpha \iff m_{02} = \left(l_{02}^{-1} + l_{11}^{-1} + l_{21}^{-1} - 1\right)\alpha - m_{11} - m_{21}.$$

Since $0 \in \mathcal{A}_P^\circ$ and X is $1/k$-log canonical, we have $0 < \not{d}^+ \le k$ and $-k \le \not{d}^- < 0$, see Proposition 4.1.14. For $\alpha = 0, k, \pm 1$, the above considerations yield

$$-2l_{01} < d_{01} < k, \qquad \not{d}^+ < 1 \;\Rightarrow\; d_{01} < 1,$$

$$-k - 2l_{02} < d_{02} < 0, \qquad \not{d}^- > -1 \;\Rightarrow\; d_{02} > -1 - 2l_{02}.$$

We bound l_{11} and l_{21} in terms of $\alpha = \min(\not{d}^+, -\not{d}^-)$. For $i = 1, 2$, consider the polygons

$$\mathcal{B}_i \;:=\; \operatorname{conv}\left(\left(0, \not{d}^+\right), (l_{i1}, d_{i1}), (0, \not{d}^-)\right) \;\subseteq\; \mathbb{R}^2.$$

Since $\mathcal{A}_P$ is almost k-hollow, both $\mathcal{B}_i$ are k-hollow. Thus, Lemma 5.2.2 applies and we obtain

$$l_{i1} \;\le\; \frac{2}{\alpha}k^2.$$

One proceeds by going through all possible constellations of the platonic triples (l_{01}, l_{11}, l_{21}) and (l_{02}, l_{11}, l_{21}).

Case (ii).1. $l_{01} = l_{02} = 1$. In this setting, there are no a priori constraints on l_{11}, l_{21} and our task is to find suitable bounds. First we specify

$$\not{d}^+ = \frac{d_{01} + m_{11} + m_{21}}{l_{11}^{-1} + l_{21}^{-1}}, \quad \not{d}^- = \frac{d_{02} + m_{11} + m_{21}}{l_{11}^{-1} + l_{21}^{-1}}, \quad \not{d}^+ - \not{d}^- = \frac{d_{01} - d_{02}}{l_{11}^{-1} + l_{21}^{-1}}.$$

In the case $\not{d}^+ \ge 1/2$ and $\not{d}^- \le -1/2$, we have $l_{i1} \le 4k^2$ for $i = 1, 2$ as just observed. Moreover, with $\beta := \not{d}^+ - \not{d}^-$, we obtain

$$1 \;\le\; d_{01} - d_{02} \;=\; \beta\left(\frac{1}{l_{11}} + \frac{1}{l_{21}}\right).$$

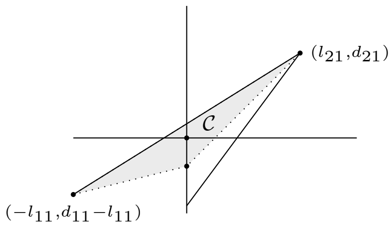
Then $\beta \le 3/2$, implies $l_{11}, l_{21} \le 6$ and thus we are left with $\beta > 3/2$. Suppose $\not{d}^+ < 1/2$. Then $d_{01} = -1$ and $\not{d}^+ > 0$ forces $m_{11} + m_{21} > 1$. Moreover,

$$\left(\frac{1}{l_{11}} + \frac{1}{l_{21}}\right) \ \ge\ \frac{1}{\beta} \ >\ \frac{1}{k+\frac{1}{2}}$$

due to $\beta = \not{d}^+ - \not{d}^- < k + \frac{1}{2}$. We conclude $\min(l_{11}, l_{21}) \le 2k$. We may assume $l_{11} \le 2k$. Then we can estimate

$$\frac{1}{2k} \ \le\ m_{11} \ \le\ \frac{2k-1}{2k}, \qquad\qquad \frac{1}{2k} \ <\ m_{21} \ \le\ 1.$$

The idea is to bound l_{21} via the volume of a suitable almost k-hollow lattice simplex. Consider the prolongated second arm of the anticanonical complex:



One directly checks that $(-l_{11}, d_{11} - l_{11})$ lies on the bounding line L through $(0, \not{d}^+)$ and (l_{21}, d_{21}). Thus, we can indeed define the polygon $\mathcal{C}$ as indicated above by

$$\mathcal{C} \ :=\ \operatorname{conv}\left((0,-1), (l_{21}, d_{21}), (-l_{11}, d_{11} - l_{11})\right).$$

We check that $\mathcal{C}$ is almost k-hollow. As a subset of $\mathcal{A}_P$, the r.h.s. part of $\mathcal{C}^\circ$ contains no k-fold lattice points except $(0,0)$. Concerning the l.h.s. part, we need

$$(-k, 0) \ \notin\ \mathcal{C}, \qquad\qquad (-k, -k) \ \notin\ \mathcal{C}.$$

Indeed, due to $\not{d}^+ < 1$ and $l_{11} \le 2k$, this gives $\mathcal{C}^\circ \cap k\mathbb{Z}^2 = \{(0,0)\}$. The slopes of L and the bounding line G through $(-l_{11}, d_{11} - l_{11})$ and $(0,-1)$ satisfy

$$m_L \ >\ m_{11} \ \ge\ \frac{1}{2k}, \qquad\qquad m_G \ =\ \frac{l_{11} - d_{11} - 1}{l_{11}} \ \le\ 1 - \frac{2}{k}.$$

Consequently, $(-k, 0)$ lies above L and $(-k, -k)$ lies below G. Altogether, $\mathcal{C}$ is almost k-hollow and we have $l_{21} \le 2\operatorname{vol}(\mathcal{C})$. Thus the case $\not{d}^+ < 1/2$ is settled. If $\not{d}^- > -1/2$, then we swap source and sink by multiplying the last row by -1. After re-adjusting, we are again in the case $\not{d}^+ < 1/2$.

Case (ii).2. $l_{01} = 1$ and $l_{02} > 1$. In this case the bounds for d_{01} and d_{02} are

$$\begin{aligned} -1 \ &\le\ d_{01} \ \le\ k - 1, \\ 1 - k - 2l_{02} \ &\le\ d_{02} \ \le\ -1. \end{aligned}$$

Case (ii).2.1. At least two of the coordinates of (l_{02}, l_{11}, l_{21}) are different from 2. Then $l_{02}, l_{11}, l_{21} \le 5$, so every entry of P is bounded.

Case (ii).2.2. Two of the coordinates of (l_{02}, l_{11}, l_{21}) are equal to 2. We have to consider the following two cases.

Case (ii).2.2.1. $(l_{02}, l_{11}, l_{21}) = (l_{02}, 2, 2)$. In this case

$$\not{d}^{+} \;=\; d_{01} + 1 \;>\; 0, \qquad \not{d}^{-} \;=\; d_{02} + l_{02} \;<\; 0.$$

So $\not{d}^{+} \geq 1$ and $\not{d}^{-} \leq -1$. By Lemma 5.2.2 we get $l_{02} \leq 2k^2$ and therefore everything is bounded.

Case (ii).2.2.2. $(l_{02}, l_{11}, l_{21}) = (2, l_{11}, 2)$. Since $(v_c^-)_3 < 0$ we get

$$d_{02} \;<\; -2m_{11} - 1 \;<\; -1$$

and thus $d_{02} \leq -2$. Equality cannot occur, because the columns of P are primitive. That means $d_{02} \leq -3 = -1 - l_{02}$ and therefore $\not{d}^{-} \leq -1$. We distinguish between the following two cases.

Case (ii).2.2.2.1. $d_{01} \geq 0$. Then additionally $\not{d}^{+} \geq 1$ and again by Lemma 5.2.2 we have $l_{11} \leq 2k^2$. Therefore, every entry of P is bounded.

Case (ii).2.2.2.2. $d_{01} \leq -1$. Then $d_{01} = -1$. Consider the line L given by

$$L(x) \;=\; \frac{d_{11}+1}{l_{11}+2} \cdot x + \frac{-l_{11} \;+\; 2d_{11}}{l_{11}+2}.$$

We have $L(0) = \not{d}^{+}$ and $L(-2) = -1$. Therefore, since the polygon

$$\operatorname{conv}\left(\left(0, \not{d}^{+}\right), (0,-1), (-2,-1)\right)$$

is k-hollow, $\not{d}^{-} \leq -1$ and X is $1/k$-log canonical, the lattice simplex

$$\mathcal{P} \;:=\; \operatorname{conv}\left((l_{11}, d_{11}), (0,-1), (-2,-1)\right)$$

is almost k-hollow. Its volume $\operatorname{vol}(\mathcal{P}) = d_{11} + 1$ is bounded. Note that l_{11} is also bounded as $l_{11} < 2d_{11}$.

Case (ii).3. $l_{01} > 1$ *and* $(l_{02}, l_{11}, l_{21}) = (1, l_{11}, l_{21})$. Multiplying the last row by -1 and re-adjusting the resulting matrix yields

$$\begin{bmatrix} -1 & -l_{01} & l_{11} & 0 \\ -1 & -l_{01} & 0 & l_{21} \\ -d_{02}-1 & -d_{01}-l_{01} & l_{11}-d_{11} & l_{21}-d_{21} \end{bmatrix}.$$

Since $d_{02} < m_{01}$ the matrix is in standard form and appears therefore in one of the cases above.

Case (ii).4. $l_{01} > 1$ *and* $l_{02} > 1$.

Case (ii).4.1. At least two of the coordinates of (l_{01}, l_{11}, l_{21}) *are different from* 2. Then $l_{01}, l_{11}, l_{21} \leq 5$. Since $l_{11} \neq 2$ or $l_{21} \neq 2$ the triple (l_{02}, l_{11}, l_{21}) contains at most one 2. So, by platonicity, $l_{02} \leq 5$ as well. Taking into account the general bounds for d_{01} and d_{02}, we see that every entry of P is bounded.

Case (ii).4.2. Two of the coordinates of (l_{01}, l_{11}, l_{21}) *are equal to* 2. The following two cases have to be considered.

Case (ii).4.2.1. $(l_{01}, l_{11}, l_{21}) = (l_{01}, 2, 2)$. We have

$$\not{d}^{+} \;=\; d_{01} + l_{01} \;>\; 0, \qquad \not{d}^{-} \;=\; d_{02} + l_{02} \;<\; 0.$$

Hence $\not{d}^{+} \geq 1$ and $\not{d}^{-} \leq -1$ and thus $l_{01}, l_{02} \leq 2k^2$ by Lemma 5.2.2 and every entry is bounded.

Case (ii).4.2.2. $(l_{01}, l_{11}, l_{21}) = (2, l_{11}, 2)$. We distinguish between the following two cases.

Case (ii).4.2.2.1. $l_{02} = 2$. Then we have

$$\begin{aligned} \not{d}^{+} &= \frac{d_{01}l_{11} + 2d_{11} + l_{11}}{2} > 0, \\ \not{d}^{-} &= \frac{d_{02}l_{11} + 2d_{11} + l_{11}}{2} < 0. \end{aligned}$$

By Lemma 5.2.2 we get $l_{11} \leq 4k^2$, so every entry of P is bounded.

Case (ii).4.2.2.2. $l_{02} > 2$. Then $(l_{02}, l_{11}, 2)$ is one of the following platonic triples.

$$(5,3,2), (3,5,2), (4,3,2), (3,4,2), (3,3,2).$$

Therefore, every entry is bounded.

Case (iii). The matrix P has the shape

$$\begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{31} \\ d_{01} & d_{02} & d_{11} & d_{21} & d_{31} \end{bmatrix},$$

where we require without loss of generality $l_{11} \geq l_{21} \geq l_{31}$. Both of the platonic tuples have to contain more than 2 entries different from 1 since P is irredundant. That means $l_{01} = l_{02} = 1$ and therefore $d_{01} > d_{02}$. Moreover,

$$\begin{aligned} \not{d}^{+} &= \frac{l_{11}l_{21}l_{31}d_{01} + l_{21}l_{31}d_{11} + l_{11}l_{31}d_{21} + l_{11}l_{21}d_{31}}{l_{11}l_{21} + l_{11}l_{31} + l_{21}l_{31} - l_{11}l_{21}l_{31}}, \\ \not{d}^{-} &= \frac{l_{11}l_{21}l_{31}d_{02} + l_{21}l_{31}d_{11} + l_{11}l_{31}d_{21} + l_{11}l_{21}d_{31}}{l_{11}l_{21} + l_{11}l_{31} + l_{21}l_{31} - l_{11}l_{21}l_{31}}. \end{aligned}$$

Since $\not{d}^{+} > 0$ and $\not{d}^{-} < 0$ we get $d_{01} \geq -2$ and $d_{02} \leq -1$. Additionally, the conditions $(v_c^+)_3 \leq k$ and $(v_c^-)_3 \geq -k$ yield

$$d_{01} < \frac{k}{2}, \qquad d_{02} > -\frac{k}{2} - 3.$$

Now we distinguish between the cases depending on the specific platonic tuples.

Case (iii).1. At least two of the coordinates of (l_{11}, l_{21}, l_{31}) *are different from* 2. Then each l_{i1} is bounded by 5, the d_{i1} are therefore bounded as well.

Case (iii).2. Two of the coordinates of (l_{11}, l_{21}, l_{31}) *are equal to* 2. Then without loss of generality $(l_{11}, l_{21}, l_{31}) = (l_{11}, 2, 2)$. We have

$$\begin{aligned} \not{d}^{+} &= d_{01}l_{11} + d_{11} + l_{11} \geq 1, \\ \not{d}^{-} &= d_{02}l_{11} + d_{11} + l_{11} \leq -1. \end{aligned}$$

By Lemma 5.2.2 there is the bound $l_{11} \leq 2k^2$.

Case (iv). The matrix P has the following shape.

$$\begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} \end{bmatrix},$$

where $l_{21} \geq 2$ since P is irredundant. The vertices $\tilde{v}^+ = \not{d}^+ v^+$ and $\tilde{v}^- = \not{d}^- v^-$ of the anticanonical complex $\mathcal{A}_P$ of X are determined by

$$\begin{aligned} \not{d}^+ &= \frac{l_{11}l_{21}d_{01} + l_{01}l_{21}d_{11} + l_{01}l_{11}d_{21}}{l_{01}l_{11} + l_{01}l_{21} + l_{11}l_{21} - l_{01}l_{11}l_{21}}, \\ \not{d}^- &= \frac{l_{12}l_{21}d_{02} + l_{02}l_{21}d_{12} + l_{02}l_{12}d_{21}}{l_{02}l_{12} + l_{02}l_{21} + l_{12}l_{21} - l_{02}l_{12}l_{21}}. \end{aligned}$$

The conditions $0 < \not{d}^+ \leq k$ and $-k \leq \not{d}^- < 0$ yield

$$\begin{aligned} -2l_{01} + 1 \;&\leq\; d_{01} &&< \; k\left(\frac{l_{01}}{2} + 1\right), \\ -k\left(\frac{l_{02}}{2} + 1\right) - 2l_{02} \;&<\; l_{12}d_{02} + l_{02}d_{12} &&\leq \; -1. \end{aligned}$$

As before, we go through the different possible constellations of the platonic tuples.

