Diffusion of regular domains in Riemannian manifolds

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Zusammenfassung in deutscher Sprache

Aus geometrischer Sicht sind Wärmeleitungsgleichungen interessant, weil sie eine Verbindung zwischen lokalen und globalen Größen einer Mannigfaltigkeit herstellen. Ein bekanntes Beispiel hierfür ist die Kurzzeit-Asymptotik des Wärmeleitungskernes einer Riemannschen Mannigfaltigkeit $(\mathrm{M}^n, \bar{\mathrm{g}})$, also der Lösung der Wärmeleitungsgleichung

$$\begin{cases} \bar{\Delta}_{\bar{\mathrm{g}}} u(x,t) = \frac{\partial}{\partial t} u(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u(x,0) = \delta_y(x) & x \in \mathrm{M}^n, \end{cases}$$

wobei $\Delta_{\bar{\mathrm{g}}}$ der Laplace-Beltrami Operator der Riemannschen Mannigfaltigkeit $(\mathrm{M}^n, \bar{\mathrm{g}})$ ist und δ_y das Dirac-Maß in $y \in \mathrm{M}^n$ bezeichne. Erstmals zeigten Plejel und Minakshisundaram in der Publikation [MP49], dass die Spur eines Wärmeleitungskernes eine Kurzzeit-Entwicklung erlaubt, deren Koeffizienten durch geometrische Invarianten, wie etwa Volumen und das Integral der Skalarkrümmung, gegeben sind.

Durch dieses Beispiel motiviert, betrachten wir im Rahmen dieser Arbeit Wärmeleitungsgleichungen auf Riemannschen Mannigfaltigkeiten, deren Anfangsbedingungen durch uniforme Verteilungen von Wärme in regulären Gebieten $\Omega \subset \mathrm{M}^n$ gegeben sind. Beschrieben werden derartige Prozesse durch Anfangswertprobleme der Form

$$\begin{cases} \bar{\Delta}_{\bar{\mathrm{g}}} u_\Omega(x,t) = \frac{\partial}{\partial t} u_\Omega(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u_\Omega(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n, \end{cases} \tag{HE}$$

wobei χ_Ω als die charakteristische Funktion von Ω definiert sei.

Wir untersuchen die Kurzzeit-Asymptotik verschiedener Quantitäten der Lösung u_Ω der Wärmeleitungsgleichung (HE), um daraus Eigenschaften der Regionen Ω abzuleiten. Wir betrachten dabei zunächst den Fall kompakter Riemannscher Mannigfaltigkeiten und untersuchen den Wärmeinhalt (HC), die Boltzmann-Entropie (Ent) und den mittleren Wärmeinhalt (MHC) von Ω, die folgt definiert sind. Wir setzen

$$\mathrm{HC}(t) := \int_{\Omega^c} u_\Omega(y,t)\, \omega(y), \tag{HC}$$

$$\mathrm{Ent}(t) := \int_{\mathrm{M}^n} u_\Omega(y,t) \log\left(u_\Omega(y,t)\right)\, \omega(y), \tag{Ent}$$

$$\mathrm{MHC}(t) := \int_{\{u_\Omega(\cdot,t) \leqslant 1/2\}} u_\Omega(y,t)\, \omega(y). \tag{MHC}$$

Ähnlich wie im Beispiel der Spur von Wärmeleitungskernen erlauben diese Quantitäten eine Kurzzeit-Entwicklung, deren Koeffizienten durch Invariante des Randes des Gebietes Ω gegeben sind. In den Theoremen **Theorem 3.2**, **Theorem 3.6** und **Theorem 5.1** werden die ersten Terme dieser Entwicklungen angegeben. Auf Zeitintervallen, die von einem Radius r abhängen, erhalten wir gleichmäßige und skalierungsinvariante Abschätzungen an die Fehlerterme. Hierbei ist der Radius r wiederum durch geometrische Größen des Randes und der umgebenden Mannigfaltigkeit bestimmt (vergleiche Abschnitt 1.3).

Um diese Entwicklungen zu bestimmen, werden wir zunächst in *Kapitel 2* die punktweise Asymptotik der Lösung u_Ω für kurze Zeiten und Punkte nahe des Randes herleiten (**Theorem 2.5**). Es ist wichtig zu beachten, dass der resultierende Fehlerterm exponentiell mit dem exakten Faktor $1/4t$ in der Distanz zum Rand abfällt. Indem wir Gaußsche Normalkoordinaten einführen, werden hieraus in Kapitel 3 die Approximationen an den Wärmeinhalt (HC) und die Boltzmann-Entropie (Ent) abgeleitet (**Theorem 3.2**, respektive **Theorem 3.6**).

Zur Bestimmung der Kurzzeit-Asymptotik des mittleren Wärmeinhaltes (MHC) nutzen wir zunächst in *Kapitel 4* die punktweise Approximation von u_Ω, um die Evolution der Niveau-Flächen,

$$t \mapsto \{u_\Omega(\cdot, t) = \lambda\}, \quad \lambda \in (0,1)$$

anzunähern. In **Theorem 4.3** zeigen wir, dass diese durch Geometrie des Randes, die Geometrie der umgebenden Mannigfaltigkeit und das Niveau λ bestimmt ist. Der exponentielle Abfall des Fehlertermes aus Theorem 2.5 wird hier genutzt, um zu zeigen, dass der entstehende Fehler in der Approximation der Evolution der Niveau-Fächen für einen beliebigen Exponenten $\alpha > 0$ nur wie $\lambda^{-\alpha}$ im Niveau λ wächst. Insbesondere ist der Fehlerterm also auf dem Intervall $(0,1)$ integrierbar. Die Approximation an die Evolution der Niveau-Mengen nutzen wir dann in *Kapitel 5* zur Konstruktion von Barriere-Mengen der Superniveau-Mengen von u_Ω. Das Volumen der Barriere-Mengen kann dann unter Verwendung der variationellen Formeln in Abschnitt 5.4 approximiert werden. Mit Hilfe des Prinzipes von Cavalieri wird dann die Asymptotik des mittleren Wärmeinhaltes (MHC) bestimmt.

Im letzten Kapitel werden analoge Resultate für Mannigfaltigkeiten mit beschränkter Geometrie beweisen. Des weiteren werden allgemeinere Anfangswertprobleme in Betracht gezogen, die einen Potentialterm und eine sich gemäß eines 2-Tensors entwickelnde Metrik beinhalten.

Introduction

Heat equations describe the diffusion of an initial distribution of heat in some underlying system. If the system is isolated, it is intuitively clear that any distribution of heat will average itself out and the information of the initial condition will be lost. For short times, on the other hand, solutions of the heat equation regularize the initial data while still preserving important properties. An everyday example of this phenomenon is the cooling process of a cup of coffee. Initially, the diffusion of heat is determined by the cup's shape, whereas, after waiting long, the cup's temperature will equal the ambient temperature.

Denoting by $\bar{\Delta}_{\bar{\mathrm{g}}}$ the Laplace-Beltrami operator of some Riemannian manifold $(\mathrm{M}^n, \bar{\mathrm{g}})$, the heat equation is given by

$$\bar{\Delta}_{\bar{\mathrm{g}}} u(x,t) = \tfrac{\partial}{\partial t} u(x,t), \quad x \in \mathrm{M}^n \text{ and } t \in (0,\infty), \tag{1}$$

subject to some initial condition $u(\cdot, 0) = f$. From a geometrical point of view, major interest in the heat equation (1) stems from the fact that it yields a relation between the local and global properties of Riemannian manifolds. A famous example concerns the isospectral problem; are two Riemannian manifolds with the same Laplace-Beltrami spectrum isometric? If one interprets the Riemannian manifold as a drum, the eigenvalues of the Laplace-Beltrami operator are the frequencies at which it can vibrate. Informally, the isospectral problem may thus be formulated as: *»Can one hear the shape of a drum?«* , [Kac66]. While the answer to this question is negative, the Laplace-Beltrami operator determines certain quantities. One way to see this, is by studying the short-time asymptotics of heat kernels $\varrho(x, \cdot, t)$, i.e. solutions of the heat equation (1) such that the initial condition is given by $\varrho(x, \cdot, 0) = \delta_x$, where δ_x is the Dirac-measure in some point $x \in \mathrm{M}^n$. It was first observed by Pleijel and Minakshisundaram in [MP49] that in the compact case, the trace of a Riemannian manifold's heat kernel admits a short-time expansion in terms of its geometric invariants. The lowest order order terms are given by

$$\int_{\mathrm{M}^n} \varrho(x,x,t)\,\omega(x) = \tfrac{1}{(4\pi t)^{n/2}} \left(|\mathrm{M}^n| + t \int_{\mathrm{M}^n} \bar{\mathrm{R}}(x)\,\omega(x) + O(t^2) \right), \tag{2}$$

where $\omega(x)$ denotes the volume form, $|\mathrm{M}^n|$ the volume and $\bar{\mathrm{R}}$ the scalar curvature of the Riemannian manifold. Since the left-hand side can be written in terms of the eigenvalues of Laplace-Beltrami operator, this implies that the volume of M^n and the integrated scalar curvature are isospectral invariants.

In this thesis we study how the short-time behaviour of the diffusion of sets is determined by their geometry. We consider solutions of the heat equations (1) starting as uniform distributions of heat in regular domains. In other words, denoting by χ_Ω the characteristic function of the set Ω, we study equations of the form

$$\begin{cases} \bar{\Delta}_{\bar{g}} u_\Omega(x,t) = \frac{\partial}{\partial t} u_\Omega(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u_\Omega(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n. \end{cases} \tag{HE}$$

Under further assumptions on the Riemannian manifold, like compactness or Ricci curvature bounded from below, the solution u_Ω of the heat-equation (HE) is uniquely given in terms of the heat kernel,

$$u_\Omega(y,t) = \int_\Omega \varrho(x,y,t)\omega(x).$$

As the example of the heat kernel's trace (formula (2)) shows, one should study integrated quantities of the solution u_Ω to characterize the global geometry of the set Ω. In the spirit of the publications by E. de-Giorgi [De 53], M. Ledoux [Led94], M. Preunkert [Pre06] and M. van der Berg [vdBG15], we therefore examine the short-time asymptotics of the *heat content* (HC), the *Boltzmann-Entropy* (Ent) and the *mean-heat content* (MHC) that are defined as below:

$$\mathrm{HC}_\Omega(t) := \int_{\Omega^c} u_\Omega(y,t)\,\omega(y), \tag{HC}$$

$$\mathrm{Ent}_\Omega(t) := \int_{\mathrm{M}^n} u_\Omega(y,t)\log\left(u_\Omega(y,t)\right)\,\omega(y), \tag{Ent}$$

$$\mathrm{MHC}_\Omega(t) := \int_{\{u_\Omega(\cdot,t)\leqslant 1/2\}} u_\Omega(y,t)\,\omega(y). \tag{MHC}$$

Furthermore, motivated by the observation in [Eva93] that the evolution of the $1/2$-level set in Euclidean space approximates the mean curvature flow of the boundary $\partial\Omega$ we study the evolution of general level sets

$$t \mapsto \{u_\Omega(\cdot,t) = \lambda\}, \quad \lambda \in (0,1).$$

Our results are summarized in the following theorems. We first consider closed Riemannian manifolds of class C^6 and domains $\Omega \subset \mathrm{M}^n$ such that the boundary $\partial\Omega$ is an embedded hypersurface with bounds on the first two derivatives of the second fundamental form. The uniform estimates are scaling invariant and are valid on time intervals and, in the case of Theorem 2.5, tubular neighbourhoods of the boundary, that depend on a radius r. The radius r is determined by geometric invariants of the boundary and the ambient space (see Section 1.3). The coefficients of pointwise formulae in $y \in \mathrm{M}^n$ are to be understood to be evaluated in a base point $x_0 \in \partial\Omega$ such that $y = \exp_{x_0} s\nu(x_0)$, where $\nu(x_0)$ is a local choice of normal vector in x_0 with respect to the boundary $\partial\Omega$. By H and $|A|^2$ we denote the mean curvature, respectively the total curvature of the boundary of Ω and by $\overline{\mathrm{Ric}}$ the Ricci-tensor of the ambient space. The function $\Phi : \mathbb{R} \to (0,1)$ appearing in the asymptotic expansions is the Gaussian error function,

$$\Phi(x) := \tfrac{1}{\sqrt{4\pi}} \int_{-\infty}^{x} \mathrm{e}^{-z^2/4} \mathrm{d}z\,,$$

while c_λ is its inverse.

Theorem 2.5 For all $y \in \mathrm{M}^n$ such that $d\,(y, \partial\Omega) \leqslant r$ and all times $0 < t < r^2$, the solution u_Ω of (HE) can be approximated by

$$\begin{aligned}\Big| u_\Omega(y,t) - \Big(\Phi\left(-\tfrac{s}{\sqrt{t}}\right) \\ + \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \Big(-\sqrt{t}H + \sqrt{t}s\left(\tfrac14 \left(2\left(\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2\right) + H^2\right)\right)\Big)\Big)\Big| \\ \leqslant \tfrac{t^{3/2}}{r^3}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{s^2}{t}\right)\tfrac{\mathrm{e}^{-s^2/4t}}{\sqrt{4\pi}}. \qquad (3)\end{aligned}$$

Theorem 3.2 For all times $0 < t < r^2$, the heat content (HC) can be approximated by

$$\begin{aligned}\Big| \mathrm{HC}(t) - \Big(\tfrac{\sqrt{t}}{\sqrt{\pi}} \,\mathrm{area}\,(\partial\Omega) + \\ - \tfrac{t^{3/2}}{12\sqrt{\pi}} \Big(\Big(2 \int_{\partial\Omega} \left(\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2\right) \mathrm{vol}_{\partial\Omega}\Big) + \Big(\int_{\partial\Omega} H^2 \,\mathrm{vol}_{\partial\Omega}\Big)\Big)\Big)\Big| \\ \leqslant t^2 \left(\tfrac{1}{r^3}\left(1 + \tfrac{|\Omega|}{r^n}\right) \mathrm{area}\,(\partial\Omega) + |\mathrm{M}^n| \,\tfrac{1}{r^4}\right). \qquad (4)\end{aligned}$$

Theorem 3.6 For all times $0 < t < r^2$, the entropy (Ent) of Ω can be approximated by

$$\left| \mathrm{Ent}_\Omega(t) + \mathrm{Ent}_1 \sqrt{t}\,\mathrm{area}\,(\partial\Omega)\right| \leqslant \tfrac{t^{1/2}}{r}\,\mathrm{area}\,(\partial\Omega)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right), \qquad (5)$$

where Ent_1 is the entropy of the Gaussian error function.

Theorem 5.1 For all times $0 < t < r^2$ the mean-heat content (MHC) can be approximated by

$$\begin{aligned}\Big| \mathrm{MHC}(t) - \Big(\tfrac{\sqrt{t}}{\sqrt{\pi}}\,\mathrm{area}\,(\partial\Omega) + \tfrac{t}{2} \int_{\partial\Omega} H \;\mathrm{vol}_{\partial\Omega} \\ + t^{\frac32}\Big(-\tfrac{1}{6\sqrt{\pi}} \int_{\partial\Omega} \left(\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2\right) \mathrm{vol}_{\partial\Omega} - \tfrac{1}{3\sqrt{\pi}} \int_{\partial\Omega} H^2 \,\mathrm{vol}_{\partial\Omega}\Big)\Big)\Big| \\ \leqslant \tfrac{t^2}{r^4}\left(|\mathrm{M}^n| + r\,\mathrm{area}\,(\partial\Omega)\right) \mathrm{e}^{\frac{|\mathrm{M}^n|}{r^n}}\,. \qquad (6)\end{aligned}$$

Theorem 4.3 Fix $\lambda \in (0,1)$. Then, for any $y \in \{u_\Omega(\cdot,t) = \lambda\}$, where

$$0 < t < r^2 \max\left(-\log\lambda, \log\left(1-\lambda\right)\right)^{-1},$$

there exists a unique base point $x_0 \in \partial\Omega$ and $d_{\lambda,t}(y) \in \mathbb{R}$ such that

$$y = \exp_{x_0} d_{\lambda,t}(y)\nu(x_0).$$

Moreover, for any $\alpha > 0$ a constant $0 < C(\alpha) < \infty$ can be chosen such that $d_{\lambda,t}(y)$ can be approximated by

$$\begin{aligned} &\left| d_{\lambda,t}(y) - \left(-c_\lambda\sqrt{t} - tH - t^{3/2}\tfrac{c_\lambda}{2}\left(\left(\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2\right)\right)\right)\right| \\ &\leqslant C(\alpha)\tfrac{t^2}{r^3}\lambda^{-\alpha}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{|\mathrm{M}^n|}{r^n}\right)^3 \mathrm{e}^{\frac{|\mathrm{M}^n|}{r^n}}. \end{aligned} \tag{7}$$

In the final chapter we prove analogous results for complete Riemannian manifolds of bounded geometry and consider how the heat content of the conjugate heat equation evolves under Ricci flow. More generally, we consider coupled systems of the form

$$\begin{cases} \frac{\partial}{\partial t}\bar{\mathrm{g}}_{\alpha\beta} = 2\bar{\mathrm{X}}_{\alpha\beta} & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ \left(\frac{\partial}{\partial t} - \bar{\Delta}_{\bar{\mathrm{g}}(t)} + \bar{\mathrm{Q}}\right) u_\Omega = 0 & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u_\Omega(\cdot,0) = \left(f\cdot\chi_\Omega\right)(\cdot) & x \in \mathrm{M}^n, \end{cases} \tag{$\star$}$$

where $\bar{\mathrm{X}}_{\alpha\beta}$ is a time-dependent symmetric 2-tensor, $\bar{\mathrm{Q}} \in C^\infty(\mathrm{M}^n \times [0,T])$ is a potential term and $f \in C^\infty(\mathrm{M}^n)$. If the initial condition is given by a Dirac-meassure, it was observed by G. Perelman in [Per02, Remark 9.6], that the infinitesimal behaviour of the solution of the conjugate heat equation characterizes Ricci flow amongst all other evolution equations. We show in **Example 6.10**, that the difference between the heat content of the conjugate heat equation with metric evolving by Ricci flow and the heat content of a static metric, Theorem 3.2, is given by the boundary integral of the scalar curvature.

Let us compare the above theorems to the results in the literature. The first paper that applies the asymptotic behaviour of the heat equation to understand the geometry of sets is due to E. De-Giorgi. In [De 53] he showed that sets of bounded variation satisfy

$$P(\Omega) = \lim_{t\to 0} \|\nabla u_\Omega(\cdot,t)\|_{L^1(\mathbb{R}^n)},$$

where $P(\Omega)$ denotes the perimeter of the set Ω. This relation was later generalized to complete and connected Riemannian manifolds with Ricci curvature bounded from below in the publication [Mir+07]. In order to find an alternative proof of the isoperimetric inequality, the heat content (HC) was for first related to the perimeter of sets by M. Ledoux in [Led94]. In Euclidean space, he proved the inequality

$$\lim_{t\to 0} \sqrt{\pi t^{-1}}\,\mathrm{HC}(t) \leqslant P(\Omega),$$

with equality in the case of balls. M. Preunkert in [Pre06] later improved this result by showing that, in fact, equality holds for any set of bounded variation. For closed and smooth Riemmanian manifolds, an asymptotic expansion of the heat-content (HC) similar to (4) was derived by M. van der Berg and P. Gilkey in [vdBG15, Theorem 1.6]. Here, using invariance theory, the authors established a complete asymptotic expansion of the heat content (HC) and determined its coefficients by studying certain example manifolds. One advantage of the formula (4) is, that it yields a lower bound on the time t such that *uniform* estimates are fulfilled. We also note that the term involving the squared mean curvature in (4) differs from the results in [vdBG15].

In order to prove the convergence of an approximation scheme of the mean curvature flow proposed by J.K. Bence, B. Merriman and S. Osher [MBO92], the asymptotics of the $1/2$-level set of u_Ω in the Euclidean case were first considered by L. Evans in [Eva93]. M. Preunkert [Pre06] later studied the evolution of general λ-level sets in Euclidean space. We note that the coefficients in our expansion formula (7) differ from a statement in the forementioned publication (Theorem 19). Moreover, the expansion formula (7) yields a clear dependence of the error term on the level $\lambda \in (0,1)$. The integrability of the error term on the interval $(0,1)$ is crucial whenever integrated quantities are to be approximated.

The thesis is structured as follows. In *Chapter 1* we introduce the notation used in the following chapters and recall some well-known results from the literature concerning heat kernels and the evolution of hypersurfaces. We also define the curvature-dependent radius $r := r_{18}(5,x)$ that appears in our estimates.

In *Chapter 2* we derive the pointwise asymptotics of the solution u_Ω of (HE) in terms of the distance to the boundary and the time $t > 0$ (**Theorem 2.5**). A major effort is made to make the error term decay in the distance with the precise exponential factor $1/4t$. Moreover, the order of derivatives of the curvature tensor appearing in the error term are independent of the dimension n. To this end, we first derive a good approximation of heat kernels integrated over some measurable set Ω in Section 2.1. Here, a Gaussian lower bound and the semigroup property of the heat kernel are applied to modify the classical proof regarding the short-time approximation of heat kernels via a parametrix (Lemma 2.15 and Lemma 2.14). The error terms in these lemmata are independent of the volume of the manifold. In Proposition 2.19 and Theorem 2.7 we then derive two types of estimate. The first one is independent of the set's volume and the second estimate, which is valid for short distances, decays in distance with the desired $1/4t$ exponential rate. In Section 2.2 we consider the Euclidean case. Combining these results, we are then able to prove Theorem 2.5 in Section 2.3 by introducing normal coordinates.

In *Chapter 3* we derive the short-time asymptotics of the heat content and the entropy (**Theorem 3.2** and **Theorem 3.6**). For this purpose, we apply the pointwise approximation of u_Ω from the previous chapter to compute the heat content near the boundary of a set Ω in tubular coordinates. To do the computations, it is important that the error term in Theorem 2.5 decays exponentially. In order to verify the expansion formula of the heat content, we also consider the special case of unit balls in Euclidean space.

In *Chapter 4* a short-time expansion of the evolution of the λ-level sets of u_Ω (**Theorem 4.3**) is derived. Here, the difficulty lies in proving that the error term does not depend to heavily on the level λ. We show that it grows at most at a rate of $\lambda^{-\alpha}$, where $\alpha > 0$ is arbitrary. For this reason, the $-1/4t$ exponential decay rate of the uniform estimate regarding the pointwise asymptotic of the solution u_Ω is crucial. Moreover, since the error term in Theorem 2.5 depends on the distance in space, it is indispensable to first find a lower bound on the times $t > 0$ such that the level sets lie in a tubular neighbourhood of the boundary and then prove upper and lower bounds on the distance between the level sets and the boundary of the initial set. We find these bounds in Section 4.1 (Lemma 4.6, respectively Lemma 4.7).

In *Chapter 5* we consider the asymptotics of the mean heat content (**Theorem 5.1**). Using Cavalieri's principle, we write it in terms of the superlevel sets of u_Ω. The short-time expansion of the evolution of the λ-level sets of u_Ω from the previous chapter is then used to construct upper and lower barrier sets of the superlevel sets of u_Ω. Having defined these barrier sets explicitly in terms of the curvature of the boundary and the ambient space, we are then able to approximate their volume by using the properties of the signed distance function and the variational formulae discussed in Section 5.4. A major difficulty in the computations is to guarantee that the error term in the approximation of the volume of the superlevel sets is integrable on the interval $(0, 1/2)$. Therefore, the upper bound on the λ-dependence of the error term from the previous section is crucial.

In the final chapter, *Chapter 6*, we study how the results from the previous chapters transfer to the case of complete Riemannian manifolds of bounded geometry. To derive estimates independent of the volume of the ambient space and the initial set Ω, we use the integral approximation from Proposition 2.19 instead of the one from Theorem 2.7. This comes at the cost of a worse exponential decaying factor in the error terms. In Section 6.2 we also consider the more general coupled system ($\star$) involving a potential term and evolving metrics. As an example, we consider the asymptotics of the backwards Ricci flow and the conjugate heat equation.

1 Notation and preliminarities

1.1 Notation

In this section we introduce the notation that will be used throughout this thesis.

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a smooth Riemannian manifold of dimension $n \geqslant 2$. We denote by a bar and Greek indices all quantities on M^n. For example, in a local coordinate system $\{\bar{x}^\alpha\}_{1\leqslant\alpha\leqslant n}$, by $\bar{\mathrm{g}} = \{\bar{\mathrm{g}}_{\alpha\beta}\}_{1\leqslant\alpha,\beta\leqslant n}$ the metric, by $\bar{\Gamma} = \{\bar{\Gamma}^\gamma_{\alpha\beta}\}_{1\leqslant\alpha,\beta,\gamma\leqslant n}$ the Levi-Civita connection and by $\bar{\nabla}$ the covariant derivative. In local coordinates, the Christoffel symbols and the Laplace-Beltrami operator $\bar{\Delta}$ are given by

$$\bar{\Gamma}^\delta_{\alpha\beta} = \tfrac{1}{2}\bar{\mathrm{g}}^{\delta\gamma}\left(\tfrac{\partial}{\partial\bar{x}^\alpha}\bar{\mathrm{g}}_{\beta\gamma} + \tfrac{\partial}{\partial\bar{x}^\beta}\bar{\mathrm{g}}_{\alpha\gamma} - \tfrac{\partial}{\partial\bar{x}^\gamma}\bar{\mathrm{g}}_{\alpha\beta}\right),$$

respectively

$$\bar{\Delta} f = \tfrac{1}{\sqrt{\det\bar{\mathrm{g}}}}\tfrac{\partial}{\partial x^\alpha}\left(\bar{\mathrm{g}}^{\alpha\beta}\sqrt{\det\bar{\mathrm{g}}}\tfrac{\partial}{\partial x^\beta}f\right), \quad f \in C^2(\mathrm{M}^n).$$

The Riemann curvature tensor acting on vector fields $\bar{\mathrm{X}}, \bar{\mathrm{Y}}, \bar{\mathrm{Z}} \in \mathcal{T}^1_0(\mathrm{M}^n)$,

$$\overline{\mathrm{Riem}}\left(\bar{\mathrm{X}}, \bar{\mathrm{Y}}\right)\bar{\mathrm{Z}} = \bar{\nabla}_{\bar{\mathrm{X}}}\bar{\nabla}_{\bar{\mathrm{Y}}}\bar{\mathrm{Z}} - \bar{\nabla}_{\bar{\mathrm{Y}}}\bar{\nabla}_{\bar{\mathrm{X}}}\bar{\mathrm{Z}} - \bar{\nabla}_{[\bar{\mathrm{X}},\bar{\mathrm{Y}}]}\bar{\mathrm{Z}},$$

is characterized in local coordinates by its components

$$\bar{\mathrm{R}}_{\alpha\beta\gamma\delta} = \bar{\mathrm{g}}_{\delta\varepsilon}\,\bar{\mathrm{R}}^{\,\varepsilon}_{\alpha\beta\gamma} = \bar{\mathrm{g}}\left(\overline{\mathrm{Riem}}\left(\tfrac{\partial}{\partial\bar{x}^\alpha}, \tfrac{\partial}{\partial\bar{x}^\beta}\right)\tfrac{\partial}{\partial\bar{x}^\delta}, \tfrac{\partial}{\partial\bar{x}^\gamma}\right),$$

and satisfies the symmetries

$$\bar{\mathrm{R}}_{\alpha\beta\gamma\delta} = -\,\bar{\mathrm{R}}_{\beta\alpha\gamma\delta} = -\,\bar{\mathrm{R}}_{\alpha\beta\delta\gamma} = \bar{\mathrm{R}}_{\gamma\delta\alpha\beta}\,. \tag{8}$$

The components of the Ricci tensor $\overline{\mathrm{Ric}} = \{\bar{\mathrm{R}}_{\alpha\beta}\}_{1\leqslant\alpha\beta\leqslant n}$ and the scalar curvature $\bar{\mathrm{R}}$ are given by $\bar{\mathrm{R}}_{\alpha\beta} = \bar{\mathrm{g}}^{\gamma\delta}\,\bar{\mathrm{R}}_{\gamma\alpha\beta\delta}$, respectively $\bar{\mathrm{R}} = \bar{\mathrm{g}}^{\alpha\beta}\,\bar{\mathrm{R}}_{\alpha\beta}$. The metric tensor extends to general tensors $\bar{\mathrm{S}}, \bar{\mathrm{T}} = \{\bar{\mathrm{T}}^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}\}, \{\bar{\mathrm{S}}^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}\} \in \mathcal{T}^k_l(\mathrm{M}^n)$ by setting

$$\bar{\mathrm{g}}\left(\bar{\mathrm{T}}, \bar{\mathrm{S}}\right) = \bar{\mathrm{g}}_{\alpha_1\gamma_1}\ldots\bar{\mathrm{g}}_{\alpha_k\gamma_k}\bar{\mathrm{g}}^{\beta_1\delta_1}\ldots\bar{\mathrm{g}}^{\beta_l\delta_l}\bar{T}^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}\bar{S}^{\gamma_1\ldots\gamma_k}_{\delta_1\ldots\delta_l},$$

and the norm of $\bar{\mathrm{T}}$ restricted to $\mathcal{U} \subset \mathrm{M}^n$ is defined by

$$\left|\bar{\mathrm{T}}\right|_{\mathcal{U}} := \sup_{x\in\mathcal{U}}\sqrt{\bar{\mathrm{g}}\left(\bar{\mathrm{T}}(x), \bar{\mathrm{T}}(x)\right)}.$$

Let $F : \Sigma \hookrightarrow \mathrm{M}^n$ be a smoothly embedded hypersurface. The geometry of Σ will be denoted by Latin indices. The induced metric on Σ is in local coordinates given by

$$\begin{aligned} g_{ij}(x) &= \bar{\mathrm{g}}\left(\tfrac{\partial F}{\partial x^i}(x), \tfrac{\partial F}{\partial x^j}(x)\right) \\ &= \bar{\mathrm{g}}_{\alpha\beta}\left(F(x)\right) \tfrac{\partial F^\alpha}{\partial x^i}(x) \tfrac{\partial F^\beta}{\partial x^j}(x), \quad x \in \Sigma. \end{aligned}$$

Furthermore, $\{\Gamma_{ij}^k\}$, $Riem = R_{ijkl}$ and ∇ describe the intrinsic geometry of the induced metric on Σ.

For a local choice of unit normal ν of the hypersurface Σ and an orthonormal basis $e_1 \ldots, e_n = \nu$ of $T_x\mathrm{M}^n$, the components of the second fundamental form $A(x) = \{h_{ij}\}_{1\leqslant i,j\leqslant n-1}$ are given by

$$h_{ij}(x) = \bar{\mathrm{g}}\left(\bar{\nabla}_i \nu, e_j\right) = -\bar{\mathrm{g}}\left(\nu, \bar{\nabla}_i e_j\right) \tag{9}$$

and the Weingarten map $W : T_x\Sigma \to T_x\Sigma$ is given by

$$\{h_j^i\}_{1\leqslant i,j\leqslant n-1} = \{g^{ik}h_{kj}\}_{1\leqslant i,j\leqslant n-1}.$$

The mean curvature H and the total curvature $|A|^2$ are defined by

$$H = g^{ij}h_{ij}, \quad \text{respectively} \quad |A|^2 = g^{ik}g^{jl}h_{ij}h_{kl}.$$

In local coordinates $\{x^i\}_{1\leqslant i\leqslant n-1}$ and $\{\bar{x}^\alpha\}_{1\leqslant \alpha\leqslant n}$, these relations are equivalent to the Weingarten equations

$$\begin{aligned} \tfrac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \tfrac{\partial F^\alpha}{\partial x^k} + \bar{\Gamma}_{\beta\delta}^\alpha \tfrac{\partial F^\beta}{\partial x^i} \tfrac{\partial F^\delta}{\partial x^j} &= -h_{ij}\nu^\alpha\,, \\ \tfrac{\partial \nu^\alpha}{\partial x^i} + \bar{\Gamma}_{\beta\delta}^\alpha \tfrac{\partial F^\beta}{\partial x^i}\nu^\delta &= h_{ij}g^{jl}\tfrac{\partial F^\alpha}{\partial x^l}\,. \end{aligned} \tag{10}$$

We say Σ satisfies a *r-ball condition* if for any $x \in \Sigma$ and all $0 \leqslant s \leqslant r$

$$B_s\left(\exp_x s\nu(x)\right) \cap \Sigma = \{x\} = B_s\left(\exp_x -s\nu(x)\right) \cap \Sigma.$$

The maximal value $r > 0$ such that Σ fulfills a r-ball condition is called the *injectivity radius* of Ω and will be denoted by inj_Σ. In this case, defining a tube around the hypersurface Σ by

$$T(\Sigma, \mathrm{inj}_\Sigma) := \{y \in \mathrm{M}^n : \text{there exists } x \in \Sigma \text{ such that } d(x,y) \leqslant \mathrm{inj}_\Sigma\},$$

the following map is a diffeomorphism;

$$\bar{F}_\Sigma : \Sigma \times [-\mathrm{inj}_\Sigma, \mathrm{inj}_\Sigma] \to T(\Sigma, \mathrm{inj}_\Sigma), \quad (x,s) \mapsto \exp_x s\nu(x).$$

The *signed distance function* $d_\Sigma : T(\Sigma, r) \to \mathbb{R}$ and the *parallel sets* $\Sigma_s \hookrightarrow \mathrm{M}^n$ of Σ are then defined by

$$d_\Sigma(y) := \left(\pi_2 \circ \bar{F}_\Sigma^{-1}\right)(y), \quad \text{respectively} \quad \Sigma_s := \bar{F}_\Sigma(\Sigma, s).$$

1.2 Heat kernels of Riemannian manifolds

We briefly summarize some properties of heat kernels that will be used in the subsequent chapters.

Definition 1.1 (Heat kernels) A *heat kernel*, or a *fundamental solution*, of a Riemannian manifold $(\mathrm{M}^n, \bar{\mathrm{g}})$ is a function $\varrho : \mathrm{M}^n \times \mathrm{M}^n \times (0, \infty) \to \mathbb{R}$ that is C^2 in the space variables x, y and C^1 in the time variable t and solves the differential equation

$$\begin{cases} \bar{\Box}_x \varrho(x,y,t) = 0 & (x,t) \in \mathrm{M}^n \times (0,\infty) \\ \varrho(x,y,0) = \delta_y(x) & x \in \mathrm{M}^n, \end{cases} \tag{11}$$

where $\bar{\Box}_x := \bar{\Delta}_x - \frac{\partial}{\partial t}$.

Remark 1.2 Heat kernels always exist and satisfy $\int_{\mathrm{M}^n} \varrho(x,y,t) \leqslant 1$. We say (M^n, g) is *stochastically complete* if

$$\int_{\mathrm{M}^n} \varrho(x,y,t)\omega(x) = 1$$

for all $y \in \mathrm{M}^n$ and $t > 0$.

Heat kernels satisfy the following properties.

Theorem 1.3 *Let* $\varrho : \mathrm{M}^n \times \mathrm{M}^n \times (0,\infty) \to \mathbb{R}$ *be a heat kernel of a Riemannian manifold* $(\mathrm{M}^n, \bar{\mathrm{g}})$.

(*i*) *The heat kernel is symmetric in the two space variables, i.e.*

$$\varrho(x,y,t) = \varrho(y,x,t).$$

(*ii*) *The heat kernel satisfies the following semigroup property;*

$$\int_{\mathrm{M}^n} \varrho(z,x,t-s)\varrho(x,y,s)\omega(x) = \varrho(x,y,t)$$

for all $t > s > 0$ *and* $x, y \in \mathrm{M}^n$.

(*iii*) *Suppose* $(\mathrm{M}^n, \bar{\mathrm{g}})$ *is geodesically complete (or compact without boundary) such that the Ricci-tensor satisfies* $\left|\overline{\mathrm{Ric}}\right| \geqslant -C$ *for some* $C < \infty$. *Then it is stochastically complete and the heat kernel is unique. Moreover, for* $f \in L^2(\mathrm{M}^n)$

$$u(y,t) := \int_{\mathrm{M}^n} f(x)\varrho(x,y,t)\omega(x)$$

is the unique solution of the heat equation $\bar{\Box} u(y,t) = 0$ *with initial data* $u(y,0) = f$.

PROOF. See e.g. [Cha84, Chapter VIII Section 2 Theorem 4]. □

Of course, the heat kernel of the Euclidean space $(\mathbb{R}^n, \delta_{\alpha\beta})$ is given by

$$\varrho_e(x,y,t) = (4\pi t)^{-n/2}\, \mathrm{e}^{-|x-y|^2/4t}. \tag{12}$$

Heat kernels of general Riemannian manifolds can be estimated in terms of the Euclidean heat kernel (12). The following lower and upper bound are proved, for example, in the paper [LX11, Theorem 1.5], respectively by Li and Yau in the publication [LY86, Corollary 3.1].

Theorem 1.4 (Lower Gaussian bound) *Let* $(\mathrm{M}^n, \bar{\mathrm{g}})$ *be a complete Riemannian manifold (or closed) such that* $\left|\overline{\mathrm{Ric}}\right| > -C$*, for some* $C < \infty$*. Then the heat kernel satisfies*

$$\varrho(x,y,t) \geqslant (4\pi t)^{-n/2}\, \mathrm{e}^{-\frac{d(x,y)^2}{4t}\left(1+\frac{1}{3}Ct\right)-\frac{n}{4}Ct}. \tag{13}$$

Theorem 1.5 (Upper Gaussian bound) *Let* $(\mathrm{M}^n, \bar{\mathrm{g}})$ *be a complete (or closed) Riemannian manifold with* $\left|\overline{\mathrm{Ric}}\right| \geqslant -C$ *for some* $C < \infty$*. Then for any* $\eta > 1$ *the Neumann heat kernel satisfies*

$$\varrho(x,y,t) \leqslant C_{14}(\eta,n)\, \left|B_{\sqrt{t}}(x)\right|^{-\frac{1}{2}} \left|B_{\sqrt{t}}(y)\right|^{-\frac{1}{2}} \mathrm{e}^{-\frac{d(x,y)^2}{4\eta t}+C(n)(\eta-1)Ct}, \tag{14}$$

where $C_{14}(\eta,n) \to \infty$ *as* $\eta \to 1$ *and* $C(n)$ *depends on the dimension* n*.*

Remark 1.6 If one considers compact Riemannian manifolds with convex boundary, the above lower and upper Gaussian bounds of the heat kernel also hold for the *Neumann heat kernel*, i.e. the solution of Equation (11) subject to the boundary condition $\bar{\nabla}_{\nu_{\partial \mathrm{M}^n}} \varrho = 0$.

For $k \geqslant 0$ and $r > 0$ we define approximate heat kernel $\varrho_{k,r} : \mathrm{M}^n \times \mathrm{M}^n \times (0,\infty) \to \mathbb{R}$ as follows (compare [Cha84, Chapter VI Section 3]).

Definition 1.7 (Approximate Heat kernels) Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a Riemannian manifold such that

$$\mathrm{inj}_{\bar{\mathrm{g}}} := \inf_{x \in \mathrm{M}^n} \mathrm{inj}_{\bar{\mathrm{g}}}(x) > 0.$$

Fix $k \geqslant 0$ and $0 < r < {}^{\mathrm{inj}_{\bar{\mathrm{g}}}}/_2$. Then, for a cut-off function $\alpha_r : \mathrm{M}^n \times \mathrm{M}^n \to \mathbb{R}$ on the diagonal of M^n, i.e. $\alpha_r \equiv 1$ for $d(x,y) \leqslant r$ and $\alpha_r \equiv 0$ for $d(x,y) > 2r$, the *approximate heat kernel* $\varrho_{k,r} : \mathrm{M}^n \times \mathrm{M}^n \times (0,\infty) \to \mathbb{R}$ is defined by

$$\varrho_{k,r}(x,y,t) := \alpha_r(x,y)\, (4\pi t)^{-n/2}\, \mathrm{e}^{-\frac{d(x,y)^2}{4t}} \sum_{i=0}^{k} \varrho_i(x,y) t^i,$$

where the $\varrho_i(x,y)$, the *heat kernel coefficients*, are smooth functions solving the differential equations

$$\bar{\Box}_x \left((4\pi t)^{-n/2} \, \mathrm{e}^{-\frac{d(x,y)^2}{4t}} \sum_{i=0}^{k} \varrho_i(x,y) t^i \right) = t^k \bar{\Delta}_y \varrho_k(x,y) \, (4\pi t)^{-n/2} \, \mathrm{e}^{-\frac{d(x,y)^2}{4t}} .$$

1.3 Frequently used constants

In the following, constants will always be labelled by the equation in which they first appear. Since our computations are often done in normal coordinates, the estimates will involve terms that depend on the derivatives of the Christoffel symbols and the metric tensor. For this reason, our estimates will always be in terms of the radii $r_{17}(k,x)$ and $r_{19}(\Sigma)$ defined below. We will always assume that these radii are strictly positive. First, we set

$$C_{15}(x) := \sqrt{\left|\overline{\mathrm{Riem}}\right|_{B_{\mathrm{inj}_{\bar{g}}}(x)}}, \quad r_{15}(x) := \min\left(\mathrm{inj}_{\bar{g}}, C_{15}(x)^{-1}\right). \tag{15}$$

Then, for $k \geqslant 0$ the following constant is well-defined

$$\begin{aligned} C_{16}(k,x) := \max \Bigg(& \left|\overline{\mathrm{Riem}}\right|^{1/2}, \left|\bar{\nabla}\overline{\mathrm{Riem}}\right|^{1/3}, \left|\bar{\nabla}^{(2)}\overline{\mathrm{Riem}}\right|^{1/4}, \ldots, \\ & \left|\bar{\nabla}^{(k-1)}\overline{\mathrm{Riem}}\right|^{1/k+1} \left(r_{15}(x) \left|\bar{\nabla}^{(k)}\overline{\mathrm{Riem}}\right|\right)^{1/k+1}, \\ & \left(r_{15}(x)^2 \left|\bar{\nabla}^{(k+1)}\overline{\mathrm{Riem}}\right|^{1/k+1}\right)\Bigg)_{B_{\mathrm{inj}_{\bar{g}}}(x)} . \end{aligned} \tag{16}$$

We now define the radius $r_{17}(k,x)$ by

$$r_{17}(k,x) := \min\left(r_{15}(x), C_{16}(k,x)^{-1}\right). \tag{17}$$

Suppose $F : \Sigma \hookrightarrow \mathrm{M}^n$ is a compactly embedded hypersurface of class C^4. Then we define for $k \geqslant 0$ the radii $r_{18}(k,y)$, $r_{19}(k,\Sigma)$ by

$$\begin{aligned} & r_{18}(k,y) := \\ & \min\left(r_{17}(k,y), \mathrm{inj}_{\Omega}, \left(\sup_{x \in B_{r_{17}(k,y)}(y) \cap \Sigma} \left(|A|, |\nabla A|^{1/2}, \left|\nabla^2 A\right|^{1/3} \right) \right)^{-1} \right), \end{aligned} \tag{18}$$

respectively

$$r_{19}(k,\Sigma) := \inf_{y \in \mathrm{M}^n} r_{18}(y). \tag{19}$$

Remark 1.8 (Definition of $r_{17}(k,x)$) In a normal coordinate system based in $x \in \mathrm{M}^n$ with respect to some orthonormal basis $e_1, \dots e_n$ of $T_x\mathrm{M}^n$, the components of the metric tensor in $sz \in \mathbb{R}^n$ with $|z| = 1$ are given by

$$s^2\bar{\mathrm{g}}_{\alpha\beta}(sz) =: s^2\bar{\mathrm{g}}_{\alpha\beta}(s) = \langle \bar{\mathrm{Y}}_\alpha(s), \bar{\mathrm{Y}}_\beta(s)\rangle, \tag{20}$$

where the $\bar{\mathrm{Y}}_\alpha(s)$ are the unique solutions of the Jacobi equation

$$\bar{\mathrm{Y}}_\alpha(s)'' + \overline{\mathrm{Riem}}\left(\gamma'(s), \bar{\mathrm{Y}}_\alpha(s)\right)\gamma'(s), \quad \bar{\mathrm{Y}}_\alpha(0) = 0 \text{ and } \bar{\mathrm{Y}}_i'(0) = e_\alpha$$

along the path $\gamma(s) = \exp_x sz$. Since the $\bar{\mathrm{Y}}_i(s)$ satisfy the above differential equations, it is possible (see e.g. [Lyt09, Lemma 2.1]) to estimate

$$|Y_\alpha(s)| \leqslant 8s \leqslant 8r_{15}(x), \quad |Y_\alpha'(s)| \leqslant 4 \quad \text{for } |s| \leqslant r_{15}(x)\ln 2. \tag{21}$$

Moreover, higher derivatives of the $\bar{\mathrm{Y}}_\alpha$ are given by

$$\bar{\mathrm{Y}}_\alpha^{(k)}(s) = -\sum_{l=0}^{k-2}\binom{k-2}{l}\left(\bar{\nabla}^{k-2-l}\overline{\mathrm{Riem}}\right)\left(\bar{\mathrm{Y}}_\alpha^{(l)}(s), \gamma'(s)\right)\gamma'(s).$$

Hence, differentiating the right-hand side of (20) $k+2$-times in 0, the estimates (21) yield a local expansion formula for the components $\bar{\mathrm{g}}_{\alpha\beta}$ of the metric tensor:

$$\begin{aligned}&\left|\bar{\mathrm{g}}_{\alpha\beta}(sz) - \left(\delta_{\alpha\beta} - s^2\tfrac{1}{3}\,\bar{\mathrm{R}}_{\alpha\gamma\delta\beta}\,z^\gamma z^\delta\right) + \sum_{l=1}^{k} s^l\,\bar{\mathrm{A}}_{\alpha\beta\alpha_1\dots\alpha_l} z^{\alpha_1}\dots z^{\alpha_l}\right|\\ &\leqslant C(n,k)\tfrac{|s|^{k+1}}{r_{17}(k,x)^{1+k}}.\end{aligned}$$

Here, the $\bar{\mathrm{A}}_{\alpha\beta\alpha_1\dots\alpha_k}$ are certain polynomials in the curvature terms of M^n evaluated in x. For functions in the metric coefficients $\bar{\mathrm{g}}_{\alpha\beta}$ one can choose $C(k,n) < \infty$ accordingly do get similar estimates. For example, the Christoffel symbols and the determinant of the metric tensor satisfy

$$\begin{aligned}\left|\bar{\Gamma}^\alpha_{\beta\gamma}(sx) - \left(-\tfrac{1}{3}s\left(\bar{\mathrm{R}}_{\alpha\gamma\beta\delta} + \bar{\mathrm{R}}_{\alpha\beta\gamma\delta}\right)\right)x_\delta\right| &\leqslant C(n)s^2 r_{17}(3,x)^{-3},\\ \left|\sqrt{\det\bar{\mathrm{g}}} - \left(1 - \tfrac{1}{6}s^2\,\bar{\mathrm{R}}_{\alpha\beta}\,x_\alpha x_\beta\right)\right| &\leqslant C(n)s^3 r_{17}(3,x)^{-3}.\end{aligned} \tag{22}$$

1.4 Evolution equations

In this section we recall the evolution equations of hypersurfaces under diffeomorphisms.

