Geometric necks in mean curvature flow of 2-convex hypersurfaces

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Deutsche Zusammenfassung

Der sogenannte mittlere Krümmungsfluss ist eine geometrische Deformation von Mannigfaltigkeiten, im speziellen von Hyperflächen. Er kann als natürliche geometrische Verallgemeinerung der klassischen Wärmeleitungsgleichung aufgefasst werden. Wir betrachten dazu eine Startfläche, gegeben durch eine Immersion

$$F_0 : \mathcal{M}^n \to \mathbb{R}^{n+1}$$

einer glatten n-dimensionalen Hyperfläche im Euklidischen Raum. Dann suchen wir eine Familie von glatten Hyperflächen $F : \mathcal{M}^n \times [0, T) \to \mathbb{R}^{n+1}$ die folgendes Gleichungssystem löst

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} F(p,t) = -H(p,t)\nu(p,t) = \Delta_t F(p,t) & \text{for all } (p,t) \in \mathcal{M}^n \times [0,T) \\ \\ F(p,0) = F_0(p) & \text{for all } p \in \mathcal{M}^n. \end{cases}$$

Solch eine Familie nennen wir dann Lösung des mittleren Krümmungsflusses. Es ist bekannt, dass dieser Fluss schon für einfache Beispiele in endlicher Zeit Singularitäten bilden kann. Das heißt, wir können den Fluss nicht mehr glatt fortsetzen, da die Krümmung unendlich groß wird. Unter der speziellen Konvexitätsannahme der 2-Konvexität an die Startfläche F_0 lässt sich die Existenz von "neck"-Singularitäten beweisen, auch degeneriertes "neck pinching" genannt, welche die approximative Form eines Zylinders annehmen. Diese Singularitäten sind von besonderem Interesse, da sie die Grundlage des Chirurgie-Algorithmus bilden, mit dem der Fluss über die Singularitäten hinaus fortgesetzt werden kann. Dazu muss aber die Geometrie eben dieser "Necks" gut analysiert werden. In der folgenden Arbeit beschäftigen wir uns mit der Tatsache, dass solche Zylinder-ähnlichen Gebiete sich unter dem Fluss glätten und sich der Form des runden Zylinders annähern. Dieses Phänomen wird auch "neck"-Verbesserung genannt. Um ein solches Resultat zu beweisen, müssen wir Abschätzungen für diverse geometrische Größen, wie Krümmung oder Gradienten von Krümmung, herleiten. Die wichtigsten Abschätzungen dazu sind die klassischen Konvexitäts-, zylindrischen und Gradienten-Abschätzungen, die auf Huisken und Sinestrari [HS09] zurückgehen. In der vorliegenden Arbeit versuchen wir diese Abschätzungen zu verbessern, in dem wir sie speziell nur auf zylindrischen Gebieten betrachten. Außerdem werden wir eine spezielle Parametrisierung dieser Singularitäten definieren, die es uns erlaubt die "Rundheit" der Region anhand der "Rundheit" von Querschnitten des Zylinders zu analysieren. Ein Resultat wird sein, dass wir solch eine Parametrisierung mit Hilfe einer Bewegung innerhalb des approximativen Zylinders in der Zeit verfolgen können und dabei die Eigenschaft, dass die Querschnitte konstante mittlere Krümmung haben, erhalten bleibt. Dabei leiten wir Evolutionsgleichungen für die Bewegung der Querschnitte kombiniert mit dem umgebenen mittleren Krümmungsfluss in einer allgemeinen Form her, sodass sie potentiell auch für anderen kombinierte Flüsse Verwendung finden können. Ausgehend von der punktweisen "Neck"-Verbesserung, die wir

aus den obigen Abschätzungen erhalten, können wir dann eine Aussage über die gesamte Parametrisierung machen und erhalten somit ein "Neck"-Verbesserungs Resultat. Dieses besagt, dass wir zu fixen $\varepsilon > 0$ und jedem kleinen $\varepsilon > \delta > 0$ ein großes Θ finden können, sodass ein "Neck", das ε nahe beim Runden Zylinder liegt und sich in einem Zeitintervall $[-\Theta, 0]$ unter dem mittleren Krümmungsfluss bewegt, bereits δ-rund ist auf einem kleineren Zeitintervall.

Chapter 1

Introduction

Geometric flows, such as the mean curvature flow, have been of great interest in mathematical research not only in the past years. Being the intersection between various fields in theoretical mathematics, their research has been fruitful as well as challenging and their applications have been enormous.
One of the major achievements in the applications of geometric flows has been Perelmans proof of Thurstons geometrization conjecture [BC10] via the Ricci flow, which has been initiated by Richard Hamilton [Ham82]. This particular flow deforms Riemannian metrics by their Ricci curvature and has many similarities with the behavior of the classical heat equation. In this thesis, however, we are interested in a deformation of hypersurfaces, the so called mean curvature flow.
The mean curvature flow is designed as a natural geometric analogue for the heat equation for surfaces. For a smooth closed hypersurface immersion $F_0 : \mathcal{M}^n \to \mathbb{R}^{n+1}$ we are looking to find a one parameter family $F : \mathcal{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ of smooth immersions such that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} F(p,t) = -H(p,t)\nu(p,t) = \Delta_t F(p,t) & \text{for all } (p,t) \in \mathcal{M}^n \times [0,T) \\ \\ F(p,0) = F_0(p) & \text{for all } p \in \mathcal{M}^n \end{cases} \tag{1.1}$$

where $H(p,t)$ is the mean curvature of the surface $\mathcal{M}_t^n := F(\mathcal{M}^n, t)$ at the point $F(p,t)$ with respect to the outer unit normal $\nu(p,t)$ at $F(p,t)$ and Δ_t is the corressponding Laplace-Beltrami operator on $\mathcal{M}_t^n$. Then we call $\mathcal{M}_t^n$ a solution to mean curvature flow with initial data $\mathcal{M}_0 := F_0(\mathcal{M}^n)$. Since mean curvature flow defines a parabolic quasilinear system of equations up to tangential diffeomorphisms, smooth short time existence follows from classical PDE theory. Long time existence however can not be expected as can be seen by an easy example. We consider the sphere $F_0(\mathcal{M}^n) = \mathbb{S}^n_{R_0}$ of radius R_0 as initial data. Then solving mean curvature flow is equivalent to solving a first order ODE, namely

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} R(t) = -\frac{n}{R(t)} \\ \\ R(0) = R_0 \end{cases}$$

with solution $R(t) = \sqrt{R_0^2 - 2nt}$. This shows that the solution becomes extinct, meaning it converges to a "round point" as $t \to T := R_0^2(2n)^{-2}$ which is the maximal existence time. This is shown in Figure 1.1.

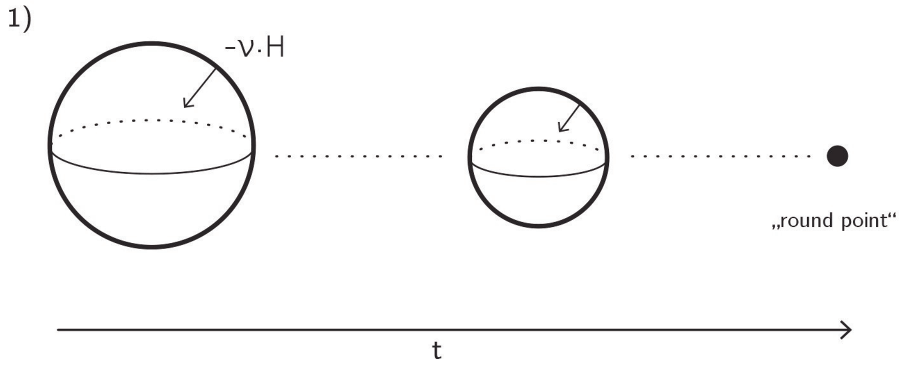

Figure 1.1: Shrinking sphere

Also at this time the total curvature $|A|^2 = nR(t)^{-2}$ becomes infinite and thus we cannot extend the flow smoothly any further. We say that the flow develops a singularity at $t = T$. For any compact closed initial data F_0 we can use the sphere as a barrier. A standard PDE comparison principle then yields that any such compact solution must develop a singularity in finite time. In general, Huisken [Hui84, Theorem 3.3] proved that if $T < \infty$ is the first singular time of a compact solution $(\mathcal{M}_t)_t$ to mean curvature flow then $\sup_{\mathcal{M}_t} |A| \to \infty$ as $t \to T$.

We can not expect to classify and or analyze solutions and their singularities in full generality. Thus it is somehow natural to assume additional geometric properties of the initial data. For example different notions of convexity have been studied but also other geometric assumptions such as local non-collapsing. The classical results in view of characterizing solutions and their singularities are the following. Huisken [Hui84] proved that any convex solution to (1.1) shrinks to a point in finite time and after rescaling looks like the shrinking sphere solution from the example above. Other results in this direction are due to Gage and Hamilton [GH86], and Grayson [Gra87] where they show that a convex closed curve as well as an embedded curve respectively in the plane shrinks to a point and becomes round. This gives a full characterization of what can happen to closed embedded curves in the plane under mean curvature flow which is also called curve shortening flow for $n = 1$. A typical question would then be, whether all singularities look like the one of the sphere or what other types of singularities are possible. One possible way in order to characterize singularities is to distinguish between different rates of blow-ups. Roughly speaking Type I singularities are those that resemble the blow-up rate of the sphere and those that do not are the singularities of Type II. Once the singularity type is known, we can perform a parabolic rescaling and look at possible limiting flows. This boils down to classify so called ancient solutions of mean curvature flow, i.e. solutions that exist on some time interval $(-\infty, T)$. Huisken [Hui90] showed that a Type I singularity of the mean curvature is asymptotically self-similar. Also a smooth limiting hypersurface with non negative mean curvature must be one of the following surfaces [Hui93]. Either a sphere $\mathbb{S}^n$ or a cylinder $\mathbb{S}^{n-k} \times \mathbb{R}^k$ or $\Gamma \times \mathbb{R}^{n-1}$ where Γ is one of the Abresch Langer curve solutions to curve shortening flow [AL86]. Recently Huisken and Sinestrari were able to prove several necessary and sufficient conditions for convex ancient solutions to be shrinking spheres [HS15].

