Weighted Koopman Semigroups on Banach Modules

Dissertation
der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
Sita Siewert
aus Rottenburg am Neckar

Tübingen
2020

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation:	23.09.2020
Stellvertretender Dekan:	Prof. Dr. József Fortágh
1. Berichterstatter:	Prof. Dr. Rainer Nagel
2. Berichterstatter:	Prof. Dr. Frank Loose
3. Berichterstatterin:	Prof. Dr. Tanja Eisner

Table of Contents

Danksagung

An dieser Stelle möchte ich mich bei allen Personen bedanken, die mich während meiner Promotion begleitet und unterstützt haben und ohne die diese Arbeit nicht möglich gewesen wäre.

Mein größter Dank gilt Rainer Nagel für die Vergabe des spannenden Themas sowie für seine ausgezeichnete Betreuung. Zu Beginn der Promotion hat er mich bei meiner Stipendiumsbewerbung beraten und unterstützt. Außerdem hat er mich früh ermutigt, Konferenzen zu besuchen und Vorträge zu halten. Er war es auch, der mir verschiedene Forschungsreisen vermittelte, welche eine große Bereicherung meiner Promotionszeit darstellten. In zahlreichen Gesprächen steckte er mich mit seinem mathematischen Optimismus an, gab mir Zuversicht und motivierte mich.

Tanja Eisner danke ich für die Betreuung und Begutachtung meiner Arbeit. Besonders danken möchte ich ihr und der Universität Leipzig für einen Forschungsaufenthalt. Der freundliche mathematische Austausch stellte eine große Inspiration dar. Außerdem bedanke ich mich bei Frank Loose für die Betreuung und Begutachtung der Arbeit. Zudem möchte ich mich bei ihm und bei Wolfgang Arendt jeweils für eine Stelle als wissenschaftliche Mitarbeiterin zu Beginn und am Ende meiner Promotion bedanken, die es mir ermöglichten, wertvolle Erfahrungen in der Lehre zu sammeln. Auch Ulrich Groh danke ich, bei dem ich ein Tutorium und ein Seminar betreuen konnte. Bei Roland Derndinger möchte ich mich für die interessanten Gespräche und seine hilfreichen Ratschläge bedanken. Bei Petra Csomós und Eszter Sikolya bedanke ich mich für einen äußerst anregenden Forschungsaufenthalt an der Eötvös-Loránd-Universität in Budapest. Marjeta Kramar Fijavž danke ich herzlich, da sie mir einen unvergessenen Forschungsaufenthalt an der Universität Ljubljana ermöglichte. Auch Philipp Kunde und der Universität Hamburg danke ich für einen sehr angenehmen Forschungsaufenthalt.

Großer Dank gebührt Nikolai Edeko für das Bereitstellen seiner LaTeX-Vorlage, die ich für das Verfassen dieser Dissertation verwenden durfte. Darüber hinaus bedanke ich mich bei ihm für zahlreiche erhellende und inspirierende mathematische Diskussionen und die stets angenehme Zusammenarbeit. Ebenso danke ich Henrik Kreidler für die vielen gemeinsamen mathematischen Gespräche und für die produktive Zusammenarbeit während seiner Promotionszeit in Tübingen und darüber hinaus. Kari Küster danke ich für unzählige anregende und motivierende (meta-)mathematische Gespräche und auch Viktoria Kühner danke ich für den angenehmen mathematischen Austausch. Darüber hinaus danke ich Nikolai Edeko, Henrik Kreidler, Kari Küster und Viktoria Kühner für das aufmerksame und gründliche Korrekturlesen der Arbeit. Allen Mitgliedern der AGFA möchte ich für die freundschaftliche und produktive Atmosphäre, für lebhafte Diskussionen aller Art, für interessante Vorträge und natürlich für die gemeinsamen Teerunden meinen Dank aussprechen. Besonders danke ich Tim Binz, Nikolai Edeko, Patrick Hermle, Retha Heymann, Tim Krake, Henrik Kreidler, Viktoria Kühner und Kari Küster!

Der Universität Tübingen danke ich für das Landesgraduiertenstipendium, welches mir die Promotion ermöglichte, sowie für die Finanzierung verschiedener Forschungsreisen.

Von ganzem Herzen danke ich meinen Freunden und meiner Familie, die mir bei all meinen Vorhaben den Rücken stärken und immer für mich da sind. Meinen Kindern danke ich für die Freude, die sie jeden Tag in mein Leben zaubern. Meinen Eltern und meinen Schwiegereltern danke ich für ihre anhaltende Unterstützung und meinem Mann danke ich dafür, dass er immer an mich glaubt.

Zusammenfassung in deutscher Sprache

Die vorliegende Arbeit befasst sich mit der globalen Linearisierung dynamischer Systeme mittels *Koopmanismus*, welcher in der Monographie *Operator Theoretic Aspects of Ergodic Theory* von T. Eisner, B. Farkas, M. Haase und R. Nagel [EFHN15] systematisch verfolgt wird.

Für topologische dynamische Systeme $(K;(\varphi_t)_{t\in\mathbb{R}})$ hat sich folgender Ansatz bewährt. Dem kompakten Raum K wird der Banachraum $\mathrm{C}(K)$ der stetigen, komplexwertigen Funktionen auf K zugeordnet. Dieser Raum ist sowohl eine kommutative C^*-Algebra mit Einselement als auch ein AM-Banachverband mit Ordnungseinheit. Der stetige Fluss $(\varphi_t)_{t\in\mathbb{R}}$ auf K induziert eine C_0-Gruppe linearer Operatoren $(T_\varphi(t))_{t\in\mathbb{R}}$ auf $\mathrm{C}(K)$ mittels

$$T_\varphi(t)f := f \circ \varphi_{-t} \quad \text{für alle } f \in \mathrm{C}(K), t \in \mathbb{R}.$$

Diese sogenannten *Koopmanoperatoren* sind sowohl Verbands- als auch Algebrahomomorphismen und der Generator der Gruppe ist eine *Derivation* auf $\mathrm{C}(K)$. Mit Hilfe dieser Eigenschaften lässt sich eine solche *Koopmangruppe* charakterisieren. Die reiche Struktur der Koopmangruppe ermöglicht es nun, wichtige Eigenschaften dynamischer Systeme mittels Verbands-, Algebra- und Halbgruppentheorie zu untersuchen. Entscheidend hierbei ist, dass alle wesentlichen Eigenschaften des dynamischen Systems ihre genaue Entsprechung im *Koopmansystem* haben.

Die zentrale Frage, mit der sich diese Arbeit auseinandersetzt, ist, welche Linearisierung sinnvoll ist, falls ein dynamisches System mit zusätzlicher Struktur betrachtet wird. Ein besonders eindrückliches Beispiel ist hierbei ein *glattes dynamisches System* $(M;(\varphi_t)_{t\in\mathbb{R}})$, d.h. ein glatter Fluss auf einer kompakten Riemannschen Mannigfaltigkeit ohne Rand. Die geometrische Struktur eines solchen Systems spiegelt sich nicht in der Koopmangruppe auf dem Raum stetiger Funktionen wider. Deshalb ist es hier nötig und auch sinnvoll, das Tangentialbündel TM der Mannigfaltigkeit und die Familie der Differentiale $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$ des Flusses miteinzubeziehen. Eine globale Linearisierung wird erzielt, indem der Raum $\Gamma(M, TM)$ der stetigen Schnitte des Tangentialbündels mit nachfolgender Dynamik betrachtet wird. Die sogenannten *Pushforwardoperatoren*, definiert durch

$$\mathcal{T}_{\mathrm{d}\varphi}(t)s := \mathrm{d}\varphi_t \circ s \circ \varphi_{-t} \quad \text{für alle } s \in \Gamma(M, TM), t \in \mathbb{R},$$

bilden eine stark stetige Einparametergruppe $(\mathcal{T}_{\mathrm{d}\varphi}(t))_{t\in\mathbb{R}}$ linearer Operatoren auf dem $\mathrm{C}(M)$-Banachmodul $\Gamma(M, TM)$, die *Pushforwardgruppe* genannt. Dieses Beispiel wird im Folgenden in einen wesentlich allgemeineren Rahmen gefasst.

Die Arbeit ist in zwei Teile gegliedert. Teil I, welcher aus zwei Kapiteln besteht, befasst sich mit der abstrakten Charakterisierung *dynamischer Banachmoduln* und entspricht im Wesentlichen dem gemeinsam mit Henrik Kreidler verfassten Artikel [KS20], welcher bei *Mathematische Zeitschrift* veröffentlicht wurde. Eine vorläufige Version des Artikels findet sich auch in [Kre19]. Im ersten Kapitel werden *dynamische Banachbündel*, für welche die oben erwähnten Differentiale $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$ auf TM ein Beispiel bilden, in einer sehr allgemeinen Situation definiert. Hierbei wird zwischen dynamischen Banachbündeln über topologischen dynamischen Systemen und solchen über messbaren dynamischen Systemen unterschieden. Überdies werden weitere typische Beispiele untersucht. Kapitel 2 bildet die Grundlage für alle weiteren Resultate der Arbeit. Hier werden dynamische Banachmoduln als das operatorentheoretische Pendant zu den Objekten des ersten Kapitels definiert. Theorem 2.22 und Theorem 2.45 liefern eine Darstellung dynamischer Banachmoduln als *gewichtete Koopmandarstellungen* auf entsprechenden Schnitträumen.

Teil II widmet sich einem wichtigen Spezialfall: stetigen Flüssen $(\varphi_t)_{t\in\mathbb{R}}$ auf kompakten Räumen K und stark stetige Einparameterhalbgruppen $(\mathcal{T}(t))_{t\geqslant 0}$ auf Banachräumen $\Gamma(K, E)$ stetiger Schnitte in Banachbündel E. Die Kapitel 3 bis 5 basieren auf der Zusammenarbeit mit Henrik Kreidler. Kapitel 6 baut auf gemeinsamer Arbeit mit Nikolai Edeko und Henrik Kreidler auf. Im dritten Kapitel werden die Darstellungsresultate des zweiten Kapitels um Charakterisierungen *gewichteter Koopmanhalbgruppen* mittels ihrer Generatoren und ihrer Resolventen ergänzt, siehe Theorem 3.8 und Theorem 3.12. Kapitel 4 untersucht spektrale Eigenschaften gewichteter Koopmanhalbgruppen auf $\Gamma(K, E)$ und ihrer Generatoren und stellt einen Bezug zum Spektrum von Koopmangruppen auf $C(K)$ und deren Generatoren her. Unter gewissen Voraussetzungen an den zugrundeliegenden Fluss und das Banachbündel gilt der *Spektrale Abbildungssatz* für gewichtete Koopmanhalbgruppen, siehe Theorem 4.13. Die Resultate aus Kapitel 4 finden ihre Anwendung im darauffolgenden Kapitel. Dort wird das asymptotische Verhalten gewichteter Koopmanhalbgruppen untersucht. Als besonders wichtige Eigenschaft wird *exponentielle Dichotomie* von Flüssen auf Banachbündeln untersucht und mittels spektraler Eigenschaften der zugehörigen gewichteten Koopmanhalbgruppe charakterisiert, siehe Theorem 5.8 and Corollary 5.9. Dies führt auch zu einem Resultat über das sogenannte *Sacker-Sell-Spektrum*, siehe Corollary 5.11. Im letzten Kapitel schließlich wird das eingangs erwähnte Beispiel glatter dynamischer Systeme $(M;(\varphi_t)_{t\in\mathbb{R}})$ eingehend untersucht. Der Generator einer gewichteten Koopmangruppe auf dem Raum stetiger Schnitte des Tangentialbündels TM der Mannigfaltigkeit M ist die additive Störung der *Lie-Ableitung* durch einen beschränkten Multiplikationsoperator, siehe Theorem 6.7 und Remark 6.9. Dadurch lassen sich qualitative Eigenschaften einer beliebigen gewichteten Koopmangruppe auf $\Gamma(M,TM)$ auf die oben definierte Pushforwardgruppe $(\mathcal{T}_{\mathrm{d}\varphi}(t))_{t\in\mathbb{R}}$ zurückführen, siehe Corollary 6.12, Corollary 6.13 und Corollary 6.14. Zuletzt wird das qualitative Verhalten glatter Flüsse auf Mannigfaltigkeiten – wie beispielsweise *Hyperbolizität* – mit Hilfe der Pushforwardgruppe untersucht, siehe Proposition 6.20 und Proposition 6.21.

Contributions

Part I

The presentation of Part I of the present thesis is essentially taken from the article [KS20], published in *Mathematische Zeitschrift*, which is joint work with Henrik Kreidler. All results of the article were formulated, discussed, and proved in cooperation. Discussions with Nikolai Edeko, Daniel Hättig, Viktoria Kühner, Philipp Kunde, Walther Paravicini, and Marco Peruzzetto inspired us when writing the article. A preliminary version of the article can also be found in [Kre19].

Part II

Section 3.3., Section 3.4, Section 4.2, Section 5.2, and Section 5.3 are joint work with Henrik Kreidler. All results of these sections were formulated, discussed, and proved in cooperation.

The formulations and proofs of the results of Chapter 6 are based on joint work with Nikolai Edeko and Henrik Kreidler.

Introduction

It is an old idea, going back to J. von Neumann and B. O. Koopman,[1] to assign to nonlinear dynamics on a (topological or measurable) *state space* corresponding linear operators on an *observable space*, i.e., a vector space of scalar-valued functions on the state space. This idea lead to the proof of the fundamental *mean and pointwise ergodic theorems* by J. von Neumann and G. D. Birkhoff,[2] around 1930 and is, now called *Koopmanism*, the leitmotif for much current research and important results. A systematic treatment of this operator theoretic approach to dynamical systems can be found in the monograph *Operator Theoretic Aspects of Ergodic Theory* by T. Eisner, B. Farkas, M. Haase, and R. Nagel [EFHN15].

We briefly sketch one of the standard mathematical situations for Koopmanism. Take a *topological dynamical system* $(K; (\varphi_t)_{t\in\mathbb{R}})$,[3] i.e., a compact state space K and a continuous flow $(\varphi_t)_{t\in\mathbb{R}}$ on K, frequently originating from a differential equation. To the compact space K corresponds the Banach space $\mathrm{C}(K)$ of all complex-valued continuous functions on K which even is a commutative unital C^*-algebra and an AM-Banach lattice with order unit. The flow $(\varphi_t)_{t\in\mathbb{R}}$ then induces a C_0-group, called *Koopman group*, of linear operators $(T_\varphi(t))_{t\in\mathbb{R}}$ on $\mathrm{C}(K)$ given by

$$T_\varphi(t)f := f \circ \varphi_{-t} \quad \text{for all } f \in \mathrm{C}(K), t \in \mathbb{R}.$$

All these operators are lattice and algebra homomorphisms and the generator of the group is a *derivation* on the algebra $\mathrm{C}(K)$. These qualities are even characteristic for Koopman groups, see Theorem 3.1. Moreover, one can recover all information about the topological dynamical system by investigating the associated Koopman

[1] cf. [vNe32b] and [Koo31]

[2] cf. [vNe32a] and [Bir31]

[3] For the theory of topological dynamical systems we refer to, e.g., [Ell69], [Bro79], [Aus88], [dVr93], [HoKr18], and [Tao09].

group and vice versa.[4] The theory of Koopman groups has been already developed systematically in [Nag86], Part B-II, and then extended by many recent results, see, e.g., [EK20], [Küs20], [Ede20], [Kre19], and [Küh19].

However, in many situations the dynamical system has, in addition to its topological properties, further structure which is not reflected by the corresponding Koopman group. In this case, it is not sufficient or adequate to consider observables whose values at a given state are (complex) numbers. Still, we aim for a C_0-semigroup on a Banach space as a global linearization in order to make tools of linear functional analysis applicable for the investigation of the dynamical system. The following simple but typical example illustrates this situation and indicates which kind of observable space might be suitable.

Consider a *smooth dynamical system* $(M;(\varphi_t)_{t\in\mathbb{R}})$,[5] i.e., a compact Riemannian manifold M without boundary and a smooth flow $(\varphi_t)_{t\in\mathbb{R}}$ on M.[6] This flow induces a Koopman group $(T_\varphi(t))_{t\in\mathbb{R}}$ on the Banach space $\mathrm{C}(M)$, which does not reflect much of the geometric structure of M. To overcome this deficit, we consider the tangent bundle TM of the manifold and the differential $\mathrm{d}\varphi_t$ of each φ_t. At each point $x \in M$ the differential is a bounded linear operator $\mathrm{d}\varphi_t(x) \in \mathscr{L}(T_xM, T_{\varphi_t(x)}M)$ which is compatible with the underlying flow via the chain rule, i.e., $\mathrm{d}\varphi_{t+r}(x) = \mathrm{d}\varphi_t(\varphi_r(x))\mathrm{d}\varphi_r(x)$ for all $t,r \in \mathbb{R}$. The family of differentials $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$ is a *flow over* $(\varphi_t)_{t\in\mathbb{R}}$ on the tangent bundle TM of M. To obtain a global rather than just a local linearization of $(\varphi_t)_{t\in\mathbb{R}}$, we pass on to a group of linear operators on a suitable Banach space still reflecting geometric information.

To this purpose, take the Banach space $\Gamma(M,TM)$ of all continuous sections of TM and define linear operators via

$$\mathcal{T}_{\mathrm{d}\varphi}(t)s := \mathrm{d}\varphi_t \circ s \circ \varphi_{-t} \quad \text{for all } s \in \Gamma(M,TM), t \in \mathbb{R},$$

called *pushforward of s by φ_t*, see [Lee13], p. 183. This yields a strongly continuous one-parameter group $(\mathcal{T}_{\mathrm{d}\varphi}(t))_{t\in\mathbb{R}}$ of linear operators on the $\mathrm{C}(M)$-Banach module $\Gamma(M,TM)$, called *pushforward group*.

[4]The category (see [Mac98]) of topological dynamical systems and the category of strongly continuous group representations as automorphisms of commutative C^*-algebras are equivalent, see, e.g., Section 1.4 of [Dix77].

[5]cf. [Sma67], [BP13], [FH19], [Mei07], [Den05], [Ma87], and [Ree80]

[6]See, e.g., [Lee13], [Lan95], [BP13], [Spi99], and [AMR83] for the theory of differential geometry.

The following questions arise:

1. Is the group $(\mathcal{T}_{\mathrm{d}\varphi}(t))_{t\in\mathbb{R}}$ strongly continuous?
2. What is the generator of this group?
3. How can we characterize such groups, their generators, and their resolvents by algebraic or by order theoretic properties?
4. What do the spectra of these operators look like and how are they related to the properties of the flow?
5. How can we describe the asymptotic behavior of the group and what are the conclusions for the behavior of the original dynamical system?

All these questions shall be discussed in this thesis in various contexts and more general situations.

Starting from a topological dynamical system $(K; (\varphi_t)_{t\in\mathbb{R}})$ we proceed as follows. To each point $x \in K$ we attach a Banach space E_x obtaining a so-called *Banach bundle* E over K. From a Banach bundle we obtain a *Banach module* over $\mathrm{C}(K)$ as the space of continuous sections $\Gamma(K, E)$ of the Banach bundle, see [Gie82], [DG83], [HoKe17], [Cun67], or [AAK92].

On such a Banach bundle, we consider a *semiflow* $(\Phi_t)_{t\geqslant 0}$ *over* $(\varphi_t)_{t\in\mathbb{R}}$ that is a family of continuous mappings on E such that each Φ_t restricted to E_x, $x \in K$, is a bounded linear operator $\Phi_t(x) := \Phi_t|_{E_x} \in \mathscr{L}(E_x, E_{\varphi_t(x)})$. Moreover, $(\Phi_t)_{t\geqslant 0}$ satisfies for all $t, r \geqslant 0$, $x \in K$ the so-called *cocycle rule*, i.e.,

$$\Phi_{t+r}(\varphi_r(x)) = \Phi_t(\varphi_r(x))\Phi_r(x). \tag{CR}$$

This semiflow on E induces a C_0-semigroup $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on the Banach module $\Gamma(K, E)$ via

$$\mathcal{T}_\Phi(t)s := \Phi_t \circ s \circ \varphi_{-t} \quad \text{for all } s \in \Gamma(K, E), t \geqslant 0.$$

This semigroup is called a *weighted Koopman semigroup* and will be treated systematically in the present thesis.

The typical examples of such semigroups on Banach modules are *evolution semigroups* corresponding to non-autonomous partial differential equations (see

[EN00], Section VI.9) and, as explained above, the pushforward operators acting on the continuous sections of a tangent bundle of a manifold. In the context of extensions of topological dynamical systems (see [Ell69], Section 5 or [EK20]) we obtain a weighted Koopman semigroup that is isomorphic to the Koopman group corresponding to the extended topological dynamical system, see Example 3.11 (iv).

The thesis is divided into two parts. Part I, which consists of two chapters, covers a very general case of dynamical systems and semigroup representations, while Part II is dedicated to the special case of continuous flows on compact spaces and to one-parameter semigroups.

In Chapter 1, we introduce dynamics on Banach bundles "over" a dynamical system and proceed as follows. In Section 1.1, we consider *topological Banach bundles* E over a locally compact space Ω, see [DG83], Definition 1.1. On such Banach bundles, we introduce a *semiflow* $\boldsymbol{\Phi}$ which is compatible with a given flow $\boldsymbol{\varphi}$ on the underlying space Ω in the sense that the cocycle rule (CR) is satisfied, see Definition 1.8. Finally, we discuss important examples for such *dynamical Banach bundles* $(E;\boldsymbol{\Phi})$, see Example 1.12. Section 1.2 treats the "measurable case", that is, semiflows $\boldsymbol{\Phi}$ on *measurable Banach bundles* (see Definition 1.18) over a *measure-preserving dynamical system*.[7] Again, we end the section with some interesting examples, see Example 1.19.

Chapter 2 establishes the operator theoretic counterpart of dynamical Banach bundles—*dynamical Banach modules*—consisting of a *Banach module* Γ over a commutative Banach algebra A and a *weighted semigroup representation* $\mathcal{T}$ on Γ "over" a *group representation* $\boldsymbol{T}$ as algebra automorphisms of A, see Definition 2.12. From the objects presented in the first chapter, we gain such dynamical Banach modules by turning to a space of sections of E and to a *weighted Koopman representation* $\mathcal{T}_{\boldsymbol{\Phi}}$, see Proposition 2.14. The main results of this chapter are representations for such dynamical Banach modules, see Theorem 2.22 and Theorem 2.45 as weighted Koopman representations on a suitable space of sections. Analogous to the "non-weighted" Koopmanism, all information about the dynamical Banach bundle correspond to information of the induced weighted Koopman representation and the other way around.

The presentation of Part I is essentially taken from the article [KS20], published

[7]cf. [EFHN15] or, e.g., [Gla03]

in *Mathematische Zeitschrift*. A preliminary version can also be found in [Kre19].

The second part treats *strongly continuous one-parameter semigroups of weighted Koopman operators* on the Banach module of *continuous sections of a Banach bundle E* over a compact space K, see Definition 3.5. In Chapter 3, we reformulate the representation theorem of the previous chapter in this setting and add algebraic and order theoretic characterizations of such weighted Koopman semigroups by their generators and their resolvents, see Theorem 3.8 and Theorem 3.12.

In the next chapter, we turn to spectral properties of weighted Koopman semigroups on Banach modules of continuous sections. For an *aperiodic flow* and a continuous bundle, the spectra of weighted Koopman semigroups and their generators are directly related to each other by the *spectral mapping theorem*, see Theorem 4.13. Afterwards, in Chapter 5, we investigate the asymptotic behavior of weighted Koopman semigroups and their associated semiflows on Banach bundles. We apply the results from the previous chapter to give a characterization of *exponential dichotomy* and *hyperbolicity*, see Theorem 5.8 and Corollary 5.9, and of the *Sacker-Sell spectrum*, see Corollary 5.11.

The last chapter deals with the introductory example of a smooth dynamical system $(M;(\varphi_t)_{t\in\mathbb{R}})$. The results from the previous chapters yield additional characterizations of weighted Koopman groups on spaces of continuous sections of a compact Riemannian manifold, see Theorem 6.7. In particular, the generator of a weighted Koopman group on $\Gamma(M,TM)$ is the additive perturbation of the *Lie derivative* by a bounded multiplication operator, see Remark 6.9. Thus, certain properties of a weighted Koopman group can be reduced to properties of the above introduced pushforward group $(\mathcal{T}_{\mathrm{d}\varphi}(t))_{t\in\mathbb{R}}$, see Corollary 6.12, Corollary 6.13, and Corollary 6.14. Finally, the qualitive behavior of smooth flows on mannifolds—like *hyperbolicity*—can be investigated by means of the pushforward group, see Proposition 6.20 and Proposition 6.21.

Chapter 3, 4, and 5 are based on joint work with Henrik Kreidler. Chapter 6 is based on joint work with Nikolai Edeko and Henrik Kreidler.

Part I

Representation of semigroups on Banach modules

In this part, we consider so-called *Banach bundles* over a locally compact or measure space with their associated *Banach modules* over a commutative Banach algebra. In analogy to the Gelfand theorem, see, e.g., Theorem 4.23 of [EFHN15] or Section 1.4 of [Dix77], we obtain a representation for such Banach modules, see Proposition 2.26. Again, relevant properties of the Banach bundle translate into algebraic and lattice-theoretic properties of the associated Banach module. Based on this result, we will introduce certain dynamics on Banach bundles compatible with the dynamics on the underlying space as well as *dynamical Banach modules*. The main results of Part I are representations of such dynamical Banach modules, see Theorem 2.22 and Theorem 2.45. First, we recall the basic definitions and results from the literature, see, e.g., [Gie82], [DG83], [HoKe17], [Cun67], or [AAK92].

The presentation is essentially taken from the article [KS20], published in *Mathematische Zeitschrift*. A preliminary version can also be found in [Kre19].

In the following all vector spaces are over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and all locally compact spaces are Hausdorff.

Chapter 1

Semiflows on Banach bundles

Starting from dynamics φ on a locally compact or measure space X, we introduce appropriate dynamics "over" φ, see Definition 1.8 and Definition 1.18, on topological or measurable *Banach bundles* over X, see Definition 1.1 and Definition 1.13. In the first section we treat the topological case and in the second section we consider the measurable case.

1.1 Semiflows on topological Banach bundles

In this section, we define dynamics on topological Banach bundles over some fixed topological dynamical system. We start with the abstract definition of a topological Banach bundle, see Definition 1.1 of [DG83], see also [HoKe17].

Definition 1.1. Let E be a topological space (*total space*), Ω a locally compact space (*base space*), and $p_E\colon E \longrightarrow \Omega$ a continuous, open, and surjective mapping (*bundle projection*). Then (E, Ω, p_E), denoted by $p_E\colon E \longrightarrow \Omega$, is called a *(topological) Banach bundle* over Ω if the following properties are satisfied.

(i) For each $x \in \Omega$ the *fiber* $E_x := p_E^{-1}(x)$ is a Banach space.

(ii) The mappings

$$+\colon E\times_\Omega E \longrightarrow E, \quad (u,v)\mapsto u+v := u +_{E_{p_E(v)}} v,$$
$$\cdot\colon \mathbb{K}\times E \longrightarrow E, \quad (\lambda,v)\mapsto \lambda\cdot v := \lambda\cdot_{E_{p_E(v)}} v$$

are continuous where $E\times_\Omega E := \bigcup_{x\in\Omega} E_x\times E_x \subseteq E\times E$ is equipped with the subspace topology.

(iii) The mapping (*bundle norm*)

$$\|\cdot\|\colon E\longrightarrow \mathbb{R}_+, \quad v\mapsto \|v\|_{E_{p_E(v)}}$$

is upper semicontinuous.

(iv) For each $x\in\Omega$ and each open set $W\subseteq E$ containing the zero $0_x\in E_x$ there exist $\varepsilon>0$ and an open neighborhood U of x such that

$$\{v\in p_E^{-1}(U) \mid \|v\|_{E_{p_E(v)}} \leqslant \varepsilon\}\subseteq W.$$

If, in addition, the mapping $\|\cdot\|$ is continuous, then $p_E\colon E\longrightarrow\Omega$ is called a *continuous Banach bundle*. If no confusion arises, we denote a Banach bundle $p_E\colon E\longrightarrow\Omega$ by $p\colon E\longrightarrow\Omega$ or simply by E.

Remark 1.2. Note that if E is a Banach bundle over a locally compact space Ω, we obtain a Banach bundle $\tilde{E}$ over the one-point compactification $K:=\Omega\,\dot\cup\,\{\infty\}$ in a canonical way by taking the space $\tilde{E} := E\,\dot\cup\,\{0\}$, the canonical mapping $p_{\tilde{E}}\colon \tilde{E}\longrightarrow K$, and the topology on $\tilde{E}$ generated by the topology on E and the sets

$$U(L,\varepsilon) := \left\{v\in p_{\tilde{E}}^{-1}(\Omega\setminus L)\,\middle|\, \|v\|<\varepsilon\right\}$$

for compact $L\subseteq\Omega$ and $\varepsilon>0$. In the following we will frequently make use of this extension.

From Banach bundles we obtain natural vector spaces.

Definition 1.3. A *continuous section* of a Banach bundle $p\colon E\longrightarrow\Omega$ over a locally compact space Ω is a continuous mapping $s\colon\Omega\longrightarrow E$ such that $p\circ s = \mathrm{id}_\Omega$. If for all $\varepsilon>0$ there exists $K\subseteq\Omega$ compact with $\|s(x)\|\leqslant\varepsilon$ for all $x\notin K$, then s is called a *continuous section vanishing at infinity*. The *space of all continuous sections of E* is denoted by $\Gamma(\Omega,E)$, while the subspace of all *continuous sections vanishing at infinity* is denoted by $\Gamma_0(\Omega,E)$.

Obviously, the space of all continuous sections $\Gamma(\Omega, E)$ endowed with pointwise addition and pointwise scalar multiplication is a vector space. The subspace $\Gamma_0(K, E)$ of continuous sections vanishing at infinity equipped with the norm $\|\cdot\|$ defined by

$$\|s\| := \sup_{x\in\Omega} \|s(x)\|, \quad s \in \Gamma_0(\Omega, E)$$

is a Banach space. The continuous sections of a Banach bundle determine its topology. We make this precise by the following lemma.

Lemma 1.4. *Let $p\colon E \longrightarrow \Omega$ be a Banach bundle over a locally compact space Ω. For $v \in E$ the sets*

$$V(s, U, \varepsilon) := \{w \in p^{-1}(U) \mid \|w - s(p(w))\| < \varepsilon\},$$

with $s \in \Gamma_0(\Omega, E)$ satisfying $s(p(v)) = v$, $U \subseteq \Omega$ an open neighborhood of $p(v)$, and $\varepsilon > 0$, form a neighborhood base of v in E.

Proof. In the case of a compact base space this follows from Consequences 1.6 (vii) and Theorem 3.2 of [Gie82] – note that by the proof of Proposition 2.2 of [Gie82] we may confine ourselves to globally defined sections. The general case can readily be reduced to this by considering $\tilde{E}$, cf. Remark 1.2. □

We now list some important examples of Banach bundles.

Example 1.5. (i) Let Z be any Banach space and Ω a locally compact space. Then $E := \Omega \times Z$ is a continuous Banach bundle over Ω, called the *trivial bundle with fiber Z* if $p\colon \Omega \times Z \longrightarrow K$ is the projection onto the first component and $\Omega \times Z$ is equipped with the product topology.

(ii) Consider a Riemannian manifold M without boundary. Then the tangent bundle TM over M equipped with the canonical projection and topology is a continuous Banach bundle over M, cf. Chapter 3 of [Lee13].

(iii) Let $\pi\colon L \longrightarrow K$ be a continuous surjection between compact spaces L and K. For each $k \in K$ let $L_k := \pi^{-1}(k)$ be the associated fiber. We define

$$E := \dot{\bigcup_{k\in K}} \mathrm{C}(L_k),$$
$$p\colon E \longrightarrow K, \quad v \in \mathrm{C}(L_k) \mapsto k$$

and endow this with the topology generated by the sets

$$W(s, U, \varepsilon) := \{v \in p^{-1}(U) \mid \|v - s|_{L_{p(v)}}\|_{\mathrm{C}(L_{p(v)})} < \varepsilon\},$$

where $U \subseteq K$ is open, $s \in \mathrm{C}(L)$, and $\varepsilon > 0$. Then E is a Banach bundle over K and the corresponding space of continuous sections $\Gamma(K, E)$ is isomorphic to $\mathrm{C}(L)$, see Theorem 4.2 of [Gie82]. Moreover, E is a continuous Banach bundle if and only if π is open. This construction is used frequently in topological dynamics, see e.g., page 30 of [Kna67] or Section 5 of [Ell87]. See also [EK20].

We now associate *morphisms* to these Banach bundles, cf. Section 1 of [DG83].

Definition 1.6. Let Ω be a locally compact space and $\varphi\colon \Omega \longrightarrow \Omega$ a continuous mapping. Consider two Banach bundles $p_E\colon E \longrightarrow \Omega$ and $p_F\colon F \longrightarrow \Omega$. A continuous mapping

$$\Phi\colon E \longrightarrow F$$

is called *(bounded) Banach bundle morphism over* φ if

(i) $p_F \circ \Phi = \varphi \circ p_E$, i.e., the diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p_F} \\ \Omega & \xrightarrow{\varphi} & \Omega \end{array}$$

commutes,

(ii) $\Phi(x) := \Phi|_{E_x} \in \mathscr{L}(E_x, F_{\varphi(x)})$ for each $x \in \Omega$,

(iii) $\|\Phi\| := \sup_{x\in\Omega} \|\Phi(x)\|_{\mathscr{L}(E_x, F_{\varphi(x)})} < \infty$.

Moreover, Φ is *isometric* if each $\Phi(x)$ is an isometry. If $\varphi = \mathrm{id}_\Omega$, we simply call a Banach bundle morphism over φ a *Banach bundle morphism*.

Remark 1.7. If $\Omega = K$ is compact, then conditions (i) and (ii) of Definition 1.6 already imply (iii), see the proof of Proposition 1.4 of [DG83].

We are interested in dynamical Banach bundles over dynamical systems induced by groups. A *topological G-dynamical system* $(\Omega; \boldsymbol{\varphi})$ is a continuous group action

$$\boldsymbol{\varphi}\colon G \times \Omega \longrightarrow \Omega, \quad (g, x) \mapsto \varphi_g(x) = gx$$

of a locally compact group G on a locally compact space Ω. We call $\boldsymbol{\varphi} = (\varphi_g)_{g\in G}$ a *(continuous) flow* on Ω. For the rest of the section, we fix such a topological G-dynamical system $(\Omega; \boldsymbol{\varphi})$ and a closed subsemigroup $S \subseteq G$ containing the neutral element e, i.e., a closed submonoid of G. Important examples of this situation are $G = \mathbb{Z}$, $S = \mathbb{N}_0$ and $G = \mathbb{R}$, $S = \mathbb{R}_+$.

Definition 1.8. An *S-dynamical Banach bundle over the topological G-dynamical system* $(\Omega; \boldsymbol{\varphi})$ is a pair $(E; \boldsymbol{\Phi})$ of a Banach bundle E over Ω and a *semigroup representation*[1]

$$\boldsymbol{\Phi}\colon S \longrightarrow E^E, \quad g \mapsto \Phi_g,$$

such that

(i) the mapping
$$\Phi_g\colon E \longrightarrow E$$
is a Banach bundle morphism over φ_g for each $g \in S$,

(ii) $\boldsymbol{\Phi}$ is *jointly continuous*, i.e., the mapping
$$S \times E \longrightarrow E, \quad (g, v) \mapsto \Phi_g v$$
is continuous,

(iii) $\boldsymbol{\Phi}$ is *locally bounded*, i.e., $\sup_{g\in K} \|\Phi_g\| < \infty$ for every compact subset $K \subseteq S$.

We call $\boldsymbol{\Phi} = (\Phi_g)_{g\in S}$ a *semiflow over $(\varphi_g)_{g\in G}$ on E over* Ω. If $S = G$, then we call $\boldsymbol{\Phi} = (\Phi_g)_{g\in G}$ a *flow* over $(\varphi_g)_{g\in G}$ on E over Ω.