Let $F_0 : \Sigma \hookrightarrow \mathrm{M}^n$ be a smoothly embedded hypersurface such that the radius $r_{19}(0, \Sigma) > 0$ is well-defined. Furthermore, let $\bar{F} : \mathrm{M}^n \times (-\varepsilon, \varepsilon) \to \mathrm{M}^n$ be a variation of M^n, i.e.

$$\bar{F}(\cdot, \tau) \text{ is a diffeomorphism}, \quad \bar{F}(\cdot, 0) = \mathrm{id}_{\mathrm{M}^n}(\cdot).$$

We denote the variational vector field of $\bar{F}$ by $\bar{\mathrm{X}}_{\bar{F}}$. If the tangential components of $\bar{\mathrm{X}}_{\bar{F}}$ vanish, i.e.

$$\bar{\mathrm{X}}_{\bar{F}}(x, \tau) = -\bar{\mathrm{g}}\left(\bar{\mathrm{X}}_{\bar{F}}, \nu\right) \nu(x, \tau) =: -\alpha(x, \tau)\nu(x, \tau),$$

the geometric quantities of the hypersurfaces $\Sigma_\tau := F(\Sigma, \tau)$ satisfy the following evolution equations (see e.g. [HP99, Theorem 3.2]).

Theorem 1.9 *On Σ_τ the following equations hold.*

$$\begin{aligned}
\tfrac{\partial}{\partial \tau} g_{ij} &= 2\alpha h_{ij}, \\
\tfrac{\partial}{\partial \tau} \mathrm{d}\mu &= \alpha H \mathrm{d}\mu, \\
\tfrac{\partial}{\partial \tau} \nu &= -\nabla \alpha, \\
\tfrac{\partial}{\partial \tau} h_{ij} &= -\nabla_i \nabla_j \alpha + \alpha \left(h_{ik} h_j^k - \bar{\mathrm{R}}_{ninj}\right), \\
\tfrac{\partial}{\partial \tau} H &= -\Delta \alpha - \alpha \left(|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu)\right).
\end{aligned} \tag{23}$$

Here, $\mathrm{d}\mu$ is the induced measure on the hypersurface and Δ is the Laplace-Beltrami operator with respect to the time-dependent induced metric on the hypersurface.

General variational vector fields satisfy the following evolution equations. For omitted details we refer to [MY02, Chapter 2] and [Hei01, Chapter 1].

Lemma 1.10 *For a variation $\bar{F} : \mathrm{M}^n \times (0, \varepsilon) \to \mathrm{M}^n$ of M^n we set*

$$\bar{\mathrm{X}} := \bar{\mathrm{X}}_{\bar{F}} = \tfrac{\partial \bar{F}}{\partial t}, \quad \alpha := \bar{\mathrm{g}}\left(\bar{\mathrm{X}}, \nu\right).$$

Then the family of hypersurfaces $\bar{F}(\Sigma, \tau) := \Sigma_\tau$ satisfies the following evolution equations;

$$\begin{aligned}
\tfrac{\partial}{\partial \tau} g_{ij} &= \bar{\mathrm{g}}\left(\bar{\nabla}_j \bar{\mathrm{X}}, e_i\right) + \bar{\mathrm{g}}\left(\bar{\nabla}_i \bar{\mathrm{X}}, e_j\right) \\
\tfrac{\partial}{\partial \tau} \nu &= -g^{ij} \bar{\mathrm{g}}\left(\nu, \bar{\nabla}_i \bar{\mathrm{X}}\right) e_j, \\
\tfrac{\partial}{\partial \tau} \mathrm{d}\mu &= \operatorname{div} \bar{\mathrm{X}}, \\
\tfrac{\partial}{\partial \tau} H &= -\Delta \alpha - \alpha \left(|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu)\right) + \bar{\mathrm{X}}^T(H).
\end{aligned} \tag{24}$$

Furthermore, for $Y(\tau) \in \mathcal{T}_0^1(\Sigma_\tau)$ and $\beta = \bar{\mathrm{g}}(Y, \nu)$

$$\tfrac{\partial}{\partial \tau} \operatorname{div} Y = Y\left(\alpha H + \operatorname{div} \bar{\mathrm{X}}^T\right) + \beta H + \operatorname{div}\left(\left(\bar{\nabla}_{\bar{\mathrm{X}}} Y\right)^T\right). \tag{25}$$

Proof. We refer to [MY02, Lemma 2.4] for the first three equalities from (24). The last equality follows from (24). In a local coordinate system around $x \in \Sigma$ we have

$$
\begin{aligned}
\tfrac{\partial}{\partial \tau} \operatorname{div} Y &= \tfrac{\partial}{\partial t}\left(\tfrac{1}{\sqrt{\det g}}\left(\tfrac{\partial}{\partial x_i}\left(\sqrt{\det g}\, Y^i\right)\right)\right) \\
&= \tfrac{1}{\sqrt{\det g}}\left(\tfrac{\partial}{\partial x_i}\left(\left(\tfrac{\partial}{\partial t}\sqrt{\det g}\right) Y^i + \sqrt{\det g}\left(\tfrac{\partial}{\partial t} Y^i\right)\right)\right) \\
&\quad + \left(\tfrac{\partial}{\partial t}\tfrac{1}{\sqrt{\det g}}\right)\left(\tfrac{\partial}{\partial x_i}\left(\sqrt{\det g}\, Y^i\right)\right) \\
&= \tfrac{1}{\sqrt{\det g}}\left(\tfrac{\partial}{\partial x_i}\left(\left(\operatorname{div} \bar{\mathrm{X}}\sqrt{\det g}\right) Y^i + \sqrt{\det g}\left(\bar{\nabla}_{\bar{\mathrm{X}}} Y^i\right)\right)\right) \\
&\quad - \operatorname{div} \bar{\mathrm{X}} \tfrac{1}{\sqrt{\det g}}\left(\tfrac{\partial}{\partial x_i}\left(\sqrt{\det g}\, Y^i\right)\right) \\
&= \operatorname{div}\left(\left(\operatorname{div} \bar{\mathrm{X}}\right) Y\right) + \operatorname{div}\left(\bar{\nabla}_{\bar{\mathrm{X}}} Y\right) - \operatorname{div} \bar{\mathrm{X}} \operatorname{div} Y.
\end{aligned}
$$

On the other hand,

$$
\operatorname{div}\left(\left(\operatorname{div} \bar{\mathrm{X}}\right) Y\right) - \operatorname{div} \bar{\mathrm{X}} \operatorname{div} Y = Y(\operatorname{div}_\Sigma \bar{\mathrm{X}}) = Y\left(\alpha H + \operatorname{div} \bar{\mathrm{X}}^T\right). \qquad \square
$$

Remark 1.11 In contrast to the case of vanishing tangential components, Theorem 1.9, the evolution equation of the mean curvature (24) and the total curvature $|A|^2$ are not first order differential equations in the the components h_{ij} of the second fundamental form A. We will make use of the following lemma to estimate the evolution of the geometric quantities in terms of the curvature of the initial hypersurface. It is derived, for example, in [Hei01, Lemma 1.5].

Lemma 1.12 *For a vector field $\bar{\mathrm{X}} \in \mathcal{T}_0^1(\mathrm{M}^n)$ let $\bar{F}^{\bar{\mathrm{X}}} : \mathrm{M}^n \times (-\varepsilon_{\bar{\mathrm{X}}}, \varepsilon_{\bar{\mathrm{X}}})$ be the corresponding global flow. Then the mean curvatures H and the total curvatures $|A|^2$ of the hypersurfaces $\Sigma_t := F^{\bar{\mathrm{X}}}(\Sigma, t)$ satisfy*

$$
\begin{aligned}
\tfrac{\partial}{\partial \tau} H = {} & \tfrac{1}{2} g^{ij} \bar{\nabla}_\nu \left(L_{\bar{\mathrm{X}}} \bar{\mathrm{g}}\right)(e_i, e_j) - g^{ij} \bar{\nabla}_i \left(L_{\bar{\mathrm{X}}} \bar{\mathrm{g}}\right)(\nu, e_j) \\
& + h^{ij} L_{\bar{\mathrm{X}}} \bar{\mathrm{g}}\,(e_i, e_j) + \tfrac{1}{2} H L_{\bar{\mathrm{X}}} \bar{\mathrm{g}}\,(e_r, e_s), \qquad (26) \\
\tfrac{\partial}{\partial \tau} |A|^2 = {} & h^{ij} \bar{\nabla}_\nu \left(L_{\bar{\mathrm{X}}} \bar{\mathrm{g}}\right)(e_i, e_j) - 2 h^{ij} \bar{\nabla}_i \left(L_{\bar{\mathrm{X}}} \bar{\mathrm{g}}\right)(\nu, e_j) \\
& + |A|^2 \left(L_{\bar{\mathrm{X}}} \bar{\mathrm{g}}\,(\nu, \nu) - g^{rs}\left(L_{\bar{\mathrm{X}}} \bar{\mathrm{g}}\,(e_r, e_s)\right)\right). \qquad (27)
\end{aligned}
$$

In particular, setting

$$
r_{28}\left(\bar{\mathrm{X}}\right) := \left(\max\left(\sqrt{\left|\nabla^2 \bar{\mathrm{X}}\right|}, \left|\nabla \bar{\mathrm{X}}\right|, \left|\bar{\mathrm{X}}\right| r_{19}(4, \Sigma)^{-1}\right)\right)^{-1} \qquad (28)
$$

there is a constant $\varepsilon(n) > 0$ that only depends on n such that for $\tau \leqslant \varepsilon(n) r_{28}\left(\bar{\mathrm{X}}\right)$

$$
|A_\tau| \leqslant 2\left(|A_0| + r_{28}\left(\bar{\mathrm{X}}\right)\right).
$$

Proof. In an orthonormal basis of $e_1, \dots e_n = \nu$ of $T_x\mathrm{M}^n$ we have

$$\begin{aligned}\tfrac{\partial}{\partial\tau}\,|A|^2 = \tfrac{\partial}{\partial\tau}\left(h^{ij}h_{ij}\right) &= 2h^{ij}\tfrac{\partial}{\partial\tau}h_{ij} + 2h^l_i h_{kl}\tfrac{\partial}{\partial\tau}g^{ik}\\ &= 2h^{ij}\tfrac{\partial}{\partial\tau}h_{ij} - 2h^{lr}h^s_l\tfrac{\partial}{\partial\tau}g_{rs}.\end{aligned}$$

Similar to [Hei01, Lemma 1.5] we compute

$$\begin{aligned}-2h^{lr}h^s_l\tfrac{\partial}{\partial\tau}g_{rs} &= -2\,|A|^2\,g^{rs}\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\left(e_r, e_s\right)\right),\\ 2h^{ij}\tfrac{\partial}{\partial\tau}h_{ij} &= h^{ij}\bar\nabla_\nu\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)\left(e_i, e_j\right) - 2h^{ij}\bar\nabla_i\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)\left(\nu, e_j\right)\\ &\quad + |A|^2\,L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\left(\nu,\nu\right).\end{aligned}$$

We conclude

$$\begin{aligned}\tfrac{\partial}{\partial\tau}\,|A|^2 &= h^{ij}\bar\nabla_\nu\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)\left(e_i, e_j\right) - 2h^{ij}\bar\nabla_i\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)\left(\nu, e_j\right)\\ &\quad + |A|^2\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\left(\nu,\nu\right) - g^{rs}\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\left(e_r, e_s\right)\right)\right)\\ &\leqslant |A|\left(\bar\nabla L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)_{\max.} + |A|^2\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)_{\max.}.\end{aligned}$$

Since in local coordinates the covariant derivatives $\bar\nabla X$ and $\bar\nabla^2 X$ can be written in terms of the partial derivatives of X, the Christoffel symbols and their derivatives, we have, for some $C(n)$ only depending on n,

$$\begin{aligned}\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)_{\alpha\beta} &= \bar{\mathrm{X}}^\gamma\tfrac{\partial}{\partial x^\gamma}\bar{\mathrm{g}}_{\alpha\beta} + \tfrac{\partial}{\partial x^\alpha}\bar{\mathrm{X}}^\gamma\bar{\mathrm{g}}_{\gamma\beta} + \tfrac{\partial}{\partial x^\beta}\bar{\mathrm{X}}^\gamma\bar{\mathrm{g}}_{\gamma\alpha}\\ &\leqslant C(n)\left(\left|\bar{\mathrm{X}}\right| r_{17}(3,x)^{-1} + \left|\bar\nabla\bar{\mathrm{X}}\right|\right),\\ \left(\bar\nabla_\lambda L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)_{\alpha\beta} &= \tfrac{\partial}{\partial x^\lambda}\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)_{\alpha\beta} - \bar\Gamma^\delta_{\lambda\alpha}\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)_{\delta\beta} - \bar\Gamma^\delta_{\lambda\beta}\left(L_{\bar{\mathrm{X}}}\bar{\mathrm{g}}\right)_{\delta\alpha}\\ &\leqslant C(n)\left(\left|\nabla^2\bar{\mathrm{X}}\right| + \left|\bar{\mathrm{X}}\right| r_{17}(3,x)^{-2} + \left|\nabla\bar{\mathrm{X}}\right| r_{17}(3)^{-1,x}\left(\bar{\mathrm{X}}\right)\right).\end{aligned}$$

In particular,

$$2\,|A|\,\tfrac{\partial}{\partial\tau}\,|A| = \tfrac{\partial}{\partial\tau}\,|A|^2 \leqslant C(n) r_{28}\left(\bar{\mathrm{X}}\right)^{-1}\left(\left|\,|A|\,r_{28}\left(\bar{\mathrm{X}}\right)^{-1} + |A|^2\right|\right).$$

Therefore, Grönwall's inequality implies for $0 \leqslant \tau \leqslant \varepsilon(n)\cdot r_{28}\left(\bar{\mathrm{X}}\right) := \left(C(n)^{-1}\ln 2\right)\cdot r_{28}\left(\bar{\mathrm{X}}\right)$,

$$|A| \leqslant \left(|A(0)| + r_{28}(\bar{\mathrm{X}})^{-1}\right)\exp\left(tC(n)r_{28}(\bar{\mathrm{X}})^{-1}\right) \leqslant 2\left(|A(0)| + r_{28}(\bar{\mathrm{X}})^{-1}\right).$$

□

2 Pointwise estimates for the solution of heat equations

In this chapter we derive a short-time and short-distance expansion of the solution $u_\Omega : \mathrm{M}^n \times (0,\infty) \to \mathbb{R}$ of the heat equation

$$\begin{cases} \bar{\Delta}_{\bar{g}} u_\Omega(x,t) = \frac{\partial}{\partial t} u_\Omega(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u_\Omega(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n. \end{cases} \tag{HE}$$

Here, we assume $(\mathrm{M}^n, \bar{g})$ to be a closed Riemannian manifold and $\Omega \subset \mathrm{M}^n$ to be a set with compact boundary $\partial\Omega$ such that embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. The asymptotic expansion of $u_\Omega(y,t)$ will be in terms of the time parameter t and the signed distance function $d_\Omega(x) : \mathrm{M}^n \to \mathbb{R}$ with respect to the set Ω. Before stating the main result of this chapter, Theorem 2.5, we start by defining the signed distance function and briefly reviewing some of its properties.

Definition 2.1 For some $\Omega \subset \mathrm{M}^n$ the *signed distance function* $d_\Omega(x) : \mathrm{M}^n \to \mathbb{R}$ is defined by

$$d_\Omega(y) = \begin{cases} \inf_{x\in\Omega} d(x,y) & \text{if } x \in \Omega^c, \\ -\inf_{x\in\Omega} d(x,y) & \text{if } x \in \Omega, \end{cases}$$

where $d : \mathrm{M}^n \times \mathrm{M}^n \to \mathbb{R}$ is the distance function of $(\mathrm{M}^n, \bar{g})$.

Remark 2.2 (Properties of the distance function) If the boundary $\partial\Omega$ satisfies a r-boundary condition, the inverse function theorem implies that the mapping

$$F_{\partial\Omega} : \partial\Omega \times [-\operatorname{inj}_\Omega, \operatorname{inj}_\Omega] \to T(\partial\Omega, \operatorname{inj}_\Omega), \quad (x,s) \mapsto \exp_x s\nu(x)$$

is a diffeomorphism. Here, $\nu(x)$ is a outward pointing local choice of a normal vector of $\partial\Omega$ in $x \in \partial\Omega$ and $\operatorname{inj}_\Omega > 0$ is called the *injectivity radius* of Ω. Moreover, if $y = \exp_x s\nu(x)$ for some unique $x \in \partial\Omega$ we have $s = d_\Omega(y)$. Furthermore, if $\partial\Omega$ is of class C^4, it satisfies an *inner-* and *outer ball condition*, i.e. there exists some $r \geqslant \operatorname{inj}_\Omega > 0$ such that for $x \in \partial\Omega$ there are some $z_{x,s} \in \Omega^c$ and $y_{x,s} \in \Omega$ such that for all $s \leqslant r$

$$B_s(z_{x,s}) \cap \Omega = \{x\} = B_s(y_{x,s}) \cap \Omega.$$

Finally, the gradient of distance function d_Ω satisfies $|\nabla d_\Omega| = 1$ for $|d_\Omega| < \operatorname{inj}_\Omega$.

In Theorem 2.5 we show that the leading term in the expansion of $u_\Omega(\cdot,t)$ will be independent of the geometry of the ambient manifold $(\mathrm{M}^n, \bar{g})$ and the set Ω. It is given in terms of the Gaussian error function Φ that is defined as follows.

Definition 2.3 (Gaussian error function) We call the function $\Phi : \mathbb{R} \to (0,1)$ defined by

$$\Phi(y) := \tfrac{1}{\sqrt{4\pi}} \int_{-\infty}^{y} \mathrm{e}^{-x^2/4}\, \mathrm{d}x$$

the *Gaussian error function.*

In Section 2.2 we discuss certain properties of the Gaussian error function that are needed to prove Theorem 2.5. Amongst other things, we need to estimate its asymptotic behaviour as $y \to \pm\infty$.

The solution of the simplest heat equation of the form (HE) is explicitly given in terms of the Gaussian error function.

Example 2.4 We consider the heat equation (HE) in the one-dimensional Euclidean case, where the initial data is given by the characteristic function of the half-space $\Omega := (-\infty, 0)$. Then its solution u_Ω explicitly given by

$$u_\Omega(y,t) = \tfrac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} \mathrm{e}^{-\frac{(x-y)^2}{4t}} \mathrm{d}x = \tfrac{1}{\sqrt{4\pi}} \int_{-\infty}^{-y/\sqrt{t}} \mathrm{e}^{-x^2/4} \mathrm{d}x = \Phi\left(-\tfrac{d_\Omega(y)}{\sqrt{t}}\right).$$

Similarly, if the initial data is the characteristic function of an interval $\Omega := (a,b)$ and $y \geqslant b$, the solution of the heat equation (HE) is given by

$$u_\Omega(y,t) = \Phi\left(-\tfrac{d_\Omega(y)}{\sqrt{t}}\right) - \Phi\left(-\tfrac{a-y}{\sqrt{t}}\right).$$

As demonstrated in Section 2.2 the second term on right-hand side is exponentially small as $t \to 0$.

We are now ready to formulate the main theorem of this chapter. Before deriving its proof in Section 2.3, we will find an approximation of the solution of (HE) in terms of the Gaussian measure in Section 2.1 and treat the Euclidean case in Section 2.2.

Theorem 2.5 *For any $n \in \mathbb{N}$ there are constants $0 < \varepsilon(n), C(n) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a closed Riemannian manifold of class C^6. Let $\Omega \subset \mathrm{M}^n$ be a set such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := r_{19}(k, \partial\Omega) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor $\overline{\mathrm{Riem}}$, the first two derivatives of the second fundamental form A of $\partial\Omega$

and the injectivity radii of M^n *and* Ω*, is positive. Then we may approximate the solution* u_Ω *of the heat equation (HE) in the following way.*

For $y \in T(\partial\Omega, \mathrm{inj}_\Omega)$ *let* $x_0 \in \partial\Omega$ *be the unique base point of* y *and* $s = d_\Omega(y)$ *be the distance of* y *to the boundary of* Ω*. The point* y *is then given by* $y = \exp_{x_0} s\nu(x_0)$*. We define the function* $v_\Omega : T(\partial\Omega, \mathrm{inj}_\Omega) \times (0, \infty) \to \mathbb{R}_+$ *by setting*

$$\begin{aligned} v_\Omega(y,t) :=& \Phi\left(-\tfrac{s}{\sqrt{t}}\right) + \tfrac{\mathrm{e}^{-s^2/4t}}{\sqrt{4\pi}}\Big(- \sqrt{t}H(x_0) + \sqrt{t}s \\ & \left(\tfrac{1}{4}\left(-2\left(\overline{\mathrm{Ric}}(x_0)(\nu(x_0),\nu(x_0)) + |A(x_0)|^2\right) + H(x_0)^2\right)\right)\Big). \end{aligned}$$

Here, H, ν*,* $|A|$ *are evaluated with respect to the curvature of the boundary* $\partial\Omega$ *and* $\Phi : \mathbb{R} \to (0,1)$ *is the Gaussian error function from Definition 2.3. The we have the uniform estimate*

$$|u_\Omega(y,t) - v_\Omega(y,t)| \leqslant C(n)\tfrac{t^{3/2}}{r^3}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{s^2}{t}\right)\tfrac{\mathrm{e}^{-s^2/4t}}{\sqrt{4\pi}}$$

for all $t \in (0, \varepsilon(n)r^2)$ *and* $y \in T(\partial\Omega, \varepsilon(n)r)$*.*

Theorem 2.5 can also be written in terms of the curvature of the parallel hypersurfaces

$$\partial\Omega_s := \{x \in \mathrm{M}^n : d_\Omega(y) = s\}.$$

Corollary 2.6 *For any* $n \in \mathbb{N}$ *there are constants* $0 < \varepsilon(n), C(n) < \infty$ *such that the following is true.*

Under the same assumptions as in Theorem 2.5 we define the function

$$\begin{aligned} v_\Omega(y,t) :=& \Phi\left(-\tfrac{s}{\sqrt{t}}\right) + \tfrac{\mathrm{e}^{-s^2/4t}}{\sqrt{4\pi}}\Big(- \sqrt{t}H_{\partial\Omega_s}(y) + \sqrt{t}s \\ & \left(\tfrac{1}{4}\left(-2\left(\overline{\mathrm{Ric}}(y)(\nu_{\partial\Omega_s}(y),\nu_{\partial\Omega_s}(y)) + |A_{\partial\Omega_s}(y)|^2\right) + H_{\partial\Omega_s}(y)^2\right)\right)\Big). \end{aligned}$$

Here, H*,* $|A|^2$ *and* ν *are evaluated in* $y \in \mathrm{M}^n$ *with respect to the curvature of the parallel hypersurfaces* $\Omega_s = \{x \in \mathrm{M}^n : d_\Omega(y) = s\}$*. Then we have the uniform estimate*

$$|u_\Omega(y,t) - v_\Omega(y,t)| \leqslant C(n)\tfrac{t^{3/2}}{r^3}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{s^2}{t}\right)\tfrac{\mathrm{e}^{-s^2/4t}}{\sqrt{4\pi}}$$

for all $t \in (0, \varepsilon(n)r^2)$ *and* $y \in T(\partial\Omega, \varepsilon(n)r)$*.*

PROOF. The hypersurfaces $\partial\Omega_s$ solve, for $F_0 : \partial\Omega \to \mathrm{M}^n$ being the inclusion mapping, the evolution equation

$$\begin{cases} \frac{\partial F}{\partial s}(x,t) = \nu(x), & x \in \partial\Omega,\, t \in [-\varepsilon(n)r, \varepsilon(n)r], \\ F(x,0) = F_0 & x \in \partial\Omega. \end{cases}$$

Therefore, the evolution equations of the geometric quantities, Theorem 1.9, imply

$$\frac{\partial}{\partial s} H = -\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu) \right).$$

Thus, along the path $\gamma(s) := \exp_{x_0} s\nu(x)$, it holds

$$\begin{aligned} &\left|(H(x_0) - H_s(y)) - s\left(-\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)\right)\right| \\ &= \int_0^s (s-\tau)\frac{\partial}{\partial r}\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)(\gamma(r))\mathrm{d}\tau. \\ &\leqslant C(n)\int_0^s (s-\tau)\left(|A|^3 + \left|\nabla \bar{\mathrm{Riem}}\right|\right)(\gamma(r))\mathrm{d}\tau. \end{aligned} \tag{29}$$

Furthermore, by the evolution equations of the total curvature

$$|h_{ij}(s)| = \left| h_{ij}(0) + \int_0^s \left(h_{ik}h_j^k - \bar{\mathrm{R}}_{ninj}\right)(c(r))\,\mathrm{d}r \right| \leqslant C(n)\tfrac{1}{r} + \int_0^s |A|\,(r)^2\mathrm{d}r.$$

Hence, since $|A| \leqslant r^{-1}$, Grönwall's lemma implies that there is a $\varepsilon(n) > 0$ such that

$$|A|\,(s) \leqslant 2C(n)r^{-1}.$$

In particular, the right-hand side of (29) is bounded by $C(n)s^2 \cdot r^{-3}$. □

2.1 Integral estimates of the heat kernel

In this section we derive an approximation of the solution of the heat equation

$$\begin{cases} \bar{\Delta}_{\bar{g}} u(x,t) = \frac{\partial}{\partial t} u(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n, \end{cases} \tag{HE}$$

where $\Omega \subset \mathrm{M}^n$ is measurable. The approximation will be in terms of the Gaussian measure

$$(4\pi t)^{-n/2}\, \mathrm{e}^{-d(x,y)^2/4t}.$$

In order to prove the main result of this section, Theorem 2.7, we modify the proof of the classical Theorem regarding the short-time asymptotics of heat kernels, Theorem 2.8. A proof of this Theorem can be found, for example in [BGM71, page 210-219], [Cha84, Section 6.4], or [Cho+10, Chapter 24].

There, approximations of heat kernels are derived via a parametrix. The parametrix is constructed by introducing normal coordinates and solving certain differential equations (see Definition 2.10 and Remark 2.11). Using the smoothing properties of approximate heat kernels, a exact solution is then given by adding a correction term. This correction term is a series of convolutions. In the Lemmata 2.12, 2.14 and 2.15 we use the Gaussian lower bound of the heat kernel, Theorem 1.4, and the semigroup property of heat kernels to deduce upper bounds on the norm of the coefficients of this series in terms of the exact heat kernel. Since we are interested in integrals of the heat kernel, these estimates imply that, in our case, the correction term can be estimated in terms of a number of derivatives of the metric tensor that does not depend on the dimension of the manifold (see Lemma 2.18 and Proposition 2.19).

The main result of this section is the following Theorem. Its proof is formulated at the end of this section.

Theorem 2.7 *Fix $k, n \in \mathbb{N}$. There exist constants $0 < \varepsilon(n,k), C(n,k) < \infty$ depending only on k and n such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a closed Riemannian manifold of class C^{k+5}. Suppose $\Omega \subset \mathrm{M}^n$ is measurable and the radius

$$r := \min_{z \in \mathrm{M}^n} r_{17}(k+4, z) > 0$$

that depends on the first $k+5$-th derivatives of the curvature tensor and the injectivity radius of M^n, is well-defined. Then there are functions

$$\varrho_i : \mathrm{M}^n \times \mathrm{M}^n \to \mathbb{R}, \quad 0 \leqslant i \leqslant k$$

such that for any $y \in \mathrm{M}^n$ with $|d_\Omega(y)| \leqslant \varepsilon(n,k) r$ and any $t \in (0, \varepsilon(n,k) r^2)$ the following estimate holds;

$$\begin{aligned}
&\left| \int_\Omega \varrho(x,y,t)\omega(x) - \int_{\Omega \cap B_{r/(2\sqrt{8})}(y)} \left((4\pi t)^{-n/2} \mathrm{e}^{-\frac{d(x,y)^2}{4t}} \sum_{i=0}^{k} \varrho_i(x,y) t^i \right) \omega(x) \right| \\
&\leqslant C(n,k) \tfrac{t^{k+1}}{r^{2k+2}} \left(\int_{\Omega \cap B_{r/(2\sqrt{8})}(y)} (4\pi t)^{-n/2} \mathrm{e}^{-\frac{d(x,y)^2}{4t}} \omega(x) + \tfrac{|\Omega|}{r^n} \mathrm{e}^{-\frac{d_\Omega(y)^2}{4t}} \right).
\end{aligned}$$

We now state the classical result concerning the short-time asymptotics of heat kernels to compare it with the above theorem.

Theorem 2.8 *Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a compact Riemannian manifold without boundary and $k > n/2 + 2$. Then there exist smooth functions*

$$\varrho_i : \mathrm{M}^n \times \mathrm{M}^n \to \mathbb{R}, \quad 0 \leqslant i \leqslant k,$$

the heat-kernel coefficients, such that for any $\eta > 1$, and $0 < T < \infty$ there exists a constant $C(\mathrm{M}^n, \eta, k, T)$ such that the uniform estimate

$$\left| \varrho(x,y,t) - \frac{\mathrm{e}^{-\frac{d(x,y)^2}{4t}}}{(4\pi t)^{n/2}} \sum_{i=0}^{k} t^i \varrho_i(x,y) \right| \leqslant C(\mathrm{M}^n, k, \eta) t^{k+1} \, (4\pi t)^{-n/2} \, \mathrm{e}^{-\frac{d(x,y)^2}{4\eta t}} \tag{30}$$

holds for all $x, y \in \mathrm{M}^n$ and $0 < t < T$. Here, $C(\mathrm{M}^n, k, \eta)$ depends on the derivatives of the curvature tensor and the volume of M^n.

Proof. See e.g. [BGM71, page 210-219], [Cha84, Section 6.4], or [Cho+10, Chapter 24]. □

Remark 2.9 In contrast to the classical result, Theorem 2.8, the error term in Theorem 2.7 decays with the exact factor 4 in the exponential. Using different approaches, estimates of this kind have been proved in [Lud19] and in [Mol75]. However, it is not obvious how the uniform estimates of the error terms depend on the curvature of the Riemannian manifold. The estimate in Theorem 2.7 is independent of the volume of M^n and only involves a number of derivatives of the metric depending on k and not on the dimension of the manifold.

We start by defining approximate heat kernels.

Definition 2.10 (Approximate Heat kernels) Fix $k \geqslant 0$ and $0 < r < \mathrm{inj}_{\bar{\mathrm{g}}}$. Then, for a cut-off function $\alpha_r : \mathrm{M}^n \times \mathrm{M}^n \to \mathbb{R}$ on the diagonal of M^n, i.e. $\alpha_r \equiv 1$ for $d(x,y) \leqslant r$ and $\alpha_r \equiv 0$ for $d(x,y) > 2r$, the *approximate heat kernel* $\varrho_{k,r} : \mathrm{M}^n \times \mathrm{M}^n \times (0,\infty) \to \mathbb{R}$ is defined by

$$\varrho_{k,r}(x,y,t) := \alpha_r(x,y) \, (4\pi t)^{-n/2} \, \mathrm{e}^{-\frac{d(x,y)^2}{4t}} \sum_{i=0}^{k} \varrho_i(x,y) t^i,$$

where the $\varrho_i(x,y)$, the *heat kernel coefficients*, are smooth functions solving the differential equations

$$\bar{\Box}_x \left((4\pi t)^{-n/2} \, \mathrm{e}^{-\frac{d(x,y)^2}{4t}} \sum_{i=0}^{k} \varrho_i(x,y) t^i \right) = t^k \bar{\Delta}_y \varrho_k(x,y) \, (4\pi t)^{-n/2} \, \mathrm{e}^{-\frac{d(x,y)^2}{4t}}. \tag{31}$$

Remark 2.11 (Heat kernel coefficients) By the classical results regarding the small-time asymptotics of the heat kernel (see e.g. [Cha84, Chapter VI Section 3]), the heat kernel coefficients $\varrho_i(x,y)$ satisfy, in a normal coordinate system centered in $y \in \mathrm{M}^n$ with respect to some orthonormal $e_1, \ldots e_n$ of $T_y\mathrm{M}^n$, for $\varphi(x,y) := \sqrt{\det \bar{\mathrm{g}}_{\alpha\beta}(x)}$, the recursion formula

$$\begin{cases} \varrho_0(x,y) = \varphi(x,y)^{-1/2} & \text{for } i = 0, \\ \varrho_i(x,y) = \varphi(x,y)^{-1/2} \int_0^1 r^{i-1} \left(\varphi^{1/2} \bar{\Delta}_x u_{i-1} \right) \left(\exp_y (rx, y) \right) \mathrm{d}r & \text{for } 1 \leqslant i \leqslant k. \end{cases} \tag{32}$$

Thus, to estimate the i-th heat kernel coefficient, the $(2i)$-th derivatives of the components $g_{\alpha\beta}(x)$ of the metric tensor must be bounded. We choose $r := r_{17}(k+4,y)$ in the definition of the cut-off function. Then, as in Remark 1.8, it holds for $0 \leqslant i \leqslant k$

$$\left|\bar{\nabla}^{(m)}\varrho_i(x,y)\right| \leqslant C(n,k) r_{17}(k+4,y)^{-2i-m}, \quad 0 \leqslant m \leqslant 2.$$

In particular, for $k=1$ we will assume the following term to be bounded;

$$\begin{aligned} C_{16}(x,5) := \max \Bigg(& \left|\overline{\mathrm{Riem}}\right|^{1/2}, \left|\bar{\nabla}\overline{\mathrm{Riem}}\right|^{1/3}, \left|\bar{\nabla}^{(2)}\overline{\mathrm{Riem}}\right|^{1/4}, \left|\bar{\nabla}^{(3)}\overline{\mathrm{Riem}}\right|^{1/5}, \\ & \left|\bar{\nabla}^{(4)}\overline{\mathrm{Riem}}\right|^{1/6}, \left(r_{15}(x) \left|\bar{\nabla}^{(5)}\overline{\mathrm{Riem}}\right|\right)^{1/6}, \\ & \left(r_{15}(x)^2 \left|\bar{\nabla}^{(6)}\overline{\mathrm{Riem}}\right|^{1/6}\right) \Bigg)_{B_{\mathrm{inj}_{\bar{g}}}(x)} . \end{aligned} \tag{33}$$

By the Recursion formula (32), it is also possible (see e.g. [Cho+10, Chapter 24.4] or [Cha84, Chapter VI.3]) to approximate the first two heat kernel coefficients;

$$\begin{aligned} \left|\varrho_0(x,y) - \left(1 + \tfrac{1}{12}\bar{\mathrm{R}}_{\alpha\beta}x^\alpha x^\beta\right)\right| &\leqslant C(n)\ |x|^3\, r_{17}(5,y)^{-3}, \\ \left|\varrho_1(x,y) - \tfrac{1}{6}\bar{\mathrm{R}}\right| &\leqslant C(n)\ |x|\, r_{17}(5,y)^{-3}\ . \end{aligned} \tag{34}$$

Equation (31) and the lower bound on the heat kernel, Theorem 1.4, yield the following estimate on $\bar{\Box}_x\varrho_{k,r}(x,y,t)$.

Lemma 2.12 *Fix $k,n \in \mathbb{N}$ and $\eta > 1$. Then, there is a constant $C(n,k,\eta) < \infty$ such that the following is true.*

Under the assumptions of Theorem 2.7 we set

$$C_{35}(T,\eta,k,n) := C(n,k,\eta)\left(1 + \tfrac{T^k}{r^{2k}}\right) \mathrm{e}^{\frac{1}{4}n\eta T r^{-2}} . \tag{35}$$

Then we have

$$|\varrho_{k,r}(x,y,t)| \leqslant C_{35}(T,\eta,k,n)\varrho(x,y,\eta t) \tag{36}$$

for any $t \in (0,T]$. Moreover, we also have the following estimate

$$\left|\bar{\Box}_x\varrho_{k,r}(x,y,t)\right| \leqslant C_{35}(T,\eta,k,n)\tfrac{t^k}{r^{2k+1}} \begin{cases} \varrho(x,y,t) & \text{for } d(x,y) \leqslant r, \\ \varrho(x,y,\eta t) & \text{for } \leqslant r \leqslant d(x,y) \leqslant 2r, \\ 0 & \text{else.} \end{cases} \tag{37}$$

Proof. The constant $C_{35}(T,\eta,k,n)$ depends on k,n and $\eta>1$. Since we are mainly interested in its dependence on the time parameter T, we will, for the sake clarity, denote it by $C_{35}(T)$. First, we note that since the support of the approximate heat kernel $\varrho_{k,r}(\cdot,y,t)$ is by definition contained in a ball of radius $2r$ around y and the Ricci curvature $\overline{\mathrm{Ric}}$ is bounded from above by r^{-2}, the lower Gaussian bound, Inequality (13), implies for any $x\in \mathrm{M}^n$

$$\begin{aligned}
|\varrho_{k,r}(x,y,t)| &\leqslant \eta^{n/2}C(k,n)\left(1+\tfrac{t^k}{r^{2k}}\right)(4\pi\eta t)^{-n/2}\mathrm{e}^{-\frac{d(x,y)^2}{4\eta t}}\chi_{[0,2r]}(d(x,y))\\
&= \eta^{n/2}C(k,n)\left(1+\tfrac{t^k}{r^{2k}}\right)\chi_{[0,2r]}(d(x,y))\mathrm{e}^{\frac{1}{12}d(x,y)r^{-2}+\frac{n}{4}\eta t r^{-2}}\\
&\quad\times(4\pi t)^{-n/2}\,\mathrm{e}^{-\frac{d(x,y)^2}{4\eta t}\left(1+\frac{1}{3}\frac{t}{r^2}\right)-\frac{n}{4}\frac{\eta t}{r^2}}\\
&\overset{(13)}{\leqslant} C(k,n)\left(1+\tfrac{\eta t^k}{r^{2k}}\right)\chi_{[0,2r]}(d(x,y))\mathrm{e}^{\frac{1}{12}d(x,y)^2r^{-2}+\frac{n}{4}tr^{-2}}\ \varrho(x,y,t)\\
&\leqslant C(k,n,\eta)\left(1+\tfrac{t^k}{r^{2k}}\right)\mathrm{e}^{\frac{1}{3}+\frac{n}{4}tr^{-2}}\ \varrho(x,y,t).
\end{aligned}$$

For the second inequality, we explicitly construct the cut-off function α_r, by choosing a function $\varphi:\mathbb{R}_+\to[0,1]$ satisfying $\varphi=1$ on $[0,1]$, $\varphi=0$ on $[2,\infty)$ and $|\varphi'|,|\varphi''|<C$ and setting $\alpha_r(x,y):=\varphi\left({}^{d(x,y)}/_{r}\right)$. For $y\in\mathrm{M}^n$ fixed, α_r then satisfies, for some $C(n)$ only depending on n, the estimates

$$\left|\bar\nabla\alpha_r\right| = \frac{\left|\bar\nabla d(\cdot,y)\right|\varphi'}{r}\leqslant\frac{C}{r},\tag{38}$$

$$\left|\bar\Delta\alpha_r\right| = \left|\left(\frac{n-1}{|x|}+\frac{\partial(\log\varphi(x,y))}{\partial|x|}\right)\frac{\varphi'}{r}+\frac{\left|\bar\nabla d(\cdot,y)\right|^2\varphi''}{r^2}\right|\leqslant\frac{C(n)}{r^2}.\tag{39}$$

On the other hand, since the heat kernel coefficients satisfy Equation (31), we have

$$\begin{aligned}
\left(\bar\Box_x\varrho_{k,r}\right)(x,y,t) = \alpha_r(x,y)t^k\,(4\pi t)^{-n/2}\,\mathrm{e}^{-\frac{d(x,y)^2}{4t}}\,\bar\Delta\varrho_k(x,y)\\
+\bar\Delta\alpha_r(x,y)\varrho_k(x,y,t)+\langle\bar\nabla\alpha_r,\bar\varrho_k\rangle(x,y,t),
\end{aligned}\tag{40}$$

where we set

$$\varrho_k(x,y,t) = (4\pi t)^{-n/2}\,\mathrm{e}^{-\frac{d(x,y)^2}{4t}}\sum_{i=0}^{k}\varrho_i(x,y)t^i.$$

For $d(x,y)>2r$, the right-hand side of (40) is identically zero. For $d(x,y)\leqslant r$, the second and third term in (40) vanish. Hence, by Remark 2.11 and Theorem

1.4

$$\left|\left(\bar{\Box}_x \varrho_{k,r}\right)(x,y,t)\right| \overset{2.11}{\leqslant} C(k,n)\exp\left(\tfrac{1}{12}+\tfrac{n}{4}\tfrac{t}{r^2}\right) \times \frac{t^k}{r^{2k+2}} \frac{\exp\left(-\frac{d(x,y)^2}{4t}\left(1+\frac{1}{3}\left|\overline{\mathrm{Ric}}\right|t\right)-\frac{n}{4}\left|\overline{\mathrm{Ric}}\right|t\right)}{(4\pi t)^{n/2}}$$

$$\overset{1.4}{\leqslant} C(k,n)\exp\left(\tfrac{1}{12}+\tfrac{n}{4}\tfrac{T}{r^2}\right)\tfrac{t^k}{r^{2k+2}}\varrho(x,y,t).$$

For $r \leqslant d(x,y) \leqslant 2r$ the first two terms are estimated similarly with (39). To estimate the last summand in (40), we first note

$$0 \leqslant \frac{\mathrm{e}^{-\frac{(\eta-1)r^2}{4\eta t}}}{t^{m/2}} \leqslant \frac{1}{r^m}\left(\frac{2\eta m}{\mathrm{e}(\eta-1)}\right)^{m/2} \quad \text{for all } m \in \mathbb{N} \text{ and } \eta > 1.$$

Thus, we have for $m = 2k+2$

$$\left|\bar{\nabla}\varrho_k(x,y,t)\right|$$

$$\leqslant \left(\frac{r\left|\bar{\nabla} d(x,y)\right|}{t}\left|\sum_{i=0}^{k}\varrho_i(x,y)t^i\right| + \left|\sum_{i=0}^{k}\left(\bar{\nabla}\varrho_i\right)(x,y)t^i\right|\right)(4\pi t)^{-n/2}\,\mathrm{e}^{-\frac{d(x,y)^2}{4t}}$$

$$\overset{2.11}{\leqslant} C(n,k,\eta)\tfrac{t^k}{r^{2k+1}}\left(1+\tfrac{t^k}{r^{2k}}\right)\exp\left(\tfrac{1}{12}+\tfrac{n}{4}\tfrac{\eta t}{r^2}\right) \times \frac{\exp\left(-\frac{d(x,y)^2}{4\eta t}\left(1+\frac{1}{3}\left|\overline{\mathrm{Ric}}\right|\eta t\right)-\frac{n}{4}\left|\overline{\mathrm{Ric}}\right|\eta t\right)}{(4\pi\eta t)^{n/2}}$$

$$\overset{1.4}{\leqslant} C(n,k,\eta)\tfrac{t^k}{r^{2k+1}}\left(1+\tfrac{t^k}{r^{2k}}\right)\exp\left(\tfrac{1}{12}+\tfrac{n}{4}\tfrac{\eta t}{r^2}\right)\varrho(x,y,\eta t).$$

We conclude by (38), (39) and (40)

$$\left|\left(\bar{\Box}_x\varrho_{k,r}\right)(x,y,t)\right| \leqslant C(n,k,\eta)\tfrac{t^k}{r^{2k+1}}\left(1+\tfrac{T^k}{r^{2k}}\right)\exp\left(\tfrac{n}{4}\tfrac{\eta T}{r^2}\right)\varrho(x,y,\eta t). \qquad \Box$$

The exact fundamental solution of (HE) is to be constructed via a parametrix. Therefore, we must study convolution of functions on $\mathrm{M}^n \times \mathrm{M}^n \times (0,\infty)$.

Definition 2.13 (Convolution) The *space-time convolution* of two functions $F, G \in C^0(\mathrm{M}^n \times \mathrm{M}^n \times (0,\infty))$ is defined by

$$(F * G)(x,y,t) := \int_0^t \int_{\mathrm{M}^n} F(x,z,s)G(z,y,t-s)\omega(z)\mathrm{d}s.$$

Furthermore, we define for $l \in \mathbb{N}$, $F^{*l} := F * \ldots * F$ and $F^{*0} = F$.

In the next step, we use the semigroup property of the heat kernel to derive the following upper bound on convolutions of approximate heat kernels.