Case (iv).1. (l_{01}, l_{11}, l_{21}) *and* (l_{02}, l_{12}, l_{21}) *are platonic triples with at most two coordinates different from* 1. Since $l_{21} \neq 1$ we can assume $(l_{01}, l_{11}, l_{21}) = (1, l_{11}, l_{21})$ without loss of generality.

Case (iv).1.1. $(l_{02}, l_{12}, l_{21}) = (1, l_{12}, l_{21})$. Thus, the matrix P looks like

$$\begin{bmatrix} -1 & -1 & l_{11} & l_{12} & 0 \\ -1 & -1 & 0 & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} \end{bmatrix}.$$

First note that by Lemma 5.2.4 (i)(b) we have

$$-m_{11} - m_{21} \;=\; d_{02} \;\in\; \mathbb{Z}.$$

Since $0 < m_{11}, m_{21} < 1$ we obtain $d_{02} = -1$. The bounds from above become

$$\begin{aligned} 0 \;&\leq\; d_{01} &&< \; \frac{3}{2}k, \\ -\frac{3}{2}k - 2 \;&<\; d_{12} - l_{12} &&\leq \; -1. \end{aligned}$$

Hence

$$l_{12}\left(-3k - 1\right) \;<\; d_{12} \;\leq\; l_{12} - 1.$$

If $l_{11} = 1$, we get $d_{11} = 0$. Again, by Lemma 5.2.4 (i)(a) we see $m_{21} = 1$, which is a contradiction. If $l_{12} = 1$, Lemma 5.2.4 (i)(a) gives

$$m_{21} \;=\; -d_{01} - d_{12} \;\in\; \mathbb{Z}.$$

This is a contradiction as well. Therefore $l_{11}, l_{12} \geq 2$. We have

$$\not{d}^+ \;=\; \frac{l_{11}l_{21}d_{01} + l_{21}d_{11} + l_{11}d_{21}}{l_{11} + l_{21}}.$$

Since $d_{01} \geq 0$ we get $\not{d}^+ \geq 1$. Using Lemma 5.2.4 (i)(a) we also obtain

$$\not{d}^- \;=\; \frac{m_{12} + m_{21} - 1}{\frac{1}{l_{12}} + \frac{1}{l_{21}}} \;=\; \frac{-d_{01} - 1}{l_{12}^{-1} + l_{21}^{-1}}.$$

Since $l_{12}, l_{21} \geq 2$ and $d_{01} \geq 0$ we have $\not{d}^- \leq -1$. Hence we can use Lemma 5.2.2 to conclude $l_{11}, l_{12}, l_{21} \leq 2k^2$, so every entry is bounded.

Case (iv).1.2. $(l_{02}, l_{12}, l_{21}) = (l_{02}, 1, l_{21})$. Using Lemma 5.2.4 (i)(a) we get

$$d_{01} + d_{12} \;=\; -m_{21}.$$

That means $m_{21} \in \mathbb{Z}$. Since $\gcd(l_{21}, d_{21}) = 1$ this yields $l_{21} = 1$ which contradicts the irredundancy of P. So this case cannot occur.

Case (iv).2. $l_{01}, l_{11} > 1$ and (l_{02}, l_{12}, l_{21}) is a platonic triple with at most two coordinates different from 1. Without loss of generality $l_{02} = 1$.

Case (iv).2.1. (l_{01}, l_{11}, l_{21}) contains two 2*'s.*

Case (iv).2.1.1. $l_{01} = l_{11} = 2$. That means the matrix P takes the form

$$P \;=\; \begin{bmatrix} -2 & -1 & 2 & l_{12} & 0 \\ -2 & -1 & 0 & 0 & l_{21} \\ d_{01} & d_{02} & 1 & d_{12} & d_{21} \end{bmatrix}.$$

By Lemma 5.2.4 (i)(b) we have

$$2m_{21} \;=\; -2d_{02} - 1 \;\in\; \mathbb{Z}.$$

Since $0 < m_{21} < 1$ this gives $\frac{2d_{21}}{l_{21}} = 1$ and therefore $d_{21} = 1$ and $l_{21} = 2$. Looking at the above equation again, we see $d_{02} = -1$. The matrix P and the values $\not{d}^+, \not{d}^-$ become

$$P \;=\; \begin{bmatrix} -2 & -1 & 2 & l_{12} & 0 \\ -2 & -1 & 0 & 0 & 2 \\ d_{01} & -1 & 1 & d_{12} & 1 \end{bmatrix}, \quad \not{d}^+ \;=\; d_{01} + 2, \quad \not{d}^- \;=\; \frac{2d_{12} - l_{12}}{l_{12} + 2}.$$

So $\not{d}^+ \geq 1$ and $d_{01} \geq -1$. Also $\not{d}^- \leq -1$ if and only if $d_{12} \leq -1$. We examine the different possible cases.

Case (iv).2.1.1.1. $d_{12} \leq -1$. In this case $l_{12} \leq 2k^2$ by Lemma 5.2.2. Hence

$$-\frac{3}{2}k \;\leq\; d_{12} \;\leq\; -1.$$

That means every entry of P is bounded.

Case (iv).2.1.1.2. $d_{12} \geq 0$. Again, by 5.2.4 (i)(a) we have

$$\frac{d_{01}}{2} + m_{12} \;=\; -\frac{1}{2}.$$

So $d_{01} \geq 0$ cannot occur in this case. Hence $d_{01} = -1$. Then $d_{12} = 0$ and $l_{12} = 1$.

Case (iv).2.1.2. $l_{01} = l_{21} = 2$. The matrix P is given by

$$P \;=\; \begin{bmatrix} -2 & -1 & l_{11} & l_{12} & 0 \\ -2 & -1 & 0 & 0 & 2 \\ d_{01} & d_{02} & d_{11} & d_{12} & 1 \end{bmatrix}.$$

By Lemma 5.2.4 (i)(b) we have

$$m_{11} \;=\; -\frac{1}{2} - d_{02}.$$

Since $0 \leq m_{11} < 1$ we obtain $d_{02} = -1$. In turn, we see $m_{11} = \frac{1}{2}$, so $l_{11} = 2$ and $d_{11} = 1$. That means we are in the setting of Case (iv).2.1.1.

Case (iv).2.1.3. $l_{11} = l_{21} = 2$. The matrix P is given by

$$P = \begin{bmatrix} -l_{01} & -1 & 2 & l_{12} & 0 \\ -l_{01} & -1 & 0 & 0 & 2 \\ d_{01} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}.$$

Without loss of generality $l_{01} \geq 3$ as $l_{01} = 1$ was covered in Case (iv).1.1 and $l_{01} = 2$ in Case (iv).2.1.1. By Lemma 5.2.4 (i)(b) we have $d_{02} = -1$. Therefore, we have

$$\cancel{d}^{+} = l_{01} + d_{01}, \qquad \cancel{d}^{-} = \frac{2d_{12} - l_{12}}{l_{12} + 2}.$$

Hence $\cancel{d}^{-} \leq -1$ if and only if $d_{12} \leq -1$. In this case we have bounds for l_{01} and l_{12} by Lemma 5.2.2. So we consider the case that $d_{12} \geq 0$. By convexity of the 1. arm of the anticanonical complex $\mathcal{A}_P$ we conclude $d_{12} = 0$ and therefore by k-hollowness $l_{12} \leq k$. This in turn allows us to use Lemma 5.2.2 again and we obtain the bound $l_{01} \leq (k+3) \cdot k^2$. Therefore, every entry of P is bounded.

Case (iv).2.2. (l_{01}, l_{11}, l_{21}) *contains exactly one* 2. Then $l_{01}, l_{11}, l_{21} \leq 5$. By Lemma 5.2.4 (i)(b) we have

$$m_{12} = -m_{01} - m_{21} = \frac{-d_{01}l_{21} - d_{21}l_{01}}{l_{01}l_{21}}.$$

Hence $l_{12} \leq l_{01}l_{21} \leq 20$. Then the value of d_{12} is bounded as well and so is therefore every entry of P.

Case (iv).3. (l_{01}, l_{11}, l_{21}) *is a platonic triple with at most two coordinates different from* 1 *and* $l_{02}, l_{12} > 1$. Without loss of generality $l_{01} = 1$. The matrix P is therefore given by

$$P = \begin{bmatrix} -1 & -l_{02} & l_{11} & l_{12} & 0 \\ -1 & -l_{02} & 0 & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} \end{bmatrix}.$$

Swapping source and sink, i.e. multiplying the last row by -1, and re-adjusting the matrix, we end up with

$$\begin{bmatrix} -l_{02} & -1 & l_{11} & l_{12} & 0 \\ -l_{02} & -1 & 0 & 0 & l_{21} \\ -d_{02} - 2l_{02} & -d_{01} - 2 & l_{11} - d_{11} & l_{12} - d_{12} & l_{21} - d_{21} \end{bmatrix}.$$

These matrices were classified in the previous case.

Case (iv).4. $l_{01}, l_{02}, l_{11}, l_{12} > 1$. We examine the following cases depending on the type of the occurring platonic tuples.

Case (iv).4.1. (l_{01}, l_{11}, l_{21}) *contains two* 2*'s.* Without loss of generality $l_{01} = 2$.

Case (iv).4.1.1. $l_{11} = 2$.

Case (iv).4.1.1.1. (l_{02}, l_{12}, l_{21}) *contains two* 2*'s.*

Case (iv).4.1.1.1.1. $l_{02} = l_{12} = 2$. Then

$$P = \begin{bmatrix} -2 & -2 & 2 & 2 & 0 \\ -2 & -2 & 0 & 0 & l_{21} \\ d_{01} & d_{02} & 1 & d_{12} & d_{21} \end{bmatrix}.$$

By Lemma 5.2.4 (i)(b) we get

$$2m_{21} \;=\; -d_{02} - 1 \;\in\; \mathbb{Z}.$$

So $2m_{21} = 1$ and hence $l_{21} = 2$ and $d_{21} = 1$. Therefore $d_{21} = -2$ which contradicts the primitivity of the columns of P.

Case (iv).4.1.1.1.2. $l_{02} = l_{21} = 2$. That means

$$P \;=\; \begin{bmatrix} -2 & -2 & 2 & l_{12} & 0 \\ -2 & -2 & 0 & 0 & 2 \\ d_{01} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}.$$

Using Lemma 5.2.4 (i)(b) again we obtain $d_{02} = -2$ which is a contradiction to the primitivity of the columns.

Case (iv).4.1.1.1.3. $l_{12} = l_{21} = 2$. Then

$$P \;=\; \begin{bmatrix} -2 & -l_{02} & 2 & 2 & 0 \\ -2 & -l_{02} & 0 & 0 & 2 \\ d_{01} & d_{02} & 1 & d_{12} & 1 \end{bmatrix}.$$

By Lemma 5.2.4 (i)(b) we have $m_{02} = -1$. Therefore $d_{02} = -1$ and $l_{02} = 1$. We get the bounds

$$\begin{aligned} -1 \;&\leq\; d_{01} \;\leq\; 0, \\ -\frac{3}{2}k - 2 \;&<\; d_{12} \;\leq\; 0. \end{aligned}$$

Case (iv).4.1.1.2. (l_{02}, l_{12}, l_{21}) *contains exactly one* 2. So there are the bounds $2 \leq l_{02}, l_{12}, l_{21} \leq 5$ and hence

$$\begin{aligned} -2 \;&\leq\; d_{01} \;<\; k\left(\frac{l_{01}}{2} + 1\right), \\ -2k - 7 \;&<\; d_{02} \;\leq\; d_{01} - 1, \\ -k - 1 \;&\leq\; d_{12} \;\leq\; 2. \end{aligned}$$

Case (iv).4.1.2. $l_{21} = 2$. In this case we have

$$P \;=\; \begin{bmatrix} -2 & -l_{02} & l_{11} & l_{12} & 0 \\ -2 & -l_{02} & 0 & 0 & 2 \\ d_{01} & d_{02} & d_{11} & d_{12} & 1 \end{bmatrix}.$$

Case (iv).4.1.2.1. (l_{02}, l_{12}, l_{21}) *contains two* 2*'s.* We examine the following cases.

Case (iv).4.1.2.1.1. $l_{02} = 2$. By Lemma 5.2.4 (i)(b) we have

$$-2m_{11} \;=\; d_{02} + 1 \;\in\; \mathbb{Z}.$$

Since $0 \leq m_{11} < 1$ we conclude

$$m_{11} \;\in\; \left\{0, \frac{1}{2}\right\}.$$

The case $m_{11} = \frac{1}{2}$ cannot occur as it leads to $d_{02} = -2$ which contradicts the primitivity of the columns of P. Hence $l_{11} = 1, d_{11} = 0$ and $d_{02} = -1$. Therefore

$$\not{d}^{+} \;=\; \frac{d_{01} + 1}{2}, \qquad \not{d}^{-} \;=\; d_{12}.$$

Using Lemma 5.2.2 we see that l_{12} is bounded. Moreover, we have

$$\begin{aligned} -3 &\le d_{01} \le 2k-1, \\ -3k^3-2k-1 &\le d_{12} < 6k^4+2k^3+3k^2. \end{aligned}$$

Hence, every entry of P is bounded.

Case (iv).4.1.2.1.2. $l_{12}=2$. Then

$$d\!\!\!/^{+} = \frac{d_{01}l_{11}+l_{11}+2d_{11}}{2}, \qquad d\!\!\!/^{-} = \frac{d_{12}l_{02}+l_{02}+2d_{02}}{2}.$$

As before, by Lemma 5.2.2 the values of l_{02} and l_{11} are bounded. In the same manner as above we get the bounds

$$\begin{aligned} -3 &\le d_{01} \le 2k-1, \\ -4k^3-6k^2-k &< d_{02} \le d_{01}-1, \\ -3k-1 &\le d_{12} \le 1. \end{aligned}$$

Case (iv).4.1.2.2. (l_{02},l_{12},l_{21}) *contains exactly one* 2. That is, $3 \le l_{02},l_{12} \le 5$. Then

$$d\!\!\!/^{+} = \frac{d_{01}l_{11}+l_{11}+2d_{11}}{2}, \qquad d\!\!\!/^{-} = \frac{2d_{02}l_{12}+2d_{12}l_{02}+l_{02}l_{12}}{2l_{02}+2l_{12}-l_{02}l_{12}}.$$

The denominator of $d\!\!\!/^{-}$ is bounded by 3. Lemma 5.2.2 says that therefore $l_{11} \le 6k^2$. Furthermore

$$\begin{aligned} -3 &\le d_{01} \le 2k-1, \\ -14k-8 &\le d_{02} \le d_{01}-1, \\ -7k-1 &\le d_{12} \le 4. \end{aligned}$$

Case (iv).4.2. (l_{01},l_{11},l_{21}) *contains exactly one* 2.