A *morphism* from an S-dynamical Banach bundle $(E; \boldsymbol{\Phi})$ over $(\Omega; \boldsymbol{\varphi})$ to an S-dynamical Banach bundle $(F; \boldsymbol{\Psi})$ over $(\Omega; \boldsymbol{\varphi})$ is a Banach bundle morphism $\Theta\colon E \longrightarrow F$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\Theta} & F \\ {\scriptstyle \Phi_g}\downarrow & & \downarrow{\scriptstyle \Psi_g} \\ E & \xrightarrow[\Theta]{} & F \end{array}$$

[1] I.e., $\Phi_{gh} = \Phi_h \circ \Phi_g$ for all $g, h \in S$ and $\Phi_e = \mathrm{Id}_E$ for the neutral element $e \in S$. This is also known as a *monoid representation*.

commutes for each $g \in S$.

Remark 1.9. The concept of a dynamical Banach bundle is closely related to the notion of cocycles and linear skew-product flows, cf. Definition 6.1 of [CL99]. In fact, if $(E; \mathbf{\Phi})$ is an S-dynamical Banach bundle over $(\Omega; \boldsymbol{\varphi})$, the operators $\Phi_g(x) := \Phi_g|_{E_x} \in \mathscr{L}(E_x, E_{\varphi_g(x)})$ for $g \in S$ and $x \in K$ satisfy the *cocycle rule*

$$\Phi_{g_1 g_2}(x) = \Phi_{g_1}(\varphi_{g_2}(x)) \circ \Phi_{g_2}(x)$$

for all $g_1, g_2 \in S$ and $x \in \Omega$.

Remark 1.10. If $\Omega = K$ is compact, then—once again—a simple adaptation of the arguments of the proof of Proposition 1.4 of [DG83] shows that the third condition in Definition 1.8 is superfluous.

Proposition 1.11. *Let $\Omega = K$ be compact. Then every semigroup representation $\mathbf{\Phi}\colon S \longrightarrow E^E$, $g \mapsto \Phi_g$ satisfying conditions (i) and (ii) of Definition 1.8 defines an S-dynamical Banach bundle over $(\Omega; \boldsymbol{\varphi})$.*

Proof. Pick $x \in L$ and $g \in S$. Since $\Phi_g 0_x = 0_{\varphi_g(x)}$ we find an open neighborhood U of x, $\varepsilon > 0$, and an open neighborhood V of g such that

$$\Phi_h u \in \{w \in E \mid \|w\| \leqslant 1\}$$

for every $h \in V$, $u \in \{v \in p^{-1}(U) \mid \|v\| \leqslant \varepsilon\}$. But then $\|\Phi_g|_{E_y}\| \leqslant \frac{1}{\varepsilon}$ for every $g \in V$ and $y \in U$. Compactness yields the claim. □

Now we consider dynamics on the Banach bundles of Example 1.5.

Example 1.12. (i) Assume that $G = \mathbb{R}$, $S = \mathbb{R}_+$, Z is a Banach space, and $E = \Omega \times Z$ is the corresponding trivial Banach bundle, cf. Example 1.5 (i). If $\{\Phi^t(x) \in \mathscr{L}(Z) \mid x \in \Omega, t \geqslant 0\}$ is a strongly continuous exponentially bounded cocycle in the sense of Definition 6.1 of [CL99], then the continuous linear skew-product flow $\Phi_t\colon \Omega \times Z \longrightarrow \Omega \times Z$ given by

$$\Phi_t(x, v) := (\varphi_t(x), \Phi^t(x)v)$$

for $x \in \Omega$, $v \in Z$, and $t \geqslant 0$ defines an $\mathbb{R}_+$-dynamical Banach bundle $(E; \mathbf{\Phi})$ over $(\Omega; \boldsymbol{\varphi})$. Conversely, each $\mathbb{R}_+$-dynamical Banach bundle $(E; \mathbf{\Phi})$ defines a strongly continuous exponentially bounded cocycle by setting

$$\Phi^t(x)v := \mathrm{pr}_2(\Phi_t(x,v))$$

for $x \in \Omega$, $v \in Z$, and $t \geqslant 0$, where $\mathrm{pr}_2 \colon \Omega \times Z \longrightarrow Z$ is the projection onto the second component.

In particular, evolution families, see Example 6.5 of [CL99] and Section IV.9 of [EN00], define $\mathbb{R}_+$-dynamical Banach bundles.

(ii) Take $G = \mathbb{R}$, $\Omega = M$ a Riemannian manifold without boundary, and $\boldsymbol{\varphi} = (\varphi_t)_{t\in\mathbb{R}}$ a smooth flow on M, cf. [Lee13], Chapter 9, i.e., $(\varphi_t)_{t\in\mathbb{R}}$ is a (continuous) flow on M such that the mapping $\boldsymbol{\varphi} \colon \mathbb{R} \times M \longrightarrow M$ is smooth, cf. [Lee13], Chapter 3. If the family of differentials $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$ is locally bounded, then, by the chain rule, $(TM; (\mathrm{d}\varphi_t)_{t\in\mathbb{R}})$ is an $\mathbb{R}$-dynamical Banach bundle over $(M; (\varphi_t)_{t\in\mathbb{R}})$, cf. [Lee13], Corollary 3.22.

(iii) Assume that $\Omega = K$ is compact and $\pi \colon (L; \boldsymbol{\psi}) \longrightarrow (K; \boldsymbol{\varphi})$ is an extension of topological G-dynamical systems, i.e., $(L; \boldsymbol{\psi})$ and $(K; \boldsymbol{\varphi})$ are topological G-dynamical systems and $\pi : L \to K$ a continuous surjection such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\psi_g} & L \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ K & \xrightarrow[\varphi_g]{} & K \end{array}$$

commutes for each $g \in G$. Assume further that E is defined as in Example 1.5 (iii). For each $g \in G$ consider

$$\Phi_g \colon E \longrightarrow E, \quad v \in \mathrm{C}(L_k) \mapsto v \circ \psi_{g^{-1}} \in \mathrm{C}(L_{\varphi_g(k)}).$$

This defines a G-dynamical Banach bundle $(E; \boldsymbol{\Phi})$ over $(K; \boldsymbol{\varphi})$.

1.2 Semiflows on measurable Banach bundles

In this section, we define Banach bundles over measure spaces as in Section II.4 of [FD88] or Appendix A.3 of [ADR00], see also [Gut93b]. A measure space X is a triple $(\Omega_X, \Sigma_X, \mu_X)$ consisting of a set Ω_X, a σ-algebra Σ_X of subsets of Ω_X, and a positive σ-finite measure $\mu_X \colon \Sigma_X \longrightarrow [0,\infty]$. We also assume that our measure spaces are *complete*, i.e., subsets of null sets are measurable.

Definition 1.13. A *(measurable) Banach bundle* over a measure space X (*base space*) is a triple $(E, p_E, \mathcal{M}_E)$ where E is a set (*total space*), $p_E : E \longrightarrow \Omega_X$ is a surjective mapping (*bundle projection*) such that the *fiber* $E_x := p_E^{-1}(x)$ is a Banach space for each $x \in \Omega_X$, and $\mathcal{M}_E$ is a linear subspace of

$$\mathcal{S}_E := \{s : \Omega_X \longrightarrow E \mid p_E \circ s = \mathrm{id}_{\Omega_X}\}$$

such that

(i) if $f : \Omega_X \longrightarrow \mathbb{K}$ is measurable and $s \in \mathcal{M}_E$, then $fs \in \mathcal{M}_E$, where

$$fs : s \longrightarrow E, \quad x \mapsto f(x)s(x),$$

(ii) for each $s \in \mathcal{M}_E$ the mapping

$$|s| : \Omega_X \longrightarrow \mathbb{R}_+, \quad x \mapsto \|s(x)\|_{E_x}$$

is measurable,

(iii) if $(s_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}_E$ converging almost everywhere to $s \in \mathcal{S}_E$, then $s \in \mathcal{M}_E$.

Elements $s \in \mathcal{S}_E$ are called *sections* and elements $s \in \mathcal{M}_E$ are called *measurable sections*.

The bundle is *separable* if, in addition,

(iv) there is a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ such that $\mathrm{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ is dense in E_x for almost every $x \in \Omega_X$.

If no confusion arises we denote the bundle projection p_E simply by p. Further, we mostly write E for a measurable Banach bundle $(E, p_E, \mathcal{M}_E)$.

Remark 1.14. Let X be a measure space and (E, p) a pair of a set E and a surjective mapping $p : E \longrightarrow \Omega_X$ such that the fiber $E_x := p^{-1}(x)$ is a Banach space for each $x \in \Omega_X$. Then by Section II.4.2 of [FD88] every linear subspace $\mathcal{M}_E$ of $\mathcal{S}_E$ satisfying condition (iii) of Definition 1.13 *generates* a measurable Banach bundle, i.e., there is a smallest linear subspace $\tilde{\mathcal{M}}_E$ of $\mathcal{S}_E$ containing $\mathcal{M}_E$ such that $(E, p, \tilde{\mathcal{M}}_E)$ is a measurable Banach bundle. Moreover, $\tilde{\mathcal{M}}_E$ consists precisely of all almost everywhere limits of sequences in $\mathrm{lin}\{\mathbb{1}_A s \mid A \in \Sigma_X, s \in \mathcal{M}_E\}$.

Remark 1.15. In the case of a separable Banach bundle E our vector space $\mathcal{M}_E$ of measurable sections becomes a *measurable Banach fibre space* in the sense of [Hey15], Definition VI.1.iii.

We briefly list some examples for measurable Banach bundles and refer to Appendix A.3 of [ADR00] for additional examples.

Example 1.16. (i) Let X be a measure space and Z a Banach space. Consider $E := \Omega_X \times Z$ with the projection p onto the first component. The space of sections $\mathcal{S}_E$ can be identified with the space of all functions from Ω_X to Z. The set of all strongly measurable functions, see Section 1.3.5 of [HP57], then defines a subset $\mathcal{M}_E$ of $\mathcal{S}_E$ which turns E into a measurable Banach bundle called the *trivial Banach bundle with fiber* Z. This coincides with the measurable Banach bundle generated by the constant sections, see Section II.5.1 of [FD88].

(ii) Let E be a topological Banach bundle over a locally compact space Ω, μ be a σ-finite regular Borel measure on Ω, and $\mathcal{B}(\Omega)$ the Borel σ-algebra of Ω. Then the space $\Gamma(\Omega, E)$, see Definition 1.3, generates a measurable Banach bundle E_μ over the completion of the measure space $(\Omega, \mathcal{B}(\Omega), \mu)$. See Section II.15 of [FD88] for a more explicit description of the measurable sections of a continuous Banach bundle.

Before introducing dynamics on measurable Banach bundles, we first define morphisms of measure spaces. A *premorphism* $\varphi\colon X \longrightarrow Y$ between measure spaces X and Y is a measurable and measure-preserving mapping $\varphi\colon \Omega_X \longrightarrow \Omega_Y$. Setting $\varphi \sim \psi$ if $\varphi(x) = \psi(x)$ for almost every $x \in \Omega_X$ defines an equivalence relation on the set of premorphisms from X to Y. The equivalence classes with respect to this equivalence relation are the *morphisms* from X to Y. As usual, given a morphism we will implicitly choose a representative and denote it by φ when there is no room for confusion. We define morphisms of measurable Banach bundles in a similar manner.

Definition 1.17. Let $\varphi\colon X \longrightarrow X$ be a morphism on a measure space X. Consider Banach bundles $(E, p_E, \mathcal{M}_E)$ and $(F, p_F, \mathcal{M}_F)$ over X. A *premorphism* Φ from E to F over φ is a mapping $\Phi\colon E \longrightarrow F$ such that

(i) $\Phi \circ \mathcal{M}_E \subseteq \mathcal{M}_F \circ \varphi$,

(ii) $p_F \circ \Phi = \varphi \circ p_E$ almost everywhere,

(iii) $\Phi|_{E_x} \in \mathscr{L}(E_x, F_{\varphi(x)})$ for almost every $x \in \Omega_X$,

(iv) $\|\Phi\| := \operatorname{ess\,sup}_{x\in\Omega_X} \|\Phi|_{E_x}\| < \infty$.

Again, we want to identify premorphisms which agree up to a null set. Set

$$\begin{aligned} \mathrm{Premor}_\varphi(E,F) &:= \{\Phi\colon E \longrightarrow F \text{ premorphism over } \varphi\}, \\ \mathcal{N}_\varphi(E,F) &:= \{\Phi \in \mathrm{Premor}_\varphi(E,F) \mid \Phi = 0 \text{ almost everywhere}\}, \end{aligned}$$

and $\mathrm{Mor}_\varphi(E,F) := \mathrm{Premor}_\varphi(E,F)/\mathcal{N}_\varphi(E,F)$ for measurable Banach bundles E and F as above.

An equivalence class $[\Phi] \in \mathrm{Mor}_\varphi(E,F)$ is called a *morphism of measurable Banach bundles over* φ. It is *isometric* if $\Phi|_{E_x}$ is isometric for almost every $x \in \Omega_X$. If $\varphi = \mathrm{id}_X$, we call a morphism over φ simply a *morphism of measurable Banach bundles*. As above, we will implicitly choose representatives of morphisms whenever necessary and denote them with the same symbol.

Now we introduce dynamical measurable Banach bundles. For the rest of this section let G be a group with neutral element $e \in G$. We call a pair $(X;\boldsymbol{\varphi})$ a *measure-preserving G-dynamical system* if X is a measure space together with a group homomorphism

$$\boldsymbol{\varphi}\colon G \longrightarrow \mathrm{Aut}(X), \quad g \mapsto \varphi_g,$$

where $\mathrm{Aut}(X)$ is the set of automorphisms of X. We call $\boldsymbol{\varphi} = (\varphi_g)_{g\in G}$ a *flow* on X. For the rest of the section we fix measure-preserving G-dynamical system $(X;\boldsymbol{\varphi})$ and a submonoid $S \subseteq G$, i.e., a subsemigroup containing $e \in G$.

Definition 1.18. An *S-dynamical Banach bundle over the measure-preserving G-dynamical system* $(X;\boldsymbol{\varphi})$ is a pair $(E;\boldsymbol{\Phi})$ of a measurable Banach bundle E over X and a family $\boldsymbol{\Phi} = (\Phi_g)_{g\in S}$ of mappings with $\Phi_g\colon E \to E$ is a morphism over φ_g for $g \in S$ such that

$$\begin{aligned} \Phi_g \circ \Phi_h &= \Phi_{gh} \quad \text{for all } g,h \in S, \\ \Phi_e &= \mathrm{Id}_E. \end{aligned}$$

We call $\mathbf{\Phi} = (\Phi_g)_{g\in S}$ a *semiflow over* $(\varphi_g)_{g\in G}$ *on* E *over* X. If $S = G$, then we call $\mathbf{\Phi} = (\Phi_g)_{g\in G}$ a *flow* over $(\varphi_g)_{g\in G}$ on E over Ω. If E is separable we call $(E;\mathbf{\Phi})$ *separable*.

A *morphism* between measurable Banach bundles $(E;\mathbf{\Phi})$ and $(F;\mathbf{\Psi})$ over $(X;\boldsymbol{\varphi})$ is a morphism $\Theta\colon E \longrightarrow F$ of Banach bundles such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \Theta\ } & F \\ {\scriptstyle \Phi_g}\downarrow & & \downarrow{\scriptstyle \Psi_g} \\ E & \xrightarrow[\ \Theta\ ]{} & F \end{array}$$

commutes for each $g \in S$.

Example 1.19. (i) Let E be the trivial bundle with fiber Z, see Example 1.16 (i). Then the S-dynamical Banach bundles correspond to *measurable cocycles*, i.e., a mapping

$$\mathbf{\Phi}\colon S \times X \longrightarrow \mathscr{L}(Z), \quad (g, x) \mapsto \Phi_g(x)$$

such that

- $\Phi_{gh}(x) = \Phi_g(\varphi_h(x))\circ\Phi_h(x)$ for almost every $x \in X$ and for all $g, h \in S$,
- $\Phi_e(x) = \mathrm{Id}_Z$ for almost every $x \in X$,
- $X \longrightarrow Z$, $x \mapsto \Phi_g(x)v$ is strongly measurable for all $g \in S$ and $v \in Z$,
- $\operatorname{ess\,sup}_{x\in\Omega_X} \|\Phi_g(x)\| < \infty$ for every $g \in S$.

(ii) Let $(E;\mathbf{\Phi})$ be a topological S-dynamical Banach bundle over a topological G-dynamical system $(\Omega;\boldsymbol{\varphi})$ with G and S discrete and let μ be a σ-finite regular Borel measure on Ω. Moreover, let E_μ be the induced measurable Banach bundle of Example 1.16 (ii). Then $(E_\mu;\mathbf{\Phi})$ is an S-dynamical measurable Banach bundle over the measure-preserving G-dynamical system induced by $(\Omega;\boldsymbol{\varphi})$.

Chapter 2

Representation of semigroups on spaces of sections

Our starting point are topological and measure-preserving dynamical systems and their corresponding *Koopman representation*, cf. Example 2.13. We show that any semiflow on a Banach bundle over a dynamical system, cf. Definition 1.8 and Definition 1.18, induces a semigroup representation on a Banach space of sections, the so-called *weighted Koopman representation* cf. Example 2.13. This weighted Koopman representation—playing a central role in the present thesis—is characterized by additional algebraic and lattice-theoretic properties which we investigate in the following. For this purpose we recall the concept of *Banach modules*, see, e.g., [Gie82], [DG83], [HoKe17], or [Cun67].

Definition 2.1. Let A be a commutative Banach algebra. A Banach space Γ which is also an A-module is a *Banach module over* A if the norm is submultiplicative, i.e., $\|fs\| \leqslant \|f\|\|s\|$ for all $f \in A$ and $s \in \Gamma$.

A *homomorphism* from a Banach module Γ over A to a Banach module Λ over A is a bounded linear operator $T \in \mathscr{L}(\Gamma, \Lambda)$ which is also an A-module homomorphism, i.e., $\mathcal{T}(f \cdot s) = f \cdot \mathcal{T} s$ for all $f \in A$, $s \in \Gamma$. It is *isometric* if T is an isometry.

In the following we always assume that Banach modules Γ over a commutative

Banach algebra A are *non-degenerate*, see [Par08], in the sense that

$$\Gamma = \overline{\text{lin}}\,\{fs \mid f \in A, s \in \Gamma\}.$$

Note that if A is a commutative C^*-algebra (if $\mathbb{K} = \mathbb{C}$) or its self-adjoint part (if $\mathbb{K} = \mathbb{R}$) and $(e_i)_{i\in I}$ is an approximate unit, see Section 1.8 of [Dix77], then this is the case if and only if $\lim_i e_i s = s$ for each $s \in \Gamma$. In particular, if A has a unit, then the module is unitary.

Here are some Banach modules associated with Banach bundles.

Example 2.2. Let E be a topological Banach bundle over a locally compact space Ω. Then the space of all continuous sections vanishing at infinity $\Gamma_0(\Omega, E)$, see Definition 1.3, is a Banach module over $C_0(\Omega)$ if equipped with the operation

$$C_0(\Omega) \times \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E), \quad (f, s) \mapsto [x \mapsto f(x)s(x)]$$

and the norm $\|\cdot\|$ defined by $\|s\| := \sup_{x\in\Omega} \|s(x)\|$ for $s \in \Gamma_0(\Omega, E)$.

Remark 2.3. Let Ω be a locally compact space and E a Banach bundle over Ω. If K is the one-point compactification of Ω and $\tilde{E}$ the extended bundle of E, see Remark 1.2, then

$$\Gamma(K, \tilde{E}) \to \Gamma_0(\Omega, E), \quad s \mapsto s|_\Omega$$

is an isometric isomorphism of Banach spaces. In particular, we can consider $\Gamma_0(\Omega, E)$ as a Banach module over $C(K)$.

Example 2.4. For a measurable Banach bundle E over a measure space X we define

$$\begin{aligned} \mathcal{N}_E &:= \{s \in \mathcal{M}_E \mid s = 0 \text{ almost everywhere}\},\\ \Gamma^1(X, E) &:= \{s \in \mathcal{M}_E \mid |s| \text{ is integrable}\}\,/\mathcal{N}_E,\\ \Gamma^\infty(X, E) &:= \{s \in \mathcal{M}_E \mid |s| \text{ is essentially bounded}\}\,/\mathcal{N}_E. \end{aligned}$$

With the natural norms and operations the spaces $\Gamma^1(X, E)$ and $\Gamma^\infty(X, E)$ are Banach modules over $L^\infty(X)$.

In order to define dynamical Banach modules we introduce first "morphisms over morphisms".

Definition 2.5. Let A be a commutative Banach algebra and $T \in \mathscr{L}(A)$ an algebra homomorphism. Moreover, let Γ and Λ be Banach modules over A. Then $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ is a *T-homomorphism* if

$$\mathcal{T}(fs) = Tf \cdot \mathcal{T}s \quad \text{for all } f \in A \text{ and } s \in \Gamma.$$

Example 2.6. (i) Let $\varphi\colon \Omega \longrightarrow \Omega$ be a homeomorphism of a locally compact space Ω. Then the *Koopman operator* $T_\varphi \in \mathscr{L}(\mathrm{C}_0(\Omega))$ defined by

$$T_\varphi f := f \circ \varphi^{-1} \quad \text{for } f \in \mathrm{C}_0(\Omega)$$

is an algebra automorphism.

A morphism Φ over φ between two Banach bundles E and F over Ω induces a T_φ-homomorphism $\mathcal{T}_\Phi \in \mathscr{L}(\Gamma_0(\Omega, E), \Gamma_0(\Omega, F))$ by

$$\mathcal{T}_\Phi s := \Phi \circ s \circ \varphi^{-1} \quad \text{for } s \in \Gamma_0(\Omega, E),$$

called the *weighted Koopman operator*.

(ii) Let $\varphi\colon X \longrightarrow X$ be an automorphism of a measure space X. Then the *Koopman operator* $T_\varphi \in \mathscr{L}(\mathrm{L}^\infty(X))$ defined by

$$T_\varphi f := f \circ \varphi^{-1} \quad \text{for } f \in \mathrm{L}^\infty(X)$$

is an algebra automorphism.

A morphism Φ over φ between two Banach bundles E and F over X induces a T_φ-homomorphism $\mathcal{T}_\Phi \in \mathscr{L}(\Gamma^1(X, E), \Gamma^1(X, F))$ by

$$\mathcal{T}_\Phi s := \Phi \circ s \circ \varphi^{-1} \quad \text{for } s \in \Gamma^1(X, E),$$

called the *weighted Koopman operator*. Similarly, Φ induces an operator $\mathcal{T}_\Phi \in \mathscr{L}(\Gamma^\infty(X, E), \Gamma^\infty(X, F))$.

Before introducing the concept of dynamical Banach modules we prove a characterization of T-homomorphisms as some sort of "locality preserving operators".

Definition 2.7. Let A be a commutative Banach algebra and Γ a Banach module over A. For $s \in \Gamma$ we call the closed ideal

$$I_s := \{f \in A \mid fs = 0\}$$

the *supporting ideal of s in A*.

If $A = C_0(\Omega)$ for some locally compact space Ω, then there is a correspondence between the concept of supporting ideals and the following notion of support, see Definition 9.3 of [AAK92].

Definition 2.8. Let Ω be a locally compact space and Γ a Banach module over $C_0(\Omega)$. For $s \in \Gamma$ we call

$$\mathrm{supp}(s) := \{x \in \Omega \mid \text{each } f \in C_0(\Omega) \text{ with } f(x) \neq 0 \text{ satisfies } fs \neq 0\} \subseteq \Omega$$

the *support of s in* Ω.

Lemma 2.9. *Let* Ω *be a locally compact space and* Γ *a Banach module over* $C_0(\Omega)$. *Then*

$$I_s = \{f \in C_0(\Omega) \mid f|_{\mathrm{supp}(s)} = 0\}$$

for every $s \in \Gamma$.

Proof. Let $s \in \Gamma$. Since I_s is a closed ideal in $C_0(\Omega)$, we find a unique closed subset M such that $f|_M = 0$ if and only if $f \in I_s$. It is clear that $\mathrm{supp}(s) \subseteq M$. On the other hand, if $x \in \Omega \setminus \mathrm{supp}(s)$, we find $f \in C_0(\Omega)$ with $f(x) \neq 0$ but $fs = 0$. Then $f|_M = 0$ which shows $x \notin M$. □

The following is a first characterization of T-homomorphisms extending Theorem 9.5 of [AAK92].

Theorem 2.10. *Let* $\varphi\colon \Omega \longrightarrow \Omega$ *be a homeomorphism of a locally compact space* Ω *and* Γ *and* Λ *Banach modules over* $C_0(\Omega)$. *For* $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ *the following assertions are equivalent.*

(a) $\mathcal{T}$ *is a* T_φ*-homomorphism.*
(b) $T_\varphi I_s \subseteq I_{\mathcal{T}s}$ *for every* $s \in \Gamma$.
(c) $\mathrm{supp}(\mathcal{T}s) \subseteq \varphi(\mathrm{supp}(s))$ *for each* $s \in \Gamma$.

For the proof we need the following lemma.

Lemma 2.11. *Let* Ω *be a locally compact space and* Γ *be a Banach module over* $C_0(\Omega)$ *and take* $K = \Omega \,\dot\cup\, \{\infty\}$ *to be the one-point compactification of* Ω. *The* *mapping*

$$\mathrm{C}(K) \times \Gamma \longrightarrow \Gamma, \quad (f, s) \mapsto (f - f(\infty)\mathbb{1})|_{\Omega} s + f(\infty) s$$

turns Γ *into a (unitary) Banach module over* $\mathrm{C}(K)$.

Proof. It is easy to check that the mapping above turns Γ into a unitary module over $\mathrm{C}(K)$. Choose an approximate unit $(e_i)_{i\in I}$ for $\mathrm{C}_0(\Omega)$. Now take $f \in \mathrm{C}(K)$ and $s \in \Gamma$ and observe that

$$\begin{aligned} \|fs\| &= \lim_i \|(f - f(\infty)\mathbb{1})|_{\Omega} e_i s + f(\infty) e_i s\| \\ &= \lim_i \|(fe_i)s\| \leqslant \limsup_i \|e_i f\| \|s\| \\ &\leqslant \|f\| \|s\|. \end{aligned}$$

This shows $\|fs\| \leqslant \|f\|\|s\|$ and therefore Γ is a Banach module over $\mathrm{C}(K)$. □

Proof (of Theorem 2.10). The equivalence of (b) and (c) is obvious by Tietze's theorem while the equivalence of (a) and (c) follows from Theorem 9.5 of [AAK92] if $K = \Omega$ is compact and $\varphi = \mathrm{id}_K$[1].

Now take Ω non-compact but still assume $\varphi = \mathrm{id}_\Omega$. We consider the one-point compactification K of Ω and the module structure of Γ over $\mathrm{C}(K)$, see Lemma 2.11. For $s \in \Gamma$ we denote the support of s with respect to this module structure by $\mathrm{supp}_K(s)$. It is easy to see that

$$\overline{\mathrm{supp}(s)}^K \subseteq \mathrm{supp}_K(s) \subseteq \mathrm{supp}(s) \cup \{\infty\}.$$

Let $(e_i)_{i\in I}$ be an approximate unit for $\mathrm{C}_0(\Omega)$. Obviously, $\infty \notin \mathrm{supp}_K(s)$ if and only if there is $g \in \mathrm{C}_0(\Omega)$ with $gs = s$. But this is the case if and only if there is $i_0 \in A$ with $(e_i g - e_i)s = 0$, i.e., $(e_i g - e_i)|_{\mathrm{supp}(s)} = 0$ for every $i \geqslant i_0$. Therefore, the result for non-compact Ω can be reduced to the compact case.

Finally let $\varphi\colon \Omega \longrightarrow \Omega$ be an arbitrary homeomorphism of a locally compact space Ω. Consider the module Λ_{T_φ} which is the space Λ equipped with the new operation $f \cdot_{T_\varphi} s := T_\varphi f \cdot s$ for $f \in \mathrm{C}_0(\Omega)$ and $s \in \Lambda$. Then $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ is a T_φ-homomorphism if and only if $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda_{T_\varphi})$ is a homomorphism of Banach modules. By the above, this is the case if and only if

$$\{x \in \Omega \mid \text{each } f \in \mathrm{C}_0(\Omega) \text{ with } f(x) \neq 0 \text{ satisfies } T_\varphi f \cdot \mathcal{T} s \neq 0\} \subseteq \mathrm{supp}(s),$$

[1]The proof also works in the real case.

i.e., $\operatorname{supp}(\mathcal{T}s) \subseteq \varphi(\operatorname{supp}(s))$ for each $s \in \Gamma$. $\square$

We now introduce dynamical Banach modules. Fix a pair $(A;\boldsymbol{T})$ of a commutative Banach algebra A and a strongly continuous group representation

$$\boldsymbol{T}\colon G \longrightarrow \mathscr{L}(A), \quad g \mapsto T_g$$

of a locally compact group G as algebra automorphisms of A. Moreover, let $S \subseteq G$ be a fixed closed submonoid, i.e., a closed subsemigroup containing the neutral element $e \in G$.

Definition 2.12. An *S-dynamical Banach module over* $(A;\boldsymbol{T})$ is a pair $(\Gamma;\boldsymbol{\mathcal{T}})$ consisting of a Banach module Γ over A and a semigroup representation[2]

$$\boldsymbol{\mathcal{T}}\colon S \longrightarrow \mathscr{L}(\Gamma), \quad g \mapsto \mathcal{T}_g$$

such that

(i) $\mathcal{T}(g) \in \mathscr{L}(\Gamma)$ is a $T(g)$-homomorphism for each $g \in S$,

(ii) $\boldsymbol{\mathcal{T}}$ is *strongly continuous*, i.e.,

$$S \longrightarrow \Gamma, \quad g \mapsto \mathcal{T}(g)s$$

is continuous for every $s \in \Gamma$.

We call $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ a *weighted semigroup representation on* Γ *over* $\boldsymbol{T}$ *on* A.

A *homomorphism* from an S-dynamical Banach module $(\Gamma;\boldsymbol{\mathcal{T}})$ over $(A;\boldsymbol{T})$ to an S-dynamical Banach module $(\Lambda;\boldsymbol{\mathcal{S}})$ over $(A;\boldsymbol{T})$ is a homomorphism $V \in \mathscr{L}(\Gamma,\Lambda)$ of Banach modules over A such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{V} & \Lambda \\ {\scriptstyle \mathcal{T}(g)}\downarrow & & \downarrow{\scriptstyle \mathcal{S}(g)} \\ \Gamma & \xrightarrow[V]{} & \Lambda \end{array}$$

commutes for each $g \in S$.

[2] I.e., $\mathcal{T}(gh) = \mathcal{T}(g)\mathcal{T}(h)$ for all $g, h \in S$ and $\mathcal{T}(e) = \mathrm{Id}_\Gamma$ for the neutral element $e \in S$.

Starting with the topological case, we now show that dynamical Banach bundles induce dynamical Banach modules.

Example 2.13. Consider an S-dynamical Banach bundle $(E;\boldsymbol{\Phi})$ over a topological G-dynamical system $(\Omega;\boldsymbol{\varphi})$. For each $g \in G$ the Koopman operator $T_\varphi(g) := T_{\varphi_g}$ is an automorphism of $\mathrm{C}_0(\Omega)$, see Example 2.6 (i), and $g \mapsto T_\varphi(g)$ defines a representation $\boldsymbol{T_\varphi} = (T_\varphi(g))_{g\in G}$ of G as operators on $\mathrm{C}_0(\Omega)$, called the *Koopman representation*. It is strongly continuous which is probably well-known, but also a special case of Proposition 2.14 below. By setting $\mathcal{T}_\Phi(g) := \mathcal{T}_{\Phi_g}$ for each $g \in S$, we obtain a $T_\varphi(g)$-homomorphism $\mathcal{T}_\Phi(g) \in \mathscr{L}(\Gamma_0(\Omega, E))$ for each $g \in S$, see Example 2.6. We call the semigroup representation $\boldsymbol{\mathcal{T}_\Phi} = (T_\Phi(g))_{g\in S}$ on the Banach module $\Gamma_0(\Omega, E)$ the *weighted Koopman representation* induced by $(E;\boldsymbol{\Phi})$.

Proposition 2.14. *Let $(\Omega;\boldsymbol{\varphi})$ be a topological G-dynamical system, $A = \mathrm{C}_0(\Omega)$ and $\boldsymbol{T} = \boldsymbol{T_\varphi}$ the Koopman representation of $(\Omega;\boldsymbol{\varphi})$.*

(i) *If $(E;\boldsymbol{\Phi})$ is an S-dynamical Banach bundle over $(\Omega;\boldsymbol{\varphi})$, then the weighted Koopman representation $\boldsymbol{\mathcal{T}_\Phi}$ defines an S-dynamical Banach module over $(\mathrm{C}_0(\Omega);\boldsymbol{T_\varphi})$.*

(ii) *For a morphism $\Theta\colon (E;\boldsymbol{\Phi}) \longrightarrow (F;\boldsymbol{\Psi})$ of S-dynamical Banach bundles over $(\Omega;\boldsymbol{\varphi})$ the operator $V_\Theta \in \mathscr{L}(\Gamma_0(\Omega, E), \Gamma_0(\Omega, F))$ defined by*

$$V_\Theta s := \Theta \circ s \quad \text{for } s \in \Gamma_0(\Omega, E)$$

is a homomorphism $V_\Theta \in \mathscr{L}(\Gamma_0(\Omega, E), \Gamma_0(\Omega, F))$ between the S-dynamical Banach modules $(\Gamma_0(\Omega, E);\boldsymbol{\mathcal{T}_\Phi})$ and $(\Gamma_0(\Omega, F);\boldsymbol{\mathcal{T}_\Psi})$.

For the proof we need the following lemma.

Lemma 2.15. *Consider an S-dynamical Banach bundle $(E;\boldsymbol{\Phi})$ over $(\Omega;\boldsymbol{\varphi})$. Let $K := \Omega \,\dot\cup\, \{\infty\}$ be the one-point compactification of Ω and $\tilde{E}$ the extended Banach bundle of Remark 1.2. Then the following assertions hold.*

(i) *The extension $\tilde{\boldsymbol{\varphi}}$ of the flow $\boldsymbol{\varphi}$ to K defined by*

$$\tilde{\boldsymbol{\varphi}}\colon G \times K \longrightarrow K, \quad (g, x) \mapsto \begin{cases} \infty & x = \infty, \\ \varphi_g(x) & x \neq \infty, \end{cases}$$

is continuous.

(ii) *Setting*

$$\tilde{\mathbf{\Phi}}\colon S \times \tilde{E} \longrightarrow \tilde{E}, \quad (g, v) \mapsto \begin{cases} 0 & v \in E_\infty, \\ \Phi_g v & v \in E, \end{cases}$$

defines an S-dynamical Banach bundle $(\tilde{E}; \tilde{\mathbf{\Phi}})$ *over* $(K; \tilde{\boldsymbol{\varphi}})$*.*

Proof. If $g \in G$ and L is a compact subset of Ω, we choose a compact neighborhood V of g and set $U := (V^{-1} \cdot L)^c$. Then U is cocompact with $hy \notin L$ for all $h \in V$ and $y \in U$. This shows (i).