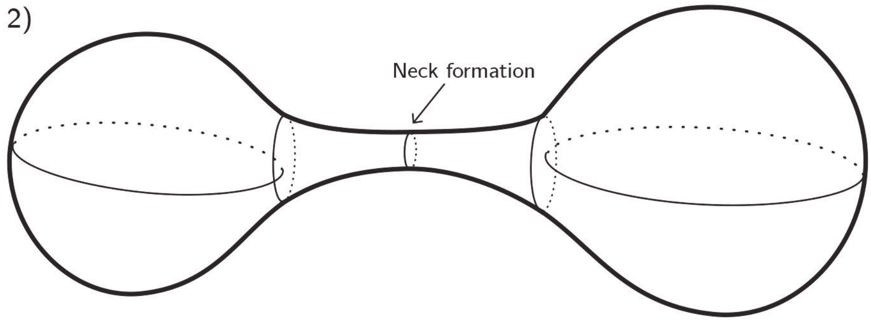

Figure 1.2: Neck formation

One phenomenon that is also present in the Ricci flow is the so called neck formation. For example, if we imagine our initial surface to look like two large balls connected with a thin tube like the sketch in Figure 1.2 shows, we expect the tube to "shrink" off before the balls collapse to a point. This is called the standard degenerate neckpinch. A rigorous construction of such a solution to the flow can be found in [Gra89] where he constructs the initial surface as a surface of revolution. Angenent formally proved the existence of such a neckpinch situation [AV97]. Once singularity profiles are established and understood the aim is to extend the flow beyond those singular points in space time. There are essentially two different approaches to this type of problem. Similar to the theory of PDE we can hope to find a notion of weak solutions to the flow. This has been done by Evans and Spruck in the so called level set approach, see [ES91], [ES92b] and [ES92a]. This technique however heavily depends on the evolution we are studying and has to be developped for each flow separately.

A different and more flexible approach which has been pioneered by Richard Hamilton for the Ricci flow [Ham97] and then later successfully applied to prove the Poincare conjecture by Perelman [Per03] [Per02], see for example [WMT07] for an exposition, is the so called surgery procedure. This construction has an analogue for mean curvature flow solutions that satisfy a certain convexity condition, namely two convexity. Huisken and Sinestrari in [HS09] succeeded in adapting the surgery procedure to mean curvature flow and thereby found a classification of all 2-convex solutions in the Euclidian space for $n \geq 3$. The case $n = 2$ [BH16] is different in the sense that some of the major estimates that make the surgery procedure possible do not carry over. Here 2-convexity is the same as mean convexity such that there is no extra information from this assumption. We will just give a brief sketch here how the surgery is done. For more details we refer to Chapter 3 and [HS09, Chapter 3 and 8]. As the word surgery suggests, we want to cut out "ill" regions of the surface,these are those regions where singularities are about to occur and the curvature will become very large. The first step is to detect these regions and to prove that the curvature blow-up only happens in regions that form a neck which means they resemble the geometry of a cylinder or in connected components of which we already know the shape of. Then we fix a curvature threshold H_0 determining when we start to do the surgery. Then once we found a region with curvature $H \geq H_0$ and an approximate cylinder structure we cut out the region of the cylinder where the curvature is very high and replace the ends with two convex caps. This is done in a way that the maximal curvature reduces to

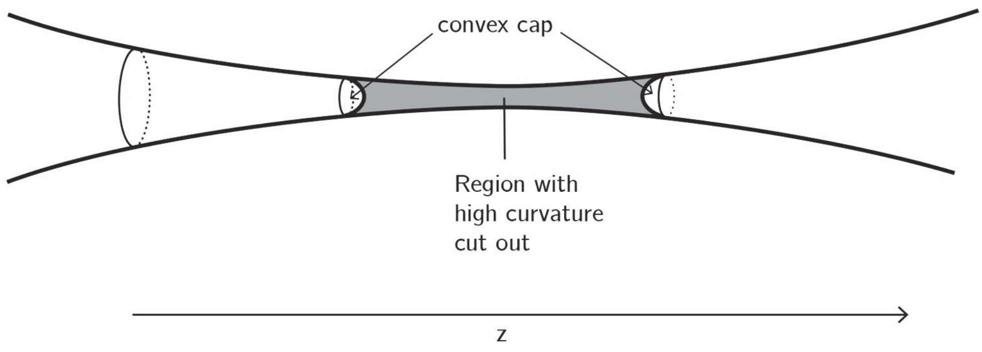

Figure 1.3: Surgery

roughly $\frac{H_1}{2}$ where $H_1 = \frac{H_0}{1000}$. Afterwards connected components whose topology is known are discarded, for examples spheres.

This procedure is indicated in Figure 1.3. It turns out that this procedure is controlled by a couple of parameters that only depend on the structure of the initial surface one of which is the curvature threshold H_0 and it ends in finitely many steps. The assumption that of 2- convexity is key here, since some of the major estimates that make the surgery construction possibly heavily rely on it. In particular, on two convex surfaces we have $H^2 \geq \frac{1}{n}|A|^2$ such that a curvature blow up near a singularity automatically implies a blow up in the mean curvature. In general this is unknown but conjectured to be true. We refer to [Hui90],[HS99b] and [LS16] for some advances towards that matter.
At first glimpse the level set approach and the surgery construction look entirely different, however there is a nice connection between the two. Head [Hea13] and Lauer [Lau10] simultaneously and independently proved that if we let the curvature threshold tend to infinity the flow actually converges to the level set flow by Evans and Spruck.

In this thesis we will mainly focus on neck-like singularities which appear naturally in mean curvature flow of 2-convex surfaces. As we have said before neck regions play a crucial role in the surgery construction. It is, thus, very useful to develop a regularity and stability theory for those regions. We know by the convergence result in [Hui84] that convex surfaces become more and more round as they approach their extinction time. Since each of the cross sections of a necklike singularity is also convex and close to a sphere, we expect a similar global smoothing behavior in our case. It was unclear whether this is actually true, when the surgery procedure was introduced. We will call this phenomenon neck improvement. To have a quantitative way to measure this improvement would not only have various applications in the classification of solutions and their singularities but could also make the surgery construction significantly easier. This is not only interesting in view of understanding the singularity behavior better but might also be necessary to generalize the surgery construction to weaker convexity assumptions. A qualitative result towards neck improvement is the canonical neighborhood theorem by Haselhofer [HK17, Theorem 4.1]. A first quantitative result this direction for mean curvature flow has been obtained by Brendle and Choi [BC19] ($n \geq 3$) and [BC17] ($n = 2$), where they showed that indeed large enough regions that have been ε close to the standard cylinder in a suitable C^k topology for a long enough time under mean curvature flow are in fact already $\frac{\varepsilon}{2}$ close if we wait long enough. It is interesting that a similar analysis carries over to the Ricci Flow case which was observed by Brendle [Bre19b] [Bre19a]. There are essentially two different ways to provide such a analysis. Brendle used an ODE type argument where he analyzes the ODE part of the reaction diffusion equation governing mean curvature flow. This is done by an analysis of the Jacobian operator and its kernel on the surface in combination with the linearization of the flow. The same techniques have been employed by Hartley [Har13] to show that cylindrical graphs that form an ancient solution and are sufficiently close to the standard cylinder actually converge exponentially to the standard cylinder as time passes. In this thesis we will explicitly use the geometric structure of necklike regions in order to obtain localized and improved estimates of all relevant geometric quantities. In comparison to the results mentioned above we are looking to find explicit quantitative estimates that provide us with information on how long we actually have to wait until certain geometric quantities have improved to a certain factor.

In the first Chapter we will introduce some basic notations and facts about general submanifold geometry. Chapter 2 is dedicated to the evolution equations that govern the flow. In particular, we will look at how the submanifold geometry changes whenever we let the ambient manifold flow. This will be needed to understand the behavior of the necklike regions and their crosssections better. In chapter 3 we will define these regions precisely and show how they can be parameterized by a certain foliation via constant mean curvature surfaces. Here we will follow the presentation of [HS09] and [Ham82]. The three key estimates that are used to control the surgery procedure, i.e. the convexity, the cylindrical and the gradient estimate, will be localized and improved in chapter 4. More precisely, we will be able to derive new quantitative estimates that show how old necklike solutions, i.e. solutions to the flow that have a neck geometry for a long time, become more round and closer to the standard shrinking cylinder. Particularly, we obtain a new estimate on the first principle curvature from above. The proof of the new improved convexity estimate includes a classification result for ancient solutions that have bounded symmetric polynomials in the principle curvatures from below. In particular, we can say that an ancient necklike solution has to be the standard shrinking cylinder. Finally, from these improved results we can proof that for fixed ε small any $\delta > 0$ as small as we like we can find a large time $\theta > 0$ depending on δ such that if a ε-necklike solution existed since $-\theta$ it will be a δ-neck like solution on a smaller time intervall. Here we can also make the smaller time intervall as large as we like just be choosing θ larger.
In chapter 5 we will fix an initial parameterization of a necklike region by constant mean curvature surfaces. In order to be able analyze how these surfaces become more and more "round" in time we have to find a way to be able to follow them along the flow in a well defined way. For this reason we construct a normal movement within the neck in such a way that we can guarantee that the crosssections still have constant mean curvature by taking account for the ambient geometry changes with this interior movement. In this way, we make sure that at any time under consideration our foliation which we obtained by following the initial foliation through time will still be a constant mean curvature one. In this context, we will also state the evolution equation for this coupled movement. We will be able to present the evolution equations in a very general setting such that they can potentially be applied to different settings, for example mean curvature flow in a Ricci flow background [Lot12]. Then we will use an implicit function theorem argument to construct a speed for this interior movement. Finally, we will prove a pointwise roundness estimate

on the second fundamental forms of these interior surfaces.