Case (iv).4.2.1. $l_{01}=2$ *or* $l_{11}=2$. Without loss of generality $l_{01}=2$. Then $3 \le l_{11},l_{21} \le 5$. Since (l_{02},l_{12},l_{21}) is also platonic with no entry equal to 1, we have $2 \le l_{02},l_{12} \le 5$. As before we additionally obtain the bounds

$$\begin{aligned} -3 &\le d_{01} \le 2k-1, \\ -14k-9 &\le d_{02} \le d_{01}-1, \\ -7k+1 &\le d_{12} \le 14k-10. \end{aligned}$$

Case (iv).4.2.2. $l_{21}=2$. Then $3 \le l_{01},l_{11} \le 5$.

Case (iv).4.2.2.1. (l_{02},l_{12},l_{21}) *contains two* 2*'s.* The following cases can occur.

Case (iv).4.2.2.1.1. $l_{02}=2$. We have

$$d\!\!\!/^{+} = \frac{2d_{01}l_{11}+2d_{11}l_{01}+l_{01}l_{11}}{2l_{01}+2l_{11}-l_{01}l_{11}}, \qquad d\!\!\!/^{-} = \frac{d_{02}l_{12}+l_{12}+2d_{12}}{2}.$$

The bounds for l_{01} and l_{11} show that the denominator of $d\!\!\!/^{+}$ is bounded by 3. Hence, by Lemma 5.2.2 we see that $l_{12} \le 6k^2$. We additionally get the bounds

$$\begin{aligned} -9 \ &\leq\ d_{01} \ \leq\ 4k-1, \\ -5k^3-k-5 \ &\leq\ d_{02} \ \leq\ d_{01}-1, \\ -12k^3+3k^2-2k-1 \ &\leq\ d_{12} \ \leq 13k^5+3k^3+13k^2-1. \end{aligned}$$

Case (iv).4.2.2.1.2. $l_{12}=2$. In this case

$$\cancel{d}^+ \ = \ \frac{2d_{01}l_{11}+2d_{11}l_{01}+l_{01}l_{11}}{2l_{01}+2l_{11}-l_{01}l_{11}}, \qquad \cancel{d}^- \ = \ \frac{d_{12}l_{02}+l_{02}+2d_{02}}{2}.$$

That means, as before, that the denominator of $\cancel{d}^+$ is bounded by 3. Therefore, by Lemma 5.2.2 we see $l_{02} \leq 6k^2$. Moreover

$$\begin{aligned} -9 \ &\leq\ d_{01} \ \leq\ 4k-1, \\ -5k^3-10k^2-k+1 \ &\leq\ d_{02} \ \leq\ d_{01}-1, \\ -6k \ &\leq\ d_{12} \ \leq 5k^3+10k^2-2. \end{aligned}$$

Every entry of P is bounded.

Case (iv).4.2.2.2. (l_{02}, l_{12}, l_{21}) contains exactly one 2. That means $3 \leq l_{02}, l_{12} \leq 5$. Hence we get the following bounds.

$$\begin{aligned} -9 \ &\leq\ d_{01} \ \leq\ 4k-1, \\ -14k-8 \ &\leq\ d_{02} \ \leq\ d_{01}-1, \\ -9k \ &\leq\ d_{12} \ \leq 25k-10. \end{aligned}$$

Case (v). The matrix P has the shape

$$\begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{31} \end{bmatrix},$$

where we require without loss of generality $l_{21} \geq l_{31}$. We look at the vertices $\tilde{v}^+ = \cancel{d}^+ v^+$ and $\tilde{v}^- = \cancel{d}^- v^-$ of the anticanonical complex $\mathcal{A}_P$. Here

$$\begin{aligned} \cancel{d}^+ \ &= \ \frac{l_{11}l_{21}l_{31}d_{01}+l_{01}l_{21}l_{31}d_{11}+l_{01}l_{11}l_{31}d_{21}+l_{01}l_{11}l_{21}d_{31}}{l_{01}l_{11}l_{21}+l_{01}l_{11}l_{31}+l_{01}l_{21}l_{31}+l_{11}l_{21}l_{31}-2l_{01}l_{11}l_{21}l_{31}}, \\ \cancel{d}^- \ &= \ \frac{l_{12}l_{21}l_{31}d_{02}+l_{02}l_{21}l_{31}d_{12}+l_{02}l_{12}l_{31}d_{21}+l_{02}l_{12}l_{21}d_{31}}{l_{02}l_{12}l_{21}+l_{02}l_{12}l_{31}+l_{02}l_{21}l_{31}+l_{12}l_{21}l_{31}-2l_{02}l_{12}l_{21}l_{31}}. \end{aligned}$$

Since $0 < \cancel{d}^+ \leq k$ and $-k \leq \cancel{d}^- < 0$ we get the conditions

$$\begin{aligned} -3l_{01}+1 \ &\leq\ d_{01} \ && \leq\ k-1, \\ -l_{02}l_{12}(k+2)+1 \ &\leq\ l_{12}d_{02}+l_{02}d_{12} \ && \leq\ -1. \end{aligned}$$

Log terminality of X yields the following cases.

Case (v).1. $(l_{01}, l_{11}, l_{21}, l_{31})$ and $(l_{02}, l_{12}, l_{21}, l_{31})$ are platonic tuples with at most two coordinates different from 1. Since $l_{21}, l_{31} \neq 1$ we have

$$l_{01} \ = \ l_{02} \ = \ l_{11} \ = \ l_{12} \ = \ 1.$$

Therefore $d_{11} = 0$ and P is given by

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 \\ -1 & -1 & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{31} \end{bmatrix}.$$

By Lemma 5.2.4 (ii)(b) we have

$$-d_{02} \;=\; m_{21} + m_{31} \;\in\; (0,2).$$

Hence $d_{02} = -1$. In this case, we obtain the bounds

$$\begin{aligned} 0 \;&\le\; d_{01} \;\le\; k-1, \\ -k \;&\le\; d_{12} \;\le\; -1. \end{aligned}$$

We have

$$\mathscr{d}^+ \;=\; \frac{d_{01}l_{21}l_{31} + d_{31}l_{21} + d_{21}l_{31}}{l_{21}+l_{31}}, \; \mathscr{d}^- \;=\; \frac{(d_{12}-1)l_{21}l_{31} + d_{31}l_{21} + d_{21}l_{31}}{l_{21}+l_{31}}.$$

Since $d_{01} \ge 0$ and $d_{21}, d_{31} \ge 1$ this yields $\mathscr{d}^+ \ge 1$. The condition $d_{12} \le 1$ gives

$$\mathscr{d}^- \;\le\; \frac{d_{31}l_{21} + d_{21}l_{31} - 2l_{21}l_{31}}{l_{21}+l_{31}}.$$

The fraction on the right is smaller or equal to -1 if and only if

$$\frac{d_{21}+1}{l_{21}} + \frac{d_{31}+1}{l_{31}} \;\le\; 2.$$

This is obviously satisfied since $d_{21} < l_{21}$ and $d_{31} < l_{31}$. Hence $\mathscr{d}^+ \ge 1$ and $\mathscr{d}^- \le -1$. By Lemma 5.2.2 we have $l_{21} \le 2k^2$ and so every entry of P is bounded since $l_{21} \ge l_{31}$.

Case (v).2. $(l_{01}, l_{11}, l_{21}, l_{31}) = (1, 1, l_{21}, l_{31})$ *and* ($l_{02} > 1$ *or* $l_{12} > 1$). Without loss of generality $l_{12} > 1$. Then

$$\begin{bmatrix} -1 & -1 & 1 & l_{12} & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 \\ -1 & -1 & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{31} \end{bmatrix},$$

where $d_{01} > d_{02}$, $d_{12} < 0$ and (l_{12}, l_{21}, l_{31}) is a platonic triple with each component greater than 1. As before, Lemma 5.2.4 (ii)(b) gives $d_{02} = -1$. The conditions $d_{01} \ge 0$ and $d_{12} \le -1$ yield $\mathscr{d}^+ \ge 1$ and $\mathscr{d}^- \le -1$, respectively. By Lemma 5.2.2 we have $l_{12}, l_{21}, l_{31} \le 2k^2$. Only d_{12} remains to be bounded. The second inequality from the beginning of the case yields

$$\begin{aligned} d_{12} \;&\ge\; -l_{12}(k+2) + 1 + l_{12} \\ &=\; -l_{12}(k+1) + 1. \end{aligned}$$

Therefore every entry of P is bounded.

Case (v).3. ($l_{01} > 1$ *or* $l_{11} > 1$) *and* $(l_{02}, l_{12}, l_{21}, l_{31}) = (1, 1, l_{21}, l_{31})$. Without loss of generality $l_{11} > 1$. Then

$$\begin{bmatrix} -1 & -1 & l_{11} & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 \\ -1 & -1 & 0 & 0 & 0 & l_{31} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{31} \end{bmatrix},$$

where $d_{01} > d_{02}$, $d_{12}l_{11} < d_{11}$ and (l_{11}, l_{21}, l_{31}) is a platonic triple with each component greater than 1. The inequalities from the beginning of the case become

$$\begin{aligned} -2 \ &\leq \ d_{01} && \leq \ k-1, \\ -k-1 \ &\leq \ d_{02} + d_{12} && \leq \ -1. \end{aligned}$$

Keeping in mind $d_{01} > d_{02}$, these give the bounds

$$\begin{aligned} -k-1 \ &\leq \ d_{02} \ \leq \ k-2, \\ -2k+1 \ &\leq \ d_{12} \ \leq \ 0. \end{aligned}$$

We examine the following cases.

Case (v).3.1. The tuple (l_{11}, l_{21}, l_{31}) *contains exactly one* 2. In this case each entry of P is bounded since $l_{i1} > d_{i1} \geq 0$ for $i \geq 1$.

Case (v).3.2. The tuple (l_{11}, l_{21}, l_{31}) *contains two* 2*'s.* Without loss of generality $l_{31} = 2$. Hence $d_{31} = 1$. We have the following two cases.

Case (v).3.2.1. $l_{11} = 2$. Then $d_{11} = 1$ and

$$\not{d}^{+} \ = \ d_{01}l_{21} + d_{21} + l_{21}, \qquad \not{d}^{-} \ = \ \frac{(2d_{02} + 2d_{12} + 1)l_{21} + 2d_{21}}{l_{21} + 2}.$$

Therefore $\not{d}^{+} \geq 1$. Furthermore we have

$$\not{d}^{-} \ \leq \ -1 \iff d_{02} + d_{12} \ \leq \ -\frac{d_{21} + 1}{l_{21}} - 1.$$

That means

$$d_{02} + d_{12} \ \leq \ -2 \implies \not{d}^{-} \ \leq \ -1.$$

Note that $d_{02} + d_{12} \leq -1$. If $d_{02} + d_{12} \leq -2$ the value of l_{21} is bounded by Lemma 5.2.2. If $d_{02} + d_{12} = -1$ we consider the line passing through $(-2, -1)$ and (l_{21}, d_{21}). It intersects the vertical axis in $(0, \not{d}^{-})$. Hence, there is the almost k-hollow lattice simplex

$$\mathcal{P} \ := \ \operatorname{conv}\left(\begin{pmatrix} l_{21} \\ d_{21} \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

Its volume

$$\operatorname{vol}(\mathcal{P}) \ = \ l_{21} - d_{21} + 1$$

is bounded by $c(k)$. Then $l_{21} \leq 2c(k) - 3$.

Case (v).3.2.2. $l_{21} = 2$. Then $d_{21} = 1$. We have

$$\not{d}^{+} \ = \ d_{01}l_{11} + d_{11} + l_{11}, \qquad \not{d}^{-} \ = \ d_{02} + d_{12} + 1.$$

Therefore, the value of l_{11} is bounded by Lemma 5.2.2.

Case (v).4. ($l_{01} > 1$ *or* $l_{11} > 1$) *and* ($l_{02} > 1$ *or* $l_{12} > 1$). Without loss of generality $l_{01} = 1$. We examine the following cases.

Case (v).4.1. $l_{02} = 1$. The bounds from the beginning of the case yield

$$\begin{aligned} -2 &\leq d_{01} \leq k-1, \\ -(k+3) &\leq d_{02} \leq k-2, \\ -2kl_{12}+1 &\leq d_{12} < l_{12}(k+3). \end{aligned}$$

Case (v).4.1.1. The tuple (l_{11}, l_{21}, l_{31}) *contains exactly one* 2. Then we have $l_{11}, l_{12}, l_{21}, l_{31} \leq 5$. Therefore every entry of P is bounded.

Case (v).4.1.2. The tuple (l_{11}, l_{21}, l_{31}) *contains two* 2*'s.* Without loss of generality $l_{31} = 2$. Hence $d_{31} = 1$.

Case (v).4.1.2.1. $l_{11} = 2$. Then $d_{11} = 1$. We distinguish between the following cases.

Case (v).4.1.2.1.1. $l_{12} \neq 2$ *and* $l_{21} \neq 2$. Since (l_{12}, l_{21}, l_{31}) is a platonic triple, we have $l_{12}, l_{21} \leq 5$. Hence every entry of P is bounded.

Case (v).4.1.2.1.2. $l_{12} = 2$. Then

$$\not{d}^+ = d_{01}l_{21} + d_{21} + l_{21}, \qquad \not{d}^- = \frac{(2d_{02} + d_{12} + 1)\, l_{21}}{2} + d_{21}.$$

By Lemma 5.2.2 we get $l_{21} \leq 4k^2$. Every entry of P is therefore bounded.

Case (v).4.1.2.1.3. $l_{21} = 2$. Then $d_{21} = 1$. In this case

$$\not{d}^+ = 2d_{01} + 3, \qquad \not{d}^- = d_{02}l_{12} + d_{12} + l_{12}.$$

We can use Lemma 5.2.2 again and we see that $l_{12} \leq 2k^2$.

Case (v).4.1.2.2. $l_{21} = 2$. Then $d_{21} = 1$ and

$$\begin{aligned} (v_c^+)_4 &= d_{01}l_{11} + d_{11} + l_{11}, \\ (v_c^-)_4 &= d_{02}l_{12} + d_{12} + l_{12}. \end{aligned}$$

Hence, l_{11} and l_{12} are bounded by $2k^2$ by Lemma 5.2.2. Therefore, every entry of P is bounded.

Case (v).4.2. $l_{12} = 1$. We get the following bounds from the bounds of the beginning of the case.

$$\begin{aligned} -2 &\leq d_{01} \leq k-1, \\ -l_{02}(k+3) &\leq d_{02} \leq k-2, \\ -2k+1 &\leq d_{12} \leq k+2. \end{aligned}$$

Case (v).4.2.1. The tuple (l_{11}, l_{21}, l_{31}) *contains exactly one* 2. Then

$$l_{11}, l_{21}, l_{31} \leq 5.$$

Since (l_{02}, l_{21}, l_{31}) is a platonic triple we also have $l_{02} \leq 5$. Every entry of P is therefore bounded.