Now take $\varepsilon > 0$ and assume that $g \in S$. Since $\mathbf{\Phi}$ is locally bounded, we find a $\delta > 0$ with $\|\Phi_h\| < \frac{1}{\delta}$ for every $h \in V \cap S$. For $v \in E$ with $\|v\| < \delta\varepsilon$, $p_E(v) \in U$, and $h \in V \cap S$ we then have $p_{\tilde{E}}(\Phi_h v) \notin L$ and $\|\Phi_h v\| < \varepsilon$, i.e., $\Phi_h v \in U(L, \varepsilon)$ in the notation of Remark 1.2. This shows that $\tilde{\mathbf{\Phi}}$ is jointly continuous. □

Proof (of Proposition 2.14). We first prove continuity of the weighted Koopman representation in the case of a compact space $\Omega = K$. Fix $s \in \Gamma(K, E)$ and let $g \in S$ and $\varepsilon > 0$. For $x \in K$ the set

$$V := V(\Phi_g \circ s \circ \varphi_{g^{-1}}, K, \varepsilon) := \{v \in E \mid \|v - \Phi_g s(g^{-1}(p(v)))\| < \varepsilon\}$$

is a neighborhood of $\Phi_g s(g^{-1}x)$. Since the mapping

$$S \times K \longrightarrow E, \quad (h, y) \mapsto \Phi_h s(y)$$

is continuous as a composition of the continuous mappings

$$S \times K \longrightarrow S \times E, \quad (h, y) \mapsto (h, s(y)),$$
$$S \times E \longrightarrow E, \quad (h, v) \mapsto \Phi_h v,$$

we find a neighborhood $O \subseteq S$ of g and a neighborhood $U \subseteq K$ of $g^{-1}x$ such that $\Phi_h s(y) \in V$ for every $h \in O$ and $y \in U$, i.e.,

$$\|\Phi_h s(y) - \Phi_g s(g^{-1}hy))\| < \varepsilon.$$

By compactness of K we thus find a neighborhood $W \subseteq S$ of g with

$$\sup_{y \in K} \|\Phi_h s(y) - \Phi_g s(g^{-1}hy))\| < \varepsilon$$

for all $h \in W$. But then

$$\sup_{y \in K} \|\Phi_h s(h^{-1}y) - \Phi_g s(g^{-1}y))\| = \sup_{y \in K} \|\Phi_h s(y) - \Phi_g s(g^{-1}hy))\| < \varepsilon$$

for each $h \in W$.

The general case of (i) now follows from Lemma 2.15 and Remark 2.3 and part (ii) is obvious. □

Example 2.16. Let G carry the discrete topology, $(X; \boldsymbol{\varphi})$ be a measure-preserving G-dynamical system, $A = \mathrm{L}^\infty(X)$ and $\boldsymbol{T} = \boldsymbol{T_\varphi}$ the induced *Koopman representation* on $\mathrm{L}^\infty(X)$, i.e., $T_\varphi(g) := T_{\varphi_g}$ for every $g \in G$.

Then every S-dynamical Banach bundle $(E; \boldsymbol{\Phi})$ over $(X; \boldsymbol{\varphi})$ induces a *weighted Koopman representation* $\boldsymbol{\mathcal{T}_\Phi}$ on $\Gamma^1(X, E)$ via $T_\Phi(g) := T_{\Phi_g}$ for $g \in S$ which defines an S-dynamical Banach module over $(\mathrm{L}^\infty(X); \boldsymbol{T_\varphi})$.

Moreover, if $\Theta\colon (E; \boldsymbol{\Phi}) \longrightarrow (F; \boldsymbol{\Psi})$ is a morphism of S-dynamical Banach bundles over $(X; \boldsymbol{\varphi})$, then $V_\Theta s := \Theta \circ s$ for $s \in \Gamma^1(X, E)$ defines a homomorphism from $(\Gamma^1(X, E); \boldsymbol{\mathcal{T}_\Phi})$ to $(\Gamma^1(X, F); \boldsymbol{\mathcal{T}_\Psi})$.

2.1 AM- and AL-modules

We have seen that topological and measurable Banach bundles induce dynamical Banach modules and that these assignments are functorial. We now describe the essential ranges of these functors.

For this we recall a connection between Banach modules and Banach lattices, observed by Kaijser in Proposition 2.1 of [Kai78] and Abramovich, Arenson, and Kitover in Lemma 4.6 of [AAK92] in the compact case. We give a new proof for the locally compact case based on Lemma 1 of [Cun67] and also provide more details on the lattice structure.

Proposition 2.17. *If Ω is a locally compact space, Γ a Banach module over $\mathrm{C}_0(\Omega)$, and $s \in \Gamma$, then the submodule $\Gamma_s := \overline{\mathrm{C}_0(\Omega) \cdot s}$ is a Banach lattice with positive cone $\overline{\mathrm{C}_0(\Omega)_+ \cdot s}$. Moreover, we obtain the following for $f, g \in \mathrm{C}_0(\Omega, \mathbb{R})$ and $h \in \mathrm{C}_0(\Omega)$,*

(i) *$fs \leqslant gs$ if and only if $f|_{\mathrm{supp}(s)} \leqslant g|_{\mathrm{supp}(s)}$,*

(ii) *$(fs \vee gs) = (f \vee g)s$,*

(iii) $(fs \wedge gs) = (f \wedge g)s$,

(iv) $|hs| = |h|s$.

If $\mathbb{K} = \mathbb{C}$, then Γ_s is the complexification of the real Banach lattice $\overline{C_0(\Omega, \mathbb{R})s}$.

Proof. Take $f, g \in C_0(\Omega)$ with $|g| \leqslant |f|$. We show that $\|fs\| \leqslant \|gs\|$. Set $N := g^{-1}(\{0\})$ and choose an approximate unit $(e_i)_{i \in I}$ for $I_N := \{h \in C_0(\Omega) \mid h|_N = 0\}$ such that e_i has compact support for every $i \in I$. Also define $h_i \in C_0(\Omega)$ for $i \in I$ by

$$h_i(x) := \begin{cases} e_i(x)\frac{g(x)}{f(x)}, & x \notin N, \\ 0, & x \in N. \end{cases}$$

Then $|h_i(x)| \leqslant 1$ for every $x \in \Omega$ and therefore

$$\|gs\| = \lim_i \|e_i gs\| = \lim_i \|h_i fs\| \leqslant \limsup_i \|h_i\| \|fs\| \leqslant \|fs\|. \tag{2.1}$$

We set $|fs| := |f|s$ for $f \in C_0(\Omega)$. By the above we obtain for $f, g \in C_0(\Omega)$

$$\||f|s - |g|s\| = \||\,|f| - |g|\,|s\| \leqslant \||f - g|s\| = \|(f - g)s\| = \|fs - gs\|. \tag{2.2}$$

This implies that $|\cdot|\colon C_0(\Omega)s \longrightarrow C_0(\Omega)s$ has a unique extension to a continuous map $|\cdot|\colon \Gamma_s \longrightarrow \Gamma_s$. The only non-trivial part in showing that this defines a modulus in the sense of Definition 1.1 of [MW74] is to check that the linear hull of the image $|\Gamma_s| = \overline{C_0(\Omega)s}$ is the whole space Γ_s. However, if $r = \lim_{n\to\infty} f_n s \in \Gamma_s$, then—using (2.1) and (2.2) as well as the formulas for the positive and negative parts of functions, see Corollary 1 of Proposition II.1.4 of [Sch74]—it is standard to check that $((\operatorname{Re} f_n)_+ s)_{n\in\mathbb{N}}$, $((\operatorname{Re} f_n)_- s)_{n\in\mathbb{N}}$, $((\operatorname{Im} f_n)_+ s)_{n\in\mathbb{N}}$, and $((\operatorname{Im} f_n)_- s)_{n\in\mathbb{N}}$ are Cauchy sequences and therefore converge in $\overline{C_0(\Omega)_+ s}$. This implies that r can be written as a linear combination of elements of $\overline{C_0(\Omega)_+ s}$. Moreover, this shows $\overline{C_0(\Omega, \mathbb{R})s} = \overline{C_0(\Omega)_+ s} - \overline{C_0(\Omega)_+ s}$.

By Proposition 1.3 of [MW74], we obtain that $\overline{C_0(\Omega)_+ s}$ is a cone and defines a partial order on $\overline{C_0(\Omega, \mathbb{R})s}$. Moreover, $\|hs\| = \||hs|\|$ for every $h \in C_0(\Omega)$ by (2.1) and thus $\|r\| = \||r|\|$ for every $r \in \Gamma_s$. If $r, u \in \Gamma_s$ with $|r| \leqslant |u|$, we find sequences $(f_n)_{n\in\mathbb{N}} \in C_0(\Omega)$ with $\lim f_n s = r$ and $(g_n)_{n\in\mathbb{N}}$ in $C_0(\Omega)_+$ with $\lim g_n s = |u| - |r|$. But then

$$\|r\| = \||r|\| = \lim_{n\to\infty} \||f_n|s\| \leqslant \lim_{n\to\infty} \|(|f_n| + g_n)s\| = \||u|\| = \|u\|.$$

By Corollary 1.4 and Theorem 2.2 of [MW74], Γ_s is a Banach lattice with positive cone $|\Gamma_s| = \overline{C_0(\Omega)s}$ and $|\cdot|$ as its modulus, and, if $\mathbb{K} = \mathbb{C}$, that Γ_s is the complexification of the real Banach lattice $\overline{C_0(\Omega, \mathbb{R})s}$, cf. Section II.11 of [Sch74]. In particular, (iv) holds and this implies (ii) and (iii) by the usual formulas for vector lattices, see Corollary 1 of Proposition II.1.4 of [Sch74]. Finally, if $f \in C_0(\Omega, \mathbb{R})$, then $fs \geqslant 0$ if and only if $|f|s = fs$, i.e., $f - |f| \in I_s$. But by Lemma 2.9 this is exactly the case when $f|_{\mathrm{supp}(s)} \geqslant 0$, showing (i). □

We use this observation to introduce different types of Banach modules.

2.1.1 AM-modules

Our first type of Banach modules is based on the concept of AM-spaces, see [Sch74], Section II.7.

Definition 2.18. Let Ω be a locally compact space. A Banach module Γ over $C_0(\Omega)$ is an *AM-module over* $C_0(\Omega)$ if each submodule $\Gamma_s = \overline{C_0(\Omega) \cdot s}$, $s \in \Gamma$, is an AM-space.

Remark 2.19. By Proposition 2.17 a Banach module over $C_0(\Omega)$ is an AM-module over $C_0(\Omega)$ if and only if

$$\max(\|f_1 s\|, \|f_2 s\|) = \|(f_1 \vee f_2)s\|$$

for all $f_1, f_2 \in C_0(\Omega)_+$ and $s \in \Gamma$.

Example 2.20. If E is a topological Banach bundle over a locally compact space Ω, then $\Gamma_0(\Omega, E)$, see Definition 1.3, is an AM-module over $C_0(\Omega)$.

Remark 2.21. (i) AM-modules are also called *locally convex Banach modules*, see Definition 7.10 in [Gie82] or Definition 1.1 of [Par08], see also [HoKe17]. By Proposition 7.14 of [Gie82] our definition is equivalent in the unital case, and using an approximate identity, even in the general setting. Our terminology leads to a duality between AM- and AL-modules, see Proposition 2.33 below.

(ii) Given a compact space K, each AM-module over $\mathrm{C}(K)$ is isometrically isomorphic to a space of sections $\Gamma(K,E)$ of some Banach bundle E over K which is unique up to isometric isomorphy, see Theorems 2.5 and 2.6 of [DG83]. A similar result holds, and is probably well-known, in the locally compact case. However, since we did not find a reference for this fact, we give a proof in Proposition 2.26 below.

We now state and prove our first representation result for dynamical Banach modules.

Theorem 2.22. *Let G be a locally compact group, $S \subseteq G$ be a closed submonoid, and $(\Omega; \boldsymbol{\varphi})$ a topological G-dynamical system. Then the assignments*

$$(E; \boldsymbol{\Phi}) \mapsto (\Gamma_0(\Omega, E); \boldsymbol{\mathcal{T}_\Phi})$$
$$\Theta \mapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of S-dynamical topological Banach bundles over $(\Omega; \boldsymbol{\varphi})$ to the category of S-dynamical AM-modules over $(\mathrm{C}_0(\Omega); \boldsymbol{T_\varphi})$.

The proof of Theorem 2.22 starts with the following simple observation.

Lemma 2.23. *Let Ω be a locally compact space, $\varphi\colon \Omega \longrightarrow \Omega$ a homeomorphism, and $p_E\colon E \longrightarrow \Omega$ be a Banach bundle over Ω. Then $p_\varphi\colon E_\varphi \longrightarrow \Omega$ with $E_\varphi := E$ and $p_\varphi := \varphi^{-1} \circ p_E$ is a Banach bundle over Ω which has the following properties.*

(i) *The identical mapping $\mathrm{id}_E\colon E \longrightarrow E_\varphi$ is a Banach bundle morphism over φ^{-1}.*

(ii) *If F is a Banach bundle over Ω, then a mapping $\Phi\colon F \longrightarrow E$ is a Banach bundle morphism over φ if and only if $\Phi\colon F \longrightarrow E_\varphi$ is a Banach bundle morphism over id_Ω.*

Using these facts, most of the proof of Theorem 2.22 can be reduced to the non-dynamical case. We first consider single operators.

Lemma 2.24. *Let E and F be Banach bundles over a locally compact space Ω. Moreover, let $\varphi\colon \Omega \longrightarrow \Omega$ be a homeomorphism and $\mathcal{T} \in \mathscr{L}(\Gamma_0(\Omega, E), \Gamma_0(\Omega, F))$*

a T_φ-module homomorphism. Then there is a unique Banach bundle morphism Φ over φ with $\mathcal{T} = \mathcal{T}_\Phi$. Moreover, $\|\Phi\| = \|\mathcal{T}\|$ and $\mathcal{T}$ is an isometry if and only if Φ is isometric.

Proof. Assume that $\Omega = K$ is compact. Consider the bundle F_φ induced by φ, see Lemma 2.23. The operator $V \in \mathscr{L}(\Gamma(K,E),\Gamma(K,F_\varphi))$ defined by $Vs := s \circ \varphi$ is an isometric and surjective $T_{\varphi^{-1}}$-homorphism. Therefore, the operator $V\mathcal{T} \in \mathscr{L}(\Gamma(K,E),\Gamma(K,F_\varphi))$ is a (non-dynamical) homomorphism of Banach modules. By Theorem 2.6 of [DG83] we thus find a unique bundle morphism $\Phi\colon E \longrightarrow F_\varphi$ over id_K with

$$V\mathcal{T}s = \Phi \circ s$$

for each $s \in \Gamma(K,E)$, i.e., $\Phi\colon E \longrightarrow F$ is the unique bundle morphism over φ with

$$\mathcal{T}s = V^{-1}(\Phi \circ s) = \Phi \circ s \circ \varphi^{-1}$$

for every $s \in \Gamma(K,E)$. Moreover, $\|\Phi\| = \|V\mathcal{T}\| = \|\mathcal{T}\|$ and Φ is isometric if and only if $V\mathcal{T}$ is an isometry, i.e., if and only if $\mathcal{T}$ is isometric, see Propositions 10.13 and 10.16 of [Gie82].

Now suppose that Ω is non-compact, but locally compact. Let K be the one-point compactification and $\tilde{\varphi}\colon K \longrightarrow K$ the canonical continuous extension of φ. The canonical mapping

$$\Gamma(K,\tilde{E}) \longrightarrow \Gamma_0(\Omega,E), \quad s \mapsto s|_\Omega$$

is an isometric isomorphism of Banach spaces, see Remark 2.3, and therefore $\mathcal{T}$ induces an operator $\tilde{\mathcal{T}} \in \mathscr{L}(\Gamma(K,\tilde{E}),\Gamma(K,\tilde{F}))$. It is easy to check that $\tilde{\mathcal{T}}$ is a $T_{\tilde{\varphi}}$-homomorphism and we can apply the first part to find a unique bundle morphism $\tilde{\Phi}\colon \tilde{E} \longrightarrow \tilde{E}$ over $\tilde{\varphi}$ with $\mathcal{T}(s|_\Omega) = (\tilde{\Phi} \circ s \circ \tilde{\varphi}^{-1})|_\Omega$ for every $s \in \Gamma(K,\tilde{E})$. Since each Banach bundle morphism of E over φ has a unique extension to a Banach bundle morphism of $\tilde{E}$ over $\tilde{\varphi}$, see Lemma 2.15, the restriction $\tilde{\Phi}|_E$ is the unique bundle morphism Φ over φ with $\mathcal{T}s := \Phi \circ s \circ \varphi^{-1}$ for all $s \in \Gamma_0(\Omega,E)$. The remaining claims are obvious. □

Lemma 2.25. *Let G be a locally compact group, $S \subseteq G$ be a closed submonoid, and $(\Omega;\boldsymbol{\varphi})$ a topological G-dynamical system. Moreover, let E be a Banach bundle over Ω and let $\boldsymbol{\mathcal{T}}\colon S \longrightarrow \mathscr{L}(\Gamma_0(\Omega,E))$ be a strongly continuous semigroup representation such that $(\Gamma_0(\Omega,E);\boldsymbol{\mathcal{T}})$ is an S-dynamical Banach module over $(\mathrm{C}_0(\Omega);\boldsymbol{T_\varphi})$. Then there is a unique S-dynamical Banach bundle $(E;\boldsymbol{\Phi})$ over $(\Omega;\boldsymbol{\varphi})$ such that $\boldsymbol{\mathcal{T}_\Phi} = \boldsymbol{\mathcal{T}}$.*

Proof. We apply Lemma 2.24 to find a unique bundle morphism Φ_g over φ_g such that $\mathcal{T}(g) = \mathcal{T}_{\Phi_g}$ for each $g \in S$. Since $\mathcal{T}(1) = \mathrm{Id}_{\Gamma_0(\Omega,E)}$, we obtain that $\Phi(1) = \mathrm{id}_E$. Moreover, for $g_1, g_2 \in S$ we obtain that $\tilde{\Phi} := \Phi_{g_1} \circ \Phi_{g_2}$ is a bundle morphism over $\varphi_{g_1 g_2}$ with

$$\mathcal{T}(g_1 g_2) = \mathcal{T}(g_1)\mathcal{T}(g_2) = \mathcal{T}_\Phi(g_1)\mathcal{T}_\Phi(g_2) = \mathcal{T}_{\tilde{\Phi}}.$$

By uniqueness of $\Phi_{g_1 g_2}$ we therefore obtain

$$\Phi_{g_1} \circ \Phi_{g_2} = \tilde{\Phi} = \Phi_{g_1 g_2}.$$

To conclude the proof we have to show that the mapping

$$\mathbf{\Phi}\colon S \longrightarrow E^E, \quad g \mapsto \Phi_g$$

is jointly continuous and that $\mathbf{\Phi}$ is locally bounded. The latter follows since $\|\Phi(g)\| = \|\mathcal{T}(g)\|$ for every $g \in S$ by Lemma 2.24 and $\mathcal{T}$ is locally bounded by strong continuity and the principle of uniform boundedness.

Now let $v \in E$ and $g \in S$. Take $s \in \Gamma_0(\Omega, E)$ with $s(gp_E(v)) = \Phi_g v$, $\varepsilon > 0$, and an open neighborhood U of $gp_E(v)$. Since Φ_g is continuous, we find $\tilde{s} \in \Gamma_0(\Omega, E)$, $\delta > 0$ and a neighborhood $\tilde{V}$ of $p_E(v)$ such that $\tilde{s}(p_E(v)) = v$ and

$$\Phi_g(V(\tilde{s}, \tilde{V}, \delta)) \subseteq V(s, U, \varepsilon),$$

see Lemma 1.4. In particular, we obtain $g(\tilde{V}) \subseteq U$ and $\|\Phi_g \tilde{s}(x) - s(gx)\| < \varepsilon$ for every $x \in \tilde{V}$. Since φ is continuous, we find a neigborhood $V \subseteq \tilde{V}$ of $p_E(v)$ and a neighborhood $\tilde{W}$ of g in S such that $hy \in g(\tilde{V})$ for every $y \in V$ and $h \in \tilde{W}$. Finally, choose a compact neighborhood $W \subseteq \tilde{W}$ of g with

$$\sup_{x\in\Omega} \|\Phi_h \tilde{s}(x) - \Phi_g \tilde{s}(g^{-1}hx)\| = \|\mathcal{T}(h)\tilde{s} - \mathcal{T}(g)\tilde{s}\| < \varepsilon.$$

for every $h \in W$. Then $M := \sup_{h\in W} \|\Phi_h\| < \infty$ and for $h \in W$ and $u \in V(\tilde{s}, V, \frac{\varepsilon}{M+1})$, we obtain $hp_E(u) \in U$ and

$$\begin{aligned}\|\Phi_h u - s(hp_E(u))\| &\leqslant \|\Phi_h\| \cdot \|u - \tilde{s}(p_E(u))\| \\ &\quad + \|\Phi_h \tilde{s}(p_E(u)) - \Phi_g \tilde{s}(g^{-1}hp_E(u))\| \\ &\quad + \|\Phi_g \tilde{s}(g^{-1}hp_E(u)) - s(hp_E(u))\| \\ &< 3\varepsilon.\end{aligned}$$

This shows $\Phi_h u \in V(s, U, 3\varepsilon)$ for each $h \in W$ and $u \in V(\tilde{s}, V, \frac{\varepsilon}{M+1})$ and thus $\mathbf{\Phi}$ is jointly continuous. □

Finally, we look at AM-modules.

Proposition 2.26. *Let Ω be a locally compact space and Γ an AM-module over $\mathrm{C}_0(\Omega)$. Then there is a Banach bundle E over Ω such that $\Gamma_0(\Omega, E)$ is isometrically isomorphic to Γ. Moreover, this bundle is unique up to isometric isomorphy.*

Proof. If Ω is compact, the claim holds by Theorem 2.6 of [DG83]. If Ω is non-compact, we consider Γ as a Banach module over $\mathrm{C}(K)$ where K is the one-point compactification of Ω, see Lemma 2.11. Using a similar argument as in Lemma 2.11 we see that Γ is then an AM-module over $\mathrm{C}(K)$ and we therefore find a Banach bundle F over K such that $\Gamma(K, F)$ is isometrically isomorphic to Γ as a Banach module over $\mathrm{C}(K)$. Moreover, by the proof of Theorem 2.6 of [DG83] we have $F_\infty \cong \Gamma/J_\infty$ with

$$J_\infty = \overline{\mathrm{lin}}\{fs \mid f \in \mathrm{C}(K) \text{ with } f(\infty) = 0 \text{ and } s \in \Gamma\}.$$

Since Γ is non-degenerate, we obtain $J_\infty = \Gamma$ and thus $F_\infty = \{0\}$. We can therefore define a Banach bundle E over Ω by setting $E := F \setminus F_\infty$ and $p_E := p_F|_E$ and it is clear that $F = \tilde{E}$. In particular, we obtain an isometric isomorphism of Banach spaces, see Remark 2.3,

$$\Gamma(K, F) \longrightarrow \Gamma_0(\Omega, E), \quad s \mapsto s|_\Omega$$

and it is then easy to check that Γ is isometrically isomorphic to $\Gamma_0(\Omega, E)$ as a Banach module over $\mathrm{C}_0(\Omega)$. Uniqueness up to isometric isomorphy follows directly from Lemma 2.24. □

Combining Proposition 2.26 with the preceding Lemmas 2.24 and 2.25 leads to the proof of Theorem 2.22.

Remark 2.27. It is not hard to construct an inverse to the functor of Theorem 2.22. In fact, if Γ is an AM-module over $\mathrm{C}_0(\Omega)$, then we obtain the fibers E_x of a Banach bundle E by setting

$$J_x := \overline{\mathrm{lin}}\{fs \mid f \in \mathrm{C}_0(\Omega) \text{ with } f(x) = 0 \text{ and } s \in \Gamma\},$$
$$E_x := \Gamma/J_x,$$

for $x \in \Omega$, see Section 2 of [DG83] or Section 7 of [Gie82]. Moreover, if $\varphi\colon \Omega \longrightarrow \Omega$ is a homeomorphism and $\mathcal{T} \in \mathscr{L}(\Gamma)$ is a T_φ-homomorphism, then $\mathcal{T} J_x \subseteq J_{\varphi(x)}$ for every $x \in \Omega$ and therefore $\mathcal{T}$ induces a bounded operator $\Phi_x \in \mathscr{L}(E_x, E_{\varphi(x)})$.

With these constructions one can assign a dynamical Banach bundle to a dynamical AM-module $(\Gamma;\mathcal{T})$. We skip the details, cf. Theorem 2.6 of [DG83].

2.1.2 AL-modules

The dual concept of AM-spaces in the theory of Banach lattices are so-called AL-spaces, see Section II.8 of [Sch74]. Again we make use of this concept to introduce a certain class of Banach modules.

Definition 2.28. Let Ω be a locally compact space. A Banach module Γ over $C_0(\Omega)$ is called an *AL-module over* $C_0(\Omega)$ if Γ_s is an AL-space for each $s \in \Gamma$.

Remark 2.29. By Proposition 2.17 a Banach module over $C_0(\Omega)$ is an AL-module over $C_0(\Omega)$ if and only if

$$\|f_1 s + f_2 s\| = \|f_1 s\| + \|f_2 s\|$$

for all $f_1, f_2 \in C_0(\Omega)_+$ and $s \in \Gamma$.

Note that if X is a measure space, then $L^\infty(X)$ is isomorphic to $C(K)$ as a Banach algebra and a Banach lattice for some compact space K. Thus, every Banach module over $L^\infty(X)$ can be seen as a Banach module over $C(K)$. In particular, we may speak of *AM- and AL-modules over* $L^\infty(X)$.

Example 2.30. Let E be a measurable Banach bundle over a measure space X. Then $\Gamma^1(X,E)$, see Example 2.4, is an AL-module over $L^\infty(X)$.

Remark 2.31. It is tempting to expect that for a measure space X every AL-module over $L^\infty(X)$ is already isomorphic to a space $\Gamma^1(X,E)$ for some measurable Banach bundle E over X. However, we will see below that this is not the case, see Example 2.43.

As in the case of Banach lattices, AM- and AL-modules over $C(K)$ are dual to each other. To formulate this result we first equip the dual space of a Banach module with a module structure.

Definition 2.32. Let K be a compact space and Γ a Banach module over $\mathrm{C}(K)$. Then the dual space Γ' equipped with the operation $(f \cdot s')(s) := s'(f \cdot s)$ for $s \in \Gamma$, $s' \in \Gamma'$, and $f \in \mathrm{C}(K)$ is the *dual Banach module of* Γ *over* $\mathrm{C}(K)$.

It is straightforward to check that the dual Banach module of a Banach module is in fact a Banach module. We can now make the duality between AM- and AL-modules precise using the following result due to Cunnigham, see Theorem 5 of [Cun67], though in somewhat different notation.

Proposition 2.33. *Let* Ω *be a locally compact space. For a Banach module* Γ *over* $\mathrm{C}_0(\Omega)$ *the following assertions hold.*

(i) Γ *is an AM-module if and only if* Γ' *is an AL-module.*

(ii) Γ *is an AL-module if and only if* Γ' *is an AM-module.*

2.2 Lattice normed modules

2.2.1 $\mathrm{U}_0(\Omega)$-normed modules

As observed in [Cun67], AM-modules admit an additional lattice theoretic structure. For a locally compact space Ω, we write

$$\begin{aligned} \mathrm{U}(\Omega) &:= \{f\colon \Omega \longrightarrow \mathbb{R} \mid f \text{ is upper semicontinuous}\}, \\ \mathrm{U}_0(\Omega) &:= \{f \in \mathrm{U}(\Omega) \mid \forall\, \varepsilon > 0\, \exists\, K \subseteq \Omega \text{ compact with } |f(x)| \leqslant \varepsilon\, \forall\, x \notin K\}, \\ \mathrm{U}_0(\Omega)_+ &:= \{f \in \mathrm{U}_0(\Omega) \mid f \geqslant 0\}, \end{aligned}$$

and introduce the following concept, see Section 6.6 of [HoKe17] for the compact case.

Definition 2.34. Let Ω be a locally compact space and Γ a Banach module over $\mathrm{C}_0(\Omega)$. A mapping

$$|\cdot|\colon \Gamma \longrightarrow \mathrm{U}_0(\Omega)_+$$

is a $\mathrm{U}_0(\Omega)$-*valued norm* if

(i) $\||s|\| = \|s\|$,

(ii) $|fs| = |f| \cdot |s|$,

(iii) $|s_1 + s_2| \leqslant |s_1| + |s_2|$,

for all $s, s_1, s_2 \in \Gamma$ and $f \in \mathrm{C}_0(\Omega)$. A Banach module over $\mathrm{C}_0(\Omega)$ together with a $\mathrm{U}_0(\Omega)$-valued norm is called a $\mathrm{U}_0(\Omega)$-*normed module*.

Example 2.35. Let E be a Banach bundle over a locally compact space Ω. Setting $|s|(x) := \|s(x)\|$ for $x \in \Omega$ and $s \in \Gamma_0(\Omega, E)$ turns $\Gamma_0(\Omega, E)$ into a $\mathrm{U}_0(\Omega)$-normed module.

Note that each $\mathrm{U}_0(\Omega)$-normed module is automatically an AM-module over $\mathrm{C}_0(\Omega)$. The converse also holds and is basically due to Cunningham in the compact case, see Lemma 3 and Theorem 2 in [Cun67].

Proposition 2.36. *Let Ω be a locally compact space. For a Banach module Γ over $\mathrm{C}_0(\Omega)$ the following are equivalent.*

(a) *Γ is an AM-module over A.*

(b) *Γ admits a $\mathrm{U}_0(\Omega)$-valued norm.*

If these assertions hold, then the $\mathrm{U}_0(\Omega)$-valued norm is unique and given by

$$|s|(x) = \inf\{\|fs\| \mid f \in \mathrm{C}_0(\Omega)_+ \text{ with } f(x) = 1\}$$

for $x \in \Omega$ and $s \in \Gamma$.

Proof. Using Lemma 2.11 and an approximate unit, existence via the desired formula of the $\mathrm{U}_0(\Omega)$-valued norm can be reduced to the compact case which is treated in Lemma 3 and Theorem 2 of [Cun67].

For uniqueness, observe that any $\mathrm{U}_0(\Omega)$-valued norm $|\cdot|\colon \Gamma \longrightarrow \mathrm{U}_0(\Omega)_+$ satisfies

$$|s|(x) \leqslant \inf\{\|fs\| \mid f \in \mathrm{C}_0(\Omega)_+ \text{ with } f(x) = 1\}$$

for every $x \in \Omega$ and $s \in \Gamma$. On the other hand, if $x \in \Omega$, $s \in \Gamma$, and $\varepsilon > 0$, we find a neighborhood U of x such that $|s|(y) \leqslant |s|(x) + \varepsilon$ for every $y \in U$ since $|s|$ is upper semicontinuous. Thus there is $f \in \mathrm{C}_0(\Omega)_+$ with $\|f\| = f(x) = 1$ and

$$\|fs\| = \sup_{y\in\Omega} |fs|(y) = \sup_{y\in\Omega} |f(y)| \cdot |s|(y) \leqslant |s|(x) + \varepsilon$$

which implies the claim. □

Remark 2.37. The representing Banach bundles of AM-modules Γ over $C_0(\Omega)$ satisfying $|s| \in C_0(\Omega) \subseteq U_0(\Omega)$ for every $s \in \Gamma$ are precisely the continuous Banach bundles, see Theorem 15.11 of [Gie82] or pages 47–48 of [DG83] for the compact case; the locally compact case can easily be reduced to this.

We can now state the main theorem of this subsection which shows that the algebraic and lattice theoretic structures of $U_0(\Omega)$-normed modules are closely related to each other. Here, we use the notation T_φ for the map $U_0(\Omega) \longrightarrow U_0(\Omega)$, $f \mapsto f \circ \varphi^{-1}$.

Theorem 2.38. *Let Ω be a locally compact space, $\varphi\colon \Omega \longrightarrow \Omega$ a homeomorphism, and Γ and Λ $U_0(\Omega)$-normed modules. For $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ the following are equivalent.*

(a) $\mathcal{T}(fs) = T_\varphi f \cdot \mathcal{T}s$ *for every* $f \in C_0(\Omega)$ *and* $s \in \Gamma$.

(b) $\operatorname{supp}(\mathcal{T}s) \subseteq \varphi(\operatorname{supp}(s))$ *for every* $s \in \Gamma$.

(c) $|\mathcal{T}s| \leqslant \|\mathcal{T}\| \cdot T_\varphi|s|$ *for every* $s \in \Gamma$.

(d) *There is* $m > 0$ *such that* $|\mathcal{T}s| \leqslant m \cdot T_\varphi|s|$ *for every* $s \in \Gamma$.

Moreover, if $\Gamma = \Gamma_0(\Omega, E)$ and $\Lambda = \Gamma_0(\Omega, F)$ for Banach bundles E and F over Ω, then the properties above are also equivalent to the following assertion.

(e) *There is a morphism Φ over φ with $\mathcal{T} = \mathcal{T}_\Phi$.*

If (e) *holds, then the morphism Φ in* (e) *is unique, $\|\Phi\| = \|\mathcal{T}\|$, and Φ is isometric if and only if $\mathcal{T}$ is isometric.*

For the proof we need the following lemma connecting the lattice-valued norm with the concept of support introduced in Definition 2.8.

Lemma 2.39. *Let Γ be a $U_0(\Omega)$-normed module. Then*

$$\operatorname{supp}(s) = \operatorname{supp}(|s|) = \overline{\{x \in \Omega \mid |s|(x) \neq 0\}}$$

for each $s \in \Gamma$.

Proof. Let $x \in \Omega$ with $|s|(x) \neq 0$ and $f \in \mathrm{C}_0(\Omega)$ with $f(x) \neq 0$. Then $|fs|(x) = |f|(x)|s|(x) \neq 0$ and therefore $\|fs\| \neq 0$.

Conversely, let $x \in \operatorname{supp}(s)$. Assume there is an open neighborhood U of x such that $|s|(y) = 0$ for every $y \in U$. We then find $f \in \mathrm{C}_0(\Omega)$ with support in U and $f(x) = 1$. But then $|fs| = |f||s| = 0$ and therefore $fs = 0$ which contradicts $x \in \operatorname{supp}(s)$. □

Proof (of Theorem 2.38). The equivalence of (a) and (b) holds by Theorem 2.10. Now assume that (a) and (b) hold and that there is $s \in \Gamma$ such that $|\mathcal{T}s| \not\leqslant \|\mathcal{T}\| \cdot T_\varphi|s|$. We then find $x \in \Omega$ with $\|\mathcal{T}\| \cdot |s|(x) < |\mathcal{T}s|(\varphi(x))$. Since $|s|$ is upper semicontinuous, we find $\varepsilon > 0$ and an open neighborhood V of x such that $\|\mathcal{T}\| \cdot |s|(z) \leqslant |\mathcal{T}s|(\varphi(x)) - \varepsilon$ for all $z \in V$. Now take a function $f \in \mathrm{C}_0(\Omega)_+$ with support in V such that $0 \leqslant f \leqslant \mathbb{1}$ and $f(x) = 1$. Setting $\tilde{s} := fs$ we obtain

$$\begin{aligned}
\|\mathcal{T}\| \cdot \|\tilde{s}\| + \varepsilon &= \sup_{z \in V} \|\mathcal{T}\| \cdot f(z) \cdot |s|(z) + \varepsilon \\
&\leqslant |\mathcal{T}s|(\varphi(x)) = (T_\varphi f)(\varphi(x)) \cdot |\mathcal{T}s|(\varphi(x)) = |\mathcal{T}(fs)|(\varphi(x)) \\
&\leqslant \|\mathcal{T}\tilde{s}\|,
\end{aligned}$$

which contradicts the definition of $\|\mathcal{T}\|$. The implication "(c) ⇒ (d)" is obvious and "(d) ⇒ (b)" follows from Lemma 2.39. The rest of the theorem follows from Lemma 2.24. □

Remark 2.40. In view of Proposition 2.36 and Theorem 2.38, the assignments of Theorem 2.22 also define an essentially surjective and fully faithful functor from the category of dynamical Banach bundles over a topological dynamical system $(\Omega; \boldsymbol{\varphi})$ to the category having as objects pairs of $\mathrm{U}_0(\Omega)$-normed modules and semigroup representations of "dominated operators", in the sense of Theorem 2.38 (c), and as morphisms operators $V \in \mathscr{L}(\Gamma, \Lambda)$ between $\mathrm{U}_0(\Omega)$-normed modules such that there is an $m > 0$ with $|Vs| \leqslant m \cdot |s|$ for all $s \in \Gamma$ which are compatible with the semigroup representations.

2.2.2 $\mathrm{L}^1(X)$-normed modules

AL-modules also admit a lattice-valued norm.