Lemma 2.14 *Fix $k, n \in \mathbb{N}$ and $\eta > 1$. Then there is a constant $C(n,k,\eta) < \infty$ such that the following is true.*

Under the assumptions of Theorem 2.7 we have for any $T < \infty$ and all $0 < t \leqslant T$ and all $l \in \mathbb{N}$,

$$\bar{\Box}_x \left(\varrho_{k,r}(x,y,t)\right)^{*l} \leqslant \tfrac{C_{35}(T,n,k,\eta)}{r^{2k+2}} \left(C_{35}(T,n,k,\eta)\tfrac{t^k}{r^{2k+2}}\right)^l \times \tfrac{t^{k+l}}{(k+1)\dots(k+l)} \varrho(x,y,\eta t), \tag{41}$$

where

$$C_{35}(T,n,k,\eta) := C(n,k,\eta)\left(1 + \tfrac{T^k}{r^{2k}}\right) \mathrm{e}^{\frac{1}{4}n\eta T r^{-2}} . \tag{42}$$

Proof. For the sake of clarity, we denote $C_{35}(T,n,k,\eta)$ by $C_{35}(T)$. We prove the claim inductively. By Lemma 2.12

$$\begin{aligned} |\varrho_{k,r}(x,y,t)| &\leqslant C_{35}(T)\varrho(x,y,\eta t), \\ \left|\bar{\Box}_x \varrho_{k,r}(x,y,t)\right| &\leqslant C_{35}(T)\tfrac{t^k}{r^{2k+1}}\varrho(x,y,\eta t). \end{aligned} \tag{43}$$

The case $l = 0$ is implied by (43). As for $l+1$, the definition of convolutions, the induction's hypothesis and Inequality (43) yield

$$\begin{aligned} &\bar{\Box}_x \left(\varrho_{k,r}(x,y,t)\right)^{*l+1} \\ &= \int_0^t \int_{\mathrm{M}^n} \bar{\Box}_x \varrho_{k,r}(x,z,s) \bar{\Box}_x \left(\varrho_{k,r}(z,y,t-s)\right)^{*l} \omega(z) \mathrm{d}s \\ &\leqslant \int_0^t \int_{\mathrm{M}^n} \Big(C_{35}(T)\tfrac{s^k}{r^{2k+2}} \varrho(x,z,\eta s) \tfrac{C_{35}(T)}{r^{2k+2}} \\ &\quad \times \left(C_{35}(T)\tfrac{t^k}{r^{2k+2}}\right)^l \tfrac{(t-s)^{k+l}}{(k+1)\dots(k+l)} \varrho(z,y,\eta(t-s)) \Big) \omega(z) \mathrm{d}s \\ &\leqslant \tfrac{C_{35}(T)}{r^{2k+2}} \left(C_{35}(T)\tfrac{t^k}{r^{2k+2}}\right)^{l+1} \int_0^t \Big(\tfrac{(t-s)^{k+l}}{(k+1)\dots(k+l)} \\ &\quad \times \Big(\int_{\mathrm{M}^n} \varrho(x,z,\eta s)\varrho(z,y,\eta(t-s))\omega(z) \Big) \mathrm{d}s \Big). \end{aligned} \tag{44}$$

Hence, by the semigroup property of the heat kernel,

$$\int_{\mathrm{M}^n} \varrho(x,z,\eta s)\varrho(z,y,\eta(t-s))\omega(z) = \varrho(x,y,\eta t),$$

the right-hand side of (44) is given by

$$\bar{\Box}_x \left(\varrho_{k,r}(x,y,t)\right)^{*l+1} \leqslant \tfrac{C_{35}(T)}{r^{2k+2}} \left(C_{35}(T)\tfrac{t^k}{r^{2k+2}}\right)^{l+1} \times \tfrac{t^{k+l+1}}{(k+1)\dots(k+l)(k+l+1)} \varrho(x,y,\eta t).$$

This completes the induction. □

We now apply the above lemma to show that, in fact, $\eta = 1$ can be chosen for close points $x, y \in \mathrm{M}^n$.

Lemma 2.15 *Fix $k, n \in \mathbb{N}$. Then there exist constants $\varepsilon(n) > 0$ and $C(n,k) < \infty$ such that the following is true.*

Under the assumptions of Theorem 2.7 we have for $d(x,y) \leqslant \varepsilon(n,k)r$, $0 \leqslant t \leqslant \varepsilon(n,k)r^2$ and any $l \in \mathbb{N}$

$$\left|\left(\bar{\Box}_x \varrho_{k,r}\right)^{*l}(x,y,t)\right| \leqslant \tfrac{C(n,k)}{r^{2k+2}}\left(\tfrac{C(n,k)}{r^{2k+2}}t^k\right)^l \tfrac{(l+1)t^{k+l}}{(k+1)\dots(k+l)}\varrho(x,y,t). \tag{45}$$

Proof. We start by rewriting the estimates from Lemma 2.12 and Lemma 2.14. We choose $\eta = 2$ in Lemma 2.14. Then $C_{35}(r^2,k,n) \leqslant C(k,n)$ for some $C(k,n) < \infty$ only depending on k and n. Hence, we have

$$|\varrho_{k,r}(x,y,t)| \leqslant C(k,n)\varrho(x,y,t) \tag{46}$$

$$\left|\bar{\Box}_x \varrho_{k,r}(x,y,t)\right| \leqslant C(k,n)\tfrac{t^k}{r^{2k+1}}\begin{cases}\varrho(x,y,t) & \text{for } d(x,y) \leqslant r,\\ \varrho(x,y,2t) & \text{for } \leqslant r \leqslant d(x,y) \leqslant 2r,\\ 0 & \text{else.}\end{cases} \tag{47}$$

and

$$\left|\bar{\Box}_x\left(\varrho_{k,r}(x,y,t)\right)^{*l}\right| \leqslant \tfrac{C(k,n)}{r^{2k+2}}\left(C(k,n)\tfrac{t^k}{r^{2k+2}}\right)^l \times \tfrac{t^{k+l}}{(k+1)\dots(k+l)}\varrho(x,y,2t). \tag{48}$$

In particular, Inequality (48) and the Gaussian upper bound of the heat kernel imply that, for any $x, y \in \mathrm{M}^n$ with $d(x,y) > r/2$ and some appropriate $C(n,k)$, it holds

$$\left|\bar{\Box}_x\left(\varrho_{k,r}(x,y,t)\right)^{*l}\right| \leqslant \tfrac{C(n,k)}{r^{2k+2}}\left(C(n,k)\tfrac{t^k}{r^{2k+2}}\right)^l \times \tfrac{t^{k+l}}{(k+1)\dots(k+l)}(4\pi r)^{-n/2}\mathrm{e}^{-\frac{r^2}{32t}}. \tag{49}$$

We now prove inductively, that for any $x, y \in \mathrm{M}^n$ with $d(x,y) \leqslant r$ the estimate

$$\left|\left(\bar{\Box}_x \varrho_{k,r}\right)^{*l}(x,y,t)\right| \leqslant \tfrac{C(n,k)}{r^{2k+2}}\left(\tfrac{C(n,k)}{r^{2k+2}}t^k\right)^l \tfrac{t^{k+l}}{(k+1)\dots(k+l)} \times \left(l\tfrac{\mathrm{e}^{-\frac{r^2}{32t}}}{(4\pi r^2)^{n/2}} + \varrho(x,y,t)\right) \tag{50}$$

holds. For $l=0$ this estimate follows from Inequality (47). Now assume the estimate is true for $l \in \mathbb{N}$. By the definition of the convolution, we have

$$\begin{aligned}
&\left(\bar{\Box}_x \varrho_{k,r}\right)^{*l+1}(x,y,t) \\
&= \int_0^t \int_{\mathrm{M}^n} \bar{\Box}_x \left(\varrho_{k,r}(x,z,s)\right) \left(\bar{\Box}_x \varrho_{k,r}\right)^{*l}(z,y,t-s)\omega(z)\mathrm{d}s \\
&= \int_0^t \int_{\mathrm{M}^n \setminus B_{r/2}(y)} \bar{\Box}_x \left(\varrho_{k,r}(x,z,s)\right) \left(\bar{\Box}_x \varrho_{k,r}\right)^{*l}(z,y,t-s)\omega(z)\mathrm{d}s \\
&\quad + \int_0^t \int_{B_{r/2}(y)} \bar{\Box}_x \left(\varrho_{k,r}(x,z,s)\right) \left(\bar{\Box}_x \varrho_{k,r}\right)^{*l}(z,y,t-s)\omega(z)\mathrm{d}s. \qquad (51)
\end{aligned}$$

The first integral can be estimated using the Inequality (47) and Inequality (49). It holds

$$\begin{aligned}
&\int_0^t \int_{\mathrm{M}^n \setminus B_{r/2}(y)} \bar{\Box}_x \left(\varrho_{k,r}(x,z,s)\right) \left(\bar{\Box}_x \varrho_{k,r}\right)^{*l}(z,y,t-s)\omega(z)\mathrm{d}s \\
&\leqslant \int_0^t \int_{\mathrm{M}^n \setminus B_{r/2}(y)} \Bigg(C(n,k) \tfrac{s^k}{r^{2k+2}} \varrho(x,z,2s) \\
&\quad \times \tfrac{C(n,k)}{r^{2k+2}} \left(C(n,k) \tfrac{(t-s)^k}{r^{2k+2}} \right)^l \tfrac{(t-s)^{k+l}}{(k+1)\dots(k+l)} (4\pi r)^{-n/2} \mathrm{e}^{-\frac{r^2}{32t}} \Bigg) \omega(z)\mathrm{d}s.
\end{aligned}$$

Due to the stochastic completeness of M^n;

$$\int_{\mathrm{M}^n \setminus B_{r/2}(y)} \varrho(x,z,2s)\omega(z) \leqslant 1,$$

we conclude

$$\begin{aligned}
&\int_0^t \int_{\mathrm{M}^n \setminus B_{r/2}(y)} \bar{\Box}_x \left(\varrho_{k,r}(x,z,s)\right) \left(\bar{\Box}_x \varrho_{k,r}\right)^{*l}(z,y,t-s)\omega(z)\mathrm{d}s \\
&\leqslant \tfrac{C(n,k)}{r^{2k+2}} \left(C(n,k) \tfrac{t^k}{r^{2k+2}} \right)^{l+1} \tfrac{t^{k+l+1}}{(k+1)\dots(k+l)(k+l+1)} (4\pi r)^{-n/2} \mathrm{e}^{-\frac{r^2}{32t}}. \qquad (52)
\end{aligned}$$

To estimate the second integral on the right-hand side of Equality (51), we note that since $d(z,x) \leqslant r$ for any $x, y \in B_{r/2}(y)$ we can apply Inequality (46).

Furthermore, the induction's hypothesis is satisfied on $B_{r/2}(y)$. Thus,

$$\begin{aligned}
&\int_0^t \int_{B_{r/2}(y)} \bar{\Box}_x \left(\varrho_{k,r}(x,z,s)\right) \left(\bar{\Box}_x \varrho_{k,r}\right)^{*l} (z,y,t-s)\omega(z)\mathrm{d}s \\
&\leqslant \int_0^t \int_{B_{r/2}(y)} \Big(C(n,k) \tfrac{s^k}{r^{2k+2}} \varrho(x,z,s) \\
&\qquad \times \tfrac{C(n,k)}{r^{2k+2}} \left(\tfrac{C(n,k)}{r^{2k+2}} (t-s)^k \right)^l \tfrac{t^{k+l}}{(k+1)\dots(k+l)} \varrho(z,y,t-s) \Big) \omega(z)\mathrm{d}s \\
&\quad + \int_0^t \int_{B_{r/2}(y)} \Big(C(n,k) \tfrac{t^k}{r^{2k+2}} \varrho(x,z,s) \\
&\qquad \times \tfrac{C(n,k)}{r^{2k+2}} \left(\tfrac{C(n,k)}{r^{2k+2}} (t-s)^k \right)^l \tfrac{t^{k+l}}{(k+1)\dots(k+l)} l (4\pi r)^{-n/2} \mathrm{e}^{-\frac{r^2}{32t}} \Big) \omega(z)\mathrm{d}s.
\end{aligned}$$

By the semigroup property, respectively the stochastic completeness of the heat kernel we can estimate the right-hand side from above by

$$\begin{aligned}
&\tfrac{C(n,k)}{r^{2k+2}} \left(\tfrac{C(n,k)}{r^{2k+2}} t^k \right)^{l+1} \tfrac{t^{k+l+1}}{(k+1)\dots(k+l)(k+l+1)} \varrho(x,y,t) \\
&+ \tfrac{C(n,k)}{r^{2k+2}} \left(\tfrac{C(n,k)}{r^{2k+2}} t^k \right)^{l+1} \tfrac{t^{k+l+1}}{(k+1)\dots(k+l)(k+l+1)} l \tfrac{\mathrm{e}^{-\frac{r^2}{32t}}}{(4\pi r^2)^{n/2}}.
\end{aligned}$$

Together with Inequality (52), this completes the induction. To conclude the lemma's claim, we note that by the Gaussian lower bound of the heat kernel, the above estimate implies that for $d(x,y) \leqslant r/\sqrt{8}$

$$(4\pi r)^{-n/2} \mathrm{e}^{-\frac{r^2}{32t}} \leqslant (4\pi t)^{-n/2} \mathrm{e}^{-\frac{d(x,y)^2}{4t}} \leqslant C(n,k)\varrho(x,y,t). \tag{53}$$

□

We will use the following lemmata regarding the regularity of convolutions to prove Theorem 2.7. The first one, Lemma 2.16, is proved by Grieser in [Gri04, Proposition 2.6 and Lemma 2.8 (a)]. A proof of the second lemma can be found in [Cho+10, Lemmata 23.25-23.29].

Lemma 2.16 *Let* $(\mathrm{M}^n, \bar{\mathrm{g}})$ *be a compact Riemannian manifold and* $A, B \in \Psi_H^{-1}(\mathrm{M}^n)$. *Here,* $\Psi_H^{-1}(\mathrm{M}^n)$ *is the set of functions on* A *on* $(0,\infty) \times \mathrm{M}^n \times \mathrm{M}^n$ *satisfying*

(*i*) *A is smooth,*

(*ii*) *if $x \neq y$ $\bar{\nabla}^{\gamma}_{t,x,y} A(t,x,y) = O(t^{\infty})$ as $t \to 0$, for all γ,*

(*iii*) *for any $p \in \mathrm{M}^n$ there is a local coordinate system $U \ni p$ and $\tilde{A} \in C^{\infty}\left([0,\infty)_{1/2} \times \mathbb{R}^n \times U\right)$ so that for $x, y \in U$*

$$A(t,x,y) = t^{-n/2} \tilde{A}(t, {x-y}/{\sqrt{t}}, y)$$

and

$$\left|\bar{\nabla}^{\gamma}_{\sqrt{t},X,y}\tilde{A}(t,X,y)\right| = O(\,|X|^{-\infty}), \quad X \to \infty.$$

The space $C^\infty\left([0,\infty)_{1/2}\right)$ *is the space of smooth functions in* $\sqrt{t}$.

Then $A * B \in \Psi^{-1}_H(\mathrm{M}^n)$. *Moreover,*

$$A(t,x) := \int_{\mathrm{M}^n} A(t,x,y)\omega(x) \in C^\infty\left([0,\infty)_{1/2} \times \mathrm{M}^n\right).$$

Lemma 2.17 *Let* $(\mathrm{M}^n, \bar{\mathrm{g}})$ *be a closed Riemannian manifold,* $k \geqslant 0$, $\varrho_{k,r} : \mathrm{M}^n \times \mathrm{M}^n \times (0,\infty)$ *be a approximate heat kernel and* $G \in C^0(\mathrm{M}^n \times \mathrm{M}^n \times [0,\infty))$. *Then, for any* $\alpha \in (1/2, 1)$, *there is some* $C_{54}(\mathrm{M}^n,\alpha)$ *and such that*

$$\int_{\mathrm{M}^n} \tfrac{\partial \varrho_{k,r}}{\partial x_i}(x,z,t-s)\omega(z)\mathrm{d}s \leqslant C_{54}(\mathrm{M}^n,\alpha)\,(t-s)^{-\alpha}. \tag{54}$$

Furthermore, $\varrho_{k,r} * G \in C^2(\mathrm{M}^n \times \mathrm{M}^n \times (0,\infty)) \cap C^1(\mathrm{M}^n \times [0,\infty))$ *and, in a local coordinate system* $(U, \{x^\alpha\}_{i=1}^n)$ *with* $x \in U$, *the space and time derivatives of* $\varrho_{k,r} * G$ *are given by*

$$\tfrac{\partial}{\partial x_i}(\varrho_{k,r} * G)(x,y,t) = \int_0^t \int_{\mathrm{M}^n} \tfrac{\partial \varrho_{k,r}}{\partial x_i}(x,z,t-s)G(z,y,s)\omega(z)\mathrm{d}s, \tag{55}$$

$$\tfrac{\partial^2}{\partial x_i \partial x_j}(\varrho_{k,r} * G)(x,y,t) = \int_0^t \int_{\mathrm{M}^n} \tfrac{\partial^2 \varrho_{k,r}}{\partial x_i \partial x_j}(x,z,t-s)G(z,y,s)\omega(z)\mathrm{d}s, \tag{56}$$

$$\tfrac{\partial}{\partial t}(\varrho_{k,r} * G)(x,y,t) = G(x,y,t) + \int_0^t \int_{\mathrm{M}^n} \tfrac{\partial}{\partial t}\varrho_{k,r}(x,z,t-s)G(z,y,s)\omega(z)\mathrm{d}s. \tag{57}$$

Moreover, for $f \in C^0(\mathrm{M}^n \times [0,T])$ *we have*

$$\tfrac{\partial^l}{\partial t^l}\tfrac{\partial^m}{\partial x^m}\int_{\mathrm{M}^n} \varrho_{k,r}(x,z,t-s)f(z,s)\omega(z) = \int_{\mathrm{M}^n} \tfrac{\partial^l}{\partial t^l}\tfrac{\partial^m}{\partial x^m}\varrho_{k,r}(x,z,t-s)f(z,s)\omega(z). \tag{58}$$

on $\mathrm{M}^n \times \{(s,t) \in \mathbb{R}^2 : 0 < t-s \leqslant T\}$.

Lemma 2.18 *Fix* $k, n \in \mathbb{N}$. *Then there is a constant* $C(k,n) < \infty$ *such that the following is true.*

Under the assumptions of Theorem 2.7, for all $l \in \mathbb{N}$ *and* $T < \infty$, *the functions*

$$G^{k,l}_\Omega(y,t) := \int_\Omega \bar{\Box}_x \left(\varrho_{k,r}(x,y,t)\right)^{*l}\omega(x) \tag{59}$$

are continuous on $\mathrm{M}^n \times [0,T]$ *with* $\lim_{t\to 0} \left|G_\Omega^{k,l}(\cdot,t)\right| = 0$ *uniformly. Furthermore, for any* $0 < t \leqslant T$*, they satisfy the estimate*

$$\left|G_\Omega^{k,l}(y,t)\right| \leqslant \tfrac{C_{35}(T,k,n)}{r^{2k+2}} \left(C_{35}(T,k,n)\tfrac{t^k}{r^{2k+2}}\right)^l \tfrac{t^{k+l}}{(k+1)\dots(k+l)}, \tag{60}$$

where

$$C_{35}(T,k,n) := C(n,k)\left(1 + \tfrac{T^k}{r^{2k}}\right) \mathrm{e}^{\frac{1}{2}nTr^{-2}}\ . \tag{61}$$

Proof. For the sake of clarity, we denote the functions $G_\Omega^{k,l}(y,t)$ by $G_\Omega^l(y,t)$. We choose $\eta = 2$ in the definition of $C_{35}(T,\eta,k,n)$ and denote it by $C_{35}(T)$. Since $\int_\Omega \varrho(x,y,t)\omega(x) \leqslant 1$ independent of $y \in \mathrm{M}^n$ and $t > 0$, Lemma 2.14 implies

$$\begin{aligned}\left|G_\Omega^l(y,t)\right| &\leqslant \tfrac{C_{35}(T)}{r^{2k+2}} \left(C_{35}(T)\tfrac{t^k}{r^{2k+2}}\right)^l \tfrac{t^{k+l}}{(k+1)\dots(k+l)} \int_\Omega \varrho(x,y,2t)\omega(x)\\ &\leqslant \tfrac{C_{35}(T)}{r^{2k+2}} \left(C_{35}(T)\tfrac{t^k}{r^{2k+2}}\right)^l \tfrac{t^{k+l}}{(k+1)\dots(k+l)}.\end{aligned}$$

The right-hand side is uniformly bounded on the interval $[0,T]$ with

$$\lim_{t\to 0} \sup_{y\in\mathrm{M}^n} \left|G_\Omega^l(y,t)\right| = 0.$$

Hence, we may extend $G_\Omega^l(y,\cdot)$ by 0 in $t = 0$. Moreover, by construction of the approximate heat kernels, we have

$$\begin{aligned}\left(\bar{\Box}_x \varrho_{k,r}\right)(x,y,t) &= \alpha_r(x,y)t^k (4\pi t)^{-n/2} \mathrm{e}^{-\frac{d(x,y)^2}{4t}} \bar{\Delta}\varrho_k(x,y)\\ &\quad + \bar{\Delta}\alpha_r(x,y)\varrho_k(x,y,t) + \langle\bar{\nabla}\alpha_r, \bar{\nabla}\varrho_k\rangle(x,y,t).\end{aligned}$$

Since the support of α_r is contained in a neighbourhood of the diagonal of $(\mathrm{M}^n \times \mathrm{M}^n)$, one may find for any $\eta > 1$ a function $q_{k,\eta} \in C(\mathrm{M}^n \times \mathrm{M}^n \times [0,T])$ such that

$$\left(\bar{\Box}_x \varrho_{k,r}\right)(x,y,t) = t^{-n/2+k} q_{k,\eta}(x,y,t)\mathrm{e}^{-\frac{d(x,y)^2}{4\eta t}}$$

Therefore, $\bar{\Box}_x\varrho_{k,r}$ satisfies the assumptions of Lemma 2.16. Hence, we inductively conclude for all $l \in \mathbb{N}$

$$\int_\Omega \bar{\Box}_x \left(\varrho_{k,r}(x,y,t)\right)^{*l} \omega(x) \in C(\mathrm{M}^n \times [0,T]). \qquad \Box$$

We are now ready to write the solution of (HE) in terms of approximate heat kernels.

Proposition 2.19 *Fix $k, n \in \mathbb{N}$ and $\eta > 1$. Then there is a constant $C(n,k,\eta)$ such that the following is true.*

Under the assumptions of Theorem 2.7 the unique solution of the heat equation (HE) on $\mathrm{M}^n \times [0,T]$, $T < \infty$ is given by

$$u_\Omega(y,t) = \int_\Omega \varrho_{k,r}(x,y,t) + \sum_{l=0}^{\infty} \int_0^t \int_{\mathrm{M}^n} \int_\Omega \bar{\Box}_x (\varrho_{k,r}(z,x,s))^{*l} \varrho_{k,r}(y,z,t-s)\omega(x)\omega(z)\mathrm{d}s.$$

Moreover, we have for $0 < t \leqslant T$

$$\left| \int_\Omega \varrho(x,y,t) - \varrho_{k,r}(x,y,t)\omega(x) \right| \leqslant C_{62}(T,n,k,\eta) \tfrac{t^{k+1}}{r^{2k+2}} \int_\Omega (4\pi t)^{-n/2} \mathrm{e}^{-\frac{d(x,y)^2}{4\eta t}} \omega(x),$$

where

$$C_{62}(T,n,k,\eta) := C(n,k,\eta) \left(1 + \tfrac{T^k}{r^{2k}}\right)^2 \exp\left(\tfrac{n\eta T}{r^2 2}\right) \exp\left(\left(1 + \tfrac{T^k}{r^{2k}}\right) \exp\left(\tfrac{n}{4}\tfrac{\eta T}{r^2}\right) \tfrac{T^{k+1}}{r^{2k+2}}\right). \tag{62}$$

Proof. By Lemma 2.18 the functions

$$G_\Omega^{k,l}(y,t) := \int_\Omega \bar{\Box}_x \left(\varrho_{k,r}(x,y,t)\right)^{*l} \omega(x), \quad l \in \mathbb{N}$$

satisfy $G^l_\Omega(y,t) \in C^0(\mathrm{M}^n \times [0,\infty])$ for all $l \in \mathbb{N}$. We set

$$\begin{aligned} F_\Omega^{k,l}(y,t) &:= \int_0^t \int_{\mathrm{M}^n} \int_\Omega \bar{\Box}_x (\varrho_{k,r}(z,x,s))^{*l} \varrho_{k,r}(y,z,t-s)\omega(x)\omega(z)\mathrm{d}s \\ &= \int_0^t \int_{\mathrm{M}^n} G^l_\Omega(z,s)\varrho_{k,r}(y,z,t-s)\omega(z)\mathrm{d}s. \end{aligned}$$

Although the functions $G_\Omega^{k,l}(y,t)$ and $F_\Omega^{k,l}(y,t)$ depend on k, we will, for the sake of clarity, denote them in the following by $G^l_\Omega(y,t)$, respectively $F^l_\Omega(y,t)$. The estimation of the approximate heat kernel (36), Lemma 2.12, and the stochastic completeness of M^n imply

$$\begin{aligned} \left|F^l_\Omega(y,t)\right| &\leqslant \int_0^t \tfrac{C_{35}(T)}{r^{2k+2}} \left(C_{35}(T)\tfrac{s^k}{r^{2k+2}}\right)^l \\ &\quad \times \tfrac{s^{k+l}}{(k+1)\dots(k+l)} \left(\int_{\mathrm{M}^n} \varrho(y,z,\eta\,(t-s))\omega(z)\right)\mathrm{d}s \\ &\leqslant \tfrac{C_{35}(T)}{r^{2k+2}} \left(C_{35}(T)\tfrac{t^k}{r^{2k+2}}\right)^l \tfrac{t^{k+l+1}}{(k+1)\dots(k+l)(k+l+1)}. \end{aligned}$$

Therefore, the sequence $\left(\sum_{l=0}^{N} F_\Omega^l(y,t)\right)_{N\in\mathbb{N}}$ converges absolutely and uniformly on $\mathrm{M}^n \times [0,T]$ for any $T>0$ with

$$\left|\sum_{l=0}^{\infty} F_\Omega^l(y,t)\right| \leqslant C_{35}(T)\tfrac{t^{k+1}}{r^{2k+2}}\mathrm{e}^{C_{35}(T)\tfrac{t^{k+1}}{r^{2k+2}}}.$$

Here, the right-hand side tends uniformly in $y\in\mathrm{M}^n$ to 0 as $t\to 0$. To show that the convergence is in $C^2(\mathrm{M}^n)\cap C^1([0,\infty)$, we note that by formula 54, Lemma 2.17,

$$\begin{aligned}&\left|\int_0^t\int_{\mathrm{M}^n}\tfrac{\partial^2\varrho_{k,r}}{\partial x_i\partial x_j}(y,z,t-s)\omega(z)\mathrm{d}s\right|,\ \left|\int_0^t\int_{\mathrm{M}^n}\tfrac{\partial}{\partial t}\varrho_{k,r}(y,z,t-s)\omega(z)\mathrm{d}s\right|\\ &\leqslant \int_0^t C_{54}(\mathrm{M}^n,\varepsilon)\,(t-s)^{-\varepsilon}<\infty,\end{aligned}$$

where $\varepsilon < 1/2$. In particular,

$$\begin{aligned}&\int_0^t\int_{\mathrm{M}^n}\tfrac{\partial^2\varrho_{k,r}}{\partial x_i\partial x_j}(y,z,t-s)\int_\Omega\bar{\Box}_x(\varrho_{k,r}(z,x,s))^{*l}\omega(x)\omega(z)\mathrm{d}s\\ &\leqslant C_{54}(\mathrm{M}^n,\varepsilon)\tfrac{C_{35}(T)}{r^{2k+2}}\left(C_{35}(T)\tfrac{t^k}{r^{2k+2}}\right)^l\tfrac{t^{k+l+1}}{(k+1)\ldots(k+l)(k+l+1)}\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathrm{M}^n}\int_\Omega\bar{\Box}_x(\varrho_{k,r}(z,x,s))^{*l}\tfrac{\partial}{\partial t}\varrho_{k,r}(y,z,t-s)\omega(x)\omega(z)\mathrm{d}s\\ &\leqslant C_{54}(\mathrm{M}^n,\varepsilon)\tfrac{C_{35}(T)}{r^{2k+2}}\left(C_{35}(T)\tfrac{t^k}{r^{2k+2}}\right)^l\tfrac{t^{k+l+1}}{(k+1)\ldots(k+l)(k+l+1)}.\end{aligned}$$

Hence, the time and spatial derivatives of the $F_\Omega^l(x,y)$ can be computed by

$$\begin{aligned}\tfrac{\partial}{\partial t}(F_\Omega^l(y,t)) &= \int_0^t\int_{\mathrm{M}^n}\int_\Omega\bar{\Box}_x(\varrho_{k,r}(z,x,s))^{*l}\tfrac{\partial}{\partial t}\varrho_{k,r}(y,z,t-s)\omega(x)\omega(z)\mathrm{d}s\\ &\quad-\int_\Omega\bar{\Box}_x(\varrho_{k,r}(x,y,t))^{*l}\omega(x),\end{aligned}$$

respectively

$$\tfrac{\partial^2}{\partial x_i\partial x_j}F_\Omega^l(y,t)=\int_0^t\int_{\mathrm{M}^n}\tfrac{\partial^2\varrho_{k,r}}{\partial x_i\partial x_j}(y,z,t-s)\int_\Omega\bar{\Box}_x(\varrho_{k,r}(z,x,s))^{*l}\omega(x)\omega(z)\mathrm{d}s.$$

In particular, the sequence $\left(\sum_{l=0}^{N} F_\Omega^l(y,t)\right)_{N\in\mathbb{N}}$ converges in $C^2(\mathrm{M}^n)\cap C^1[0,\infty)$ to some function F on any compact subset of $\mathrm{M}^n\times[0,\infty)$ and the differentiation of the series can be carried out term-by-term. Moreover,

$$\begin{aligned}\bar{\Box}F_\Omega^l(y,t) &= \int_0^t\int_{\mathrm{M}^n}\int_\Omega\bar{\Box}_x(\varrho_{k,r}(z,x,s))^{*l}\bar{\Box}_x\varrho_{k,r}(y,z,t-s)\omega(x)\omega(z)\mathrm{d}s\\ &\quad-\int_\Omega\bar{\Box}_x(\varrho(x,y,t))^{*l}\omega(x)\\ &= \int_\Omega\bar{\Box}_x\left(\varrho_{k,r}(x,y,t)\right)^{*l+1}\omega(x)-\int_\Omega\bar{\Box}_x\left(\varrho_{k,r}(x,y,t)\right)^{*l}\\ &= G_\Omega^{l+1}(y,t)-G_\Omega^l(y,t).\end{aligned}$$

Thus, $\sum_{l=0}^{\infty} F^l_\Omega(y,t)$ is an absolutely convergent telescoping series and we conclude that

$$u_\Omega(y,t) := \int_\Omega \varrho_{k,r}(x,y,t) + \sum_{l=0}^{\infty} F^l_\Omega(y,t)$$

solves the heat equation with initial data $u_\Omega(y,0) = \chi_\Omega(y)$. Therefore,

$$\left|\int_\Omega \varrho(x,y,t) - \varrho_{k,r}(x,y,t)\omega(x)\right| \leqslant \sum_{l=1}^{\infty} \left|F^l_\Omega(y,t)\right|.$$

By Lemma 2.14 and the semigroup property of the heat kernel we may estimate the right-hand side by

$$\begin{aligned}
&\sum_{l=1}^{\infty} \left|F^l_\Omega(y,t)\right| \\
&\leqslant \sum_{l=1}^{\infty} \Bigg(\int_0^t \int_{\mathrm{M}^n} \int_\Omega \Bigg(\tfrac{C_{35}(T)}{r^{2k+2}} \left(C_{35}(T) \tfrac{s^k}{r^{2k+2}}\right)^l \tfrac{s^{k+l}}{(k+1)\ldots(k+l)} \varrho(z,x,\eta s) \\
&\quad \times C_{35}(T)\varrho(y,z,\eta(t-s)) \Bigg) \omega(x)\omega(z)\mathrm{d}s \Bigg) \\
&\leqslant C_{35}(T)^2 \tfrac{t^k}{r^{2k+2}} \left(\int_\Omega \varrho(x,y,\eta t)\omega(x)\right) \\
&\quad \times \sum_{l=1}^{\infty} \left(C_{35}(T)\tfrac{T^k}{r^{2k+2}}\right)^l \times \left(\int_0^t \tfrac{s^l}{(k+1)\ldots(k+l)} \mathrm{d}s\right) \\
&= C_{35}(T)^2 \tfrac{t^{k+1}}{r^{2k+2}} \left(\int_\Omega \varrho(x,y,\eta t)\omega(x)\right) \\
&\quad \times \sum_{l=1}^{\infty} \left(C_{35}(T)\tfrac{T^k}{r^{2k+2}}\right)^l \tfrac{t^l}{(k+1)\ldots(k+l)(k+l+1)} \\
&\leqslant C_{35}(T)^2 \exp\left(C_{35}(T)\tfrac{T^{k+1}}{r^{2k+2}}\right) \tfrac{t^{k+1}}{r^{2k+2}} \left(\int_\Omega \varrho(x,y,\eta t)\omega(x)\right).
\end{aligned}$$

Finally, we observe that the Gaussian upper bound of the heat kernel implies

$$\int_\Omega \varrho(x,y,\eta t)\omega(x) \leqslant 2C_{14}(\eta,n)\mathrm{e}^{C(n,\eta)\frac{T}{r}} \int_\Omega (4\pi t)^{-n/2}\, \mathrm{e}^{-\frac{d(x,y)^2}{4\eta^2 t}}\, \omega(x)$$

Thus,

$$\left|\int_\Omega \left(\varrho(x,y,t) - \varrho_{k,r}(x,y,t)\right)\omega(x)\right| \leqslant C_{63}(T) \int_\Omega (4\pi t)^{-n/2}\, \mathrm{e}^{-\frac{d(x,y)^2}{4\eta^2 t}}\, \omega(x)$$

where

$$C_{63}(T) := 2C_{35}(T)^2 C_{14}(\eta,n)\mathrm{e}^{C_{35}(T)\frac{T^{k+1}}{r^{2k+2}} + C(n,\eta)\frac{T}{r}}. \tag{63}$$

□

Now we are ready to prove Theorem 2.7.

Proof of Theorem 2.7. For $y \in \mathrm{M}^n$ fixed we define $\Omega_1 := \Omega \cap B_{r/2\sqrt{8}}(y)$. By the previous lemma, the solution of the heat-equation (HE) with initial data $u_{\Omega_1}(\cdot, 0) = \chi_{\Omega_1}(\cdot)$ is given by

$$u_{\Omega_1}(y,t) = \int_{\Omega_1} \varrho_{k,r}(x,y,t) + \sum_{l=0}^{\infty} F_{\Omega_1}^{k,l}(y,t),$$

where

$$F_{\Omega_1}^{k,l}(y,t) = \int_0^t \int_{\mathrm{M}^n} \int_{\Omega_1} \bar{\Box}_x (\varrho_{k,r}(z,x,s))^{*l} \varrho_{k,r}(y,z,t-s)\omega(x)\omega(z)\mathrm{d}s.$$

The functions $F_{\Omega_1}^{k,l}$ will be denoted by $F_{\Omega_1}^{l}$ in the following. Since by Lemma 2.11, for $d(x,z) \leqslant r/\sqrt{8}$,

$$\left| \left(\bar{\Box}_x \varrho_{k,r} \right)^{*l} (x,z,s) \right| \leqslant \tfrac{C(n,k)}{r^{2k+2}} \left(\tfrac{C(n,k)}{r^{2k+2}} s^k \right)^l \tfrac{(l+1)s^{k+l}}{(k+1)\ldots(k+l)} \varrho(x,z,s), \tag{64}$$

we estimate

$$\begin{aligned}
&\left| F_{\Omega_1}^{l}(y,t) \right| \\
&\leqslant \int_0^t \int_{\mathrm{M}^n} \int_{\Omega_1} \Big(\tfrac{C(n,k)}{r^{2k+2}} \left(\tfrac{C(n,k)}{r^{2k+2}} s^k \right)^l \tfrac{(l+1)s^{k+l}}{(k+1)\ldots(k+l)} \varrho(x,z,s) \\
&\qquad \times C(n,k) \varrho(y,z,t-s) \Big) \omega(x)\omega(z)\mathrm{d}s \\
&= C(n,k) \int_0^t \int_{\Omega_1} \Big(\tfrac{C(n,k)}{r^{2k+2}} \left(\tfrac{C(n,k)}{r^{2k+2}} s^k \right)^l \tfrac{(l+1)s^{k+l}}{(k+1)\ldots(k+l)} \varrho(x,y,t) \Big) \omega(x)\mathrm{d}s \\
&\leqslant C(n,k) \tfrac{C(n,k)}{r^{2k+2}} \left(\tfrac{C(n,k)}{r^{2k+2}} t^k \right)^l \tfrac{(l+1)t^{k+l+1}}{(k+1)\ldots(k+l)(k+l+1)} \\
&\qquad \times \int_{\Omega_1} \varrho(x,y,t)\omega(x) \\
&\leqslant C(n,k)^2 \tfrac{t^{k+1}}{r^{2k+2}} \tfrac{C(n,k)^l}{(k+1)\ldots(k+l)} \int_{\Omega_1} \varrho(x,y,t)\omega(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left| u_{\Omega_1}(y,t) - \int_{\Omega_1} \varrho_{k,r}(x,y,t) \right| &= \left| \sum_{l=0}^{\infty} F_{\Omega_1}^{l}(y,t) \right| \\
&\leqslant C(n,k)^2 \exp\left(C(n,k)\right) \tfrac{t^{k+1}}{r^{2k+2}} \int_{\Omega_1} \varrho(x,y,t)\omega(x).
\end{aligned} \tag{65}$$

Finally, we observe

$$\left|\int_{\Omega} \varrho(x,y,t)\omega(x) - \int_{\Omega_1} \varrho_{k,r}(x,y,t)\omega(x)\right| \tag{66}$$
$$\leqslant \int_{\Omega\setminus\Omega_1} \varrho(x,y,t)\omega(x) + \left|\int_{\Omega_1} \varrho(x,y,t)\omega(x) - \int_{\Omega_1} \varrho_{k,r}(x,y,t)\omega(x)\right|.$$

On the other hand, since by assumption $|d_\Omega(y)| \leqslant {}^{r}/_{2\sqrt{8}}$, we can choose η in the upper Gaussian bound of the heat kernel, Theorem 1.5, appropriately such that

$$\int_{\Omega\setminus\Omega_1} \varrho(x,y,t)\omega(x) \leqslant C(n,k)t^k \tfrac{|\Omega|}{r^{n+2k}} \mathrm{e}^{-\frac{r^2}{2\sqrt{8}\cdot 4t}} \leqslant C(n,k)t^k \tfrac{|\Omega|}{r^{n+2k}} \mathrm{e}^{-\frac{d_\Omega(y)^2}{4t}}. \tag{67}$$

We complete the proof by taking the maximum of the constants in (65), (67) and observing that Inequality (66) implies

$$\left(1 - C(n,k)\tfrac{t^{k+1}}{r^{2k+2}}\right) \int_{\Omega} \varrho(x,y,t)\omega(x)$$
$$\leqslant C(n,k) \left(\int_{\Omega_1} (4\pi t)^{-n/2}\, \mathrm{e}^{-\frac{d^2_{(x,y)}}{4t}} \omega(x) + \tfrac{|\Omega|}{r^n} \mathrm{e}^{-\frac{d_\Omega(y)^2}{4t}}\right). \qquad \square$$

2.2 Computations of Gaussian type integrals

In this section we will estimate integrals of Gaussian type, i.e. integrals of the form

$$\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{y+\varphi(x)} \mathrm{d}\gamma_n((x,z)) A\left((x,z),(x,z)\right),$$

where $A = \{A_{\alpha\beta}\}_{1\leqslant\alpha,\beta\leqslant n}$ is some bilinear form, $\mathrm{d}\gamma_n(z)$ is the Gaussian measure,

$$\mathrm{d}\gamma_n(z) := (4\pi)^{-n/2}\, \mathrm{e}^{-\frac{|z|^2}{4}} \mathrm{d}z,$$

and $\varphi \in C^4\left(\mathbb{R}^{n-1}\right)$ is a perturbation of the $n-th$ coordinate. The estimates will be used in the subsequent chapter to compute the short-time asymptotics of heat equations.

In the one-dimensional case, the above integrals can be written in terms of the Gaussian error function $\Phi : \mathbb{R} \to (0,1)$.

$$\Phi(x) := \int_{-\infty}^{x} \tfrac{1}{\sqrt{4\pi}} \mathrm{e}^{-z^2/4} \mathrm{d}z.$$

We will need the following lemma regarding the growth of the Gaussian error function.

Lemma 2.20 (Properties of the Gaussian error function) *For $x \leqslant 0$, the Gaussian error function $\Phi : \mathbb{R} \to (0,1)$ can be bounded from above and below by*

$$\frac{\mathrm{e}^{-x^2/4}}{\sqrt{\pi}\left(\frac{|x|}{2}+\sqrt{2+\frac{x^2}{4}}\right)} \leqslant \Phi(x) \leqslant \frac{\mathrm{e}^{-x^2/4}}{\sqrt{\pi}\left(\frac{|x|}{2}+\sqrt{\frac{4}{\pi}+\frac{x^2}{4}}\right)} \leqslant \frac{\mathrm{e}^{-x^2/4}}{2}. \tag{68}$$

Hence, the Gaussian error function Φ tends to 0 as $x \to -\infty$ at an exponential rate.

Proof. See e.g. [AS64, 7.1.13]. □

Before we can compute the perturbations of Gaussian integrals, we will need the following integral formulae. They are derived as in [[GR07], Chapter 8.5] and integration by parts.

Lemma 2.21 *For $x \in \mathbb{R}$ and $i = 1, 2, 3, 4$ we have*

$$\int_{-\infty}^{x} z^i \mathrm{d}\gamma_1(z) = \begin{cases} -2\frac{\mathrm{e}^{-x^2/4}}{\sqrt{4\pi}} & \text{for } i = 1, \\ -2x\frac{\mathrm{e}^{-x^2/4}}{\sqrt{4\pi}} + 2\Phi(x) & \text{for } i = 2, \\ -2(x^2+4)\frac{\mathrm{e}^{-x^2/4}}{\sqrt{4\pi}} & \text{for } i = 3, \\ -2x(x^2+6)\frac{\mathrm{e}^{-x^2/4}}{\sqrt{4\pi}} + 12\Phi(x) & \text{for } i = 4, \\ -2(x^4+8x^2+32)\frac{\mathrm{e}^{-x^2/4}}{\sqrt{4\pi}} & \text{for } i = 5. \end{cases}$$

To compute Gaussian integrals of any dimension n, we apply the following lemma.

Lemma 2.22 (Symmetry of Gaussian Integrals) *For $p_1, \ldots, p_n \in \mathbb{N}$ we have*

$$\int_{\mathbb{R}^n} x_1^{p_1} \cdot \ldots \cdot x_n^{p_n} \mathrm{d}\gamma_n(x) = \begin{cases} \prod_{i=1}^n \frac{p_i!}{(\frac{p_i}{2})!}, & p_1, \ldots, p_n \text{ all even,} \\ 0, & \text{else.} \end{cases}$$

Proof. See e.g. [Pre06, Corollary 22]. □

We now estimate how the above lemma behaves under a perturbation of some function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$.