Case (v).4.2.2. The tuple (l_{11}, l_{21}, l_{31}) *contains two* 2*'s.* Without loss of generality $l_{31} = 2$ and hence $d_{31} = 1$.

Case (v).4.2.2.1. $l_{11} = 2$. Then $d_{11} = 1$. Since $(l_{02}, l_{21}, 2)$ has to be a platonic triple, there are the following cases.

Case (v).4.2.2.1.1. $l_{02} = 2$. Then

$$\not{d}^{+} \;=\; l_{21}d_{01} + d_{21} + l_{21}, \not{d}^{-} \;=\; \frac{(d_{02} + 2d_{12} + 1)\, l_{21}}{2} + d_{21}.$$

Thus, using Lemma 5.2.2 again, we see that l_{21} is bounded by $4k^2$.

Case (v).4.2.2.1.2. $l_{02} > 2$. In this case $l_{02}, l_{21} \le 5$. So, every entry of P is bounded.

Case (v).4.2.2.2. $l_{21} = 2$. Then $d_{21} = 1$ and

$$\not{d}^{+} \;=\; d_{01}l_{11} + d_{11} + l_{11}, \qquad \not{d}^{-} \;=\; d_{12}l_{02} + d_{02} + l_{02}.$$

By Lemma 5.2.2 we have $l_{02}, l_{11} \le 2k^2$. So every entry of P is bounded.

Case (vi). The matrix P has the shape

$$\begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 0 & l_{31} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 0 & 0 & l_{41} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} & d_{31} & d_{41} \end{bmatrix},$$

where we require without loss of generality $l_{21} \ge l_{31} \ge l_{41}$. Since P is irredundant, we have $l_{21}, l_{31}, l_{41} \ne 1$ and therefore

$$l_{01} \;=\; l_{02} \;=\; l_{11} \;=\; l_{12} \;=\; 1.$$

Hence $d_{11} = 0$ and the matrix becomes

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & l_{31} & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & l_{41} \\ d_{01} & d_{02} & 0 & d_{12} & d_{21} & d_{31} & d_{41} \end{bmatrix}.$$

By Lemma 5.2.4 (iii)(b) we have

$$m_{21} + m_{31} + m_{41} \;=\; -d_{02} \;\in\; \mathbb{Z}.$$

There are the following possible cases.

$$(l_{21}, l_{31}, l_{41}) \;\in\; \{(5,3,2), (4,3,2), (3,3,2), (l_{21},2,2)\}.$$

Depending on the specific values we get one of the following conditions.

$$\frac{d_{21}}{5} + \frac{d_{31}}{3} + \frac{1}{2} \;\in\; \mathbb{Z}, \qquad \frac{d_{21}}{4} + \frac{d_{31}}{3} + \frac{1}{2} \;\in\; \mathbb{Z},$$
$$\frac{d_{21}}{3} + \frac{d_{31}}{3} + \frac{1}{2} \;\in\; \mathbb{Z}, \qquad \frac{d_{21}}{l_{21}} + \frac{1}{2} + \frac{1}{2} \;\in\; \mathbb{Z},$$

where $1 \le d_{i1} < l_{i1}$. The fourth condition yields a contradiction. The others respectively lead to

$$30 \mid 6d_{21} + 10d_{31} + 15, \qquad 12 \mid 3d_{21} + 4d_{31} + 6, \qquad 6 \mid 2d_{21} + 2d_{31} + 3.$$

The third condition cannot hold. For the other two we have respectively

$$6d_{21} + 10d_{31} \;=\; 15, \qquad 3d_{21} + 4d_{31} \;=\; 6.$$

Going through the cases we see that these equations lead to contradictions as well. So this type of matrix cannot occur at all. $\square$

For each k, Theorem 5.2.1 provides a finite list of defining matrices P that deliver in particular all the non-toric combinatorially minimal $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces. For the final classification, we reduce the list by checking the required properties computationally and using the normal form from Definition 4.2.26. Altogether we arrive at the following results.

Theorem 5.2.5. *We obtain the following statements on non-toric combinatorially minimal ε-log canonical del Pezzo $\mathbb{K}^*$-surfaces.*

- $\varepsilon = 1$: *There are exactly* 13 *sporadic and* 2 *one-parameter families of non-toric combinatorially minimal canonical del Pezzo $\mathbb{K}^*$-surfaces.*
- $\varepsilon = \frac{1}{2}$: *There are exactly* 62 *sporadic and* 5 *one-parameter families of non-toric combinatorially minimal* $1/2$*-log canonical del Pezzo $\mathbb{K}^*$-surfaces.*
- $\varepsilon = \frac{1}{3}$: *There are exactly* 318 *sporadic and* 14 *one-parameter families of non-toric combinatorially minimal* $1/3$*-log canonical del Pezzo $\mathbb{K}^*$-surfaces.*

5.3. Classifying $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces

We are ready for the classification of non-toric $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces. Theorem 5.3.9 shows basic data of the classsification for $k = 1, 2, 3$. The full list of defining data will be made available in [**25**]. The idea is to build up the defining P-matrices from the almost k-hollow polygons classified in Theorem 2.3.10 on the one hand and the defining data of the non-toric combinatorially minimal $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces classified in Theorem 5.2.5 on the other.

Construction 5.3.1. Fix $k \in \mathbb{Z}_{\geq 1}$. With any almost k-hollow LDP-polygon $\mathcal{A} \subseteq \mathbb{R}^2$, we associate a defining matrix $P_{\mathcal{A}}$ with $r = 1$, having the vertices of $\mathcal{A}$ as its columns:

$$P_{\mathcal{A}} \ := \ [v_{01}, \ldots, v_{0n_0}, v_{11}, \ldots, v_{1n_1}, v^{\pm}],$$

where $v_{0j} = (-l_{0j}, d_{0j})$ and $v_{1j} = (l_{1j}, d_{1j})$ and, if present, $v^{\pm} = (0, \pm 1)$. Suitable numbering of the columns ensures that $P_{\mathcal{A}}$ is slope-ordered. Conversely, setting

$$\mathcal{A}_{P} \ := \ \mathrm{conv}(v; \ v \text{ column of } P)$$

for every slope-ordered defining matrix P with $r = 1$ of a toric $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surface gives an almost k-hollow LDP-polygon.

Definition 5.3.2. Fix $k \in \mathbb{Z}_{\geq 1}$ and consider the lattice vectors $v^+ = (0, 1)$ and $v^- = (0, -1)$.

(i) We denote by $\mathfrak{A}_k$ the set of all almost k-hollow LDP-polygons $\mathcal{A}$ such that $\mathrm{conv}(\mathcal{A} \cup \{v^+\})$ or $\mathrm{conv}(\mathcal{A} \cup \{v^-\})$ is almost k-hollow.

(ii) We write $\mathfrak{P}_k = \{P_{\mathcal{A}}; \ \mathcal{A} \in \mathfrak{A}_k\}$ for the set of defining matrices associated with the polygons from $\mathfrak{A}_k$.

Remark 5.3.3. The set $\mathfrak{P}_k$ is invariant under admissible operations of types (i), (ii), (iii), (iv) and (v).

Proposition 5.3.4. *Let $X_1 \to X_2$ be a contraction of non-toric $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces as in Construction 5.1.11, where X_1 is non-toric and X_2 is toric and combinatorially minimal. Then the defining matrix P_2 stems from $\mathfrak{P}_k$.*

PROOF. Let (A_i, P_i) be the defining data of X_i. Then P_1 has three arms, at most six columns. We may assume that a column of the third arm of P_1 is contracted. Then the arms of P_2 are the first two of P_1 and Proposition 4.1.4 (iv) guarantees that P_2 belongs to $\mathfrak{P}_k$. □

Construction 5.3.5. Let $\mathcal{B}$ be an almost k-hollow LDP-polygon with $q \le 5$ vertices. We obtain a finite set $\mathfrak{A}(\mathcal{B}) \subseteq \mathfrak{A}_k$ of almost k-hollow LDP-polygons with at most 4 vertices via the following procedure.

- For each primitive lattice point $v \in \mathcal{B}$ fix unimodular matrices M_v^+, M_v^- such that
$$M_v^+ \cdot v \;=\; v^+, \qquad \det(M_v^+) \;=\; 1,$$
$$M_v^- \cdot v \;=\; v^-, \qquad \det(M_v^-) \;=\; -1.$$
- Start with $\mathfrak{A}(\mathcal{B}) := \emptyset$. With each primitive vector $v \in \mathcal{B}$ perform the following steps.
  - If $4 \le q \le 5$ and v is a vertex of $\mathcal{B}$ but $-v$ is not, then consider the convex hull $\mathcal{B}_v$ over all vertices of $\mathcal{B}$ except v. If $\mathcal{B}_v$ is an LDP-polygon, then add $M_v^+ \cdot \mathcal{B}_v$ and $M_v^- \cdot \mathcal{B}_v$ to $\mathfrak{A}(\mathcal{B})$.
  - If $3 \le q \le 4$ and neither v nor $-v$ is a vertex of $\mathcal{B}$, then set $\mathcal{B}_v := \mathcal{B}$ and add $M_v^+ \cdot \mathcal{B}_v$ and $M_v^- \cdot \mathcal{B}_v$ to $\mathfrak{A}(\mathcal{B})$.

Moreover, given any (finite) set $\mathfrak{B}$ of almost k-hollow LDP-polygons with at most 5 vertices, we obtain a (finite) set of almost k-hollow LDP-polygons with at most 4 vertices by setting
$$\mathfrak{A}(\mathfrak{B}) \;:=\; \bigcup_{\mathcal{B} \in \mathfrak{B}} \mathfrak{A}(\mathcal{B}).$$

Proposition 5.3.6. *Let the set $\mathfrak{B}$ represent all almost k-hollow LDP-polygons with at most 5 vertices up to unimodular equivalence. Then every matrix $P \in \mathfrak{P}_k$ with at most four columns arises via admissible operations of types (i), (ii) and (iv) from some matrix $P_{\mathcal{C}}$ with $\mathcal{C} \in \mathfrak{A}(\mathfrak{B})$.*

PROOF. Given $P \in \mathfrak{P}_k$, consider the associated LDP-polygon $\mathcal{A}_P \in \mathfrak{A}_k$. By definition of $\mathfrak{A}_k$, one of the following holds.
$$v^+ \;\in\; \mathcal{B}^+ \;:=\; \operatorname{conv}(\mathcal{A}_P \cup \{v^+\}), \qquad v^- \;\in\; \mathcal{B}^- \;:=\; \operatorname{conv}(\mathcal{A}_P \cup \{v^-\}).$$
In either case, by the choice of $\mathfrak{B}$, we find a unimodular matrix U with $U \cdot \mathcal{B}^\pm \in \mathfrak{B}$. Set $v := U \cdot v^\pm$. Then, with $M_v^\pm$ and $\mathcal{B}_v^\pm$ from Construction 5.3.5, we have
$$\mathcal{C}^+ \;:=\; M_v^+ \cdot \mathcal{B}_v^+ \;\in\; \mathfrak{A}(\mathfrak{B}), \qquad \mathcal{C}^- \;:=\; M_v^- \cdot \mathcal{B}_v^- \;\in\; \mathfrak{A}(\mathfrak{B}).$$
The matrix $P_{\mathcal{C}^\pm}$ associated with $\mathcal{C}^\pm$ equals $M_v^\pm \cdot U \cdot P$. As $M_v^\pm \cdot U$ fixes $v^\pm$ up to sign, P arises from $P_{\mathcal{C}^\pm}$ via admissible operations of types (i), (ii) and (iv). □

The main result of [**1**] shows that for given $\varepsilon > 0$ there are up to deformation only finitely many ε-log terminal surfaces. Here is an effective version of this statement for the case of non-toric $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces.

Theorem 5.3.7. *Consider a non-toric $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surface $X(A,P)$, where P is irredundant, slope-ordered and adapted to the source. Fix $\alpha > 0$ such that $d_Y^+ > \alpha$ and $d_Y^- < -\alpha$ for any combinatorially minimal $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surface Y and any toric surface Y arising from Proposition 5.3.6. Set $\ell := 2\alpha^{-1}k^2$.*

(i) *The number $r+1$ of arms of P is bounded by $4k$ in the case of two elliptic fixed points and by $2k+2$ in the case of only one elliptic fixed point.*

(ii) *We have $n_i \le 2\ell + 1$ for $i = 0, \ldots, r$. Moreover, the entries of P are bounded by $l_{ij} \le \ell$ and $-l_{ij}(4k + r - 1) < d_{ij} < l_{ij}$.*

Proof. Assertion (i) is clear by Proposition 4.1.15. We show (ii). Proposition 5.1.16 yields $d^+ > \alpha$ and $d^- < -\alpha$ for any defining matrix P of a $1/k$-log canonical del Pezzo $\mathbb{K}$-surface. Thus, Lemma 5.2.2 provides the desired common bound for all the l_{ij}. Now, by convexity of the i-th arm of $\mathcal{A}_P$, there can be at most $2\ell + 1$ columns in the i-th arm of P. That means that we have $n_i \le 2\ell + 1$.

Finally, we have to bound the numbers d_{ij}. From Theorem 4.1.14, we infer $d^+ \le k$ and $d^- \ge -k$. As P is adapted to the source, we have $0 \le m_{i1} < 1$ for $i = 1, \ldots, r$. Remark 3.3.6 says $\ell^+ \le 2$ and $\ell^- \le 2$. Altogether, this yields

$$m_{01} \ \le k\ \ell^+ - \sum_{i=1}^{r} m_{i1} \ \le\ 2k, \qquad m_{0n_0} \ \ge\ -k\ell^- - \sum_{i=1}^{r} m_{in_i} \ \ge\ -2k - r.$$

Thus, besides $d_{i1}, \ldots, d_{ir}$ also d_{01} and d_{0n_0} are bounded as claimed. Due to slope-orderedness, the remaining d_{ij} are bounded once we have bounded m_{in_i} from below for $i = 1, \ldots, r$. This is seen as follows.

$$m_{in_i} \ \le\ -\ell^- - m_{0n_0} - \sum_{q \ne i,0} m_{qn_q} \ \le\ -2k - 2k - r + 1 \ =\ -4k - r + 1.$$

□

Algorithm 5.3.8. Let $k \in \mathbb{Z}_{\ge 1}$. The input is a finite set $\mathfrak{B}$ representing up to unimodular equivalence all almost k-hollow lattice polygons and a finite set $\mathfrak{M}$ containing up to isomorphism all defining matrices P of non-toric combinatorially minimal $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces.