Definition 2.41. Let Ω be a locally compact space and Γ a Banach module over $\mathrm{C}_0(\Omega)$. A mapping

$$|\cdot|\colon \Gamma \longrightarrow \mathrm{C}_0(\Omega)'_+$$

is an $\mathrm{C}_0(\Omega)'$-*valued norm* if

(i) $\| |s| \| = \|s\|$,

(ii) $|fs| = |f| \cdot |s|$,

(iii) $|s_1 + s_2| \leqslant |s_1| + |s_2|$,

for all $s, s_1, s_2 \in \Gamma$ and $f \in \mathrm{C}_0(\Omega)$. A Banach module over A together with a $\mathrm{C}_0(\Omega)'$-valued norm is called a $\mathrm{C}_0(\Omega)'$-*normed module*.

Again the main part of the following result is due to Cunningham in the compact case, see Theorem 4 of [Cun67]. We give a new proof in the general case and also provide an explicit formula for the lattice-valued norm.

Proposition 2.42. *Let Ω be a locally compact space. For a Banach module Γ over $\mathrm{C}_0(\Omega)$ the following are equivalent.*

(a) Γ *is an AL-module over* $\mathrm{C}_0(\Omega)$.

(b) Γ *admits a* $\mathrm{C}_0(\Omega)'$-*valued norm.*

If these assertions hold, then the $\mathrm{C}_0(\Omega)'$-*valued norm is unique and given by* $|s|(f) := \|fs\|$ *for all* $s \in \Gamma$ *and* $f \in \mathrm{C}_0(\Omega)_+$.

Proof. It is clear that (b) implies (a) since $\mathrm{C}_0(\Omega)'$ is an AL-space, cf. Proposition 9.1 of [Sch74]. If (a) holds, we define $|s|(f) = \|fs\|$ for all $s \in \Gamma$ and $f \in \mathrm{C}_0(\Omega)_+$. For every $s \in \Gamma$ the map $|s|\colon \mathrm{C}_0(\Omega)_+ \to \mathbb{R}_{\geqslant 0}$ is additive and positively homogeneous and therefore has a unique positive extension $|s| \in A'$ by Lemma 1.3.3 of [MN91], which obviously also holds in the complex case. Now take an approximate unit $(e_i)_{i\in I}$ for $\mathrm{C}_0(\Omega)$. Then

$$\|s\| = \lim_i \|e_i s\| = \lim_i |s|(e_i) = \| |s| \|.$$

It is clear that $|s_1 + s_2| \leqslant |s_1| + |s_2|$ for all $s_1, s_2 \in \Gamma$. Finally, let $f \in \mathrm{C}_0(\Omega)$ and $s \in \Gamma$. Then

$$|fs|(g) = \|gfs\| = \| |gf| s\| = |s|(|f| g) = (|f| \cdot |s|)(g)$$

for every $g \in C_0(\Omega)_+$. This shows $|f \cdot s| = |f| \cdot |s|$.

To prove uniqueness, let $|\cdot|$ be any $C_0(\Omega)'$-valued norm on Γ and let $(e_i)_{i\in I}$ be an approximate unit for $C_0(\Omega)$. Then

$$\|fs\| = \lim_i |fs|(e_i) = \lim_i |s|(fe_i) = |s|(f)$$

for each $s \in \Gamma$ and $f \in C_0(\Omega)_+$, showing the claim. □

Given a measure space X, we can consider $L^\infty(X)$ as a space $C(K)$ for some compact space K. If Γ is an AL-module over $L^\infty(X)$, Proposition 2.42 then yields a lattice-valued norm $|\cdot|\colon \Gamma \longrightarrow L^\infty(X)'_+$. On the other hand, if E is a measurable Banach bundle over X, then the mapping

$$|\cdot|\colon \Gamma^1(X,E) \longrightarrow L^1(X)_+, \quad s \mapsto \|s(\cdot)\|$$

satisfies properties (i) – (iii) of Definition 2.41 and since $L^1(X)$ embeds canonically (as a Banach lattice and as a Banach module over $L^\infty(X)$) into $L^\infty(X)'$, this already defines the unique $L^\infty(X)'$-valued norm. In particular, an AL-module over $L^\infty(X)$ can only be isometrically isomorphic to $\Gamma^1(X,E)$ for some measurable Banach bundle E over X if the $L^\infty(X)'$-valued norm takes values in (the canonical image of) $L^1(X)$. This is not always the case as the following example shows.

Example 2.43. Let X be any measure space and consider $\Gamma := L^\infty(X)'$ as a Banach module over $L^\infty(X)$. Then Γ is an AL-module over $L^\infty(X)$ by Proposition 2.33 since $L^1(X)$ is an AL-module over $L^\infty(X)$. The usual modulus $|\cdot|\colon L^\infty(X)' \to L^\infty(X)'$ is given by

$$|s|(f) = \sup\{|s(g)| \mid 0 \leqslant |g| \leqslant f\}$$

for $f \in L^\infty(X)_+$ and $s \in L^\infty(X)'$, see Corollary 1 to Proposition II.4.2 of [Sch74]. It is easy to see that

$$\sup\{|s(g)| \mid 0 \leqslant |g| \leqslant f\} = \sup\{|s(gf)| \mid 0 \leqslant |g| \leqslant \mathbb{1}\} = \|fs\|$$

for $f \in L^\infty(X)_+$ and $s \in L^\infty(X)'$ and therefore $|\cdot|$ is the unique $L^\infty(X)'$-valued norm. If $L^1(X)$ is not finite-dimensional, then $L^1(X)$ is not reflexive, see Corollary 2 of Theorem II.9.9 in [Sch74]. By Proposition 8.3 (iii) and (v) of [Sch74] there are also positive elemnents in $L^\infty(X)'$ which are not contained in (the canonical image of) $L^1(X)$, i.e., there is $s \in \Gamma$ with $|s| \in L^\infty(X)' \setminus L^1(X)$.

Definition 2.44. Let X be a measure space. An $\mathrm{L}^\infty(X)'$-normed module Γ is called an $\mathrm{L}^1(X)$-*normed module* if $|s| \in \mathrm{L}^1(X)$ for every $s \in \Gamma$.

We now state and prove our second main result. Here a measure space X is *separable* if there is a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of Ω_X such that for every $B \in \Sigma_X$ and every $\varepsilon > 0$ there is an $n \in \mathbb{N}$ with $\mu_X(A_n \Delta B) < \varepsilon$.

Theorem 2.45. *Let G be a (discrete) group, $S \subseteq G$ be a submonoid, and $(X;\boldsymbol{\varphi})$ a measure preserving G-dynamical system with X separable. Then the assignments*

$$(E;\mathbf{\Phi}) \mapsto (\Gamma^1(X,E);\mathcal{T}_{\mathbf{\Phi}})$$
$$\Theta \mapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of S-dynamical separable measurable Banach bundles over $(X;\boldsymbol{\varphi})$ to the category of S-dynamical separable $\mathrm{L}^1(X)$-normed modules over $(\mathrm{L}^\infty(X);\boldsymbol{T_\varphi})$.

We start by showing that separable Banach bundles over separable measure spaces in fact induce separable spaces of sections.

Proposition 2.46. *Let E be a separable measurable Banach bundle over a separable measure space X. Then $\Gamma^1(X,E)$ is separable.*

The proof of the following lemma is based on the proof of Proposition 4.4 of [FD88], see also Lemma A.3.5 of [ADR00] for a similar result.

Lemma 2.47. *Let E be a separable Banach bundle over a measure space X and $(s_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ such that $\mathrm{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ is dense in E_x for almost every $x \in \Omega_X$. Then $\mathrm{lin}\{s_n \mid n \in \mathbb{N}\}$ generates E, i.e., every $s \in \mathcal{M}_E$ is an almost everywhere limit of a sequence in $\mathrm{lin}\{\mathbb{1}_A s_n \mid A \in \Sigma_X, n \in \mathbb{N}\}$.*

Proof. By the set $\{s_n \mid n \in \mathbb{N}\}$ with its linear hull over $\mathbb{Q}$ (if $\mathbb{K} = \mathbb{R}$) or $\mathbb{Q} + i\mathbb{Q}$ (if $\mathbb{K} = \mathbb{C}$), we may assume that $\{s_n(x) \mid n \in \mathbb{N}\}$ is dense in E_x for almost every $x \in \Omega_X$. Now let $s \in \mathcal{M}_E$, $\varepsilon > 0$ and set

$$A_n := \{x \in \Omega_X \mid \|s(x) - s_n(x)\| \leqslant \varepsilon\} \in \Sigma_X$$

for every $n \in \mathbb{N}$. Then

$$\Omega_X \setminus \left(\bigcup_{n \in \mathbb{N}} A_n \right)$$

is a nullset. Therefore, $\|s(x) - \tilde{s}(x)\| \leqslant \varepsilon$ for almost every $x \in \Omega_X$ where

$$\tilde{s}(x) = \begin{cases} s_n(x) & x \in A_n \setminus \bigcup_{k=1}^{n-1} A_k, n \in \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

Since $\tilde{s}$ is a measurable section with respect to the Banach bundle generated by $\mathrm{lin}\{s_n \mid n \in \mathbb{N}\}$, see Remark 1.14, this shows the claim. □

Lemma 2.48. *Let E be a separable Banach bundle over a measure space X. Then there is a sequence $(s_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_E$ such that*

(i) $\mathrm{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ *is dense in E_x for almost every $x \in \Omega_X$,*

(ii) $\mu_X(\{|s_n| \neq 0\}) < \infty$ *for every $n \in \mathbb{N}$,*

(iii) $|s_n| = \mathbb{1}_{\{|s_n| \neq 0\}}$ *almost everywhere for every $n \in \mathbb{N}$,*

Moreover, for any sequence $(s_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_E$ with properties (i) *and* (ii)*, the set*

$$\mathrm{lin}\{\mathbb{1}_A s_n \mid A \in \Sigma_X, n \in \mathbb{N}\} \subseteq \Gamma^1(X, E)$$

is dense in $\Gamma^1(X, E)$.

Proof. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}_E$ satisfying (i). Replacing s_n by $\tilde{s}_n$ defined as

$$\tilde{s}_n(x) := \begin{cases} \frac{1}{\|s_n(x)\|} s_n(x) & s_n(x) \neq 0, \\ 0 & s_n(x) = 0, \end{cases}$$

for every $n \in \mathbb{N}$ we may assume that (i) and (iii) hold. Now pick a sequence $(A_n)_{n \in \mathbb{N}}$ of measurable subsets of Ω_X of finite measure such that

$$\Omega_X = \bigcup_{m \in \mathbb{N}} A_m.$$

Then $\mu_X(\{|\mathbb{1}_{A_m} s_n| \neq 0\}) < \infty$ for all $m, n \in \mathbb{N}$. Replacing $(s_n)_{n \in \mathbb{N}}$ once again, we may assume that properties (i) – (iii) are fulfilled.

Now assume that $(s_n)_{n\in\mathbb{N}}$ is a sequence $\mathcal{M}_E$ satisfying (i) and (ii) and let $s \in \mathcal{M}_E$ with $\int |s| \,\mathrm{d}\mu_X < \infty$. By Lemma 2.47 and Lemma 4.3 of [FD88] we find a sequence $(r_n)_{n\in\mathbb{N}}$ in

$$M := \operatorname{lin}\{\mathbb{1}_A s_n \mid A \in \Sigma_X, n \in \mathbb{N}\} \subseteq \mathcal{M}_E$$

such that $\lim_{n\to\infty} r_n = s$ almost everywhere and $|r_n| \leqslant |s|$ almost everywhere for all $n \in \mathbb{N}$. By Lebesgue's theorem we therefore obtain that the canonical image of M in $\Gamma^1(X, E)$ is dense in $\Gamma^1(X, E)$. □

Proof (of Proposition 2.46). Using the separability of X, we pick a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of Ω_X such that for every $B \in \Sigma_X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ with $\mu_X(A_n \Delta B) < \varepsilon$. Moreover, take a sequence $(s_n)_{n\in\mathbb{N}}$ as in Lemma 2.48. For each $n \in \mathbb{N}$ and every $A \in \Sigma_X$ we then find an $m \in \mathbb{N}$ with

$$\|\mathbb{1}_A s_n - \mathbb{1}_{A_m} s_n\| \leqslant \mu(A \Delta A_m) < \varepsilon.$$

This implies that $\{\mathbb{1}_{A_m} s_n \mid n, m \in \mathbb{N}\}$ is total in $\Gamma^1(X, E)$. □

The following result characterizes weighted Koopman operators induced by measurable dynamical Banach bundles similarly to the topological setting, cf. Theorem 2.38.

Theorem 2.49. *Let X be a measure space, $\varphi\colon X \longrightarrow X$ an automorphism, and Γ and Λ $\mathrm{L}^1(X)$-normed modules. For an operator $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ the following are equivalent.*

(a) *$\mathcal{T}(fs) = T_\varphi f \cdot \mathcal{T}s$ for all $f \in \mathrm{L}^\infty(X)$ and every $s \in \Gamma$.*

(b) *$|\mathcal{T}s| \leqslant \|\mathcal{T}\| \cdot T_\varphi|s|$ for every $s \in \Gamma$.*

(c) *There is an $m > 0$ such that $|\mathcal{T}s| \leqslant m \cdot T_\varphi|s|$ for every $s \in \Gamma$.*

Moreover, if $\Gamma = \Gamma^1(X, E)$ and $\Lambda = \Gamma^1(X, F)$ for Banach bundles E and F over X with E separable, then the above are also equivalent to the following assertion.

(d) *There is a morphism $\Phi\colon E \longrightarrow F$ over φ such that $\mathcal{T} = \mathcal{T}_\Phi$.*

If (d) *holds, then the morphism Φ in* (d) *is unique,*

$$|\Phi|\colon \Omega_X \longrightarrow [0, \infty), \quad x \mapsto \|\Phi_x\|$$

defines an element of $\mathrm{L}^\infty(X)$ and

- $\sup\{|\mathcal{T}_\Phi s| \mid s \in \Gamma^\infty(X,E)$ *with* $|s| \leqslant \mathbb{1}\} = T_\varphi|\Phi| \in \mathrm{L}^\infty(X)$,
- $\|\Phi\| = \|\mathcal{T}_\Phi\|_{\Gamma^\infty(X,E)} = \|\mathcal{T}_\Phi\|_{\Gamma^1(X,E)}$,
- Φ *is an isometry if and only* $\mathcal{T}_\Phi \in \mathscr{L}(\Gamma^1(X,E),\Gamma^1(X,F))$ *is an isometry.*

Proof. We write $\langle\,\cdot\,,\,\cdot\,\rangle$ for the canonical duality between $\mathrm{L}^1(X)$ and $\mathrm{L}^\infty(X)$. Now assume that (a) is valid and take $s \in \Gamma$. For each $f \in \mathrm{L}^\infty(X)$ with $f \geqslant 0$ we obtain

$$\begin{aligned}\langle|\mathcal{T}s|, f\rangle &= \|f\mathcal{T}s\| = \|\mathcal{T}((T_{\varphi^{-1}}f)\cdot s)\| \\ &\leqslant \|\mathcal{T}\|\cdot\|T_{\varphi^{-1}}f\cdot s\| = \|\mathcal{T}\|\cdot\langle|s|, T_{\varphi^{-1}}f\rangle = \langle\|\mathcal{T}\|\cdot T_\varphi|s|, f\rangle\end{aligned}$$

since φ is measure-preserving. Thus, $|\mathcal{T}s| \leqslant \|\mathcal{T}\|\cdot T_\varphi|s|$.

The implication "(b) $\Rightarrow$ (c)" is clear. Now assume that (c) holds. Since X is σ-finite, we find measurable and pairwise disjoint sets $A_n \in \Sigma_X$ with finite measure for $n \in \mathbb{N}$ such that

$$\Omega_X = \bigcup_{n\in\mathbb{N}} A_n.$$

For fixed $n \in \mathbb{N}$ consider the submodules

$$\begin{aligned}\Gamma_n &:= \{s \in \Gamma \mid \mathbb{1}_{A_n}|s| = |s| \in \mathrm{L}^\infty(X)\} \subseteq \Gamma,\\ \Lambda_n &:= \{s \in \Lambda \mid \mathbb{1}_{\varphi(A_n)}|s| = |s| \in \mathrm{L}^\infty(X)\} \subseteq \Lambda.\end{aligned}$$

We define $\|s\|_\infty := \||s|\|_{\mathrm{L}^\infty(X)}$ for $s \in \Gamma_n$ and $s \in \Lambda_n$, respectively. We show that this turns Γ_n and Λ_n into Banach modules over $\mathrm{L}^\infty(X)$. If $(s_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in Γ_n with respect to the norm $\|\cdot\|_\infty$, then it is also a Cauchy sequence with respect to the norm of Γ. By completeness of Γ there is $s \in \Gamma$ such that $\lim_{m\to\infty} s_m = s$ in Γ. Using that there is a subsequence $(s_{m_k})_{k\in\mathbb{N}}$ of $(s_m)_{m\in\mathbb{N}}$ such that $|s_{m_k} - s| \to 0$ almost everywhere, it follows that $s \in \Gamma_n$ and $\lim_{m\to\infty} s_m = s$ with respect to $\|\cdot\|_\infty$. Thus, Γ_n—and likewise Λ_n—is a Banach module over $\mathrm{L}^\infty(X)$. Moreover, $\mathcal{T}|_{\Gamma_n} \in \mathscr{L}(\Gamma_n, \Lambda_n)$ by (c). Choose a compact space K and an isomorphism $V \in \mathscr{L}(\mathrm{L}^\infty(X), \mathrm{C}(K))$ of Banach algebras and lattices. We then consider Γ_n and Λ_n as Banach modules over $\mathrm{C}(K)$ via V^{-1} and see that the mappings

$$\begin{aligned}\Gamma_n &\longrightarrow \mathrm{C}(K), \quad s \mapsto V|s|,\\ \Lambda_n &\longrightarrow \mathrm{C}(K), \quad s \mapsto V|s|\end{aligned}$$

turn Γ_n and Λ_n into $\mathrm{U}(K)$-normed modules. Moreover, since every algebra isomorphism on $\mathrm{C}(K)$ is induced by a homeomorphism on K, we can apply Theorem 2.38 to the $VT_\varphi V^{-1}$-homomorphism $\mathcal{T}|_{\Gamma_n} \in \mathscr{L}(\Gamma_n, \Lambda_n)$. This shows that $\mathcal{T}(fs) = (T_\varphi f)\cdot\mathcal{T}s$ for all $f \in \mathrm{L}^\infty(X)$ and $s \in \Gamma_n$.

Take $f \in \mathrm{L}^\infty(X)$ and $s \in \Gamma$ with $|s| = \mathbb{1}_{A_n}|s|$. Then $s = \lim_{m\to\infty} \mathbb{1}_{\{|s|\leqslant m\}} s$ in Γ and therefore

$$\mathcal{T}(fs) = \lim_{m\to\infty} \mathcal{T}(f\mathbb{1}_{\{|s|\leqslant m\}}s) = \lim_{m\to\infty} (T_\varphi f)\cdot \mathcal{T}(\mathbb{1}_{\{|s|\leqslant m\}}s) = (T_\varphi f)\cdot \mathcal{T}s.$$

Finally, we obtain for arbitrary $s \in \Gamma$ and $f \in \mathrm{L}^\infty(X)$

$$\begin{aligned}\mathcal{T}(fs) &= \lim_{N\to\infty} \mathcal{T}\left(f\sum_{n=1}^N \mathbb{1}_{A_n}s\right) = \lim_{N\to\infty}\sum_{n=1}^N \mathcal{T}(f\mathbb{1}_{A_n}s)\\ &= \lim_{N\to\infty}\sum_{n=1}^N T_\varphi f\cdot \mathcal{T}\mathbb{1}_{A_n}s = T_\varphi f\cdot\mathcal{T}s.\end{aligned}$$

This shows (a).

Now assume that $\Gamma = \Gamma^1(X,E)$ and $\Lambda = \Gamma^1(X,F)$ for measurable Banach bundles E and F over X with E separable. We let $Q := \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $Q := \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$. Now take a sequence $(s_n)_{n\in\mathbb{N}}$ as in $\mathcal{M}_E$ satisfying conidtions (i) – (iii) of Lemma 2.48 and set

$$H_x := \mathrm{lin}_Q\{s_k(x) \mid k \in \mathbb{N}\}$$

for every $x \in \Omega_X$.

Let $\mathcal{T}$ be a T_φ-homomorphism. Choose a representative for φ (which we again denote by φ) and a representative $r_n \in \mathcal{M}_F$ of $\mathcal{T}s_n \in \Gamma^1(X,F)$ for each $n \in \mathbb{N}$. By (b) we obtain

$$\left\|\left(\sum_{k=1}^N q_k r_k\right)(\varphi(x))\right\| \leqslant \|\mathcal{T}\|\cdot\left\|\left(\sum_{k=1}^N q_k s_k\right)(x)\right\| \tag{2.3}$$

for all $(q_1,\dots,q_N) \in Q^N$, $N \in \mathbb{N}$, and almost every $x \in \Omega_X$. For almost every $x \in \Omega_X$ we therefore find a unique Q-linear map $\Phi_x\colon H_x \longrightarrow H_{\varphi(x)}$ such that $\Phi_x s_n(x) = (r_n)(\varphi(x))$ for every $n \in \mathbb{N}$. By (2.3) and property (i) of Lemma 2.48 it has a unique extension to a bounded operator $\Phi_x \in \mathscr{L}(E_x, F_x)$ for almost every $x \in \Omega_X$. We set $\Phi_x := 0 \in \mathscr{L}(E_x, F_{\varphi(x)})$ for the remaining points $x \in \Omega_X$ and obtain a mapping

$$\Phi\colon E \longrightarrow F, \quad v \mapsto \Phi_{p_E(v)}v.$$

Since $\Phi\circ(\mathbb{1}_A\cdot s_n) = (\mathbb{1}_{\varphi(A)}\cdot r_n)\circ\varphi$ almost everywhere for every $n \in \mathbb{N}$ and every set $A \in \Sigma_X$, we can apply Lemma 2.47 to see that for each $s \in \mathcal{M}_E$ there is a $r \in \mathcal{M}_F$

with $\Phi \circ s = r \circ \varphi$ almost everywhere. This shows that Φ defines a morphism of measurable Banach bundles over φ and we denote this again by Φ. Moreover, $\mathcal{T}_\Phi s_n = \mathcal{T} s_n$ and, since $\{\mathbb{1}_A s_n \mid n \in \mathbb{N}, A \in \Sigma_X\}$ defines a total subset of $\Gamma^1(X, E)$ by Lemma 2.48, we obtain $\mathcal{T} = \mathcal{T}_\Phi$. Thus (a), (b) and (c) imply (d). The converse implication is obvious.

Now let $\Phi\colon E \longrightarrow F$ be a morphism over φ. As usual, we pick a representing premorphism whenever necessary. Using Lemma 2.48 and standard arguments we find a sequence $(\tilde{s}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ such that

- $|\tilde{s}_n| \leqslant \mathbb{1}$ almost everywhere for every $n \in \mathbb{N}$,
- $\mu_X(\{|\tilde{s}_n| \neq 0\}) < \infty$ for every $n \in \mathbb{N}$,
- $\{\tilde{s}_n(x) \mid n \in \mathbb{N}\}$ is dense in the unit ball B_{E_x} of E_x for almost every $x \in \Omega_X$.

Then

$$\|\Phi|_{E_x}\| = \sup_{n\in\mathbb{N}} \|\Phi|_{E_x} \tilde{s}_n(x)\|$$

for almost every $x \in \Omega_X$. Thus, $\Omega_X \longrightarrow \mathbb{R}$, $x \mapsto \|\Phi_{E_x}\|$ is measurable and $|\Phi|$ defines an element of $\mathrm{L}^\infty(X)$ of norm $\|\Phi\|$.

Clearly, $|\mathcal{T}_\Phi s| \leqslant T_\varphi|\Phi|$ for every $s \in \Gamma^\infty(X, E)$ with $|s| \leqslant \mathbb{1}$. On the other hand,

$$T_\varphi|\Phi|(x) = \|\Phi|_{E_{\varphi^{-1}(x)}}\| = \sup_{n\in\mathbb{N}} \|\Phi|_{E_{\varphi^{-1}(x)}} \tilde{s}_n(\varphi^{-1}(x))\| = \sup_{n\in\mathbb{N}} \|(\mathcal{T}_\Phi \tilde{s}_n)(x)\|$$

for almost every $x \in \Omega_X$. This shows

$$T_\varphi|\Phi| = \sup\{|\mathcal{T}_\Phi s| \mid s \in \Gamma^\infty(X, E) \text{ with } |s| \leqslant \mathbb{1}\}. \tag{2.4}$$

Moreover,

$$\begin{aligned} \|\Phi\| &= \operatorname{ess\,sup}_{x\in\Omega_X} \sup_{n\in\mathbb{N}} \|(\mathcal{T}_\Phi \tilde{s}_n)(x)\| = \sup_{n\in\mathbb{N}} \operatorname{ess\,sup}_{x\in\Omega_X} \|(\mathcal{T}_\Phi \tilde{s}_n)(x)\| \\ &= \sup_{n\in\mathbb{N}} \|\mathcal{T}_\Phi \tilde{s}_n\|_{\Gamma^\infty(X,E)} \leqslant \|\mathcal{T}_\Phi\|_{\Gamma^\infty(X,E)}, \end{aligned}$$

and $\|\mathcal{T}_\Phi\|_{\Gamma^\infty(X,E)} \leqslant \|\Phi\|$ is clear, hence $\|\mathcal{T}_\Phi\|_{\Gamma^\infty(X,E)} = \|\Phi\|$.

Now pick $s \in \Gamma^\infty(X, E)$ with $|s| \leqslant \mathbb{1}$. For every measurable set $A \in \Sigma_X$ with finite measure

$$\mathbb{1}_A|\mathcal{T}_\Phi s| = |\mathcal{T}_\Phi(T_\varphi^{-1}\mathbb{1}_A \cdot s)| \leqslant \|\mathcal{T}_\Phi\|_{\Gamma^1(X,E)} \cdot T_\varphi|(T_\varphi^{-1}\mathbb{1}_A \cdot s)| \leqslant \|\mathcal{T}_\Phi\|_{\Gamma^1(X,E)} \cdot \mathbb{1}_A$$

by (b). Since X is σ-finite, we obtain $\|\Phi\| \leqslant \|\mathcal{T}_\Phi\|_{\Gamma^1(X,E)}$ by (2.4), and the inequality $\|\mathcal{T}_\Phi\|_{\Gamma^1(X,E)} \leqslant \|\Phi\|$ is obvious. Therefore,

$$\|\Phi\| = \|\mathcal{T}_\Phi\|_{\Gamma^\infty(X,E)} = \|\mathcal{T}_\Phi\|_{\Gamma^1(X,E)}$$

and, since the difference of premorphisms over φ is again a premorphism over φ, this equality also proves the uniqueness of Φ in (d).

If Φ is an isometry, then clearly $\mathcal{T}_\Phi \in \mathscr{L}(\Gamma^1(X,E), \Gamma^1(X,F))$ is an isometry. Assume conversely that $\mathcal{T}_\Phi$ is an isometry. We already know that $\Phi|_{E_x}$ is a contraction for almost every $x \in \Omega_X$. Assume that there is a set $A \in \Sigma_X$ with positive measure such that $\Phi|_{E_x}$ is not an isometry for every $x \in A$. We then find an $n \in \mathbb{N}$ and a set $B \in \Sigma_X$ with positive measure such that $\|\Phi|_{E_x}\tilde{s}_n(x)\| < \|\tilde{s}_n(x)\|$ for every $x \in B$. This implies

$$\|\mathcal{T}_\Phi \tilde{s}_n\| = \int_X \|\Phi|_{E_x}\tilde{s}_n(x)\| \, \mathrm{d}\mu_X < \int_X \|\tilde{s}_n(x)\| \, \mathrm{d}\mu_X = \|\tilde{s}_n\|,$$

a contradiction. □

Since we have not employed any continuity assumptions on dynamical measurable Banach bundles, we immediately obtain the following consequence of Theorem 2.49.

Corollary 2.50. *Let G be a (discrete) group, $S \subseteq G$ be a submonoid, and $(X;\boldsymbol{\varphi})$ a measure-preserving G-dynamical system. Moreover let E be a separable Banach bundle over X and let $\boldsymbol{\mathcal{T}}\colon S \longrightarrow \mathscr{L}(\Gamma^1(X,E))$ be a semigroup representation such that $(\Gamma^1(X,E);\boldsymbol{\mathcal{T}})$ is an S-dynamical Banach module over $(\mathrm{L}^\infty(X);\boldsymbol{T}_{\boldsymbol{\varphi}})$. Then there is a unique dynamical Banach bundle $(E;\boldsymbol{\Phi})$ over $(X;\boldsymbol{\varphi})$ such that $\boldsymbol{\mathcal{T}}_{\boldsymbol{\Phi}} = \boldsymbol{\mathcal{T}}$.*

Finally, we use a result of Gutmann [Gut93b] to represent $\mathrm{L}^1(X)$-normed modules.

Proposition 2.51. *Let X be a measure space and Γ an $\mathrm{L}^1(X)$-normed module. Then the following assertions hold.*

(i) *There is a measurable Banach bundle E over X such that $\Gamma^1(X,E)$ is isometrically isomorphic to Γ.*

(ii) *If Γ is separable, then there is a separable Banach bundle E over X such that $\Gamma^1(X, E)$ is isometrically isomorphic to Γ. Moreover, E is unique up to isometric isomorphy.*

Proof. In the real case, 7.1.3 of [Kus00] shows that the space Γ is in particular a Banach–Kantorovich space over $\mathrm{L}^1(X)$, see Chapter 2 of [Kus00] for this concept, and we find a measurable Banach bundle E over X such that Γ is isometrically isomorphic to $\Gamma^1(X, E)$ as a lattice normed space by Theorem 3.4.8 of [Gut93b][3]. If we start with a complex $\mathrm{L}^1(X)$-normed module, the proof of this theorem reveals that the constructed Banach bundle E is canonically a Banach bundle of complex Banach spaces and that the isomorphism of Γ and $\Gamma^1(X, E)$ is $\mathbb{C}$-linear, see Theorem 3.3.4 of [Gut93b] and Theorems 2.1.5 and 2.4.2 of [Gut93a]. In any case, we can apply Theorem 2.49 to see that this isomorphism is an isometric Banach module isomorphism.

Now assume that Γ and therefore $\Gamma^1(X, E)$ is separable. Let $(s_n)_{n\in\mathbb{N}}$ be dense in $\Gamma^1(X, E)$ and choose a representative in $\mathcal{M}_E$ for each s_n which we also denote by s_n. We define a new measurable Banach bundle by setting $F_x := \overline{\mathrm{lin}\{s_n(x) \mid n \in \mathbb{N}\}}$ for every $x \in \Omega_X$ and

$$\mathcal{M}_F := \{s \in \mathcal{M}_E \mid s(x) \in F_x \text{ for every } x \in \Omega_X\}.$$

Then

$$V : \Gamma^1(X, F) \longrightarrow \Gamma^1(X, E), \quad s \mapsto s$$

is an isometric module homomorphism. However, since $s_n \in \Gamma^1(X, F)$ for every $n \in \mathbb{N}$, V is in fact an isometric isomorphism. Clearly, F is separable. Uniqueness up to isometric isomorphy follows immediately from Theorem 2.49. □

Combining Proposition 2.46, Corollary 2.50, Theorem 2.49, and Proposition 2.51 now readily yields Theorem 2.45.

Remark 2.52. Note that in contrast to the topological setting, the construction of the representing separable measurable Banach bundle is not canonical and involves choices.

[3]Note that the definition of measurable Banach bundles by Gutmann slightly differs from ours. However, every measurable Banach bundle in the sense of Gutmann canonically defines a measurable Banach bundle in our sense having the same space $\Gamma^1(X, E)$.

Part II

One-parameter semigroups of weighted Koopman operators

In this part, we investigate an important special case of the objects presented in Part I: *strongly continuous one-parameter semigroups of weighted Koopman operators* on Banach modules of continuous sections, cf. Definition 3.5. We adapt the results from the previous part and give additional characterizations of *weighted Koopman semigroups*, cf. Theorem 3.8 and Theorem 3.12. In Chapter 4 we investigate the spectrum of weighted Koopman semigroups and of their generators, cf. Theorem 4.13, which leads to a characterization of *hyperbolicity* in Chapter 5, cf. Theorem 5.8. Part II constitutes a mostly self-contained presentation of the topic and may therefore be read independently of Part I.

Our main references are [EN00] for C_0-*semigroups*, [EFHN15] for *Koopmanism*, and [Gie82], [DG83], [AAK92], and Part I for *Banach bundles* and *Banach modules*.

The results in Chapter 3, 4, and 5 are based on joint work with Henrik Kreidler. Chapter 6 is based on joint work with Nikolai Edeko and Henrik Kreidler.

Chapter 3

Weighted Koopman semigroups on spaces of continuous sections

In Part I we introduced weighted semigroup representations on AM-modules, see Definition 2.12, and gave several characterizations of such semigroup representations, cf. Theorem 2.22, Theorem 2.45, and Theorem 2.38. In this chapter, we turn to the special case of C_0-semigroups and include the generator and the resolvent into the characterization of weighted C_0-semigroups, see Theorem 3.8 and Theorem 3.12.

The results in Section 3.3 and 3.4 are joint work with Henrik Kreidler.

3.1 Koopmanism

We start from a compact space K and consider the associated Banach space $\mathrm{C}(K)$ of all scalar-valued continuous functions on K. Pointwise multiplication turns $\mathrm{C}(K)$ into a commutative C^*-algebra with unit.

Conversely, the Gelfand theorem states that for each commutative C^*-algebra A with unit there exists, up to isomorphy, a unique compact space K such that

$$A \cong \mathrm{C}(K),$$

see, e.g., Theorem 4.23 of [EFHN15] or Section 1.4 of [Dix77].

Furthermore, under the natural pointwise order, $\mathrm{C}(K)$ becomes a Banach lattice and even an AM-space with unit. By Kakutani's theorem each such space, in particular, the dual of any AL-space, is isomorphic to some $\mathrm{C}(K)$, see Section II.7 and II.9 of [Sch74].

As a consequence of the Gelfand and the Kakutani theorem, all properties of the topological space K correspond to algebraic and lattice-theoretic properties of $\mathrm{C}(K)$ and vice versa, cf., e.g., [Ede20], Proposition 2.2.

For certain dynamics on K and on $\mathrm{C}(K)$ we obtain a similar correspondence. A *topological dynamical system* $(K;\boldsymbol{\varphi})$ is a continuous group action

$$\boldsymbol{\varphi}\colon \mathbb{R}\times K \longrightarrow K, \quad (t,x)\mapsto \varphi_t(x) = \boldsymbol{\varphi}(t,x)$$

of the group $\mathbb{R}$ on a compact space K. We call $\boldsymbol{\varphi} = (\varphi_t)_{t\in\mathbb{R}}$ a *(continuous) flow* on K. To each topological dynamical system corresponds a C_0-group $(T_\varphi(t))_{t\in\mathbb{R}}$ on $\mathrm{C}(K)$ defined by

$$T_\varphi(t)f = f\circ\varphi_{-t} \quad \text{ for all } f\in\mathrm{C}(K), t\in\mathbb{R}.$$

This global linearization $(T_\varphi(t))_{t\in\mathbb{R}}$ of the flow is called a *Koopman group* and its generator is denoted by $(\delta, D(\delta))$. This change of perspective enables an elegant translation of properties of topological dynamics into functional analytic poperties, see, e.g., [EFHN15], Theorem 16.36. Koopman groups are systematically treated in, e.g., Part B of [Nag86] or Chapter 16 of [BKR17] and the time discrete case in Chapter 4 of [EFHN15].

As a basic result we recall that such Koopman groups on $\mathrm{C}(K)$ can be characterized in various ways, cf. Part B-II of [Nag86], Theorem 3.4.