1.1 Basic definitions

Let $F : \mathcal{M}^n \to \mathbb{R}^{n+1}$ be a smooth immersion of an n-dimensional hypersurface into the $n+1$ dimensional Euclidian space. For any point $q \in F(\mathcal{M}^n)$ and local coordinate system $x = (x_1, \ldots, x_n)$ around q the tangent vectors

$$\frac{\partial F}{\partial x_i}; i = 1, \ldots, n$$

form a Basis of $T_q F(\mathcal{M}^n)$. To make the notations more simple we will from now on omit to differ between $\mathcal{M}^n$ and $F(\mathcal{M}^n)$ by an abuse of notation. Let us denote by δ_{Eucl} the standard metric on $\mathbb{R}^{n+1}$. Then the pullback metric via F of δ, or equivalently the induced metric on $\mathcal{M}^n$, is given by $g := F_* \delta$ which we can write as

$$g_{ij}(p) = \delta_{\text{Eucl}} \left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right) = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle_{\mathbb{R}^{n+1}}$$

for $i, j = 1 \ldots, n$ in local coordinates around $p \in \mathcal{M}^n$. The inverse of the metric will be denoted by $g_{ij}^{-1} =: g^{ij}$. The area element of the surface can then be defined as

$$\mathrm{d}\mu := \sqrt{\det(\mathrm{g_{ij}})}\, \mathrm{d}x.$$

We will also denote by $\nu(p)$ the outward unit normal to the hypersurface $\mathcal{M}^n$ such that $\langle \nu(p), \frac{\partial F}{\partial x_i} \rangle = 0$ for all $i = 1, \ldots, n$. This choice of the unit normal vectorfield induces the second fundamental form A on $\mathcal{M}^n$ which again in local coordinates around a point p can be written as

$$h_{ij}(p) = \left\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle_{\mathbb{R}^{n+1}} = - \left\langle \nu, \frac{\partial^2 F}{\partial x_i \partial x_j} \right\rangle_{\mathbb{R}^{n+1}}$$

for $i, j = 1, \ldots, n$. By taking the trace with respect to the induced metric g we obtain the mean curvature

$$H = g^{ij} h_{ij}.$$

Another important geometric tensor on $\mathcal{M}^n$ is the so called shape operator or Weingarten map $W_p : T_p\mathcal{M}^n \to T_p\mathcal{M}^n$ which is given by $W_p(x) = -\,\mathrm{d}_p \nu(x)$ for $x \in T_p\mathcal{M}$ which we can obtain by raising one index in the second fundamental form

$$h^i_j = g^{ik} h_{kj}$$

for $i, j = 1, \ldots, n$. In general we use the Einstein convention of taking the sum over repeated indices such that the above is actually a sum over k from 1 to n. The principle curvatures of $\mathcal{M}^n$ are the Eigenvalues of W which we denote by

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

For the analysis of mean curvature flow another geometric quantity is very important, namely the squared norm of the second fundamental form which is given by

$$|A|^2 = g^{ik} g^{jl} h_{ij} h_{kl} = \sum_{i=1}^{n} \lambda_i^2.$$

From these definitions we can easily derive the following very useful identify

$$|A|^2 - \frac{1}{n} H^2 = \frac{1}{n} \sum_{i<j} (\lambda_i - \lambda_j)^2 \geq 0.$$

In particular, if we denote by $\overset{\circ}{A}$ the traceless part of A given by

$$\overset{\circ}{h}_{ij} := h_{ij} - \frac{H}{n} g_{ij},$$

then we can easily see that $|\overset{\circ}{A}|^2 = |A|^2 - \frac{1}{n}H^2 = 0$ if and only if $\lambda_i = \lambda_j$ for all i and j. Let $u : \mathcal{M}^n \to \mathbb{R}$ be a smooth function. Then its tangential gradient on $\mathcal{M}^n$ is given by

$$\nabla u = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial F}{\partial x_j}.$$

Similarly, for a smooth vectorfield $X = \{X^i\}$ on $\mathcal{M}^n$ we define its covariant derivative to be

$$\nabla X^j = \frac{X^i}{\partial x_j} + \Gamma^j_{ik} X^k$$

and for a 0-2 tensor T with components T_{ij}.

$$\nabla_i T_{kl} = \frac{\partial T_{kl}}{\partial x_i} - \Gamma^m_{ik} T_{ml} - \Gamma^m_{il} T_{lm}$$

where the Christoffel-symbols Γ can be computed via

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right).$$

In particular, the Codazzi equations tell us that the covariant derivative of A is totally symmetric, in coordinates:

$$\nabla_i h_{jk} = \nabla_k h_{ij} = \nabla_j h_{ik}.$$

The Riemannian curvature tensor is now given by the Gauss equations

$$\mathrm{Rm}_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}$$

for all $1 \leq 1, j, k, l \leq n$. By taking the trace in the first and third entry we obtain the Ricci tensor and by taking the trace once more we get the scalar curvature

$$\mathrm{Ric}_{ik} = H h_{ik} - h_{il} g^{lj} h_{jk}, \quad R = H^2 - |A|^2.$$

When we exchange second covariant derivatives of a vector $X = \{X^k\}$ and a co-vector $Y = \{Y_k\}$ respectively the Riemannian curvature tensor comes into play

$$\nabla_i \nabla_j Y_k - \nabla_j \nabla_i Y_k = \mathrm{Rm}_{ijkl} g^{lm} Y_m$$
$$\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = \mathrm{Rm}^k_{ijm} X^m.$$

Finally the Laplace-Beltrami operator of a tensor is given by

$$\Delta T^i_{jk} = g^{kl} \nabla_k \nabla_l T^i_{jk}.$$

For a $(2,0)$ tensor T, we further have

$$\nabla_i \nabla_j T_{kl} - \nabla_j \nabla_i T_{kl} = -\mathrm{Rm}_{ijk}{}^m T_{ml} - \mathrm{Rm}_{ijl}{}^m T_{km} \tag{1.2}$$

see for example [BC10, Chapter 1.3.3]. Last but not least we will also state the commutator identity for the second fundamental form which we will prove in the upcoming section for a more general setting.

Proposition 1.1 (Simons Identity)**.**

$$\Delta h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij} \tag{1.3}$$

$$\frac{1}{2}\Delta |A| = g^{li} g^{mj} h_{lm} \nabla_i \nabla_j H + |A|^2 + H \mathrm{tr}_{\mathcal{M}}(A) - |A|^4. \tag{1.4}$$

Here we will also write

$$g^{li} g^{mj} h_{lm} \nabla_i \nabla_j H = \langle h_{ij}, \nabla_i \nabla_j H \rangle$$

to make things easier to read.

1.2 Some Submanifold Geometry

We consider a submanifold $\Sigma^{n-1} \subset \mathcal{M}$ of co-dimension 1. We can split the tangent space of $\mathcal{M}$ at any point $p \in \Sigma$ with the help of the metric g on $\mathcal{M}$ into components that are tangential on Σ^{n-1} and into those that are normal to Σ, that is

$$T_p \mathcal{M} = T_p \Sigma^{n-1} \oplus \left(T_p \Sigma^{n-1}\right)^{\perp}.$$

We will choose a normal direction $\eta(p)$ such that $\left(T_p \Sigma^{n-1}\right)^{\perp} = \mathrm{Span}(\eta(p))$. In this way the covariant derivative ∇ on $\mathcal{M}$ induces a covariant derivative ∇^{Σ} defined by

$$\nabla^{\Sigma}_X Y(p) := \pi^T (\nabla_X Y(p)) := \nabla_X Y(p) - g(p) \left(\nabla_X Y(p), \eta(p)\right) \eta(p)$$

for any vectors $X(p), Y(p)$ in $T_p \Sigma^{n-1}$ and any $p \in \Sigma^{n-1}$, where

$$\pi^T : T_p \mathcal{M} \to T_p \Sigma^{n-1}$$

is the projection onto the tangent space of Σ. In order to be really precise here, we should also extend the vector fields X, Y locally to give sense to $\nabla_X Y$, which does not depend on the choice of the extension. This connection agrees with the induced connection by the restriction of the metric on $T_p \Sigma^{n-1}$ and the difference between the two connections gives rise to the second fundamental form l on Σ^{n-1} via

$$l(X, Y)(p)\eta(p) := \nabla^{\Sigma}_X Y(p) - \nabla_X Y(p).$$

If $Y = \eta$, however we need a different connection since $\nabla^{\Sigma}_X \eta(p)$ is not well defined. We look at the normal projection

$$\pi^{\perp} : T_p \mathcal{M} \to \left(T_p \Sigma^{n-1}\right)^{\perp}$$

which induces a connection on the normal bundle of Σ^{n-1} by

$$\nabla^{\perp}_Z \eta(p) := \pi^{\perp} \left(\nabla_Z \eta(p)\right) = \nabla_Z \eta(p) - \pi^T (\nabla_Z \eta)(p).$$

Of course in our case when the co-dimension is 1 we notice that since $g(\eta, \eta) = 1$, we have that $\nabla_X \eta(p) \in T_p \Sigma^{n-1}$ and therefore $\nabla^{\perp}_X \eta = 0$ for all vector fields X on Σ^{n-1}. In order to be able to write coordinate expressions we will use local coordinates $x = (x_1, \ldots, x_n)$ around a point $p \in \Sigma^{n-1}$ such that for $a = 1, \ldots, n-1$ $\frac{\partial}{\partial x_a}$ form a basis of $T_p \Sigma^{n-1}$ and $\frac{\partial}{\partial x_n} = \eta$ corresponds to the normal direction to Σ^{n-1} in $\mathcal{M}^n$. We will need formulas that relate the ambient covariant derivative for tensors with the one on Σ^{n-1}. For a derivation of the following formulas in full generality including a proof of Proposition 1.1 we refer to [SSY75, Formulae (1.14) and (1.19)].

$$\nabla^\Sigma_a \mathrm{Rm}_{nbcd} = \nabla_a \mathrm{Rm}_{nbcd} - l_{ac}\mathrm{Rm}_{nbnd} - l_{ad}\mathrm{Rm}_{nbcn} + l_a{}^e \mathrm{Rm}_{ebcd}. \tag{1.5}$$

Here the curvature tensor $\nabla_m \mathrm{Rm}_{ijkl}$ on $\mathcal{M}^n$ is restricted to the submanifold Σ^{n-1}. The formula basically accounts for the mistake we make when we only consider the tangential covariant derivative for $\mathrm{Rm}(\eta, \cdot, \cdot, \cdot)$. By taking the trace in the second and last argument we obtain the formula for the Ricci curvature.

$$\nabla^\Sigma_a \mathrm{Ric}_{nb} = \nabla_a \mathrm{Ric}_{nb} - l_{ab}\mathrm{Ric}_{nn} + l^c_a \mathrm{Ric}_{cb}. \tag{1.6}$$

Note that we only take the trace on Σ and therefore we get correction terms in normal direction of each individual term of (1.5). However, due to the symmetries of Rm they all either vanish or cancel out. We will frequently use the Gauß-Codazzi-Mainardi equations between Σ and the ambient manifold $\mathcal{M}$ as well as $\mathcal{M}$ and the ambient Euclidian space, which we stated above, in coordinate expressions

$$\begin{aligned} \nabla^\Sigma_a l_{bc} &= \nabla^\Sigma_c l_{ab} + \mathrm{Rm}_{nbca} \\ \mathrm{Ric}_{nc} &= \nabla^\Sigma_b l_{bc} - \nabla^\Sigma_c L \end{aligned}$$

where as usual a, b, c run from 1 to $n-1$. By exchanging second covariant derivatives of h_{ij} we can further obtain the following commutator identities which are slightly more complicated than the ones with Euclidian ambient space. We will use the notation Rm^Σ for the Riemmanian curvature tensor on Σ^{n-1}.