- Compute the set $\mathfrak{A}(\mathfrak{B})$ of defining matrices P with $r = 1$ from Construction 5.3.5.
- Compute the set $\mathfrak{S}_0$ of all irredundant defining matrices P with $r = 2$ and entries bounded according to Theorem 5.3.7 that arise from $\mathfrak{A}(\mathfrak{B})$ via a redundant extension followed by a proper one.
- Compute the union $\mathfrak{S}_1 := \mathfrak{S}_0 \cup \mathfrak{M}$, bring the matrices of $\mathfrak{S}_1$ into normal form and, this way, remove all doubled entries $\mathfrak{S}_1$.
- Compute the set $\mathfrak{S}$ of all irredundant defining matrices P with data bounded according to Theorem 5.3.7 that arise from $\mathfrak{S}_1$ via a series of redundant and proper extensions.
- Bring the matrices of $\mathfrak{S}$ into normal form and, in this way, remove all entries of $\mathfrak{S}$ that appear multiple times.

The output set $\mathfrak{S}$ contains precisely one member for each equivalence class of defining matrices P of non-toric $1/k$-log canonical del Pezzo $\mathbb{K}^*$-surfaces.

Theorem 5.3.9. *We obtain the following statements on non-toric ε-log canonical del Pezzo $\mathbb{K}^*$-surfaces.*

- $\varepsilon = 1$: *There are exactly* 30 *sporadic and* 4 *one-parameter families of non-toric canonical del Pezzo $\mathbb{K}^*$-surfaces. The maximal Picard number is* 4*, realized by* 1 *sporadic and* 1 *one-parameter family.*
- $\varepsilon = \frac{1}{2}$: *There are exactly* 998 *sporadic,* 184 *one-parameter families,* 40 *two-parameter families,* 12 *three-parameter families,* 2 *four-parameter families and* 1 *five-parameter family of non-toric* $1/2$*-canonical del Pezzo $\mathbb{K}^*$-surfaces. The maximal Picard number is* 8*, realized by the unique five-parameter family.*
- $\varepsilon = \frac{1}{3}$: *There are exactly* 65022 *sporadic,* 12402 *one-parameter families,* 3190 *two-parameter families,* 917 *three-parameter families,* 254 *four-parameter families,* 64 *five-parameter families,* 14 *six-parameter families,* 6 *seven-parameter families,* 2 *eight-parameter families and* 1 *nine-parameter family of non-toric* $1/3$*-canonical del Pezzo $\mathbb{K}^*$-surfaces. The maximal Picard number is* 12*, realized by the unique nine-parameter family.*

Proposition 5.3.10 ([**34**],5.4)**.** *Let X be a log del Pezzo $\mathbb{K}^*$-surface of Gorenstein index ι. Then X is $1/\iota$-log canonical.*

Note that for $\iota > 1$, the converse is false in general. Now, using appropriate algorithms we obtain classifications of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index up to 3 from Theorem 5.3.9.

Theorem 5.3.11. *We have the following statements on non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index ι.*

- $\iota = 1$: *There are exactly* 30 *sporadic and* 4 *one-parameter families of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index* 1*. The maximal Picard number is 4, realized by* 1 *sporadic and* 1 *one-parameter family.*
- $\iota = 2$: *There are exactly* 53 *sporadic,* 17 *one-parameter families,* 7 *two-parameter families,* 3 *three-parameter families,* 1 *four-parameter family and* 1 *five-parameter family of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index* 2*. The maximal Picard number is* 8*, realized by the unique five-parameter family.*
- $\iota = 3$: *There are exactly* 268 *sporadic,* 123 *one-parameter families,* 67 *two-parameter families,* 36 *three-parameter families,* 18 *four-parameter families,* 10 *five-parameter families,* 5 *six-parameter families,* 3 *seven-parameter families,* 1 *eight-parameter family and* 1 *nine-parameter family of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index* 3*. The maximal Picard number is* 12*, realized by the unique nine-parameter family.*

Example 5.3.12. The unique family of non-toric ε-log canonical del Pezzo $\mathbb{K}^*$-surfaces having maximal number of relations and maximal Picard number listed in Theorem 5.3.9 under $\varepsilon = 2, 3$ is part of the following example series. Let $k \in \mathbb{Z}_{\geq 1}$ and consider the defining matrix

$$P_k = \begin{bmatrix} -1 & -1 & 1 & 1 & \dots & 0 & 0 \\ -1 & -1 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & 0 \\ -1 & -1 & 0 & 0 & & 1 & 1 \\ 2k & 2k-1 & 0 & -1 & \dots & 0 & -1 \end{bmatrix} \in \operatorname{Mat}(4k, 8k; \mathbb{Z}).$$

Note that the corresponding $\mathbb{K}^*$-surface X_k is del Pezzo by the Kleiman conditions. Furthermore we have

$$d^+ = \frac{m^+}{\ell^+} = \frac{2k}{4k - (4k-2)} = k,$$

$$d^- = \frac{m^-}{\ell^-} = \frac{2k - 1 - (4k-1)}{4k - (4k-2)} = -k.$$

Hence, the relative interiors of the lineality part and the arms of $\mathcal{A}_{X_k}$ do not contain k-fold lattice points except 0. So X_k is $1/k$-log canonical. This shows that the bound for the number of relations from Proposition 4.1.15 is sharp.

We take a look at the minimal resolution of singularities of X_k. The first step of the canonical resolution from Summary 3.4.2 yields a matrix P_k' by adding the vectors v^+ and v^- to P_k. The corresponding surface X_k' is smooth. The self intersection numbers of the corresponding parabolic fixed point curves are

$$(D^+_{X_k'})^2 = -m^+ = -2k, \qquad (D^-_{X_k'})^2 = m^- = -2k.$$

That means, they are not contractible and thus the canonical morphism $X_k' \to X_k$ is the minimal resolution. Since $D^+_{X_k'}$ and $D^-_{X_k'}$ do not intersect, there are precisely 2 singularities. The respective resolution graphs each consist of exactly one smooth rational curve having self intersection number $-2k$.

Using Proposition 3.4.9 we see that X_k has Gorenstein index k. Hence, the maximal number of relations in the Cox Ring of $1/k$-log canonical $\mathbb{K}^*$-surfaces is attained by ones of Gorenstein index k. This is notable since by Proposition 5.3.10 the class of log terminal $\mathbb{K}^*$-surfaces of Gorenstein index k forms a proper subclass of $1/k$-log canonical $\mathbb{K}^*$-surfaces.

CHAPTER 6

K-polystability and Ricci-flat Kähler cone metrics

6.1. Toric degenerations and K-polystability

Given a rational variety X with torus action of complexity one, we discuss certain families $\mathcal{X}_\kappa \to \mathbb{K}$ having general fiber X and a not necessarily normal toric variety as a special fiber. Each family itself also comes with a torus action of complexity one and the map $\mathcal{X}_\kappa \to \mathbb{K}$ is equivariant. The precise construction is performed in terms of defining data.

Construction 6.1.1. Consider defining data (A, P, Σ). Given an integer κ with $0 \le \kappa \le r$, we obtain new defining data $(A, P_\kappa, \Sigma_\kappa)$ with an $(r+s+1) \times (n+1+m)$ matrix P_κ and a fan Σ_κ in $\mathbb{Z}^{r+s+1}$ via the following procedure.

(i) The matrix P_κ arises from P by first appending a zero row at the bottom and then inserting $(v_{\kappa n_\kappa}, 1)$ as a new column at the place ij with $i = \kappa$ and $j = n_\kappa + 1$.
(ii) The fan Σ_κ in $\mathbb{Z}^{r+s+1}$ has the maximal cones $(\sigma \times 0) + \varrho_{\kappa n_\kappa+1}$, where σ runs through the maximal cones of Σ and $\varrho_{\kappa n_\kappa+1}$ denotes the ray through the new column.

We will denote by Z the toric variety defined by Σ and by $X \subseteq Z$ the $\mathbb{T}^s$-variety arising from (A, P, Σ). Similarly, Z_κ is the toric variety defined by Σ_κ and $\mathcal{X}_\kappa \subseteq Z_\kappa$ the $\mathbb{T}^{s+1}$-variety arising from $(A, P_\kappa, \Sigma_\kappa)$.

We take a look at the geometry of the Construction. First, let us see in detail how the involved ambient toric varieties Z_κ and Z interact.

Remark 6.1.2. In the setting of Construction 6.1.1, let $F_\kappa \colon \mathbb{Z}^{r+s+1} \to \mathbb{Z}^{r+s+1}$ be the linear isomorphism keeping e_i fixed for $i = 1, \ldots, r+s$ and sending e_{r+s+1} to the vector $(-v_{\kappa n_\kappa}, 1)$. Then we have a commutative diagram, where both downward arrows represent the projection onto the $(r+s+1)$-th coordinate:

$$
\begin{tikzcd}
\mathbb{Z}^{r+s+1} \arrow[rr, "F_\kappa", "\cong"'] \arrow[dr] & & \mathbb{Z}^{r+s+1} \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}
$$

The map F_κ is an isomorphism of fans from Σ_κ in $\mathbb{Z}^{r+s+1}$ to the fan product of Σ in $\mathbb{Z}^{r+s}$ and the fan of faces of $\mathbb{Q}_{\ge 0}$ in $\mathbb{Z}$. Accordingly, we have a commutative diagram of the associated toric morphisms, involving the ambient toric varieties Z_κ of $\mathcal{X}_\kappa$ and Z of X:

$$\begin{tikzcd} Z_\kappa \arrow[rr, "\varphi_\kappa", "\cong"'] \arrow[dr, "\Psi_\kappa"'] & & Z \times \mathbb{K} \arrow[dl, "\mathrm{pr}_{\mathbb{K}}"] \\ & \mathbb{K} & \end{tikzcd}$$

Observe that Ψ_κ is given in Cox coordinates by $[z_{ij}, z_k] \mapsto z_{\kappa n_\kappa+1}$. Moreover, in terms of the acting tori $\mathbb{T}^{r+s+1}$ of Z_κ and $\mathbb{K}^*$ of $\mathbb{K}$, the map Ψ_κ sends an element $t = (t_1, \ldots, t_{r+s+1})$ to its last coordinate t_{r+s+1}. Consequently, for all points $z \in Z_\kappa$ and all $t \in \mathbb{T}^{r+s+1}$, we have

$$\Psi_\kappa(t \cdot z) \; = \; t_{r+s+1}\Psi_\kappa(z).$$

Finally, the fiber $\Psi_\kappa^{-1}(0)$ equals the toric prime divisor of Z_κ defined by the ray through $v_{\kappa n_\kappa+1}$ and thus, being a toric orbit closure, it comes with the structure of a toric variety. We will identify the toric variety $\Psi_\kappa^{-1}(0)$ with Z via the toric morphism given by

$$\mathbb{Z}^{r+s+1}/\mathbb{Z}v_{\kappa n_\kappa+1} \;\to\; \mathbb{Z}^{r+s}, \qquad v + \mathbb{Z}v_{\kappa n_\kappa+1} \;\mapsto\; \mathrm{pr}_{\mathbb{Z}^{r+s}} \circ F_\kappa(v).$$

Now we examine the family $\mathcal{X}_\kappa \to \mathbb{K}$. The first of the subsequent two Remarks relates the Cox ring of $\mathcal{X}_\kappa$ to that of X. In the second one, we take a look at the fibers and at the equivariance properties of the family.

Remark 6.1.3. Consider the $\mathbb{T}^s$-variety X defined by (A, P, Σ) with a complete fan Σ. Recall that we have $\mathrm{Cl}(X) = K$ for the divisor class group and that the Cox ring is given by

$$\mathcal{R}(X) \; = \; \mathbb{K}[T_{ij}, S_k]/\langle g_0, \ldots, g_{r-2}\rangle, \qquad g_\iota = \det \begin{bmatrix} T_\iota^{l_\iota} & T_{\iota+1}^{l_{\iota+1}} & T_{\iota+2}^{l_{\iota+2}} \\ a_\iota & a_{\iota+1} & a_{\iota+2} \end{bmatrix},$$

where the K-degrees of the T_{ij} and S_k are the classes of the basis vectors e_{ij} and e_k in $K = \mathbb{Z}^{m+n}/\mathrm{im}(P^*)$, respectively. Now consider the $\mathbb{T}^{s+1}$-variety $\mathcal{X}_\kappa$ arising from the data $(A, P_\kappa, \Sigma_\kappa)$ as in Construction 6.1.1. Then we have

$$\mathrm{Cl}(\mathcal{X}_\kappa) \; = \; K, \qquad \mathcal{R}(\mathcal{X}_\kappa) \; = \; \mathbb{K}[T_{ij}, S_k]/\langle g_{\kappa,0}, \ldots, g_{\kappa,r-2}\rangle,$$

where the new variable $T_{\kappa n_\kappa+1}$ is of K-degree zero and all other variables T_{ij} and S_k have the same K-degree in $\mathcal{R}(\mathcal{X}_\kappa)$ as they have in $\mathcal{R}(X)$. Moreover, the defining relations $g_{\kappa,\iota}$ are related to the g_ι via

$$g_{\kappa,\iota} = g_\iota(\tilde{T}_{ij};\, 0 \le i \le r,\, 1 \le j \le n_i), \quad \tilde{T}_{ij} := \begin{cases} T_{\kappa n_\kappa} T_{\kappa n_\kappa+1}, & i = \kappa,\ j = n_\kappa, \\ T_{ij}, & \text{else.} \end{cases}$$

Remark 6.1.4. Consider $X \subseteq Z$ and $\mathcal{X}_\kappa \subseteq Z_\kappa$ as in Construction 6.1.1. Restricting $\Psi_\kappa \colon Z_\kappa \to \mathbb{K}$ from Remark 6.1.2 gives a morphism $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$. The vanishing ideal of the fiber over $\zeta \in \mathbb{K}$ in Cox coordinates is given as

$$I(\psi_\kappa^{-1}(\zeta)) \; = \; \langle g_{\kappa,0}, \ldots, g_{\kappa,r-2}\rangle + \langle T_{\kappa n_\kappa+1} - \zeta\rangle.$$

Moreover, the morphism $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$ is compatible with the $\mathbb{T}^{s+1}$-action on $\mathcal{X}_\kappa$ and the multiplication on $\mathbb{K}$ in the sense that for every point $z \in \mathcal{X}_\kappa$ and every element $t = (t_1, \ldots, t_{s+1}) \in \mathbb{T}^{s+1}$ we have

$$\psi_\kappa(t \cdot z) \; = \; t_{s+1}\psi_\kappa(z).$$

In particular, for any $v \in \mathbb{Z}^{s+1}$ of the form $v = (v_1, \dots, v_s, 1)$, the corresponding one parameter subgroup $\lambda_v \colon \mathbb{K}^* \to \mathbb{T}^{s+1}$ gives a $\mathbb{K}^*$-action on $\mathcal{X}_\kappa$ such that for all $z \in \mathcal{X}_\kappa$ and $t \in \mathbb{K}^*$ we have

$$\psi_\kappa(\lambda_v(t) \cdot x) \;=\; t\psi_\kappa(z).$$

Proposition 6.1.5. *Consider $X = X(A, P, \Sigma)$ with complete Σ and $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$ as provided by Remark 6.1.4. For $\zeta \in \mathbb{K}$ write $\mathcal{X}_{\kappa,\zeta} := \psi_\kappa^{-1}(\zeta)$. Then, the following assertions hold.*

(i) *The variety $\mathcal{X}_\kappa$ is irreducible and normal and the morphism $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$ is a proper flat family.*
(ii) *For $\zeta \neq 0$, we have $\mathcal{X}_{\kappa,\zeta} \cong X$. The fiber $\mathcal{X}_{\kappa,0}$ is the closure of a $\mathbb{T}^{s+1}$-orbit in $\mathcal{X}_\kappa$ and hence an irreducible toric variety.*
(iii) *The special fiber $\mathcal{X}_{\kappa,0}$ is normal if and only if at most one of the monomials $T_i^{l_i}$, where $i \neq \kappa$, has exponents l_{ij} strictly bigger than 1.*

Remark 6.1.6. Note that the normality condition from Proposition 6.1.5 (ii) implies quasismoothness of X.