Lemma 2.23 *For any $n \in \mathbb{N}$ there is a constant $C(n) < \infty$ such that the following is true.*

Let $\varphi \in C^4(\mathbb{R}^{n-1})$ be a function satisfying $\varphi(0) = 0$ and $|D\varphi(0)| = 0$. Furthermore, assume

$$C := \sup\left(\left|D^2\varphi\right|, \sqrt{|D^3\varphi|}, \left(\left|D^3\varphi\right|\right)^{1/3}\right)$$

is bounded in a neighbourhood of 0*. Then, for $y \in \mathbb{R}$ and $t > 0$ with $\sqrt{t}$, $|y| \leqslant 1/4C$ the perturbation of Gaussian Integrals by φ can be estimated by*

$$\Bigg|\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{y+\varphi(x)} \frac{\mathrm{e}^{-\frac{|x|^2+z^2}{4t}}}{(4\pi t)^{n/2}}\mathrm{d}z\mathrm{d}x$$
$$-\left(\Phi\left(\tfrac{y}{\sqrt{t}}\right) + \frac{\mathrm{e}^{-\frac{y^2}{4t}}}{\sqrt{4\pi}}\sqrt{t}\left(\Delta\varphi(0) - y\left(\tfrac{1}{2}\left|D^2\varphi(0)\right|^2 + \tfrac{1}{4}(\Delta\varphi)^2\right)\right)\right)\Bigg|$$
$$\leqslant C(n)C^3t^{3/2}\left(1 + \tfrac{|y|^2}{t}\right)\frac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}.$$

Proof. First note that after a change of variables $(x, z) := (x,z)/\sqrt{t}$

$$\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{y+\varphi(x)} \frac{\mathrm{e}^{-\frac{|x|^2+z^2}{4t}}}{(4\pi t)^{n/2}}\mathrm{d}z\mathrm{d}x = \int_{\mathbb{R}^{n-1}} \mathrm{d}\gamma_{n-1}(x)\Phi\left(\tfrac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}\right).$$

On the other hand, a Taylor expansion of the Gaussian error function yields

$$\left|\Phi\left(\tfrac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}\right) - \left(\Phi\left(\tfrac{y}{\sqrt{t}}\right) + \frac{\mathrm{e}^{-\frac{y^2}{4t}}}{\sqrt{4\pi}}\left(\tfrac{\varphi(\sqrt{t}x)}{\sqrt{t}} - \tfrac{y}{\sqrt{t}}\tfrac{\varphi(\sqrt{t}x)^2}{4t}\right)\right)\right|$$
$$\leqslant C^3\left||x|^6\, t^{3/2} \sup_{z\in[y/\sqrt{t},(y+\varphi(\sqrt{t}x))/\sqrt{t}]} \frac{(-2+z^2)}{24}\frac{\mathrm{e}^{-z^2/4}}{\sqrt{4\pi}}\right|.$$

Since by assumption $|\varphi| \leqslant Ct\,|x|^2$ and $y, \sqrt{t} \leqslant \frac{1}{4C}$, the error on the right-hand side can be estimated by

$$\sup_{z\in[y/\sqrt{t},(y+\varphi(\sqrt{t}x))/\sqrt{t}]} \frac{(-2+z^2)}{24}\frac{\mathrm{e}^{-z^2/4}}{\sqrt{4\pi}}$$
$$\leqslant \frac{2+\frac{|y|^2}{t}+C^2t\,|x|^4+2\,|y|C\,|x|^2}{24}\frac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}\mathrm{e}^{\frac{C\,|y|\,|x|^2}{2}}$$
$$\leqslant \left(1 + \tfrac{|y|^2}{t}\right)\frac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}\left(1 + |x|^4\right)\mathrm{e}^{\frac{|x|^2}{8}}.$$

Hence, after integrating over $\mathbb{R}^{n-1}$;

$$\Bigg| \int_{\mathbb{R}^{n-1}} \mathrm{d}\gamma_{n-1} \Phi\left(\tfrac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}\right) - \left(\Phi\left(\tfrac{y}{\sqrt{t}}\right) + \tfrac{\mathrm{e}^{-\frac{y^2}{4t}}}{\sqrt{4\pi}}\left(\tfrac{\varphi(\sqrt{t}x)}{\sqrt{t}} - \tfrac{y}{\sqrt{t}}\tfrac{\varphi(\sqrt{t}x)^2}{4t}\right)\right)\Bigg|$$
$$\leqslant \left(\tfrac{1}{6}\int_{\mathbb{R}^{n-1}} \mathrm{d}\gamma_{n-1}(x)(|x|^6 + |x|^{10})\mathrm{e}^{\frac{|x|^2}{8}}\right) C^3 t^{3/2}\left(1+\tfrac{|y|^2}{t}\right)\tfrac{\mathrm{e}^{-\frac{y^2}{4t}}}{\sqrt{4\pi}}$$
$$\leqslant C(n)C^3 t^{3/2}\left(1+\tfrac{|y|^2}{t}\right)\tfrac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}.$$

Moreover, a Taylor expansion of φ and φ^2 yields

$$\Bigg|\left(\tfrac{\varphi(\sqrt{t}x)}{\sqrt{t}} - \tfrac{y}{\sqrt{t}}\tfrac{\varphi(\sqrt{t}x)^2}{4t}\right) - \left(\tfrac{1}{2}\varphi_{ij}(0)x_i x_j\sqrt{t} + \tfrac{1}{6}\varphi_{ijk}(0)x_i x_j x_k t - \tfrac{1}{16}\left(\varphi_{ij}x_i x_j\right)^2 y\sqrt{t}\right)\Bigg|$$
$$\leqslant \left|D^4\varphi\right| |x|^4 t^{3/2} + \tfrac{|y|}{\sqrt{t}}\left(\left|D^2\varphi\right|\left|D^3\varphi\right| |x|^5 t^{3/2} + \left|D^3\varphi\right|^3 |x|^6 t^2\right)$$
$$\leqslant C^3 t^{3/2}\left(1+\tfrac{|y|}{\sqrt{t}}\right)\left(|x|^4 + |x|^5 + |x|^6\right).$$

Since the Gaussian measure of polynomials is bounded, there exists some $C(n) < \infty$ only depending on n such that

$$\Bigg|\int_{\mathbb{R}^{n-1}} \mathrm{d}\gamma_{n-1}(x)\Phi\left(\tfrac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}\right) - \int_{\mathbb{R}^{n-1}} \mathrm{d}\gamma_{n-1}(x)\Phi\left(\tfrac{y}{\sqrt{t}}\right) - \tfrac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}\left(\tfrac{1}{2}\varphi_{ij}(0)x_i x_j\sqrt{t} + \varphi_{ijk}(0)x_i x_j x_k \tfrac{t}{6} - y\tfrac{\sqrt{t}(\varphi_{ij}x_i x_j)^2}{16}\right)\Bigg|$$
$$\leqslant C(n)C^3 t^{3/2}\left(1+\tfrac{|y|}{\sqrt{t}}\right)\tfrac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}.$$

It only remains to compute second integral on the right-hand side. Due to the symmetry of the Gaussian integrals 2.22, the first three integrands can be computed as

$$\int_{\mathbb{R}^{n-1}} \mathrm{d}\gamma_{n-1}(x)\Phi\left(\tfrac{y}{\sqrt{t}}\right) = \Phi\left(\tfrac{y}{\sqrt{t}}\right),$$
$$\tfrac{\sqrt{t}}{2}\int_{\mathbb{R}^{n-1}} \mathrm{d}\gamma_{n-1}(x)\varphi_{ij}(0)x_i x_j = \Delta\varphi(0)\sqrt{t},$$
$$\tfrac{t}{6}\int_{\mathbb{R}^{n-1}} \mathrm{d}\gamma_{n-1}(x)\varphi_{ijk}(0)x_i x_j x_k = 0.$$

Now, we compute the integrals

$$\sum_{i,j,k,l=1}^{n-1} \tfrac{1}{(4\pi)^{n-1/2}}\int_{\mathbb{R}^{n-1}} \mathrm{e}^{-\frac{|x|^2}{4}}\varphi_{ij}(0)\varphi_{kl}(0)x_i x_j x_k x_l \,\mathrm{d}x\,.$$

By the symmetry of the Gaussian integrals 2.22, this sum is equal to

$$\sum_{\substack{\text{two pairs of}\\ \text{indices}}} \varphi_{ij}(0)\varphi_{kl}(0)\frac{1}{(4\pi)^{n-1/2}}\int_{\mathbb{R}^{n-1}} \mathrm{e}^{-\frac{|x|^2}{4}} x_i x_j x_k x_l \,\mathrm{d}x\,.$$

Terms with two identical pairs of indices appear only once in this sum. By Corollary 2.22, these terms satisfy

$$\frac{1}{(4\pi)^{n-1/2}}\varphi_{ii}(0)^2\int_{\mathbb{R}^{n-1}} \mathrm{e}^{-\frac{|x|^2}{4}} x_i^4 \,\mathrm{d}x = 12\varphi_{ii}(0)^2.$$

The terms with two different pairs of indices satisfy

$$\begin{aligned}&\sum_{\substack{\text{two pairs of}\\ \text{different indices}}} \varphi_{ij}(0)\varphi_{kl}(0)\frac{1}{(4\pi)^{n-1/2}}\int_{\mathbb{R}^{n-1}} \mathrm{e}^{-|x|^2/4} x_i x_j x_k x_l \,\mathrm{d}x\\ &= 4\sum_{\substack{\text{two pairs of}\\ \text{different indices}}} \varphi_{ij}(0)\varphi_{kl}(0)\,.\end{aligned}$$

The terms with two different pairs of indices are either terms of the form $\varphi_{ii}(0)\varphi_{kk}(0)$, $\varphi_{ik}(0)\varphi_{ik}(0)$ or $\varphi_{ik}(0)\varphi_{ki}(0)$, where $1 \leqslant i \neq k \leqslant n-1$. The terms $\varphi_{ii}(0)\varphi_{kk}(0)$ appear twice and terms of the form $\varphi_{ik}(0)\varphi_{ik}(0)$ or $\varphi_{ik}(0)\varphi_{ki}(0)$ once. Moreover, we have $\varphi_{ki}(0) = \varphi_{ik}(0)$. Therefore, we have

$$\begin{aligned}&\sum_{i,k,j,l}\frac{1}{(4\pi)^{n-1/2}}\int_{\mathbb{R}^{n-1}} \mathrm{e}^{-|x|^2/4}\varphi_{ij}(0)\varphi_{kl}(0) x_i x_j x_k x_l \,\mathrm{d}x\\ &= 12\sum_{i=1}^{n-1}\varphi_{ii}(0)^2 + 8\sum_{i=1}^{n-1}\sum_{k\neq i}^{n-1}\varphi_{ii}(0)\varphi_{kk}(0) + 8\sum_{i=1}^{n-1}\sum_{k\neq i}^{n-1}\varphi_{ik}(0)^2\\ &= 8\left|D^2\varphi(0)\right|^2 + 4\sum_{i=1}^{n-1}\varphi_{ii}(0)^2 + 8\sum_{i=1}^{n-1}\sum_{k\neq i}^{n-1}\varphi_{ii}(0)\varphi_{kk}(0)\\ &= 8\left|D^2\varphi(0)\right|^2 + 4\left(\Delta\varphi\right)^2.\end{aligned}$$

We conclude

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}}\Phi\left(\tfrac{y}{\sqrt{t}}\right) + \frac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}\left(\tfrac{1}{2}\varphi_{ij}(0)x_i x_j\sqrt{t} + \varphi_{ijk}(0)x_i x_j x_k\tfrac{t}{6} - y\tfrac{\sqrt{t}(\varphi_{ij}x_i x_j)^2}{16}\right)\\ &\qquad \mathrm{d}\gamma_{n-1}(x)\\ &= \Phi\left(\tfrac{y}{\sqrt{t}}\right) + \frac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}\sqrt{t}\left(\Delta\varphi(0) - y\left(\tfrac{1}{2}\left|D^2\varphi(0)\right|^2 + \tfrac{1}{4}\left(\Delta\varphi\right)^2\right)\right).\end{aligned}$$

□

The last computation needed for the next section concerns Gaussian integrals involving some bilinear form.

Lemma 2.24 *For any $n \in \mathbb{N}$ there is a constant $C(n) < \infty$ such that the following is true.*

Let $A = \{A_{\alpha\beta}\}_{1\leqslant\alpha,\beta\leqslant n}$ be a symmetric bilinear operator on $\mathbb{R}^n$ and $\varphi \in C^2(\mathbb{R}^{n-1})$. Then, for

$$C := \sup\left(\left|D^2\varphi\right|, \sqrt{|A|}\right)$$

and $|y| \leqslant C^{-1}/4$ we have

$$\begin{aligned}\Bigg|\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{y+\varphi(x)} & A((x,z),(x,z))\frac{\mathrm{e}^{-\frac{|x|^2+z^2}{4t}}}{(4\pi t)^{n/2}}\mathrm{d}z\mathrm{d}x \\ & - t\left(\Phi\left(\tfrac{y}{\sqrt{t}}\right)2\operatorname{trace}(A) - A_{nn}2\tfrac{y}{\sqrt{t}}\tfrac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}\right)\Bigg| \\ \leqslant{} & C(n)C^3t^{3/2}\left(1+\tfrac{y^2}{t}\right)\tfrac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}.\end{aligned}$$

Proof. After a change of variables $(x,z) := {}^{(x,z)}/\sqrt{t}$ and setting $\bar{x} := (x,z)$ we compute

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{y+\varphi(x)} A((x,z),(x,z))\frac{\mathrm{e}^{-\frac{|x|^2+z^2}{4t}}}{(4\pi t)^{n/2}}\mathrm{d}z\mathrm{d}x \\ &= t\int_{\mathbb{R}^{n-1}}\mathrm{d}\gamma_{n-1}(x)\int_{-\infty}^{\frac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}} A_{\alpha\beta}\bar{x}_\alpha\bar{x}_\beta\frac{\mathrm{e}^{-z^2/4t}}{\sqrt{4\pi}}\mathrm{d}z\mathrm{d}x.\end{aligned}$$

By Lemma 2.21, we have

$$\begin{aligned}&A_{\alpha\beta}\int_{-\infty}^{\frac{y+\sqrt{t}\varphi(x)}{\sqrt{t}}}\bar{x}_\alpha\bar{x}_\beta\frac{\mathrm{e}^{-z^2/4t}}{\sqrt{4\pi}}\mathrm{d}z \\ &= \begin{cases} A_{\alpha\beta}x_\alpha x_\beta\Phi\left(\frac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}\right) & \text{if } \alpha,\beta < n, \\ -2A_{\alpha\beta}x_\alpha\frac{\mathrm{e}^{-\frac{\left(y+\varphi(\sqrt{t}x)\right)^2}{4t}}}{\sqrt{4\pi}} & \text{if } \alpha\neq\beta = n, \\ -2A_{\alpha\beta}x_\beta\frac{\mathrm{e}^{-\frac{\left(y+\varphi(\sqrt{t}x)\right)^2}{4t}}}{\sqrt{4\pi}} & \text{if } \beta\neq\alpha = n, \\ A_{nn}\left(-2\frac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}\frac{\mathrm{e}^{-\frac{\left(y+\varphi(\sqrt{t}x)\right)^2}{4t}}}{\sqrt{4\pi}} + 2\Phi\left(\frac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}\right)\right) & \text{if } \alpha = \beta = n.\end{cases}\end{aligned}$$

We expand $\Phi(x)$ in $y/\sqrt{t}$ and get

$$\begin{aligned}\left|\Phi\left(\tfrac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}\right) - \Phi\left(\tfrac{y}{\sqrt{t}}\right)\right| &\leqslant \tfrac{\varphi(\sqrt{t}x)}{\sqrt{t}}\sup_{z\in[y/\sqrt{t},(y+\varphi(\sqrt{t}x))/\sqrt{t}]}\tfrac{\mathrm{e}^{-z^2/4}}{\sqrt{4\pi}} \\ &\leqslant Ct^{1/2}\,|x|\,\mathrm{e}^{|x|^2/8}\tfrac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}.\end{aligned}$$

Similarly, expansions of

$$f(x) := -\frac{\mathrm{e}^{-x^2/4}}{\sqrt{\pi}} \text{ and } g(x) := -x\frac{\mathrm{e}^{-x^2/4}}{\sqrt{\pi}}$$

in ${}^{y}\!/\!\sqrt{t}$ yield

$$\begin{aligned}\left|-2\frac{\mathrm{e}^{-\frac{\left(y+\varphi(\sqrt{t}x)\right)^2}{4t}}}{\sqrt{4\pi}} - \left(-2\frac{\mathrm{e}^{-\frac{y^2}{4t}}}{\sqrt{4\pi}}\right)\right| &\leqslant Ct^{1/2}\,|x|^2 \sup_{z\in[y/\sqrt{t},(y+\varphi(\sqrt{t}x))/\sqrt{t}]} \frac{|x|\mathrm{e}^{-x^2/4}}{\sqrt{4\pi}} \\ &\leqslant Ct^{1/2}\left(|x|^2+\frac{|y|}{\sqrt{t}}\right)\frac{\mathrm{e}^{-\frac{y^2}{4t}}}{\sqrt{4\pi}}\mathrm{e}^{\frac{|x|^2}{8}},\end{aligned}$$

respectively

$$\begin{aligned}&\left|-2\frac{y+\varphi(\sqrt{t}x)}{\sqrt{t}}\frac{\mathrm{e}^{-\frac{(y+\varphi(\sqrt{t}x))^2}{4t}}}{\sqrt{4\pi}} - \left(-2\frac{y}{\sqrt{t}}\frac{\mathrm{e}^{-\frac{y^2}{4t}}}{\sqrt{4\pi}}\right)\right| \\ &\leqslant Ct^{1/2}\,|x|^2 \sup_{z\in[y/\sqrt{t},(y+\varphi(\sqrt{t}x))/\sqrt{t}]} \left(x^2-2\right)\frac{\mathrm{e}^{-x^2/4}}{\sqrt{4\pi}} \\ &\leqslant Ct^{1/2}\frac{\mathrm{e}^{-\frac{y^2}{4t}}}{\sqrt{4\pi}}\left(|x|^4+2\,|x|^2+\frac{|y|}{\sqrt{t}}\right)\mathrm{e}^{\frac{|x|^2}{8}}.\end{aligned}$$

In particular, since $|A| \leqslant C^2$ by assumption, by integrating over $\mathbb{R}^{n-1}$ with respect to the Gaussian measure $\mathrm{d}\gamma_{n-1}(x)$ and using the symmetries of the Gaussian measure 2.22 we get

$$\begin{aligned}&\Bigg|\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{y+\varphi(x)} A((x,z),(x,z))\frac{\mathrm{e}^{-\frac{|x|^2+z^2}{4t}}}{(4\pi t)^{n/2}}\mathrm{d}z\mathrm{d}x \\ &\quad - \left(2\,\mathrm{trace}\,(A)\,\Phi\left(\frac{y}{\sqrt{t}}\right) - A_{nn}2\frac{y}{\sqrt{t}}\frac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}}\right)\Bigg| \\ &\leqslant C(n)C^3t^{3/2}\left(1+\frac{y^2}{t}\right)\frac{\mathrm{e}^{-y^2/4t}}{\sqrt{4\pi}},\end{aligned}$$ □

for some constant $C(n)$ only depending on n.

2.3 Proof of Theorem 2.5

With the results from the Sections 2.1 and 2.2 we are now ready to prove Theorem 2.5. The strategy of the proof will be the following

- Reduce the proof to the case $d_\Omega(y) \geqslant 0$.

- Choose a local coordinate system to do the computations in.
- Use Theorem 2.7 and the lemmata in Section 2.2 to derive a approximation $v_\Omega(y,t)$ of $u_\Omega(y,t)$.
- Estimate the error $|v_\Omega(y,t) - u_\Omega(y,t)|$.

Proof of Theorem 2.5 **Reduction to the case $d_\Omega(y) \geqslant 0$.**

For any $y \in \mathrm{M}^n$ contained in a tubular neighbourhood of the boundary $\partial\Omega$ there is some unique $x_0 \in \partial\Omega$ such that $y = \exp_{x_0} d_\Omega(y)\nu_{\partial\Omega}(x_0)$, where $\nu_{\partial\Omega}(x_0)$ is a choice of a local outer normal vector of $\partial\Omega$ in x_0. Suppose $d_\Omega(y) < 0$. Since the solution of (HE) is given by

$$u_\Omega(y,t) = \int_\Omega \varrho(x,y,t)\omega(x),$$

the stochastic completeness of M^n implies

$$u_\Omega(y,t) = 1 - \int_{\Omega^c} \varrho(x,y,t)\omega(x) = 1 - u_{\Omega^c}(y,t).$$

Choosing the characteristic function χ_Ω^c as initial data in (HE), we then have $d_{\Omega^c}(y) = -d_\Omega(y) \geqslant 0$. Furthermore, choosing local outward pointing normal vectors $\nu_{\partial\Omega}(x_0)$, respectively $\nu_{\partial\Omega^c}(x_0)$ we have $H_{\partial\Omega}(x_0) = -H_{\partial\Omega^c}(x_0)$. Hence,

$$\begin{aligned} v_{\Omega^c}(y,t) &= \Phi\left(-\tfrac{d_{\Omega^c}(y)}{\sqrt{t}}\right) + \frac{\mathrm{e}^{-\frac{d_{\Omega^c}(y)^2}{4t}}}{\sqrt{4\pi}}\Bigg(-\sqrt{t}H_{\partial\Omega^c}(x_0) + \sqrt{t}d_{\Omega^c}(y) \\ &\quad \times \left(\tfrac{1}{4}\left(2\left(\bar{\mathrm{R}}(\nu_{\partial\Omega^c},\nu_{\partial\Omega^c})(x_0) + |A_{\partial\Omega^c}|(x_0)^2\right) + H_{\partial\Omega^c}^2\right)\right)\Bigg) \\ &= \Phi\left(\tfrac{d_\Omega(y)}{\sqrt{t}}\right) + \frac{\mathrm{e}^{-\frac{d_\Omega(y)^2}{4t}}}{\sqrt{4\pi}}\Bigg(\sqrt{t}H_{\partial\Omega}(x) \\ &\quad - \sqrt{t}d_\Omega(y)\left(\tfrac{1}{4}\left(2\left(\bar{\mathrm{R}}(\nu_{\partial\Omega},\nu_{\partial\Omega}) + |A_{\partial\Omega}|^2\right) + H_{\partial\Omega}^2\right)\right)\Bigg). \end{aligned}$$

Since the Gaussian error function satisfies $\Phi(x) = 1 - \Phi(-x)$, we therefore have

$$1 - v_{\Omega^c}(y,t) = v_\Omega(y,t).$$

Choice of local coordinate system

To abbreviate the computations, we write $s := d_\Omega(y)$ in the following. We may choose an orthonormal basis $x_1, \ldots, x_n$ of $T_y\mathrm{M}^n$ such that the n-th basis vector is given by the parallel transport of $\nu_{\partial\Omega}(x_0)$ along the geodesic $c(\tau) := \exp_{x_0} \tau\nu_{\partial\Omega}(x_0)$; i.e. $e_n := \nu_y := T_{s,0}\nu_{\partial\Omega}(x_0)$. Moreover, the outer normal ν of the hypersurface

$$\exp_y^{-1}\left(\partial\Omega \cap B_{2r}(y)\right)$$

intersected with $[-r,r]^n \subset B_{2r}(0)$ is given by

$$\nu\left(0,\dots,-s\right) = \left(0,\dots,1\right)$$

in the point $\exp_y^{-1}(x_0) = (0,\dots,-s)$. In particular, for some $r_1 > 0$, the hypersurface can be expressed as a graph of a function $\tilde{\varphi} : [-r_1,r_1]^{n-1} \to \mathbb{R}$ satisfying

$$\tilde{\varphi}(0) = -d_\Omega(y) \quad \text{and} \quad |D\tilde{\varphi}(0)| = 0,$$

i.e.,

$$\exp_y^{-1}\left(\partial\Omega \cap B_r(y)\right) \cap [-r_1,r_1]^n = \{(\bar{x},x_n) \in [r_1,r_1]^n : x_n = \tilde{\varphi}(x)\}. \tag{69}$$

We write $\varphi(x) := \tilde{\varphi}(x) + s$. The derivatives of φ in 0 can be expressed in terms of the curvature of M^n and $\partial\Omega$. Indeed, by the Weingarten equations (10) we have

$$\varphi_{ij}(0) = -h_{ij} - \bar{\Gamma}^n_{ij}\left((0,-s)\right).$$

Additionally, by Remark 1.8 and Remark 2.11, the Christoffel symbols $\bar{\Gamma}^\gamma_{\alpha\beta}(x)$, the volume element and the heat kernel coefficients $\varrho_0(x,y)$, $\varrho_1(x,y)$ satisfy, for some appropriate constant $C(n) < \infty$,

$$\begin{aligned}
\left|\bar{\mathrm{g}}_{\alpha\beta}(x) - \left(\delta_{\alpha\beta} - \tfrac{1}{3}\bar{\mathrm{R}}_{\alpha\gamma\delta\beta}x_\gamma x_\delta\right)\right| &\leqslant C(n)\ |x|^3\, r^{-3},\\
\left|\bar{\Gamma}^\alpha_{\beta\gamma}(x) - \left(-\tfrac{1}{3}\left(\bar{\mathrm{R}}_{\alpha\gamma\beta\delta} + \bar{\mathrm{R}}_{\alpha\beta\gamma\delta}\right)\right)x_\delta\right| &\leqslant C(n)\ |x|^2\, r^{-3},\\
\left|\sqrt{\det \bar{\mathrm{g}}} - \left(1 - \tfrac{1}{6}\bar{\mathrm{R}}_{\alpha\beta}x_\alpha x_\beta\right)\right| &\leqslant C(n)\ |x|^3\, r^{-3}.\\
\left|\varrho_0(x,y) - \left(1 + \tfrac{1}{12}\bar{\mathrm{R}}_{\alpha\beta}x_\alpha x_\beta\right)\right| &\leqslant C(n)\ |x|^3\, r^{-3},\\
\left|\varrho_1(x,y) - \tfrac{1}{6}\bar{\mathrm{R}}\right| &\leqslant C(n)\ |x|\, r^{-3}\ .
\end{aligned} \tag{70}$$

The local expansion of the Christoffel symbols $\bar{\Gamma}^n_{\alpha\beta}$ (70) then yields

$$\left|\varphi_{ij}(0) - \left(-h_{ij} + \tfrac{1}{3}s\left(\bar{\mathrm{R}}_{innj}(y) + \bar{\mathrm{R}}_{jnni}(y)\right)\right)\right| \leqslant C(n)s^2r^{-3},$$

where $1 \leqslant i,j \leqslant n-1$. In particular, it holds

$$\left|\Delta\varphi(0) - \left(-H_{\partial\Omega}(x_0) + s\tfrac{2}{3}\bar{\mathrm{R}}\left(\nu_y,\nu_y\right)\right)\right| \leqslant C(n)\ |s|^2\, r^{-3}\,, \tag{71}$$

$$\left|(\Delta\varphi(0))^2 - H_{\partial\Omega}(x_0)^2\right| \leqslant C(n)\ |s|\, r^{-3}\,, \tag{72}$$

$$\left|\left|D^2\varphi(0)\right|^2 - |A_{\partial\Omega}(x_0)|^2\right| \leqslant C(n)\ |s|\, r^{-3}. \tag{73}$$

for some $C(n) < \infty$.

We now bound the derivatives of φ in terms of the radius r. This guarantees that, for some $\varepsilon(n) > 0$ only depending on the dimension n, the radius $r_1 = \varepsilon(n)r$ can be chosen in (69).

By our choice of coordinates and the first Weingarten equation (10), we have

$$\bar{g}_{n\beta}\,\nu^{\beta}\varphi_{ij} = -\,\bar{g}_{\alpha\beta}\,\nu^{\beta}\left(\sum_{\beta\neq n}\sum_{\delta\neq n}\bar{\Gamma}^{\alpha}_{\beta\delta} + 2\bar{\Gamma}^{\alpha}_{ni}\varphi_i + \bar{\Gamma}^{\alpha}_{nn}\varphi_i\varphi_i\right) - h_{ij}. \tag{74}$$

We estimate the right-hand side from above. By our choice of r and Remark 1.8, there is for any $\varepsilon_1(n) > 0$ a constant $\delta_1(n) > 0$ such that

$$\left|\bar{g}_{\alpha\beta}(x) - \delta^{\beta}_{\alpha}\right| \leqslant \varepsilon(n) \quad \text{for all } x \in B_{\delta_1(n)r}((0,s)) \text{ and } 1 \leqslant \alpha,\beta \leqslant n.$$

Moreover, by (70) the Christoffel symbols satisfy $|\bar{\Gamma}^{\alpha}_{\beta\gamma}| \leqslant \varepsilon_1(n)r^{-1}$. Hence, for any $\varepsilon_2(n) > 0$, we may choose $\varepsilon_1(n) > 0$ small enough to estimate the terms involving the Christoffel symbols $\bar{\Gamma}^{\alpha}_{\beta\gamma}$ by

$$\left|\sum_{\beta\neq n}\sum_{\delta\neq n}\bar{\Gamma}^{\alpha}_{\beta\delta} - 2\bar{\Gamma}^{\alpha}_{ni}\varphi_i - \bar{\Gamma}^{\alpha}_{nn}\varphi_i\varphi_i\right| \leqslant \varepsilon_2(n)r^{-1}\left(1 + |D\varphi|^2\right).$$

To estimate the components of the normal vector, we note that by the second Weingarten equation (10)

$$\begin{aligned} \tfrac{\partial\nu^{\alpha}}{\partial x^i} &= -\bar{\Gamma}^{\alpha}_{\beta\delta}\tfrac{\partial F^{\beta}}{\partial x^i}\nu^{\delta} + h_{ij}g^{jl}\tfrac{\partial F^{\alpha}}{\partial x^l} \\ &\leqslant C(n)r^{-1}\left(1 + |D\varphi|\right)\left(1 + |\nu|\right). \end{aligned}$$

Therefore, Grönwall's lemma implies that, for some $\delta_2(n) > 0$, in a ball of radius $r_2 := \delta_2(n)r^{-1}(1 + |D\varphi|)$, the components of the normal vector ν satisfy

$$\left|\nu^i\right| \leqslant \varepsilon_2(n) \text{ for } 1 \leqslant i \leqslant n-1, \quad \text{and } \ |\nu^n - 1| \leqslant \varepsilon_2(n). \tag{75}$$

Thus, taking the square of both sides of Equality (74) and summing over $1 \leqslant i,j \leqslant n-1$ implies

$$\left|D^2\varphi\right| \leqslant C(n)r^{-1}\left(1 + |D\varphi|^2\right).$$

For a fixed $\bar{x} \in B_r(0)$ we set $f(\tau) := |D\varphi(\tau\bar{x})|$. Then, since $f'(\tau) = |D\varphi|^{-1}D^2\varphi(\tau\bar{x})(\bar{x}, D\varphi)$, we have

$$\tfrac{1}{1+|f(t)|^2}f'(t) \leqslant C(n)r^{-1}\,|\bar{x}|.$$

By definition, we have $f(0) = 0$. Thus, integrating both sides from 0 to τ yields, for $|\tau\bar{x}| \leqslant \left(C(n)r^{-1}\right)^{-1} \tan^{-1}(\varepsilon_2(n))$, the estimation

$$f(\tau) \leqslant \tan\Big(C(n)r^{-1}\tau\ |\bar{x}|\Big) \leqslant \varepsilon_2(n).$$

Using this estimate on the gradient of φ and the Weingarten equations (10), one can show that there exists a constant $C(n)$ depending only on the dimension n such that

$$\Big|D^i\varphi\Big| \leqslant C(n)r^{-i+1}, \quad i = 2, 3, 4. \tag{76}$$

Approximation of $u_\Omega(y,t)$.

We now estimate $u(y,t)$. Since by assumption $\sqrt{t} \leqslant \varepsilon(n)r$, choosing $k = 1$ in Theorem 2.7 yields the estimation

$$\begin{aligned}
&\left|\int_\Omega \varrho(x,y,t)\omega(x) - \int_{\Omega\cap B_{2r}(y)} \varrho_1(x,y,t)\omega(x)\right| \\
&\leqslant C(n)\tfrac{t^2}{r^4}\left(\int_{\Omega\cap B_{2r}(y)} (4\pi t)^{-n/2}\, \mathrm{e}^{-\frac{d(x,y)^2}{4t}}\omega(x) + |\Omega|\left(4\pi r^2\right)^{-n/2} \mathrm{e}^{-\frac{s^2}{4t}}\right) \\
&\leqslant C(n)\tfrac{t^{3/2}}{r^3}\left(\int_{\Omega\cap B_{2r}(y)} (4\pi t)^{-n/2}\, \mathrm{e}^{-\frac{d(x,y)^2}{4t}}\omega(x) + |\Omega|\left(4\pi r^2\right)^{-n/2} \mathrm{e}^{-\frac{s^2}{4t}}\right),
\end{aligned} \tag{77}$$

where $\varrho_1(x,y,t)$ is the approximate heat kernel from Definition 2.10 and $C(n) < \infty$ only depends on n. Taking the sum of the local expansion formulae (70), we have

$$\begin{aligned}
&\left|\varrho_1(x,y,t)\sqrt{\det\bar{\mathrm{g}}} - (4\pi t)^{-n/2}\, \mathrm{e}^{-\frac{|x|^2}{4t}}\left(\left(1 + \tfrac{1}{6}t\bar{\mathrm{R}}\right) - \tfrac{1}{12}\bar{\mathrm{R}}_{\alpha\beta}x^\alpha x^\beta\right)\right| \\
&\leqslant C(n)r^{-3}\left(|x|^3 + t\ |x|\right).
\end{aligned}$$

Thus, by substituting $z := x/\sqrt{t}$ we conclude

$$\begin{aligned}
&\left|\int_\Omega \varrho(x,y,t)\omega(x) - \int_{\exp_y^{-1}(\Omega\cap B_{2r}(y))} \frac{\mathrm{e}^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\left(\left(1 + \tfrac{1}{6}t\bar{\mathrm{R}}\right) - \tfrac{1}{12}\bar{\mathrm{R}}_{\alpha\beta}x^\alpha x^\beta\right)\mathrm{d}x\right| \\
&\leqslant C(n)\tfrac{t^{3/2}}{r^3}\int_{\exp_y^{-1}(\Omega\cap B_{2r}(y))/\sqrt{t}} \frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(1 + |z|^3\right)\mathrm{d}z
\end{aligned} \tag{78}$$

for some $C(n) < \infty$. Since for $[r,r]^n \subset B_{2r}(y)$

$$\exp_y^{-1}(\Omega) \cap [r,r]^n = \{(\bar{x}, x_n) \in [r,r]^n : x_n \leqslant \varphi(\bar{x}) - d_{\Omega(y)}\}$$

the second integral on the left-hand side of (78) can be expressed as

$$\int_{[-r,r]^{n-1}}\int_{-r}^{\varphi(\bar{x})-s}\frac{\mathrm{e}^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\left(\left(1+\tfrac{1}{6}t\bar{\mathrm{R}}\right)-\tfrac{1}{12}\bar{\mathrm{R}}_{\alpha\beta}x^\alpha x^\beta\right)\mathrm{d}x_n\mathrm{d}\bar{x}$$
$$=\int_{[-r,r]^{n-1}/\sqrt{t}}\int_{-r/\sqrt{t}}^{\varphi(\sqrt{t}\bar{x})-s/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(\left(1+\tfrac{1}{6}t\bar{\mathrm{R}}\right)-\tfrac{1}{12}t\bar{\mathrm{R}}_{\alpha\beta}z^\alpha z^\beta\right)\mathrm{d}z_n\mathrm{d}\bar{z},$$

where we substituted $(\bar{z},z_n):=(\bar{x},x_n)/t^{1/2}$. In order to apply the lemmata from Section 2.2, we note

$$\int_{[-r,r]^{n-1}/\sqrt{t}}\int_{-r/\sqrt{t}}^{\varphi(\sqrt{t}\bar{x})-s/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(\left(1+\tfrac{1}{6}t\bar{\mathrm{R}}\right)-\tfrac{1}{12}\bar{\mathrm{R}}_{\alpha\beta}z^\alpha z^\beta\right)\mathrm{d}z_n\mathrm{d}\bar{z}$$
$$-\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\varphi(\sqrt{t}\bar{x})-s/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(\left(1+\tfrac{1}{6}t\bar{\mathrm{R}}\right)-\tfrac{1}{12}t\bar{\mathrm{R}}_{\alpha\beta}z^\alpha z^\beta\right)\mathrm{d}z_n\mathrm{d}\bar{z}$$
$$=\int_{\frac{[-r,r]^{n-1}}{\sqrt{t}}}\int_{-\infty}^{-r/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(\left(1+\tfrac{1}{6}t\bar{\mathrm{R}}\right)-\tfrac{1}{12}t\bar{\mathrm{R}}_{\alpha\beta}z^\alpha z^\beta\right)\mathrm{d}z_n\mathrm{d}\bar{z}$$
$$+\int_{\left([-r,r]^{n-1}/\sqrt{t}\right)^c}\int_{-r/\sqrt{t}}^{\varphi(\sqrt{t}\bar{x})-s/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(\left(1+\tfrac{1}{6}t\bar{\mathrm{R}}\right)-\tfrac{1}{12}t\bar{\mathrm{R}}_{\alpha\beta}z^\alpha z^\beta\right)\mathrm{d}z_n\mathrm{d}\bar{z}. \quad (79)$$

Using Estimate (76),

$$\sup\left(\left|D^2\varphi\right|,\sqrt{|D^3\varphi|},\left(\left|D^3\varphi\right|\right)^{1/3}\right)\leqslant C(n)r^{-1},$$

we can apply Lemma 2.23. For $t\leqslant(4C(n))^{-1}r$, we have

$$\left|\left(1+\tfrac{1}{6}t\bar{\mathrm{R}}\right)\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\varphi(\sqrt{t}\bar{x})-s/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\mathrm{d}z_n\mathrm{d}\bar{z}-\left(\left(1+\tfrac{1}{6}t\bar{\mathrm{R}}\right)\Phi\left(-\tfrac{s}{\sqrt{t}}\right)\right.\right.$$
$$\left.\left.+\frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}\sqrt{t}\left(\Delta\varphi(0)+s\left(\tfrac{1}{2}\left|D^2\varphi(0)\right|^2+\tfrac{1}{4}\left(\Delta\varphi\right)^2\right)\right)\right)\right|$$
$$\leqslant C(n)r^{-3}t^{3/2}\left(1+\tfrac{s^2}{t}\right)\frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}. \quad (80)$$

By Lemma 2.24, we have

$$\left|\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\varphi(\sqrt{t}\bar{x})-s/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(-\tfrac{1}{12}t\bar{\mathrm{R}}_{\alpha\beta}z^\alpha z^\beta\right)\mathrm{d}z_n\mathrm{d}\bar{z}\right.$$
$$\left.-t\left(-\tfrac{1}{6}\Phi\left(-\tfrac{s}{\sqrt{t}}\right)\mathrm{tr}\left(\bar{\mathrm{R}}_{\alpha\beta}\right)-\tfrac{1}{6}\bar{\mathrm{R}}_{nn}\tfrac{s}{\sqrt{t}}\frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}\right)\right|$$
$$\leqslant C(n)r^{-3}t^{3/2}\left(1+\tfrac{s^2}{t}\right)\frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}. \quad (81)$$

Since $\operatorname{tr}\left(\bar{\mathrm{R}}_{\alpha\beta}\right) = \bar{\mathrm{R}}$, taking the sum of (80) and (81) yields

$$\begin{aligned}
&\Phi\left(-\tfrac{s}{\sqrt{t}}\right) + \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}\sqrt{t} \\
&\quad \times \left(\Delta\varphi(0) + s\left(-\tfrac{1}{6}\bar{\mathrm{R}}\left(\nu,\nu\right) + \tfrac{1}{4}\left(2\left|D^2\varphi(0)\right|^2 + \left(\Delta\varphi(0)\right)^2\right)\right)\right) \\
&=: \bar{v}_{\Omega,\varphi}(y,t).
\end{aligned}$$

Finally, since by (73) the derivatives of φ can be expressed in curvature terms, the function

$$\begin{aligned}
v_\Omega(y,t) &:= \Phi\left(-\tfrac{s}{\sqrt{t}}\right) \\
&\qquad + \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}\sqrt{t}\left(-H + s\left(\tfrac{1}{4}\left(2\left(\bar{\mathrm{R}}_{nn} + |A|^2\right) + H^2\right)\right)\right)
\end{aligned}$$

satisfies

$$\left|\bar{v}_{\Omega,\varphi}(y,t) - v_\Omega(y,t)\right| \leqslant C(n) r^{-3}\sqrt{t} s^2 \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}. \tag{82}$$

It remains to estimate the summed error terms.

Estimating the error terms.

In the final step of the proof, we show that the sum of the error terms in the inequalities (77)-(82) may be estimated from above by

$$C(n)\tfrac{t^{3/2}}{r^3}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{s^2}{t}\right)\tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}, \tag{83}$$

where $C(n)$ is a constant only dependent on n. We will not be interested in the exact value of $C(n)$. Furthermore, it will be increased appropriately to fulfill our estimates. The error terms in (80), (81) and (82) are clearly bounded by (83). Note that Lemma 2.20 and Lemma 2.21 imply for $k \leqslant 3$

$$\int_{-\infty}^{x} x^k \mathrm{e}^{-x^2/4}\mathrm{d}x \leqslant C(k)\left(1 + |x|^{k-1}\right)\mathrm{e}^{-x^2/4}, \quad x \leqslant 0. \tag{84}$$

Hence, we may estimate the first integral in (79) by

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{-r/\sqrt{t}} \tfrac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(\left(1 + \left|\bar{\mathrm{R}}\right|\tfrac{t}{6}\right) + \tfrac{t}{12}\left|\bar{\mathrm{R}}\right||z|^2\right)\mathrm{d}z_n\mathrm{d}\bar{z} \\
&\leqslant \int_{\mathbb{R}^{n-1}}\int_{-\infty}^{-r/\sqrt{t}} \tfrac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(2 + |\bar{z}|^2 + z_n^2\right)\mathrm{d}z_n\mathrm{d}\bar{z} \\
&\overset{(84)}{\leqslant} C\left(1 + \tfrac{r}{\sqrt{t}}\right)\mathrm{e}^{-r^2/4t}.
\end{aligned}$$

Since, by definition, the distance of y to the boundary is strictly smaller than r, we may choose C appropriately such that

$$\frac{r}{\sqrt{t}}\mathrm{e}^{-\frac{r^2}{4t}} \leqslant Ct^{3/2}r^{-3}\mathrm{e}^{-\frac{s^2}{4t}}.$$

In particular, the first integral in (79) is dominated by the error term (83). The second integral in (79) is estimated similarly. Just note that

$$\begin{aligned}&\int_{\left([-r,r]^{n-1}/\sqrt{t}\right)^c}\int_{-r/\sqrt{t}}^{\varphi(\sqrt{t}\bar{x})-s/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(2+|z|^2\right)\mathrm{d}z_n\mathrm{d}\bar{z}\\ \leqslant &\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{-r/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(2+|z|^2\right)\mathrm{d}z_n\mathrm{d}\bar{z}.\end{aligned}$$

It remains to estimate the error terms (77) and (78). Since $\sqrt{\det \bar{g}} \leqslant 2$ by our choice of r, their sum is bounded by

$$C(n)\frac{t^{3/2}}{r^3}\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\varphi(\sqrt{t}\bar{x})-s/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z|^2}{4}}}{(4\pi)^{n/2}}\left(1+|z|^3\right)\mathrm{d}z. \tag{85}$$

The first integrand can be estimated by (83) using Lemma 2.23. To estimate the term integrand involving the factor $|z|^3$ we observe

$$|x|^3 \leqslant \left(\sum_{\alpha=1}^n |z_\alpha|\right)^3 \leqslant n^3\sum_{\alpha=1}^n |z_\alpha|^3.$$

Integrands involving $|z_\alpha|^3$ may again be estimated in a similar way as in Lemma 2.23. Furthermore, by (84)

$$\begin{aligned}&\int_{-\infty}^{\varphi(\sqrt{t}\bar{x})-s/\sqrt{t}}\frac{\mathrm{e}^{-\frac{|z_n|^2}{4}}}{(4\pi)^{n/2}}z_n^3\mathrm{d}z_n\\ \leqslant\ & C\left(1+\frac{\left(\varphi(\sqrt{t}\bar{x})-s\right)^2}{t}\right)\exp\left(-\frac{1}{4t}\left(\varphi(\sqrt{t}\bar{z})-s\right)^2\right)\\ \leqslant\ & 2C\left(1+\frac{s^2}{t}\right)\mathrm{e}^{-\frac{s^2}{4t}}\mathrm{e}^{\frac{|\bar{z}|^2}{8}}.\end{aligned}$$

After integrating the right-hand side we see that (85) is also dominated by the error term (83). □

3 Short-time asymptotics for the heat content

In this chapter we find, up to an error that decays like $O(t^2)$, the asymptotic expansion of the heat content outside of a compact set $\Omega \subset \mathrm{M}^n$, i.e. of the quantity

$$\mathrm{HC}_\Omega(t) := \int_{\Omega^c} u_\Omega(y,t)\omega(y)\,. \tag{HC}$$

Here, like in the previous chapters, $u_\Omega : \mathrm{M}^n \times (0,\infty) \to (0,\infty)$ is the solution of the heat equation

$$\begin{cases} \Delta_g u(x,t) = \frac{\partial}{\partial t} u(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n. \end{cases} \tag{HE}$$

To this end, we use the pointwise asymptotic expansion of u_Ω derived in Theorem 2.5 and the co-area formula to integrate over the level-sets of the distance function. Similar arguments are then used in Theorem 3.6 in order to approximate the Boltzmann entropy $\mathrm{Ent}_\Omega(t)$ of Ω that is defined by

$$\mathrm{Ent}_\Omega(t) := \int_{\mathrm{M}^n} u_\Omega(y,t) \log\left(u_\Omega(y,t)\right)\omega(y). \tag{Ent}$$

Since the expansion formula from Theorem 3.2 differs from a similar result in the publication [vdBG15, Theorem 1.6, Corollary 1.7], we begin this chapter with an example. The computations can be carried out by using, for example, Mathematica [Wol].