Proposition 1.2 (Simons' identities)**.**
Let $\Sigma^{n-1} \subset \mathcal{M}^n$ be a smooth hypersurface in an n dimensional Riemannian manifold. Then we have the following identities

$$\begin{aligned} \nabla^\Sigma_a \nabla^\Sigma_b l_{cd} &= \nabla^\Sigma_c \nabla^\Sigma_d l_{ab} + l_{ab} l_c{}^e l_{ed} - l_a{}^e l_{cb} l_{ed} + l_{ad} l_c{}^e l_{eb} - l_a{}^e l_{cd} l_{eb} + \mathrm{Rm}_{acb}{}^e l_{ed} + \mathrm{Rm}_{acd}{}^e l_{eb} \\ &\quad + \nabla_a \mathrm{Rm}_{ndcb} + l_{ab}\mathrm{Rm}_{ndcn} - l_a{}^e \mathrm{Rm}_{edcb} \\ &\quad + \nabla_c \mathrm{Rm}_{nbda} + l_{cd}\mathrm{Rm}_{nbna} - l_c{}^e \mathrm{Rm}_{ebda} \end{aligned} \tag{1.7}$$

$$\begin{aligned} \frac{1}{2}\Delta^\Sigma |l|^2 &= l^{ab}\nabla^\Sigma_a \nabla^\Sigma_b L + |\nabla^\Sigma l|^2 + L l_{ac} l^c{}_b l^{ab} - |l|^4 + L l^{ab}\mathrm{Rm}_{nanb} - |l|^2 \mathrm{Rm}_{ncn}{}^c \\ &\quad + 2 l^{ab} l_{bc} \mathrm{Rm}^c{}_{da}{}^d - 2 l^{ab} l^{cd} \mathrm{Rm}_{cadb} + l^{ab}\left(\nabla_b \mathrm{Rm}_{nca}{}^c + \nabla_c \mathrm{Rm}_{nab}{}^c\right). \end{aligned} \tag{1.8}$$

Proof. First we recall the rule for interchanging the order of second covariant derivatives for $(2,0)$-tensors T

$$\nabla^\Sigma_a \nabla^\Sigma_b T_{cd} - \nabla^\Sigma_b \nabla^\Sigma_a T_{cd} = {\mathrm{Rm}^\Sigma}_{abc}{}^e T_{ed} + {\mathrm{Rm}^\Sigma}_{abd}{}^e T_{ec}$$

By using the Gauß Codazzi equations we obtain then

$$\begin{aligned} \nabla^\Sigma_a \nabla^\Sigma_b l_{cd} &= \nabla^\Sigma_a \nabla^\Sigma_c l_{bd} + \nabla^\Sigma_a \mathrm{Rm}_{ndcb} \\ &= \nabla^\Sigma_c \nabla^\Sigma_a l_{bd} + {\mathrm{Rm}^\Sigma}_{acb}{}^e l_{ed} + {\mathrm{Rm}^\Sigma}_{acd}{}^e l_{eb} + \nabla^\Sigma_a \mathrm{Rm}_{ndcb} \\ &= \nabla^\Sigma_c \nabla^\Sigma_d l_{ab} + {\mathrm{Rm}^\Sigma}_{acb}{}^e l_{ed} + {\mathrm{Rm}^\Sigma}_{acd}{}^e l_{eb} + \nabla^\Sigma_a \mathrm{Rm}_{ndcb} + \nabla^\Sigma_c \mathrm{Rm}_{nbda} \\ &= \nabla^\Sigma_c \nabla^\Sigma_d l_{ab} + l_{ab} l_c{}^e l_{ed} - l_a{}^e l_{cb} l_{ed} + l_{ad} l_c{}^e l_{eb} - l_a{}^e l_{cd} l_{eb} + \mathrm{Rm}_{acb}{}^e l_{ed} + \mathrm{Rm}_{acd}{}^e l_{eb} \\ &\quad + \nabla^\Sigma_a \mathrm{Rm}_{ndcb} + \nabla^\Sigma_c \mathrm{Rm}_{nbda} \end{aligned}$$

Now we use formula (1.5) to change from the derivative on Σ^{n-1} to the one on $\mathcal{M}^n$ to get

$$
\begin{aligned}
&= \nabla^\Sigma_c \nabla^\Sigma_d l_{ab} + l_{ab} l_c{}^e l_{ed} - l_a{}^e l_{cb} l_{ed} + l_{ad} l_c{}^e l_{eb} - l_a{}^e l_{cd} l_{eb} + \mathrm{Rm}_{acb}{}^e l_{ed} + \mathrm{Rm}_{acd}{}^e l_{eb} \\
&+ \nabla_a \mathrm{Rm}_{ndcb} + l_{ac} \mathrm{Rm}_{ndnb} + l_{ab} \mathrm{Rm}_{ndcn} - l_a{}^e \mathrm{Rm}_{edcb} \\
&+ \nabla_c \mathrm{Rm}_{nbda} + l_{cd} \mathrm{Rm}_{nbna} + l_{ca} \mathrm{Rm}_{nbdn} - l_c{}^e \mathrm{Rm}_{ebda}.
\end{aligned}
$$

By the symmetries of Rm we have

$$
l_{ac} \mathrm{Rm}_{ndnb} + l_{ca} \mathrm{Rm}_{nbdn} = 0
$$

which proves the first formula (1.7). Now we take the trace in the indices a and b in this formula and use the second Bianchi Identity

$$
\begin{aligned}
\Delta^\Sigma l_{cd} &= \nabla^\Sigma_c \nabla^\Sigma_d L + L l_c{}^e l_{ed} - |l|^2 l_{cd} - 2\mathrm{Rm}^a{}_c{}^b{}_d l_{ab} \\
&+ \nabla_e \mathrm{Rm}_{ncd}{}^e - L \mathrm{Rm}_{ndcn} + l_{de} \mathrm{Rm}^e{}_{fc}{}^f \\
&+ \nabla_c \mathrm{Rm}_{ned}{}^e - l_{cd} \mathrm{Rm}_{nen}{}^e + l_{ce} \mathrm{Rm}^e{}_{fd}{}^f.
\end{aligned}
$$

The result follows then by computing

$$
\Delta^\Sigma \left(l^{ab} l_{ab} \right) = 2 l^{ab} \Delta^\Sigma l_{ab} + 2 |\nabla^\Sigma l|^2.
$$

$\square$

Chapter 2

Evolution Equations

In the analysis of geometric flows the evolutions equations are crucial as they show how the movement of the surface translates into partial differential equations for certain geometric quantities such as the ones we introduced in the previous section. Not only is the theory of PDE a powerful tool and provides a lot of measures to analyze the flow but also predictions on how the flow will behave for short / long time can be directly made by looking at the type of the PDEs. The prime example is the reaction diffusion system coming from the evolution equation of the Weingarten operator in mean curvature flow. We will state the classic evolution equations for mean curvature flow here. For this chapter we will assume $F : \mathcal{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ is a smooth and closed solution to mean curvature flow. All formulas are stated in local coordinates around a point $p \in \mathcal{M}^n_t := F(\mathcal{M}^n, t)$ for some $t \in [0,T)$.

Proposition 2.1 (Evolution Equations for mean curvature flow)**.**

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{ij} = -2Hh_{ij} \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}\mu = -H^2\,\mathrm{d}\mu \tag{2.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} h^i_j = \Delta h^i_j + |A|^2 h^i_j \tag{2.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} H = \Delta H + |A|^2 H \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4. \tag{2.5}$$

Proof. A derivation of the evolution equations in full generality can for example be found in [HP99]. □

As we have seen in the example of the introduction the cubic reaction term $|A|^2 h^i_j$ in the evolution of the Weingarten operator for example implies a blow up of the curvature in finite time such that it is interesting to examine high curvature regions more closely in order to understand the singularity behavior of the flow. Also from the evolution of the area element we immediately see that the total area is decreasing along the flow,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\mathcal{M}_t) = -\int_{\mathcal{M}_t} H^2\,\mathrm{d}\mu.$$

This is no coincidence as mean curvature flow is the steepest flow for the area functional with stationary points being the minimal surfaces.

It will be particularly interesting for us how submanifolds of $\mathcal{M}_t^n$ evolve as time passes. Therefore we will compute the evolution equations for the geometric quantities for submanifolds. We will consider a more general setting. Let $(\mathcal{M}^n, g^t)$ be a Riemannian manifold with a metric g^t depedending on a parameter t. We want this metric to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} g^t(X, Y) = T(X, Y) \tag{2.6}$$

for some smooth symmetric tensor-field T on $\mathcal{M}^n$ and any smooth vector fields X and Y. Of course in view of (2.1) and our application later T will be given by $T = -2Hh$. We consider a smooth submanifold immersion $G : \Sigma^{n-1} \to \mathcal{M}^n$ with constant mean curvature with respect to the induced metric coming from g^0. By abuse of notation we will identify Σ^{n-1} with its image under G. In order to understand how the mean curvature of Σ^{n-1} changes when we have a perturbation of the metric $g^t = g^0 + tT + o(t^2)$ for small parameters t, we have to understand how all geometric factors of Σ^{n-1} change. How certain geometric quantities behave under variation of the metric is well known. For example, it is used for the so called Ricci-mean curvature flow in [Lot12]. For the sake of completeness, we will derive them here once again. The perturbation of the metric will also cause a change in the covariant derivative. In particular, the Christoffel-symbols $\Gamma_{ij}^k = \Gamma_{ij}^k(t)$ of $\mathcal{M}^n$ will be influenced. Here we denote by $\nabla_i := \nabla_{e_i}$ for the coordinate system $\{x^i\}$ around the point p under consideration.