Lemma 6.1.7. *Consider $\Psi_\kappa \colon Z_\kappa \to \mathbb{K}$ and its restriction $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$. Then we have $\mathcal{X}_{\kappa,0} = \psi_\kappa^{-1}(0) \subseteq \Psi_\kappa^{-1}(0) = Z$. Moreover, with suitable $b_i \in \mathbb{K}^*$ the vanishing ideal $\mathcal{I}_{\kappa,0}$ of $\mathcal{X}_{\kappa,0}$ in Cox coordinates of Z is the K-prime binomial ideal*

$$\mathcal{I}_{\kappa,0} \;=\; \langle T_1^{l_1} + b_i T_i^{l_i};\ i = 2, \dots, r\rangle, \qquad \text{if } \kappa = 0,$$

$$\mathcal{I}_{\kappa,0} \;=\; \langle T_0^{l_0} + b_i T_i^{l_i};\ i = 1, \dots, r,\ i \neq \kappa\rangle, \qquad \text{if } \kappa \neq 0.$$

Proof. The first statement is clear by construction. Moreover using the specific nature of the defining trinomial relations $g_{\kappa,\iota}$ we obtain the shape of $\mathcal{I}_{\kappa,0}$, see also the proof of [**27**, Prop. 10.7]. Finally, K-primeness is ensured by [**27**, Prop. 10.7]. □

Proof of Proposition 6.1.5. Assertion (i) is clear by construction. We prove (ii). From Remarks 6.1.3 and 6.1.4 we infer $\mathcal{X}_{\kappa,1} \cong X$. Using equivariance of ψ_κ, we see $\mathcal{X}_{\kappa,\zeta} \cong X$ for all $\zeta \in \mathbb{K}^*$. The fiber $\mathcal{X}_{\kappa,0}$ is the intersection of $\mathcal{X}_\kappa$ with the toric divisor of Z_κ given by the ray through $v_{\kappa n_\kappa + 1}$. Thus, $\mathcal{X}_\kappa$ is the closure of a $\mathbb{T}^{s+1}$-orbit in $\mathcal{X}_\kappa$. Assertion (iii) is an application of the Jacobian criterion for normality of complete intersections. Take the Jacobian J of the binomial generator system from Lemma 6.1.7 and see that J is of full rank outside a closed subset of codimension at least two in $\mathcal{X}_\kappa$ if and only if at most one of the monomials $T_i^{l_i}$, where $i \neq \kappa$, has exponents l_{ij} strictly bigger than 1. □

Corollary 6.1.8. *Let X arise from (A, P, Σ) with polytopal Σ and let L be an ample line bundle on the ambient toric variety Z defined by Σ. Then we have the following.*

(i) *The line bundle L extends to an ample line bundle $\mathcal{L}$ on the ambient toric variety $Z_\kappa \cong Z \times \mathbb{K}$ of $\mathcal{X}_\kappa$.*
(ii) *Any $\mathbb{K}^*$-action λ_v as in Remark 6.1.4 turns $(\mathcal{X}_\kappa, \mathcal{L})$ into a test configuration in the sense of* [**13**, Def. 1.1].

(iii) *A test configuration as in (ii) is special if and only if at most one of the monomials* $T_0^{l_0}, \ldots, T_r^{l_r}$ *has exponents* l_{ij} *strictly bigger than* 1.

Construction 6.1.9. Consider defining data (A, P, Σ), where the fan Σ is complete. The leaves of the associated tropical variety are

$$\tau_i \;=\; \operatorname{cone}(e_i) + \operatorname{lin}(e_{r+1}, \ldots, e_{r+s}), \qquad\qquad i \;=\; 0, \ldots, r,$$

where $e_1, \ldots, e_{r+1} \in \mathbb{Q}^{r+1}$ are the canonical basis vectors and $e_0 = -e_1 - \cdots - e_r$. Then, for $\kappa = 0, \ldots, r$, we obtain lattice fans $(\Sigma_\kappa, N_\kappa)$ by setting

$$V_\kappa \;:=\; \operatorname{lin}(\tau_\kappa), \qquad N_\kappa \;:=\; \mathbb{Z}^{r+s} \cap V_\kappa, \qquad \Sigma_\kappa \;:=\; \{\sigma \cap V_\kappa;\ \sigma \in \Sigma\}.$$

Proposition 6.1.10. *Consider* $X = X(A, P, \Sigma)$ *with complete* Σ *and a family* $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$ *as provided by Remark 6.1.4.*

(i) *The toric degeneration* $\mathcal{X}_{\kappa,0}$ *of* X *has* $(\Sigma_\kappa, N_\kappa)$ *as its convergency fan.*

(ii) *The toric variety* X_κ *associated with* $(\Sigma_\kappa, N_\kappa)$ *is the normalization of* $\mathcal{X}_{\kappa,0}$.

Proof. We look at $\mathcal{X}_{\kappa,0} = \psi_\kappa^{-1}(0) \subseteq \Psi_\kappa^{-1}(0) = Z$. Moreover, the coordinate functions $\chi_1, \ldots, \chi_r$ and $\eta_1, \ldots, \eta_s$ on the acting torus $\mathbb{T}^{r+s} \subseteq Z$ satisfy

$$p^*(\chi_i) \;=\; \frac{T_i^{l_i}}{T_0^{l_0}}, \quad i = 1, \ldots, r, \qquad p^*(\eta_j) \;=\; S_j, \quad j = 1, \ldots, s.$$

Using Lemma 6.1.7, we conclude that $X_{\kappa,0} := \mathcal{X}_{\kappa,0} \cap \mathbb{T}^{r+s}$ is described by binomials of the form $\chi_1 + b_i\chi_i$ if $\kappa = 0$ and $1 + b_i\chi_i$ for $\kappa \neq 0$. Thus, applying a suitable element $t(\kappa) \in \mathbb{T}^{r+s}$, we obtain

$$\begin{aligned} t(\kappa) \cdot X_{\kappa,0} &\;=\; V(\chi_1 - \chi_i;\ i = 2, \ldots, r), \\ t(\kappa) \cdot X_{\kappa,0} &\;=\; V(\chi_i - 1;\ i = 1, \ldots, r,\ i \neq \kappa), \end{aligned}$$

according to $\kappa = 0$ and $\kappa \neq 0$. For every $\kappa = 0, \ldots, r$, this gives a subtorus $\mathbb{T}_\kappa \subseteq \mathbb{T}^{r+s}$ corresponding to the sublattice $N_\kappa \cap \mathbb{Z}^{r+s} \subseteq \mathbb{Z}^{r+s}$. By construction,

$$\mathcal{X}_{\kappa,0} \;=\; \overline{t(\kappa)^{-1} \cdot \mathbb{T}_\kappa} \;=\; t(\kappa)^{-1} \cdot \overline{\mathbb{T}_\kappa} \;\subseteq\; Z.$$

$\square$

Definition 6.1.11. Consider $X = X(A, P, \Sigma)$, the family $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$ and the lattice N_κ as in Construction 6.1.9.

(i) The *antitropical coordinates* on N_κ are given by the isomorphism

$$\eta_\kappa \colon \mathbb{Z}^{s+1} \;\to\; N_\kappa, \qquad e_i \;\mapsto\; \begin{cases} e_{r+i}, & i = 1, \ldots, s, \\ -e_\kappa, & i = s+1. \end{cases}$$

(ii) The *antitropical half space* in $\mathbb{Q}^{s+1}$ is $\mathcal{H}_\kappa := \{v \in \mathbb{Q}^{s+1};\ v_{s+1} \geq 0\}$.

Remark 6.1.12. Consider $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$. Let $\lambda \colon \mathbb{K}^* \to \mathbb{T}^{s+1}$ be a one-parameter subgroup acting on $\mathcal{X}_\kappa$ such that $\psi_\kappa(\lambda(t) \cdot z) = t\psi_\kappa(z)$.

(i) On $\Psi_\kappa^{-1}(0) = Z$, the one-parameter subgroup λ is given by $v' - v_{\kappa n_\kappa} \in \mathbb{Z}^{r+s}$ with $v' \in \{0\} \times \mathbb{Z}^s$.

(ii) On $\mathcal{X}_{\kappa,0} = \psi_\kappa^{-1}(0)$, the one-parameter subgroup is given by $(d_1, \ldots, d_s, l_{\kappa n_\kappa})$ in antitropical coordinates.

Conversely, every $(d_1, \ldots, d_s, l_{\kappa n_\kappa}) \in \mathbb{Z}^{s+1}$ is the antitropical coordinate vector of a one-parameter subgroup $\lambda\colon \mathbb{K}^* \to \mathbb{T}^{s+1}$ acting on $\mathcal{X}_\kappa$ such that $\psi_\kappa(\lambda(t) \cdot z) = t\psi_\kappa(z)$.

We remind the reader of the definition of the Donaldson-Futaki invariant from [**8**].

Definition 6.1.13. Let $X = X(A, P, \Sigma)$ with complete Σ such that X is Fano. Suppose $(\mathcal{X}_\kappa, \mathcal{L})$ is a test configuration as in Corollary 6.1.8 with $L = \mathcal{O}(-\mathcal{K}_X)$ and associated $\mathbb{K}^*$-action λ_v. Consider $\psi_\kappa\colon \mathcal{X}_\kappa \to \mathbb{K}$ from Remark 6.1.4 and the special toric fiber $\mathcal{X}_{\kappa,0}$ (with acting torus T'). We call κ *valid*, if $\mathcal{X}_{\kappa,0}$ is normal. Let $\ell \in \mathbb{Z}_{\geq 1}$ be minimal such that $-\ell\mathcal{K}_{\mathcal{X}_{\kappa,0}}$ is Cartier. We denote by M' and N' the lattices of characters and one-parameter subgroups of T', respectively. Consider the canonical linearization for $\mathcal{L}_{\kappa,0} = \mathcal{O}(-\ell\mathcal{K}_{\mathcal{X}_{\kappa,0}})$ and write $l_k = \dim(H^0(X, \mathcal{L}_{\kappa,0}^{\otimes k}))$. We set

$$w_k(v) \ := \ \sum_{u \in M'} \langle u, v\rangle \cdot \dim(H^0(X, \mathcal{L}_{\kappa,0}^{\otimes k}))_u$$

and define the linear form

$$\mathrm{DF}(\mathcal{X}_\kappa, \mathcal{L}) \ := \ F_{\mathcal{X}_{\kappa,0}}(v) \ := \ -\lim_{k \to \infty} \frac{w_k(v)}{k \cdot l_k \cdot \ell} \ \in \ N'_{\mathbb{R}}.$$

It is called the *Donaldson-Futaki invariant* of the test configuration $(\mathcal{X}_\kappa, \mathcal{L})$. Furthermore, the variety X is called *K-polystable* if $\mathrm{DF}(\mathcal{X}_\kappa, \mathcal{L}) \geq 0$ for each valid κ.

Construction 6.1.14. Consider a Fano variety X arising from the data (A, P, Σ), where Σ is polytopal. Fix (α_{ij}, α_k) such that

$$\sum \alpha_{ij} w_{ij} + \sum \alpha_k w_k \ \in \ K \ = \ \mathrm{Cl}(X)$$

is an anticanonical class of X. For instance $\alpha_{0j} = (1 + l_{0j} - rl_{0j})w_{0j}$ and $\alpha_{ij} = \alpha_k = 1$ in the remaining cases. As in Proposition 3.1.7, let

$$P' \ = \ \left[\begin{array}{c} P \\ \alpha \end{array}\right], \qquad \alpha \ = \ (\alpha_{ij}, \alpha_k).$$

For $\kappa = 0, \ldots, r$ consider P'_κ and Σ'_κ from Construction 6.1.9, where the latter hosts the faces of the cone σ'_κ over the columns of P'_κ. Set

$$\tau'_\kappa \ := \ \eta_\kappa^{-1}(\sigma'_\kappa) \subseteq \mathbb{Q}^{s+2}$$

and denote by $\omega'_\kappa \subseteq \mathbb{Q}^{s+2}$ the dual cone of σ'_κ. Then we obtain the *moment polytope* of $\mathcal{X}'_{\kappa,0}$ as

$$\mathcal{B}_\kappa \ := \ \omega'_\kappa \cap (\mathbb{Q}^s \times \{1\} \times \mathbb{Q}) \ \subseteq \ \mathbb{Q}^s \times \{1\} \times \mathbb{Q} \ \cong \ \mathbb{Q}^{s+1}.$$

Under the latter identification, the antitropical half space is again given as $\mathcal{H}_\kappa := \{v \in \mathbb{Q}^{s+1};\ v_{s+1} \geq 0\}$.

Remark 6.1.15. Consider a Fano variety X arising from (A, P, Σ) with polytopal Σ, the family $\psi_\kappa\colon \mathcal{X}_\kappa \to \mathbb{K}$ and the lattice N_κ as in Construction 6.1.9. If $\mathcal{X}_{\kappa,0}$ is normal, then the moment polytope $\mathcal{B}_\kappa \subseteq \mathbb{Q}^{s+1}$ of $\mathcal{X}_{\kappa,0}$ is the dual polytope of the fan polytope

$$\mathcal{A}_\kappa \ := \ \mathrm{conv}(\eta_\kappa^{-1}(v_\varrho); \varrho \in \Sigma_\kappa^{(1)}) \ \subseteq \ \mathbb{Q}^{s+1}.$$

6.2. K-polystability and Ricci-flat Kähler cone metrics in the surface case

We take a look at the notions of K-polystability and Ricci-flat Kähler cone metrics in the surface case. Implementing corresponding algorithms and applying them to the classified surfaces from previous chapters, we obtain the results listed in Theorems 6.2.3 and 6.2.7.

First, we present a combinatorial characterization for K-polystability using the moment polytope from Construction 6.1.14.