Example 3.1 (Heat contents of balls in Euclidean space) We consider the case, where $(\mathrm{M}^n, g) = (\mathbb{R}^n, \delta_{ij})$ is the Euclidean space of dimension $n = 3, 5$ and the sets $\Omega^n := B_1^n(0) \subset \mathbb{R}^n$ are unit balls. The radial symmetry of the solutions $u_{B_1^n(0)}(y,t)$ of the heat equation (HE) can be used to compute the explicit formulae of the heat contents. In fact, in terms of the error function,

$$\mathrm{Erf}(z) = \tfrac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-x^2}\,\mathrm{d}x = \tfrac{1}{2}\left(\Phi\left(2z\right) - 1\right),$$

the solutions $u_{\Omega^n} : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ of (HE) are explicitly given by

$$\begin{aligned} &\int_{B_1^3(0)} \int_{B_1^3(0)} \tfrac{1}{(4\pi t)^{3/2}} \mathrm{e}^{-|x-y|^2/4t} \mathrm{d}x\,\mathrm{d}y \\ &= \tfrac{4}{3}\mathrm{e}^{-\frac{1}{t}}\sqrt{\pi}\left(\sqrt{t}\left(1 - 2t + \mathrm{e}^{\frac{1}{t}}\left(-3+2t\right)\right) + \mathrm{e}^{\frac{1}{t}}\sqrt{\pi}\,\mathrm{Erf}\left(\tfrac{1}{\sqrt{t}}\right)\right) \\ &= \tfrac{4}{3}\pi - 4\sqrt{\pi}\sqrt{t} + \tfrac{8}{3}\sqrt{\pi}t^{3/2} + O(t^{5/2}), \end{aligned} \tag{86}$$

respectively for $n = 5$ by

$$
\begin{aligned}
&\int_{B_1^5(0)} \int_{B_1^5(0)} \tfrac{1}{(4\pi t)^{5/2}} \mathrm{e}^{-|x-y|^2/4t} \mathrm{d}x \, \mathrm{d}y \\
&= \tfrac{8}{15} \mathrm{e}^{-\frac{1}{t}} \pi^{3/2} \\
&\quad \times \left(\sqrt{t} - 5\mathrm{e}^{\frac{1}{t}} \sqrt{t} + 2t^{3/2} + 10\mathrm{e}^{\frac{1}{t}} t^{3/2} + 12t^{5/2} - 12\mathrm{e}^{\frac{1}{t}} t^{5/2} + \mathrm{e}^{\frac{1}{t}} \sqrt{\pi} \operatorname{Erf}\left(\tfrac{1}{\sqrt{t}}\right) \right) \\
&= \tfrac{8\pi^2}{15} - \tfrac{8}{3} \pi^{3/2} \sqrt{t} + \tfrac{16}{3} \pi^{3/2} t^{3/2} + O(t^{5/2}).
\end{aligned} \tag{87}
$$

It is natural to assume that for short times t the above asymptotic expansions should reflect the geometry of the unit balls $B_1^n(0)$. In view of the left-hand side's scaling, the obvious candidates for an expansion in $\sqrt{t}$ are

$$
\begin{aligned}
&\int_{B_1^n(0)} \int_{B_1^n(0)} \tfrac{1}{(4\pi t)^{n/2}} \mathrm{e}^{-|x-y|^2/4t} \mathrm{d}x \, \mathrm{d}y \\
&= c_0 \, |B_1^n(0)| + c_1 \sqrt{t} \operatorname{area}(\partial B_1^n(0)) + c_2 t \int_{\partial B_1^n(0)} H_{B_1^n(0)} \mathrm{d}\mu \\
&\quad + t^{3/2} \left(c_{3,1} \int_{\partial B_1^n(0)} H^2_{B_1^n(0)} \mathrm{d}\mu + c_{3,2} \int_{\partial B_1^n(0)} \left| A_{B_1^n(0)} \right|^2 \mathrm{d}\mu \right) + O(t^2).
\end{aligned}
$$

For $n = 3, 5$ the mean curvature, the total curvature, the volume and the surface area of unit balls in Euclidean space are given by

$$
\begin{gathered}
H_{\partial B_1^n(0)} = n - 1 = |A|^2, \\
|B_1^n(0)| = 4/3\pi, 8/15\pi^2 \text{ and } |\partial B_1^n(0)| = 4\pi, 8/3\pi^2.
\end{gathered}
$$

Assuming the constants $c_{i,j}$ are universal, Equation (86) and Equation (87) imply that the first two coefficients are given by $c_0 = 1$, $c_1 = -1/\sqrt{\pi}$ and $c_2 = 0$. Moreover, they yield the following linear equation system for $c_{3,1}$ and $c_{3,2}$;

$$
\begin{cases}
16\pi c_{3,1} + 8\pi c_{3,2} & = \tfrac{8}{3}\sqrt{\pi}, \\
\tfrac{128}{3}\pi^2 c_{3,1} + \tfrac{32}{3}\pi^2 c_{3,2} & = \tfrac{16}{3}\pi^{3/2}.
\end{cases}
$$

The unique solution of this system is given by $c_{3,1} = 1/12\sqrt{\pi}$ and $c_{3,2} = 1/6\sqrt{\pi}$. Hence, we get, for this special case of Theorem 3.2, the following asymptotic expansion;

$$
\begin{aligned}
&\int_{B_1^n(0)} \int_{B_1^n(0)} 1/(4\pi t)^{n/2} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \mathrm{d}x \, \mathrm{d}y \\
&= |B_1^n(0)| - \tfrac{\sqrt{t}}{\sqrt{\pi}} \operatorname{area}(\partial B_1^n(0)) + 0t \int_{\partial B_1^n(0)} H_{B_1^n(0)} \mathrm{d}\mu \\
&\quad + t^{3/2} \left(\tfrac{1}{12\sqrt{\pi}} \int_{\partial B_1^n(0)} H^2_{B_1^n(0)} \mathrm{d}\mu + \tfrac{1}{6\sqrt{\pi}} \int_{\partial B_1^n(0)} \left| A_{B_1^n(0)} \right|^2 \mathrm{d}\mu \right) + O(t^2).
\end{aligned}
$$

We now return to the general case and derive the following theorem.

Theorem 3.2 (Asymptotics of the heat content) *For any $n \in \mathbb{N}$ there are constants $0 < \varepsilon(n), C(n) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a closed Riemannian manifold of class C^6 and $\Omega \subset \mathrm{M}^n$ a set such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := \min_{x \in \mathrm{M}^n} r_{18}(5, x) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor of the ambient space M^n, the first two derivatives of the second fundamental form of $\partial\Omega$ and the injectivity radius of M^n and Ω, is positive.

We define the function

$$\begin{aligned} Z_\Omega(t) := &\tfrac{\sqrt{t}}{\sqrt{\pi}} \operatorname{area}(\partial\Omega) + \\ &- \tfrac{t^{3/2}}{12\sqrt{\pi}} \left(\left(2 \int_{\partial\Omega} \left(\bar{\mathrm{Ric}}(\nu,\nu) + |A|^2 \right) \mathrm{d}\mu \right) + \left(\int_{\partial\Omega} H^2 \mathrm{d}\mu \right) \right). \end{aligned}$$

Then the heat content $\mathrm{HC}_\Omega(t)$ defined in (HC) satisfies for all $t \in (0, \varepsilon(n) r^2)$ the uniform estimate

$$|\mathrm{HC}_\Omega(t) - Z_\Omega(t)| \leqslant C(n) t^2 \left(\tfrac{1}{r^3} \left(1 + \tfrac{|\Omega|}{r^n} \right) \operatorname{area}(\partial\Omega) + |\mathrm{M}^n| \tfrac{1}{r^4} \right). \tag{88}$$

Remark 3.3 (i) We note that Formula (88) coincides with the computations in Example 3.1.

(ii) There is a discrepancy between the Expansion formula (88) and [vdBG15, Corollary 1.7] in regard to the term involving the squared mean curvature;

$$-\tfrac{1}{12\sqrt{\pi}} \left(\int_{\partial\Omega} H^2 \mathrm{d}\mu \right) \neq -\tfrac{5}{12\sqrt{\pi}} \left(\int_{\partial\Omega} H^2 \mathrm{d}\mu \right).$$

The following integral formulae involving the Gaussian error function will be used in the computations of the heat content. They are derived by partial integration and the table bases in [GR07, Section 6.28].

Lemma 3.4 *The Gaussian error function $\Phi : \mathbb{R} \to (0,1)$ satisfies the following integral equalities. We have*

$$\int_0^\infty s^i \Phi(-s)\,\mathrm{d}\,s = \begin{cases} \frac{1}{\sqrt{\pi}} & \textit{for } i = 0, \\ \frac{1}{2} & \textit{for } i = 1, \\ \frac{4}{3\sqrt{\pi}} & \textit{for } i = 2, \\ \frac{3}{2} & \textit{for } i = 3, \\ \frac{32}{5\sqrt{\pi}} & \textit{for } i = 4. \end{cases}$$

Moreover, we have

$$\int_0^\infty s^i \frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \,\mathrm{d}\, s = \begin{cases} \frac{1}{2} & \textit{for } i = 0, \\ \frac{1}{\sqrt{\pi}} & \textit{for } i = 1, \\ 1 & \textit{for } i = 2, \\ \frac{4}{\sqrt{\pi}} & \textit{for } i = 3. \end{cases}$$

We are now ready to determine the short-time asymptotics of the heat content (HC).

Proof of Theorem 3.2. Since the solution of the heat equation (HE) is given by $u_\Omega(y,t) = \int_\Omega \varrho(x,y,t)\omega(x)$, the heat content can be written as

$$\mathrm{HC}(t) = \int_{\Omega^c} \int_\Omega \varrho(x,y,t)\omega(x)\omega(y).$$

We start by separating the integral into the heat content near the boundary and the heat content away from the boundary, i.e.

$$\begin{aligned} \int_{\Omega^c} \int_\Omega \varrho(x,y,t)\omega(x)\omega(y) &= \int_{T(\partial\Omega,r)^+} \int_\Omega \varrho(x,y,t)\omega(x)\omega(y) \\ &\quad + \int_{\mathrm{M}^n \setminus T(\partial\Omega,r)^+} \int_\Omega \varrho(x,y,t)\omega(x)\omega(y), \end{aligned} \tag{89}$$

where $T(\partial\Omega,r)^+$ is a tube of radius r intersected with the complement of Ω. Then the distance of the integral domains in the second term is given by r. Hence, by the Davies inequality (see e.g. e.g. [Gri99, Theorem 3.2]),

$$\int_A \int_B \varrho(x,y,t)\omega(x)\omega(y) \leqslant \sqrt{|A|\;|B|}\mathrm{e}^{-\frac{d(x,y)^2}{4t}},$$

the second integral can be estimated from above by

$$\begin{aligned} &\int_{\mathrm{M}^n \setminus \Omega_{r^+}} \int_\Omega \varrho(x,y,t)\omega(x)\omega(y) \\ &\leqslant |\mathrm{M}^n| \exp\left(-\frac{d\left(\mathrm{M}^n \setminus \Omega_{r^+},\Omega\right)^2}{4t}\right) < \frac{64}{\sqrt{2}\mathrm{e}^2}\; |\mathrm{M}^n|\, \frac{t^2}{r^4}. \end{aligned}$$

By our choice of r the hypersurfaces $\partial\Omega_s$ are C^4 and the gradient of the distance function is equal to one for $|s| \leqslant r$. Therefore, the co-area formula then implies

$$\begin{aligned} \int_{\Omega_{r^+}} \int_\Omega \varrho(x,y,t)\omega(x)\omega(y) &= \int_0^r \int_{\partial\Omega_s} \frac{1}{\left|\bar\nabla d_\Omega(y)\right|} \int_\Omega \varrho(x,y,t)\omega(x)\mathrm{d}\mu_s(y)\,\mathrm{d}\, s \\ &= \int_0^r \int_{\partial\Omega_s} \int_\Omega \varrho(x,y,t)\omega(x)\mathrm{d}\mu_s(y)\,\mathrm{d}\, s. \end{aligned}$$

Here, $\mathrm{d}\mu_s$ is the induced volume form on the hypersurfaces $\partial\Omega_s$. Recall that by the asymptotic expansion of $\int_\Omega \varrho(x,y,t)\omega(x)$ for $d_\Omega(y) = s$, Corollary 2.6, we have

$$\Bigg| \int_\Omega \varrho(x,y,t)\omega(x) - \Bigg(\Phi\left(-\tfrac{s}{\sqrt{t}}\right) + \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \Bigg(-\sqrt{t} H_{\partial\Omega_s}(y)$$

$$+ \sqrt{t}s \left(\tfrac{1}{4}\left(-2\left(\bar{\mathrm{Ric}}(\nu_{\partial\Omega_s}(y), \nu_{\partial\Omega_s}(y)) + |A_{\partial\Omega_s}(y)|^2 \right) + H_{\partial\Omega_s}(y)^2 \right) \right) \Bigg) \Bigg) \Bigg|$$

$$\leqslant C(n) \tfrac{t^{3/2}}{r^3} \left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{s^2}{t} \right) \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}.$$

Hence, the first integral in (89) can be approximated by

$$\Bigg| \int_0^r \int_{\partial\Omega_s} \int_\Omega \varrho(x,y,t)\omega(x) \mathrm{d}\mu_s(y)\, \mathrm{d}\, s - \int_0^r \Bigg(\Phi\left(-\tfrac{s}{\sqrt{t}}\right) \mathrm{area}\,(\partial\Omega_s)$$

$$+ \sqrt{t} \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \left(\int_{\partial\Omega_s} -H_s \mathrm{d}\mu_s \right)$$

$$+ \tfrac{\sqrt{t}}{2} s \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \left(\int_{\partial\Omega_s} -\left(\bar{\mathrm{Ric}}(\nu_s, \nu_s) + |A_s|^2 \right) + \tfrac{1}{2} H_s^2 \mathrm{d}\mu_s \right) \Bigg) \mathrm{d}s \Bigg|$$

$$\leqslant C(n) \tfrac{t^{3/2}}{r^3} \mathrm{area}\,(\partial\Omega) \left(1 + \tfrac{|\Omega|}{r^n} \right) \int_0^r \left(1 + \tfrac{s^2}{t} \right) \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \mathrm{d}s. \tag{90}$$

Since the one-parameter family $\partial\Omega_s$ satisfies the initial value problem

$$\begin{cases} \frac{\partial F}{\partial s}(x,t) = \nu(x), & x \in \partial\Omega,\ t \in [0,r], \\ F(x,0) = F_0 & x \in \partial\Omega, \end{cases}$$

where $\nu(x,t)$ is the outer normal of the hypersurfaces $\partial\Omega_s$ and $F_0 : \partial\Omega \to (\mathrm{M}^n, g)$ is the inclusion mapping, the evolution equations of the geometric quantities, Theorem 1.9, and a Grönwall-type estimation then yield the following expansion formulae for the integrals in (90). It holds

$$\begin{aligned}
&\Bigg|\operatorname{area}(\partial\Omega_s) \\
&\quad - \left(\operatorname{area}(\partial\Omega) + s\int_{\partial\Omega} H \mathrm{d}\mu + \tfrac{s^2}{2}\int_{\partial\Omega} -\left(\bar{\operatorname{Ric}}(\nu,\nu) + |A|^2\right) + H^2\right)\mathrm{d}\mu\Bigg| \\
&= \left|\int_0^s \tfrac{(\tau-r)^2}{2}\tfrac{\partial}{\partial\tau}\left(\int_{\partial\Omega_\tau}\left(\left(\bar{\operatorname{Ric}}(\nu_\tau,\nu_\tau) + |A_\tau|^2\right) + H_\tau^2\right)\mathrm{d}\mu_\tau\right)\mathrm{d}\tau\right| \\
&\leqslant C(n)\tfrac{s^3}{r^3}\operatorname{area}(\partial\Omega), \\
&\left|\int_{\partial\Omega_s} -H\mathrm{d}\mu_s - \left(\int_{\partial\Omega} -H\mathrm{d}\mu + s\int_{\partial\Omega} -H^2 + \left(|A|^2 + \bar{\operatorname{Ric}}(\nu,\nu)\right)\right)\mathrm{d}\mu\right|, \\
&= \left|\int_0^s (s-\tau)\tfrac{\partial}{\partial\tau}\left(\int_{\partial\Omega_\tau} H^2\mathrm{d}\mu_\tau\right)\mathrm{d}\tau\right| \\
&\leqslant C(n)\tfrac{s^2}{r^3}\operatorname{area}(\partial\Omega), \\
&\Bigg|\tfrac{1}{2}\int_{\partial\Omega_s}\left(-\left(\bar{\operatorname{Ric}}(\nu_s,\nu_s) + |A_s|^2\right) + \tfrac{1}{2}H^2\right)\mathrm{d}\mu_s \\
&- \tfrac{1}{2}\int_{\partial\Omega}\left(-\left(\bar{\operatorname{Ric}}(\nu,\nu) + |A|^2\right) + \tfrac{1}{2}H^2\right)\mathrm{d}\mu\Bigg| \\
&= \left|\int_0^s \tfrac{\partial}{\partial\tau}\left(\int_{\partial\Omega_\tau}\left(2\left(\bar{\operatorname{Ric}}(\nu_\tau,\nu_\tau) + |A_\tau|^2\right) + H_\tau^2\right)\mathrm{d}\mu_\tau\right)\mathrm{d}\tau\right| \\
&\leqslant C(n)\tfrac{s}{r^3}\operatorname{area}(\partial\Omega).
\end{aligned}$$

Substituting the above expansions into (90), we have, after a change of variables,

$$\begin{aligned}
&\Bigg|\int_0^r\int_{\partial\Omega_s}\int_\Omega \varrho(x,y,t)\omega(x)\mathrm{d}\mu_s(y)\,\mathrm{d}\,s - \Bigg(\sqrt{t}\operatorname{area}(\partial\Omega)\int_0^{\frac{r}{\sqrt{t}}}\Phi(-s)\,\mathrm{d}s \\
&\quad + t\left(\int_{\partial\Omega} H\mathrm{d}\mu\right)\left(\int_0^{\frac{r}{\sqrt{t}}}\left(-\frac{\mathrm{e}^{-\frac{s^2}{4}}}{\sqrt{4\pi}} + s\Phi(-s)\right)\mathrm{d}s\right) \\
&\quad + t^{3/2}\left(\int_{\partial\Omega}\left(\bar{\operatorname{Ric}}(\nu,\nu) + |A|^2\right)\mathrm{d}\mu\right)\left(\tfrac{1}{2}\int_0^{\frac{r}{\sqrt{t}}}\left(-s^2\Phi(-s) + s\frac{\mathrm{e}^{-\frac{s^2}{4}}}{\sqrt{4\pi}}\right)\mathrm{d}s\right) \\
&\quad + t^{3/2}\left(\int_{\partial\Omega} H^2\mathrm{d}\mu\right)\left(\tfrac{1}{2}\int_0^{\frac{r}{\sqrt{t}}}\left(s^2\Phi(-s) - \tfrac{3}{2}s\frac{\mathrm{e}^{-\frac{s^2}{4}}}{\sqrt{4\pi}}\right)\mathrm{d}s\right)\Bigg)\Bigg| \\
&\leqslant C(n)\tfrac{t^{3/2}}{r^3}\left(1 + \tfrac{|\Omega|}{r^n}\right)\operatorname{area}(\partial\Omega)\int_0^r\left(\left(1 + \tfrac{s^2}{t}\right)\frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} + \tfrac{s^3}{t^{3/2}}\Phi\left(-\tfrac{s}{\sqrt{t}}\right)\right)\mathrm{d}s. \quad (91)
\end{aligned}$$

Performing another change of variables $s := s/t^{1/2}$, we see that the error term on the right-hand side can be bounded by

$$
\begin{aligned}
&C(n)\tfrac{t^{3/2}}{r^3}\left(1+\tfrac{|\Omega|}{r^n}\right)\operatorname{area}(\partial\Omega)\int_0^r\left(\left(1+\tfrac{s^2}{t}\right)\frac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}}+\tfrac{s^3}{t^{3/2}}\Phi\left(-\tfrac{s}{\sqrt{t}}\right)\right)\mathrm{d}s\\
&\leqslant C(n)\tfrac{t^2}{r^3}\left(1+\tfrac{|\Omega|}{r^n}\right)\operatorname{area}(\partial\Omega)\int_0^\infty\left(\left(1+s^2\right)\frac{\mathrm{e}^{-\frac{s^2}{4}}}{\sqrt{4\pi}}+s^3\Phi(-s)\right)\mathrm{d}s\\
&<10C(n)\tfrac{t^2}{r^3}\left(1+\tfrac{|\Omega|}{r^n}\right)\operatorname{area}(\partial\Omega).
\end{aligned}
$$

Furthermore, we have by Lemma 2.20, for $k\geqslant 0$ and some $C<\infty$

$$
\begin{aligned}
\int_{\frac{r}{\sqrt{t}}}^\infty\left(s^2+s+1\right)\left(\mathrm{e}^{-\frac{s^2}{4}}+\Phi(-s)\right)\mathrm{d}s &\leqslant\left(\max_{s\geqslant 0}\left(s^2+s+1\right)\mathrm{e}^{-\frac{s^2}{8}}\right)\int_{\frac{r}{\sqrt{t}}}^\infty \mathrm{e}^{-\frac{s^2}{8}}\\
&= C\sqrt{4\pi}\Phi\left(-\tfrac{r}{2\sqrt{t}}\right)\mathrm{d}s\\
&\leqslant C\sqrt{\pi}\mathrm{e}^{-\frac{r^2}{8t}}\leqslant C\sqrt{\pi}2^k\mathrm{e}^{-\frac{k}{2}}k^{k-2}\tfrac{t^{k/2}}{r^k}.
\end{aligned}
$$

Thus, we may integrate over all positive numbers in the (91) at the price of an error-term bounded by

$$
C(n)\left(1+\tfrac{|\Omega|}{r^n}\right)t^2r^{-3}\operatorname{area}(\partial\Omega).
$$

We complete the proof by using the integral formulae in Lemma 3.4. We have for some constant $C(n)$ only depending on n

$$
\begin{aligned}
&\Bigg|\int_{\Omega^c}\int_\Omega\varrho(x,y,t)\omega(x)\omega(y)-\left(\sqrt{t}\operatorname{area}\tfrac{1}{\sqrt{\pi}}+t\left(\int_{\partial\Omega}H\mathrm{d}\mu\right)0\right.\\
&\qquad\left.-\tfrac{t^{3/2}}{6\sqrt{\pi}}\left(\int_{\partial\Omega}\left(\bar{\mathrm{Ric}}(\nu,\nu)+|A|^2\right)\mathrm{d}\mu\right)-\tfrac{t^{3/2}}{12\sqrt{\pi}}\left(\int_{\partial\Omega}H^2\mathrm{d}\mu\right)\right)\Bigg|\\
&\leqslant C(n)t^2\left(\tfrac{1}{r^3}\left(1+\tfrac{|\Omega|}{r^n}\right)\operatorname{area}(\partial\Omega)+|\mathrm{M}^n|\tfrac{1}{r^4}\right).
\end{aligned}
$$

□

Remark 3.5 Similarly, it is possible to compute the entropy of sets, i.e. the quantity

$$
\mathrm{Ent}_\Omega(t):=\int_{\mathrm{M}^n}u_\Omega(y,t)\log u_\Omega(y,t)\omega(y). \tag{Ent}
$$

It turns out that, like in the pointwise asymptotic expansion of u_Ω and the approximation formula of the heat content (HC), the lowest order term in the expansion can be written in terms of the one-dimensional problem, Example 2.4. Here, the entropy is explicitly given by

$$
\mathrm{Ent}_{(0,\infty)}(t)=-\sqrt{t}\,\mathrm{Ent}_1,
$$

where the constant Ent_1 is the entropy of the Gaussian error function,

$$\mathrm{Ent}_1 := \int_{-\infty}^{\infty} \Phi(s) \log \Phi(s) \mathrm{d}s \approx 1.27731.$$

Theorem 3.6 (Entropy of sets) *For any $n \in \mathbb{N}$ there are constants $0 < \varepsilon(n), C(n) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a closed Riemannian manifold of class C^6 and $\Omega \subset \mathrm{M}^n$ a set such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := \min_{x \in \mathrm{M}^n} r_{18}(5, x) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor of the ambient space M^n, the first two derivatives of the second fundamental form of $\partial\Omega$ and the injectivity radius of M^n and Ω, is positive.

Then the entropy (Ent) of Ω satisfies for all $t \in (0, \varepsilon(n) r^2)$ the uniform estimate

$$\left| \mathrm{Ent}_\Omega(t) + \sqrt{t}\, \mathrm{area}\,(\partial\Omega)\, \mathrm{Ent}_1 \right| \leqslant C(n) \tfrac{t^{1/2}}{r}\, \mathrm{area}\,(\partial\Omega) \left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right).$$

Here, Ent_1 *is the entropy of the Gaussian error function.*

Proof. The contribution of entropy away from the boundary is exponentially small and may be estimated using the lower and upper bound on the heat kernel. Now fix a point $y \in \mathrm{M}^n$ in a yet to be defined tubular neighbourhood of the boundary of Ω. Define s by $y = \exp_{x_0} s\nu$, where $x_0 \in \partial\Omega$ is its unique base point. Using the pointwise approximation of u_Ω from Theorem 2.5, we note

$$\begin{aligned} &\left| (u_\Omega \log u_\Omega)_{|\partial\Omega_s} - \Phi\left(-\tfrac{s}{\sqrt{t}}\right) \log \Phi\left(-\tfrac{s}{\sqrt{t}}\right) \right| \\ &\leqslant C(n) \tfrac{t^{1/2}}{r} \left(1 + \tfrac{|\Omega|}{r^n}\right) \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \left(\sup_{\tau \in [u_\Omega(y,t), \Phi(-s/\sqrt{t})]} (1 + \log(\tau)) \right). \end{aligned}$$

We must show that the term on the right-hand side is integrable on $\mathbb{R}$, i.e.

$$\int_{-\infty}^{\infty} \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \left(\sup_{\tau \in [u_\Omega(y,t), \Phi(-s/\sqrt{t})]} (1 + \log(\tau)) \right) \mathrm{d}s \tag{92}$$

exists. The term in the bracket of Inequality (92) attains its maximimum in the minimum of the functions $\Phi\left(-s/\sqrt{t}\right)$ and $u_\Omega(y,t)$. Since the maximum of two measureable functions is measurable it remains to show that the functions

$$\begin{aligned} f(s) &:= \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \left((1 + \log(\Phi\left(-s/\sqrt{t}\right)))\right), \\ g(s) &:= \tfrac{\mathrm{e}^{-\frac{s^2}{4t}}}{\sqrt{4\pi}} \left(\left(1 + \log(u_\Omega(\exp_{x_0} s\nu, t))\right)\right) \end{aligned}$$

are integrable. The integrability of f follows from the lower bound of the Gaussian error function, Lemma 2.20. As for the integrability of g, we may assume without loss of generality that $s \geqslant 0$. By our choice of r, a ball B of radius $s/2$ about the point $\exp_{x_0}(-s\nu)$ is then contained in the set Ω. Its volume may be estimated from below in terms of the volume ω_n of Euclidean balls. Therefore, the Gaussian lower Gaussian bound of the heat kernel, Theorem 1.4, implies

$$\begin{aligned} u_\Omega(y,t) &\geqslant 2^{-1}\left(4\pi t\right)^{-n/2}\int_B \mathrm{e}^{-\frac{d(x,y)^2}{4t}}\omega(x) \\ &\geqslant \left(2^{-2n+2}\pi^{-n/2}\omega_n\right)\tfrac{s^n}{t^{n/2}}\mathrm{e}^{-\frac{s^2}{4t}}. \end{aligned}$$

Thus, after a change of coordinates $s \mapsto \tau/t^{1/2}$, we note that (92) is bounded from above by

$$\sqrt{t}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\frac{\tau^2}{4}}}{\sqrt{4\pi}}\left(\log\left(2^{-2n+2}\pi^{-n/2}\omega_n\right)+n\log\left(\tau\right)-\tfrac{\tau^2}{4}\right)\mathrm{d}\tau. \tag{93}$$

Since the above integrand tends only logarithmically to ∞ as $\tau \to 0$, we conclude that (92) is integrable. Analogously to the proof of Theorem 3.2, introducing tubular and a change of variables $s \mapsto \tau/t^{1/2}$ now imply

$$\begin{aligned} &\left|\int_{\mathrm{M}^n} u_\Omega(y,t)\log u_\Omega(y,t)\omega(y) + \sqrt{t}\,\mathrm{area}\left(\partial\Omega\right)\int_{-\infty}^{\infty}\Phi(s)\log\Phi(s)\mathrm{d}s.\right| \\ &\leqslant C(n)\tfrac{t^{1/2}}{r^2}\,\mathrm{area}\left(\partial\Omega\right)\left(1+\tfrac{|\Omega|}{r^n}\right). \end{aligned}$$

□

4 Evolution of level sets of solutions of the heat equation

In this chapter we always assume $(\mathrm{M}^n, \bar{\mathrm{g}})$ to be a closed Riemannian manifold. For a set $\Omega \subset \mathrm{M}^n$ with smooth boundary we define $u_\Omega : \mathrm{M}^n \times (0, \infty) \to \mathbb{R}$ as the solution of the heat equation

$$\begin{cases} \Delta_g u(x,t) = \frac{\partial}{\partial t} u(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n. \end{cases} \tag{HE}$$

We study the evolution of the level sets of u_Ω that are denoted by

$$\Sigma_\lambda(t) := \{u_\Omega(y,t) = \lambda\}, \quad \lambda \in (0,1),\ t > 0.$$

The resulting short-time expansion of the level sets, Theorem 4.3, is to be understood in terms of the *normal distance function*, see Definition 4.1. Note that we cannot infer regularity of the sets $\Sigma_\lambda(t)$. In fact, for any fixed time t, for λ very close to 0 and 1 the level sets are empty. For this reason, we will first make sure in Section 4.1, Lemma 4.5 that a constant $C(n) < \infty$ exists, such that for $\lambda \in (0,1)$ fixed, the normal distance function is well-defined on the time interval $(0, t_{94}(\lambda))$, where

$$t_{94}(\lambda) := \frac{r^2}{16\Big(\max\Big(-\log\lambda, -\log 1-\lambda, -\log\Big(C(n)^{-1}\frac{r^n}{|\mathrm{M}^n|}\Big)\Big)\Big)}. \tag{94}$$

Theorem 4.3 will then be proved in Section 4.2 by applying the pointwise estimates on u_Ω from Chapter 2. The error terms appearing in the proof will involve quantities of the normal distance function itself, namely its exponential. Therefore, we first need to bound it from above and below in Section 4.1 (see Lemma 4.6, respectively Lemma 4.7).

We begin this chapter by stating the necessary definitions to formulate its main result.

Definition 4.1 (Normal distance function) Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a closed Riemannian manifold of class C^6 and $\Omega \subset \mathrm{M}^n$ a set such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := \min r_{19}(\partial\Omega, 5)$$

that depends on the first six derivatives of the curvature tensor and the injectivity radius of M^n and Ω as defined in Section 1.3, is positive. Fix $\lambda \in (0,1)$ let $t_{94}(\lambda) > 0$ be defined as above. Then, the mapping

$$F_{\lambda,t,\Omega} : \Sigma_\lambda(t) \to \partial\Omega \times (-r, r), \quad y \mapsto (x_0, s) \text{ such that } \exp_{x_0} s\nu(x_0) = y.$$

is well-defined. The *normal distance function* is then defined by

$$d_{\lambda,t}(y) := (\pi_2 \circ F_{\lambda,t,\Omega})(y), \tag{95}$$

where π_2 is the projection onto the second component.

Remark 4.2 (The one-dimensional case) First, we consider the Euclidean space. By Example 2.4 the solution of the heat equation (HE) with initial data being the characteristic function of the half-space $\Omega := (-\infty, 0)$ is given by

$$u_\Omega(y,t) = \Phi\left(-\tfrac{y}{\sqrt{t}}\right),$$

where Φ is the Gaussian error function (compare Example 2.4). Solving for $\lambda = u_\Omega(y,t)$ we therefore have

$$d_{\lambda,t}(y) = -\Phi^{-1}(\lambda)\sqrt{t} =: c_\lambda\sqrt{t}.$$

In particular, the normal distance function is explicitly given in terms of the inverse of the Gaussian error function which we will denote by $c_\lambda : (0,1) \to \mathbb{R}$. The following theorem will show that, regardless of the geometry of the ambient space M^n and the set Ω, the leading term in the expansion of the normal distance function is the same as in the one-dimensional case.

The following theorem is to be understood in the sense that it is possible to „squeeze“ the level-sets $\Sigma_\lambda(t)$ between two sufficiently regular hypersurfaces $\Sigma_\lambda(t)^+$ and $\Sigma_\lambda(t)^-$ satisfying

$$d\left(\Sigma_\lambda(t)^+, \Sigma_\lambda(t)^-\right) = O(t^{2-\alpha})$$

with $\alpha > 0$ arbitrarily small.

Theorem 4.3 (Distance to the level-sets) *For any $n \in \mathbb{N}$ and $\alpha > 0$ there are constants $0 < \varepsilon(n), C(n,\alpha) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a closed Riemannian manifold of class C^6 and $\Omega \subset \mathrm{M}^n$ a set such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := \min_{x \in \mathrm{M}^n} r_{18}(5, x) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor of the ambient space M^n, the first two derivatives of the second fundamental form of $\partial\Omega$ and the injectivity radius of M^n and Ω, is positive. Furthermore, let $0 < t_{94}(\lambda)$ be defines as in 4.1;

$$t_{94}(\lambda) := \frac{r^2}{16\left(\max\left(-\log\lambda, -\log 1-\lambda, -\log\left(\varepsilon(n)\frac{r^n}{|\mathrm{M}^n|}\right)\right)\right)}.$$

Then, for $y \in \Sigma_\lambda(t)$ *with* $y = \exp_{x_0} d_{\lambda,t}(y)\nu(x_0)$ *we set*

$$\tilde{d}_{\lambda,t}(y) := -c_\lambda\sqrt{t} - tH(x_0) - t^{3/2}\tfrac{c_\lambda}{2}\left(\left(\mathrm{Ric}(x_0)(\nu(x_0),\nu(x_0)) + |A|(x_0)^2\right)\right),$$

where the curvature terms are evaluated with respect to the boundary $\partial\Omega$ *and* c_λ *is the inverse Gaussian error function. Then we have the uniform estimate*

$$\begin{aligned}
&\left|d_{\lambda,t}(y) - \tilde{d}_{\lambda,t}(y)\right| \\
&\leqslant C(n,\alpha)\tfrac{t^2}{r^3}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{|\mathrm{M}^n|}{r^n}\right)^3 \lambda^{-\alpha}\exp\left(C(n,\alpha)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right)\right) \\
&:= C_{96}(\mathrm{M}^n,\lambda)\tfrac{t^2}{r^3} \qquad (96)
\end{aligned}$$

for all $t \in (0, t_{94}(\lambda))$.

Remark 4.4 (*i*) The proof of this theorem is a modification of the proof of Theorem 19 in [Pre06]. A preliminary result of Estimate (96) was proved in [Bar19, Theorem 2.1.6]. However, these results do not consider the λ-dependence of the error term. It is important to note that the exponent $\alpha > 0$ in Estimate (96) is arbitrary. In particular, for any power $k \in \mathbb{N}$ of the error term, a sufficiently small $\alpha(k) > 0$ can be chosen such that it is integrable on the interval $(0,1)$.

(*ii*) In the special case of $(\mathrm{M}^n, \bar{\mathrm{g}}) = (\mathbb{R}^n, \delta_{ij})$, the coefficient of order $O(t^{3/2})$ in Estimate (96) differs from the corresponding coefficient in [Pre06],

$$\tfrac{c_\lambda}{2}|A|^2 \neq 0 \quad \text{for } \lambda \neq \tfrac{1}{2}.$$

However, in Example 5.3 we prove that the Formula (96) is consistent with Example 3.1 and Theorem 3.2.

(*iii*) One interpretation of the above theorem is that the superlevel sets $\Omega_\lambda(t)$ of $u_\Omega(y,t)$ can be approximated by certain barrier sets,

$$\Omega_\lambda(t)^- \subset \Omega_\lambda(t) = \{u_\Omega(y,t) \geqslant \lambda\} \subset \Omega_\lambda(t)^+,$$

where $\Omega_\lambda(t)^\pm := \bar{F}^\pm_{\lambda,t}(\Omega)$ and the $\bar{F}^\pm_{\lambda,t} : \mathrm{M}^n \to \mathrm{M}^n$ are families of diffeomorphisms that are defined on the boundary of Ω by

$$\bar{F}^\pm_{\lambda,t}(\Omega)_{|\partial\Omega}(x_0) := \exp_{x_0}\left(\left(-c_\lambda\sqrt{t} - tH - t^{3/2}\tfrac{c_\lambda}{2}\left(\left(\mathrm{Ric}(\nu,\nu) + |A|^2\right)\right) \pm C_{96}(\mathrm{M}^n,\lambda)\tfrac{t^2}{r^3}\right)\nu\right).$$

In Chapter 5, this will be applied to derive the asymptotics of the *mean-heat content*;

$$\mathrm{MHC}_\Omega(t) := \int_{\{u_\Omega(\cdot,t)\leqslant 1/2\}} u_\Omega(y,t)\omega(x).$$

4.1 Bounds on the distance to level sets

In this section, we derive upper and lower bound of the distance between the level-sets $\Sigma_\lambda(t)$ and the boundary of Ω. We set

$$d_\lambda(t) := \begin{cases} \sup_{y\in\Sigma_\lambda(t)} \inf_{x\in\partial\Omega} d(x,y) & \text{if } y \in \Omega^c, \\ -\sup_{y\in\Sigma_\lambda(t)} \inf_{x\in\partial\Omega} d(x,y) & \text{if } y \in \Omega, \end{cases}$$

The estimates will be in terms of $\lambda \in (0,1)$ and $t > 0$. They will be used in the subsequent section to derive a pointwise asymptotic expansion of the distance $d_\Omega(y)$ for any $y \in \Sigma_\lambda(t)$.

First, we note that for points not contained in a tube of radius $\mathrm{inj}_{\partial\Omega}$ around the boundary of Ω it is not necessarily possible to find a unique point $x_0 \in \partial\Omega$ such that $y = \exp_{x_0} d_\Omega(y)\nu(x_0)$. For this reason, we apply the Gaussian upper bound of heat kernels to estimate the value of solutions of (HE) away from the boundary.

Lemma 4.5 *For any $n \in \mathbb{N}$ there is a constant $0 < C(n) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a compact Riemannian manifold with $\mathrm{Ric} > -K$ and $u_\Omega : \mathrm{M}^n \times (0,\infty) \to \mathbb{R}$ the solution of the heat equation (HE). Then we have

$$\Omega^c \backslash T(\partial\Omega, r) \subset \{u_\Omega(\cdot,t) \leqslant C(n)\, |\mathrm{M}^n|\, r^{-n} \mathrm{e}^{-\frac{r^2}{8t}}\}$$
$$\Omega \backslash T(\partial\Omega, r) \subset \{u_\Omega(\cdot,t) \geqslant 1 - C(n)\, |\mathrm{M}^n|\, r^{-n} \mathrm{e}^{-\frac{r^2}{8t}}\}.$$

Furthermore, the level sets $\Sigma_\lambda(t)$ are contained in a tube of radius r around $\partial\Omega$ for all times

$$0 < t \leqslant t_{97}(\lambda) := \frac{r^2}{16\Big(\max\Big(-\log\lambda, -\log(1-\lambda), \log\Big(C(n)\frac{|\mathrm{M}^n|}{r^n}\Big)\Big)\Big)}. \tag{97}$$

Proof. For $y \in \Omega^c \cap \Sigma_\lambda(t)$ with $r > d_\Omega(y)$ the Gaussian upper bound of the heat kernel (14) implies for $\eta = \sqrt{2}$

$$\lambda = \int_\Omega \varrho(x,y,t)\omega(x) \leqslant C(n) \int_{\mathrm{M}^n\backslash B_r(y)} \frac{\mathrm{e}^{-\frac{d(x,y)^2}{\sqrt{2}4t}}}{(4\pi t)^{n/2}} \leqslant C(n)\frac{|\mathrm{M}^n|}{r^n}\mathrm{e}^{-\frac{r^2}{8t}}.$$

For the second part, suppose there is some $y \in \Sigma_\lambda(t)$ with $d_\Omega(y) > r$. Then applying the logarithm to the above inequality yields

$$t \geqslant \frac{r^2}{8\Big(-\log\lambda - \log\Big(C(n)^{-1}\frac{r^n}{|\mathrm{M}^n|}\Big)\Big)}.$$ □

We now bound $d_\lambda(t)$ from above for points close to the boundary.

Lemma 4.6 *For any $n \in \mathbb{N}$ there is a constant $C(n) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a closed Riemannian manifold such that

$$r := \min_{x \in \mathrm{M}^n} r_{17}(5, x) > 0.$$

Fix $\lambda \in (0, 1/2)$, set

$$t_{97}(\lambda) := \frac{r^2}{16\Big(\max\Big(-\log\lambda, -\log(1-\lambda), -\log\Big(C(n)^{-1}\frac{r^n}{|\mathrm{M}^n|}\Big)\Big)\Big)}$$

and define

$$f(\lambda) := \sqrt{-\log(\lambda) + \log\left((-\log(\lambda))^{n/4+2}\right)}.$$

Then we have

$$d_\lambda(t) \leqslant 2\sqrt{t}\max\left(f(\lambda), C(n)\sqrt{1 + \tfrac{|\mathrm{M}^n|}{r^n}}\right)$$

for all $t \in (0, t_{97}(\lambda))$. Similarly, we have for $\lambda \in (1/2, 1)$

$$|d_\lambda(t)| \leqslant 2\sqrt{t}\max\left(f(1-\lambda), C(n)\sqrt{1 + \tfrac{|\mathrm{M}^n|}{r^n}}\right).$$

Proof. We first consider the Euclidean case $(\mathbb{R}^n, \delta_{\alpha\beta})$. Suppose for some $y \in \Sigma_\lambda(t) \cap \Omega^c$ there is some (yet to be defined) function

$$f : (0,1) \to \mathbb{R}_+, \quad \text{with } f \geqslant 2 \quad \text{and } d_\Omega(y) > 2\sqrt{t}\sqrt{f(\lambda)}.$$

In particular, the ball $B_{2\sqrt{t f(\lambda)}}(y)$ is contained in the complement of Ω. Therefore, we may estimate

$$\begin{aligned}
\lambda = \int_\Omega \frac{\mathrm{e}^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{n/2}}\mathrm{d}x &\leqslant \int_{\mathbb{R}^n \setminus B_{2\sqrt{t f(\lambda)}}(y)} \frac{\mathrm{e}^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\mathrm{d}x \\
&\leqslant \sum_{k=1}^{\infty} \left|B_{2(k+1)\sqrt{t f(\lambda)}}(0)\right| \frac{\mathrm{e}^{-k^2 f(\lambda)}}{(4\pi t)^{n/2}} \\
&\leqslant \omega_n \frac{f(\lambda)^{n/2}}{(\pi)^{n/2}} \sum_{k=1}^{\infty} (k+1)^n \,\mathrm{e}^{-k^2 f(\lambda)}, \qquad (98)
\end{aligned}$$

where ω_n is the volume of the n-dimensional unit ball. Since by assumption $f(\lambda) \geqslant 2$, we have

$$(k+1)^n \,\mathrm{e}^{-k^2 f(\lambda)} \leqslant \mathrm{e}^{-k f(\lambda)} \sup_{x \geqslant 0}\left(\mathrm{e}^{-2x}\,(k+1)^n\right) = (n/2)^n \,\mathrm{e}^{-n+2}\,\mathrm{e}^{-k f(\lambda)}$$

for all $k \geqslant 2$. Thus, the sum on the right-hand side of (98) is dominated by the geometric series. We conclude, for some $C_{99}(n) < \infty$,

$$\lambda \leqslant C_{99}(n) f(\lambda)^{n/2} \mathrm{e}^{-f(\lambda)} \,. \tag{99}$$

We define f by

$$f(\lambda) := -\log\lambda + \log\left((-\log\lambda)^{n/4+2}\right).$$

Then Inequality (99) becomes

$$\lambda \leqslant \left(C_{99}(n)\frac{\left(-\log\lambda+\log\left((-\log\lambda)^{n/4+2}\right)\right)^{n/2}}{(-\log\lambda)^{n/4+1}}\right)\frac{\lambda}{-\log\lambda}.$$

The bracket on the right-hand side is uniformly bounded by some constant $C_{100}(n) < \infty$ for $\lambda \in (0,1)$. Therefore,

$$\lambda \leqslant \exp(-2C_{100}(n)) \tag{100}$$

implies the contradiction $\lambda \leqslant \lambda/2$. Hence,

$$d_\Omega(y) \leqslant 2\sqrt{t}\left(-\log\lambda + \log\left((-\log\lambda)^{n/4+2}\right)\right) \qquad \text{for } \lambda \leqslant \exp(-2C_{100}(n)).$$

On the other hand, if we define f by

$$f(\lambda) := \sqrt{2C_{100}(n) + \log\left(4C_{100}(n)\right)^{n/4}} := C_{101}(n) \tag{101}$$

Inequality (99) also implies

$$\lambda \leqslant \exp(-2C_{100}(n)).$$

Thus, we may estimate the distance of the point y to the boundary by the maximum of the functions f in (17) and (101), i.e.

$$d_\Omega(y) \leqslant 2\sqrt{t}\max\left(\left(-\log\lambda + \log\left((-\log\lambda)^{n/4+2}\right)\right), C_{101}(n)\right). \tag{102}$$

Due to the stochastic completeness of $(\mathbb{R}^n, \delta_{ij})$ similar arguments do also imply

$$|d_\Omega(y)| \leqslant 2\sqrt{t}\max\left(\left(\left(-\log(1-\lambda) + \log\left((-\log(1-\lambda))^{n/4+2}\right)\right)\right), C_{101}(n)\right)$$

for $y \in \Omega$.