Theorem 3.1. *For a C_0-group $(T(t))_{t\in\mathbb{R}}$ on $\mathrm{C}(K)$ with generator $(\delta, D(\delta))$ the following assertions are equivalent.*

(i) *There is a topological dynamical system $(K;(\varphi_t)_{t\in\mathbb{R}})$ such that $T(t) = T_\varphi(t)$ for all $t\in\mathbb{R}$.*

(ii) *Each operator $T(t)$ is a *-homomorphism with $T(t)\mathbb{1} = \mathbb{1}$.*

(iii) *Each operator $T(t)$ is a Markov lattice homomorphism.*

(iv) *The generator* $(\delta, D(\delta))$ *is a* derivation on $\mathrm{C}(K)$*, i.e.,* $D(\delta)$ *is a subalgebra of* $\mathrm{C}(K)$ *with* $1 \in D(\delta)$ *such that*

$$\delta(fg) = \delta f \cdot g + f \cdot \delta g \quad \text{for all } f, g \in D(\delta).$$

3.2 Spaces of continuous sections

In the following, we consider Banach bundles E over a compact space K, cf. Definition 1.1, and Banach modules Γ over the C*-algebra $\mathrm{C}(K)$, cf. Definition 2.1, as introduced in Part I. In this section, we recall the basic properties of Banach modules induced by Banach bundles, i.e., spaces $\Gamma(K, E)$ of continuous sections of E, see Definition 1.3. Furthermore, we restate a representation theorem from Part I in the present situation, see Theorem 3.3. For the theory of Banach bundles and Banach modules we refer to [Gie82], [DG83], [HoKe17], or [Cun67] and Part I.

Endowing the space of continuous sections $\Gamma(K, E)$ with the operation

$$\cdot\colon \mathrm{C}(K) \times \Gamma(K, E) \longrightarrow \Gamma(K, E), \quad (f, s) \mapsto f \cdot s := [x \mapsto f(x)s(x)],$$

the norm $\|\cdot\|$ defined by

$$\|s\| := \sup_{x \in K} \|s(x)\|, \quad s \in \Gamma(K, E),$$

and the mapping

$$\begin{aligned} |\cdot| : \Gamma(K, E) &\longrightarrow \mathrm{U}(K)_+ := \{f\colon K \longrightarrow \mathbb{R} \mid f \text{ is upper semicontinuous, } f \geqslant 0\} \\ s &\mapsto [x \mapsto \|s(x)\|], \end{aligned}$$

we obtain the following properties.

Proposition 3.2. *The space of continuous sections* $\Gamma(K, E)$ *is an AM-module, see Definition 2.18, and a* $\mathrm{U}(K)$*-normed module over* $\mathrm{C}(K)$*, see Definition 2.34. Moreover, the following holds.*

(i) *For each* $v \in E$ *there exists* $s \in \Gamma(K, E)$ *such that* $s(p(v)) = v$*.*

(ii) *For each* $v_1, v_2 \in E$ *with* $v_1 \neq v_2$ *and* $p(v_1) \neq p(v_2)$ *there exists* $s \in \Gamma(K, E)$ *such that* $s(p(v_1)) \neq s(p(v_2))$*.*

(iii) *The lattice-valued norm* $|\cdot|$ *satisfies* $|s| \in \mathrm{C}(K)$ *for all* $s \in \Gamma(K, E)$ *if and only if* E *is continuous.*

Proof. The structure properties of $\Gamma(K, E)$ are obvious, cf. Example 2.20 and Example 2.35 of Part I. Assertion (i) follows by Corollary 2.10 of [Gie82] and implies (ii). For (iii), see Remark 2.37. □

In Part I we proved in a more general setting that each AM-module and each lattice-normed module over $\mathrm{C}(K)$ can be represented as a space of continuous sections, see Proposition 2.26 and Proposition 2.36. We recall this result in this situation.

Theorem 3.3. *For a Banach module* Γ *over* $\mathrm{C}(K)$ *the following assertions are equivalent.*

(a) Γ *is an AM-module over* $\mathrm{C}(K)$*.*

(b) Γ *is a* $\mathrm{U}(K)$*-normed module over* $\mathrm{C}(K)$*.*

(c) *There exists, up to isometric isomorphy, a unique Banach bundle* E *over* K *such that* Γ *is isometrically isomorphic to* $\Gamma(K, E)$*.*

If these assertions hold, then the $\mathrm{U}(K)$*-valued norm in* (b) *is unique and given by*

$$|s|(x) = \inf\{\|fs\| \mid f \in \mathrm{C}(K)_+ \text{ with } f(x) = 1\}$$

for $s \in \Gamma$*,* $x \in K$*.*

3.3 Algebraic characterization of weighted Koopman semigroups

In this section, we show that the dynamics on the "bundle side" corresponds to the dynamics on the "module side". On the bundle side we start from a semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on E over K, see Definition 1.8. We then introduce a corresponding C_0-semigroup $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on the Banach module $\Gamma(K, E)$ of continuous sections of E over the Koopman group on $\mathrm{C}(K)$, see Definition 3.5. The main result is an algebraic characterization of this C_0-semigroup and an additional characterization by means of its generator, see Theorem 3.8. Finally, we discuss typical examples.

We start with the definition of dynamics on the Banach module $\Gamma(K, E)$ over the Koopman group $(T_\varphi(t))_{t\in\mathbb{R}}$ on $\mathrm{C}(K)$, cf. Definition 2.12.

Definition 3.4. A C_0-semigroup $(\mathcal{T}(t))_{\geqslant 0}$ on $\Gamma(K, E)$ is called *weighted semigroup over* $(T_\varphi(t))_{t\in\mathbb{R}}$ if each operator $\mathcal{T}(t)$ is a $T_\varphi(t)$*-homomorphism*, i.e,

$$\mathcal{T}(t)(fs) = T_\varphi(t)f \cdot \mathcal{T}(t)s \quad \text{for all } s \in \Gamma(K, E), f \in \mathrm{C}(K), \text{ and } t \geqslant 0.$$

We now define a C_0-semigroup on $\Gamma(K, E)$ induced by a semiflow over $(\varphi_t)_{t\in\mathbb{R}}$ on a Banach bundle E over K. To this end, we reformulate Example 2.13 in the context of C_0-semigroups.

Definition 3.5. Let Φ be a Banach bundle morphism over a homeomorphism φ on a Banach bundle E over K. The *weighted Koopman operator* $\mathcal{T}_\Phi$ *on* $\Gamma(K, E)$ *induced by* Φ *and* φ is defined by

$$\mathcal{T}_\Phi s := \Phi \circ s \circ \varphi^{-1}, \quad s \in \Gamma(K, E).$$

An operator family $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on $\Gamma(K, E)$ is called *weighted Koopman semigroup* if there is a semiflow $(\Phi_t)_{t\geqslant 0}$ over a flow $(\varphi_t)_{t\in\mathbb{R}}$ on E over K such that

$$\mathcal{T}_\Phi(t)s := \Phi_t \circ s \circ \varphi_{-t}, \quad s \in \Gamma(K, E), \ t \geqslant 0.$$

The following proposition justifies this terminology and states the main properties of such operator families.

Proposition 3.6. *The family* $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ *of linear operators on* $\Gamma(K, E)$ *induced by a semiflow* $(\Phi_t)_{t\geqslant 0}$ *over* $(\varphi_t)_{t\in\mathbb{R}}$ *on* E *over* K *has the following properties.*

(i) $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ *is a* C_0*-semigroup on* $\Gamma(K, E)$*.*

(ii) *The operators* $\mathcal{T}_\Phi(t)$ *are* $T_\varphi(t)$*-homomorphisms.*

(iii) *The generator* $(\mathcal{A}, D(\mathcal{A}))$ *of* $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ *is a* δ-derivation *on* $\Gamma(K, E)$*, i.e.,* $D(\mathcal{A})$ *is a* $D(\delta)$*-submodule of* $\Gamma(K, E)$ *and*

$$\mathcal{A}(fs) = \delta f \cdot s + f \cdot \mathcal{A}s \quad \textit{for all } f \in D(\delta), s \in D(\mathcal{A}).$$

Remark 3.7. Assertion (i) and (ii) of the above proposition are a special case of Proposition 2.14 and Example 2.6 of Part I. However, we give a direct proof in this special situation.

Proof. For (i), it suffices to show that

$$t \mapsto \mathcal{T}_\Phi(t)s = \Phi_t \circ s \circ \varphi_{-t}$$

is continuous in 0 for all $s \in \Gamma(K, E)$, see [EN00], Proposition I.5.3. Since the mappings

$$\begin{aligned}
\mathbb{R}_+ \times K &\longrightarrow \mathbb{R}_+ \times K, \quad && (t, x) \mapsto (t, \varphi_{-t}(x)),\\
\mathbb{R}_+ \times K &\longrightarrow \mathbb{R}_+ \times E, \quad && (t, x) \mapsto (t, s(x)),\\
\mathbb{R}_+ \times E &\longrightarrow E, \quad && (t, v) \mapsto \Phi_t v,
\end{aligned}$$

are jointly continuous, their composition

$$\mathbb{R}_+ \times K \longrightarrow E, \quad (t, x) \mapsto \Phi_t s(\varphi_{-t}(x)),$$

is jointly continuous, too. Therefore, and since $\|\cdot\|$ is upper semicontinuous,

$$\|\Phi_t \circ s \circ \varphi_{-t} - s\| = \sup_{x \in K} \|\Phi_t s(\varphi_{-t}(x)) - s(x)\|$$

tends to zero as $t \to 0$.

For assertion (ii), let $f \in \mathrm{C}(K)$, $s \in \Gamma(K, E)$. We have

$$\begin{aligned}
\mathcal{T}_\Phi(t)(fs)(x) &= \Phi_t(\varphi_{-t}(x))(f(\varphi_{-t}(x))s(\varphi_{-t}(x))\\
&= f(\varphi_{-t}(x)) \cdot \Phi_t(\varphi_{-t}(x))s(\varphi_{-t}(x))\\
&= (T_\varphi(t)f)(x) \cdot (\mathcal{T}_\Phi(t)s)(x)
\end{aligned}$$

for all $x \in K$, $t \geqslant 0$.

For (iii), take $f \in D(\delta)$, $s \in D(\mathcal{A})$. Then

$$\frac{\mathcal{T}_\Phi(t)(fs) - fs}{t} \overset{\text{(ii)}}{=} \frac{T_\varphi(t)f - f}{t} \cdot \mathcal{T}_\Phi(t)s + f \cdot \frac{\mathcal{T}_\Phi(t)s - s}{t}$$

converges to $\delta f \cdot s + f \cdot \mathcal{A}s$ for $t \to 0$. □

We recall that a morphism Θ from a semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on a Banach bundle E over K to a semiflow $(\Psi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on a Banach bundle F over

K, cf. Definition 1.6, induces a homomorphism V_Θ from the weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on $\Gamma(K,E)$ to the weighted Koopman semigroup $(\mathcal{T}_\Psi(t))_{t\geqslant 0}$ on $\Gamma(K,F)$, cf. Definition 2.12, via

$$V_\Theta s := \Theta \circ s \quad \text{for all } s \in \Gamma(K,E),$$

cf. Proposition 2.14.

The "bundle dynamics" and the "module dynamics" correspond to each other. More precisely, each weighted semigroup over $(T_\varphi(t))_{t\in\mathbb{R}}$ on $\Gamma(K,E)$ over $\mathrm{C}(K)$ can be uniquely represented as a weighted Koopman semigroup. We reformulate Theorem 2.22 for C_0-semigroups and give an additional algebraic characterization via the generator of the semigroup.

Theorem 3.8. *For a C_0-semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on $\Gamma(K,E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$ the following assertions are equivalent.*

(a) *$(\mathcal{T}(t))_{t\geqslant 0}$ is a weighted Koopman semigroup on $\Gamma(K,E)$, i.e., there exists a unique semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on E over K such that $\mathcal{T}(t) = \mathcal{T}_\Phi(t)$ for all $t \geqslant 0$.*

(b) *The operators $\mathcal{T}(t)$ are $T_\varphi(t)$-homomorphisms for all $t \geqslant 0$.*

(c) *The generator $(\mathcal{A}, D(\mathcal{A}))$ is a δ-*derivation *on $\Gamma(K,E)$, i.e., $D(\mathcal{A})$ is a $D(\delta)$-submodule of $\Gamma(K,E)$ and*

$$\mathcal{A}(fs) = \delta f \cdot s + f \cdot \mathcal{A}s$$

for all $f \in D(\delta)$, $s \in D(\mathcal{A})$.

Moreover, if these assertions hold, then the semiflow $(\Phi_t)_{t\geqslant 0}$ in (a) *is unique, satisfies $\|\mathcal{T}_\Phi(t)\| = \|\Phi_t\|$ for all $t \geqslant 0$, and $\mathcal{T}_\Phi(t)$ is an isometry if and only if Φ_t is isometric.*

Proof. By Proposition 3.6 (ii) assertion (a) implies (b) and the proof of Proposition 3.6 (iii) yields the implication "(b) $\Rightarrow$ (c)" .

Assume that (c) holds and take $f \in D(\delta)$, $s \in D(\mathcal{A})$, and $t > 0$. We define $\xi(r) := \mathcal{T}(t-r)(T_\varphi(r)f \cdot \mathcal{T}(r)s)$ for $r \in (0,t)$. By Lemma B.16 of [EN00] the function ξ is differentiable on $(0,t)$ with

$$\begin{aligned}\xi'(r) &= -\mathcal{T}(t-r)\mathcal{A}(T_\varphi(r)f\cdot\mathcal{T}(r)s)\\ &\quad+\mathcal{T}(t-r)(\delta T_\varphi(r)f\cdot\mathcal{T}(r)s+T_\varphi(r)f\cdot\mathcal{A}\mathcal{T}(r)s)\\ &=0\end{aligned}$$

for every $r\in(0,t)$. Thus, $T_\varphi(t)f\mathcal{T}(t)s=\xi(t)=\xi(0)=\mathcal{T}(t)(fs)$. Since $D(\delta)$ is dense in $\mathrm{C}(K)$ and $D(\mathcal{A})$ is dense in $\Gamma(K,E)$, assertion (b) follows.

For the implication "(b) $\Rightarrow$ (a)" we refer to Part I, Lemma 2.25. The remaining assertions of the theorem follow by Lemma 2.24. □

Remark 3.9. Starting from a weighted semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ over $(T_\varphi(t))_{t\in\mathbb{R}}$ on some AM-module Γ over $\mathrm{C}(K)$, see Definition 2.18, we construct the semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on some Banach bundle E over K such that

$$\mathcal{T}(t)\cong\mathcal{T}_\Phi(t)\qquad\text{for each } t\geqslant 0$$

on $\Gamma\cong\Gamma(K,E)$. Recall that $E=\bigcup_{x\in K}E_x$ with $E_x=\Gamma/J_x$ and

$$J_x=\overline{\mathrm{lin}}\{fs\mid f\in\mathrm{C}(K)\text{ with } f(x)=0\text{ and } s\in\Gamma\}$$

as in Remark 2.27 is the, up to isometric isomorphy, unique Banach bundle such that $\Gamma\cong\Gamma(K,E)$. Then $\mathcal{T}(t)J_x\subseteq J_{\varphi_t(x)}$ for each $x\in K$. For the canonical quotient map $q_x\colon\Gamma\longrightarrow\Gamma/J_x$, each operator $\mathcal{T}(t)$ induces a bounded operator $\Phi_t(x)\in\mathscr{L}(E_x,E_{\varphi_t(x)})$ via

$$\Phi_t(x)q_x(s):=q_{\varphi_t(x)}(\mathcal{T}(t)s).$$

This yields the unique semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on E over K with $\mathcal{T}(t)\cong\mathcal{T}_\Phi(t)$, see [DG83], Section 2, [Gie82], Section 7, or Remark 2.27 of Part I.

Remark 3.10. The equivalence "(a) $\Leftrightarrow$ (b)" of Theorem 3.8 also holds for a single operator $\mathcal{T}\in\mathscr{L}(\Gamma(K,E))$. Let $\varphi\colon K\longrightarrow K$ be a homeomorphism, then the following assertions are equivalent.

(a) The operator $\mathcal{T}$ is a T_φ-homomorphism, i.e., $\mathcal{T}(fs)=T_\varphi f\cdot\mathcal{T}s$ for all $f\in\mathrm{C}(K)$, $s\in\Gamma(K,E)$.

(b) There is a unique Banach bundle morphism Φ over φ, see Definition 1.6, such that $\mathcal{T}=\mathcal{T}_\Phi$, i.e., $\mathcal{T}s=\Phi\circ s\circ\varphi^{-1}$ for all $s\in\Gamma(K,E)$, cf. Example 2.13 and Lemma 2.24.

Here come four classes of typical examples for the above objects, see also Example 1.12.

Example 3.11. (i) We start with the classical case of an invertible, scalar-valued cocycle $(\Phi_t)_{t\in\mathbb{R}}$ over a flow $(\varphi_t)_{t\in\mathbb{R}}$ on a compact space K, i.e., a family $(\Phi_t)_{t\in\mathbb{R}} \subseteq \mathrm{C}(K)$ of continuous functions on K such that

$$\begin{aligned} \Phi_{t+r} &= (\Phi_t \circ \varphi_r) \cdot \Phi_s && \text{for all } t, r \in \mathbb{R}, \\ \Phi_0(x) &= 1 && \text{for all } x \in K \end{aligned}$$

and the mapping

$$\mathbb{R} \times K \longrightarrow \mathbb{R}, \quad (t, x) \mapsto \Phi_t(x)$$

is continuous, see, e.g., [Nag86], Section B.II.3 and the references therein.

Consider the weighted Koopman group $(\mathcal{T}_\Phi(t))_{t\in\mathbb{R}}$ on $\mathrm{C}(K)$ defined by

$$\mathcal{T}_\Phi(t)f := \Phi_t \cdot (f \circ \varphi_{-t}) \quad \text{for all } t \in \mathbb{R}, f \in \mathrm{C}(K).$$

By Proposition 3.8 of Part B.II. of [Nag86] the weighted Koopman group is a C_0-group. It is a group of positive operators if and only if the cocycle $(\Phi_t)_{t\in\mathbb{R}}$ consists of positive functions, see Theorem 3.6 of [AG84] and Part B.II. of [Nag86], Proposition 3.9.

Such scalar-valued cocycles and the associated weighted Koopman groups occur in many different situations, cf., for example, *disjointness preserving operators* [AH86], *Lamperti operators* [Are83], or *weighted endomorphisms* [Uhl86].

(ii) Consider a topological dynamcial system $(K, (\varphi_t)_{t\in\mathbb{R}})$, a Banach space Z, and the *trivial Banach bundle* $E = K \times Z$ over K, with p the projection onto the first component. Let $(\Phi^t)_{t\geqslant 0}$ be a family of bounded operators $\{\Phi^t(x) \in \mathscr{L}(Z) \mid x \in K, t \geqslant 0\}$ such that

(a) the mapping $K \times \mathbb{R}_+ \longrightarrow Y, (x, t) \mapsto \Phi^t(x)v$ is continuous for all $v \in Z$,

(b) $\Phi^{t+r}(x) = \Phi^t(\varphi_r(x))\Phi^r(x)$ for all $t, r \geqslant 0$ and $\Phi^0(x) = \mathrm{id}_Y$ for all $x \in K$.

By the principle of uniform boundedness, $(\Phi^t)_{t\geqslant 0}$ is *exponentially bounded*, i.e., there exists $M > 0$ and $\omega > 0$ such that $\|\Phi^t(x)\| \leqslant M\mathrm{e}^{\omega t}$ for all $t \geqslant 0$, $x \in K$.

The *linear skew-product flow* $(\Phi_t)_{t\geqslant 0}$ *on* $K \times Z$ *associated with* $(\Phi^t)_{t\in\mathbb{R}}$, defined by

$$\mathbb{R}_+ \times K \times Z \longrightarrow K \times Z, \quad (x, v) \mapsto \Phi_t(x, v) := (\varphi_t(x), \Phi^t(x)v),$$

is a semiflow over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on $K\times Z$ over K, cf. [CL99], Section 6.2. The family $(\Phi^t)_{t\geqslant 0}$ is called *cocycle over* $(\varphi_t)_{t\in\mathbb{R}}$. In this situation, there is a one-to-one correspondence between a cocycle and a linear skew-product flow.

The cocycle then induces a weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on $\Gamma(K, E) \cong \mathrm{C}(K, Z)$—also called *evolution semigroup*—via

$$\mathcal{T}_\Phi(t)s := \Phi^t \circ s \circ \varphi_{-t}$$

for all $s \in \mathrm{C}(K, Z)$, $t \geqslant 0$, see [CL99], Section 6.2.

As a particular case, we can take $K := \mathbb{R} \mathbin{\dot\cup} \{\infty\}$ and the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K defined by

$$\varphi_t(x) := \begin{cases} x + t, & x \in \mathbb{R}, \\ \infty, & x = \infty, \end{cases}$$

for all $t \in \mathbb{R}$. Then, an *exponentially bounded evolution family* $(U(t, r))_{t\geqslant r}$ on a Banach space Z, see Definition VI.9.1 of [EN00], defines a semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on $K \times \big(Z \mathbin{\dot\cup} \{0\}\big)$ over K by

$$\Phi_t(x) := \begin{cases} U(x + t, x), & x \in \mathbb{R}, t \geqslant 0, \\ 0, & x = \infty, t \geqslant 0, \end{cases}$$

see Remark 1.2 and Lemma 2.15.

The associated evolution semigroup on $\Gamma(K, Z \mathbin{\dot\cup} \{0\}) \cong \mathrm{C}_0(\mathbb{R}, Z)$ is a weighted Koopman semigroup, see, e.g., [EN00], Section VI.9, or [Nic97], Section 1, and Remark 2.3.

For more results on evolution semigroups, their application to non-autonomous abstract Cauchy problems, and further examples we refer to, e.g., [Rau94], [BV19], [LS06], or [RRS96].

(iii) Let $(\varphi_t)_{t\in\mathbb{R}}$ be a smooth flow on a compact Riemannian manifold M without boundary, $E = TM$ the tangent bundle of M, and $\Phi_t = \mathrm{d}\varphi_t$ the differential of φ_t, $t \in \mathbb{R}$, see [Lee13], Chapter 3, p. 68. The weighted Koopman operators $\mathcal{T}_{\mathrm{d}\varphi}(t)$ on $\Gamma(M, TM)$ are *pushforward operators*, see [Lee13], Chapter 8,

p. 183, for the definition of a *pushforward of a vector field*. We call the group $(\mathcal{T}_{\mathrm{d}\varphi}(t))_{t\in\mathbb{R}}$ the *pushforward group*. In Chapter 6 we investigate pushforward groups in more detail.

(iv) Finally, we consider a construction from topological dynamics, see, e.g., page 30 of [Kna67] or Section 5 of [Ell87] or [EK20]. Let

$$\pi\colon (L;(\psi_t)_{t\in\mathbb{R}}) \longrightarrow (K;(\varphi_t)_{t\in\mathbb{R}})$$

be an extension of the topological dynamical system $(K;(\varphi_t)_{t\in\mathbb{R}})$, i.e., $(L;(\psi_t)_{t\in\mathbb{R}})$ is another topological dynamical system and $\pi : L \to K$ a continuous surjection such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\psi_t} & L \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ K & \xrightarrow[\varphi_t]{} & K \end{array}$$

commutes for each $t \in \mathbb{R}$. We consider $L_x := \pi^{-1}(x)$ for each $x \in K$, define

$$E := \bigcup_{x\in K} \mathrm{C}(L_x),$$
$$p : E \longrightarrow K, \quad \mathrm{C}(L_x) \ni f \mapsto x,$$

and endow this with the topology generated by the sets

$$W(s,U,\varepsilon) := \left\{ f \in p^{-1}(U) \mid \|f - s|_{L_{p(f)}}\|_{\mathrm{C}(L_{p(f)})} < \varepsilon \right\}$$

where $U \subseteq K$ is open, $s \in \mathrm{C}(L)$, and $\varepsilon > 0$. Then $p\colon E \longrightarrow K$ is a Banach bundle and the induced Banach module $\Gamma(K,E)$ is isomorphic to $\mathrm{C}(L)$, see Theorem 4.2 of [Gie82].

For each $t \in \mathbb{R}$ consider

$$\Phi_t\colon E \longrightarrow E, \quad \mathrm{C}(L_x) \ni f \mapsto f \circ \psi_{-t} \in \mathrm{C}(L_{\varphi_t(x)}).$$

This defines a flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{t\in\mathbb{R}}$. The induced weighted Koopman group $(\mathcal{T}_\Phi(t))_{t\in\mathbb{R}}$ is isomorphic to the Koopman group $(T_\psi(t))_{t\in\mathbb{R}}$ induced by the flow $(\psi_t)_{t\in\mathbb{R}}$ on L.

3.4 Lattice-theoretic characterization of weighted Koopman semigroups

Inspired by the scalar-valued case, see the paper *Resolvent positive operators* by W. Arendt, [Are87], we add order-theoretic characterizations of weighted Koopman semigroups.

Let $(K;(\varphi_t)_{t\in\mathbb{R}})$ be a topological dynamical system, $(T_\varphi(t))_{t\in\mathbb{R}}$ the corresponding Koopman group on $\mathrm{C}(K)$ with generator $(\delta, D(\delta))$, and $\Gamma(K,E)$ the space of continuous sections of a Banach bundle E over K on which we consider a weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$.

We reformulate Theorem 2.38 for a C_0-semigroup and give an additional characterization via its resolvent analogous to the scalar-valued case, cf. [Are87], Theorem 2.6.

Theorem 3.12. *Let $(\mathcal{T}(t))_{t\geqslant 0}$ be a C_0-semigroup on $\Gamma(K,E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. If the Banach bundle E over K is continuous, then the following assertions are equivalent.*

(a) $(\mathcal{T}(t))_{t\geqslant 0}$ *is a weighted Koopman semigroup, i.e., there exists a unique semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on E over K such that $\mathcal{T}(t) = \mathcal{T}_\Phi(t)$ for all $t \geqslant 0$.*

(b) *The operators $\mathcal{T}(t)$ are $T_\varphi(t)$-homomorphisms for all $t \geqslant 0$.*

(c) $\mathrm{supp}(\mathcal{T}(t)s) \subseteq \varphi_t(\mathrm{supp}(s))$ *for all $s \in \Gamma(K,E)$, $t \geqslant 0$.*

(d) $|\mathcal{T}(t)s| \leqslant \|\mathcal{T}(t)\| \cdot T_\varphi(t)|s|$ *for all $s \in \Gamma(K,E)$, $t \geqslant 0$.*

(e) *For each $t \geqslant 0$ there is $m_t > 0$ such that $|\mathcal{T}(t)s| \leqslant m_t \cdot T_\varphi|s|$ for all $s \in \Gamma(K,E)$.*

(f) *There is $\omega \in \mathbb{R}$ and $M \geqslant 1$ such that $(\omega,\infty) \subseteq \rho(\mathcal{A})$ with*

$$|R(\lambda,\mathcal{A})^n s| \leqslant M \cdot R(\lambda-\omega,\delta)^n |s|$$

for all $s \in \Gamma(K,E)$, $\lambda > \omega$, and $n \in \mathbb{N}$.

Proof. For the first part of the theorem we refer to Theorem 2.38.

It remains to show that for a continuous Banach bundle E over K the assertions (a)—(e) are equivalent to (f). For "(d) $\Rightarrow$ (f)" let $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|\mathcal{T}(t)\| \leqslant M\mathrm{e}^{\omega t}$ for all $t \geqslant 0$. By Corollary II.1.11 of [EN00] we obtain

$$\begin{aligned}
|R(\lambda, \mathcal{A})^n s| &\leqslant \frac{1}{(n-1)!} \int_0^\infty |r^{n-1} \mathrm{e}^{-\lambda r} \mathcal{T}(r)s| \,\mathrm{d}r \\
&\leqslant \frac{1}{(n-1)!} \int_0^\infty r^{n-1} \mathrm{e}^{-\lambda r} \|\mathcal{T}(t)\| T_\varphi(r)|s| \,\mathrm{d}r \\
&\leqslant \frac{M}{(n-1)!} \int_0^\infty r^{n-1} \mathrm{e}^{-(\lambda-\omega) r} T_\varphi(r)|s| \,\mathrm{d}r \\
&= M \cdot R(\lambda - \omega, \delta)^n |s|
\end{aligned}$$

for all $s \in \Gamma(K, E)$, $t \geqslant 0$, and $n \in \mathbb{N}$.

Conversely, assume that (f) is true. Then, by the Post-Widder inversion formula, see Part III, Corollary 5.5 of [EN00],

$$\begin{aligned}
|\mathcal{T}(t)s| &= \left| \lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, \mathcal{A} \right) \right)^n s \right| \\
&\leqslant \lim_{n\to\infty} M \cdot \left(\frac{n}{t} R\left(\frac{n}{t}, \delta + \omega \right) \right)^n |s| = M\mathrm{e}^{\omega t} T_\varphi(t)|s|
\end{aligned}$$

for all $t \geqslant 0$, $s \in \Gamma(K, E)$. □

Chapter 4

Spectral theory for weighted Koopman semigroups

Spectral theory plays a key role for the investigation of the qualitative behavior of a C_0-semigroup on a Banach space. In this chapter, we prove surprising symmetry properties of the spectrum of weighted Koopman semigroups on Banach modules. In particular, we obtain a strong *spectral mapping theorem* (cf. [EN00], Section IV.3) for such semigroups. We refer to [EN00], Chapter IV, for the spectral theory for C_0-semigroups.

The chapter is organized in the following way. First, we recall results for the "non-weighted case", i.e., spectral properties of Koopman groups on scalar-valued function spaces and spectral properties of their generators, cf. [Sch74], [Der79], [AG84], [AH86], and [Nag86]. Based on these results we investigate the spectrum in the "weighted case", i.e., the spectrum of weighted Koopman semigroups and their generators. We first consider the time-discrete case and then pass on to the time-continuous case.

The results in Section 4.2 are joint work with Henrik Kreidler.

4.1 The non-weighted case

First, we collect properties of the spectrum of a single Koopman operator on the space of continuous functions $\mathrm{C}(K)$ on a compact space K. Then, we consider the spectrum of a Koopman group and its generator.

4.1.1 Koopman operators

We start from the classical Perron-Frobenius spectral theory for positive operators as developed in [Sch74] and consider the Koopman operator T_φ on $\mathrm{C}(K)$ induced by a homeomorphism φ on a compact space K. We recall results on the spectrum of a general Markov lattice homomorphism T on $\mathrm{C}(K)$, cf. [Sch74], [AG84], and [AH86], which we specialize to the Koopman operator T_φ.

For the spectral radius of a Markov lattice homomorphism T we have $\mathrm{r}(T) = 1$ and even $1 \in \sigma_{\mathrm{p}}(T)$. Hence,

$$\sigma(T) \subseteq D := \{\lambda \in \mathbb{C} \mid |\lambda| \leqslant 1\}.$$

The spectrum of T and its point spectrum are *cyclic*, i.e.,

$$\begin{aligned} \lambda = |\lambda|\gamma \in \sigma(T) \ &\Rightarrow\ |\lambda|\gamma^k \in \sigma(T), \quad k \in \mathbb{Z}, \\ \lambda = |\lambda|\gamma \in \sigma_{\mathrm{p}}(T) \ &\Rightarrow\ |\lambda|\gamma^k \in \sigma_{\mathrm{p}}(T), \quad k \in \mathbb{Z}, \end{aligned}$$

see Proposition V.4.2 and Theorem V.4.4 of [Sch74]. If T is bijective, then $\sigma(T) \subseteq \mathbb{T}$ and $\sigma(T) = \sigma_{\mathrm{ap}}(T)$.

We specialize these results to the case of a Koopman operator on $\mathrm{C}(K)$.

Proposition 4.1. *For the Koopman operator T_φ on $\mathrm{C}(K)$ induced by a homeomorphism φ on K the approximate point spectrum $\sigma_{\mathrm{ap}}(T_\varphi)$ and the point spectrum $\sigma_{\mathrm{p}}(T_\varphi)$ are cyclic subsets of $\mathbb{T}$. In other words, they are a union of subgroups of $\mathbb{T}$. Moreover, $\sigma(T_\varphi) = \sigma_{\mathrm{ap}}(T_\varphi)$.*

If T_φ is *topologically ergodic*, i.e., $\operatorname{fix} T_\varphi := \{f \in \mathrm{C}(K) \mid T_\varphi f = f\}$ is one-dimensional, then $\sigma_{\mathrm{p}}(T_\varphi)$ is even a group, cf. Theorem 4.21 of [EFHN15]. This occurs, e.g., if φ is minimal, see [EFHN15], Definition 3.1. See [Küs20] for a characterization of topological ergodicity.

Using the following "aperiodicity" property of the homeomorphism φ on K, we are able to further describe the spectrum of the Koopman operator T_φ.

Definition 4.2. We call a point $x \in K$ a *periodic point* of φ if there exists $n \in \mathbb{N}$ such that $\varphi^n(x) = x$. It is called an *aperiodic point* if $\varphi^n(x) \neq x$ for all $n \in \mathbb{N}$. We consider the *prime period function* $\nu\colon K \longrightarrow \mathbb{N} \cup \{\infty\}$ of φ defined by

$$\nu(x) := \begin{cases} \inf\{n \in \mathbb{N} \mid \varphi^n(x) = x\}, & x \quad \text{periodic}, \\ \infty, & x \quad \text{aperiodic}, \end{cases}$$

and the set

$$\mathcal{B}(K) := \{x \in K \mid \nu \text{ is bounded in some neighborhood of } x\}.$$

The homeomorphism φ is called *aperiodic* if $\mathcal{B}(K) = \emptyset$. It is called *strictly aperiodic* if each $x \in K$ is aperiodic. If $\nu(x) < \infty$ for all $x \in K$, then φ is called *periodic*.

With this property we obtain a more precise description of the spectrum.

Proposition 4.3. *For an aperiodic homeomorphism φ, we have*

$$\sigma(T_\varphi) = \mathbb{T}.$$

If φ is periodic, i.e., $\nu(x) < \infty$ for all $x \in K$, then

$$\sigma(T_\varphi) = \overline{\bigcup_{x \in K} \Gamma_{\nu(x)}},$$

where $\Gamma_n := \{z \in \mathbb{C} \mid z^n = 1\}$ is the group of all n-th roots of unity for $n \in \mathbb{N}$.

If $\nu(x) = n$ for all $x \in K$ and a fixed $n \in \mathbb{N}$, then

$$\sigma(T_\varphi) = \Gamma_n.$$

Proof. The first assertion follows by Lemma 2.6 of [AG84] and by Proposition 4.1. The second assertion follows by Theorem 2.7 of [AG84] and implies the last assertion. □

4.1.2 Koopman groups

We now collect spectral properties of a Koopman group $(T_\varphi(t))_{t\in\mathbb{R}}$ on $\mathrm{C}(K)$ and its generator $(\delta, D(\delta))$. While the spectrum $\sigma(T_\varphi(t))$ is described in Proposition 4.1 and Proposition 4.3, the spectrum $\sigma(\delta)$, the approximate point spectrum $\sigma_{\mathrm{ap}}(\delta)$, and the point spectrum $\sigma_{\mathrm{p}}(\delta)$ are *additive cyclic* subsets of $\mathbb{C}$, i.e.,

$$\begin{aligned}
\lambda \in \sigma(\delta) \quad &\Rightarrow \ \operatorname{Re}\lambda + \mathrm{i}k \operatorname{Im}\lambda \in \sigma(\delta), \qquad k \in \mathbb{Z},\\
\lambda \in \sigma_{\mathrm{ap}}(\delta) \quad &\Rightarrow \ \operatorname{Re}\lambda + \mathrm{i}k \operatorname{Im}\lambda \in \sigma_{\mathrm{ap}}(\delta), \quad k \in \mathbb{Z},\\
\lambda \in \sigma_{\mathrm{p}}(\delta) \quad &\Rightarrow \ \operatorname{Re}\lambda + \mathrm{i}k \operatorname{Im}\lambda \in \sigma_{\mathrm{p}}(\delta), \quad k \in \mathbb{Z},
\end{aligned}$$

see Theorem 4.1 of Chapter B.III of [Nag86] or Theorem 3.4 of [Der79]. Since each $T_\varphi(t)$ is bijective, we even have $\sigma(\delta) \subseteq \mathrm{i}\mathbb{R}$ and $\sigma(\delta) = \sigma_{\mathrm{ap}}(\delta)$.