Lemma 2.2.
Under the pertubation $\frac{\mathrm{d}}{\mathrm{d}t} g_{ij}^t = T_{ij}$ we have the following change of the Christoffel symbols

$$\frac{\mathrm{d}}{\mathrm{d}t} \Gamma_{ij}^k = \frac{1}{2} g_t^{kl} \left(\nabla_i^t T_{jl} + \nabla_j^t T_{il} - \nabla_l^t T_{ij} \right). \tag{2.7}$$

Here T_{ij} is the coordinate expression of T, $g_t^{ij} := ((g^t)^{-1})_{ij}$ is the inverse to the metric and ∇ is the connection belonging to g^t.

Proof. We use the usual expression of the Christoffel symbols in terms of the metric g^t

$$\Gamma_{ij}^k = \frac{1}{2} g_t^{kl} \left(\frac{\partial}{\partial x_i} g_{jl}^t + \frac{\partial}{\partial x_j} g_{il}^t - \frac{\partial}{\partial x_l} g_{ij}^t \right).$$

Furthermore, the covariant derivative of a T is given by

$$\nabla_i T_{kl} = \frac{\partial}{\partial x_i} T_{kl} - \Gamma_{ik}^m T_{ml} - \Gamma_{il}^m T_{km}.$$

Now we replace the coordinate derivatives using this formula

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \Gamma_{ij}^k(t) &= \frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}t} g_g^{kl} \right) \left(\frac{\partial}{\partial x_i} g_{jl}^t + \frac{\partial}{\partial x_j} g_{il}^t - \frac{\partial}{\partial x_l} g_{ij}^t \right) \\
&\quad + \frac{1}{2} g_t^{kl} \left(\frac{\partial}{\partial x_i} T_{jl} + \frac{\partial}{\partial x_j} T_{il} - \frac{\partial}{\partial x_l} T_{ij} \right) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t} g_g^{kl} \right) g_{lm}^t \Gamma_{ij}^m + \frac{1}{2} g_t^{kl} \left(\nabla_i T_{jl} + \nabla_j T_{il} - \nabla_l T_{ij} \right) + 2 \cdot \frac{1}{2} g_t^{kl} \Gamma_{ij}^m T_{ml} \\
&= \frac{1}{2} g_t^{kl} \left(\nabla_i T_{jl} + \nabla_j T_{il} - \nabla_l T_{ij} \right) + \frac{\mathrm{d}}{\mathrm{d}t} \left(g_t^{kl} g_{lm}^t \right) \Gamma_{ij}^m
\end{aligned}$$

and the statement follows.

Remark 2.3. We note that since both sides of (2.7) are tensors, so they are independent of the choice of coordinates. That is why we could have made the same calculations above in normal coordinates around the point p such that all Christoffel symbols vanish at this point. In these coordinates we have

$$\nabla_i T_{kl} = \frac{\partial}{\partial x_i} T_{kl}$$

which makes the computations slightly more efficient.

□

Now we know how the covariant differentiation changes with the parameter t. For a basis $\{e_i\}_i$ of $T_p\mathcal{M}^n$ at a point $p \in \mathcal{M}^n$ we can compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla_{e_i} e_j\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\nabla\right)(e_i, e_j) + \nabla_{\frac{\mathrm{d}}{\mathrm{d}t} e_i} e_j + \nabla_{e_i} \frac{\mathrm{d}}{\mathrm{d}t} e_j \tag{2.8}$$

where $g^t\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\nabla\right)(e_i, e_j), e_k\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Gamma^k_{ij}$. A coordinate free expression of the previous computations is the following. Let X, Y and Z be smooth vectorfields independent of t on $\mathcal{M}^n$, then we have

$$g^t\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\nabla\right)(X, Y), Z\right) = \frac{1}{2}\left((\nabla_X T)(Y, Z) + (\nabla_Y T)(X, Z) - (\nabla_Z T)(X, Y)\right). \tag{2.9}$$

Now we care for the submanifold geometry. Here we use a different coordinate system to distinguish between normal and tangential directions on Σ. At a point $p \in \Sigma^{n-1} \subset \mathcal{M}^n$, we consider local coordinates $(x^1, \ldots, x^n)$, such that the $e_a := \frac{\partial}{\partial x^a}$ for $a = 1, \ldots, n-1$ form a basis of $T_p\Sigma^{n-1}$ and $e_n^0 := \eta^0$ is the unit normal with respect to g^0. For this purpose whenever we use the indices a, b, c, d we want them to run between 1 and $n-1$ while the indices i, j, k, l, m run from 1 to n. So that in these coordinates we can write $g^t_{ij} = g^t(e_i, e_j)$ and $T_{ij} := T(e_i, e_j)$.

Lemma 2.4.
Under the perturbation (2.6) the unit normal of Σ^{n-1} behaves as

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta^t = -\frac{1}{2}T(\eta^t, \eta^t)\eta^t - T_(\eta, e_a)g_t^{ab}e_b \tag{2.10}$$

at any point $p \in \Sigma^{n-1}$ where $e_1, \ldots, e_{n-1}$ is a Basis for $T_p\mathcal{M}^n$ and η^t is the chosen orthonormal with respect to g^t at the given time t_0.

Proof. By the our choise of η^t being orthonormal to the fixed basis e_a we have the usual identities $g^t(\eta^t, \eta^t) = 1$ and $g^t(\eta_t, e_a) = 0$ we deduce

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}g^t(\eta^t, \eta^t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}g^t\right)(\eta^t, \eta^t) + 2g^t(\eta^t, \frac{\mathrm{d}}{\mathrm{d}t}\eta^t) \\
&= T(\eta^t, \eta^t) + 2g^t(\eta^t, \frac{\mathrm{d}}{\mathrm{d}t}\eta^t) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}g^t(\eta^t, e_a) = \left(\frac{\mathrm{d}}{\mathrm{d}t}g^t\right)(\eta^t, e_a) + g^t(\eta^t, e_a) \\
&= T(\eta^t, e_a) + g^t(\frac{\mathrm{d}}{\mathrm{d}t}\eta^t, e_a).
\end{aligned}$$

The result then follows by splitting $\frac{\mathrm{d}}{\mathrm{d}t}\eta^t$ in normal and tangential components by noting that at a given point p and a given time t, we can always arrange the coordinates in a way such that $g^t(e_a, e_a)(p) = 1$. □

We also need to know how the perturbation affects the second fundamental form. For the following computations we assume that for fixed t the coordinates are chosen orthonormal at the point under consideration such that we do not have to worry about lower or upper indices.

Proposition 2.5.
In the same situation as above the second fundamental form t_{ab} with respect to g^t and η_t changes according to

$$\frac{\mathrm{d}}{\mathrm{d}t} l_{ab} = -\frac{1}{2}\left(\nabla_a T_{bn} + \nabla_b T_{an} - \nabla_n T_{ab} - T_{nn} l_{ab}\right). \tag{2.11}$$

This implies the change of the mean curvature of Σ^{n-1}

$$\frac{\mathrm{d}}{\mathrm{d}t} L = -g_t^{ab}\nabla_a T_{bn} + \frac{1}{2}\nabla_n \mathrm{Tr}_{\Sigma^{n-1}}(T) - T^{ab} l_{ab} + \frac{1}{2} L T_{nn}. \tag{2.12}$$

Proof. The second fundamental form l^t with respect to g^t is given by

$$l_{ab} = -g^t\left(\nabla_{e_a} e_b, \eta^t\right)$$

for $a, b = 1, \dots, n-1$. From this we compute using Lemma 2.2 and 2.4

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} l_{ab} &= -\left(\frac{\mathrm{d}}{\mathrm{d}t} g^t\right)\left(\nabla_{e_a} e_b, \eta^t\right) - g^t\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla_{e_a} e_b\right), \eta^t\right) - g^t\left(\nabla_{e_a} e_b, \frac{\mathrm{d}}{\mathrm{d}t}\eta^t\right)\\ &=: -A - B - C\end{aligned}$$

We continue to compute the terms individually. First we get

$$A = T(\nabla_{e_a} e_b, \eta^t) = \Gamma_{ab}^c T_{cn} + \Gamma_{ab}^n T_{nn}.$$

For the second term we obtain

$$\begin{aligned}B &= g^t\left(\frac{1}{2} g_t^{kl}\left(\nabla_a T_{bl} + \nabla_b T_{al} - \nabla_l T_{ab}\right) e_k, \eta^t\right)\\ &= \frac{1}{2} g_t^{kl} g_{kn}^t \left(\nabla_a T_{bl} + \nabla_b^t T_{al} - \nabla_l T_{ab}\right)\\ &= \frac{1}{2}\left(\nabla_a T_{bn} + \nabla_b^t T_{an} - \nabla_n T_{ab}\right)\end{aligned}$$

Finally we compute the last term which gives

$$\begin{aligned}C &= g^t\left(\nabla_{e_a} e_b, -\frac{1}{2} T_{nn}\eta^t - T_{nc} e_c\right)\\ &= +\frac{1}{2} l_{ab} T_{nn} - \Gamma_{ab}^c T_{nc}.\end{aligned}$$

Combining these identities and using $\Gamma_{ab}^n = -l_{ab}^t$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} l_{ab} &= -\Gamma_{ab}^c T_{nc} - \Gamma_{ab}^n T_{nn} - \frac{1}{2}\left(\nabla_a T_{bn} + \nabla_b T_{an} - \nabla_n T_{ab}\right) - \frac{1}{2} l_{ab}^t T_{nn} + \Gamma_{ab}^c T_{nc}\\ &= -\frac{1}{2}\left(\nabla_a T_{bn} + \nabla_b T_{an} - \nabla_n T_{ab} - l_{ab} T_{nn}\right).\end{aligned}$$

Last but not least we take the trace with respect to g^t to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(l_{ab} g_t^{ab}\right) &= -g_t^{ab}\frac{1}{2}\left(\nabla_a T_{bn} + \nabla_b T_{an} - \nabla_n T_{ab} - l_{ab} T_{nn}\right) - g_t^{ap} g_t^{bq} T_{pq} l_{ab}\\ &= -g_t^{ab}\nabla_a T_{bn} + \frac{1}{2}\nabla_n \mathrm{Tr}_{\Sigma^{n-1}}(T) + \frac{1}{2} L T_{nn} - T^{ab} l_{ab}.\end{aligned}$$

$\square$

As a first consequence of these computations we can recover the change of certain geometric quantities in the case of $(\mathcal{M}^n, (g^t)_{t\in[0,M]})$ being a solution to the Ricci flow. They can also be found in [BC10, Corollary 2.29] and [Lot12, Section 2].