We now consider the case of $(\mathrm{M}^n, \bar{\mathrm{g}})$ being a compact Riemannian manifold. For $y \in \mathrm{M}^n$ with $u_\Omega(y,t) = \lambda$ we have

$$\lambda = \int_\Omega \varrho(x,y,t)\omega(x) \leqslant \int_{B_r(y)\cap\Omega} \varrho(x,y,t)\omega(x) + \int_{\mathrm{M}^n\setminus B_r(y)} \varrho(x,y,t)\omega(x). \tag{103}$$

We estimate the second integral using the upper Gaussian bound of the heat kernel (14);

$$\int_{\mathrm{M}^n\setminus B_r(y)\cap\Omega} \varrho(x,y,t)\omega(x) \leqslant C(n)\tfrac{|\mathrm{M}^n|}{r^n}\mathrm{e}^{-\frac{d_\lambda(t)^2}{4t}}.$$

Since $t \leqslant t_{97}(\lambda)$ by assumption, Lemma 4.5 implies $y \in T(\partial\Omega, r)$. Hence, we can apply Theorem 2.7 with $k = 0$ to estimate the first integral in (103). We have

$$\int_{B_r(y)\cap\Omega} \varrho(x,y,t) \leqslant C(n)\left(\int_{B_r(y)\cap\Omega} (4\pi t)^{-n/2}\,\mathrm{e}^{-\frac{d(x,y)^2}{4t}}\omega(x) + \tfrac{|\Omega|}{r^n}\mathrm{e}^{-\frac{d_\lambda(t)^2}{4t}}\right).$$

By our choice of r, we have $\sqrt{\det g} \leqslant 2$, so that, in a normal coordinate system in y, the integral on the right-hand side is bounded by

$$\begin{aligned}\int_{B_r(y)\cap\Omega} (4\pi t)^{-n/2}\,\mathrm{e}^{-\frac{d(x,y)^2}{4t}}\omega(x) &= \int_{\exp_y^{-1}(B_r(y)\cap\Omega)} \frac{\mathrm{e}^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\sqrt{\det g(x)}\mathrm{d}x \\ &\leqslant 2\int_{\exp_y^{-1}(B_r(y)\cap\Omega)} \frac{\mathrm{e}^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\mathrm{d}x.\end{aligned}$$

Like in the Euclidean case (98) assuming

$$d_\lambda(t) \geqslant -\log\lambda + \log\left((-\log\lambda)^{n/4+2}\right)$$

yields the estimation

$$\begin{aligned}&2\int_{\exp_y^{-1}(B_r(y)\cap\Omega)} \frac{\mathrm{e}^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\mathrm{d}x \\ &\leqslant \left(2C_{99}(n)\tfrac{\left(-\log\lambda+\log\left((-\log\lambda)^{n/4+2}\right)\right)^{n/2}}{(-\log\lambda)^{n/4+1}}\right)\tfrac{\lambda}{-\log\lambda}.\end{aligned}$$

Hence, for some $C(n) < \infty$ depending only on n,

$$\lambda \leqslant \tfrac{\lambda}{-\log\lambda}C(n)\sqrt{1+\tfrac{|\mathrm{M}^n|}{r^n}},$$

which is a contradiction for

$$\lambda \leqslant \exp\left(-C(n)\left(1+\tfrac{|\mathrm{M}^n|}{r^n}\right)\right) := \varepsilon_{104}(n,\mathrm{M}^n,r). \tag{104}$$

To estimate $d_\lambda(t)$ for $\lambda > \varepsilon_{104}(n,\mathrm{M}^n)$ observe, that a constant $C(n)$ may be chosen such that

$$f(\lambda) := C_{105}(n)\left(1+\tfrac{|\mathrm{M}^n|}{r^n}\right) \tag{105}$$

yields a contradiction for $\lambda > \varepsilon_{104}(n, \mathrm{M}^n)$. We conclude,

$$d_\lambda(t) \leqslant 2\sqrt{t} \max\left(C_{105}(n)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right), f(\lambda)\right).$$

Due to the stochastic completeness of M^n, we have a similar bound for $y \in \Omega$;

$$|d_\lambda(t)| \leqslant 2\sqrt{t} \max\left(C_{105}(n)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right), f(1-\lambda)\right). \qquad \square$$

We also need the following lower bound of $d_\lambda(t)$.

Lemma 4.7 *For any $n \in \mathbb{N}$ there are constants $0 < \varepsilon(n), C(n) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$, and $r > 0$ be as in the Lemma 4.6. We define functions f, g in the level λ by

$$f(\lambda) := -\log\lambda \quad \textit{and} \quad g(\lambda) := \sqrt{(\log(-\log\lambda))}.$$

Fix $\lambda \in (0, \varepsilon(n))$ and set

$$0 < t < t_{97}(\lambda) := \frac{r^2}{16\left(\max\left(-\log\lambda, -\log 1-\lambda, -\log\left(C(n)^{-1}\frac{r^n}{|\mathrm{M}^n|}\right)\right)\right)}.$$

Then the distance of the level-set $\Sigma_\lambda(t)$ to the boundary of Ω can be bounded from below by

$$d_\lambda(t) \geqslant \sqrt{4tf(\lambda)} - 2\sqrt{t}g(\lambda).$$

Similarly we have for $\lambda \in (1-\varepsilon(n), 1)$

$$d_\lambda(t) \leqslant -2\sqrt{t}\left(\sqrt{f(1-\lambda)} + g(1-\lambda)\right).$$

Proof. We first consider the Euclidean case $(\mathbb{R}^n, \delta_{\alpha\beta})$. We set $y = x_0 + d_\Omega(y)\nu \in \Omega^c \cap \Sigma_\lambda(t)$. Suppose that it holds

$$d_\Omega(y) < \sqrt{4tf(\lambda)} - 2\sqrt{t}g(\lambda).$$

In particular,

$$\lambda = \int_\Omega \frac{\mathrm{e}^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{n/2}} \mathrm{d}x \geqslant \int_{\Omega \cap B_{\sqrt{4tf(\lambda)}}(y)} \frac{\mathrm{e}^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{n/2}} \mathrm{d}x.$$

By definition of $f(\lambda)$, any $x \in B_{\sqrt{4tf(\lambda)}}(y)$ satisfies

$$\mathrm{e}^{-\frac{|x-y|^2}{4t}} \geqslant \mathrm{e}^{-f(\lambda)} = \lambda.$$

On the other hand, since r is smaller than the injectivity radius of the boundary of Ω, a ball of radius $\sqrt{t}g(\lambda)$ about the point $z := \exp_{x_0} -2g\,(\lambda)\,\nu$ is contained in Ω. In particular, for ω_n being the volume of a unit ball, it holds

$$\left|\Omega \cap B_{2\sqrt{t}f(\lambda)}(y)\right| \geqslant \left|\Omega \cap B_{\sqrt{t}g(\lambda)}(z)\right| \geqslant \omega_n t^{n/2}\,(g(\lambda))^n\,.$$

By definition of g we then have

$$\lambda \geqslant \min_{x \in B_{2\sqrt{t}f(\lambda)}(y)} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \frac{\left|\Omega \cap B_{2\sqrt{t}f(\lambda)}(y)\right|}{(4\pi t)^{n/2}} \geqslant \lambda\,(\log\,(-\log\lambda))^{n/2}\,\frac{\omega_n}{(4\pi)^{n/2}}.$$

The above inequality is a contradiction for

$$\lambda \leqslant \exp\left(-\exp\left(\frac{4\pi}{(\omega_n)^{2/n}}\right)\right) := \varepsilon_{106}(n). \tag{106}$$

Due to the stochastic completeness of the Euclidean space, similarly one concludes

$$d_\Omega(y) \geqslant 2\sqrt{t}\sqrt{f(1-\lambda)} - 2\sqrt{t}g(\lambda),$$

for $\lambda \in (1-\varepsilon_{106}(n), 1)$.

In the case where $(\mathrm{M}^n, \bar{\mathrm{g}})$ is a closed Riemannian manifold, we recall that the Gaussian upper bound of the heat kernel, Theorem 13, implies

$$\int_\Omega \varrho(x,y,t)\omega(x) \geqslant \tfrac{1}{3}\int_{\Omega \cap B_{2r}(y)} (4\pi t)^{-n/2}\,\mathrm{e}^{-\frac{d(x,y)^2}{4t}}\omega(x). \tag{107}$$

By our choice of r we may assume that in a normal coordinates system in y, $\sqrt{\det g(x)} \geqslant 1/2$ uniformly in $B_r(y)$. Then, as in the Euclidean case,

$$d_\Omega(y) < 2\sqrt{t}\sqrt{(-\log\lambda)} - -2\sqrt{t}\sqrt{\log\,(-\log\lambda)} := 2\sqrt{t}\sqrt{(f\,(\lambda) - g\,(\lambda))}$$

implies

$$\lambda \geqslant \int_{\Omega \cap B_{2r}(y)} (4\pi t)^{-n/2}\,\mathrm{e}^{-\frac{d(x,y)^2}{4t}}\omega(x) \geqslant \tfrac{1}{6}\lambda\,(\log\,(-\log\lambda))^{n/2}\,\frac{\omega_n}{(4\pi)^{n/2}}.$$

This, on the other hand, is a contradiction for

$$\lambda \leqslant \exp\left(-\exp\left(\frac{4\pi}{(6\omega_n)^{2/n}}\right)\right) := \varepsilon_{108}(n). \tag{108}$$

□

4.2 Proof of Theorem 4.3

Before we prove Theorem 4.3, we need the following lemma regarding the growth of the inverse of the Gaussian error function. It can be understood as a special case of the lower and upper bounds of the normal distance function from the previous section.

Lemma 4.8 *Let $\Phi : \mathbb{R} \to (0,1)$ be the Gaussian error function and $c_\lambda : (0,1) \to \mathbb{R}$ its inverse. Then c_λ can be bounded from above and below in terms of the logarithm. There is some some $C_1 < \infty$ such that for $\lambda \in (0, 1/2)$*

$$-4\log\lambda - 4\log\left(\sqrt{\log\lambda}\right) - C_1 \leqslant c_\lambda^2 \leqslant -4\log\lambda. \tag{109}$$

Proof. By Lemma 2.20

$$\lambda = \Phi(c_\lambda) \leqslant \mathrm{e}^{-\frac{c_\lambda^2}{4}}.$$

Applying the logarithm to both sides and multiplying by -4 gives the upper bound. For the lower bound, applying the logarithm to (68) yields

$$\log\lambda \geqslant -\tfrac{c_\lambda^2}{4} - \left(\log\left(\tfrac{|c_\lambda|}{2} + \sqrt{2 + \tfrac{c_\lambda^2}{4}}\right)\right) - \log\sqrt{\pi}.$$

On the other hand, since the term in the logarithm is strictly bigger than 2, the log sum inequality and the upper bound on c_λ imply that the lower bound on c_λ holds for $C_1 := 24\log\sqrt{2} + 4\log\sqrt{\pi}$;

$$\begin{aligned}\left(\log\left(\tfrac{|c_\lambda|}{2} + \sqrt{2 + \tfrac{c_\lambda^2}{4}}\right)\right) &\leqslant \log\left(|c_\lambda| + \sqrt{2}\right)\\ &\leqslant \tfrac{|c_\lambda|}{|c_\lambda|+\sqrt{2}}\log 2\,|c_\lambda| + \tfrac{\sqrt{2}}{|c_\lambda|+\sqrt{2}}\log\sqrt{2}2\\ &\leqslant \log\sqrt{-\log\lambda} + 6\log\sqrt{2}.\end{aligned}$$

□

We are now ready to prove Theorem 4.3. For this purpose, we will

- use the estimates from the previous section to determine the expansion of $d_{\lambda,t}(y)$ up to order $O\left(\sqrt{t}\right)$,
- recursively compute the coefficients of order $O\left(t\right)$ and so forth.

Proof of Theorem 4.3. **Expansion formula of order** $O\left(\sqrt{t}\right)$**.**

As in Theorem 2.5 we set

$$\begin{aligned} v_\Omega(y,t) &:= \Phi\left(-\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right) \\ &\quad + \frac{\mathrm{e}^{-\frac{d_{\lambda,t}(y)^2}{4t}}}{\sqrt{4\pi}}\left(-\sqrt{t}H + \sqrt{t}d_{\lambda,t}(y)\left(\tfrac{1}{4}\left(2\left(\mathrm{Ric}(\nu,\nu)+|A|^2\right)+H^2\right)\right)\right), \end{aligned}$$

where the curvature terms are evaluated in $x_0 \in \partial\Omega$ defined by $\exp_{x_0} d_{\lambda,t}(y)\nu = y$. By assumption, Theorem 2.5 implies that $v_\Omega(y,t)$ satisfies

$$|u_\Omega(y,t) - v_\Omega(y,t)| \leqslant C(n)\tfrac{t^{3/2}}{r^3}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{d_{\lambda,t}(y)^2}{t}\right)\frac{\mathrm{e}^{-\frac{d_\Omega(y)^2}{4t}}}{\sqrt{4\pi}}.$$

In particular, since $y \in \Sigma_\lambda(t)$

$$\begin{aligned} \left|\lambda - \Phi\left(-\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)\right| &= \left|u_\Omega(y,t) - \Phi\left(-\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)\right| \\ &\leqslant C(n)\tfrac{\sqrt{t}}{r}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{d_{\lambda,t}(y)^2}{t}\right)\frac{\mathrm{e}^{-\frac{d_{\lambda,t}(y)^2}{4t}}}{\sqrt{4\pi}}. \end{aligned} \tag{110}$$

On the other hand, by definition of the Gaussian error function

$$\lambda = \int_{-\infty}^{c_\lambda} \frac{\mathrm{e}^{-x^2/4}}{\sqrt{4\pi}} = \Phi(c_\lambda).$$

Inequality (110) now implies that $\lim_{t\to 0}|-d_{\lambda,t}(y) - \sqrt{t}c_\lambda| = 0$. To get a more useful estimate, we first observe that (110) yields the estimation

$$\begin{aligned} &\left(\min_{s\in\left[c_\lambda, \frac{-d_{\lambda,t}(y)}{\sqrt{t}}\right]} \frac{\mathrm{e}^{-\frac{s^2}{4}}}{\sqrt{4\pi}}\right)\left|-d_{\lambda,t}(y) - \sqrt{t}c_\lambda\right| \\ &\leqslant C(n)\tfrac{t}{r}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{d_{\lambda,t}(y)^2}{t}\right)\frac{\mathrm{e}^{-\frac{d_{\lambda,t}(y)^2}{4t}}}{\sqrt{4\pi}}. \end{aligned}$$

In order to estimate the error term

$$\mathrm{e}^{-\frac{d_{\lambda,t}(y)^2}{4t}} \max_{s\in\left[c_\lambda, \frac{-d_{\lambda,t}(y)}{\sqrt{t}}\right]} \mathrm{e}^{\frac{s^2}{4}} \tag{111}$$

in the above inequality, we need to consider the following four cases.

Case 1 : $\operatorname{sign} c_\lambda = 1 = \operatorname{sign}\left(d_{\lambda,t}(y)\right)$,

Case 2 : $\operatorname{sign} c_\lambda = 1$ and $\operatorname{sign}(-d_{\lambda,t}(y)) = 1$,

Case 3 : $\operatorname{sign} c_\lambda = -1$ and $\operatorname{sign}(-d_{\lambda,t}(y)) = -1$,

Case 4 : $\operatorname{sign} c_\lambda = 1$ and $\operatorname{sign}(-d_{\lambda,t}(y)) = 1$.

The cases **2** and **4** are proved analogously to the cases **1**, respectively **3**. Hence, it suffices to prove case **1** and case **3**.

To case 1: By definition of c_λ we have $\lambda \geqslant 1/2$ and $y \in \Omega^c$. In particular,

$$d_{\lambda,t}(y) \geqslant 0 > -2\sqrt{t}\sqrt{(-\log(1-\lambda) - \log(-\log(1-\lambda)))}.$$

Hence, Lemma 4.7 implies

$$\lambda < 1 - \varepsilon_{108}(n) = 1 - \exp\left(-\exp\left(\frac{4\pi}{(6\omega_n)^{2/n}}\right)\right).$$

Due to the monotonicity of the inverse of the Gaussian error function on $(1/2, 1)$, we may bound (111) uniformly by

$$C_{112}(n) := \exp\left(\frac{1}{4}\Phi\left(1 - \exp\left(-\exp\left(\frac{4\pi}{(6\omega_n)^{2/n}}\right)\right)\right)\right). \tag{112}$$

To case 3: By assumption $\lambda \leqslant 1/2$ and $y \in \Omega^c$. Since the exponential function is monotone on for $x \leqslant 0$, (111) may be computed as

$$\max_{s \in \left[c_\lambda, \frac{-d_{\lambda,t}(y)}{\sqrt{t}}\right]} \left(\frac{e^{\frac{s^2}{4}}}{\sqrt{4\pi}}\right) \frac{e^{\frac{d_{\lambda,t}(y)^2}{4t}}}{\sqrt{4\pi}} = \max\left(\frac{e^{\frac{c_\lambda^2}{4}}}{\sqrt{4\pi}}, \frac{e^{-\frac{d_{\lambda,t}(y)^2}{4t}}}{\sqrt{4\pi}}\right) \frac{e^{-\frac{d_{\lambda,t}(y)^2}{4t}}}{\sqrt{4\pi}}.$$

If $d_{\lambda,t}(y)^2 \geqslant c_\lambda^2$, the above term is given by 1. If $d_{\lambda,t}(y)^2 \leqslant c_\lambda^2$, we recall that by Lemma 4.8

$$c_\lambda^2 \leqslant \max(-4\log\lambda, -4\log(1-\lambda)).$$

On the other hand, by Lemma 4.7

$$d_{\lambda,t}(y)^2 \geqslant 4t\left(-\log\lambda - \sqrt{-\log\lambda}\log(-\log\lambda)\right) \quad \text{for all } \varepsilon_{108}(n) > \lambda.$$

Therefore, for any $\alpha > 0$, a constant can be chosen such that

$$\exp\left(\frac{1}{4}\left(c_\lambda^2 - d_{\lambda,t}(y)^2\right)\right) \leqslant \max C(\alpha, n)\lambda^{-\alpha}.$$

Hence, Inequality (4) and the upper bound on $d_{\lambda,t}(y)$, Lemma 4.6 imply

$$\begin{aligned} \left|-d_{\lambda,t}(y) - \sqrt{t}c_\lambda\right| &\leqslant C(\alpha, n)\frac{t}{r}\left(1 + \frac{|\Omega|}{r^n} + \frac{d_{\lambda,t}(y)^2}{t}\right)\lambda^{-\alpha}. \\ &\leqslant C(\alpha, n)\frac{t}{r}\left(1 + \frac{|\Omega|}{r^n} + \frac{|\mathrm{M}^n|}{r^n}\right)\lambda^{-2\alpha}. \end{aligned} \tag{113}$$

Higher order Expansion formula of $d_{\lambda,t}(y)$.

A Taylor expansion of the Gaussian error function Φ yields

$$\begin{aligned}
&\Bigg|\left(\Phi\left(c_\lambda\right)-\Phi\left(-\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)\right)\\
&\quad-\frac{\mathrm{e}^{-\frac{d_{\lambda,t}(y)}{4t}}}{\sqrt{4\pi}}\left(\left(c_\lambda+\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)+\tfrac{1}{4}\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\left(c_\lambda+\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)^2\right)\Bigg|\\
&\leqslant\left(c_\lambda+\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)^3\max_{s\in\left(c_\lambda,-\frac{d_{\lambda,t}(y)}{\sqrt{t}}\right)}\frac{\left(-2+s^2\right)}{24}\frac{\mathrm{e}^{-s^2/4}}{\sqrt{4\pi}}.
\end{aligned}$$

By (113) and the upper bound on the normal distance function $d_{\lambda,t}(y)$, Lemma 4.6, the right-hand side is bounded from above by

$$\begin{aligned}
&\left(c_\lambda+\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)^3\max_{s\in\left(c_\lambda,-\frac{d_{\lambda,t}(y)}{\sqrt{t}}\right)}\frac{\left(-2+s^2\right)}{24}\frac{\mathrm{e}^{-s^2/4}}{\sqrt{4\pi}}\\
&\leqslant C(\alpha,n)\tfrac{t^{3/2}}{r^3}\left(1+\tfrac{|\Omega|}{r^n}+\tfrac{|\mathrm{M}^n|}{r^n}\right)^3\lambda^{-5\alpha}\max\left(\frac{\mathrm{e}^{-\frac{c_\lambda^2}{4}}}{\sqrt{4\pi}},\frac{\mathrm{e}^{-\frac{d_{\lambda,t}(y)^2}{4t}}}{\sqrt{4\pi}}\right).
\end{aligned}$$

Therefore, Theorem 2.5 and Inequality (4) imply

$$\begin{aligned}
&\Bigg|\left(c_\lambda+\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)+\tfrac{1}{4}\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\left(c_\lambda+\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)^2\\
&\quad-\left(-\sqrt{t}H+\sqrt{t}d_{\lambda,t}(y)\left(\tfrac{1}{4}\left(2\left(\mathrm{Ric}(\nu,\nu)+|A|^2\right)+H^2\right)\right)\right)\Bigg|\\
&\leqslant C(\alpha,n)\tfrac{t^{3/2}}{r^3}\left(1+\tfrac{|\Omega|}{r^n}+\tfrac{|\mathrm{M}^n|}{r^n}\right)^3\lambda^{-5\alpha}\max\left(\frac{\mathrm{e}^{-\frac{c_\lambda^2}{4}}}{\sqrt{4\pi}},\frac{\mathrm{e}^{-\frac{d_{\lambda,t}(y)^2}{4t}}}{\sqrt{4\pi}}\right)\frac{\mathrm{e}^{\frac{d_{\lambda,t}(y)^2}{4t}}}{\sqrt{4\pi}}.
\end{aligned}$$

Without loss of generality we may assume $\lambda\leqslant 1/2$. If $c_\lambda^2\geqslant d_{\lambda,t}(y)^2$, the term in the second column may be estimated by 1. Else if $c_\lambda^2\leqslant d_{\lambda,t}(y)^2$, we use the lower bound on c_λ, Lemma 4.8 and the upper bound on $d_{\lambda,t}(y)$, Lemma 4.6 to estimate

$$\begin{aligned}
&-c_\lambda^2+\tfrac{d_{\lambda,t}(y)^2}{t}\\
&\leqslant 4\log\lambda+2\log\left(\sqrt{-\log\lambda}\right)+C_1\\
&\quad+4\max\left(\left(-\log\lambda+\log\left((-\log\lambda)^{n/4+2}\right)\right),C_{105}(n)\left(1+\tfrac{|\mathrm{M}^n|}{r^n}\right)\right).\\
&\leqslant 4\max\left(\left(+\log\left((-\log\lambda)^{n/4+2}\right)\right),C_{105}(n)\left(1+\tfrac{|\mathrm{M}^n|}{r^n}\right)\right).
\end{aligned}$$

Hence, we may choose $C(\alpha, n) < \infty$ sufficiently large such that

$$\begin{aligned}
&\left|\left(c_\lambda + \tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right) + \tfrac{1}{4}\tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\left(c_\lambda + \tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right)^2\right| \\
&\leqslant C(\alpha, n)\tfrac{t^{3/2}}{r^3}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{|\mathrm{M}^n|}{r^n}\right)^3 \lambda^{-\alpha} \exp\left(C_{105}(n)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right)\right).
\end{aligned}$$

We now recursively determine the coefficients in the expansion (96). Together with (113) and the upper bound on $d_{\lambda,t}(y)$, Lemma 4.6, the above inequality implies

$$\begin{aligned}
\left|\left(c_\lambda + \tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right) - \left(-\sqrt{t}H\right)\right| \leqslant C(\alpha, n)\tfrac{t}{r^2}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{|\mathrm{M}^n|}{r^n}\right)^3 \lambda^{-6\alpha} \\
\times \exp\left(C_{105}(n)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right)\right)
\end{aligned}$$

for some adequate $C(n) < \infty$. Finally, substituting the above expansion once again in (96) we conclude

$$\begin{aligned}
&\left|\left(c_\lambda + \tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right) - \Bigg(- \sqrt{t}H - t\tfrac{c_\lambda}{2}\left(\left(\mathrm{Ric}(\nu,\nu) + |A|^2\right)\right)\right| \\
&= \Bigg|\left(c_\lambda + \tfrac{d_{\lambda,t}(y)}{\sqrt{t}}\right) - \tfrac{c_\lambda}{4}tH^2 - \Big(- \sqrt{t}H \\
&\quad - tc_\lambda\left(\tfrac{1}{4}\left(2\left(\mathrm{Ric}(\nu,\nu) + |A|^2\right) + H^2\right)\right)\Big)\Bigg| \\
&\leqslant C(\alpha, n)\tfrac{t^{3/2}}{r^3}\left(1 + \tfrac{|\Omega|}{r^n} + \tfrac{|\mathrm{M}^n|}{r^n}\right)^3 \lambda^{-7\alpha} \exp\left(C_{105}(n)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right)\right)
\end{aligned}$$

Multiplying both sides by $\sqrt{t}$ completes the proof. □

5 Short-time asymptotics of the mean heat content

In this chapter, we derive a short-time expansion of the *mean heat content*

$$\mathrm{MHC}(t) := \int_{\{u_\Omega(\cdot,t)\geqslant 1/2\}} u_\Omega(y,t)\,\omega(y), \tag{MHC}$$

where, for a closed Riemannian manifold and subset $\Omega \subset \mathrm{M}^n$ with a sufficiently regular boundary, $u_\Omega : \mathrm{M}^n \times \mathbb{R} \to \mathbb{R}$ is the solution of the heat equation

$$\begin{cases} \Delta_g u(x,t) = \frac{\partial}{\partial t} u(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n. \end{cases} \tag{HE}$$

To this end, the asymptotic expansion of the evolution of level sets from Chapter 4 is used in Section 5.1 to approximate the volume of the superlevel sets

$$\Omega_\lambda(t) := \{u_\Omega(\cdot,t) \geqslant \lambda\}\,, \quad \lambda \in (0,1),\, t > 0.$$

In Proposition 5.8 we find barrier sets of the sets $\Omega_\lambda(t)$ whose volume may be computed using the well-known variational formulae from Lemma 5.4. Applying Cavalieri's principle, a short-time expansion of the mean heat content (MHC) is then established in Theorem 5.1. We verify our result in Remark 5.3 by deriving an alternative proof of the short-time expansion of the heat content (HC) (Theorem 3.2) in the case of sets whose boundary has constant mean curvature.

We start by stating the main result of this chapter.

Theorem 5.1 *For any $n \in \mathbb{N}$ there are constants $0 < \varepsilon(n)$ and $C(n) < \infty$ only depending on n such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a closed Riemannian manifold of class C^6 and $\Omega \subset \mathrm{M}^n$ a set such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := \min_{x\in \mathrm{M}^n} r_{18}(5,x) > 0 \tag{114}$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor of the ambient space M^n, the first two derivatives of the second fundamental form of $\partial\Omega$ and the injectivity radius of M^n and Ω, is positive. Then the following function approximates the mean heat content (MHC). Set

$$G_\Omega(t) := \tfrac{\sqrt{t}}{\sqrt{\pi}}\,\mathrm{area}\,(\partial\Omega) + \tfrac{t}{2}\int_{\partial\Omega} H\ \mathrm{vol}_{\partial\Omega} + t^{\frac{3}{2}}\left(-\tfrac{1}{6\sqrt{\pi}}\int_{\partial\Omega}\left(\mathrm{Ric}(\nu,\nu) + |A|^2\right)\mathrm{vol}_{\partial\Omega} - \tfrac{1}{3\sqrt{\pi}}\int_{\partial\Omega} H^2\,\mathrm{vol}_{\partial\Omega}\right).$$

Then the uniform estimate

$$\left|\int_{\{u_\Omega(\cdot,t)\geqslant 1/2\}} u_\Omega(y,t)\,\omega(y) - G_\Omega(t)\right| \leqslant C_{115}(\mathrm{M}^n,\Omega,r)\tfrac{t^2}{r^4}\left(|\mathrm{M}^n| + r\,\mathrm{area}\,(\partial\Omega)\right)$$

holds for all $t \in (0, \varepsilon(n)r^2)$. *Here, the constant on the right hand side is given by*

$$C_{115}(\mathrm{M}^n,\Omega,r) := \exp\left(C(n)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right)\right). \tag{115}$$

The following integral equalities will be used for the computation of the coefficients in the expansion formula of the mean heat content (MHC).

Lemma 5.2 *The inverse of the Gaussian error function* c_λ *satisfies*

$$\int_0^{1/2} (c_\lambda)^k \mathrm{d}\lambda = \begin{cases} -\frac{1}{\sqrt{\pi}} & \text{if } k=1,\\ 1 & \text{if } k=2,\\ -\frac{4}{\sqrt{\pi}} & \text{if } k=3. \end{cases} \tag{116}$$

Proof. The integrals can be computed using the formulae in [GR07, Chapter 8.5] or [Wol]. □

Proof of Theorem 5.1 By Cavalieri's principle, we can write the mean heat content (MHC) in terms of the superlevel sets $\Omega_\lambda(t)$ in the following way. The monotonicity of the superlevel sets and the heat kernel's positivity imply

$$\begin{aligned}\int_{\{u_\Omega(\cdot,t)\geqslant 1/2\}} u_\Omega(y,t)\,\omega(y) &= \int_{\mathrm{M}^n} \left(u_\Omega(y,t)\chi_{\{u_\Omega(\cdot,t)\geqslant 1/2\}}(y)\right)\,\omega(y)\\ &= \int_0^1 \left|\{u_{\Omega(y,t)}\chi_{\{u_\Omega(\cdot,t)\geqslant 1/2\}}(y) \geqslant \lambda\}\right| \mathrm{d}\lambda\\ &= \int_0^{1/2} \left(|\Omega_\lambda(t)| - |\Omega_{1/2}(t)|\right)\mathrm{d}\lambda. \end{aligned} \tag{117}$$

In Section 5.1 we will prove that for any $\alpha > 0$ a constant $C(n,\alpha)$ can be chosen such that for all times t in the range

$$0 < t \leqslant \left(C(n,\alpha)C_{131}(\mathrm{M}^n,\Omega,r)\right)^{-2} r^2\lambda^\alpha := C_{118}(M,\Omega,r)^{-2}r^2\lambda^\alpha \tag{118}$$

the function

$$\begin{aligned} G_\Omega(\lambda,t) :=\ & |\Omega| - \sqrt{t}c_\lambda\,\mathrm{area}\,(\partial\Omega) + t\int_{\partial\Omega}\left(-H + \tfrac{c_\lambda^2}{2}H\right)\mathrm{d}\mu\\ &+ \tfrac{t^{3/2}}{6}\int_{\partial\Omega}\Big(\left(-3c_\lambda + c_\lambda^3\right)\left(\mathrm{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2\right)\\ &\quad + \left(6c_\lambda - c_\lambda^3\right)H^2\Big)\mathrm{d}\mu, \end{aligned}$$

satisfies

$$\left| G_\Omega(\lambda,t) - |\Omega_\lambda(t)| \right| \leqslant C(n,\alpha) C_{118}(M,\Omega,r) \lambda^{-\alpha} \tfrac{t^2}{r^3} \operatorname{area}(\partial\Omega)\,. \tag{119}$$

When $t > 0$ is fixed, we may therefore subdivide the interval $[0, 1/2]$ into a part where the Formula (119) holds and a part where it is unclear whether Formula (119) holds. We have

$$\begin{aligned}\int_0^{1/2} \Big(|\Omega_\lambda(t)| - |\Omega_{1/2}(t)| \Big)\, \mathrm{d}\lambda &= \int_{C_{118}(M,\Omega,r)^{2/\alpha} r^{-2/\alpha} t^{1/\alpha}}^{1/2} \Big(|\Omega_\lambda(t)| - |\Omega_{1/2}(t)| \Big)\, \mathrm{d}\lambda \\ &\quad + \int_0^{C_{118}(M,\Omega,r)^{2/\alpha} r^{-2/\alpha} t^{1/\alpha}} \Big(|\Omega_\lambda(t)| - |\Omega_{1/2}(t)| \Big)\, \mathrm{d}\lambda.\end{aligned}$$

After choosing $\alpha = 1/2$, we crudely estimate the volume of the superlevel sets in the second integral on the right-hand side by the volume of M, i.e.

$$\int_0^{C_{118}(M,\Omega,r)^4 r^{-4} t^2} \Big(|\Omega_\lambda(t)| - |\Omega_{1/2}(t)| \Big)\, \mathrm{d}\lambda \leqslant C_{118}(M,\Omega,r)^4 r^{-4} t^2\; |M|\,.$$

Hence, we may approximate the right-hand side of (117) by

$$\begin{aligned}&\left| \int_0^{1/2} |\Omega_\lambda(t)| - |\Omega_{1/2}(t)| \mathrm{d}\lambda - \int_0^{1/2} G_\Omega(\lambda,t) - G_\Omega(1/2,t)\, \mathrm{d}\lambda \right| \\ &\leqslant C_{118}(M,\Omega,r)^2 r^{-4} t^2 + \sqrt{2} C(n) \tfrac{t^2}{r^3} \operatorname{area}(\partial\Omega) + \int_0^{C_{118}(M,\Omega,r)^4 r^{-4} t^2} G_\Omega(\lambda,t)\, \mathrm{d}\lambda.\end{aligned}$$

We now estimate the last integral on the right-hand side. Using the Gaussian bound of the inverse Gaussian error function, Lemma 4.8, we have

$$\begin{aligned}&\int_0^{C_{118}(M,\Omega,r)^4 r^{-4} t^2} G_\Omega(\lambda,t)\, \mathrm{d}\lambda \\ &\leqslant \sqrt{t} C(n) \operatorname{area}(\partial\Omega) \int_0^{C_{118}(M,\Omega,r)^4 r^{-4} t^2} |c_\lambda|^3 \\ &\leqslant \sqrt{t} C(n) \operatorname{area}(\partial\Omega) \int_0^{C_{118}(M,\Omega,r)^4 r^{-4} t^2} (-\log\lambda)^2 \mathrm{d}\lambda \\ &= \sqrt{t} C(n) \operatorname{area}(\partial\Omega) \Big(C_{118}(M,\Omega,r)^2 r^{-2} t \Big)^2 \\ &\quad \times \Big(\Big(2\log\Big(C_{118}(M,\Omega,r)^2 r^{-2} t \Big) - 2 \Big) 2\log\Big(C_{118}(M,\Omega,r)^2 r^{-2} t \Big) + 2 \Big)\,.\end{aligned}$$

Since the last term grows logarithmically, a constant $C < \infty$ can be chosen such that

$$\begin{aligned}&\Big(C_{118}(M,\Omega,r)^2 r^{-2} t \Big)^{1/2} \\ &\times \Big(\Big(2\log\Big(C_{118}(M,\Omega,r)^2 r^{-2} t \Big) - 2 \Big) 2\log\Big(C_{118}(M,\Omega,r)^2 r^{-2} t \Big) + 2 \Big) \leqslant C.\end{aligned}$$

Therefore, we conclude

$$\int_0^{C_{118}(M,\Omega,r)^{2/\alpha} r^{-2/\alpha} t^{1/\alpha}} G_\Omega(\lambda,t)\,\mathrm{d}\lambda \leqslant \sqrt{t} C(n)\,\mathrm{area}\,(\partial\Omega)\left(C_{118}(M,\Omega,r)^2 r^{-2} t\right)^{3/2}$$
$$= C(n) C_{118}(M,\Omega,r)^3\,\mathrm{area}\,(\partial\Omega)\, r\frac{t^2}{r^4}.$$

Thus, adding up the error terms, we have

$$\left|\int_0^{1/2} |\Omega_\lambda(t)| - \left|\Omega_{1/2}(t)\right| \mathrm{d}\lambda - \int_0^{1/2} G_\Omega(\lambda,t) - G_\Omega(1/2,t)\,\mathrm{d}\lambda\right|$$
$$\leqslant C_{131}(\mathrm{M}^n,\Omega,r)^4 \tfrac{t^2}{r^4}\left(|\mathrm{M}^n| + r\,\mathrm{area}\,\partial\Omega\right).$$

Noting that $c_{1/2} = 0$, the proof is now completed by using formulas in Lemma 5.2. By definition of the function $G_\Omega(\lambda,t)$, it holds

$$\int_0^{1/2} G_\Omega(\lambda,t) - G_\Omega(1/2,t)\,\mathrm{d}\lambda$$
$$= -\sqrt{t}\left(\int_0^{1/2} c_\lambda \mathrm{d}\lambda\right)\mathrm{area}\,(\partial\Omega) + t\left(\tfrac{1}{2}\int_0^{1/2} c_\lambda^2\mathrm{d}\lambda\right)\int_{\partial\Omega} H\,\mathrm{d}\mu$$
$$+ t^{3/2}\left(\tfrac{1}{6}\int_0^{1/2}\left(-3c_\lambda + c_\lambda^3\right)\mathrm{d}\lambda\right)\int_{\partial\Omega}\left(\mathrm{Ric}(\nu,\nu) + |A|^2\right)\mathrm{d}\mu$$
$$+ t^{3/2}\left(\tfrac{1}{6}\int_0^{1/2}\left(6c_\lambda - c_\lambda^3\right)\mathrm{d}\lambda\right)\int_{\partial\Omega} H^2\mathrm{d}\mu$$
$$= \tfrac{\sqrt{t}}{\sqrt{\pi}}\,\mathrm{area}\,(\partial\Omega) + \tfrac{t}{2}\int_{\partial\Omega} H\,\mathrm{d}\mu - t^{3/2}\left(\tfrac{1}{3\sqrt{\pi}}\int_{\partial\Omega} H^2\mathrm{d}\mu\right.$$
$$\left. - \tfrac{1}{6\sqrt{\pi}}\int_{\partial\Omega}\left(\mathrm{Ric}(\nu,\nu) + |A|^2\right)\mathrm{d}\mu\right).$$

□

Remark 5.3 In the case of sets with constant mean curvature boundaries, the same method as in the proof above yields an approximation of the heat content (HC)

$$\mathrm{HC}(t) = \int_{\Omega^c}\int_\Omega \varrho(x,y,t)\omega(x)\omega(y). \qquad \text{(HC)}$$

Since the expansion of the normal distance function in Theorem 4.3 differs from the results in [Pre06], we now give, for the purpose of verification of Theorem 4.3, an outline of this proof. The asymptotic expansion of $d_{\lambda,t}(y)$ in Theorem 4.3 implies that $y \in \mathrm{M}^n$ lies in the complement of Ω^c whenever

$$0 \leqslant d_{\lambda,t}(y) = -c_\lambda\sqrt{t} - t\cdot H + O(t^2).$$

The above expansion is identically 0 whenever the inverse of the Gaussian error function c_λ is of scale $\sqrt{t}$. Hence, it vanishes for levels λ near $1/2$. On the other hand, using for example Mathematica [Wol], one may verify that the Taylor expansion of c_λ in $\lambda = 1/2$ is given by

$$-c_\lambda = -2\sqrt{\pi}(\lambda - 1/2) + O\left((1/2 - \lambda)^2\right). \tag{120}$$

Therefore, we have $0 = d_{\lambda,t}(y)$ whenever the level λ satisfies

$$-2\sqrt{\pi}\,(\lambda - 1/2) - \sqrt{t}H + O(\lambda - 1/2)^2 = 0.$$

Since H is assumed to be constant, we therefore have

$$\Omega_\lambda(t) \subset \Omega, \quad \text{for } \lambda \geqslant \tfrac{1}{2} - \tfrac{H\sqrt{t}}{2\sqrt{\pi}} + O(t).$$

Thus, by Cavalieri's principle

$$\int_{\Omega^c}\int_\Omega \varrho(x,y,t)\omega(x)\omega(y) = \int_0^1 |\Omega_\lambda(t)\backslash\Omega|\,\mathrm{d}\lambda = \int_0^{1/2 - H\sqrt{t}/2\sqrt{\pi}} |\Omega_\lambda(t)| - |\Omega|\,\mathrm{d}\lambda.$$

In a similar fashion to the proof of Theorem 5.1, one can then integrate the functions $G_\Omega(\lambda, t)$ that approximate the volume of the superlevel sets to derive the asymptotic behaviour of the heat content,

$$\int_{\Omega^c}\int_\Omega \varrho(x,y,t)\omega(x)\omega(y) - \int_0^{1/2 - H\sqrt{t}/2\sqrt{\pi}} (G_\Omega(\lambda,t) - |\Omega|)\,\mathrm{d}\lambda = O(t^2).$$

The integral equalities in (116) then yield

$$\begin{aligned}
&\int_0^{1/2 - H\sqrt{t}/2\sqrt{\pi}} (G_\Omega(\lambda,t) - |\Omega|)\,\mathrm{d}\lambda \\
&= \int_0^{1/2} (G_\Omega(\lambda,t) - |\Omega|)\,\mathrm{d}\lambda - \int_{1/2 - H\sqrt{t}/2\sqrt{\pi}}^{1/2} G_\Omega(\lambda,t) - |\Omega|\,\mathrm{d}\lambda \\
&= \tfrac{\sqrt{t}}{\sqrt{\pi}}\,\mathrm{area}\,(\partial\Sigma) - \tfrac{t^{3/2}}{6\sqrt{\pi}} \int_{\partial\Omega} \left(\mathrm{Ric}(\nu,\nu) + |A|^2\right) \mathrm{d}\mu \\
&\quad - \tfrac{1}{3\sqrt{\pi}} \int_{\partial\Omega} H^2 \mathrm{d}\mu \\
&\quad - \int_{1/2 - H\sqrt{t}/2\sqrt{\pi}}^{1/2} \left(-\sqrt{t}c_\lambda\,\mathrm{area}\,(\partial\Omega) - tH + O(\lambda^2 t)\right) \mathrm{d}\lambda,
\end{aligned} \tag{121}$$

It only remains to compute the second integral on the right-hand side. By Formula (120) it holds

$$\begin{aligned}
&- \int_{1/2 - H\sqrt{t}/2\sqrt{\pi}}^{1/2} \left(-\sqrt{t}c_\lambda\,\mathrm{area}\,(\partial\Omega) - tH + O(\lambda^2 t)\right) \mathrm{d}\lambda \\
&= - \int_{1/2 - H\sqrt{t}/2\sqrt{\pi}}^{1/2} \left(-2\sqrt{\pi}(\lambda - 1/2)\,\mathrm{area}\,(\partial\Omega) - tH\,\mathrm{area}\,(\partial\Omega)\right) \mathrm{d}\lambda + O(t^2) \\
&= \tfrac{1}{4\sqrt{\pi}} t^{3/2}\,\mathrm{area}\,(\partial\Omega)\,H^2 + O(t^2).
\end{aligned} \tag{122}$$

In particular, adding (121) and (122) yields the same expansion for the heat content as Theorem 3.2.

5.1 The volume of super-level sets

In this section we derive the asymptotic expansion of the volume of the superlevel sets

$$\Omega_\lambda(t) := \{u_\Omega(\cdot,t) \geqslant \lambda\}, \quad \lambda \in (0,1),\, t>0,$$

where $u_\Omega : \mathrm{M}^n \times (0,\infty)$ solves the heat equation

$$\begin{cases} \Delta_g u(x,t) = \frac{\partial}{\partial t} u(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n. \end{cases} \tag{HE}$$

The expansion formula is used in the proof of Theorem 5.1. To this end, we first recall variational formulae and apply them to barrier sets for the superlevel sets. For the construction of these sets we use Theorem 4.3. To estimate the resulting error terms, we will also need to discuss some properties of the signed distance function.

By the evolution equations in Section 1.4 we conclude the following equations. For emitted details we refer to [MY02, Chapter 2].