Proposition 4.4. *Let $(T_\varphi(t))_{t\in\mathbb{R}}$ be a Koopman group on $\mathrm{C}(K)$ with generator $(\delta, D(\delta))$. For each $t \geqslant 0$*

$$\sigma(T_\varphi(t)) = \sigma_{\mathrm{ap}}(T_\varphi(t)) \subseteq \mathbb{T}$$

is the union of subgroups of $\mathbb{T}$.

Furthermore,

$$\sigma(\delta) = \sigma_{\mathrm{ap}}(\delta) \subseteq \mathrm{i}\mathbb{R}$$

is the union of additive subgroups of $\mathrm{i}\mathbb{R}$.

Proof. The first part follows by Section 3.1.1, the second part follows by Theorem 2.9 of [Der79]. □

We now define aperiodicity for a flow $(\varphi_t)_{t\in\mathbb{R}}$ on K as in the time-discrete case.

Definition 4.5. We call a point $x \in K$ *periodic point* of a flow $(\varphi_t)_{t\in\mathbb{R}}$ on K if there exists $t > 0$ such that $\varphi_t(x) = x$. It is called an *aperiodic point* if $\varphi_t(x) \neq x$ for all $t > 0$. We consider the *prime period function* $\nu\colon K \longrightarrow [0, \infty]$ of $(\varphi_t)_{t\in\mathbb{R}}$ defined by

$$\nu(x) := \begin{cases} \inf\{t > 0 \mid \varphi_t(x) = x\}, & x \ \text{ periodic},\\ \infty, & x \ \text{ aperiodic},\end{cases}$$

and the set

$$\mathcal{B}(K) := \{x \in K \mid \nu \text{ is bounded in some neighborhood of } x\}.$$

The flow $(\varphi_t)_{t\in\mathbb{R}}$ is called *aperiodic* if $\mathcal{B}(K) = \emptyset$. It is called *strictly aperiodic* if each $x \in K$ is aperiodic. If $\nu(x) < \infty$ for all $x \in K$, then $(\varphi_t)_{t\in\mathbb{R}}$ is called *periodic*.

Proposition 4.6. *If the flow $(\varphi_t)_{t\in\mathbb{R}}$ is aperiodic, then*

$$\sigma(T_\varphi(t)) = \mathbb{T} \quad \textit{for all } t \in \mathbb{R},$$
$$\sigma(\delta) = \mathrm{i}\mathbb{R}.$$

Proof. See Proposition 4.3 and Theorem 2.12 of [Der79] or Theorem 4.9 of [Nag86]. □

Apparently, for a Koopman group associated with an aperiodic flow the *spectral mapping theorem* holds, i.e.,

$$\sigma(T_\varphi(t)) = \mathrm{e}^{t\sigma(\delta)} \quad \text{for all } t \in \mathbb{R}.$$

For a periodic flow, at least the following holds.

Proposition 4.7. *If* $0 < \nu(x) < \infty$ *for all* $x \in K$*, then the* weak spectral mapping theorem *holds, i.e.,*

$$\sigma(T_\varphi(t)) = \overline{\mathrm{e}^{t\sigma(\delta)}} \quad \textit{for all } t \in \mathbb{R}.$$

Proof. See Theorem 4.4 of [AG84]. □

4.2 The weighted case

In this section, we consider the spectrum of a single weighted Koopman operator on the space of continuous sections $\Gamma(K, E)$ of a Banach bundle E over K and then investigate spectral properties of a weighted Koopman semigroup and its generator.

4.2.1 Weighted Koopman operators

We consider a T_φ-homomorphism $\mathcal{T}$ on a Banach module $\Gamma(K,E)$ over $\mathrm{C}(K)$, see Definition 3.4. We show that the spectrum of such an operator has a symmetry related to the spectrum of T_φ.

Proposition 4.8. *If $f \in \mathrm{C}(K)$ is an eigenvector of T_φ with respect to the eigenvalue $\lambda \in \mathbb{C}$ such that $|f|$ is* strictly positive*, i.e., $|f|(x) = |f(x)| > 0$ for all $x \in K$, then*

$$\lambda \cdot \sigma_{\mathrm{p}}(\mathcal{T}) \subseteq \sigma_{\mathrm{p}}(\mathcal{T}),$$
$$\lambda \cdot \sigma_{\mathrm{ap}}(\mathcal{T}) \subseteq \sigma_{\mathrm{ap}}(\mathcal{T}).$$

In particular, if $\dim(\mathrm{fix}\ T_\varphi) = 1$*, then*

$$\sigma_{\mathrm{p}}(T_\varphi) \cdot \sigma_{\mathrm{p}}(\mathcal{T}) \subseteq \sigma_{\mathrm{p}}(\mathcal{T}),$$
$$\sigma_{\mathrm{p}}(T_\varphi) \cdot \sigma_{\mathrm{ap}}(\mathcal{T}) \subseteq \sigma_{\mathrm{ap}}(\mathcal{T}).$$

Proof. For the first part, take $\mu \in \sigma_{\mathrm{p}}(\mathcal{T})$ with corresponding eigenvector $s \in \Gamma(K,E)$ and $\lambda \in \sigma_{\mathrm{p}}(T_\varphi)$ with corresponding eigenvector $f \in \mathrm{C}(K)$ such that $|f|$ is strictly positive. Then $f \cdot s \neq 0$ and

$$\mathcal{T}(f \cdot s) = T_\varphi f \cdot \mathcal{T} s = \lambda f \cdot \mu s = \lambda\mu(f \cdot s),$$

hence $\lambda\mu \in \sigma_{\mathrm{p}}(\mathcal{T})$. The second inclusion follows by the same argument.

If $\mathrm{fix}(T_\varphi)$ is one-dimensional, then for every eigenvalue $\lambda \in \mathbb{C}$ of T_φ the corresponding eigenfunction has constant absolute value, see Theorem 4.21 of [EFHN15]. Thus, the second part follows. □

The question remains whether the above inclusion holds for the entire spectrum, i.e.,

$$\sigma(T_\varphi) \cdot \sigma(\mathcal{T}) \subseteq \sigma(\mathcal{T}).$$

In particular, if φ is aperiodic, one could expect that

$$\mathbb{T} \cdot \sigma(\mathcal{T}) \subseteq \sigma(\mathcal{T}). \tag{4.1}$$

In this case, the spectrum $\sigma(\mathcal{T})$ is just the union of annuli centered at the origin. However, for an arbitrary T_φ-homomorphism $\mathcal{T}$ on the Banach module $\Gamma(K,E)$ for a general Banach bundle E this is not true.

Example 4.9. Let Z be a Banach space and $\mathcal{T} \in \mathscr{L}(Z)$ such that $\mathbb{T}\cdot\sigma(\mathcal{T}) \not\subset \sigma(\mathcal{T})$. We realize the Banach space Z as an AM-module over $\mathrm{C}(K)$ for some compact space K and $\mathcal{T}$ as a weighted Koopman operator on Z. For this purpose, we consider the one-point compactification of the integers $K := \mathbb{Z} \cup \{\infty\}$ and the following aperiodic homeomorphism on K

$$\varphi(x) := \begin{cases} x+1, & x \in \mathbb{Z}, \\ \infty, & x = \infty. \end{cases}$$

The Banach space Z equipped with the following operation

$$\cdot\colon \mathrm{C}(K) \times Z \longrightarrow Z, \quad (f, s) \mapsto \left(\lim_{x\to\infty} f(x)\right) \cdot s$$

is an AM-module over $\mathrm{C}(K)$. Hence, Z has a lattice-valued norm given by

$$\begin{aligned} |s|(x) &= \inf\{\|fs\| \mid f \in \mathrm{C}(K)_+ \text{ with } f(x) = 1\} \\ &= \inf\left\{\left(\lim_{z\to\infty} f(z)\right)\|s\| \mid f \in \mathrm{C}(K)_+ \text{ with } f(x) = 1\right\} \\ &= \begin{cases} 0, & x \in \mathbb{Z}, \\ \|s\|, & x = \infty, \end{cases} \end{aligned}$$

for $s \in Z$, see Theorem 3.3. Obviously, $|\cdot|$ is upper semicontinuous, but not continuous. Consequently, the Banach bundle associated with Z is not continuous, see Proposition 3.2. The operator $\mathcal{T}$ is a T_φ-homomorphism on the AM-module Z, where T_φ is the Koopman operator on $\mathrm{C}(K)$ induced by the aperiodic homeomorphism φ on K.

We construct the unique Banach bundle E over K and the unique Banach bundle homomorphism Φ over φ on E over K such that $Z \cong \Gamma(K, E)$ and $\mathcal{T} \cong \mathcal{T}_\Phi$. To each $x \in \mathbb{Z}$ we attach the fiber $E_x := \{0\}$ and at $x = \infty$ we attach the Banach space $E_\infty := Z$. Then, $E = \bigcup_{x\in K} E_x$ is the Banach bundle over K such that $Z \cong \Gamma(K, E)$. Moreover,

$$\Phi(x) := \begin{cases} 0, & x \in \mathbb{Z}, \\ \mathcal{T}, & x = \infty. \end{cases}$$

defines the Banach bundle homomorphism Φ over φ on E over K such that $\mathcal{T} \cong \mathcal{T}_\Phi$.

The following proposition shows which extra assumptions suffice to obtain (4.1).

Proposition 4.10. *Let $\mathcal{T}$ be a T_φ-homomorphism on $\Gamma(K, E)$ for some Banach bundle E. Assume that one of the following assumptions holds.*

(i) *φ is aperiodic and E is continuous.*

(ii) *φ is strictly aperiodic.*

Then $\mathbb{T} \cdot \sigma_{\mathrm{ap}}(\mathcal{T}) \subseteq \sigma_{\mathrm{ap}}(\mathcal{T})$ and $\mathbb{T} \cdot \sigma(\mathcal{T}) \subseteq \sigma(\mathcal{T})$. Thus, $\sigma(\mathcal{T})$ is invariant under rotation by complex numbers of modulus one.

Proof. Let $\mu \in \mathbb{C}$ be an approximate eigenvalue with approximate eigenvector $(s_n)_{n\in\mathbb{N}}$ in $\Gamma(K, E)$ and $\lambda \in \mathbb{T}$. Each of the two assumptions (i) and (ii) implies that we find $x_n \in K$ with $\nu(x_n) \geqslant 2n + 1$ and $\|s_n(x_n)\| \geqslant \frac{1}{2}$ for each $n \in \mathbb{N}$, where ν is the prime period function of φ, see Definition 4.5. By Lemma 2.6 of [AG84] we find $f_n \in \mathrm{C}(K)$ with $f_n(x_n) = \|f_n\| = 1$ and $\|T_\varphi f_n - \lambda f_n\| \leqslant \frac{1}{n}$ for each $n \in \mathbb{N}$. But then $\|(f_n s_n)(x_n)\| \geqslant \frac{1}{2}$ for each $n \in \mathbb{N}$ and

$$\begin{aligned}(\lambda\mu - \mathcal{T}) f_n s_n &= \lambda f_n \cdot \mu s_n - T_\varphi f_n \cdot \mathcal{T} s_n \\ &= (\lambda f_n - T_\varphi f_n) \cdot \mu s_n - T_\varphi f_n \cdot (\mu s_n - \mathcal{T} s_n) \to 0\end{aligned}$$

for $n \to \infty$. Hence, $\mathbb{T} \cdot \sigma_{\mathrm{ap}}(\mathcal{T}) \subseteq \sigma_{\mathrm{ap}}(\mathcal{T})$.

Since the boundary $\partial\sigma(\mathcal{T})$ is contained in the approximate point spectrum $\sigma_{\mathrm{ap}}(\mathcal{T})$, see Section V.1 of [Sch74], we have $\mathbb{T} \cdot \sigma(\mathcal{T}) \subseteq \sigma(\mathcal{T})$. □

Example 4.11. We consider a scalar-valued weighted Koopman operator $\mathcal{T} \in \mathscr{L}(\mathrm{C}(K))$, i.e., there is a homeomorphism φ on a compact space K and a continuous, invertible function $\Phi\colon K \longrightarrow \mathbb{R}_+$ such that

$$\mathcal{T} f = \Phi \cdot f \circ \varphi \qquad \text{for all } f \in \mathrm{C}(K),$$

cf. Section 2 of [AG84]. If φ is aperiodic, then $\mathbb{T} \cdot \sigma(\mathcal{T}) \subseteq \sigma(\mathcal{T})$, cf. Remark 2.8 of [AG84].

4.2.2 Weighted Koopman semigroups

We now extend the previous results on the spectrum of a single operator to the spectrum of a weighted Koopman semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on $\Gamma(K, E)$ with generator

$(\mathcal{A}, D(\mathcal{A}))$. Our goal is to obtain a *spectral mapping theorem* of the form

$$\sigma(\mathcal{T}(t)) \setminus \{0\} = \mathrm{e}^{t\sigma(\mathcal{A})} \quad \text{for every } t \geqslant 0.$$

However, the following example, which is a time-continuous version of Example 4.9, shows that extra assumptions are needed.

Example 4.12. Let $(\mathcal{T}(t))_{t\geqslant 0}$ be a C_0-semigroup on a Banach space Z such that the spectrum $\sigma(\mathcal{T}(t))$, $t \geqslant 0$, is not invariant under rotation. Again, we realize the Banach space Z as an AM-module over $\mathrm{C}(K)$ for some compact space K and $(\mathcal{T}(t))_{t\geqslant 0}$ as a weighted Koopman semigroup on Z. To this end, we consider the one-point compactification of the real numbers $K := \mathbb{R} \cup \{\infty\}$ and the following aperiodic flow on K

$$\varphi_t(x) := \begin{cases} x + t, & x \in \mathbb{R}, \\ \infty, & x = \infty. \end{cases}$$

Analogous to Example 4.9 the Banach space Z can be realized as an AM-module over $\mathrm{C}(K)$ and the Banach bundle associated with Z is not continuous. Each operator $\mathcal{T}(t)$ is a $T_\varphi(t)$-homomorphism on the AM-module Z, where $T_\varphi(t)$ is the Koopman operator induced by the aperiodic flow $(\varphi_t)_{t\in\mathbb{R}}$.

The construction of the Banach bundle E over K and the semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on E such that $Z \cong \Gamma(K, E)$ and $\mathcal{T}(t) \cong \mathcal{T}_\Phi(t)$, $t \geqslant 0$ is analogous to the time-discrete case.

The following condition on the flow $(\varphi(t))_{t\in\mathbb{R}}$ on K or the regularity of the Banach bundle E lead to the following relation between the spectra of weighted Koopman semigroups and their generators.

Theorem 4.13. *Let $(\mathcal{T}(t))_{t\geqslant 0}$ be a weighted Koopman semigroup on $\Gamma(K, E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. Assume that one of the following conditions holds.*

(i) $(\varphi_t)_{t\in\mathbb{R}}$ *is aperiodic and E has a continuous norm.*

(ii) $(\varphi_t)_{t\in\mathbb{R}}$ *is strictly aperiodic.*

Then the spectral mapping theorem *holds, i.e.,*

$$\sigma(\mathcal{T}(t)) \setminus \{0\} = \mathrm{e}^{t\sigma(\mathcal{A})} \quad \text{for each } t \geqslant 0.$$

Moreover, $\sigma(\mathcal{A}) = \sigma(\mathcal{A}) + \mathrm{i}\mathbb{R}$ and $\sigma(\mathcal{T}(t)) = \mathbb{T} \cdot \sigma(\mathcal{T}(t))$, $t \geqslant 0$.

Remark 4.14. In the situation of the above theorem, the spectrum of the generator consists of vertical stripes and the spectrum of each weighted Koopman operator consists of annuli centered at the origin.

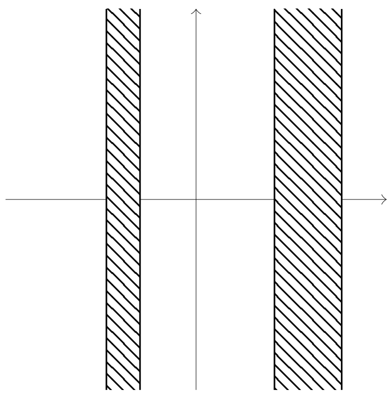

Figure 4.1: Typical spectrum of $\mathcal{A}$

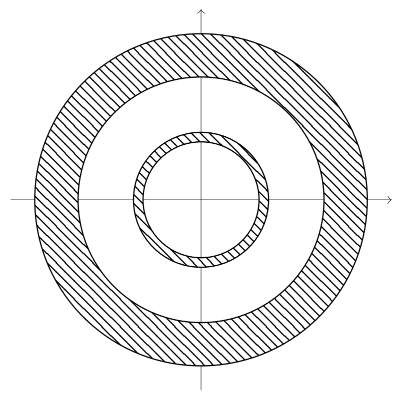

Figure 4.2: Typical spectrum of $\mathcal{T}(t)$

For the proof of the theorem we need the following lemma, cf. Lemma 6.31 of [CL99] in the case of a locally compact metric space.

Lemma 4.15. *Let λ be the Lebesgue measure on $\mathbb{R}$ and $(\varphi_t)_{t\in\mathbb{R}}$ a flow on K. If $t_0 \in (0,1)$, $N \in \mathbb{N}$, and $x \in K$ with prime period $\nu(x) \geqslant 5N+1$, then there is a neighborhood U of x and $f \in \mathrm{C}(K)$ with $0 \leqslant f \leqslant \mathbb{1}$ such that*

(i) $\operatorname{supp} f \subseteq U$,

(ii) $f(\varphi_t(x)) = 1$ *for every* $t \in [-\frac{t_0}{4}, \frac{t_0}{4}]$,

(iii) $f(\varphi_t(x)) = 0$ *for every* $t \in [-2N, -t_0] \cup [t_0, 2N]$,

(iv) $\lambda(\{t \in [-N,N] \mid \varphi_t(x) \in U\}) \leqslant 2t_0$ *for every* $x \in K$.

Proof. We first prove that there is a compact neighborhood W of x such that

$$W \cap \varphi_t(W) = \emptyset$$

for each $t \in A := \{\tilde{t} \in \mathbb{R} \mid \frac{t_0}{2} \leqslant |\tilde{t}| \leqslant 5N\}$.

Suppose that for each compact neighborhood W of x we have $x \in W \cap \varphi_t(W)$ for all $t \in A$. Denote the filter of all compact neighborhoods of x by $\mathcal{K}(x)$. Since K

is compact, we have, by the finite intersection property,

$$\emptyset \neq \bigcap_{W\in\mathcal{K}(x)} W \cap \varphi(A\times W) \subseteq \bigcap_{W\in\mathcal{K}(x)} W = \{x\}.$$

Consequently,

$$x \in \varphi\left(\bigcap_{W\in\mathcal{K}(x)} A\times W\right) = \varphi(A\times\{x\}),$$

i.e., $\varphi_t(x) = x$ for some $t \in A$. But then $\nu(x) \leqslant 5N$ contradicting the assumption.

Now take $W \in \mathcal{K}(x)$ such that $W \cap \varphi_t(W) = \emptyset$. Choose an open neighborhood V of x such that $\overline{V} \subseteq W$. Define

$$U := \bigcap_{|t|\leqslant \frac{t_0}{4}} \varphi_t(\overline{V}), \quad O := \bigcap_{|t|\leqslant \frac{t_0}{4}} \varphi_t(V).$$

Then, $\overline{O} \subseteq U$ which implies that there exists a continuous function $f \in \mathrm{C}(K)$ with $0 \leqslant f \leqslant \mathbb{1}$ such that $f(x) = 1$ for all $x \in O$ and $f(x) = 0$ for all $x \notin U$. Since $x \in V$, we have $\varphi_t(x) \in O$ for all $|t| \leqslant \frac{t_0}{4}$. Consequently, $f(\varphi_t(x)) = 1$ for every $t \in [-\frac{t_0}{4}, \frac{t_0}{4}]$.

We now show that

$$U \cap \varphi_t(U) = \emptyset$$

for each $t_0 \leqslant |t| \leqslant 2N$.

Assume there is some $x_0 \in U$ and $t \in \mathbb{R}$ with $t_0 \leqslant |t| \leqslant 2N$ such that $\varphi_t(x_0) \in U$. Then there exist $x_1, x_2 \in \overline{V}$ and $t_1, t_2 \in \mathbb{R}$ with $\max\{|t_1|, |t_2|\} \leqslant \frac{t_0}{4}$ such that $x_0 = \varphi_{t_1}(x_1)$ and $\varphi_t(x_0) = \varphi_{t_2}(x_2)$. This implies $\varphi_{t+t_1-t_2}(x_1) = x_2 \in \overline{V}$. But $\frac{t_0}{2} \leqslant |t| - |t_1| - |t_2| \leqslant |t + t_1 - t_2| \leqslant 2N + \frac{t_0}{2} \leqslant 5N$ which contradicts $W \cap \varphi_t(W) = \emptyset$ for each $t \in A$. Hence $U \cap \varphi_t(U) = \emptyset$ for each $t_0 \leqslant |t| \leqslant 2N$. Since $x \in U$ assertion (iii) follows.

Finally, we show (iv). Fix $x_1 \in K$ and consider $\{\varphi_t(x_1) \mid t \in [-N, N]\}$. Then $\lambda(\{t \in [-N, N] \mid \varphi_t(x_1) \in U\}) = 0$ if $\{\varphi_t(x_1) \mid t \in [-N, N]\} \cap U = \emptyset$. Hence, we may assume that $x \in \{\varphi_t(x_1) \mid t \in [-N, N]\} \cap U$, i.e., there exists some $t_1 \in \mathbb{R}$ with $|t_1| \leqslant N$ such that $x = \varphi_{t_1}(x_1)$. Now let $r \in \{t \in [-N, N] \mid \varphi_t(x_1) \in U\}$. Then $|r - t_1| \leqslant 2N$ and $\varphi_r(x_1) = \varphi_{r-t_1}(x) \in U$. Since $x \in U$, it follows that $|r - t_1| < t_0$. Thus, $\{t \in [-N, N] \mid \varphi_t(x_1) \in U\} \subseteq (t_1 - t_0, t_1 + t_0)$. □

We now prove the above theorem.

Proof (of Theorem 4.13). Since the spectral mapping theorem always holds for the residual and the point spectrum, see [EN00], Section IV.3, Theorem 3.7, we only have to show the assertion for the approximate point spectrum. By the spectral inclusion theorem, see [EN00], Section IV.3, Theorem 3.6, we have $\mathrm{e}^{t\sigma_{\mathrm{ap}}(\mathcal{A})} \subseteq \sigma_{\mathrm{ap}}(\mathcal{T}(t))$ for all $t \geqslant 0$. By rescaling arguments, it suffices to show that if $1 \in \sigma_{\mathrm{ap}}(\mathcal{T}(1))$ then $0 \in \sigma_{\mathrm{ap}}(\mathcal{A})$. We will show, that $1 \in \sigma_{\mathrm{ap}}(\mathcal{T}(1))$ even implies $\mathrm{i}\mathbb{R} \subseteq \sigma_{\mathrm{ap}}(\mathcal{A})$. Take $1 \in \sigma_{\mathrm{ap}}(\mathcal{T}(1))$ and choose a corresponding approximate eigenvector $(s_n)_{n\in\mathbb{N}}$. Let further $N \in \mathbb{N}$ with $N \geqslant 2$ and $\varepsilon \in (0,1)$. Since $1 \in \sigma_{\mathrm{ap}}(\mathcal{T}(1))$, there exists an $s \in \Gamma(K, E)$ with $\|s\| = 2$ and

$$\|\mathcal{T}(j)s - s\| = \left\| \sum_{k=0}^{j-1} \mathcal{T}(k) \cdot (\mathcal{T}(1)s - s) \right\| \tag{4.2}$$

$$\leqslant \sum_{k=0}^{j} \|\mathcal{T}(k)\| \cdot \|\mathcal{T}(1)s - s\| \leqslant \varepsilon < 1 \tag{4.3}$$

for $j \in \{0, \ldots, 2N\}$. Note that this implies

$$\begin{aligned}
\sup_{t\in[0,2N]} \|\mathcal{T}(t)s\| &\leqslant \sup_{t\in[0,1]} \|\mathcal{T}(t)\| \sup_{j\in\{0,\ldots,2N\}} \|\mathcal{T}(j)s\| \\
&\leqslant \sup_{t\in[0,1]} \|\mathcal{T}(t)\| \left(\sup_{j\in\{0,\ldots,2N\}} \|\mathcal{T}(j)s - s\| + \|s\| \right) \\
&\leqslant 3 \sup_{t\in[0,1]} \|\mathcal{T}(t)\| =: M.
\end{aligned}$$

Since $(\mathcal{T}(t))_{t\geqslant 0}$ is strongly continuous, we find $t_0 \in (0,1)$ with

$$\|\mathcal{T}(t+N)s - \mathcal{T}(N)s\| \leqslant \varepsilon \tag{4.4}$$

for each $t \in (-t_0, t_0)$. By either of the two assumptions of the theorem we find $x \in K$ with $\|s(x)\| \geqslant 1$ and $\nu(x) \geqslant 5N + 1$.

Define $\gamma \in \mathrm{C}([-N, N])$ by

$$\gamma(t) := \begin{cases} \frac{N+t}{N-1} & t \in [-N, -1], \\ 1 & t \in (-1, 1), \\ \frac{N-t}{N-1} & t \in [1, N], \end{cases}$$

and f and U as in Lemma 4.15. Now take any $h \in \mathrm{C}(K)$ with $\|h\| = 1$ and $h(\varphi_t(x)) = \mathrm{e}^{\mathrm{i}rt}$ for $r \in \mathbb{R}$ and $t \in [-N, N]$ and set $g := T_\varphi(-N)f \cdot h$. Then, as in the proof of Proposition 1.8 of [EN00], Section II.1, it follows that

$$\tilde{s} := \frac{1}{t_0} \int_{-N}^{N} \gamma(t) \mathrm{e}^{-\mathrm{i}rt} \mathcal{T}(t+N)(gs)\, \mathrm{d}t \in D(\mathcal{A})$$

with

$$\begin{aligned}\|(\mathcal{A} - \mathrm{i}r)\tilde{s}\| &= \frac{1}{t_0} \left\| \int_{-N}^{N} \gamma'(t) \mathrm{e}^{-\mathrm{i}rt} \mathcal{T}(t+N)(gs)\, \mathrm{d}t \right\| \\ &\leqslant \frac{M}{(N-1)t_0} \cdot \int_{-N}^{N} \|T_\varphi(t) f\|\, \mathrm{d}t \leqslant \frac{2M}{(N-1)}\end{aligned}$$

by Lemma 4.15 (i) and (iv).

On the other hand, we obtain

$$\begin{aligned}\|\tilde{s}\| \geqslant \|\tilde{s}(x)\| &= \frac{1}{t_0} \left\| \int_{-N}^{N} \gamma(t) f(\varphi_t(x)) \mathcal{T}(t+N) s(x)\, \mathrm{d}t \right\| \\ &= \frac{1}{t_0} \left\| \int_{-t_0}^{t_0} f(\varphi_t(x)) \mathcal{T}(t+N) s(x)\, \mathrm{d}t \right\| \\ &\geqslant \frac{1}{t_0} \left(\left\| \int_{-t_0}^{t_0} f(\varphi_t(x)) s(x)\, \mathrm{d}t \right\| - 2t_0 \cdot 2\varepsilon \right) \\ &= \left\| \frac{1}{t_0} \int_{-t_0}^{t_0} f(\varphi_t(x)) s(x)\, \mathrm{d}t \right\| - 4\varepsilon,\end{aligned}$$

where the inequality follows from (4.2) and (4.4). Further,

$$\left\| \frac{1}{t_0} \int_{-t_0}^{t_0} f(\varphi_t(x)) s(x)\, \mathrm{d}t \right\| = \|s(x)\| \cdot \frac{1}{t_0} \int_{-t_0}^{t_0} f(\varphi_t(x))\, \mathrm{d}t \geqslant \frac{1}{2}.$$

In conclusion, we found $\tilde{s} \in D(\mathcal{A})$ with $\|\tilde{s}\| > 0$ and $\|(\mathcal{A} - \mathrm{i}r)\tilde{s}\| \leqslant \frac{2M}{(N-1)} \to 0$ as $N \to \infty$ for all $r \in \mathbb{R}$, i.e., $\mathrm{i}\mathbb{R} \subseteq \sigma_{\mathrm{ap}}(\mathcal{A})$. □

Finally, we give the following examples.

Example 4.16. (i) We consider an invertible, scalar-valued cocycle $(\Phi_t)_{t\in\mathbb{R}}$ over a flow $(\varphi_t)_{t\in\mathbb{R}}$ on a compact space K, see Example 3.11 (i), and the induced weighted Koopman group $(\mathcal{T}_\Phi(t))_{t\in\mathbb{R}}$ on $\mathrm{C}(K)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. If the flow is aperiodic, then the spectral mapping theorem holds, i.e.,

$$\sigma(\mathcal{T}_\Phi(t)) = \mathrm{e}^{t\sigma(\mathcal{A})} \quad \text{for every } t \in \mathbb{R}.$$

Moreover, $\sigma(\mathcal{A}) = \sigma(\mathcal{A}) + \mathrm{i}\mathbb{R}$ while $\sigma(\mathcal{T}(t)) = \mathbb{T} \cdot \sigma(\mathcal{T}(t))$, $t \geqslant 0$, cf. Theorem 5.4 of [AG84].

For a periodic flow only the weak spectral mapping theorem holds, i.e.,

$$\sigma(\mathcal{T}_\Phi(t)) = \overline{\mathrm{e}^{t\sigma(\mathcal{A})}} \quad \text{for all } t \in \mathbb{R},$$

see Theorem 4.4 of [AG84].

(ii) Consider an evolution semigroup $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on $\Gamma(K, E) \cong \mathrm{C}(K, E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$, see Example 3.11 (ii). If the underlying flow is aperiodic, the spectral mapping theorem holds, i.e.,

$$\sigma(\mathcal{T}_\Phi(t)) = \mathrm{e}^{t\sigma(\mathcal{A})} \quad \text{for every } t \in \mathbb{R}_+.$$

Moreover, $\sigma(\mathcal{A}) = \sigma(\mathcal{A}) + \mathrm{i}\mathbb{R}$ while $\sigma(\mathcal{T}(t)) = \mathbb{T} \cdot \sigma(\mathcal{T}(t))$, $t \geqslant 0$, see Theorem 6.30 of [CL99].

In particular, this is true for an evolution semigroup $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on $\Gamma(K, Z \,\dot\cup\, \{0\}) \cong \mathrm{C}_0(\mathbb{R}, Z)$, see Example 3.11 (ii), induced by an exponentially bounded evolution family $(U(t,r))_{t\geqslant r}$ on a Banach space Z, see Theorem VI.9.15 of [EN00].

Chapter 5

Asymptotics of weighted Koopman semigroups

In this chapter, we apply the theory of C_0-semigroups, see [EN00], in particular, their spectral theory, to investigate stability concepts for semiflows on Banach bundles and weighted Koopman semigroups on spaces of continuous sections.

The results in Section 5.2 and Section 5.3 are joint work with Henrik Kreidler.

5.1 Hyperbolicity for C_0-semigroups

First, we recall some stability concepts for C_0-semigroups. For proofs and examples we refer to Section V.1 of [EN00], Section V.3 of [EN06], and Chapter III of [Eis10]. A C_0-semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on a Banach space Z is called *uniformly exponentially stable* if there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} \mathrm{e}^{\varepsilon t}\|\mathcal{T}(t)\| = 0.$$

We characterize this property via the growth bound ω_0 and the spectral radii $\mathrm{r}(\mathcal{T}(t))$ of the semigroup operators.

Proposition 5.1. *For a C_0-semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on a Banach space Z with generator $(\mathcal{A}, D(\mathcal{A}))$ the following assertions are equivalent.*

(a) $(\mathcal{T}(t))_{t\geqslant 0}$ *is uniformly exponentially stable.*

(b) $(\mathcal{T}(t))_{t\geqslant 0}$ *is* uniformly stable*, i.e.,*

$$\lim_{t\to\infty} \|\mathcal{T}(t)\| = 0.$$

(c) $(\mathcal{T}(t))_{t\geqslant 0}$ *is* strongly exponentially stable*, i.e., there exists $\varepsilon > 0$ such that*

$$\lim_{t\to\infty} \mathrm{e}^{\varepsilon t}\|\mathcal{T}(t)z\| = 0 \quad \textit{for all } z \in Z.$$

(d) $\omega_0 < 0$.

(e) $\mathrm{r}(\mathcal{T}(t)) < 1$ *for one/all $t > 0$.*

(f) *There exist $M \geqslant 1$ and $\omega < 0$ such that*

$$\|\mathcal{T}(t)\| \leqslant M\mathrm{e}^{\omega t} \quad \textit{for all } t \geqslant 0.$$

Moreover, if the growth bound ω_0 and the spectral bound $\mathrm{s}(\mathcal{A})$ of the generator $\mathcal{A}$ coincide, then the properties above are also equivalent to

(g) $\mathrm{s}(\mathcal{A}) < 0$.

We use this result to obtain a decomposition of a C_0-semigroup into a stable and an unstable part, see Section V.1.c, Definition 1.14 of [EN00].

Definition 5.2. A C_0-semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on a Banach space Z is *hyperbolic* if there exist two closed, $(\mathcal{T}(t))_{t\geqslant 0}$-invariant Banach subspaces Z_s and Z_u of Z such that

$$Z = Z_s \oplus Z_u$$

and the restricted semigroups $(\mathcal{T}_s(t))_{t\geqslant 0}$ on Z_s and $(\mathcal{T}_u(t))_{t\geqslant 0}$ on Z_u satisfy the following.

(i) The semigroup $(\mathcal{T}_s(t))_{t\geqslant 0}$ is uniformly exponentially stable on Z_s.

(ii) The semigroup $(\mathcal{T}_u(t))_{t\geqslant 0}$ extends to a group $(\mathcal{T}_u(t))_{t\in\mathbb{R}}$ on Z_u and the semigroup $(\mathcal{T}_u(-t))_{t\geqslant 0}$ is uniformly exponentially stable on Z_u.

The above characterization of exponential stability of a C_0-semigroup leads to the following characterization of hyperbolicity.

Proposition 5.3. *For a C_0-semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on Z the following assertions are equivalent.*

(a) $(\mathcal{T}(t))_{t\geqslant 0}$ *is hyperbolic.*

(b) *There exists a projection P on Z such that each $\mathcal{T}(t)$ commutes with P, $\mathcal{T}(t)\ker P = \ker P$, and there are constants $M \geqslant 1$, $\varepsilon > 0$ such that*

 (i) $\|\mathcal{T}(t)z\| \leqslant M\mathrm{e}^{-\varepsilon t}\|z\|$ *for all* $t \geqslant 0$, $z \in \operatorname{rg} P$,

 (ii) $\|\mathcal{T}(t)z\| \geqslant \frac{1}{M}\mathrm{e}^{\varepsilon t}\|z\|$ *for all* $t \geqslant 0$, $z \in \ker P$.

(c) $\sigma(\mathcal{T}(t)) \cap \mathbb{T} = \emptyset$ *for one/all* $t > 0$.

Moreover, if the weak spectral mapping theorem *(see [EN00], Section IV.3.a) or the* circular spectral mapping theorem *(see [EN06], Section V.3, Definition 3.14) holds, then the properties above are also equivalent to*

(d) $\sigma(\mathcal{A}) \cap \mathrm{i}\mathbb{R} = \emptyset$.

5.2 Direct sum of weighted Koopman semigroups

In order to define hyperbolicity for weighted Koopman semigroups we need the concepts of direct sums of Banach bundles and Banach modules, which are covered in Construction A.6. We now introduce the direct sum of semiflows on Banach bundles and of weighted Koopman semigroups on spaces of continuous sections.

Construction 5.4. We take two semiflows $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on E over K and $(\Psi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on F over K. Setting

$$(\Phi \oplus \Psi)_t\,(u, v) := (\Phi_t u, \Psi_t v) \quad \text{for all } t \geqslant 0, (u, v) \in E \oplus F,$$

defines a semiflow $((\Phi \oplus \Psi)_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on $E \oplus F$ over K.