Proposition 6.2.1. *Consider a del Pezzo $\mathbb{K}^*$-surface $X = X(A, P)$, the moment polytopes $\mathcal{B}_\kappa$ from Construction 6.1.14 and their barycenters $b_\kappa \in \mathbb{Q}^2$. The surface X is*

(i) K-polystable *if and only if $b_{\kappa,1} = 0$ and $b_{\kappa,2} > 0$ for each valid κ.*
(ii) K-semistable *if and only if $b_{\kappa,1} = 0$, $b_{\kappa,2} \geq 0$ for each valid κ and $b_{\kappa,2} = 0$ for at least one κ among them.*

Remark 6.2.2. For a Gorenstein log del Pezzo $\mathbb{K}^*$-surface X, the condition of K-polystability is equivalent to X admitting a so-called *Kähler-Einstein metric*, see Corollary 6.1 in [**40**]. In higher Gorenstein indices this no longer holds true.

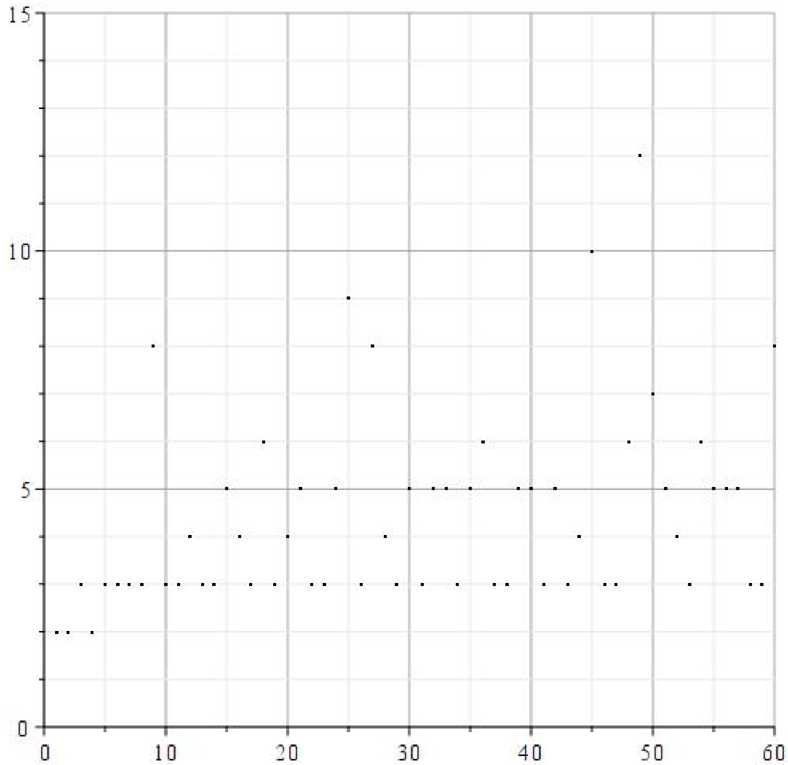
We are able to algorithmically test del Pezzo $\mathbb{K}^*$-surfaces coming from defining data (A, P) for K-poly- and -semistability. The following results are obtained for the classified surfaces from Theorems 4.3.14, 5.3.9 and 5.3.11.

Theorem 6.2.3. *We have the following statements on K-polystability of non-toric log del Pezzo $\mathbb{K}^*$-surfaces.*

- *There are exactly* 231 *sporadic and* 33 *one-parameter families of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Picard number* 1 *and Gorenstein index $\iota \leq 60$ that are K-polystable.*
- *There are exactly* 1 *sporadic and* 3 *one-parameter families of non-toric canonical del Pezzo $\mathbb{K}^*$-surfaces that are K-polystable.*
- *There are exactly* 25 *sporadic,* 23 *one-parameter families,* 5 *two-parameter families,* 4 *three-parameter families,* 1 *four-parameter family and* 1 *five-parameter family of non-toric* 1/2*-log canonical del Pezzo $\mathbb{K}^*$-surfaces that are K-polystable.*
- *There are exactly* 227 *sporadic,* 177 *one-parameter families,* 56 *two-parameter families,* 33 *three-parameter families,* 12 *four-parameter families,* 9 *five-parameter families,* 4 *six-parameter families,* 3 *seven-parameter families,* 1 *eight-parameter family and* 1 *nine-parameter family of non-toric* 1/3*-log canonical del Pezzo $\mathbb{K}^*$-surfaces that are K-polystable.*
- *There are exactly* 3 *sporadic,* 3 *one-parameter families,* 2 *two-parameter families,* 2 *three-parameter families,* 1 *four-parameter family and* 1 *five-parameter family of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index* 2 *that are K-polystable.*
- *There are exactly* 6 *sporadic,* 13 *one-parameter families,* 14 *two-parameter families,* 13 *three-parameter families,* 7 *four-parameter families,* 6 *five-parameter families,* 3 *six-parameter families,* 2

seven-parameter families, 1 *eight-parameter family and* 1 *nine-parameter family of non-toric log del Pezzo* $\mathbb{K}^*$*-surfaces of Gorenstein index* 3 *that are* K*-polystable.*

Remark 6.2.4. According to their Gorenstein indices on the horizontal axis, the numbers of surfaces from the first bullet point in Theorem 6.2.3 are distributed as follows.



Now we look at *Ricci-flat Kähler cone metrics*. They are defined on the affine cone over a Fano variety and exist if the solvability of certain equations is guaranteed, see [**37**]. We explain how to explicitly determine if they exist in the special case of del Pezzo $\mathbb{K}^*$-surfaces $X = X(A, P)$.

Algorithm 6.2.5. The input is a defining matrix of a del Pezzo $\mathbb{K}^*$-surface $X = X(A, P)$. Let

$$K \;=\; \{\kappa_1, \ldots \kappa_l\}$$

be the set of valid κ. Do the following steps starting with $i = 1$.

- Lift the moment polytope $\mathcal{B}_{\kappa_i}$ from Construction 6.1.14 to height 1 in $\mathbb{R}^3$ by adding a 1 at the end of each of its vertices and let σ be the cone whose rays pass through these vertices.
- Denote by $\mathrm{vol} = \mathrm{vol}(\sigma(\xi))$ the volume function of the truncation of σ at height 1, where $\xi = (x_1, x_2, 1) \in \mathbb{R}^3$ is a generic choice.
- Regard $f := \frac{\partial \mathrm{vol}}{\partial x_1}|_{x_2=0}$ as a function of x_1. Let z be its unique root satisfying $(z, 0, 1) \in \sigma^\vee$.
- Go to the first step with $i = i{+}1$ if $i < l$ and $\frac{\partial \mathrm{vol}}{\partial x_2}(z, 0) < 0$. Return *true* if $i = l$ and $\frac{\partial \mathrm{vol}}{\partial x_2}(z, 0) < 0$. Return *false* if $\frac{\partial \mathrm{vol}}{\partial x_2}(z, 0) \geq 0$.

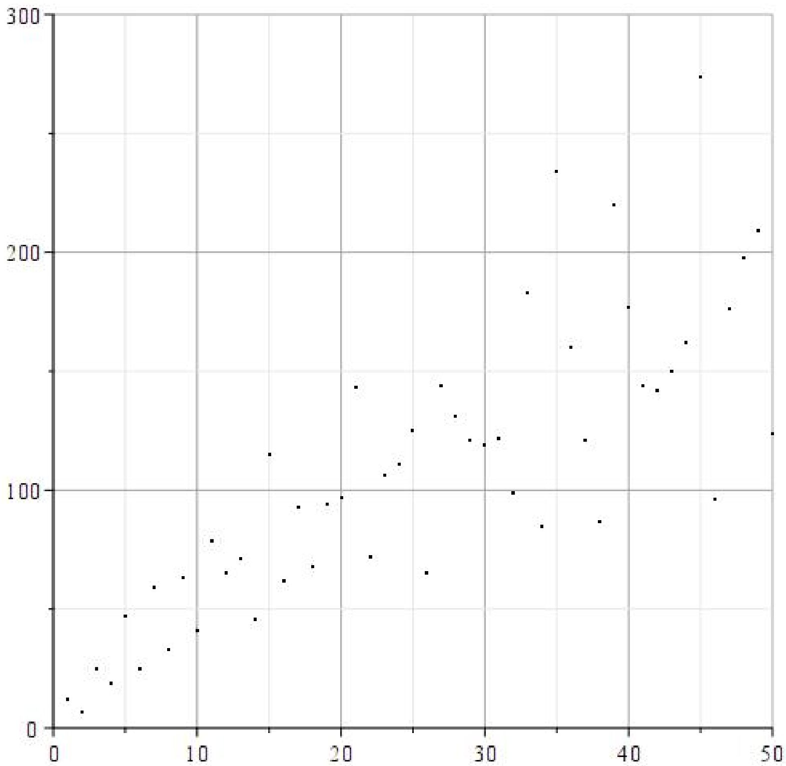
The affine cone over X from Proposition 3.1.7 admits a Ricci-flat Kähler cone metric if and only if the output is *true*.

Remark 6.2.6. Based on experimental evidence, it seems that the affine cone over a K-polystable del Pezzo surface always admits a Ricci-flat Kähler cone metric.

Theorem 6.2.7. *We have the following statements on the existence of Ricci-flat Kähler cone metrics on the affine cones over non-toric log del Pezzo $\mathbb{K}^*$-surfaces.*

- *There are exactly* 5138 *sporadic and* 283 *one-parameter families of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Picard number* 1 *and Gorenstein index $\iota \leq 50$ whose affine cone admits a Ricci-flat Kähler cone metric.*
- *There are exactly* 23 *sporadic and* 3 *one-parameter families of non-toric canonical del Pezzo $\mathbb{K}^*$-surfaces whose affine cone admits a Ricci-flat Kähler cone metric.*
- *There are exactly* 460 *sporadic,* 76 *one-parameter families,* 14 *two-parameter families,* 6 *three-parameter families,* 2 *four-parameter family and* 1 *five-parameter family of non-toric* 1/2*-log canonical del Pezzo $\mathbb{K}^*$-surfaces whose affine cone admits a Ricci-flat Kähler cone metric.*
- *There are exactly* 20247 *sporadic,* 3041 *one-parameter families,* 473 *two-parameter families,* 111 *three-parameter families,* 34 *four-parameter families,* 18 *five-parameter families,* 9 *six-parameter families,* 5 *seven-parameter families,* 2 *eight-parameter family and* 1 *nine-parameter family of non-toric* 1/3*-log canonical del Pezzo $\mathbb{K}^*$-surfaces whose affine cone admits a Ricci-flat Kähler cone metric.*
- *There are exactly* 22 *sporadic,* 8 *one-parameter families,* 5 *two-parameter families,* 3 *three-parameter families,* 1 *four-parameter family and* 1 *five-parameter family of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index* 2 *whose affine cone admits a Ricci-flat Kähler cone metric.*
- *There are exactly* 105 *sporadic,* 49 *one-parameter families,* 34 *two-parameter families,* 22 *three-parameter families,* 12 *four-parameter families,* 8 *five-parameter families,* 4 *six-parameter families,* 2 *seven-parameter families,* 1 *eight-parameter family and* 1 *nine-parameter family of non-toric log del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index* 3 *whose affine cone admits a Ricci-flat Kähler cone metric.*

Remark 6.2.8. According to their Gorenstein indices on the horizontal axis, the numbers of surfaces from the first bullet point in Theorem 6.2.7 are distributed as follows.


300
200
100
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10
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30
40
50

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Deutsche Zusammenfassung

Die vorliegende Arbeit trägt zur Klassifikation von log del Pezzo Flächen mit Toruswirkung bei.

Eine *del Pezzo Fläche* ist eine normale projektive algebraische Fläche X über einem algebraisch abgeschlossenen Körper von Charakteristik 0 mit einem amplen antikanonischen Divisor $-\mathcal{K}_X$. Die glatten del Pezzo Flächen können mittels klassischer Methoden klassifiziert werden. Diese sind die folgenden: das Produkt $\mathbb{P}_1 \times \mathbb{P}_1$ der projektiven Gerade mit sich selbst, die projektive Ebene $\mathbb{P}_2$ und die Blow-ups von $\mathbb{P}_2$ in bis zu acht Punkten in allgemeiner Lage. Ist X eine glatte del Pezzo Fläche, so besteht nach Noether's Formel die Relation

$$\mathcal{K}_X^2 + \rho(X) \;=\; 10.$$

Die Selbstschnittzahl $\mathcal{K}_X^2$ ist der *Grad* von X und $\rho(X)$ ist die Picardzahl von X. Die del Pezzo Flächen von mindestens Grad 4 können als Durchschnitte von Quadriken in einem projektiven Raum beschrieben werden. Diejenigen von Grad 3 sind Kubiken in $\mathbb{P}_3$, solche von Grad 2 Quartiken in $\mathbb{P}_{1,1,1,2}$ und jene von Grad 1 Sextiken in $\mathbb{P}_{1,1,2,3}$.

Erlaubt man Singularitäten auf einer del Pezzo Fläche X, so erhält man ein übliches Maß für deren Mildheit durch Betrachten einer Auflösung $\pi\colon X' \to X$ von Singularitäten. Die zugehörige Verzweigungsformel ist

$$\mathcal{K}_{X'} \;=\; \pi^*\mathcal{K}_X + \sum a(E)E.$$

Dabei durchläuft E die exzeptionellen Primdivisoren und die $a(E)$ sind die *Diskrepanzen* von π. Die Fläche X nennt man

- *log terminal*, falls $a(E) > -1$ für jedes E,
- *ε-log terminal*, falls $a(E) > -1 + \varepsilon$ für jedes E,
- *ε-log kanonisch*, falls $a(E) \geq -1 + \varepsilon$ für jedes E,
- *terminal*, falls sie 1-log terminal ist,
- *kanonisch*, falls sie 1-log kanonisch ist.

Dies hängt nicht von der Wahl von π ab. Eine log terminale del Pezzo Fläche nennt man auch *log del Pezzo Fläche*. Nach [**38**, Prop. 3.6] sind log del Pezzo Flächen stets rational. Alexeev zeigte, dass zu gegebenem ε nur endlich viele Familien ε-log terminaler del Pezzo Flächen existieren, siehe [**1**].

Eine weitere wichtige Invariante einer del Pezzo Fläche ist ihr *Gorensteinindex*. Dieser ist definiert durch die kleinste positive ganze Zahl ι_X mit der Eigenschaft, dass $\iota_X\mathcal{K}_X$ ein Cartierdivisor ist. In diesem Zusammenhang bilden *Gorenstein del Pezzo Flächen* X, d.h. diejenigen mit $\iota_X = 1$, die einfachste Klasse. Dabei liefern Kegel über elliptischen Kurven die einzigen nichtrationalen Beispiele, siehe [**32**, Thm. 2.2]. Die Gorenstein del Pezzo Flächen X mit höchstens rationalen Singularitäten sind genau diejenigen, welche höchstens ADE-Singularitäten, d.h. rationale Doppelpunkte, erlauben. Dies ist wiederum äquivalent dazu, dass X nur kanonische Singularitäten besitzt. Die minimalen Auflösungen dieser Flächen liefern genau die *schwachen del Pezzo Flächen*, d.h. die glatten rationalen Flächen mit big und nef antikanonischem Divisor. Die schwachen del Pezzo Flächen sind genau die iterierten Blow-ups der projektiven Ebene in bis zu acht Punkten in fast allgemeiner Lage. Dies führt letztlich zur Klassifikation der rationalen Gorenstein del Pezzo Flächen, siehe [**17**, **18**, **32**].