Lemma 5.4 *Let $\bar{F} : \Omega \times (0,\varepsilon) \to \mathrm{M}^n$ be a variation of $\Omega \subset \mathrm{M}^n$, i.e. a one-parameter family of diffeomorphisms such that $\bar{F}(\cdot,0) = \mathrm{id}_{\mathrm{M}^n}(\cdot)$. Further suppose that $r_{19}(\partial\Omega, 2) > 0$ is well-defined. We set $\Omega_\tau := \bar{F}(\Omega,\tau)$, $\Sigma_\tau := \bar{F}(\partial\Omega,\tau)$ and*

$$\bar{\mathrm{X}} := \tfrac{\partial \bar{F}}{\partial \tau}, \quad \alpha := \bar{\mathrm{g}}\left(\bar{\mathrm{X}}, \nu\right), \quad \beta := \bar{\mathrm{g}}\left(\bar{\nabla}_{\bar{\mathrm{X}}}\bar{\mathrm{X}}, \nu\right).$$

The first three variation of volume formulae are given by

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\, |\Omega_\tau| = \int_{\partial\Omega} \alpha \,\mathrm{d}\mu, \tag{123}$$

$$\tfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}\, |\Omega_\tau| = \int_{\partial\Omega} \left(\tfrac{\partial}{\partial\tau}\alpha + \alpha \operatorname{div} \bar{\mathrm{X}}\right) \mathrm{d}\mu, \tag{124}$$

$$\tfrac{\mathrm{d}^3}{\mathrm{d}\tau^3}\, |\Omega_t| = V^N(\tau) + V^T(\tau), \tag{125}$$

where

$$
\begin{aligned}
V^N(\tau) := \int_{\partial\Omega} \Big(\tfrac{\partial^2}{\partial\tau^2}\alpha + \Big(3\alpha \tfrac{\partial}{\partial\tau}\alpha + \beta\Big) H \\
&+ \alpha^2 \Big(-\Delta\alpha - \alpha\left(|A|^2 + \bar{\mathrm{R}}(\nu,\nu)\right) - H^2\Big)\Big)\mathrm{d}\mu, \\
V^T(t) := \int_{\partial\Omega} \Big(\operatorname{div}\bar{\mathrm{X}}^T \Big(2\tfrac{\partial}{\partial\tau}\alpha + 2H\alpha^2 - \alpha H + \bar{\mathrm{X}}^T(\alpha)\Big) \\
&- H\bar{\mathrm{X}}^T(\alpha) + \alpha \operatorname{div}\Big(\bar{\nabla}_{\bar{\mathrm{X}}}\bar{\mathrm{X}}^T\Big)^T\Big)\Big)\,\mathrm{d}\mu.
\end{aligned}
$$

Proof. We refer to [MY02, Theorem 2.2, respectively Theorem 2.5] for the Equalities (123) and (124). To derive Equality (125), we use (124) and conclude

$$
\begin{aligned}
\tfrac{\mathrm{d}^3}{\mathrm{d}\tau^3}\,|\Omega_t| &= \tfrac{\partial}{\partial\tau}\left(\int_{\Sigma_\tau}\Big(\tfrac{\partial}{\partial\tau}\alpha + \alpha \operatorname{div}_\Sigma \bar{\mathrm{X}}\Big)\,\mathrm{d}\mu_\tau\right) \\
&= \left(\int_{\partial\Omega}\Big(\tfrac{\partial}{\partial\tau}\alpha + \alpha \operatorname{div}\bar{\mathrm{X}}\Big)\operatorname{div}\bar{\mathrm{X}}\mathrm{d}\mu\right) \\
&\quad + \left(\int_{\partial\Omega}\Big(\tfrac{\partial^2}{\partial\tau^2}\alpha + \Big(\tfrac{\partial}{\partial\tau}\alpha\Big)\operatorname{div}\bar{\mathrm{X}} + \alpha\Big(\tfrac{\partial}{\partial\tau}\operatorname{div}\bar{\mathrm{X}}\Big)\Big)\,\mathrm{d}\mu\right).
\end{aligned}
$$

To compute $\frac{\partial}{\partial\tau}\operatorname{div}\bar{\mathrm{X}}$ we decompose the vector field $\bar{\mathrm{X}}$ into its tangential and normal parts, $\bar{\mathrm{X}} = \bar{\mathrm{X}}^T + \alpha\nu$. By Formula (24), differentiating the second term yields

$$
\begin{aligned}
\tfrac{\partial}{\partial\tau}\left(\operatorname{div}_\Sigma(\alpha\nu)\right) &= \tfrac{\partial}{\partial\tau}\left(\alpha \operatorname{div}_\Sigma \nu\right) \\
&= \tfrac{\partial}{\partial\tau}(\alpha H) \\
&= \Big(\tfrac{\partial}{\partial\tau}\alpha\Big) H + \alpha\Big(-\Delta_\Sigma\alpha - \alpha\left(|A|^2 + \bar{\mathrm{R}}(\nu,\nu)\right) + \bar{\mathrm{X}}^T(H)\Big).
\end{aligned}
$$

Applying (25), Formula (125) now follows by computing the tangential part,

$$
\tfrac{\partial}{\partial\tau}\left(\operatorname{div}\bar{\mathrm{X}}^T\right) = \bar{X}\left(\alpha H + \operatorname{div}\bar{\mathrm{X}}^T\right) + \beta H + \operatorname{div}\left(\Big(\tfrac{\partial}{\partial\tau}\bar{\mathrm{X}}\Big)^T\right). \qquad \square
$$

Corollary 5.5 *Suppose* $\bar{F} : \mathrm{M}^n \times (-\varepsilon,\varepsilon) \to \mathrm{M}^n$ *is a variation of* M^n *and* $\Omega \subset \mathrm{M}^n$*. We set*

$$
r_{126}(\bar{\mathrm{X}}) := \min\left(r_{19}(4,\partial\Omega), \left|\bar{\nabla}\bar{\nabla}\bar{\nabla}\bar{\mathrm{X}}\right|^{-1/3}, \left|\bar{\nabla}\bar{\nabla}\bar{\mathrm{X}}\right|^{-1/2}, \left|\bar{\nabla}\bar{\mathrm{X}}\right|\right). \tag{126}
$$

Then there are constants $\varepsilon(n) > 0$ *and* $C(n) < \infty$ *only depending on* n *such that for all* $|\tau| \leqslant \varepsilon_n r_{126}\left(\bar{X}\right)$ *the forth variation of volume of* Ω *satisfies*

$$
\tfrac{\mathrm{d}^4}{\mathrm{d}\tau^4}\,|\Omega_\tau| \leqslant C(n)\,\mathrm{area}\,(\partial\Omega)\, r_{126}(\bar{\mathrm{X}})^{-3}\left(1 + \left|\bar{\mathrm{X}}\right|^4\right).
$$

PROOF. Firstly, we note that, for $\tau \leqslant \varepsilon(n) r_{28}\left(\bar{X}\right)$, Lemma 1.12 and the first variation of area imply

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{area}(\Sigma_\tau) = \int_{\Sigma_\tau} \alpha H \mathrm{d}\mu_\tau \leqslant |\alpha| \, |H| \operatorname{area}(\Sigma_\tau) \leqslant 2 \left|\bar{\mathrm{X}}\right| \, |A_0| \operatorname{area}(\Sigma_\tau).$$

Hence, by Grönwall's inequality a constant $\varepsilon(n) > 0$ exists such that for $\tau \leqslant \varepsilon(n) r_{28}\left(\bar{X}\right)$

$$\operatorname{area}(\Sigma_\tau) \leqslant 2 \operatorname{area}(\partial\Omega). \tag{127}$$

Therefore, it only remains to show that any term in the t differential of $V^N(\tau)$ and $V^N(\tau)$ can be estimated by $C(n) r_{126}(\bar{\mathrm{X}})^{-3}$. It suffices to estimate the highest τ derivatives of α, β and the curvature terms occurring in the τ-differential of $V^N(\tau)$. The cross terms may be estimated by (126) due to structural reasoning. To this end, we compute the differential in local coordinates and show that only derivatives of $\bar{\mathrm{X}}$ as in (126) appear. Derivatives of the metric tensor and Christoffel symbols are dealt with the curvature terms in the definition of $r_{17}(3, x)$. Since covariant derivatives can be written in terms of the partial derivatives, Christoffel symbols and their derivatives, it therefore suffices to find estimates in terms of partial derivatives.

For the terms involving differentials of the curvature terms H and $|A|^2$ this follows from Lemma 1.12. By the evolution equation of the normal vector (24), for some orthonormal basis $e_1, \dots e_{n-1}$ of $T_x\Sigma$

$$\begin{aligned}
\tfrac{\partial^2}{\partial\tau^2}\nu = {} & g^{jk} g^{il} \bar{\mathrm{g}}\left(\nu, \bar{\nabla}_i \bar{\mathrm{X}}\right) \left(\bar{\mathrm{g}}\left(\bar{\nabla}_j \bar{\mathrm{X}}, e_l\right) + \bar{\mathrm{g}}\left(\bar{\nabla}_i \bar{\mathrm{X}}, e_k\right)\right) e_j \\
& + g^{ij} e_j \left(\bar{\mathrm{g}}\left(g^{kl} \bar{\mathrm{g}}\left(\nu, \bar{\nabla}_k \bar{\mathrm{X}}\right) e_l, \bar{\nabla}_i \bar{\mathrm{X}}\right) - \bar{\mathrm{g}}\left(\nu, \bar{\nabla}_{\bar{\mathrm{X}}}\left(\bar{\nabla}_i \bar{\mathrm{X}}\right)\right)\right) \\
& - g^{ij} \bar{\mathrm{g}}\left(\nu, \bar{\nabla}_i \bar{\mathrm{X}}\right) \bar{\nabla}_j \bar{\mathrm{X}}.
\end{aligned}$$

Hence, it is clear that the highest order covariant derivatives of $\bar{X}$ appearing in the third t-derivative of ν can be bounded by $\left|\bar{\nabla}^3 \bar{\mathrm{X}}\right|$. In particular,

$$\tfrac{\partial}{\partial\tau}\left(\overline{\mathrm{Ric}}(\nu, \nu)\right) \leqslant \left|\bar{\nabla}\overline{\mathrm{Ric}}\right| + 2\left|\overline{\mathrm{Ric}}\right| \left|\tfrac{\partial}{\partial\tau}\nu\right| \leqslant C(n)\left(\left|\bar{X}\right| + 1\right) r_{126}(\bar{\mathrm{X}})^{-3}.$$

Differentiating α yields

$$\begin{aligned}
\tfrac{\partial^3}{\partial\tau^3}\alpha = {} & \bar{\mathrm{g}}\left(\bar{\nabla}_{\bar{\mathrm{X}}}\left(\bar{\nabla}_{\bar{\mathrm{X}}}\left(\bar{\nabla}_{\bar{\mathrm{X}}}\bar{\mathrm{X}}\right)\right), \nu\right) + \bar{\mathrm{g}}\left(\tfrac{\partial^3}{\partial\tau^3}\nu, \bar{\mathrm{X}}\right) \\
& + 3\bar{\mathrm{g}}\left(\tfrac{\partial^2}{\partial\tau^2}\nu, \bar{\nabla}_{\bar{\mathrm{X}}}\bar{\mathrm{X}}\right) + 3\bar{\mathrm{g}}\left(\tfrac{\partial}{\partial\tau}\nu, \bar{\nabla}_{\bar{\mathrm{X}}}\left(\bar{\nabla}_{\bar{\mathrm{X}}}\bar{\mathrm{X}}\right)\right). \\
\leqslant {} & C(n)\left(1 + \left|\bar{\mathrm{X}}\right|^4\right) r_{126}(\bar{\mathrm{X}})^{-3}.
\end{aligned}$$

The differential of β is estimated similarly. The τ-derivatives of the terms in $V^T(\tau)$ are controlled analogously. For example, by (25) setting $Y := \left(\bar{\nabla}_{\bar{X}}\bar{X}^T\right)^T$

$$\tfrac{\partial}{\partial\tau}\operatorname{div} Y = Y\left(\alpha H + \operatorname{div}\bar{X}^T\right) + \bar{g}\left(\bar{\nabla}_{\bar{X}}Y, \nu\right) H + \operatorname{div}\left(\bar{\nabla}_{\bar{X}}Y\right)^T. \qquad (128)$$

It is clear that the first two terms is controlled by $r_{126}(\bar{X})^{-3}$. The last term can be estimated from above by

$$\operatorname{div}\left(\bar{\nabla}_{\bar{X}}\left(\bar{\nabla}_{\bar{X}}\bar{X}^T\right)^T\right)^T \leqslant (n-1)\left|\bar{\nabla}\left(\bar{\nabla}_{\bar{X}}\left(\bar{\nabla}_{\bar{X}}\bar{X}^T\right)^T\right)^T\right|.$$

On the other hand,

$$\begin{aligned}\left(\bar{\nabla}_{\bar{X}}\left(\bar{\nabla}_{\bar{X}}\bar{X}^T\right)^T\right)^T &= \bar{\nabla}_{\bar{X}}\left(\bar{\nabla}_{\bar{X}}\bar{X}\right) - \tfrac{\partial}{\partial\tau}\alpha\tfrac{\partial}{\partial\tau}\nu - \alpha\tfrac{\partial^2}{\partial\tau^2}\nu - \beta\tfrac{\partial}{\partial\tau}\nu \\ &\quad - \bar{g}\left(\bar{\nabla}_{\bar{X}}\left(\bar{\nabla}_{\bar{X}}\bar{X}\right), \nu\right)\nu + \alpha\bar{g}\left(\tfrac{\partial^2}{\partial\tau^2}\nu, \nu\right)\nu.\end{aligned}$$

Hence, it is easy to see that (128) is controlled by (126). □

To apply the estimate from the previous corollary to our special case, we need the following estimates involving the signed distance function. We extend the parallel translation of hypersurfaces to a variation on M^n.

Lemma 5.6 *Suppose $F_0 : \Sigma \hookrightarrow \mathrm{M}^n$ is a hypersurface and that the map*

$$F : \mathrm{M}^n \times (-r, r) \to \mathrm{M}^n \quad (x, \tau) \mapsto \exp_x \tau\nu(x)$$

is a diffeomorphism. Set $r := r_{19}(\Sigma, 2)$ and let $\eta : \mathbb{R} \to [0, 1]$ be a cut-off function satisfying $\eta \equiv 1$ for $|x| \leqslant 1$ and $\eta \equiv 0$ for $|x| \geqslant 2$. Then the mapping

$$\bar{F}_\Sigma : \mathrm{M}^n \times [-r/2, r/2], \quad (y, \tau) \mapsto \exp_y\left(\tau\varphi\left(4d_\Sigma(y)/r\right)\nu_{d_\Sigma(y)}(y)\right),$$

is a variation of M^n. Furthermore, for some $\varepsilon(n) > 0$ and $C(n) < \infty$ it satisfies

$$\sup_{|\tau|\leqslant\varepsilon(n)r}\left(\sup_{y\in T(\Sigma,\varepsilon(n)r)}\left|\bar{\nabla}^k\left(\tfrac{\partial}{\partial\tau}\bar{F}_\Sigma(\tau, y)\right)\right|\right) \leqslant C(n)r^{-k}, \quad 0 \leqslant k \leqslant 3.$$

Proof. By definition of F, the hypersufaces $\Sigma_\tau := F(\Sigma, \tau)$ satisfy the initial value problem

$$\begin{cases} \frac{\partial F}{\partial\tau}(x, \tau) = -\nu(x, \tau) & x \in \Sigma, s \in [-r, r] \\ F(x, 0) = F_0 & x \in \Sigma. \end{cases}$$

Hence, by the evolution equations from Lemma 1.9, the second fundamental form satisfies a first order differential equation. Grönwall's lemma therefore implies that one may choose some $\varepsilon(n) > 0$ such that

$$\sup_{|\tau|\leqslant\varepsilon_n r}\left(\sup_{x\in\Sigma_\tau}|A(x)|\right)\leqslant 2\sup_{x\in\Sigma}|A(x)|\,.$$

We note that $\bar{F}_\Sigma(\cdot,0)=\mathrm{id}_{\mathrm{M}^n}(\cdot)$ and $\bar{F}_\Sigma(\cdot,\tau)$ is a diffeomorphism for $|d_\Sigma(y)|\geqslant r/2$. In particular, $\bar{F}_\Sigma$ is a variation of M^n. Furthermore, for $\gamma_y(\tau):=\exp_y\tau\nu(y)$

$$\tfrac{\partial}{\partial\tau}\bar{F}_\Sigma(s,y)=\nu_{h(\tau,y)}(\gamma_y(\tau)),$$

where ν_h is the normal of the parallel surface Σ_h in y and $h(\tau,y)=d_\Sigma(y)+\tau$. The evolution equation of the normal vector, Lemma 1.9, implies

$$0=\tfrac{\partial}{\partial\tau}\nu_h\left(\gamma_y(\tau)\right)=\bar{\nabla}_{\nu_{h(\tau,y)}}\nu_h\left(\gamma_y(\tau)\right)=\tfrac{\partial}{\partial\tau}\left(\tfrac{\partial}{\partial\tau}\bar{F}_\Sigma(\tau,y)\right).$$

Together with the Weingarten equations (10) and the evolution equations from Theorem 1.9 this implies, in local coordinates,

$$\begin{aligned}
\bar{\nabla}_i\,\nu &= h_{il}\tfrac{\partial\bar{F}_\Sigma}{\partial x^l},\\
\bar{\nabla}_j\left(\bar{\nabla}_i\,\nu\right) &= \left(\bar{\nabla}_j\tfrac{\partial\bar{F}_\Sigma}{\partial x^l}\right)h_{il}+\left(\nabla_j h_{il}\right)\tfrac{\partial\bar{F}_\Sigma}{\partial x^l}\\
&= -h_{il}h_{jl}\nu+\left(\nabla_j h_{il}\right)\tfrac{\partial\bar{F}_\Sigma}{\partial x^l},\\
\bar{\nabla}_n\left(\bar{\nabla}_i\,\nu\right) &= \left(\tfrac{\partial}{\partial s}h_{il}\right)\tfrac{\partial\bar{F}_\Sigma}{\partial x^l}+h_{il}\left(\tfrac{\partial}{\partial s}\tfrac{\partial\bar{F}_\Sigma}{\partial x^l}\right)\\
&= \left(h_{ik}h^k_m-\bar{\mathrm{R}}_{ninm}+h_{il}h_{lk}g^{km}\right)\tfrac{\partial\bar{F}_\Sigma}{\partial x^m},\\
\bar{\nabla}_k\left(\bar{\nabla}_j\left(\bar{\nabla}_i\,\nu\right)\right) &= -\left(\left(\nabla_k h_{il}\right)h_{jl}+\left(\nabla_k h_{jl}\right)h_{il}+\left(\nabla_j h_{il}\right)h_{kl}\right)\nu\\
&\quad+\left(\nabla_k\left(\nabla_j h_{im}\right)-h_{il}h_{jl}h_{km}\right)\tfrac{\partial\bar{F}_\Sigma}{\partial x^m},\\
\bar{\nabla}_n\left(\bar{\nabla}_j\left(\bar{\nabla}_i\,\nu\right)\right) &= -\tfrac{\partial}{\partial s}\left(h_{il}h_{jl}\right)\nu\\
&\quad+\left(\left(\nabla_j h_{il}\right)h^m_l+\tfrac{\partial}{\partial s}\left(\nabla_j h_{il}\right)\right)\tfrac{\partial\bar{F}_\Sigma}{\partial x^m},\\
\bar{\nabla}_n\left(\bar{\nabla}_n\left(\bar{\nabla}_i\,\nu\right)\right) &= \Big(\left(h_{ik}h^k_m-\bar{\mathrm{R}}_{ninm}+h_{il}h_{lk}g^{km}\right)h^l_m+\tfrac{\partial}{\partial s}\Big(h_{ik}h^k_l\\
&\quad+h_{ir}h_{rk}g^{kl}\Big)-\left(\bar{\nabla}_\nu\bar{\mathrm{R}}\right)_{ninl}-h^r_i\bar{\mathrm{R}}_{nrnl}-h^r_l R_{ninr}\Big)\tfrac{\partial\bar{F}_\Sigma}{\partial x^l}.
\end{aligned}$$

In particular, all terms occurring in above formulae of the first, second, and third covariant derivatives of ν_s can be bounded by r^{-1}, respectively r^{-2} and r^{-3}. Hence,

$$\sup_{|\tau|\leqslant\varepsilon(n)r}\left(\sup_{y\in T(\Sigma,\varepsilon(n)r)}\left|\bar{\nabla}^k\left(\tfrac{\partial}{\partial\tau}\bar{F}_\Sigma(\tau,y)\right)\right|\right)\leqslant C(n)r^{-k},\quad 0\leqslant k\leqslant 3.\qquad\square$$

We also need to estimate the norm of extensions $\bar{f}$ of functions $f : \Sigma \to \mathbb{R}$ that we define as follows.

Lemma 5.7 *Let $F_0 : \Sigma \hookrightarrow \mathrm{M}^n$, $r > 0$ be as in Lemma 5.6. Furthermore, let $f : \Sigma \to \mathbb{R}$ satisfy*

$$\left|\nabla^{(k)} f\right| \leqslant C r^{-k} \qquad \text{for } k = 1, 2, 3,$$

where $C < \infty$. In a neighbourhood of the hypersurface Σ, we define the extension $\bar{f}$ of f by

$$\bar{f} : T(\Sigma, r) \to \mathbb{R}, \quad y \mapsto f\left(\pi_1 \circ F_\Sigma(y)^{-1}\right).$$

Then there is some $\varepsilon(n) > 0$ such that for $k = 1, 2$

$$\left|\bar{\nabla}^{(k)} \bar{f}\right|_{T(\varepsilon(n)r, \Sigma)} \leqslant 2C(n) r^{-k}.$$

Proof. We set $F := F_\Sigma$. By construction, the covariant derivative of $\bar{f}$ in direction ν vanishes. In particular,

$$\left|\bar{\nabla} \bar{f}\right| = \left|\nabla_{\Sigma_s} \bar{f}_{|\Sigma_s}\right|.$$

Along the geodesic $\gamma_x(s) := \exp_x s\nu(x)$ we then have, in local coordinate,

$$\begin{aligned}
\tfrac{\partial}{\partial \tau}\left(g^{ij} \tfrac{\partial F^\alpha}{\partial x^i} \tfrac{\partial f}{\partial x^j}\right) &= -2h^{ij} \tfrac{\partial F^\alpha}{\partial x^i} \tfrac{\partial f}{\partial x^j} + g^{ij} \tfrac{\partial f}{\partial x^j}\left(-\bar{\Gamma}^\alpha_{\gamma\delta} \tfrac{\partial F^\gamma}{\partial x^i} \nu^\delta + h_i^l \tfrac{\partial F^\alpha}{\partial x^l}\right) \\
&\quad + \tfrac{\partial}{\partial x^j}\left(\tfrac{\partial f}{\partial s}\right) g^{ij} \tfrac{\partial F^\alpha}{\partial x^i} \\
&= -h^{ij} \tfrac{\partial F^\alpha}{\partial x^i} \tfrac{\partial f}{\partial x^j} - g^{ij} \tfrac{\partial f}{\partial x^j} \bar{\Gamma}^\alpha_{\gamma\delta} \tfrac{\partial F^\gamma}{\partial x^i} \nu^\delta. \qquad (129)
\end{aligned}$$

Thus,

$$\begin{aligned}
\tfrac{\partial}{\partial \tau}\left|\nabla \bar{f}\right|^2 &= \tfrac{\partial}{\partial s}\left(g_{\alpha\beta}\left(g^{ij} \tfrac{\partial F^\alpha}{\partial x^i} \tfrac{\partial f}{\partial x^j}\right)\left(g^{kl} \tfrac{\partial F^\beta}{\partial x^k} \tfrac{\partial f}{\partial x^l}\right)\right) \\
&= \left(\tfrac{\partial}{\partial x^\varepsilon} g_{\alpha\beta}\right) \nu^\varepsilon \left(\left(g^{ij} \tfrac{\partial F^\alpha}{\partial x^i} \tfrac{\partial f}{\partial x^j}\right)\left(g^{kl} \tfrac{\partial F^\beta}{\partial x^k} \tfrac{\partial f}{\partial x^l}\right)\right) \\
&\quad + 2g_{\alpha\beta}\left(-h^{ij} \tfrac{\partial F^\alpha}{\partial x^i} \tfrac{\partial f}{\partial x^j} - g^{ij} \tfrac{\partial f}{\partial x^j} \bar{\Gamma}^\alpha_{\gamma\delta} \tfrac{\partial F^\gamma}{\partial x^i} \nu^\delta\right)\left(g^{kl} \tfrac{\partial F^\beta}{\partial x^k} \tfrac{\partial f}{\partial x^l}\right) \\
&\leqslant C(n) r^{-1} \left|\nabla \bar{f}\right|^2.
\end{aligned}$$

Hence,

$$\left|\bar{\nabla} \bar{f}\left(\gamma_x(\tau)\right)\right| \leqslant \left|\nabla f(x)\right| \mathrm{e}^{C(n) r^{-1} \tau}.$$

For the second bound, we observe by (129) and the Weingarten equations (10)

$$\begin{aligned}
\bar{\nabla}_\nu \bar{\nabla}_k \bar{\nabla} \bar{f} &= \bar{\nabla}_k \bar{\nabla}_\nu \bar{\nabla} \bar{f} - \overline{\mathrm{Riem}}\left(\nu, \tfrac{\partial F}{\partial x^i}\right) \bar{\nabla} \bar{f} \\
&= -\left(\nabla_k h^{ij}\right) \tfrac{\partial F}{\partial x^i} \tfrac{\partial f}{\partial x^j} + h^{ij} h_{ik} \nu \tfrac{\partial f}{\partial x^j} - h^{ij} \tfrac{\partial F}{\partial x^i} \tfrac{\partial^2 f}{\partial x^k \partial x^j} \\
&\quad - \overline{\mathrm{Riem}}\left(\nu, \tfrac{\partial F}{\partial x^i}\right) \bar{\nabla} \bar{f}.
\end{aligned}$$

By the definition of r we therefore have

$$\tfrac{\partial}{\partial \tau}\left(\left|\bar{\nabla}_k \bar{\nabla} \bar{f}\right|^2\right) \leqslant C(n) r^{-1}\left(r^{-2}+\left|\bar{\nabla}_k \bar{\nabla} \bar{f}\right|^2\right).$$

The bound on the second covariant derivatives of $\bar{f}$ now follows from Grönwall's inequality. □

We are now ready to apply Theorem 4.3 to approximate the volume of the superlevel sets $\Omega_\lambda(t)$.

Proposition 5.8 *For any $n \in \mathbb{N}$ and $\beta > 0$ there are constants $0 < \varepsilon(n, \beta)$ and $C(n, \beta) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a closed Riemannian manifold of class C^6 and $\Omega \subset \mathrm{M}^n$ a set such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := \min_{x \in \mathrm{M}^n} r_{18}(5, x) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor of the ambient space M^n, the first two derivatives of the second fundamental form of $\partial\Omega$ and the injectivity radius of M^n and Ω, is positive. Fix $\lambda \in (0, 1)$ and let $\Omega_\lambda(t)$ be the corresponding superlevel set of the solution $u_\Omega : \mathrm{M}^n \times (0, \infty)$ of the heat equation (HE). Then the following function is an approximation of the volume of $\Omega_\lambda(t)$. Set

$$\begin{aligned} G_\Omega(\lambda, t) := {} & |\Omega| - \sqrt{t} c_\lambda \operatorname{area}(\partial\Omega) + t \int_{\partial\Omega} \left(-H + \tfrac{c_\lambda^2}{2} H\right) \mathrm{d}\mu \\ & + \tfrac{t^{\frac{3}{2}}}{6} \int_{\partial\Omega} \Big(\left(-3c_\lambda + c_\lambda^3\right)\left(\operatorname{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2\right) \\ & \qquad + \left(6c_\lambda - c_\lambda^3\right) H^2 \Big) \mathrm{d}\mu, \end{aligned}$$

and

$$C_{131}(\mathrm{M}^n, \Omega, r) := \exp\left(C(n,\beta)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right)\right).$$

Then we have for all times t satisfying

$$0 \leqslant t \leqslant t_{130}(\lambda, \beta) := \varepsilon(n,\beta) r^2 C_{131}(\mathrm{M}^n, \Omega, r)^{-2} \max\left(1, c_\lambda, \lambda^\beta\right)^{-1} \tag{130}$$

the estimation

$$\left|G_\Omega(\lambda, t) - |\Omega_\lambda(t)|\right| \leqslant C(n,\beta) C_{131}(\mathrm{M}^n, \Omega, r) \lambda^{-\beta} \tfrac{t^2}{r^3} \operatorname{area}(\partial\Omega).$$

Proof. For the sake of simplicity, we approximate the volume of the superlevel sets $\Omega_\lambda(\tau)$, where $\tau = \sqrt{t}$. Fix $\lambda \in (0,1)$. With Theorem 4.3 we construct barrier sets $\bar{F}^+_\lambda(\Omega,\tau)$ and $\bar{F}^-_\lambda(\Omega,\tau)$ satisfying

$$\bar{F}^+_\lambda(\Omega,\tau) \subset \Omega_\lambda(\tau) \subset \bar{F}^+_\lambda(\Omega,\tau).$$

We then use the lemmata from Section 1.4 to compute the evolution of the geometric quantities of the sets $\bar{F}^+_\lambda(\partial\Omega,\tau)$. The variational formulae, Lemma 5.4, will be applied to approximate the volume of the barrier sets. Finally, we will estimate the occurring error terms from above.

Construction of the barrier sets $\bar{F}^+_\lambda(\Omega,\tau)$ and $\bar{F}^-_\lambda(\Omega,\tau)$.

Let $\bar{F}_{\partial\Omega}$ be defined as in Lemma 5.6. To construct the upper barrier set we first define the function $f_\lambda : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} f^+_\lambda(x_0,\tau) :=& - c_\lambda\tau - \tau^2 H(x_0) - \tau^3 \tfrac{c_\lambda}{2}\left(\left(\mathrm{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2\right)\right) \\ &+ \tfrac{\tau^4}{r^3} C_{131}(\mathrm{M}^n,\Omega,r)\lambda^{-\alpha}, \end{aligned}$$

where

$$C_{131}(\mathrm{M}^n,\Omega,r) := \exp\left(C_{105}(n)\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right)\right). \tag{131}$$

In view of Lemma 5.7, we extend f^+_λ to a neighbourhood of $\partial\Omega$ by setting

$$\bar{f}^+_\lambda : T(\Sigma,r) \times \mathbb{R} \to \mathbb{R}, \quad (y,\tau) \mapsto f^+_\lambda\left(\pi_1 \circ F_{\partial\Omega}(y)^{-1},\tau\right)$$

and define

$$C_{132}(\lambda) := 96 \max\left(1,\lambda^{-\alpha},c_\lambda\right). \tag{132}$$

By our choice of r the τ-derivatives of $\bar{f}^+_\lambda$ then satisfy for $i = 1,2,3$ and

$$0 \leqslant \tau \leqslant T_{133}(\lambda) := rC_{132}(\lambda)^{-1}C_{131}(\mathrm{M}^n,\Omega,r)^{-1} \tag{133}$$

the estimates

$$\begin{aligned} \left|\bar{f}^+_\lambda(y,\tau)\right| \leqslant r, \quad \left|\tfrac{\partial^i}{\partial\tau^i}\bar{f}^+_\lambda(y,\tau)\right| &\leqslant C_{132}(\lambda)r^{1-i}, \\ \left|\tfrac{\partial^4}{\partial\tau^4}\bar{f}^+_\lambda(y,\tau)\right| &\leqslant C_{131}(\mathrm{M}^n,\Omega,r)C_{132}(\lambda)r^{-3}. \end{aligned} \tag{134}$$

Finally, we define the mappings

$$\bar{F}^+_\lambda : T\left(\partial\Omega, r/2\right) \times (0,T_{133}(\lambda)) \to T\left(\partial\Omega,r\right), \quad (y,\tau) \mapsto \left(\bar{F}_{\partial\Omega} \circ \bar{f}^+_\lambda\right)(y,\tau).$$

The mappings $\bar{F}^+_\lambda(\cdot,\tau)$ are indeed well-defined since we have $|f^+_\lambda| \leqslant r/2$. Theorem 4.3 implies $d_\Omega(y) \leqslant \bar{f}^+_\lambda(y,\tau)$ for any $y \in \Omega_\lambda(\tau^2)$, whence we conclude

$$\Omega_\lambda(\tau^2) \subset \bar{F}^+_\lambda(\Omega,\tau).$$

Moreover, since $\bar{F}^+_\lambda(\cdot,0)$ is the identity map on M^n the inverse mapping theorem implies that $\{\bar{F}^+_\lambda(\cdot,t)\}_{0\leqslant t\leqslant T_{133}(\lambda)}$ is a family of diffeomorphisms.

Analogously, we construct lower barrier sets $\bar{F}^-_\lambda(\Omega,\tau)$ by instead defining

$$\begin{aligned} f^-_\lambda(x_0,t) :=& - c_\lambda\tau - \tau^2 H(x_0) - \tau^3 \tfrac{c_\lambda}{2}\left(\left(\mathrm{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2\right)\right) \\ &- \tfrac{\tau^4}{r^3} C_{131}(\mathrm{M}^n,\Omega,r)\lambda^{-\alpha}. \end{aligned}$$

Evolution equations for the hypersurfaces $\partial\bar{F}^+_\lambda(\Omega,t)$.

By construction, we have $\partial\left(\bar{F}^+_\lambda(\Omega,\tau)\right) = \bar{F}^+_\lambda(\partial\Omega,\tau)$. Writing $\gamma_y(\tau) := \exp_y \tau\nu$, the chain rule and Lemma 5.6 imply

$$\tfrac{\partial}{\partial\tau}\bar{F}^+_\lambda(y,\tau) = \tfrac{\partial}{\partial\tau}\bar{f}^+_\lambda(y,\tau)\cdot\nu_{h(y,\tau)}\left(\gamma_y\left(\bar{f}^+_\lambda(y,\tau)\right)\right), \tag{135}$$

where $h(y,t) := d_\Omega(y) + \bar{f}^+_\lambda(y,t)$ and ν_h is the normal of the parallel set $\partial\Omega_s$ in y. On the other hand, since $\frac{\partial}{\partial\tau}\nu_\tau \equiv 0$ we have

$$\left(\tfrac{\partial^k}{\partial\tau^k}\bar{F}^+_\lambda(x_0,t)_{|\partial\Omega}\right)_{|\tau=0} = \nu(x_0)\cdot\begin{cases} -c_\lambda & k=1,\\ -2H(x_0) & k=2,\\ -3c_\lambda\left(\left(\mathrm{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2\right)\right) & k=3,\\ 24\frac{1}{r^3}C_{131}(\mathrm{M}^n,\Omega,r)\lambda^{-\alpha} & k=4.\end{cases} \tag{136}$$

To compute the variation of the normal vector ν on the hypersurfaces $\bar{F}^+_\lambda(\partial\Omega,t)$, we use the Weingarten equations (10):

$$\begin{aligned} \left(\tfrac{\partial}{\partial\tau}\nu_\tau\right)_{|\tau=0} &= \left(\bar{\mathrm{g}}\left(\tfrac{\partial}{\partial\tau}\nu_\tau,\tfrac{\partial\bar{F}}{\partial x_i}\right)\tfrac{\partial\bar{F}}{\partial x_j}g^{ij}\right)_{|\tau=0} \\ &= \left(-\bar{\mathrm{g}}\left(\nu_\tau,\tfrac{\partial}{\partial x_i}X_\tau\right)\tfrac{\partial\bar{F}}{\partial x_j}g^{ij}\right)_{|\tau=0} \\ &= \left(c_\lambda\bar{\mathrm{g}}\left(\nu_0,\tfrac{\partial}{\partial x_i}\nu_0\right)\tfrac{\partial\bar{F}}{\partial x_j}g^{ij}\right) \\ &\overset{(10)}{=} \left(c_\lambda\bar{\mathrm{g}}\left(\nu_0,h_{kl}g^{lm}\tfrac{\partial\bar{F}}{\partial x^m}\right)\tfrac{\partial\bar{F}}{\partial x_j}g^{ij}\right) \\ &= 0. \end{aligned}$$

In particular, since

$$0 = \tfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\bar{\mathrm{g}}\left(\nu_\tau,\nu_\tau\right)\right)_{|\tau=0} = 2\bar{\mathrm{g}}\left(\tfrac{\partial}{\partial\tau}\nu_\tau,\tfrac{\partial}{\partial\tau}\nu_\tau\right)_{|\tau=0} + 2\bar{\mathrm{g}}\left(\nu_0,\left(\tfrac{\partial^2}{\partial\tau^2}\nu_\tau\right)_{|\tau=0}\right)$$

we have

$$\left(\tfrac{\partial^2}{\partial\tau^2}\nu_\tau\right)_{|\tau=0} \in T_{x_0}\partial\Omega. \tag{137}$$

Now we can compute the first and second derivative of the normal components of the variation,

$$\begin{aligned}
\left(\tfrac{\partial}{\partial\tau}\alpha_\tau\right)_{|\tau=0} &= \left(\tfrac{\partial}{\partial\tau}\left(\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,\tau))X_\tau^\alpha\nu^\beta\right)\right)_{|\tau=0} \\
&= \left(\tfrac{\partial}{\partial x^\varepsilon}\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,\tau))X_\tau^\varepsilon X_\tau^\alpha\nu^\beta\right)_{|\tau=0} \\
&\quad + \left(\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,\tau))\tfrac{\partial}{\partial\tau}X_\tau^\alpha\nu^\beta\right)_{|\tau=0} + \left(\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,\tau))X_\tau^\alpha\tfrac{\partial}{\partial\tau}\nu^\beta\right)_{|\tau=0} \\
&= -2H(x_0), \qquad (138)
\end{aligned}$$

respectively

$$\begin{aligned}
\left(\tfrac{\partial^2}{\partial\tau^2}\alpha_\tau\right)_{|t=0} &= \tfrac{\partial}{\partial\tau}\Bigg(\left(\tfrac{\partial}{\partial x^\varepsilon}\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,\tau))X_\tau^\varepsilon X_\tau^\alpha\nu^\beta\right) \\
&\quad + \left(\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,\tau))\tfrac{\partial}{\partial\tau}X_\tau^\alpha\nu^\beta\right) + \left(\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,\tau))X_\tau^\alpha\tfrac{\partial}{\partial\tau}\nu^\beta\right)\Bigg)_{|t=0} \\
&= -c_\lambda^3\Bigg(\tfrac{\partial}{\partial x^\varepsilon}\tfrac{\partial}{\partial x^\delta}\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,0))\nu^\delta\nu^\varepsilon\nu^\alpha\nu^\beta \\
&\qquad\qquad - 4H(x_0)\left(\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,0))\nu^\alpha\tfrac{\partial}{\partial\tau}\nu^\beta\right)\Bigg) \\
&\quad - 3c_\lambda\left(\left(\mathrm{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2\right)\right)\left(\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,0))\nu^\alpha\nu^\beta\right) \\
&\quad - c_\lambda\left(\bar{\mathrm{g}}_{\alpha\beta}(\bar F(x_0,0))\nu^\alpha\tfrac{\partial^2}{\partial\tau^2}\nu^\beta\right) \\
&\overset{(137)}{=} -3c_\lambda\left(\left(\mathrm{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2\right)\right). \qquad (139)
\end{aligned}$$

Approximating the volume of the barrier sets.

By the above formulas and Lemma 5.4 we may compute the first, second and third variation of volume of the sets $\Omega_\tau := \bar F_\lambda^+(\Omega,\tau)$. By (123) we have

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}\tau}\,|\Omega_t|\right)_{|\tau=0} = \int_{\partial\Omega}\bar g\,(X_0,\nu_0)\,\mathrm{d}\mu = -c_\lambda\,\mathrm{area}\,(\partial\Omega)\,.$$

Since $H = \operatorname{div}\nu$, Equalities (123), (124) and (136) yield that the second variation of volume of Ω under $\bar F_\lambda^+(\cdot,t)$ is given by

$$\begin{aligned}
\left(\tfrac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,|\Omega_\tau|\right)_{|\tau=0} &= \int_{\partial\Omega}\left(\tfrac{\mathrm{d}}{\mathrm{d}\tau}\bar{\mathrm{g}}\,(X_\tau,\nu_\tau)_{|\tau=0} + \bar{\mathrm{g}}\,(X_0,\nu_0)\operatorname{div}X_0\right)\mathrm{d}\mu \\
&= \int_{\partial\Omega}\left(-2H + \bar{\mathrm{g}}\,(-c_\lambda\nu_0,\nu_0)\operatorname{div}(-c_\lambda\nu_0)\right)\mathrm{d}\mu \\
&= \int_{\partial\Omega}\left(-2H + c_\lambda^2 H\right)\mathrm{d}\mu.
\end{aligned}$$

Furthermore, Equalities (137), (139) and (136) imply

$$\frac{\mathrm{d}^3}{\mathrm{d}\tau^3}\ |\Omega_\tau|$$
$$= \int_{\partial\Omega} \Bigg(-3c_\lambda \left(\left(\mathrm{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2 \right) \right) + (-2H)\left(-c_\lambda H - 2c_\lambda H\right)$$
$$+ \left(c_\lambda^2 \left(c_\lambda \left(|A|^2 + \bar{\mathrm{Ric}}\left(\nu,\nu\right) \right) \right) \right) - c_\lambda^3 H^2 - c_\lambda 0 \Bigg) \,\mathrm{d}\mu$$
$$= \int_{\partial\Omega} \left(\left(-3c_\lambda + c_\lambda^3 \right) \left(\mathrm{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2 \right) + \left(6c_\lambda - c_\lambda^3 \right) H^2 \right) \mathrm{d}\mu.$$

Setting

$$G_{25}\left(\lambda,\tau\right) :=$$
$$|\Omega| - \tau c_\lambda \,\mathrm{area}\left(\partial\Omega\right) + \tau^2 \int_{\partial\Omega} \left(-H + \tfrac{c_\lambda^2}{2} H \right) \mathrm{d}\mu$$
$$+ \tfrac{\tau^3}{6} \int_{\partial\Omega} \left(\left(-3c_\lambda + c_\lambda^3 \right) \left(\mathrm{Ric}(\nu,\nu)(x_0) + |A(x_0)|^2 \right) + \left(6c_\lambda - c_\lambda^3 \right) H^2 \right) \mathrm{d}\mu,$$

a Taylor expansion of $\left|\bar{F}^+_\lambda(\Omega,\tau)\right|$ yields

$$\left| \left|\bar{F}^+_\lambda(\Omega,\tau)\right| - G_{25}\left(\lambda,\tau\right) \right| \leqslant \tfrac{\tau^4}{24} \sup_{0\leqslant s\leqslant T_{133}(\lambda)} \left| \tfrac{\mathrm{d}^4}{\mathrm{d}s^4} \left|\bar{F}^+_\lambda(\Omega,s)\right| \right|.$$

Estimating the error term $\left| \frac{\mathrm{d}^4}{\mathrm{d}s^4} \left|\bar{F}^+_\lambda(\Omega,s)\right| \right|$ **.**

Using Corollary (5.5) and (135) the error term on the right-hand side may be estimated from above by

$$\sup_{0\leqslant s\leqslant T_{133}(\lambda)} \left| \tfrac{\mathrm{d}^4}{\mathrm{d}s^4} \left|\bar{F}^+_\lambda(\Omega,s)\right| \right| \leqslant C(n)\,\mathrm{area}\left(\partial\Omega\right) r_{126}(\left|\tfrac{\partial}{\partial\tau}\bar{F}^+_\lambda\right|)^{-3} \left(1 + \left|\tfrac{\partial}{\partial\tau}\bar{F}^+_\lambda\right|^4 \right)$$
$$\leqslant C(n)\,\mathrm{area}\left(\partial\Omega\right) r_{126}(\left|\tfrac{\partial}{\partial\tau}\bar{F}^+_\lambda\right|)^{-3} C_{132}(\lambda)^4.$$

It remains to estimate

$$r_{126}(\left|\tfrac{\partial}{\partial\tau}\bar{F}^+_\lambda\right|)$$
$$= \min\left(r_{19}(\partial\Omega,4), \left|\bar\nabla\bar\nabla\bar\nabla\tfrac{\partial}{\partial\tau}\bar{F}^+_\lambda\right|^{-1/3}, \left|\bar\nabla\bar\nabla\tfrac{\partial}{\partial\tau}\bar{F}^+_\lambda\right|^{-1/2}, \left|\bar\nabla\tfrac{\partial}{\partial\tau}\bar{F}^+_\lambda\right| \right).$$

After differentiating (135) once more in τ, Lemma 5.6 implies

$$\begin{aligned}
\left|\bar\nabla_\nu \tfrac{\partial}{\partial\tau} F^+_\lambda(y,\tau)\right| &= \left|\bar\nabla_\nu \tfrac{\partial}{\partial\tau}\left(\bar F_{\partial\Omega}\circ f^+_\lambda\right)(y,\tau)\right| \\
&= \left|\tfrac{\partial}{\partial\tau}\left(\tfrac{\partial}{\partial\tau} f^+_\lambda(y,\tau)\tfrac{\partial}{\partial\tau}\bar F_{\partial\Omega}\left(\gamma_y(f^+_\lambda(y,\tau))\right)\right)\right| \\
&= \left|\tfrac{\partial^2}{\partial\tau^2} f^+_\lambda(y,\tau)\tfrac{\partial}{\partial\tau}\bar F_{\partial\Omega}\left(\gamma_y(f^+_\lambda(y,\tau))\right)\right| \\
&\overset{(134)}{\leqslant} C_{132}(\lambda) r^{-1}.
\end{aligned}$$

Similarly, we have

$$\left|\bar{\nabla}_\nu\left(\bar{\nabla}_\nu \tfrac{\partial}{\partial\tau} F_\lambda^+(y,\tau)\right)\right| \leqslant C_{132}(\lambda)^2 r^{-2},$$
$$\left|\bar{\nabla}_\nu\left(\bar{\nabla}_\nu\left(\bar{\nabla}_\nu \tfrac{\partial}{\partial\tau} F_\lambda^+(y,\tau)\right)\right)\right| \leqslant C_{131}(\mathrm{M}^n,\Omega,r) C_{132}(\lambda)^3 r^{-3}.$$

To estimate other covariant derivatives, observe that applying Lemma 5.7 to $f = f_\lambda^+(x,\tau)$ yields

$$\left|\bar{\nabla}^{(k)} \bar{f}_\lambda^+(x,\tau)\right| \leqslant C(n) r^{-k}, \quad k = 1,2. \tag{140}$$

In local coordinates, Lemma 5.6 then implies

$$\begin{aligned}
&\left|\bar{\nabla}_\alpha\left(\tfrac{\partial}{\partial\tau}\bar{F}_\lambda^+(y,\tau)\right)\right| \\
&= \left|e_\alpha\left(\tfrac{\partial}{\partial\tau}\bar{f}_\lambda^+(y,\tau)\right) + \tfrac{\partial}{\partial t}\bar{f}_\lambda^+(y,\tau)\left(\bar{\nabla}_\alpha \tfrac{\partial}{\partial\tau}\bar{F}_{\partial\Omega}\right)(\gamma_y(\tfrac{\partial}{\partial\tau} f_\lambda^+(y,\tau)))\right| \\
&\leqslant \left(\left|\bar{\nabla}\tfrac{\partial}{\partial\tau}\bar{f}_\lambda^+(y,\tau)\right| + \left|\tfrac{\partial}{\partial\tau}\bar{f}_\lambda^+(y,\tau)\right| \left|\left(\bar{\nabla}\tfrac{\partial}{\partial\tau}\bar{F}_{\partial\Omega}\right)\right|\right) \\
&\overset{140}{\leqslant} C(n) C_{132}(\lambda) r^{-1} + C(n) C_{132}(\lambda) \left|\left(\bar{\nabla}\tfrac{\partial}{\partial\tau}\bar{F}_{\partial\Omega}\right)\right| \\
&\overset{5.6}{\leqslant} C(n) C_{132}(\lambda) r^{-1}.
\end{aligned}$$

Higher order derivatives are estimated similarly by using the bounds on $|\bar{\nabla}^k \bar{F}_{\partial\Omega}|$ from Lemma 5.6 and the bounds on $|\bar{\nabla}^{(k)} \frac{\partial}{\partial\tau}\bar{f}_\lambda^+(x,\tau)|$ from Lemma 5.7. The occurring derivatives of the Christoffel symbols are dominated by the curvature terms in the definition of r. Carrying out the differentiation will yield that for some $k \in \mathbb{N}$

$$r_{126}(\left|\tfrac{\partial}{\partial\tau}\bar{F}_\lambda^+\right|)^{-1} \leqslant C(n) C_{132}(\lambda)^k C_{131}(\mathrm{M}^n,\Omega,r) r^{-1}.$$

Estimation of the volume of the superlevel-sets $\Omega_\lambda(t)$.