Two weighted Koopman semigroups $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on $\Gamma(K, E)$ and $(\mathcal{T}_\Psi(t))_{t\geqslant 0}$ on $\Gamma(K, F)$ define a weighted Koopman semigroup on $\Gamma(K, E) \oplus \Gamma(K, F)$ via

$$(\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi)\,(t) := \mathcal{T}_\Phi(t) \oplus \mathcal{T}_\Psi(t) \quad \text{for all } t \geqslant 0.$$

With the isometric module isomorphism of Proposition A.7 we have

$$(\mathcal{T}_\Phi(t))_{t\geqslant 0} \oplus (\mathcal{T}_\Psi(t))_{t\geqslant 0} \cong (\mathcal{T}_{\Phi\oplus\Psi}(t))_{t\geqslant 0}.$$

Consequently, a (non-trivial) decomposition of a semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on E over K yields a (non-trivial) decomposition of the corresponding weighted Koopman semigroup. In other words, if there are two non-trivial Banach subbundles $E_1, E_2 \subseteq E$ and two semiflows $(\Phi_t^1)_{t\geqslant 0}$ on E_1 and $(\Phi_t^2)_{t\geqslant 0}$ on E_2 over $(\varphi_t)_{t\in\mathbb{R}}$ on K such that

$$(\Phi_t)_{t\geqslant 0} \cong (\Phi_t^1)_{t\geqslant 0} \oplus (\Phi_t^2)_{t\geqslant 0}$$

on $E \cong E_1 \oplus E_2$, then $\Gamma(K, E_1)$ and $\Gamma(K, E_2)$ are non-trivial Banach submodules of $\Gamma(K, E)$, see Proposition A.8, and

$$\begin{aligned}(\mathcal{T}_\Phi(t))_{t\geqslant 0} &\cong (\mathcal{T}_{\Phi^1\oplus\Phi^2}(t))_{t\geqslant 0}\\ &\cong (\mathcal{T}_{\Phi^1}(t))_{t\geqslant 0} \oplus (\mathcal{T}_{\Phi^2}(t))_{t\geqslant 0}\end{aligned}$$

on the AM-module

$$\begin{aligned}\Gamma(K, E) &\cong \Gamma(K, E_1 \oplus E_2)\\ &\cong \Gamma(K, E_1) \oplus \Gamma(K, E_2).\end{aligned}$$

As the following result shows, the converse is also true.

Proposition 5.5. *If there are two non-trivial Banach submodules $\Gamma_1, \Gamma_2 \subseteq \Gamma(K, E)$ and two weighted semigroups $(\mathcal{T}_1(t))_{t\geqslant 0}$ on Γ_1 and $(\mathcal{T}_2(t))_{t\geqslant 0}$ on Γ_2 over $(T_\varphi(t))_{t\in\mathbb{R}}$ on $\mathrm{C}(K)$ such that*

$$(\mathcal{T}_\Phi(t))_{t\geqslant 0} \cong (\mathcal{T}_1(t))_{t\geqslant 0} \oplus (\mathcal{T}_2(t))_{t\geqslant 0}$$

on $\Gamma(K, E) \cong \Gamma_1 \oplus \Gamma_2$, then there are two non-trivial Banach subbundles E_1, $E_2 \subseteq E$ and two semiflows $(\Phi_t^1)_{t\geqslant 0}$ on E_1 and $(\Phi_t^2)_{t\geqslant 0}$ on E_2 over $(\varphi_t)_{t\in\mathbb{R}}$ on K such that

$$(\Phi_t)_{t\geqslant 0} \cong (\Phi_t^1)_{t\geqslant 0} \oplus (\Phi_t^2)_{t\geqslant 0} \text{ on } E \cong E_1 \oplus E_2.$$

Moreover, $\|\mathcal{T}_1(t)\| = \|\Phi_t^1\|$ and $\|\mathcal{T}_2(t)\| = \|\Phi_t^2\|$ for all $t \geqslant 0$.

Proof. By Proposition A.8 there are two non-trivial Banach subbundles $E_1, E_2 \subseteq E$ such that $\Gamma_1 \cong \Gamma(K, E_1)$ and $\Gamma_2 \cong \Gamma(K, E_2)$. We then find by Theorem 3.8

semiflows $(\Phi_t^1)_{t\geqslant 0}$ on E_1 and $(\Phi_t^2)_{t\geqslant 0}$ on E_2, which are, up to isometric isomorphy, unique such that

$$
\begin{aligned}
(\mathcal{T}_1(t))_{t\geqslant 0} &\cong (\mathcal{T}_{\Phi^1}(t))_{t\geqslant 0} \text{ on } \Gamma_1 \cong \Gamma(K, E_1),\\
(\mathcal{T}_2(t))_{t\geqslant 0} &\cong (\mathcal{T}_{\Phi^2}(t))_{t\geqslant 0} \text{ on } \Gamma_2 \cong \Gamma(K, E_2).
\end{aligned}
$$

We now obtain

$$
\begin{aligned}
(\mathcal{T}_\Phi(t))_{t\geqslant 0} &\cong (\mathcal{T}_1(t))_{t\geqslant 0} \oplus (\mathcal{T}_2(t))_{t\geqslant 0}\\
&\cong (\mathcal{T}_{\Phi^1}(t))_{t\geqslant 0} \oplus (\mathcal{T}_{\Phi^2}(t))_{t\geqslant 0}\\
&\cong (\mathcal{T}_{\Phi^1\oplus\Phi^2}(t))_{t\geqslant 0}
\end{aligned}
$$

on the AM-module

$$
\begin{aligned}
\Gamma(K, E) &\cong \Gamma_1 \oplus \Gamma_2\\
&\cong \Gamma(K, E_1) \oplus \Gamma(K, E_2)\\
&\cong \Gamma(K, E_1 \oplus E_2).
\end{aligned}
$$

Thus, by uniqueness (up to isometric isomorphy) of the representation of AM-modules and semiflows, we have $E \cong E_1 \oplus E_2$ and $(\Phi_t)_{t\geqslant 0} \cong (\Phi_t^1)_{t\geqslant 0} \oplus (\Phi_t^2)_{t\geqslant 0}$. Finally, $\|\mathcal{T}_1(t)\| = \|\Phi_t^1\|$ and $\|\mathcal{T}_2(t)\| = \|\Phi_t^2\|$ for all $t \geqslant 0$ by Theorem 3.8. □

5.3 Exponential dichotomy and the Sacker-Sell spectrum

In this section, we introduce an important property of semiflows on Banach bundles and of weighted Koopman semigroups on Banach modules that describes their asymptotic behavior: *exponential dichtotomy* and *hyperbolicity*.

Definition 5.6. A weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on a Banach module $\Gamma(K, E)$ has *exponential dichotomy* (or is *hyperbolic*) if there are $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$-invariant Banach submodules Γ_s and Γ_u of $\Gamma(K, E)$ such that

$$
\Gamma(K, E) = \Gamma_s \oplus \Gamma_u
$$

and the restricted semigroups $(\mathcal{T}_s(t))_{t\geqslant 0}$ on Γ_s and $(\mathcal{T}_u(t))_{t\geqslant 0}$ on Γ_u satisfy the following.

(i) The semigroup $(\mathcal{T}_s(t))_{t\geqslant 0}$ is uniformly exponentially stable on Γ_s.

(ii) Each $T_\varphi(t)$-homomorphism $\mathcal{T}_u(t)$ is invertible on Γ_u and the semigroup $(\mathcal{T}_u(-t))_{t\geqslant 0}$ is uniformly exponentially stable on Γ_u.

We generalize exponential dichotomy of linear skew-product flows, cf. Definition 6.13 of [CL99], to exponential dichotomy of semiflows on Banach bundles.

Definition 5.7. A semiflow $(\Phi_t)_{t\geqslant 0}$ on a Banach bundle E *has exponential dichotomy* if there are $(\Phi_t)_{t\geqslant 0}$-invariant Banach subbundles E_s, E_u such that

$$E = E_s \oplus E_u$$

and the restricted semiflows $(\Phi_t^s)_{t\geqslant 0}$ on E_s and $(\Phi_t^u)_{t\geqslant 0}$ on E_u satisfy the following.

(i) The semiflow $(\Phi_t^s)_{t\geqslant 0}$ is *uniformly exponentially stable* on E_s, i.e., there are constants $M \geqslant 1$, $\varepsilon > 0$ such that $\|\Phi_t^s\| \leqslant M\mathrm{e}^{-\varepsilon t}$ for all $t \geqslant 0$.

(ii) The semiflow $(\Phi_t^u)_{t\geqslant 0}$ extends to a flow $(\Phi_t^u)_{t\in\mathbb{R}}$ on E_u, see Definition 1.8, and $(\Phi_{-t}^u)_{t\geqslant 0}$ is uniformly exponentially stable on E_u.

We call E_s the *stable Banach subbundle* and E_u the *unstable Banach subbundle* of E under $(\Phi_t)_{t\geqslant 0}$ while $(\Phi_t^s)_{t\geqslant 0}$ is the *stable part* and $(\Phi_t^u)_{t\geqslant 0}$ is the *unstable part* of $(\Phi_t)_{t\geqslant 0}$.

Exponential dichotomy of semiflows on Banach bundles can be characterized via a spectral property of the associated weighted Koopman semigroup.

Theorem 5.8. *For a weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ on $\Gamma(K, E)$ the following assertions are equivalent.*

(a) $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ *has exponential dichotomy.*

(b) *The associated semiflow $(\Phi_t)_{t\geqslant 0}$ on E has exponential dichotomy.*

(c) $\sigma(\mathcal{T}_\Phi(t)) \cap \mathbb{T} = \emptyset$ *for all/one $t > 0$.*

Proof. Obviously, each weighted Koopman semigroup that admits an exponential dichotomy is, in particular, a hyperbolic C_0-semigroup. Thus, assertion (a) implies (c), see Proposition 5.3.

To show the converse implication, assume that $\sigma(\mathcal{T}_\Phi(t_0)) \cap \mathbb{T} = \emptyset$ for some $t_0 > 0$.

Then $\mathbb{T} \subseteq \varrho(\mathcal{T}_\Phi(t_0))$ and we obtain a decomposition of the spectrum $\sigma(\mathcal{T}_\Phi(t_0)) = K_1 \,\dot\cup\, K_2$ with

$$K_1 := \sigma(\mathcal{T}_\Phi(t_0)) \cap \{z \in \mathbb{C} \mid |z| < 1\},$$
$$K_2 := \sigma(\mathcal{T}_\Phi(t_0)) \cap \{z \in \mathbb{C} \mid |z| > 1\},$$

which yields a *spectral decomposition*, see [EN00], p. 244. Let $\mathcal{P}$ be the corresponding *spectral projection*, $\mathcal{Q} := \mathrm{Id} - \mathcal{P}$, and $\mathcal{T}_u(t_0) := \mathcal{T}_\Phi(t_0)|_{\ker \mathcal{P}} = \mathcal{T}_\Phi(t_0)|_{\mathrm{rg}\,\mathcal{Q}}$, $\mathcal{T}_s(t_0) := \mathcal{T}_\Phi(t_0)|_{\mathrm{rg}\,\mathcal{P}}$ the induced $T_\varphi(t_0)$-homomorphisms on the $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$-invariant subspaces $\mathrm{rg}\,\mathcal{P}$ and $\ker \mathcal{P} = \mathrm{rg}\,\mathcal{Q}$ of $\Gamma(K,E)$. We have $\sigma(\mathcal{T}_s(t_0)) = K_1$, see [EN00], hence $\mathrm{r}(\mathcal{T}_s(t_0)) < 1$. By Proposition 5.1 it follows that $(\mathcal{T}_s(t))_{t\geqslant 0}$ is uniformly exponentially stable on $\mathrm{rg}\,\mathcal{P}$. Further, $\sigma(\mathcal{T}_u(t_0)) = K_2$, which implies that $(\mathcal{T}_u(t))_{t\geqslant 0}$ extends to a group on $\mathrm{rg}\,\mathcal{Q}$. Moreover, we have $\mathrm{r}(\mathcal{T}_u(-t_0)) < 1$. Thus, $(\mathcal{T}_u(-t))_{t\geqslant 0}$ is uniformly exponentially stable on $\mathrm{rg}\,\mathcal{Q}$, see Proposition 5.1. In other words, $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ is a hyperbolic C_0-semigroup on the Banach space $\Gamma(K,E)$. It remains to show that the spectral projection $\mathcal{P}$ is a module homomorphism.

By definition of a uniformly exponentially stable C_0-semigroup, it follows that there are constants $M \geqslant 1$ and $\varepsilon > 0$ such that for $s \in \Gamma(K,E)$, $t \geqslant 0$

$$\frac{1}{M}\mathrm{e}^{\varepsilon t}\|\mathcal{Q}s\| \leqslant \|\mathcal{T}(t)\mathcal{Q}s\| = \|\mathcal{T}(t)(\mathrm{Id} - \mathcal{P})s\| \leqslant \|\mathcal{T}(t)s\| + M\mathrm{e}^{-\varepsilon t}\|\mathcal{P}s\|.$$

Consequently,

$$\{s \in \Gamma(K,E) \mid \mathcal{T}(t)s \to 0 \text{ as } t \to \infty\} \subseteq \mathcal{P}\Gamma(K,E).$$

The inclusion "$\supseteq$" is also true since $(\mathcal{T}(t))_{t\in\mathbb{R}}$ is uniformly exponentially stable on $\mathrm{rg}\,\mathcal{P}$.

Furthermore, $\|\mathcal{T}(t)f\mathcal{P}s\| \leqslant \|f\|\|\mathcal{T}(t)\mathcal{P}s\|$ for all $s \in \Gamma(K,E)$, $f \in \mathrm{C}(K)$. This yields $f\mathcal{P}s \in \mathcal{P}\Gamma(K,E)$ for all $s \in \Gamma(K,E)$, $f \in \mathrm{C}(K)$. In addition, since $\mathcal{T}(t)$ is a $T_\varphi(t)$-homomorphism,

$$\begin{aligned}
\|\mathcal{P}f\mathcal{Q}s\| &= \|\mathcal{P}f\mathcal{T}(t)\mathcal{T}_u(-t)\mathcal{Q}s\| \\
&= \|\mathcal{T}(t)\mathcal{P}T_\varphi(-t)f\mathcal{T}_u(-t)\mathcal{Q}s\| \\
&\leqslant M\mathrm{e}^{-\varepsilon t}\|\mathcal{P}T_\varphi(-t)f\mathcal{T}_u(-t)\mathcal{Q}s\| \\
&\leqslant M\mathrm{e}^{-\varepsilon t}\|f\|\|\mathcal{T}_u(-t)\mathcal{Q}s\| \\
&\leqslant M^2\mathrm{e}^{-2\varepsilon t}\|f\|\|s\|
\end{aligned}$$

for all $s \in \Gamma(K,E)$, $f \in \mathrm{C}(K)$, and $t \geqslant 0$. Hence, $\mathcal{P}f\mathcal{Q}s = 0$ for all $s \in \Gamma(K,E)$ and $f \in \mathrm{C}(K)$. We conclude

$$\mathcal{P}fs = \mathcal{P}f(\mathcal{P} + \mathcal{Q})s = \mathcal{P}f\mathcal{P}s + \mathcal{P}f\mathcal{Q}s = f\mathcal{P}s$$

for all $s \in \Gamma(K, E)$ and $f \in \mathrm{C}(K)$, i.e., $\mathcal{P}$ is a module homomorphism.

As a consequence, the closed subspaces $\operatorname{rg}\mathcal{P}$ and $\ker\mathcal{P}$ are Banach submodules of $\Gamma(K, E)$. Hence, the spectral decomposition of the Banach space $\Gamma(K, E)$ into $\ker(\mathcal{P})$ and $\operatorname{rg}(\mathcal{P})$ yields an exponential dichotomy of the weighted Koopman semigroup, i.e., (c) implies (a).

"(a) $\Rightarrow$ (b)": By Proposition 5.5, the decomposition of the weighted Koopman semigroup yields the desired decomposition of the semiflow into a stable and an unstable part. Hence, (b) is true.

"(b) $\Rightarrow$ (a)": Assume that $(\Phi_t)_{t\geqslant 0}$ decomposes into a stable part $(\Phi_t^s)_{t\geqslant 0}$ and an unstable part $(\Phi_t^u)_{t\geqslant 0}$. Again, this leads to a decomposition of the weighted Koopman semigroup, see Section 5.2. Since the norm of a semiflow is equal to the norm of the corresponding weighted Koopman semigroup by Theorem 3.8, assertion (a) follows. □

As a direct consequence of the above theorem, we obtain the following characterization of exponential dichotomy.

Corollary 5.9. *Let $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ be a weighted Koopman semigroup on $\Gamma(K, E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. If one of the assertions of Theorem 4.13 is satisfied, then the following assertions are equivalent.*

- (a) $(\mathcal{T}_\Phi(t))_{t\geqslant 0}$ *has exponential dichotomy.*
- (b) *The associated semiflow $(\Phi_t)_{t\geqslant 0}$ on E has exponential dichotomy.*
- (c) $1 \notin \sigma(\mathcal{T}_\Phi(t))$ *for all/one $t > 0$.*
- (d) $0 \notin \sigma(\mathcal{A})$.

The following definition is based on Definition 6.17 of [CL99] and goes back to R. J. Sacker and G. R. Sell, see [SS74], [SS76a], [SS76b], and [SS78].

Definition 5.10. For a semiflow $(\Phi_t)_{t\geqslant 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on a Banach bundle E, $t \geqslant 0$, and $\lambda \in \mathbb{R}$ we define the rescaled semiflow by $(\Phi_t(x))^\lambda := \mathrm{e}^{-\lambda t}\Phi_t(x)$ for all $x \in K$. The set

$$\Sigma = \left\{\lambda \in \mathbb{R} \mid (\Phi_t^\lambda)_{t\geqslant 0} \text{ on } E \text{ admits an exponential dichotomy}\right\}$$

is called the *Sacker-Sell spectrum* of the semiflow $(\Phi_t)_{t \geqslant 0}$.

Using Theorem 5.8, we are able to give an explicit description of the Sacker-Sell spectrum.

Corollary 5.11. *Let* $(\mathcal{T}_\Phi(t))_{t \geqslant 0}$ *be a weighted Koopman semigroup on* $\Gamma(K, E)$ *with generator* $(\mathcal{A}, D(\mathcal{A}))$. *Then*

$$\Sigma = \ln |\sigma(\mathcal{T}_\Phi(1)) \setminus \{0\}|.$$

Moreover, if one of the assertions of Theorem 4.13 is fulfilled, then

$$\Sigma = \sigma(\mathcal{A}) \cap \mathbb{R}.$$

Chapter 6

An example from differential geometry

In this chapter, we specialize the previous theory and investigate weighted Koopman groups on the space of continuous sections of the tangent bundle of a compact smooth manifold. Furthermore, we obtain additional characterizations of weighted Koopman groups in this situation.

The results are based on joint work with Nikolai Edeko and Henrik Kreidler.

In the following let $(M; (\varphi_t)_{t\in\mathbb{R}})$ be a smooth dynamical system, i.e., M is a compact Riemannian manifold without boundary with smooth structure, see, e.g., [Lee13], Chapter 13, and $(\varphi_t)_{t\in\mathbb{R}}$ is a smooth flow on M, see, e.g., [Lee13], Chapter 9.

Each smooth flow $(\varphi_t)_{t\in\mathbb{R}}$ on M defines a smooth vector field $V^\varphi\colon M \to TM$ by

$$V^\varphi(x) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \varphi_t(x), \quad x \in M,$$

see, for instance, [Lee13], Proposition 9.7. On the other hand, to each vector field corresponds a unique flow on M, see [Lee13], Theorem 9.12.

A flow $(\varphi_t)_{t\in\mathbb{R}}$ induces a C_0-group $(T_\varphi(t))_{t\in\mathbb{R}}$ on $\mathrm{C}(M)$ by

$$T_\varphi(t)f = f \circ \varphi_{-t}, \quad f \in \mathrm{C}(M),\ t \in \mathbb{R},$$

called *Koopman group*, with generator $(\delta, D(\delta))$, see Section 3.1.

Its generator is defined as

$$\delta f = \lim_{t \to 0} \frac{f \circ \varphi_{-t} - f}{t} \quad \text{for all } f \in D(\delta),$$

where the limit is taken in the norm in $\mathrm{C}(M)$. Recall that the *Lie derivative* $\mathcal{L}_{V^\varphi} f$ of a smooth function f with respect to V^φ is the pointwise limit

$$\mathcal{L}_{V^\varphi} f(x) = \lim_{t \to 0} \frac{f(\varphi_t(x)) - f(x)}{t} \quad \text{for all } f \in \mathrm{C}^\infty(M),\ x \in M,$$

see [Lan95], Section V, §2., p. 121. The following lemma shows that the pointwise limit $\mathcal{L}_{V^\varphi} f(x)$ even converges uniformly in $x \in M$ and that $\mathrm{C}^\infty(M)$ is a core for $(\delta, D(\delta))$.

Lemma 6.1. *For a Koopman group $(T_\varphi(t))_{t\in\mathbb{R}}$ induced by a smooth flow $(\varphi_t)_{t\in\mathbb{R}}$ on M the space of smooth functions $\mathrm{C}^\infty(M)$ is a core for its generator $(\delta, D(\delta))$ and*

$$\delta f = -\mathcal{L}_{V^\varphi} f \quad \textit{for all } f \in \mathrm{C}^\infty(M).$$

Proof. As a consequence of the mean value theorem we know that the (pointwise defined) difference quotient converges locally uniformly in local coordinates. Since M is compact, uniform convergence follows, i.e., $\delta f = -\mathcal{L}_{V^\varphi} f$ for all $f \in \mathrm{C}^\infty(M)$.

Moreover, the space $\mathrm{C}^\infty(M)$ is invariant under $(T_\varphi(t))_{t\in\mathbb{R}}$ and dense in $\mathrm{C}(M)$, cf. [PM82], Poposition 2.7. It follows that $\mathrm{C}^\infty(M)$ is a core for the generator, see [EN00], Section II.1, Proposition 1.7. □

The geometric structure of the smooth dynamical systems suggests to consider the tangent bundle TM and flows $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on TM, see Definition 1.8. For more results on so-called *cocycles* and *linear-skew product flows* we refer to, e.g., [Sma67], [JPS87], [Ree80], or [CL99], Section 6.2. We now investigate the corresponding weighted Koopman groups $(\mathcal{T}_\Phi(t))_{t\in\mathbb{R}}$ on the AM-module $\Gamma(M, TM)$ of continuous sections of the tangent bundle TM induced by $(\Phi_t)_{t\in\mathbb{R}}$, see Definition 3.5. We recall from Section 3.3, Theorem 3.8, that a C_0-group $(\mathcal{T}_\Phi(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$ is a weighted Koopman group if and only if its generator $(\mathcal{G}, D(\mathcal{G}))$ is a δ-derivation, i.e., the domain $D(\mathcal{G})$ is a $D(\delta)$-submodule of $\Gamma(M, TM)$ and the generator satisfies the functional equation

$$\mathcal{G}(fs) = \delta f \cdot s + f \cdot \mathcal{G}s \quad \text{for all } f \in D(\delta), s \in D(\mathcal{G}). \tag{FE}$$

We now consider an important example of a weighted Koopman group where the associated flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{t\in\mathbb{R}}$ has the following property.

Definition 6.2. A flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{t\in\mathbb{R}}$ is called *smooth* if the mapping

$$\mathbf{\Phi}\colon \mathbb{R} \times TM \to TM, \quad (t, x) \mapsto \Phi_t(x)$$

is smooth.

Lemma 6.3. *Take a weighted Koopman group* $(\mathcal{T}_\Phi(t))_{t\in\mathbb{R}}$ *induced by a smooth flow* $(\Phi_t)_{t\in\mathbb{R}}$. *Then the space of smooth sections* $\Gamma^\infty(M, TM)$ *is a core for its generator* $(\mathcal{G}, D(\mathcal{G}))$.

Proof. First, we show that $\Gamma^\infty(M, TM)$ is contained in $D(\mathcal{G})$, i.e.,

$$\lim_{t\to 0} \frac{\Phi_t \circ s \circ \varphi_{-t} - s}{t}$$

exists in $\Gamma(M, TM)$ for all $s \in \Gamma^\infty(M, TM)$. In local coordinates $\mathcal{T}_\Phi(t)s = \Phi_t \circ s \circ \varphi_{-t}$ is just the compositions of smooth mappings on an open subset of $\mathbb{R}^n$ (where n is the dimension of M) which is obviously smooth. Thus, the above limit exists for all smooth sections of TM.

Further, in local coordinates we know that each continuous vector-valued function can be approximated by a smooth vector-valued function. Hence, $\Gamma^\infty(M, TM)$ is even norm dense in $\Gamma(M, TM)$. It remains to show that $\Gamma^\infty(M, TM)$ is invariant under $(\mathcal{T}_\Phi(t))_{t\in\mathbb{R}}$. This follows by the smoothness of the flow $(\Phi_t)_{t\in\mathbb{R}}$. Consequently, $\Gamma^\infty(M, TM)$ is a core, see [EN00], Section II.1, Proposition 1.7. □

Consider the differential $\mathrm{d}\varphi_t$ of the smooth mapping φ_t, $t \in \mathbb{R}$, as in Example 3.11 (iii). By [Lee13], Proposition 3.21 and Corollary 3.22, it follows that $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$ is a smooth flow over $(\varphi_t)_{t\in\mathbb{R}}$. The induced weighted Koopman operators are the *pushforward operators*, cf. [Lee13], Chapter 8, p. 183. We call the corresponding weighted Koopman group $(\mathcal{T}(t))_{t\in\mathbb{R}} := (\mathcal{T}_{\mathrm{d}\varphi}(t))_{t\in\mathbb{R}}$ *pushforward group* and denote its generator by $(\mathcal{A}, D(\mathcal{A}))$ defined as

$$\mathcal{A}s = \lim_{t\to 0} \frac{\mathrm{d}\varphi_t \circ s \circ \varphi_{-t} - s}{t} \quad \text{for all } s \in D(\mathcal{A}),$$

where the limit is taken in the norm in $\Gamma(M, TM)$. Recall that the *Lie derivative* $\mathcal{L}_{V^\varphi} s$ of a smooth section s with respect to V^φ is the pointwise limit

$$\mathcal{L}_{V^\varphi} s(x) = \lim_{t\to 0} \frac{\mathrm{d}\varphi_{-t}(\varphi_t(x))s(\varphi_t(x)) - s(x)}{t}$$

for all $s \in \Gamma^\infty(M, TM)$, $x \in M$, see [Lee13], p. 228. Again, the pointwise limit $\mathcal{L}_{V^\varphi} s(x)$ even converges uniformly in $x \in M$ and $\Gamma^\infty(M, TM)$ is a core for $(\mathcal{A}, D(\mathcal{A}))$.

Lemma 6.4. *Consider the pushforward group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. The space of smooth sections $\Gamma^\infty(M, TM)$ is a core for the generator with*

$$\mathcal{A}s = -\mathcal{L}_{V^\varphi} s \quad \text{for all } s \in \Gamma^\infty(M, TM).$$

Proof. Since $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$ is a smooth flow over $(\varphi_t)_{t\in\mathbb{R}}$, Lemma 6.3 implies that the space of smooth sections $\Gamma^\infty(M, TM)$ is a core for $(\mathcal{A}, D(\mathcal{A}))$. As in the proof of Lemma 6.1 we consider for $s \in \Gamma^\infty(M, TM)$, $x \in M$, the limit $\mathcal{L}_{V^\varphi} s(x)$ in local coordinates. Again, by the mean value theorem, local uniform convergence follows which implies, since M is compact, uniform convergence, i.e., $\mathcal{A}s = -\mathcal{L}_{V^\varphi} s$ for all $s \in \Gamma^\infty(M, TM)$. □

6.1 The generator of a weighted Koopman group as a perturbation of the Lie derivative

Starting from the pushforward group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$, we obtain every other weighted Koopman group in the following way.

Proposition 6.5. *Let $(\mathcal{S}(t))_{t\in\mathbb{R}}$ be a C_0-group on $\Gamma(M, TM)$. Then the following assertions are equivalent.*

(a) *$(\mathcal{S}(t))_{t\in\mathbb{R}}$ is a weighted Koopman group.*

(b) $(\mathcal{S}(t))_{t\in\mathbb{R}}$ *is a multiplicative perturbation of the pushforward group* $(\mathcal{T}(t))_{t\in\mathbb{R}}$ *by* $\mathrm{C}(M)$*-module homomorphisms* $C(t)$*,* $t \in \mathbb{R}$*, see Definition 2.1, i.e., we have for each* $t \in \mathbb{R}$

$$\mathcal{S}(t) = C(t)\mathcal{T}(t).$$

Proof. "(b) ⇒ (a)": Let $t \in \mathbb{R}$, $f \in \mathrm{C}(M)$, and $s \in \Gamma(M, TM)$. Then, by definition of a module homomorphism, see Definition 2.1, and since each weighted Koopman operator $\mathcal{T}(t)$ is a $T_\varphi(t)$-homomorphism, see Definition 3.4, we have

$$\begin{aligned}
\mathcal{S}(t)(fs) &= (C(t)\mathcal{T}(t))(fs) \\
&= C(t)(T_\varphi f \cdot \mathcal{T}(t)s) \\
&= T_\varphi f \cdot C(t)\mathcal{T}(t)s \\
&= T_\varphi f \cdot \mathcal{S}(t)s.
\end{aligned}$$

Thus, by Theorem 3.8, $(\mathcal{S}(t))_{t\in\mathbb{R}}$ is a weighted Koopman group.

"(a) ⇒ (b)": Let $(\mathcal{S}(t))_{t\in\mathbb{R}}$ be a weighted Koopman group and let $(\Phi_t)_{t\in\mathbb{R}}$ be the corresponding flow on TM over $(\varphi_t)_{t\in\mathbb{R}}$. For each $x \in M$ we define a family of bounded linear operators $(C^x_\Phi(t))_{x\in M}$ on the tangent space T_xM by

$$C^x_\Phi(t) := \Phi_t(\varphi_{-t}(x))\mathrm{d}\varphi_{-t}(x) \in \mathscr{L}(T_xM) \quad \text{for all } t \in \mathbb{R}.$$

Further, we set for all $s \in \Gamma(M, TM)$, $x \in M$, and $t \in \mathbb{R}$

$$\big(C_\Phi(t)s\big)(x) := C^x_\Phi(t)s(x). \tag{6.1}$$

This defines a linear bounded operator $C_\Phi(t)$ on $\Gamma(M, TM)$ such that for all $s \in \Gamma(M, TM)$, $x \in M$, $f \in \mathrm{C}(M)$, and $t \in \mathbb{R}$

$$\begin{aligned}
C_\Phi(t)(f \cdot s)(x) &= \Phi_t(\varphi_{-t}(x))\mathrm{d}\varphi_{-t} f(x)s(x) \\
&= f(x)\Phi_t(\varphi_{-t}(x))\mathrm{d}\varphi_{-t} s(x) \\
&= (f \cdot C_\Phi(t)s)\,(x)
\end{aligned}$$

and

$$\begin{aligned}
\big(C_\Phi(t)\mathcal{T}(t)s\big)(x) &= \big(C^x_\Phi(t)\mathcal{T}(t)s\big)(x) \\
&= \big(C^x_\Phi(t)\mathrm{d}\varphi_t(\varphi_{-t}(x)s\big)(\varphi_{-t}(x)) \\
&= \big(\Phi_t(\varphi_{-t}(x))\mathrm{d}\varphi_{-t}(x)\mathrm{d}\varphi_t(\varphi_{-t}(x))s\big)(\varphi_{-t}(x)) \\
&= \Phi_t(\varphi_{-t}(x))s(\varphi_{-t}(x)) \\
&= \big(\mathcal{S}(t)s\big)(x).
\end{aligned}$$

□

From the above defined family of bounded linear operators $(C_\Phi(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$ we obtain that the generator of each weighted Koopman group with smooth flow is the additive perturbation of the generator $(\mathcal{A}, D(\mathcal{A}))$ of the pushforward group.

Proposition 6.6. *Consider a weighted Koopman group* $(\mathcal{T}_\Phi(t))_{t\in\mathbb{R}}$ *on* $\Gamma(M, TM)$ *with smooth flow* $(\Phi_t)_{t\in\mathbb{R}}$ *over* $(\varphi_t)_{\in\mathbb{R}}$ *and with generator* $(\mathcal{G}, D(\mathcal{G}))$. *Let* $(C_\Phi(t))_{t\in\mathbb{R}}$ *be the family of bounded linear operators on* $\Gamma(M, TM)$ *as in (6.1). Then we have for all* $s \in \Gamma^\infty(M, TM)$

$$\mathcal{M}_\Phi s := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} C_\Phi(t)s = -\mathcal{A}s + \mathcal{G}s.$$

Moreover, $\mathcal{M}_\Phi$ *is a* bounded multiplication operator*, i.e., it is a bounded linear operator on* $\Gamma(M, TM)$ *which satisfies for all* $s \in \Gamma(M, TM)$*,* $f \in D(\delta)$

$$\mathcal{M}_\Phi(f \cdot s) = f \cdot \mathcal{M}_\Phi s.$$

Proof. Let $s \in \Gamma^\infty(M, TM)$ and $x \in M$. Then

$$\begin{aligned}
&\frac{\big(\Phi_t(\varphi_{-t}(x))\mathrm{d}\varphi_{-t}(x)\big)s(x) - s(x)}{t} \\
&= \frac{\Phi_t(\varphi_{-t}(x))\big(\mathrm{d}\varphi_{-t}(x)s(x) - s(\varphi_{-t}(x))\big) + \big(\Phi_t(\varphi_{-t}(x))s(\varphi_{-t}(x)) - s(x)\big)}{t} \\
&= \Phi_t(\varphi_{-t}(x))\mathrm{d}\varphi_{-t}(x) \cdot \frac{s(x) - \mathrm{d}\varphi_t(\varphi_{-t}(x))s(\varphi_{-t}(x))}{t} \\
&\qquad + \frac{\Phi_t(\varphi_{-t}(x))s(\varphi_{-t}(x)) - s(x)}{t} \\
&= C^x_\Phi(t) \cdot \left(-\frac{\mathrm{d}\varphi_t(\varphi_{-t}(x))s(\varphi_{-t}(x)) - s(x)}{t}\right) \\
&\qquad + \frac{\Phi_t(\varphi_{-t}(x))(s(\varphi_{-t}(x))) - s(x)}{t}.
\end{aligned}$$

Each operator $C^x_\Phi(t)$ converges to id_{T_xM} as t goes to 0. Further, we have for all $s \in \Gamma^\infty(M, TM) \subseteq D(\mathcal{A}) \cap D(\mathcal{G})$ and $x \in M$

$$\begin{aligned}
-\mathcal{A}s(x) &= -\lim_{t\to 0} \frac{\mathrm{d}\varphi_t(\varphi_{-t}(x))s(\varphi_{-t}(x)) - s(x)}{t} \\
&= \mathcal{L}_{V^\varphi} s(x),
\end{aligned}$$

see Lemma 6.4, and

$$\mathcal{G}s(x) = \lim_{t\to 0} \frac{\Phi_t(\varphi_{-t}(x))(s(\varphi_{-t}(x))) - s(x)}{t}.$$

This implies that for all $s \in \Gamma^\infty(M, TM)$ and $x \in M$

$$\mathcal{M}_\Phi s(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} C_\Phi(t)s(x) = -\mathcal{A}s(x) + \mathcal{G}s(x).$$

Considering the above limit in local coordinates implies again, by the mean value theorem and by compactness of M, uniform convergence, i.e.,

$$\mathcal{M}_\Phi s = -\mathcal{A}s + \mathcal{G}s \quad \text{for all } s \in \Gamma^\infty(M, TM).$$

Since $\mathcal{A}$ and $\mathcal{G}$ are generators of weighted Koopman semigroups, we have by the functional equation (FE) for all $s \in \Gamma^\infty(M, TM)$, $f \in D(\delta)$

$$\begin{aligned} \mathcal{M}_\Phi(fs) &= -\mathcal{A}(fs) + \mathcal{G}(fs) \\ &= -(\delta f \cdot s + f \cdot \mathcal{A}s) + (\delta f \cdot s + f \cdot \mathcal{G}s) \\ &= f\mathcal{M}_\Phi s. \end{aligned}$$

By the *Tensor Characterization Lemma*, see [Lee13], Lemma 12.24, it follows that $M \ni x \mapsto \mathcal{M}_\Phi s(x)$ is smooth for all $s \in \Gamma^\infty(M, TM)$ and can be extended to the space of continuous sections $\Gamma(M, TM)$.