Corollary 2.6.
If the metric of $\mathcal{M}^n$ solves $\frac{\mathrm{d}}{\mathrm{d}t} g^t_{ij} = -2\mathrm{Ric}_{ij}$ then we have the following evolution equations.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^k_{ij} &= -g^{kl}_t\left(\nabla_i \mathrm{Ric}_{jl} + \nabla_j \mathrm{Ric}_{il} - \nabla_l \mathrm{Ric}_{ij}\right)\\
\frac{\mathrm{d}}{\mathrm{d}t} l_{ab} &= \nabla_a \mathrm{Ric}_{bn} + \nabla_b \mathrm{Ric}_{an} - \nabla_n \mathrm{Ric}_{ab} - \mathrm{Ric}_{nn} l_{ab}\\
\frac{\mathrm{d}}{\mathrm{d}t} L &= 2g^{ab}_t \nabla_a \mathrm{Ric}_{bn} - \nabla_n R - \nabla_n \mathrm{Ric}_{nn} - L\mathrm{Ric}_{nn} + 2l^{ab}\mathrm{Ric}_{ab}
\end{aligned}$$

where R is the scalar curvature.

The for us relevant case is when the ambient manifold $\mathcal{M}^n$ is isometrically immersed or embedded in $\mathbb{R}^{n+1}$ and evolves under mean curvature flow. Then the perturbation for our metric is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{ij} = -2Hh_{ij}.$$

Since the motion of Σ^{n-1} is determined by the motion of $\mathcal{M}^n_t$ we expect a similar reaction diffusion behavior in the equations. However written in this form it is not as obvious as in the corresponding evolutions for the surrounding manifold. Therefore we need to write the evolutions in terms of geometric quantities on Σ. For the sake of clear presentation we assume that at each point p under consideration we pick orthonormal coordinates e_i of $T_p\mathcal{M}^n$ such that $e_n = \eta$ and we do not have to worry about lower or upper indices.

Theorem 2.7 (Evolution Equations for Σ)**.**

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} l_{ab} = \Delta^\Sigma l_{ab} &- \nabla^\Sigma_b \nabla^\Sigma_a L - L l_{ad} l_{db} + l_{ab}|l|^2 + l_{ab}(\mathrm{Ric}_{nn} - Hh_{nn})\\
&- l_{bc}\mathrm{Ric}_{ca} - L\mathrm{Rm}_{nanb} - l_{cb}\mathrm{Rm}_{nacn} + l_{da}\mathrm{Rm}_{bccd} + 2l_{dc}\mathrm{Rm}_{bcad}\\
&+ \nabla_b A^2_{an} - \nabla_n H \cdot h_{ab} + \nabla_a H \cdot h_{nb} + \nabla_c \mathrm{Rm}_{nacb}
\end{aligned}$$

Additionally, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(|l|^2\right) = \Delta^\Sigma |l|^2 &- 2|\nabla^\Sigma l|^2 + 2l_{ab}\nabla^\Sigma_b \nabla^\Sigma_a L + 2|l|^2(\mathrm{Ric}_{nn} - Hh_{nn}) + 2(|l|^4 - L\mathrm{tr}_\Sigma(l^3))\\
&+ 4Hh_{ab}l_{bc}l_{ca} - 2l_{ab}l_{bc}\mathrm{Ric}_{ca} - 2Ll_{ab}\mathrm{Rm}_{nanb} - 2l_{ab}l_{cb}\mathrm{Rm}_{nacn} + 2l_{ab}l_{da}\mathrm{Rm}_{bccd}\\
&+ 4l_{ab}l_{dc}\mathrm{Rm}_{bcad} + 2l_{ab}\nabla_b A^2_{an} - 2\nabla_n H \cdot h_{ab} + 2l_{ab}\nabla_a H \cdot h_{nb} + 2l_{ab}\nabla_c \mathrm{Rm}_{nacb}.
\end{aligned} \tag{2.13}$$

Proof. We start from

$$\frac{\mathrm{d}}{\mathrm{d}t} l_{ab} = \nabla_a (Hh)_{bn} + \nabla_b (Hh)_{an} - \nabla_n (Hh)_{ab} - l_{ab}Hh_{nn}. \tag{2.14}$$

Then for any point $p \in \Sigma$ by the Gauß equation for the Ricci Curvature and the Codazzi equation between Σ and $\mathcal{M}^n$ we obtain

$$\begin{aligned}
\nabla_a (Hh)_{bn} &= \nabla_a\left(Hh_{bn} - h_{bk}h^k_n\right) + \nabla_a (h_{bk}h^k_n)\\
&= \nabla_a \mathrm{Ric}_{bn} + \nabla_a (h_{bk}h^k_n)\\
&= \nabla^\Sigma_a \mathrm{Ric}_{nb} + l_{ab}\mathrm{Ric}_{nn} - l^c_a \mathrm{Ric}_{cb} + \nabla_a (h_{bk}h^k_n)\\
&= \nabla^\Sigma_a \nabla^\Sigma_c l_{cb} - \nabla^\Sigma_a \nabla^\Sigma_b L + l_{ab}\mathrm{Ric}_{nn} - l^c_a \mathrm{Ric}_{cb} + \nabla_a (h_{bk}h^k_n).
\end{aligned}$$

Here we used (1.6) in the second to last step and Codazzi equation in the last step. Also by the Codazzi equation between $\mathcal{M}^n$ and $\mathbb{R}^{n+1}$ we have

$$\nabla_n(Hh)_{ab} = \nabla_a(Hh)_{bn} + \nabla_n H \cdot h_{ab} - \nabla_a H \cdot h_{nb}.$$

Therefore, we can compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} l_{ab} &= \nabla^\Sigma_b \nabla^\Sigma_c l_{ca} - \nabla^\Sigma_b \nabla^\Sigma_a L + l_{ab}(\mathrm{Ric}_{nn} - Hh_{nn}) - l_{ac}\mathrm{Ric}_{bc} \\ &+ \nabla_b A^2_{an} - \nabla_n H \cdot h_{ab} + \nabla_a H \cdot h_{nb}\end{aligned}$$

where $A^2_{ij} = h_{ik}h^k_j$. Now we exchange covariant derivatives in the first term

$$\nabla^\Sigma_b \nabla^\Sigma_c l_{ca} = \nabla^\Sigma_c \nabla^\Sigma_b l_{ca} + \mathrm{Rm}^\Sigma_{bccd} l_{da} + \mathrm{Rm}^\Sigma_{bcad} l_{dc}$$

like in (1.2), see again [BC10, Chapter 1.3.3], and employ the Codazzi equations for $l = l_{ab}$

$$2l_{ab}\left(\nabla^\Sigma_c \nabla^\Sigma_a l_{cb}\right) = 2l_{ab}\Delta^\Sigma l_{ab} + 2l_{ab}\nabla^\Sigma_c \mathrm{Rm}_{nacb}$$

which gives us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} l_{ab} &= \Delta^\Sigma l_{ab} - \nabla^\Sigma_b \nabla^\Sigma_a L + l_{ab}\mathrm{Ric}_{nn} - l_{ac}\mathrm{Ric}_{bc} + \nabla_b A^2_{an} \\ &- \nabla_n H \cdot h_{ab} + \nabla_a H \cdot h_{nb} + \nabla^\Sigma_c \mathrm{Rm}_{nacb} + \mathrm{Rm}^\Sigma_{bccd} l_{da} + \mathrm{Rm}^\Sigma_{bcad} l_{dc} \\ &= \Delta^\Sigma l_{ab} - \nabla^\Sigma_b \nabla^\Sigma_a L - L l_{ad} l_{db} + l_{ab}|l|^2 + l_{ab}(\mathrm{Ric}_{nn} - Hh_{nn}) \\ &- l_{ac}\mathrm{Ric}_{bc} - L\mathrm{Rm}_{nanb} - l_{cb}\mathrm{Rm}_{nacn} + l_{da}\mathrm{Rm}_{bccd} + 2l_{dc}\mathrm{Rm}_{bcad} \\ &+ \nabla_b A^2_{an} - \nabla_n H \cdot h_{ab} + \nabla_a H \cdot h_{nb} + \nabla_c \mathrm{Rm}_{nacb}.\end{aligned}$$

Now we compute the time derivative of $|l|^2$. We note that since l_{ab} is symmetric and we contract in both indices we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(|l|^2) &= \frac{\mathrm{d}}{\mathrm{d}t}(l_{ab}l_{ab}) \\ &= 4Hh_{ab}l_{bc}l_{ca} + 2l_{ab}\left(\frac{\mathrm{d}}{\mathrm{d}t} l_{ab}\right).\end{aligned}$$

We use the previous computation to get

$$\begin{aligned}l_{ab}\left(\frac{\mathrm{d}}{\mathrm{d}t} l_{ab}\right) &= l_{ab}\Delta^\Sigma l_{ab} + l_{ab}\nabla^\Sigma_c \mathrm{Rm}_{nacb} + l_{ab}\nabla^\Sigma_b \nabla^\Sigma_a L + |l|^2(\mathrm{Ric}_{nn} - Hh_{nn}) - l_{ab}l_{bc}\mathrm{Ric}_{ca} - L l_{ab}\mathrm{Rm}_{nanb} \\ &- l_{ab}l_{cb}\mathrm{Rm}_{nacn} + l_{ab}l_{da}\mathrm{Rm}_{bccd} + 2l_{ab}l_{dc}\mathrm{Rm}_{bcad} + |l|^4 - L\mathrm{tr}_\Sigma(l^3) \\ &+ l_{ab}\nabla_b A^2_{an} - \nabla_n H \cdot h_{ab} + l_{ab}\nabla_a H \cdot h_{nb} + l_{ab}\nabla_c \mathrm{Rm}_{nacb}.\end{aligned}$$