We substitute $t := \sqrt{\tau}$ and choose $\alpha = {}^{\beta}/_{k}$. Then it holds for all $0 \leqslant t \leqslant r^2 C_{132}(\lambda)^{-2} C_{131}(\mathrm{M}^n,\Omega,r)^{-2}$

$$\begin{aligned}
G_{25}(\lambda,t) - C(n,\beta) C_{131}(\mathrm{M}^n,\Omega,r)\lambda^{-\beta}\tfrac{t^2}{r^3}\,\mathrm{area}(\partial\Omega) &\leqslant \\
\leqslant \left|\bar{F}_\lambda^-(\Omega,t)\right| \leqslant \left|\Omega_\lambda(t^2)\right| &\leqslant \left|\bar{F}_\lambda^+(\Omega,t)\right| \leqslant \\
\leqslant G_{25}(\lambda,t) + C(n,\beta) C_{131}(\mathrm{M}^n,\Omega,r)\lambda^{-\beta}\tfrac{t^2}{r^3}\,\mathrm{area}(\partial\Omega)&.
\end{aligned}$$

□

6 Generalization to non-compact manifolds and heat equations with potentials

In the previous chapters we assumed the Riemannian manifold $(\mathrm{M}^n, \bar{\mathrm{g}})$ to be closed and $u_\Omega : \mathrm{M}^n \times (0,\infty)$ to be a solution for the heat equation

$$\begin{cases} \bar{\Delta}_{\bar{\mathrm{g}}} u_\Omega(x,t) = \frac{\partial}{\partial t} u(x,t) & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u_\Omega(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n. \end{cases} \tag{HE}$$

where $\Omega \subset \mathrm{M}^n$. In this chapter, we consider generalizations of this equation. The proofs of the results will be almost analogous to the results in chapters 2, 3, 4 and 5. For this reason, we omit details that are treated in these chapters and focus only on parts of the proofs where a modification is needed. We examine the following problems.

(*i*) In Section 6.1 we replace the compactness assumption of (M^n, g) by the assumption of bounded geometry. We derive expansion formulae for the pointwise behaviour of u_Ω in Theorem 6.2, the heat content (HC) in Theorem 6.3, the mean heat content (MHC) in Theorem 6.5 and the evolution of the level sets of u_Ω in Theorem 6.4.

(*ii*) In Section 6.8 we assume that $\bar{\mathrm{g}}(t)$ is a time-dependent metric on a compact manifold that satisfies for a time-dependent symmetric 2-tensor $\{\bar{\mathrm{X}}_{\alpha\beta}\}_{1\leqslant\alpha,\beta\leqslant n}$ the evolution equation

$$\frac{\partial}{\partial t}\bar{\mathrm{g}}_{\alpha\beta} = 2\bar{\mathrm{X}}_{\alpha\beta}.$$

For a function $f \in C^\infty(\mathrm{M}^n)$ and a potential term $\bar{\mathrm{Q}} \in C^\infty(\mathrm{M}^n \times [0,T])$ we then consider differential equations of the form

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_{g(t)} + \bar{\mathrm{Q}}\right) u_{\Omega,f,\bar{\mathrm{Q}}} = 0 & (x,t) \in \mathrm{M}^n \times (0,\infty) \\ u_{\Omega,f,\bar{\mathrm{Q}}}(x,0) = (f \cdot \chi_\Omega)(x) & x \in \mathrm{M}^n. \end{cases} \tag{$\star$}$$

In Theorem 6.8 we derive expansion formulae of the pointwise behaviour solutions of $u_{\Omega,f,\bar{\mathrm{Q}}}$ and in Theorem 3.2 of the generalized heat content, i.e. of the quantity $\int_{\Omega^c} u_{\Omega,f,\bar{\mathrm{Q}}}(y)\omega_t(y)$. At the end of this section we consider the most interesting system of this form, Ricci-flow coupled with the adjoint heat equation, in Example 6.10.

6.1 Non-compact manifolds

In the following we assume that $(\mathrm{M}^n, \bar{\mathrm{g}})$ is a complete Riemannian manifold of bounded geometry, i.e. all sectional curvatures are bounded. The set $\Omega \subset \mathrm{M}^n$ is assumed to be a compact set such that the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4 and the radius

$$r := \min_{x \in \mathrm{M}^n} r_{18}(5, x) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor $\overline{\mathrm{Riem}}$, the first two derivatives of the second fundamental form of $\partial\Omega$ and the injectivity radius of M^n and Ω, is positive. Furthermore, suppose $u_\Omega : \mathrm{M}^n \times (0, \infty) \to \mathbb{R}$ is the solution of the heat equation (HE). We study how the results regarding the pointwise expansion of $u_\Omega(\cdot, t)$ (Theorem 2.5), the heat content (Theorem 3.2), the evolution of the level-sets (Theorem 4.3) and the mean heat content (Theorem 3.2) transfer to this setting. The results in this section will be almost analogous to the compact setting. We only need to adjust the steps where the compactness is used.

To this end, we need the following lemma regarding the volume growth and integral bounds of the heat kernel. After a slight modification, a proof of this lemma is given in [CC03, Proposition B.7.3].

Lemma 6.1 *For any $n \in \mathbb{N}$ and $\eta > 1$ there are constants $0 < \varepsilon(n, \eta)$ and $C(n, \eta) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a complete Riemannian manifold with bounded geometry. Then there exist constants $K_1, K_2 < \infty$ such that for any $x \in \mathrm{M}^n$ the volume of any ball $B_s(x)$ of radius $s \geqslant 0$ around x satisfies the following exponential bound,

$$|B_s(x)| \leqslant K_1 \mathrm{e}^{K_2 s} .$$

Furthermore, for all times $0 < t < \varepsilon(n, \eta) K_2^{-1} r$

$$\int_{\mathrm{M}^n \setminus B_r(y)} \varrho(x, y, t) \leqslant C(n, \eta) K_1 \mathrm{e}^{8K_2^2 t}\, r^{-n/2} \mathrm{e}^{-\frac{r^2}{4\eta t}} .$$

In the construction of the solution of the heat equation (HE) via a parametrix the compactness assumption is used in order to apply Lemma 2.16 and Lemma 2.17 that infer regularity of the Volterra series

$$\sum_{l=0}^{\infty} \int_0^t \int_{\mathrm{M}^n} \int_\Omega \bar{\Box}_x (\varrho_{k,r}(z, x, s))^{*l} \varrho_{k,r}(y, z, t-s) \omega(x) \omega(z) \mathrm{d}s,$$

where $k \geqslant 1$ is fixed. It seems clear from the way these lemmata are proved in [Gri04], respectively [Cho+10, Chapter 23], that only the finiteness of the integrals

$$\int_\Omega \bar{\square}_x \left(\varrho_{k,r}(x,y,t)\right)^{*l} \omega(x), \quad l \in \mathbb{N}$$

is needed to conclude their assertions. Choosing $k = 1$ this, on the other hand, is guaranteed by Lemma 2.18 whenever the heat kernel coefficients ϱ_0, ϱ_1 and their second derivatives are bounded. Moreover, by Lemma 2.19, for any $\eta > 1$ the estimate

$$\left|\int_\Omega \varrho(x,y,t) - \varrho_{k,r}(x,y,t)\omega(x)\right| \leqslant C_{62}(T,\eta) \int_\Omega \varrho(x,y,\eta t)\omega(x), \quad 0 < t < T,$$

is independent of the volume of Ω and M^n. Thus, we may use Lemma 2.19 to prove the non-compact version of Theorem 2.5. We note that in contrast to the compact case, an exponential decay factor $1/4\eta$ with $\eta > 1$ has to be chosen.

Theorem 6.2 *For any $n \in \mathbb{N}$ and $\eta > 1$ there are constants $0 < \varepsilon(n,\eta)$ and $C(n,\eta) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a complete Riemannian manifold with bounded geometry and $\Omega \subset \mathrm{M}^n$ a set with compact boundary such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := \min_{x \in \mathrm{M}^n} r_{18}(5, x) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor $\overline{\mathrm{Riem}}$, the first two derivatives of the second fundamental form of $\partial\Omega$ and the injectivity radius of M^n and Ω, is positive. Then we may approximate the solution u_Ω of the heat equation (HE) in the following way.

For $y \in T(\partial\Omega, \mathrm{inj}_\Omega)$ let $x_0 \in \partial\Omega$ be the unique base point of y and $s = d_\Omega(y)$ the distance of y to the boundary of Ω. The point y is then given by $y = \exp_{x_0} s\nu(x_0)$. We define the function $v_\Omega : T(\partial\Omega, \mathrm{inj}_\Omega) \times (0,\infty) \to \mathbb{R}_+$ by setting

$$\begin{aligned} v_\Omega(y,t) :=& \Phi\left(-\tfrac{s}{\sqrt{t}}\right) + \tfrac{\mathrm{e}^{-s^2/4t}}{\sqrt{4\pi}}\Big(- \sqrt{t}H(x_0) + \sqrt{t}s \\ & \left(\tfrac{1}{4}\left(-2\left(\overline{\mathrm{Ric}}(x_0)(\nu(x_0),\nu(x_0)) + |A(x_0)|^2\right) + H(x_0)^2\right)\right)\Big). \end{aligned}$$

Here, H, ν, $|A|$ are evaluated with respect to the curvature of the boundary $\partial\Omega$ and $\Phi : \mathbb{R} \to (0,1)$ is the Gaussian error function from Definition 2.3. Then we have the uniform estimate

$$|u_\Omega(y,t) - v_\Omega(y,t)| \leqslant C(n,\eta)\tfrac{t^{3/2}}{r^3}\left(1 + \tfrac{s^2}{t}\right)\tfrac{\mathrm{e}^{-\frac{s^2}{4\eta t}}}{\sqrt{4\pi}}$$

for all $t \in (0, \varepsilon(n)r^2)$ and $y \in T(\partial\Omega, \varepsilon(n)r)$.

The above Lemma also yields an asymptotic expansion of the heat content.

Theorem 6.3 (Heat content) *For any $n \in \mathbb{N}$ there are constants $0 < \varepsilon(n)$ and $C(n) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a complete Riemannian manifold with bounded geometry and $K_1, K_2 < \infty$ such that

$$|B_s(x)| \leqslant K_1 \mathrm{e}^{K_2 s}\,, \quad s \geqslant 0.$$

Suppose $\Omega \subset \mathrm{M}^n$ is a compact set such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := \min_{x \in \mathrm{M}^n} r_{18}(5, x) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor of the ambient space M^n, the first two derivatives of the second fundamental form of $\partial\Omega$ and the injectivity radius of M^n and Ω, is positive. We define the function

$$Z_\Omega(t) := \tfrac{\sqrt{t}}{\sqrt{\pi}} \operatorname{area}(\partial\Omega) + \\ - \tfrac{t^{3/2}}{12\sqrt{\pi}} \left(\left(2 \int_{\partial\Omega} \left(\bar{\mathrm{Ric}}(\nu, \nu) + |A|^2 \right) \mathrm{d}\mu \right) + \left(\int_{\partial\Omega} H^2 \mathrm{d}\mu \right) \right).$$

Then the heat content $\mathrm{HC}_\Omega(t)$ defined by

$$\mathrm{HC}_\Omega(t) := \int_{\Omega^c} \int_\Omega \varrho(x, y, t) \omega(x) \omega(y)$$

satisfies the uniform estimate

$$|\mathrm{HC}_\Omega(t) - Z_\Omega(t)| \leqslant C(n) t^2 \left(\tfrac{1}{r^3} \operatorname{area}(\partial\Omega) + |\Omega|\, K_1 \mathrm{e}^{8K_2^2 t} \tfrac{1}{r^{n+4}} \right)$$

for all $t \in (0, \varepsilon(n) r^2)$.

Proof. As in the compact case we separate the heat around the boundary and the heat away from the boundary;

$$\int_{\Omega^c} \int_\Omega \varrho(x, y, t) \omega(x) \omega(y) = \int_{T(\partial\Omega, r) \cap \Omega^c} \int_\Omega \varrho(x, y, t) \omega(x) \omega(y) \\ + \int_{\Omega^c \setminus T(\partial\Omega, r)} \varrho(x, y, t) \omega(x) \omega(y).$$

Note that the exponential decay factor $1/4\eta$ does not affect the integrability of the error term. Thus, the first integral is computed exactly as in Theorem 3.2.

It only remains to estimate the second integral, i.e. the heat content away from the boundary of Ω. Observe that for any $y \in \Omega^c \backslash T(\partial\Omega, r)$ we have $B_r(y) \subset \Omega^c$. Therefore, by choosing $\eta = 2$ in Lemma 6.1

$$\begin{aligned}\int_{\Omega^c \backslash T(\partial\Omega,r)} \varrho(x,y,t)\omega(x)\omega(y) &= \int_\Omega \int_{\Omega^c \backslash T(\partial\Omega,r)} \varrho(x,y,t)\omega(y)\omega(x) \\ &\leqslant \int_{\mathrm{M}^n \backslash B_r(y)} \int_\Omega \varrho(x,y,t)\omega(x)\omega(y) \\ &\overset{6.1}{\leqslant} |\Omega|\, C(n) K_1 \mathrm{e}^{8K_2^2 t}\, r^{-n} \mathrm{e}^{-\frac{r^2}{8t}} \\ &\leqslant |\Omega|\, C(n) K_1 \mathrm{e}^{8K_2^2 t}\, r^{-n-4} t^2. \qquad \square\end{aligned}$$

We now generalize Theorem 4.3, the evolution of the level-sets

$$\Sigma_\lambda(t) := \{y \in \mathrm{M}^n : u_\Omega(y,t) = \lambda\},$$

to the non-compact case. Recall that the normal distance function for $y \in \Sigma_\lambda(t)$ was defined to be the value $d_{\lambda,t}(y)$ such that $y = \exp_{x_0} d_{\lambda,t}(y)\nu(x_0)$ for some $x_0 \in \partial\Omega$. To derive an expansion of $d_{\lambda,t}(y)$ with an error term that behaves nicely as $\lambda \to 0$, it was important to have logarithmic bounds on the normal distance function (see the Expansion of order $O(\sqrt{t})$ in the proof of Theorem 4.3). The lower logarithmic bound in Lemma 4.7 transfers to the non-compact case. However, for the upper bound, Lemma 4.6, the compactness was used. But, using Lemma 6.1, it possible to prove in a similar fashion a bound of the form

$$|d_{\lambda,t}(y)|^2 \leqslant -4\eta \max\left(\log\lambda, \log 1-\lambda, C(\eta, K_1, K_2, n)\right) t \tag{141}$$

for any $\eta > 1$. Using this estimate we derive the following Theorem.

Theorem 6.4 (Distance to the level-sets) *For any $n \in \mathbb{N}$ and $\alpha > 0$ there are constants $0 < \varepsilon(n,\alpha)$ and $C(n,\alpha) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ be a complete Riemannian manifold with bounded geometry and $K_1, K_2 < \infty$ such that

$$|B_s(x)| \leqslant K_1 \mathrm{e}^{K_2 s}, \quad s \geqslant 0.$$

Suppose $\Omega \subset \mathrm{M}^n$ is a compact set such the embedding $F : \partial\Omega \to \mathrm{M}^n$ is of class C^4. Further suppose, that the radius

$$r := \min\left(\min_{x \in \mathrm{M}^n} r_{18}(5,x), K_2^{-1}\right) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor of the ambient space M^n*, the first two derivatives of the second fundamental form of* $\partial\Omega$ *and the injectivity radius of* M^n *and* Ω*, is positive. Fix* $\lambda \in (0,1)$ *and set (compare with Estimate (141))*

$$t(\lambda) := \tfrac{r^2}{-16\max(\log\lambda, \log 1-\lambda, C(K_1,K_2,n))}.$$

Then we set for $y \in \Sigma_\lambda(t)$ *with* $y = \exp_{x_0} d_{\lambda,t}(y)\nu$

$$\tilde{d}_{\lambda,t}(y) := -c_\lambda\sqrt{t} - tH - t^{3/2}\tfrac{c_\lambda}{2}\left(\left(\mathrm{Ric}(\nu,\nu) + |A|^2\right)\right),$$

where the curvature terms are to be understood evaluated in x_0 *and* c_λ *is the inverse Gaussian error function. Then* $\tilde{d}_{\lambda,t}(y)$ *is an approximation of the normal distance function* $d_\lambda(t)$*. For some constant* $C(K_1,K_2) < \infty$ *depending on* K_1 *and* K_2 *and all* $t \in (0, \varepsilon(n)t_{94}(\lambda))$ *we have*

$$\left|d_{\lambda,t}(y) - \tilde{d}_{\lambda,t}(y)\right| \leqslant C(\alpha,n)C(K_1,K_2)\tfrac{t^2}{r^3}\lambda^{-\alpha}. \tag{142}$$

Proof. Since Theorem 2.5 is used to prove Theorem 4.3, the only difference to the compact case is the estimation of the term

$$\max_{s\in\left[c_\lambda, \frac{-d_{\lambda,t}(y)}{\sqrt{t}}\right]} \left(\frac{\mathrm{e}^{\frac{s^2}{4}}}{\sqrt{4\pi}}\right) \frac{\mathrm{e}^{-\frac{d_{\lambda,t}(y)^2}{4\eta t}}}{\sqrt{4\pi}} \quad , \eta > 1$$

if $|{}^{-d_{\lambda,t}(y)}/\sqrt{t}| \geqslant |c_\lambda|$. But Estimate (141) implies that, by choosing $\eta > 1$ such that $\alpha = \eta - 1$, we have

$$\max_{s\in\left[c_\lambda, \frac{-d_{\lambda,t}(y)}{\sqrt{t}}\right]} \left(\frac{\mathrm{e}^{\frac{s^2}{4}}}{\sqrt{4\pi}}\right) \frac{\mathrm{e}^{-\frac{d_{\lambda,t}(y)^2}{4\eta t}}}{\sqrt{4\pi}} \leqslant C(\eta, K_1, K_2, n)\lambda^{-(\eta-1)}. \qquad \square$$

Since the exponent $\alpha > 0$ in the above theorem can be chosen arbitrarily small, we are able to deduce the expansion formula of the mean heat content

$$\mathrm{MHC}_\Omega(t) = \int_{\{u_\Omega(\cdot,t)\geqslant 1/2\}} u_\Omega(y,t)\,\omega(y), \tag{MHC}$$

Theorem 5.1, in the non-compact case.

Theorem 6.5 *For any* $n \in \mathbb{N}$ *there are constants* $0 < \varepsilon(n)$ *and* $C(n) < \infty$ *such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}})$ *be a complete Riemannian manifold with bounded geometry and* $K_1, K_2 < \infty$ *such that*

$$|B_s(x)| \leqslant K_1 \mathrm{e}^{K_2 s}, \quad s \geqslant 0.$$

Suppose $\Omega \subset \mathrm{M}^n$ *is a compact set such the embedding* $F : \partial\Omega \to \mathrm{M}^n$ *is of class* C^4. *Further suppose, that the radius*

$$r := \min\left(\min_{x \in \mathrm{M}^n} r_{18}(5, x), K_2^{-1}\right) > 0$$

defined in Section 1.3, that depends on the first six derivatives of the curvature tensor of the ambient space M^n*, the first two derivatives of the second fundamental form of* $\partial\Omega$ *and the injectivity radius of* M^n *and* Ω*, is positive. We define the function*

$$\begin{aligned} G_\Omega(t) &:= \tfrac{\sqrt{t}}{\sqrt{\pi}} \operatorname{area}(\partial\Omega) + \tfrac{t}{2} \int_{\partial\Omega} H \operatorname{vol}_{\partial\Omega} \\ &\quad + t^{\frac{3}{2}} \left(-\tfrac{1}{6\sqrt{\pi}} \int_{\partial\Omega} \left(\operatorname{Ric}(\nu, \nu) + |A|^2 \right) \operatorname{vol}_{\partial\Omega} - \tfrac{1}{3\sqrt{\pi}} \int_{\partial\Omega} H^2 \operatorname{vol}_{\partial\Omega} \right). \end{aligned}$$

Then there is a constant $C(n, K_1, K_2) < \infty$ *such that for all* $t \in (0, \varepsilon(n) r^2)$

$$|\mathrm{MHC}_\Omega(t) - G_\Omega(t)| \leqslant \left(1 + C(n, K_1, K_2) \mathrm{e}^{K_2 \operatorname{diam}(\Omega)}\right) \tfrac{t^2}{r^4}.$$

Proof. Fix $\alpha > 0$, $t > 0$ and denote the superlevel sets of u_Ω by $\Omega_\lambda(t)$. By the previous Theorem, an analogous expansion of the volume of the superlevel sets as in the compact case, Proposition 5.8, is valid for all $\lambda \in (0, 1/2]$ satisfying

$$C(n, \alpha) \tfrac{t}{r^2} \leqslant \lambda^{\alpha/2}.$$

Choosing $\alpha = 1/2$, Cavalieri's principle yields

$$\begin{aligned} \int_{\{u_\Omega(\cdot,t) \geqslant 1/2\}} u_\Omega(y, t)\, \omega(y) &= \int_{C(n,\alpha)^4 \frac{t^4}{r^8}}^{1/2} \left(|\Omega_\lambda(t)| - |\Omega_{1/2}(t)| \right) \mathrm{d}\lambda \\ &\quad + \int_0^{C(n,\alpha)^4 \frac{t^4}{r^8}} \left(|\Omega_\lambda(t)| - |\Omega_{1/2}(t)| \right) \mathrm{d}\lambda. \end{aligned}$$

As in the compact case, the error term in the expansion of superlevel sets is integrable on the interval $[0, 1]$. Therefore, the first integral may be computed as in the compact case. It remains to estimate the second integral from above. There is some $z \in \mathrm{M}^n$ such that the set Ω is contained in a ball B around z of diameter $\operatorname{diam}(\Omega)$ and therefore

$$u_\Omega(\cdot, t) \leqslant u_B(\cdot, t),$$

where u_B is the solution of (HE) with initial data χ_B. In particular, $\Omega_\lambda(t) \subset B_\lambda(t)$. Esimate (141) now implies

$$B_\lambda(t) \subset B_{\operatorname{diam}(\Omega)+\sqrt{t}\left(4\sqrt{-\log\lambda}+C(n,K_1,K_2)\right)}.$$

By the upper bound on the volume of balls from Lemma 6.1 we therefore have

$$\begin{aligned}|\Omega_\lambda(t)| &\leqslant K_1 \mathrm{e}^{K_2 \operatorname{diam}(\Omega)+K_2\sqrt{t}C(n,K_1,K_2)}\, \mathrm{e}^{4K_2\sqrt{t}\sqrt{-\log\lambda}} \\ &\leqslant C(K_1,K_2)\mathrm{e}^{K_2\operatorname{diam}(\Omega)}\, \lambda^{-1/2}.\end{aligned}$$

Hence, we may complete the proof by estimating

$$\begin{aligned}\int_0^{C(n)\frac{t^4}{r^8}} \left(|\Omega_\lambda(t)| - |\Omega_{1/2}(t)|\right) \mathrm{d}\lambda &\leqslant C(n,K_1,K_2)\mathrm{e}^{K_2\operatorname{diam}(\Omega)} \int_0^{C(n)\frac{t^4}{r^8}} \lambda^{-1/2}\mathrm{d}\lambda \\ &= C(n,K_1,K_2)\mathrm{e}^{K_2\operatorname{diam}(\Omega)}\, \tfrac{t^2}{r^4}.\end{aligned}$$

□

6.2 Heat equations with potentials and evolving metrics

In this section we discuss the short-time asymptotics of solutions of heat equations of the following type. Suppose $(\mathrm{M}^n, \bar{\mathrm{g}}(t))$ is a one-parameter family of closed Riemannian manifolds such that for a time-dependent symmetric 2-tensor $\bar{\mathrm{X}}_{\alpha\beta}$, the metric components satisfy the evolution equation

$$\tfrac{\partial}{\partial t}\bar{\mathrm{g}}_{\alpha\beta} = 2\bar{\mathrm{X}}_{\alpha\beta}. \tag{143}$$

Let $\bar{\mathrm{L}}_t$ be a heat operator with a potential term $\bar{\mathrm{Q}} \in C^\infty(\mathrm{M}^n \times [0,T])$;

$$\bar{\mathrm{L}}_t := \tfrac{\partial}{\partial t} - \bar{\Delta}_{\bar{\mathrm{g}}(t)} + \bar{\mathrm{Q}}. \tag{$\bar{\mathrm{L}}_t$}$$

Finally, for some $f \in C^\infty(\mathrm{M}^n)$ we consider differential equations of the form

$$\begin{cases} \bar{\mathrm{L}}_t u_{\Omega,f,\bar{\mathrm{Q}}} = 0 & \text{for all } (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u_{\Omega,f,\bar{\mathrm{Q}}}(x,0) = f(x)\cdot\chi_\Omega(x) & \text{for all } x \in \mathrm{M}^n. \end{cases} \tag{$\star$}$$

In the previous chapters, error terms involved, for some fixed $k \geqslant 1$, the radius $r_{19}(k, \partial\Omega)$. It depends on the curvature of the boundary of Ω and the ambient space (see from Section 1.3). In this section the error terms involve radii $r_{146}(k, \bar{\mathrm{X}}, \bar{\mathrm{Q}})$ and $r_{148}(k, \bar{\mathrm{X}}, \bar{\mathrm{Q}}, \partial\Omega, f)$ that additionally depend on $\bar{\mathrm{X}}$, $\bar{\mathrm{Q}}$, f and the injectivity radii of the family of Riemannian manifolds $(\mathrm{M}^n, \bar{\mathrm{g}}(t))_{0\leqslant t\leqslant T}$ for some $T > 0$. It is defined in the following way and its positivity will be assumed.

Remark 6.6 (The radii $r_{146}(k, \bar{\mathrm{X}}, \bar{\mathrm{Q}})$ and $r_{148}(k, \bar{\mathrm{X}}, \bar{\mathrm{Q}}, \partial\Omega, f)$) We assume that there is a $\varepsilon(n) > 0$ and a radius $r_{144} > 0$ that bounds the injectivity radii of the family of manifolds $(\mathrm{M}^n, \bar{\mathrm{g}}(t))$ from below, i.e.

$$\mathrm{inj}_{\bar{\mathrm{g}}(t)} > r_{144} \quad \text{for all } 0 \leqslant t \leqslant \varepsilon(n) r_{144}^2. \tag{144}$$

Next, we fix $k \geqslant 1$ and $x \in \mathrm{M}^n$ and define the radius $r(k, x, \bar{\mathrm{X}}, t)$ with respect to the metric $\bar{\mathrm{g}}(t)$ analogous to $r_{17}(k, x)$ by replacing $\overline{\mathrm{Riem}}$ in its definition by $\bar{\mathrm{X}}(t)$. By the evolution equations (143) there is a constant $0 < \varepsilon(n, k)$ such that

$$\left|\bar{\nabla}^i \overline{\mathrm{Riem}}(t)\right| \leqslant 2 \left|\bar{\nabla}^i \overline{\mathrm{Riem}}(0)\right|$$

for all $0 \leqslant t \leqslant \varepsilon(n, k) r(k, \bar{\mathrm{X}})^2$ and $0 \leqslant i \leqslant k$. Hence, we may estimate the radii $r_{17}(k, x, t)$ with respect to the metric $\bar{\mathrm{g}}(t)$ from below by

$$r_{17}(k, x, t) \geqslant \min\left(\tfrac{1}{2} r_{17}(k, x, 0), r_{144}, r(k, \bar{\mathrm{X}})\right) =: r_{145}(k, x, \bar{\mathrm{X}}, t) > 0. \tag{145}$$

Moreover, Inequality (145) implies that there is a $0 < \varepsilon(n, k)$ such that

$$r_{145}(k-1, x, \bar{\mathrm{X}}, t) \geqslant \tfrac{1}{2} r_{145}(k, x, \bar{\mathrm{X}}, 0)$$

for all $0 \leqslant t \leqslant \varepsilon(n, k) r_{145}(k, x, \bar{\mathrm{X}}, t)^2$. In the same way we define the radius $r(k, x, \bar{\mathrm{Q}}, t)$ and set

$$r_{146}(k, \bar{\mathrm{X}}, \bar{\mathrm{Q}}) := \min_{x \in \mathrm{M}^n} \min\left(r_{145}(k+1, x, \bar{\mathrm{X}}, 0), r_{145}(k+1, x, \bar{\mathrm{Q}}, 0)\right). \tag{146}$$

Now let $\Omega \subset \mathrm{M}^n$ be a set such that $\partial\Omega$ is an embedded hypersurface of the Riemannian manifolds $(\mathrm{M}^n, \bar{\mathrm{g}}(t))$ and suppose there is a $r_{147}(\partial\Omega) > 0$ satisfying

$$\mathrm{inj}_{\partial\Omega} > r_{147}(\partial\Omega) \quad \text{for all } 0 \leqslant t \leqslant \varepsilon(n) r_{147}(\partial\Omega)^2. \tag{147}$$

Finally, for a function $f \in C^{k+1}(\mathrm{M}^n)$ we define the radius $r(k, x, t, f)$ like $r_{145}(k, x, \bar{\mathrm{X}}, t)$ and set

$$r_{148}(k, \bar{\mathrm{X}}, \bar{\mathrm{Q}}, \partial\Omega, f) := \min\left(\min_{x \in \mathrm{M}^n} r_{145}(k+1, x, \bar{\mathrm{X}}, 0), r_{147}(\partial\Omega), r_{146}(k, \bar{\mathrm{X}}, \bar{\mathrm{Q}})\right). \tag{148}$$

Together with the results in [Cho+10, Chapter 24], techniques from Section 2.1 show that the corresponding fundamental solution $\varrho_{\bar{\mathrm{Q}}}(x, y, t)$ of $(\star)$ can be approximated in the following way.

Theorem 6.7 *For any $n \in \mathbb{N}$, $k > {}^{n}\!/\!_{2} + 2$ and $\eta > 1$ there are constants $0 < \varepsilon(n,\eta), C(n,\eta,k), C(n) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}}(t))_{0\leqslant t\leqslant T}$, be a family of closed Riemannian manifolds satisfying the evolution equation $\frac{\partial}{\partial t}\bar{\mathrm{g}}_{\alpha\beta} = 2\bar{\mathrm{X}}_{\alpha\beta}$, let $\bar{\mathrm{Q}} \in C^\infty(\mathrm{M}^n \times [0,T])$ be a potential term and set

$$\bar{\mathrm{L}}_t := \tfrac{\partial}{\partial t} - \bar{\Delta}_{\bar{\mathrm{g}}(t)} + \bar{\mathrm{Q}}. \qquad (\bar{\mathrm{L}}_t)$$

Furthermore, we assume the radius

$$r := r_{146}(k, \bar{\mathrm{X}}, \bar{\mathrm{Q}}) \qquad (149)$$

from Remark 6.6 to be positive. Fix $y \in \mathrm{M}^n$. Then, for all $0 \leqslant t \leqslant \varepsilon(n\eta)r^2$, the solution of the equation

$$\begin{cases} \bar{\mathrm{L}}_t \varrho_{\bar{\mathrm{Q}}}(x,y,t) = 0 & \text{for all } (x,t) \in \mathrm{M}^n \times (0,\infty) \\ \varrho_{\bar{\mathrm{Q}}}(x,y,0) = \delta_y(x) & \text{for all } x \in \mathrm{M}^n \end{cases}$$

satisfies for some smooth functions $\varrho_{\bar{\mathrm{Q}},i} : \mathrm{M}^n \times \mathrm{M}^n \times [0, \varepsilon(n,\eta)r^2] \to \mathbb{R}$ the following estimate:

$$\left| \varrho_{\bar{\mathrm{Q}}}(x,y,t) - \frac{\mathrm{e}^{-\frac{d_t(x,y)^2}{4t}}}{(4\pi t)^{n/2}} \sum_{i=0}^{k} t^i \varrho_{\bar{\mathrm{Q}},i}(x,y,t) \right| \leqslant C(n,\eta,k) \frac{t^{k+1}}{r^{2k+2}} \frac{\mathrm{e}^{-\frac{d(x,y)^2}{4\eta t}}}{(4\pi\eta t)^{n/2}}.$$

Furthermore, the first two kernel coefficients satisfy, in a normal coordinate system in y with respect to the metric $g(t)$,

$$\left| \varrho_{\bar{\mathrm{Q}},0}(x,y,t) - \left(1 + \left(\tfrac{1}{12}\,\bar{\mathrm{R}}_{\alpha\beta} + \tfrac{1}{4}\bar{\mathrm{X}}_{\alpha\beta}\right) x_t^\alpha x_t^\beta\right) \right| \leqslant C(n) \tfrac{d_t(x,y)^3}{r^{-3}},$$
$$\left| \varrho_{\bar{\mathrm{Q}},1}(x,y,t) - \left(\tfrac{1}{6}\,\bar{\mathrm{R}} + \tfrac{1}{2}\operatorname{tr}\bar{\mathrm{X}} - \bar{\mathrm{Q}}\right) \right| \leqslant C(n) \tfrac{d_t(x,y)}{r^{-3}}.$$

Using this Theorem, we may compute the asymptotics of solutions of the Equation ($\star$).

Theorem 6.8 *For any $n \in \mathbb{N}$ and $\eta > 1$ there are constants $\varepsilon(n,\eta) > 0$ and $C(n,\eta) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}}(t))_{0\leqslant t\leqslant T}$, be a family of closed Riemannian manifolds satisfying the evolution equation $\frac{\partial}{\partial t}\bar{\mathrm{g}}_{\alpha\beta} = 2\bar{\mathrm{X}}_{\alpha\beta}$, let $\bar{\mathrm{Q}} \in C^\infty(\mathrm{M}^n \times [0,T])$ be a potential term and set

$$\bar{\mathrm{L}}_t := \tfrac{\partial}{\partial t} - \bar{\Delta}_{\bar{\mathrm{g}}(t)} + \bar{\mathrm{Q}}. \qquad (\bar{\mathrm{L}}_t)$$

For a set $\Omega \subset \mathrm{M}^n$ with smoothly embedded boundary and a function $f \in C^\infty(\mathrm{M}^n)$ we assume the radius

$$r := r_{148}({}^{n}\!/\!_{2} + 2, \bar{\mathrm{X}}, \bar{\mathrm{Q}}, \partial\Omega, f)$$

from Remark 6.6 to be positive.

For $y = \exp_{x_0} s_t \nu(x_0, t)$ *with* $|s_t| \leqslant \varepsilon(n,\eta) r$ *and* $t \in (0, \varepsilon(n,\eta) r^2)$ *we define the function*

$$\begin{aligned} v_\Omega(y,t) := f(y)\Phi\left(-\tfrac{s_t}{\sqrt{t}}\right) + \sqrt{t}\frac{\mathrm{e}^{-\frac{s_t^2}{4t}}}{\sqrt{4\pi}}\left(-2\bar{\nabla}_\nu f(y) - f(y) H\right) \\ + t\Phi\left(-\tfrac{s_t}{\sqrt{t}}\right)\left(\tfrac{1}{2} f(y)\operatorname{tr}\bar{\mathrm{X}} + \Delta_{g(t)} f(y) - f(y)\bar{\mathrm{Q}}(y)\right) \\ + \sqrt{t}s_t \frac{\mathrm{e}^{-\frac{s_t^2}{4t}}}{\sqrt{4\pi}} \\ \times \left(f(y)\left(\tfrac{1}{2}\bar{\mathrm{X}}(\nu,\nu) - \tfrac{1}{2}\left(|A|^2 + \overline{\mathrm{Ric}}\,(\nu,\nu)\right) + \tfrac{1}{4}H^2\right)\right. \\ \left. + \left(\bar{\nabla}_\nu f(y)\right) H + \bar{\nabla}_\nu \bar{\nabla}_\nu f(y)\right), \end{aligned}$$

where all quantities are evaluated in (y,t) *with respect to the parallel hypersurfaces* $\partial\Omega_{s_t}$*. Then the solution* $u_{\Omega,f,\bar{\mathrm{Q}}}$ *of the Equation* $(\star)$ *satisfies*

$$|u_\Omega(y,t) - v_\Omega(y,t)| \leqslant C(n,\eta)\tfrac{t^2}{r^4}\mathrm{e}^{-\frac{d(x,y)^2}{4\eta t}}\left(1 + \tfrac{|\mathrm{M}^n|}{r^n}\right)$$

Proof. The proof of this theorem is almost analogous to the proof of Theorem 2.5. Since the solution of $(\star)$ is given by

$$u_{\Omega,f,\bar{\mathrm{Q}}}(y,t) = \int_\Omega \varrho_{\bar{\mathrm{Q}}}(x,y,t) f(x)\omega_t(x), \tag{150}$$

the same arguments as in Theorem 2.5 and a Taylor expansion of f in y will then imply that (150), up to an error of order

$$O(\sqrt{t}s_t^2 + t^{3/2})\mathrm{e}^{-\frac{d(x,y)^2}{4\eta t}} \leqslant C(n,\eta) r^{-3}\left(\sqrt{t}s_t^2 + t^{3/2}\right)\mathrm{e}^{-\frac{s_t^2}{4\eta t}},$$

is given by

$$\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\frac{\varphi(\sqrt{t}\bar{x}) - d_\Omega(y)}{\sqrt{t}}} \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{(4\pi t)^{n/2}}\Big(f(y) + f_\alpha x^\alpha + \\ \left(f(y)\left(-\tfrac{1}{12}\bar{\mathrm{R}}_{\alpha\beta} + \tfrac{1}{4}\bar{\mathrm{X}}_{\alpha\beta}\right) + \tfrac{1}{2} f_{\alpha\beta}\right) x^\alpha x^\beta + t f(y)\left(\tfrac{1}{6}\bar{\mathrm{R}} + \tfrac{1}{2}\operatorname{tr}\bar{\mathrm{X}} - \bar{\mathrm{Q}}\right)\Big)\mathrm{d}x_n \mathrm{d}\bar{x}.$$

By Lemma 2.23 and 2.24 and the arguments in Corollary 2.6, this integral is given by

$$v_\Omega(y,t) + O(t^{3/2} + s_t^2\sqrt{t})\mathrm{e}^{-\frac{s_t^2}{4t}},$$

where

$$\left|O(t^{3/2} + s_t^2\sqrt{t})\right| \leqslant C(n) r^{-3}\left(t^{3/2} + s_t^2\sqrt{t}\right).$$

□

The above pointwise expansion implies the asymptotic expansion of the heat-content; i.e. of the quantity

$$\mathrm{HC}_\Omega(t) := \int_{\Omega^c}\int_\Omega \varrho_{\bar{\mathrm{Q}}}(x,y,t) f(x)\omega_t(x)\omega_t(y). \tag{151}$$

Theorem 6.9 *For any $n \in \mathbb{N}$ there are constants $\varepsilon(n) > 0$ and $C(n) < \infty$ such that the following is true.*

Let $(\mathrm{M}^n, \bar{\mathrm{g}}(t))_{0\leqslant t\leqslant T}$, be a family of closed Riemannian manifolds satisfying the evolution equation $\frac{\partial}{\partial t}\bar{\mathrm{g}}_{\alpha\beta} = 2\bar{\mathrm{X}}_{\alpha\beta}$, let $\bar{\mathrm{Q}} \in C^\infty(\mathrm{M}^n \times [0,T])$ be a potential term and set

$$\bar{\mathrm{L}}_t := \tfrac{\partial}{\partial t} - \bar{\Delta}_{\bar{\mathrm{g}}(t)} + \bar{\mathrm{Q}}. \tag{$\bar{\mathrm{L}}_t$}$$

For a set $\Omega \subset \mathrm{M}^n$ with smoothly embedded boundary and a function $f \in C^\infty(\mathrm{M}^n)$ we assume the radius

$$r := r_{148}(n/2 + 2, \bar{\mathrm{X}}, \bar{\mathrm{Q}}, \partial\Omega, f)$$

from Remark 6.6 to be positive. For $t \in (0, \varepsilon(n)r^2)$ we define the function

$$\begin{aligned} Z_\Omega(t) := {} & \tfrac{\sqrt{t}}{\sqrt{\pi}}\int_{\partial\Omega} f\,\mathrm{d}\mu_t - \tfrac{t}{2}\int_{\partial\Omega}\tfrac{\partial f}{\partial\nu}\mathrm{d}\mu_t \\ & + \tfrac{t^{3/2}}{\sqrt{\pi}}\Big(- \int_{\partial\Omega} f\left(\tfrac{1}{6}\left(\bar{\mathrm{Ric}}(\nu,\nu) + |A|^2\right) + \tfrac{1}{12}H^2\right)\mathrm{d}\mu_t \\ & \qquad + \int_{\partial\Omega}\left(\tfrac{1}{3}\bar{\nabla}_\nu\bar{\nabla}_\nu f - \tfrac{2}{3}H\bar{\nabla}_\nu f\right)\mathrm{d}\mu_t \\ & \qquad + \int_{\partial\Omega}\left(\tfrac{1}{2} f\,\mathrm{tr}\,\bar{\mathrm{X}} + \tfrac{1}{2}\bar{\mathrm{X}}(\nu,\nu) - f\bar{\mathrm{Q}} + \bar{\Delta} f\right)\mathrm{d}\mu_t\Big). \end{aligned}$$

Then, for $u_{\Omega,f,\bar{\mathrm{Q}}}$ being the solution of Equation ($\star$), the generalized heat content

$$\mathrm{HC}_\Omega(t) := \int_{\Omega^c} u_{\Omega,f,\bar{\mathrm{Q}}}(y)\omega_t(y)$$

satisfies

$$|\mathrm{HC}_\Omega(t) - Z_\Omega(t)| \leqslant C(n)t^2\left(\tfrac{1}{r^3}\left(1 + \tfrac{|\Omega|}{r^n}\right)\mathrm{area}\,(\partial\Omega) + |\mathrm{M}^n|\,\tfrac{1}{r^4}\right).$$

Example 6.10 (The adjoint heat equation) The most interesting example of a system ($\star$) is the Ricci flow coupled with the conjugate heat equation. We consider the analogous problem, backwards Ricci flow and the differential operator $\bar{\Box}^* := \frac{\partial}{\partial t} - \Delta_{\bar{\mathrm{g}}(t)} + \bar{\mathrm{R}}$, i.e. the coupled system

$$\begin{cases} \frac{\partial}{\partial t}\bar{\mathrm{g}}_{\alpha\beta} = 2\overline{\mathrm{Ric}}_{\alpha\beta} & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ \bar{\Box}^* u_\Omega = 0 & (x,t) \in \mathrm{M}^n \times (0,\infty), \\ u_\Omega(x,0) = \chi_\Omega(x) & x \in \mathrm{M}^n, \end{cases}$$

for some $\Omega \subset \mathrm{M}^n$ with sufficiently smooth boundary. Denoting by s_t the distance from $y \in \mathrm{M}^n$ to the boundary, Theorem 6.8 then implies that the pointwise short-time expansion of u_Ω is then given by

$$\begin{aligned}
&\Phi\left(-\tfrac{s_t}{\sqrt{t}}\right) + \sqrt{t}\frac{\mathrm{e}^{-\frac{s_t^2}{4t}}}{\sqrt{4\pi}}\Big(-H \\
&+ \sqrt{t}s_t\frac{\mathrm{e}^{-\frac{s_t^2}{4t}}}{\sqrt{4\pi}}\left(\overline{\mathrm{Ric}}(\nu,\nu) - \tfrac{1}{2}\left(|A|^2 + \overline{\mathrm{Ric}}\,(\nu,\nu)\right) + \tfrac{1}{4}H^2\right)\Big).
\end{aligned}$$

Moreover, by Theorem 6.9 the heat content $\mathrm{HC}_\Omega(t) = \int_\Omega u_\Omega(y,t)\omega_t(x)$ is approximated by

$$\tfrac{\sqrt{t}}{\sqrt{\pi}}\,\mathrm{area}\,(\partial\Omega)_t + \tfrac{t^{3/2}}{\sqrt{\pi}}\int_{\partial\Omega}\bar{\mathrm{Ric}}(\nu,\nu) - \tfrac{1}{6}\left(|A|^2 + \bar{\mathrm{Ric}}(\nu,\nu)\right) + \tfrac{1}{12}H^2\mathrm{d}\mu_t.$$

The curvature terms after short times may be estimated by the initial curvature. Furthermore, and the evolution equation of surface integrals implies

$$\left(\tfrac{\partial}{\partial t}\,\mathrm{area}\,(\partial\Omega)_t\right)_{|t=0} = \int_{\partial\Omega}\left(\bar{\mathrm{R}} - \overline{\mathrm{Ric}}\,(\nu,\nu)\right)\mathrm{d}\mu.$$

We can therefore approximate the heat content in terms of the curvature at time $t = 0$. We have

$$\begin{aligned}
&\left|\mathrm{HC}_\Omega(t) - \left(\tfrac{\sqrt{t}}{\sqrt{\pi}}\,\mathrm{area}\,(\partial\Omega) + \tfrac{t^{3/2}}{\sqrt{\pi}}\int_{\partial\Omega}\bar{\mathrm{R}} - \tfrac{1}{6}\left(|A|^2 + \bar{\mathrm{Ric}}(\nu,\nu)\right) + \tfrac{1}{12}H^2\mathrm{d}\mu\right)\right| \\
&\leqslant C(n)t^2\left(\tfrac{1}{r^3}\left(1 + \tfrac{|\Omega|}{r^n}\right)\mathrm{area}\,(\partial\Omega) + |\mathrm{M}^n|\,\tfrac{1}{r^4}\right).
\end{aligned}$$

Hence, up to an error of order $O(t^2)$, the difference to the static metric case, Theorem 3.2, is characterized by the surface integral

$$\tfrac{t^{3/2}}{\sqrt{\pi}}\int_{\partial\Omega}\bar{\mathrm{R}}\;\mathrm{d}\mu.$$

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