Alexeev und Nikulin präsentierten alle möglichen Schnittgraphen einer gewissen Auflösung von Singularitäten von log del Pezzo Flächen von Gorensteinindex 2 in [**2**]. Dadurch werden diese bis auf äquisinguläre Deformation klassifiziert. Die Theorie von K3 Flächen spielte dabei eine wesentliche Rolle. Nakayama gelang es ebenfalls, unabhängig davon und mittels eines anderen Ansatzes, log del Pezzo Flächen von Gorensteinindex 2 zu klassifizieren, siehe [**38**]. Fujita und Yasutake nutzten Nakayamas Herangehensweise, um die entsprechenden Flächen von Gorensteinindex 3 zu klassifizieren.

In der vorliegenden Dissertation konzentrieren wir uns auf log del Pezzo Flächen X, die eine effektive Wirkung $\mathbb{T} \times X \to X$ eines nichttrivialen algebraischen Torus $\mathbb{T}$ erlauben, die durch einen Morphismus gegeben ist.

Ist $\mathbb{T} \cong \mathbb{K}^* \times \mathbb{K}^*$, so befinden wir uns in der Theorie der *torischen Flächen*. Nutzt man deren kombinatorische Beschreibung mittels Fächern, sind diese besonders zugänglich. Jede torische Fläche besitzt höchstens zyklische Quotientensingularitäten. Insbesondere sind diese stets log terminal. Außerdem kann der Gorensteinindex am definierenden Fächer explizit abgelesen werden.

Torische (log) del Pezzo Flächen korrespondieren zu *LDP-Polygonen*, d.h. zu zwei-dimensionalen konvexen Polygonen in $\mathbb{Q}^2$, die den Ursprung als einzigen inneren Gitterpunkt und nur primitive Gitterpunkte als Ecken besitzen. Diese Korrespondenz erlaubte explizite Klassifikationen bis zu Gorensteinindex 17, siehe [**10**, **35**]. Eine torische del Pezzo Fläche X ist ε-log kanonisch, falls das zugehörige LDP-Polygon $\mathcal{P}_X$ die Bedingung

$$\varepsilon \mathcal{P}_X^\circ \cap \mathbb{Z}^2 \;=\; \{0\}$$

erfüllt. Unter Verwendung dieses Kriteriums entwickeln wir Methoden zur expliziten Klassifikation und erhalten die folgenden Ergebnisse.

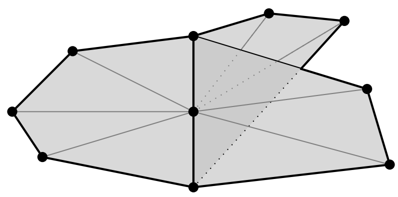
Theorem 1. *Wir erhalten die folgenden Aussagen zu torischen ε-log kanonischen del Pezzo Flächen.*

- $\varepsilon = 1$: *Bis auf Isomorphie existieren genau* 16 *torische kanonische del Pezzo Flächen. Dies sind die wohlbekannten torischen Gorenstein del Pezzo Flächen. Die maximale Picardzahl ist* 4*, realisiert durch genau eine Fläche.*
- $\varepsilon = \frac{1}{2}$: *Bis auf Isomorphie existieren genau* 505 *torische* $1/2$*-log kanonische del Pezzo Flächen. Die maximale Picardzahl ist* 6*, realisiert durch genau eine Fläche.*

$\varepsilon = \frac{1}{3}$: *Bis auf Isomorphie existieren genau* 48032 *torische* 1/3*-log kanonische del Pezzo Flächen. Die maximale Picardzahl ist* 10*, realisiert durch genau eine Fläche.*

Die andere Möglichkeit für den operierenden Torus ist $\mathbb{T} \cong \mathbb{K}^*$. Das heißt, wir betrachten sogenannte $\mathbb{K}^*$-Flächen. Ähnlich wie torische Flächen werden $\mathbb{K}^*$-Flächen seit langer Zeit intensiv untersucht, siehe beispielsweise [**19**–**21**, **41**–**44**]. Man beachte, dass alle log terminalen Flächensingularitäten als Quotienten der affinen Ebene $\mathbb{K}^2$ und einer endlichen Untergruppe der allgemeinen linearen Gruppe $\mathrm{GL}_2(\mathbb{K})$ entstehen und daher auf natürliche Weise mit einer $\mathbb{K}^*$-Wirkung ausgestattet sind. Dies macht $\mathbb{K}^*$-Flächen besonders interessant für das allgemeine Studium von log del Pezzo Flächen. Darüber hinaus wird die Klasse der $\mathbb{K}^*$-Flächen durch verschiedene kombinatorische Zugänge aus [**3**, **5**, **27**, **30**] sehr zugänglich. Dies wurde von Huggenberger in [**34**] für die Klassifikation der Gorenstein log del Pezzo $\mathbb{K}^*$-Flächen und von Süß für den Fall von Picardzahl 1 und höchstens Gorensteinindex 3 genutzt, siehe [**45**].

Um diese Klassifikationsergebnisse auszubauen, werden wir das kombinatorische Werkzeug des *antikanonischen Komplexes* nutzen. Dieser wurde erstmals in [**6**] präsentiert. Es handelt sich um einen polytopalen Komplex, der das LDP-Polygon aus der torischen Situation verallgemeinert.



Antikanonischer Komplex.

Wie bei torischen Flächen und deren zugehörigen LDP-Polygonen sind alle geometrischen Eigenschaften einer log del Pezzo $\mathbb{K}^*$-Fläche im entsprechenden antikanonischen Komplex kodiert. Wir nutzen ausschließlich diese Sprache, um das folgende Resultat zu erhalten.

Theorem 2. *Es gibt genau* 154161 *Isomorphieklassen nichttorischer log del Pezzo* $\mathbb{K}^*$*-Flächen von Picardzahl* 1 *und Gorensteinindex* $\iota \leq 200$.

Eine Tabelle mit den spezifischen Anzahlen von Isomorphieklassen zu gegebenem Gorensteinindex ist in Proposition 4.3.14 zu finden.

Wir weiten unseren Blick auf ε-log kanonische del Pezzo $\mathbb{K}^*$-Flächen. In diesem Kontext gibt es Charakterisierungen analog zum torischen Fall. Für eine log del Pezzo $\mathbb{K}^*$-Fläche X und ihrem zugehörigen antikanonischen Komplex $\mathcal{A}_X$ gilt:

- X ist genau dann ε-log terminal, wenn 0 der einzige Gitterpunkt in $\varepsilon\mathcal{A}_X$ ist.
- X ist genau dann ε-log kanonisch, wenn 0 der einzige Gitterpunkt in $\varepsilon\mathcal{A}_X^\circ$ ist.

Um explizite Klassifikationen für diese Klassen von Flächen zu erhalten, benötigen wir die Begriffe der Kontraktion und kombinatorischer Minimalität. Eine normale vollständige Fläche X ist kombinatorisch minimal, wenn jede Kontraktion $X \to Y$ ein Isomorphismus ist. Dies kann mithilfe von antikanonischen Komplexen ausgedrückt werden und liefert das folgende Theorem.

Theorem 3. *Wir erhalten die folgenden Aussagen zu nichttorischen kombinatorisch minimalen ε-log kanonischen del Pezzo $\mathbb{K}^*$-Flächen.*

- $\varepsilon = 1$: *Es gibt genau* 13 *sporadische und* 2 *Ein-Parameterfamilien nichttorischer kombinatorisch minimaler kanonischer del Pezzo $\mathbb{K}^*$-Flächen.*
- $\varepsilon = \frac{1}{2}$: *Es gibt genau* 62 *sporadische und* 5 *Ein-Parameterfamilien nichttorischer kombinatorisch minimaler* $1/2$*-log kanonischer del Pezzo $\mathbb{K}^*$-Flächen.*
- $\varepsilon = \frac{1}{3}$: *Es gibt genau* 318 *sporadische und* 14 *Ein-Parameterfamilien nichttorischer kombinatorisch minimaler* $1/3$*-log kanonischer del Pezzo $\mathbb{K}^*$-Flächen.*

Wir entwickeln einen Prozess, antikanonische Komplexe ε-log kanonischer del Pezzo $\mathbb{K}^*$-Flächen aus Komplexen kombinatorisch minimaler Flächen und LDP-Polygonen systematisch aufzubauen. Es wurden Algorithmen implementiert, welche die folgenden Resultate liefern:

Theorem 4. *Wir erhalten die folgenden Aussagen zu nichttorischen ε-log kanonischen del Pezzo $\mathbb{K}^*$-Flächen.*

- $\varepsilon = 1$: *Es gibt genau* 30 *sporadische und* 4 *Ein-Parameterfamilien nichttorischer kanonischer del Pezzo $\mathbb{K}^*$-Flächen. Die maximale Picardzahl ist* 4*, realisiert durch genau eine Fläche.*
- $\varepsilon = \frac{1}{2}$: *Es gibt genau* 998 *sporadische,* 184 *Ein-Parameterfamilien,* 40 *Zwei-Parameterfamilien,* 12 *Drei-Parameterfamilien,* 2 *Vier-Parameterfamilien und* 1 *Fünf-Parameterfamilie nichttorischer* $1/2$*-log kanonischer del Pezzo $\mathbb{K}^*$-Flächen. Die maximale Picardzahl ist* 8*, realisiert durch die eindeutige Fünf-Parameterfamilie.*
- $\varepsilon = \frac{1}{3}$: *Es gibt genau* 65022 *sporadische,* 12402 *Ein-Parameterfamilien,* 3190 *Zwei-Parameterfamilien,* 917 *Drei-Parameterfamilien,* 254 *Vier-Parameterfamilien,* 64 *Fünf-Parameterfamilien,* 14 *Sechs-Parameterfamilien,* 6 *Sieben-Parameterfamilien,* 2 *Acht-Parameterfamilien und* 1 *Neun-Parameterfamilien nichttorischer* $1/3$*-log kanonischer del Pezzo $\mathbb{K}^*$-Flächen. Die maximale Picardzahl ist* 12*, realisiert durch die eindeutige Neun-Parameterfamilie.*

Da $1/k$-log kanonische del Pezzo Flächen sämtliche log del Pezzo Flächen von Gorensteinindex k enthalten, erhalten wir das folgende Ergebnis durch Filtern der vorangegangenen Klassifikationen.

Korollar 5. *Wir erhalten die folgenden Aussagen zu nichttorischen log del Pezzo $\mathbb{K}^*$-Flächen von Gorensteinindex ι.*

$\iota = 1$: *Es gibt genau* 30 *sporadische und* 4 *Ein-Parameterfamilien nichttorischer log del Pezzo* $\mathbb{K}^*$*-Flächen von Gorensteinindex* 1*. Die maximale Picardzahl ist* 4*, realisiert durch eine sporadische und eine Ein-Parameterfamilie.*

$\iota = 2$: *Es gibt genau* 53 *sporadische,* 17 *Ein-Parameterfamilien,* 7 *Zwei-Parameterfamilien,* 3 *Drei-Parameterfamilien,* 1 *Vier-Parameterfamilie und* 1 *Fünf-Paramterfamilie nichttorischer log del Pezzo* $\mathbb{K}^*$*-Flächen von Gorensteinindex* 2*. Die maximale Picardzahl ist* 8*, realisiert durch die eindeutige Fünf-Parameterfamilie.*

$\iota = 3$: *Es gibt genau* 268 *sporadische,* 123 *Ein-Parameterfamilien,* 67 *Zwei-Paramterfamilien,* 36 *Drei-Parameterfamilien,* 18 *Vier-Parameterfamilien,* 10 *Fünf-Paramterfamilien,* 5 *Sechs-Parameterfamilien,* 3 *Sieben-Parameterfamilien,* 1 *Acht-Parameterfamilie und* 1 *Neun-Parameterfamilie nichttorischer log del Pezzo* $\mathbb{K}^*$*-Flächen von Gorensteinindex* 3*. Die maximale Picardzahl ist* 12*, realisiert durch die eindeutige Neun-Parameterfamilie.*

Die definierenden Daten aller klassifizierten log del Pezzo $\mathbb{K}^*$-Flächen aus den Theoremen 2 und 4 werden in einer Datenbank zugänglich gemacht, siehe [**25**]. Zugehörige Invarianten wie Picardzahl, Grad, Gorensteinindex, Anzahl der Singularitäten etc. werden dort ebenfalls zu finden sein.

Die vorliegende Dissertation ist auf folgende Weise organisiert. Das erste Kapitel behandelt zweidimensionale Gitterpolytope. Insbesondere solche, die keine k-fachen Gitterpunkte, d.h. Elemente aus $k\mathbb{Z}^n$, enthalten. Es wird eine Standardform für derartige Gitterdreiecke präsentiert. Desweiteren werden *Fareyfolgen* verwendet, um diese Dreiecke zu klassifizieren. Dieser Zugang fand Anwendung in [**11**]. Das zweite Kapitel ist torischen Flächen gewidmet. Wir geben einen kurzen Überblick zu torischen Varietäten im Allgemeinen und präsentieren alles Notwendige für den Flächenfall. Insbesondere die Methoden, die für obig erwähnte Klassifikationen nötig sind. Kapitel 3 stellt den allgemeinen Hintergrund zu $\mathbb{K}^*$-Flächen und deren kombinatorischer Behandlung bereit. Wir zeigen, wie man Invarianten wie Divisorenklassengruppe, Cox Ring, Picardgruppe, antikanonischen Divisor, Singularitäten und die Schnitttheorie der Fläche anhand definierender Daten erhält. Außerdem behandeln wir Details der Berechnung von Auflösungen von Singularitäten sowie des Gorensteinindexes. Im vierten Kapitel spezifizieren wir auf (nichttorische) log del Pezzo $\mathbb{K}^*$-Flächen und führen den antikanonischen Komplex ein. Wir präsentieren Algorithmen zur Klassifikation von log del Pezzo $\mathbb{K}^*$-Flächen ohne quasiglatte elliptische Fixpunkte und solcher von Picardzahl 1. Das fünfte Kapitel hat $1/k$-log kanonische del Pezzo $\mathbb{K}^*$-Flächen zum Gegenstand. Zunächst werden Kontraktionen und kombinatorische Minimalität diskutiert. Darauf werden mithilfe dieser Konzepte Details der Klassifikation für $k = 1, 2, 3$ präsentiert. Letztlich werden im sechsten Kapitel K-Polystabilität und Ricci-flache Kähler Kegelmetriken behandelt, allgemein im Hinblick auf Varietäten mit Toruswirkung sowie insbesondere auf $\mathbb{K}^*$-Flächen. Die in den vorigen Kapiteln klassifizierten Flächen werden algorithmisch auf diese Eigenschaften geprüft.

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