We combine this with the fact that

$$2l_{ab}\Delta^\Sigma l_{ab} = \Delta^\Sigma |l|^2 - 2|\nabla^\Sigma l|^2$$

such that we finally we obtain the evolution equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(|l|^2\right) &= \Delta^\Sigma |l|^2 - 2|\nabla^\Sigma l|^2 + 2l_{ab}\nabla^\Sigma_b \nabla^\Sigma_a L + 2|l|^2(\mathrm{Ric}_{nn} - Hh_{nn}) + 2(|l|^4 - L\mathrm{tr}_\Sigma(l^3)) \\ &+ 4Hh_{ab}l_{bc}l_{ca} - 2l_{ab}l_{bc}\mathrm{Ric}_{ca} - 2L l_{ab}\mathrm{Rm}_{nanb} - 2l_{ab}l_{cb}\mathrm{Rm}_{nacn} + 2l_{ab}l_{da}\mathrm{Rm}_{bccd} \\ &+ 4l_{ab}l_{dc}\mathrm{Rm}_{bcad} + 2l_{ab}\nabla_b A^2_{an} - 2\nabla_n H \cdot h_{ab} + 2l_{ab}\nabla_a H \cdot h_{nb} + 2l_{ab}\nabla_c \mathrm{Rm}_{nacb}.\end{aligned} \tag{2.15}$$

□

Chapter 3

Necklike Regions

In this section we will briefly repeat the for us necessary definitions regarding necks in mean curvature flow. Also for the sake of completeness we will give the definition of mean curvature flow with surgeries. The surgery algorithm is used in order to extend solutions of the mean curvature flow beyond singular times. The procedure is done in a way that it depends on a certain set of parameters that control the entire surgery procedure and only depend on the initial surface. In the present setting we consider a smooth n-dimensional 2-convex initial surface $F_0 : \mathcal{M}^n \to \mathbb{R}^{n+1}$ with the following three properties

(i) The first two principle curvatures λ_1 and λ_2 satisfy $\lambda_1 + \lambda_2 \geq \gamma_0 H$ which means the surfaces is uniformly 2-convex.

(ii) The mean curvature H satisfies $H \geq \gamma_1 R^{-1}$.

(iii) The volume of $\mathcal{M}_0^n := F_0(\mathcal{M})$ is bounded, namely $\mathrm{Vol}(\mathcal{M}_0^n) \leq \gamma_2 R^n$.

Here $R, \gamma := (\gamma_0, \gamma_1, \gamma_2)$ is a set of positive constants. Any such surface is said to be in the class of smooth $\mathcal{C}(R,\gamma)$-hypersurface immersions. It is convenient to choose R such that $|A|^2 \leq R^{-2}$, for example $R := \max_{M_0^n} |A|$. The class of $\mathcal{C}(R,\gamma)$ hypersurfaces is preserved as shows the following Proposition

Proposition 3.1.

1. Given any smooth closed, weakly 2-convex hypersurface immersion $\mathcal{M}_0^n$, the solution of mean curvature flow $\mathcal{M}_t^n$ with initial data $\mathcal{M}_0^n$ is strictly 2-convex for any $t > 0$.
2. For every strictly 2-convex, smooth closed hypersurface $\mathcal{M}^n$ we can choose R and γ such that $\mathcal{M}^n \in \mathcal{C}(R,\gamma)$ and $|A| \leq R^{-2}$ holds everywhere on $\mathcal{M}^n$.
3. Each class $\mathcal{C}(R,\gamma)$ is invariant under smooth mean curvature flow.
4. We have the estimate $\frac{1}{n}H^2 \leq |A|^2 \leq nH^2$.
5. The Eigenvalues satisfy $\lambda_i \geq \frac{\gamma_0}{2} H$ for all $i = 2, \ldots, n$.

Proof. The proof can be found in [HS09, Proposition 2.6] for 1., 2. and 3. and [HS09, Proposition 2.7] for 4. and 5.. □

Mean curvature flow with surgeries for some initial data $\mathcal{M}_1^n$ in a class $\mathcal{C}(R,\gamma)$ consists of a sequence of intervals $[0, T_1], \ldots, [T_{N-1}, T_N]$, a sequence of manifolds $\mathcal{M}_i^n$, $1 \leq i \leq N$ and a sequence of smooth mean curvature flows $F^i : [T_{i-1}, T_i] \times \mathcal{M}_i^n \to \mathbb{R}^{n+1}$ such that

1. The initial surface for the family F^1 is given by $F_0 : \mathcal{M}_1^n \to \mathbb{R}^{n+1}$.
2. The initial surface for the flow F^i for $i = 2, \dots, N$ is obtained from $F^{i-1}(T_{i-1}, \cdot) : \mathcal{M}_i^n \to \mathbb{R}^{n+1}$ via the following 2 step procedure. In a first step the standard surgery is used to replace a finite number of so called necks by two spherical caps. Then in a second step finitely many connected components whose topology is already known are removed resulting in a initial hypersurface $F^N_{T_{i-1}}$ for the smooth flow F^i.

The surgery algorithm is said to terminate after finitely many steps at time T_N if at time T_N before removing the connected components all of them are known to be diffeomorphic to either $\mathbb{S}^n$ or to $\mathbb{S}^{n-1} \times \mathbb{S}^n$. We will not present the surgery construction in full detail here, as we will not need it in the rest of this thesis. We will however state the main result in [HS09].

Theorem 3.2 (mean curvature flow with surgeries)**.**
For any given initial surface $\mathcal{M}_0^n \in \mathcal{C}(R, \gamma)$ there exists a mean curvature flow with surgeries starting from $\mathcal{M}_0^n$ which terminates after finitely many steps . All surfaces of the flow satisfy uniform curvature bounds determined by the class parameters R and γ and the length of smooth time intervals is bounded from below by a uniform constant depending on R and γ.

Proof. The proof can be found in [HS09, Section 8]. □

3.1 Geometric Necks

The standard cylinder of dimension n with radius 1 is given by $\mathbb{S}^{n-1} \times \mathbb{R} \subset \mathbb{R}^{n+1}$. By standard we mean it is equipped with the standard metric $\bar{g}$, induced by the Euclidian metric. We begin to define the notion of "necks". First we speak about the extrinsic curvature which is natural in our context. In order to be consistent with Hamiltons presentation of necks (see [Ham97, Chapter 3]) and their special parameterizations we will later state that due to the Gauss-equations these properties on the extrinsic curvature will also imply the corresponding properties for the intrinsic curvature.

Definition 3.3 (Extrinsic Curvature necks)**.**
Let $N^n \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and $p \in N^n$. Furthermore, let $\varepsilon > 0$ and $k \in \mathbb{N}$.

(a) We say the extrinsic curvature is ε-**cylindrical** at p if the Shape operator $W_p : T_pN \to T_pN$ satisfies
$$|W_p - \bar{W}|_{\bar{g}} \leq \varepsilon, \tag{3.1}$$
where $\bar{W}$ is the Shape operator of the standard cylinder with respect to the standard metric $\bar{g}$.

(b) We say the extrinsic curvature is (ε, k)-**parallel** at p if
$$|\nabla^l W_p|_{\bar{g}} \leq \varepsilon, \quad 1 \leq l \leq k \tag{3.2}$$

(c) We say the extrinsic curvature is $(\varepsilon, k, \Lambda)$-**cylindrical** around p if it satisfies (a) and (b) for every q in the intrinsic ball $B_\Lambda(p) \subset N$ with respect to the metric g on N^n.

(d) We say the extrinsic curvature is $(\varepsilon, k, \Lambda)$-**homothetically cylindrical** around p if there is a scaling constant σ such that σN satisfies the property in (c). We also say p lies in the center of an (extrinsic) $(\varepsilon, k, \Lambda)$-curvature neck if this is true.

Due to the Gauss equations, these statements can be related to the intrinsic curvature necks in the sense of Hamilton [Ham97, Chapter 3.3]. The definition of an intrinsic curvature neck is essentially the same as in Definition 3.3 (d) but the shape operator W is replaced by the Riemmanian curvature tensor Rm.

Proposition 3.4. For any $\varepsilon > 0$ there exists a $\varepsilon' = \varepsilon'(\varepsilon, n)$ such that every point that lies in the center of an $(\varepsilon', k, \Lambda)$-extrinsic curvature neck also lies at the center of an $(\varepsilon, k, \Lambda)$ intrinsic curvature neck.

Proof. The proof follows from the Gauß equations which relate Rm and W. □

The assumptions (3.1) and (3.2) tell us that in a curvature neck the curvature pointwise resembles the curvature of a cylinder. The natural question is whether this pointwise condition can be extended to a statement on the whole parameterization of a local area. This is in fact true for $n \geq 3$ by the following proposition.

Proposition 3.5.
Let $k \geq 1$ and $n \geq 3$. Then for all $\Lambda \geq 10$ there exists a parameter $\varepsilon = \varepsilon(\Lambda, k)$ and a constant $c = c(n, \Lambda) > 0$ such that any point p that lies in the center of $(\varepsilon', k, \Lambda)$-curvature neck with $0 < \varepsilon' \leq \varepsilon$ has a neighborhood with after appropriate rescaling can be written as the graph of a function

$$u : \mathbb{S}^{n-1} \times [-(\Lambda - 1), \Lambda - 1] \to \mathbb{R}$$

over some standard cylinder in $\mathbb{R}^{n+1}$. Furthermore, u satisfies the estimate

$$\|u\|_{C^{k+2}} \leq c(n, \Lambda)\varepsilon.$$

Proof. The proof can be found in [HS09, Proposition 3.5]. □

Remark 3.6. The dimension restriction $n \geq 3$ is absolutely necessary. The proof relies on Myer's Theorem applied to the crosssections where we need $n - 1 \geq 2$. For lower dimensions a cylindrical region could a priori be a non closed curve the wraps along the surface such that Myer's result does not apply.