Finally, we show that $\mathcal{M}_\Phi$ is bounded. We consider $\mathcal{M}_x s := (\mathcal{M}_\Phi s)(x)$ for all $x \in M$ and $s \in \Gamma(M, TM)$. Then $m_x := \|\mathcal{M}_x\|$ depends smoothly on x. Since M is compact, $\sup_{x\in M} m_x =: m < \infty$. Consequently, $\|\mathcal{M}_\Phi\| = \sup_{\|s\|=1} \|\mathcal{M}_\Phi s\| = \sup_{\|s\|=1} \sup_{x\in M} \|(\mathcal{M}s)(x)\| = \sup_{x\in M} \sup_{\|s\|=1} \|(\mathcal{M}s)(x)\| = m < \infty$. $\square$

We summarize the above results in the following theorem.

Theorem 6.7. *Let $(\mathcal{T}(t))_{t\in\mathbb{R}}$ be the pushforward group with generator $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{S}(t))_{t\in\mathbb{R}}$ be a C_0-group on $\Gamma(M, TM)$ with generator $(\mathcal{G}, D(\mathcal{G}))$. If the smooth sections $\Gamma^\infty(M, TM)$ are a core for $(\mathcal{G}, D(\mathcal{G}))$, then the following assertions are equivalent.*

(a) *$(\mathcal{S}(t))_{t\in\mathbb{R}}$ is a weighted Koopman group.*

(b) *$(\mathcal{S}(t))_{t\in\mathbb{R}}$ is a multiplicative perturbation of $(\mathcal{T}(t))_{t\in\mathbb{R}}$ by* $\mathrm{C}(M)$*-module homomorphisms $C(t)$, $t \in \mathbb{R}$, i.e., we have for each $t \in \mathbb{R}$*

$$\mathcal{S}(t) = C(t)\mathcal{T}(t).$$

(c) *The generator* $(\mathcal{G}, D(\mathcal{G}))$ *is the additive perturbation of* $(\mathcal{A}, D(\mathcal{A}))$ *by a bounded multiplication operator* $\mathcal{M}$*, i.e.,*

$$\mathcal{G} = \mathcal{A} + \mathcal{M} \quad \textit{with } D(\mathcal{G}) = D(\mathcal{A}).$$

Proof. The equivalence "(a) ⇔ (b)" follows by Proposition 6.5.

"(a) ⇒ (c)": First, we abbreviate $\Gamma := \Gamma(M, TM)$. We know that $\mathcal{G} = \mathcal{A} + \mathcal{M}$ on $D := \Gamma^\infty(M, TM)$ by Proposition 6.6. Moreover, as $\mathcal{M}$ is bounded, we have $D(\mathcal{A} + \mathcal{M}) = D(\mathcal{A}) \cap D(\mathcal{M}) = D(\mathcal{A})$. Since D is a core for $(\mathcal{G}, D(\mathcal{G}))$ (by assumption) as well as for $(\mathcal{A}, D(\mathcal{A}))$ (by Lemma 6.4) it follows that

$$\begin{aligned}
&\{(s, \mathcal{G}s) \mid s \in D(\mathcal{G})\} \\
&= \overline{\{(s, \mathcal{G}s) \mid s \in D\}} \\
&= \overline{\{(s, (\mathcal{A} + \mathcal{M})s) \mid s \in D\}} \\
&= \{(s, \tilde{s}) \in \Gamma \times \Gamma \mid \exists (s_n, (\mathcal{A} + \mathcal{M})s_n) \to (x, y), (s_n)_{n\in\mathbb{N}} \subseteq D\} \\
&= \{(s, \tilde{s} + \mathcal{M}s) \in \Gamma \times \Gamma \mid \exists (s_n, \mathcal{A}s_n) \to (x, y), (s_n)_{n\in\mathbb{N}} \subseteq D\} \\
&= \{(s, (\mathcal{A} + \mathcal{M})s) \mid s \in D(\mathcal{A})\},
\end{aligned}$$

i.e., $\mathcal{G} = \overline{\mathcal{G}_{|D}} = \overline{(\mathcal{A} + \mathcal{M})_{|D}} = \mathcal{A} + \mathcal{M}$ with $D(\mathcal{G}) = D(\mathcal{A})$.

"(c) ⇒ (a)": For $f \in D(\delta)$ and $s \in D(\mathcal{G})$ we have

$$\begin{aligned}
\mathcal{G}(fs) &= \mathcal{A}(fs) + \mathcal{M}(fs) \\
&= \delta f \cdot s + f \cdot \mathcal{A}sf \cdot \mathcal{M}s \\
&= \delta f \cdot s + f \cdot \mathcal{G}s.
\end{aligned}$$

Theorem 3.8 implies that $(\mathcal{S}(t))_{t\in\mathbb{R}}$ is a weighted Koopman group. □

Remark 6.8. Each bounded additive perturbation of the generator of a C_0-group generates a C_0-group, see [EN00], paragraph II.3.11 and Section III.1, Theorem 1.3. Thus, each additive perturbation of the generator of the pushforward group by a bounded multiplication operator is the generator of a weighted Koopman group by Theorem 6.7.

Remark 6.9. The equivalence "(a) ⇔ (c)" of the above theorem can be—in view of Lemma 6.4—formulated in the following way: Each generator of a weighted Koopman group induced by a smooth flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{\in\mathbb{R}}$ is an additive perturbation of the Lie derivative $\mathcal{L}_{V^\varphi}$ on $\Gamma^\infty(M, TM)$ by a bounded multiplication operator and each such perturbation yields a weighted Koopman group.

Example 6.10. Consider for each $x \in M$, $t \in \mathbb{R}$, the *parallel transport* $P_t(x)$ *along* $\varphi_t(x)$ *with respect to the Levi-Civita connection* ∇, cf. [Spi99], Chapter 6, p. 238 and 240. Then $(P_t)_{t\in\mathbb{R}}$ is a smooth flow over $(\varphi_t)_{t\in\mathbb{R}}$, cf. [Lee18], Theorem 4.31 and Theorem 4.32. We call the corresponding weighted Koopman group $(\mathcal{P}(t))_{t\in\mathbb{R}}$ *parallel transport group* and denote its generator by $(\mathcal{B}, D(\mathcal{B}))$. The above theorem implies for all $s \in D(\mathcal{B})$

$$\mathcal{B}s = \mathcal{A}s + \mathcal{M}s$$

for a bounded multiplication operator $\mathcal{M}$ and the generator $(\mathcal{A}, D(\mathcal{A}))$ of the pushforward group $(\mathcal{T}(t))_{t\in\mathbb{R}}$. This can be reformulated to

$$-\mathcal{A}s = -\mathcal{B}s + \mathcal{M}s$$

for all $s \in D(\mathcal{B})$. By Lemma 6.4 we have for $s \in \Gamma^\infty(M, TM)$ that $-\mathcal{A}s = \mathcal{L}_{V^\varphi}s$. Moreover, for all $s \in \Gamma^\infty(M, TM)$

$$\mathcal{L}_{V^\varphi}s = \nabla_{V^\varphi}s - \nabla_s V^\varphi,$$

cf. [Spi99], Chapter 5, p. 224 and Chapter 6, p. 238. Thus, we have for all $s \in \Gamma^\infty(M, TM)$

$$\begin{aligned} -\mathcal{B}s + \mathcal{M}s &= -\mathcal{A}s \\ &= \mathcal{L}_{V^\varphi}s \\ &= \nabla_{V^\varphi}s - \nabla_s V^\varphi. \end{aligned}$$

By Lemma 6.3 the space $\Gamma^\infty(M, TM)$ is a core for the generator $(\mathcal{B}, D(\mathcal{B}))$ of the parallel transport group. For all $s \in \Gamma^\infty(M, TM)$ we have for all $x \in M$

$$\mathcal{B}s(x) = -\nabla_{V^\varphi}s(x),$$

see [Spi99], Chapter 6, Proposition 3. Again, using local coordinates and the mean value theorem, compactness of M yields for all

$$\mathcal{B}s = -\nabla_{V^\varphi}s.$$

The bounded multiplication operator $\mathcal{M}$ is given by $\mathcal{M}s = \nabla_s V^\varphi$ for all $s \in \Gamma^\infty(M, TM)$.

Remark 6.11. One could also start with the pushforward group $(\mathcal{P}(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$ to obtain every other weighted Koopman group induced by a smooth flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{\in\mathbb{R}}$. In particular, the additive perturbation of the generator $(\mathcal{B}, D(\mathcal{B}))$ of the pushforward group by a bounded multiplication operator $\mathcal{M}$ also results in a generator of a weighted Koopman group. The proofs of Proposition 6.6 and Theorem 6.7 can simply be adapted to that situation.

We now apply semigroup theory to obtain corollaries of Theorem 6.7 which are useful to compare qualitive properties of a weighted Koopman group $(\mathcal{S}(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$ induced by a smooth flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{\in\mathbb{R}}$ and the pushforward group $(\mathcal{T}(t))_{t\in\mathbb{R}}$. In the following take the pushforward group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ and let $\mathcal{M}$ be the bounded multiplication operator corresponding to $(\mathcal{S}(t))_{t\in\mathbb{R}}$ as in Theorem 6.7.

Corollary 6.12. *For a weighted Koopman group $(\mathcal{S}(t))_{t\in\mathbb{R}}$ induced by a smooth flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{\in\mathbb{R}}$ the following* variation of parameters formula *holds*

$$\mathcal{S}(t)s = \mathcal{T}(t)s + \int_0^t \mathcal{T}(t-r)\mathcal{M}\mathcal{S}(r)s \,\mathrm{d}r$$

for every $t \in \mathbb{R}$ and $s \in \Gamma(M, TM)$, cf. [EN00], Section III.1, Corollary 1.7.

Corollary 6.13. *Each weighted Koopman group $(\mathcal{S}(t))_{t\in\mathbb{R}}$ induced by a smooth flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{\in\mathbb{R}}$ can be obtained as a so-called* Dyson-Phillips series*, i.e.,*

$$\mathcal{S}(t) = \sum_{n=0}^{\infty} \mathcal{S}_n(t),$$

where $\mathcal{S}_0(t) := \mathcal{T}(t)$ and for all $n \in \mathbb{N}$

$$\mathcal{S}_{n+1}(t) := \int_0^t \mathcal{T}(t-r)\mathcal{M}\mathcal{S}_n(r) \,\mathrm{d}r,$$

see [EN00], Section III.1, Theorem 1.10.

Corollary 6.14. *Let $(\mathcal{S}(t))_{t\in\mathbb{R}}$ be a weighted Koopman group induced by a smooth flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{\in\mathbb{R}}$. Then there exists a constant $M \geqslant 0$ such that*

$$\|\mathcal{T}(t) - \mathcal{S}(t)\| \leqslant tM$$

for all $t \in [0, 1]$, see [EN00], Section III.1, Corollary 1.11.

Corollary 6.15. *We consider the parallel transport group $(\mathcal{P}(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$ with generator $(\mathcal{B}, D(\mathcal{B}))$. For each weighted Koopman group $(\mathcal{S}(t))_{t\in\mathbb{R}}$ induced by a smooth flow $(\Phi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{\in\mathbb{R}}$ there exists a bounded multiplication operator $\mathcal{M}$ on $\Gamma(M, TM)$ such that the* Lie-Trotter product formula *holds, i.e.,*

$$\mathcal{S}(t)s = \lim_{n\to\infty} \left[\mathcal{P}\left(\frac{t}{n}\right) \mathrm{e}^{\frac{t}{n}\mathcal{M}}\right]^n s, \quad s \in \Gamma(M, TM), t \in \mathbb{R},$$

with uniform convergence for t in compact intervals.

Proof. By Remark 6.11, the generator $(\mathcal{G}, D(\mathcal{G}))$ of $(\mathcal{S}(t))_{t\in\mathbb{R}}$ is of the form $\mathcal{G} = \mathcal{B} + \mathcal{M}$ on $D(\mathcal{B})$. Further, the parallel transport P_t, $t \in \mathbb{R}$, along the flow $(\varphi_t)_{t\in\mathbb{R}}$ is an isometry, see [Lee18], Proposition 5.5. Hence, each $\mathcal{P}(t)$ satisfies $\|\mathcal{P}(t)\| \leqslant 1$, see Theorem 3.8. Moreover, the group $(\mathrm{e}^{t\mathcal{M}})_{t\in\mathbb{R}}$ generated by $\mathcal{M}$ satisfies $\left\|\mathrm{e}^{t\mathcal{M}}\right\| \leqslant \mathrm{e}^{t\|\mathcal{M}\|}$ for all $t \in \mathbb{R}$. Thus, for all $t \in \mathbb{R}$, $n \in \mathbb{N}$,

$$\left\|\left[\mathcal{P}\left(\frac{t}{n}\right)\mathrm{e}^{\frac{t}{n}\mathcal{M}}\right]^n\right\| \leqslant \left\|\mathcal{P}\left(\frac{t}{n}\right)\right\|^n \cdot \left\|\left(\mathrm{e}^{\frac{t}{n}\mathcal{M}}\right)^n\right\| \leqslant \mathrm{e}^{t\|\mathcal{M}\|}.$$

Then the claim follows by [EN00], Section III.5, Corollary 5.8. □

Remark 6.16. For matrices A and B the product formula

$$\mathrm{e}^{A+B} = \lim_{n\to\infty}\left[\mathrm{e}^{\frac{A}{n}}\mathrm{e}^{\frac{B}{n}}\right]^n$$

goes back to Lie and has been extended to unbounded operators by Trotter [Tro59].

Example 6.17. For the above Lie-Trotter product formula in the case of evolution semigroups induced by evolution families as in Example 3.11(ii), we refer to [EN00], Section III.5, Example 5.9.

6.2 Hyperbolic flows

In this section, we characterize *hyperbolic flows*—also known as *Anosov flows*—on a compact Riemannian manifold M without boundary via the corresponding pushforward group on $\Gamma(M, TM)$. We start with the definition of a hyperbolic flow, cf. [FH19], Definition 5.1.1.

Definition 6.18. Let $(\varphi_t)_{t\in\mathbb{R}}$ be a smooth flow on M with corresponding vector field V^φ. Consider for each $x \in M$ the linear span $E_0(x) := \langle V^\varphi(x)\rangle := \mathbb{R}V^\varphi(x)$ with corresponding Banach subbundle $E_0 = \dot{\bigcup}_{x\in M} E_0(x) \subseteq TM$. We call $(\varphi_t)_{t\in\mathbb{R}}$ *hyperbolic* or *Anosov* if there are $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$-invariant Banach subbundles E_s and E_u of TM such that

$$TM = E_s \oplus E_u \oplus E_0$$

and the flow $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$ over $(\varphi_t)_{t\in\mathbb{R}}$ restricted to the Banach bundle $E := E_s \oplus E_u$ has exponential dichotomy on E with stable Banach subbundle E_s and unstable

Banach subbundle E_u, see Definition 5.7. We call E the *hyperbolic Banach bundle of* $(\varphi_t)_{t\in\mathbb{R}}$.

Remark 6.19. In the situation of the above definition the hyperbolic Banach bundle E of $(\varphi_t)_{t\in\mathbb{R}}$ is, indeed, a Banach bundle, see Construction A.6.

Proposition 6.20. *Let $(\varphi_t)_{t\in\mathbb{R}}$ be a smooth flow on M with corresponding pushforward group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$. Assume that there exists a decomposition $TM = E \oplus E_0$ into $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$-invariant Banach subbundles E and E_0 of TM with E_0 as in Definition 6.18. Then the following assertions are equivalent.*

(a) *The flow $(\varphi_t)_{t\in\mathbb{R}}$ is hyperbolic, i.e., $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$ restricted to E has exponential dichotomy.*

(b) *The pushforward group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ restricted to $\Gamma(M, E)$ is hyperbolic.*

Proof. The space of sections of E is a Banach submodule of $\Gamma(M, TM)$, see Proposition A.4. Since E is $(\mathrm{d}\varphi_t)_{t\in\mathbb{R}}$-invariant, the AM-module $\Gamma(M, E)$ is invariant under the associated pushforward group $(\mathcal{T}(t))_{t\in\mathbb{R}}$, see Section 5.2, p. 90. But then the assertion follows directly by Theorem 5.8. □

Since the tangent bundle of a smooth manifold is a continuous Banach bundle, see Example 1.5 (ii), we know by Corollary 5.9 that for a smooth aperiodic flow $(\varphi_t)_{t\in\mathbb{R}}$ on M the associated pushforward group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$ has exponential dichotomy if and only if its generator $\mathcal{A}$ is invertible. In the following situation exponential dichotomy is stable under the "small" additive perturbation of $\mathcal{A}$ by a bounded multiplication operator $\mathcal{M}$, i.e., exponential dichotomy is a robust property.

Proposition 6.21. *Let $\mathcal{G} = \mathcal{A} + \mathcal{M}$ the generator of a weighted Koopman group $(\mathcal{S}(t))_{t\in\mathbb{R}}$ on $\Gamma(M, TM)$ (cf. Theorem 6.7) such that the corresponding flow $(\varphi_t)_{t\in\mathbb{R}}$ is aperiodic. If $(\mathcal{T}(t))_{t\in\mathbb{R}}$ has exponential dichotomy (i.e., $\mathcal{A}$ is invertible) and $\|\mathcal{M}\| \cdot \|\mathcal{A}^{-1}\| < 1$, then $(\mathcal{S}(t))_{t\in\mathbb{R}}$ has exponential dichotomy.*

Proof. [Kat80], Chapter IV, Theorem 1.16 implies that $\mathcal{G}$ is invertible. Hence, $(\mathcal{S}(t))_{t\in\mathbb{R}}$ has exponential dichotomy, see Corollary 5.9. □

Remark 6.22. For a similar result for hyperbolic evolution semigroups as in Example 3.11 (ii) we refer to [EN00], Section VI.9, Theorem 9.24, [CL96], Theorem 4.3, [LS99], Corollary 2.10, or [Huy07], Theorem 6.1.

The results of the present chapter indicate that there is great potential in this operator theoretic approach to smooth flows on manifolds.

Appendix A

Standard constructions for Banach bundles and Banach modules

In this appendix we briefly recall the definitions and standard constructions for Banach bundles and Banach modules, cf. [Gie82], [DG83], [FD88], or [AAK92], which we need for our investigation of hyperbolicity of weighted Koopman semigroups in Chapter 5.

A.1 Banach subbundles and Banach submodules

This section is inspired by Section 8 of [Gie82]. We consider a Banach bundle $p\colon E \to K$ over a compact space K, see Definition 1.1, and the corresponding AM-module $\Gamma(K, E)$ over $\mathrm{C}(K)$, see Definition 1.3. In this context, we recall the definition of Banach subbundles and Banach submodules and the correspondence between them.

Definition A.1. A subspace $F \subseteq E$ is called *Banach subbundle* if the following properties are satisfied.

(i) For each $x \in K$ the set $F_x := p^{-1}(x) \cap F$ is a closed subspace of the fiber $E_x = p^{-1}(x)$.

(ii) The restriction of the bundle projection $p_{|F}\colon F \to K$ is open.

Proposition A.2. *A Banach subbundle $F \subseteq E$ is a Banach bundle over K with the bundle projection and bundle norm of E restricted to F.*

Proof. For the proof, we refer to Proposition 8.2 of [Gie82]. □

Definition A.3. A closed subspace $\Gamma \subseteq \Gamma(K, E)$ is called *Banach submodule* if Γ is a submodule of $\Gamma(K, E)$.

Obviously, each Banach submodule of $\Gamma(K, E)$ is, again, an AM-module over $\mathrm{C}(K)$. By Theorem 3.3 there exists, up to isometric isomorphy, a unique Banach bundle F over K such that Γ is isometrically isomorphic to $\Gamma(K, F)$. This Banach bundle F can be identified with a Banach subbundle of E over K. The following proposition describes the correspondence between Banach subbundles and Banach submodules.

Proposition A.4. *The following statements are true.*

(i) *For each Banach subbundle $F \subseteq E$ over K the induced AM-module $\Gamma(K, F)$ over $\mathrm{C}(K)$ is a Banach submodule of $\Gamma(K, E)$.*

(ii) *Consider the evaluation map $\mathrm{e}_x\colon \Gamma \to E_x$, $s \mapsto s(x)$, $x \in K$. For each Banach submodule $\Gamma \subseteq \Gamma(K, E)$ over $\mathrm{C}(K)$ the induced Banach bundle $F := \bigcup_{x\in K} \mathrm{e}_x(\Gamma)$ over K is a Banach subbundle of E.*

Moreover, $F \mapsto \Gamma(K, F)$ is a bijection of Banach subbundles and Banach submodules. The inverse is given by $\Gamma \mapsto \bigcup_{x\in K} \mathrm{e}_x(\Gamma)$.

Proof. Cf. Theorem 8.6 and Remark 8.7 of [Gie82]. □

We conclude this section with a remark concerning the kernel and the image of a homomorphism of Banach modules, see Definition 2.1.

Remark A.5. Let $\mathcal{T}$ be a homomorphism between two Banach modules $\Gamma(K, E_1)$ and $\Gamma(K, E_2)$. Then $\ker \mathcal{T} \subseteq \Gamma(K, E_1)$ and the closure of $\mathrm{rg}\, \mathcal{T} \subseteq \Gamma(K, E_2)$ are Banach submodules.

A.2 Direct sum of Banach bundles and Banach modules

We now discuss the direct sum of two Banach bundles E and F over the compact space K.

Construction A.6. For each $x \in K$ we consider the direct sum $E_x \oplus F_x$ of the Banach spaces E_x and F_x equipped with the norm $\|(u, v)\| := \max(\|u\|, \|v\|)$ for $(u, v) \in E_x \oplus F_x$, which induces the product topology of E_x and F_x on $E_x \oplus F_x$. We then endow the direct sum

$$E \oplus F := \bigcup_{x \in K} E_x \oplus F_x \subseteq E \times F$$

with the subspace topology induced by the product topology on $E \times F$. Equipped with the canonical projection, addition, and scalar multiplication, the direct sum $E \oplus F$ of two Banach bundles E and F over K is, again, a Banach bundle over K, see, e.g., Chapter II, Section 13 of [FD88].

For two AM-modules $\Gamma(K, E)$ and $\Gamma(K, F)$ over $\mathrm{C}(K)$ we equip the Banach space direct sum

$$\Gamma(K, E) \oplus \Gamma(K, F)$$

with the canonical $\mathrm{C}(K)$-module structure and with the norm

$$\|(s_1, s_2)\| := \max(\|s_1\|, \|s_2\|)$$

for $s_1 \in \Gamma(K, E)$, $s_2 \in \Gamma(K, E)$.

By the following proposition, the direct sum $\Gamma(K, E) \oplus \Gamma(K, F)$ of two AM-modules $\Gamma(K, E)$ and $\Gamma(K, F)$ over $\mathrm{C}(K)$ is, again, an AM-module over $\mathrm{C}(K)$.

Proposition A.7. *In the situation above the mapping*

$$\Gamma(K, E) \oplus \Gamma(K, F) \longrightarrow \Gamma(K, E \oplus F), \quad (s_1, s_2) \mapsto s_1 \oplus s_2$$

with $(s_1 \oplus s_2)(x) := (s_1(x), s_2(x))$ *for* $s_1 \in \Gamma(K, E), s_2 \in \Gamma(K, F)$*, and* $x \in K$ *defines an isometric isomorphism of AM-modules.*

Proof. It is obvious that the mapping is a module isomorphism. We also have

$$\begin{aligned}\|s_1 \oplus s_2\| &= \sup_{x\in K}(\max(\|s_1(x)\|, \|s_2(x)\|) \\ &= \max(\sup_{x\in K}\|s_1(x)\|, \sup_{x\in K}\|s_2(x)\|) \\ &= \max(\|s_1\|, \|s_2\|) \\ &= \|(s_1, s_2)\|\end{aligned}$$

for $s_1 \in \Gamma(K, E)$, $s_2 \in \Gamma(K, F)$. Thus, the mapping is isometric. □

The results above yield a correspondence between decompositions of Banach bundles and decompositions of AM-modules.

Proposition A.8. *Let* $\Gamma(K, E)$ *be an AM-module over* $\mathrm{C}(K)$ *corresponding to the Banach bundle* E *over* K. *Then the following assertions hold.*

(i) *If there are two non-trivial Banach subbundles* E_1, $E_2 \subseteq E$ *such that* $E \cong E_1 \oplus E_2$, *then we have that* $\Gamma(K, E_1)$ *and* $\Gamma(K, E_2)$ *are non-trivial Banach submodules of* $\Gamma(K, E)$ *and* $\Gamma(K, E) \cong \Gamma(K, E_1) \oplus \Gamma(K, E_2)$.

(ii) *If there are two non-trivial Banach submodules* Γ_1, $\Gamma_2 \subseteq \Gamma(K, E)$ *such that* $\Gamma(K, E) \cong \Gamma_1 \oplus \Gamma_2$, *then there are two non-trivial Banach subbundles* E_1, $E_2 \subseteq E$ *such that* $E \cong E_1 \oplus E_2$ *and* $\Gamma_1 \cong \Gamma(K, E_1)$ *and* $\Gamma_2 \cong \Gamma(K, E_2)$.

Proof. By Proposition A.4 each Banach submodule corresponds to a Banach subbundle. The assertions then follow by the previous proposition and Theorem 3.3. □

Bibliography

[AAK92] Y. A. Abramovich, E. L. Arenson, and A. K. Kitover, *Banach $C(K)$-Modules and Operators Preserving Disjointness*, Longman, 1992.

[ADR00] C. Anantharaman-Delaroche and J. Renault, *Amenable Groupoids*, L'Enseignement Mathématique, 2000.

[AG84] W. Arendt and G. Greiner, The spectral mapping theorem for one-parameter groups of positive operators on $\mathrm{C}_0(X)$, *Semigroup Forum* 30 (1984), 297–330.

[AH86] W. Arendt and D. R. Hart, The spectrum of quasi-invertible disjointness preserving operators, *J. Funct. Anal.* 68 (1986), 149–167.

[AMR83] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Addison-Wesley, 1983.

[Are83] W. Arendt, Spectral properties of Lamperti operators, *Indiana University Mathematics Journal* 32 no. 2 (1983), 199–215.

[Are87] W. Arendt, Resolvent positive operators, *Proc. London Math. Soc.* s3-54 no. 2 (1987), 321–349.

[Aus88] J. Auslander, *Minimal Flows and their Extensions*, Elsevier, 1988.

[Bir31] G. D. Birkhoff, Proof of the ergodic theorem, *Proc. Natl. Acad. Sci. USA* 17 no. 12 (1931), 656–660.

[BKR17] A. Bátkai, M. Kramar Fijavž, and A. Rhandi, *Positive Operator Semigroups*, Birkhäuser, 2017.

[BP13] L. Barreira and Y. B. Pesin, *Introduction to Smooth Ergodic Theory*, American Mathematical Society, 2013.

[Bro79] I. U. Bronšteĭn, *Extensions of Minimal Transformation Groups*, Sijthoff & Noordhoff, 1979.

[BV19] L. Barreira and C. Valls, Hyperbolicity via evolution semigroups on L^p, *Qualitative Theory of Dynamical Systems* (2019), 1–22.

[CL96] S.-N. Chow and H. Leiva, Unbounded perturbation of the exponential dichotomy for evolution equations, *J. Differential Equations* 129 (1996), 509–531.

[CL99] C. Chicone and Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, American Mathematical Society, 1999.

[Cun67] F. Cunningham, M-structure in Banach spaces, *Proc. Camb. Phil. Soc.* 63 (1967), 613–629.

[Den05] M. Denker, *Einführung in die Analysis dynamischer Systeme*, Springer-Lehrbuch, Springer, 2005.

[Der79] R. Derndinger, "Über das Spektrum positiver Generatoren". Dissertation. Universität Tübingen, 1979.

[DG83] M. J. Dupré and R. M. Gillette, *Banach Bundles, Banach Modules and Automorphisms of C*-Algebras*, Longman, 1983.

[Dix77] J. Dixmier, *C*-Algebras*, North-Holland, 1977.

[dVr93] J. de Vries, *Elements of Topological Dynamics*, Springer, 1993.

[Ede20] N. Edeko, On equicontinuous factors of flows on locally path-connected compact spaces, *Ergodic Theory Dynam. Systems* (2020). To appear.

[EFHN15] T. Eisner, B. Farkas, M. Haase, and R. Nagel, *Operator Theoretic Aspects of Ergodic Theory*, Springer, 2015.

[Eis10] T. Eisner, *Stability of Operators and Operator Semigroups*, Birkhäuser, 2010.

[EK20] N. Edeko and H. Kreidler, *Uniform enveloping semigroupoids for groupoid actions*. Preprint. 2020.

[Ell69] R. Ellis, *Lectures on Topological Dynamics*, Benjamin, 1969.

[Ell87] R. Ellis, Topological Dynamics and Ergodic Theory, *Ergodic Theory Dynam. Systems* 7 (1987), 25–47.

[EN00] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, 2000.

[EN06] K.-J. Engel and R. Nagel, *A Short Course on Operator Semigroups*, Universitext - Springer-Verlag, Springer, 2006.

[FD88] J. M. G. Fell and R. S. Doran, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles. Volume I. Basic Representation Theory of Groups and Algebras*, Academic Press, 1988.

[FH19] T. Fisher and B. Hasselblatt, *Hyperbolic Flows*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2019.

[Gie82] G. Gierz, *Bundles of Topological Vector Spaces and Their Duality*, Springer, 1982.

[Gla03] E. Glasner, *Ergodic Theory via Joinings*, American Mathematical Society, 2003.

[Gut93a] A. E. Gutman, Banach bundles in the theory of lattice normed spaces. I. Continuous Banach bundles, *Siberian Adv. Math.* 3 (1993), 1–55.

[Gut93b] A. E. Gutman, Banach bundles in the theory of lattice normed spaces. II. Measurable Banach bundles, *Siberian Adv. Math.* 4 (1993), 8–40.

[Hey15] R. Heymann, "Multiplication Operators on Bochner Spaces and Banach Fibre Spaces". Dissertation. Universität Tübingen, 2015.

[HoKe17] K. H. Hofmann and K. Keimel, "Sheaf theoretical concepts in analysis: Bundles and sheaves of Banach spaces, Banach $C(X)$–modules". *Applications of Sheaves*. Ed. by M. Fourman, C. Mulvey, and D. Scott. 1977, 415–441.

[HoKr18] B. Host and B. Kra, *Nilpotent Structures in Ergodic Theory*, American Mathematical Society, 2018.

[HP57] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Revised. American Mathematical Society, 1957.

[Huy07] N. T. Huy, Existence and robustness of exponential dichotomy of linear skew-product semiflows over semiflows, *J. Math. Anal. Appl.* 333 (2007), 731–752.

[JPS87] R. A. Johnson, K. J. Palmer, and G. R. Sell, Ergodic properties of linear dynamical systems, *SIAM J. Math. Anal.* 18 no. 1 (1987), 1–33.

[Kai78] S. Kaijser, Some representation theorems for Banach lattices, *Ark. Mat.* 16 (1978), 179–193.

[Kat80] T. Kato, *Perturbation Theory for Linear Operators*, 2nd edition. Vol. 132 of Grundlehren Math. Wiss. Springer, 1980.

[Kna67] A. W. Knapp, Distal functions on groups, *Trans. Amer. Math. Soc.* 128 (1967), 1–40.

[Koo31] B. O. Koopman, Hamiltonian systems and transformation in Hilbert space, *Proc. Natl. Acad. Sci. USA* 17 no. 5 (1931), 315–318.

[Kre19] H. Kreidler, "Contributions to the Koopman Theory of Dynamical Systems". Dissertation. Universität Tübingen, 2019.

[KS20] H. Kreidler and S. Siewert, Gelfand-type theorems for dynamical Banach modules, *Math. Z.* (2020). doi: `https://doi.org/10.1007/s00209-020-02560-2`.

[Kus00] A. G. Kusraev, *Dominated Operators*, Springer, 2000.

[Küh19] V. Kühner, What can Koopmanism do for attractors in dynamical systems?, *The Journal of Analysis* (2019).

[Küs20] K. Küster, Decompositions of dynamical systems induced by the Koopman operator, *Analysis Mathematica* (2020). To appear.

[Lan95] S. Lang, *Differential and Riemannian Manifolds*, 3rd edition. Springer, 1995.

[Lee13] J. M. Lee, *Introduction to Smooth Manifolds*, 2nd edition. Springer Graduate Texts in Mathematics, 2013.

[Lee18] J. M. Lee, *Introduction to Riemannian Manifolds*, 2nd edition. Springer Graduate Texts in Mathematics, 2018.

[LS06] Y. Latushkin and A. Stepin, "Linear skew-product flows and semigroups of weighted composition operators". Vol. 1486. 2006, 98–111.

[LS99] Y. Latushkin and R. Schnaubelt, Evolution semigroups, translation algebras, and exponential dichotomy of cocycles, *J. Differential Equations* 159 (1999), 321–369.

[Ma87] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, trans. by S. Levy. Springer, 1987.

[Mac98] S. MacLane, *Categories for the Working Mathematician*, 2nd edition. Springer, 1998.

[Mei07] J. Meiss, *Differential Dynamical Systems*, Society for Industrial and Applied Mathematics, 2007.

[MN91] P. Meyer-Nieberg, *Banach Lattices*, Springer, 1991.

[MW74] G. Mittelmeyer and M. Wolff, Über den Absolutbetrag auf komplexen Vektorverbänden, *Math. Z.* 137 (1974), 87 –92.

[Nag86] R. Nagel, ed., *One-parameter Semigroups of Positive Operators*, Springer, 1986.

[Nic97] G. Nickel, Evolution semigroups for nonautonomous Cauchy problems, *Abstr. Appl. Anal.* 2 (1997), 73–95.

[Par08] W. Paravicini, A note on Banach $C_0(X)$-modules, *Münster J. Math.* 1 (2008), 267–278.

[PM82] J. Palis and W. de Melo, *Geometric Theory of Dynamical Systems: An Introduction*, Springer, 1982.

[Rau94] R. Rau, Hyperbolic evolution semigroups on vector valued function spaces, *Semigroup Forum* 48 (1994), 107–118.

[Ree80] M. Rees, Tangentially distal flows, *Isr. J. Math.* 35 no. 1–2 (1980), 9–31.

[RRS96] F. Räbiger, A. Rhandi, and R. Schnaubelt, Perturbation and an abstract characterization of evolution semigroups, *Journal of Mathematical Analysis and Applications* 198 no. 2 (1996), 516–533.

[Sch74] H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer, 1974.

[Sma67] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* 73 no. 6 (1967), 747–817.

[Spi99] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, 3rd edition. Vol. 2 of Publish or Perish, Inc., 1999.

[SS74] R. Sacker and G. Sell, Existence of dichotomies and invariant splittings for linear differential systems, I, *J. Differential Equations* 15 no. 3 (1974), 429–458.

[SS76a] R. Sacker and G. Sell, Existence of dichotomies and invariant splittings for linear differential systems, II, *J. Differential Equations* 22 no. 2 (1976), 478–496.

[SS76b] R. Sacker and G. Sell, Existence of dichotomies and invariant splittings for linear differential systems, III, *J. Differential Equations* 22 (1976), 497–522.

[SS78] R. Sacker and G. Sell, A spectral theory for linear differential systems, *J. Differential Equations* 27 (1978), 320–358.

[Tao09] T. Tao, *Poincaré's Legacies, Part I*, American Mathematical Society, 2009.

[Tro59] H. Trotter, On the product of semi-groups of operators, *Proc. Amer. Math. Soc.* 10 (1959), 545–551.

[Uhl86] H. Uhlig, The eigenfunctions of compact weighted endomorphisms of C(X), *Proc. Amer. Math. Soc.* 98 no. 1 (1986), 89–93.

[vNe32a] J. von Neumann, Proof of the quasi-ergodic hypothesis, *Proc. Natl. Acad. Sci. USA* 18 no. 1 (1932), 70–82.

[vNe32b] J. von Neumann, Zur Operatorenmethode in der klassischen Mechanik, *Ann. of Math.* 33 no. 3 (1932), 587–642.