Inspired by this result, we define the notion of hypersurface necks which are now defined via the parameterization.

Definition 3.7 (Geometric necks)**.**
Let $a, b \in \mathbb{R}$ with $a < b$ then for a local diffeomorphism $\mathcal{N} : \mathbb{S}^{n-1} \times [a, b] \to \mathcal{M}$ we denote by $r : [a, b] \to \mathbb{R}$ the mean radius function. For the cross sections $\Sigma_z := \mathcal{N}(\mathbb{S}^{n-1} \times \{z\})$ it is defined implicitly through

$$\mathrm{vol}_g(\Sigma_z) = \sigma_{n-1} r(z)^{n-1}$$

where σ_{n-1} is the volume of the unit $(n-1)$-sphere and g is the induced metric on $\mathcal{M}$. In this setup we say $\mathcal{N}$ is an (ε, k)-**cylindrical geometric neck** for some parameters $\varepsilon > 0$ and $1 \leq k \in \mathbb{N}$ if it satisfies the following conditions

1. The conformal metric $\hat{g} := r(z)^{-2} g$ satisfies the estimates

$$|\hat{g} - \bar{g}|_{\bar{g}} \leq \varepsilon, \quad |\bar{\nabla}^j \hat{g}|_{\bar{g}} \leq \varepsilon \text{ for } 1 \leq j \leq k. \tag{3.3}$$

uniformly on $\mathbb{S}^{n-1} \times [a, b]$.

2. The change of the mean radius function r is controlled

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^j \log\left(r\left(z\right)\right)\right| \leq \varepsilon \tag{3.4}$$

   for all $z \in [a, b]$ and $1 \leq j \leq k$.

We call an (ε, k)-cylindrical geometric neck with induced metric g and Shape-operator W an (ε, k)-**hypersurface neck** if additionally, we have

$$\begin{aligned} |W(q) - r(z)^{-1}\bar{W}|_{\bar{g}} &\leq \varepsilon r(z)^{-1} \quad \text{and} \\ |\bar{\nabla}^l W(q)|_{\bar{g}} &\leq \varepsilon r(z)^{-l-1}, \quad 1 \leq l \leq k \end{aligned} \tag{3.5}$$

for any $q \in \mathbb{S}^{n-1} \times \{z\}$ and $z \in [a, b]$. Here $\bar{W}$ is the Shape Operator of the standard cylinder in $\mathbb{R}^{n+1}$.

Remark 3.8.

1. Rather than checking this property directly, it is enough to find a suitable scaling parameter r_0 such that the rescaled hypersurface

$$\tilde{F} = r_0^{-1} F : \mathcal{M} \to \mathbb{R}^{n+1}$$

   satisfies the conditions

$$\begin{aligned} |\tilde{g} - \bar{g}|_{\bar{g}} \leq \varepsilon', &\quad |\bar{\nabla}^j \tilde{g}|_{\bar{g}} \leq \varepsilon' \\ |\tilde{W} - \bar{W}|_{\bar{g}} \leq \varepsilon', &\quad |\tilde{\nabla}^l \tilde{W}|_{\bar{g}} \leq \varepsilon' \end{aligned} \tag{3.6}$$

   for $1 \leq j, k \leq k'$ and suitable parameters (ε', k'). This, however, has disadvantages when we try to combine different necks to a new longer neck, since they might have different scale parameters.

2. We also remark that for $n \geq 3$ Proposition 3.5 can be seen as an existence result for Definition 3.7

As in Hamillton [Ham97] we will now introduce a certain parameterization for those necks that allows us to have a unique z-coordinate along the neck and other handy properties.

Definition 3.9 (Normal parametrization)**.**
A local diffeomorphism $\mathcal{N} : \mathbb{S}^{n-1} \times [a, b] \to (\mathcal{M}, g)$ is called normal if it has the following properties

1. Every cross section $\Sigma_z := \mathcal{N}(\mathbb{S}^{n-1} \times \{z\}) \subset (\mathcal{M}, g)$ has constant mean curvature (CMC).

2. The restriction of $\mathcal{N}$ to each $\mathbb{S}^{n-1} \times \{z\}$ quipped with the standard metric is a harmonic map to Σ_z equipped with the induced metric of g.

3. The volume of any sub-cylinder with respect to the pullback of g is given by

$$\mathrm{vol}(\mathbb{S}^{n-1} \times [v, w]) = \sigma_{n-1} \int_v^w r(z)^n \, \mathrm{d}z.$$

4. For any infinitesimal rotation (Killing vector field) $\bar{V}$ on $\mathbb{S}^{n-1} \times \{z\}$ we have

$$\int_{\mathbb{S}^{n-1}\times\{z\}} \bar{g}(\bar{V}, W)\,\mathrm{d}\mu_{\bar{g}} = 0$$

where W is the unit normal vector field of $\Sigma_z \subset (\mathcal{M}, g)$.

Furthermore, such a diffeomorphism is called maximal, whenever N^* is another such neck with $\mathcal{N} = \mathcal{N}^* \circ G$ for a map G, then the map G is surjective.

We will now state the corresponding main result for the existence of maximal normal parameterizations which holds for $n \geq 3$.

Theorem 3.10 ([HS09],[Ham97])**.**

(a) For any $\delta > 0$ we can find parameters $\varepsilon > 0$ and $k \in \mathbb{N}$ such that for any (ε, k) cylindrical hypersurface neck $\mathcal{N} : \mathbb{S}^{n-1} \times [a, b] \to \mathcal{M}^n$ with length $b - a > 3\delta$ we can find a maximal normal neck $\mathcal{N}^* : \mathbb{S}^{n-1} \times [a^*, b^*] \to \mathcal{M}^n$ and a diffeomorphism

$$G : \mathbb{S}^{n-1} \times [a^*, b^*] \to \Omega \subset \mathbb{S}^{n-1} \times [a, b]$$

for some region Ω which contains all points which have distance at least δ from the ends, such that

$$\mathcal{N}^* = \mathcal{N} \circ G.$$

This neck is contained in a maximal (ε, k)-hypersurface neck unless the manifold $\mathcal{M}^n$ is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

(b) For any $\delta > 0$ and (ε', k') we can choose the parameters (ε, k) such that $\mathcal{N}^*$ is a (ε', k')-hypersurface neck.

(c) For any $k \geq 1$ and any $\Lambda_0 > 0$ there exists a parameter $\tilde{\varepsilon}(\Lambda_0, k)$ such that any two normal hypersurface necks $\mathcal{N}_1$ and $\mathcal{N}_2$ with $0 < \varepsilon < \tilde{\varepsilon}(\Lambda_0, k)$ that overlap on a collar $\mathbb{S}^{n-1} \times [z_0, z_0 + \Lambda_0]$, for example

$$\mathcal{N}_1(\mathbb{S}^{n-1} \times [z_0, z_0 + \Lambda_0]) \subset \mathcal{N}_2(\mathbb{S}^{n-1} \times [a, b]),$$

then they agree up to isometries of the isometries of the standard cylinder and can be combined to form a single normal neck.

Proof. We want to give a proof for (a) for the CMC property, since we will use the same argument later to construct a similar parameterization that evolves in time. The argument we will present is based on the implicit function theorem. Let $\delta_0 \in (0, \frac{b-a}{2})$ be a small constant and $\beta \in (0, 1)$. In view of the definition of (ε, k) necks we will later look at a family of perturbed metrics g^t of the standard metric $g = g_0$ on $\mathbb{S}^{n-1} \times [a, b]$. We recall that the spheres $\mathbb{S}^{n-1} \times \{z\} \subset \mathbb{S}^{n-1} \times [a, b]$ are constant mean curvature, even minimal, hypersurfaces with respect to g_0 for $z \in [a, b]$. Now we define the hypersurfaces of spherical graph type

$$\Sigma_u := \Big\{(x, u(x)) \,:\, x \in \mathbb{S}^{n-1}\Big\} \subset \mathbb{S}^{n-1} \times [a, b]$$

for some function $u \in C^{2,\beta}(\mathbb{S}^{n-1}; (a, b))$. When $u \equiv z_0$ for some z_0 we will write $\Sigma_{z_0} := \Sigma \times \{z_0\}$. Let us choose any metric g on $\mathbb{S}^{n-1} \times [a, b]$. In view of our goal to find constant mean curvature surfaces we need to solve

$$L^g(\Sigma_{z+u}) + C = 0$$

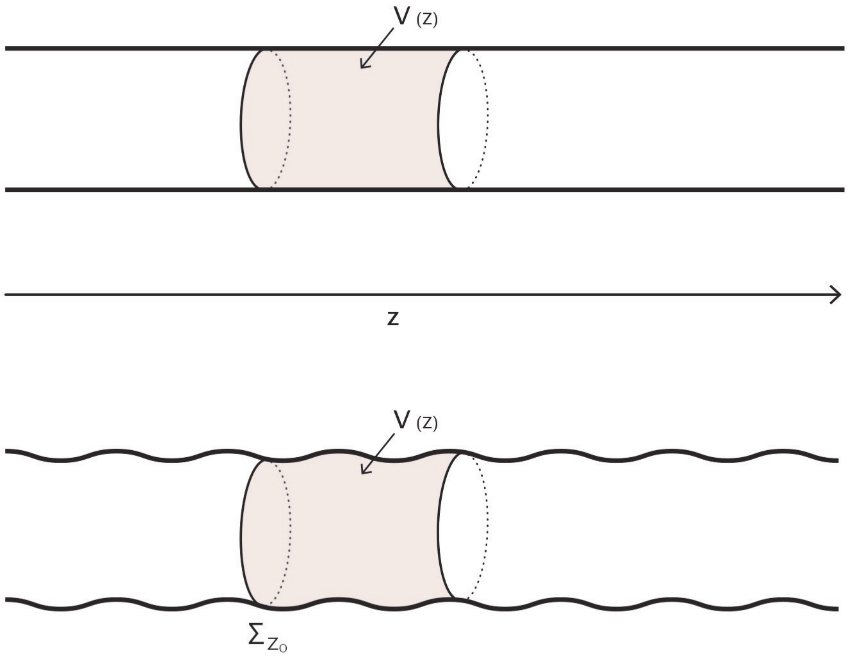

Figure 3.1: Finding a CMC parameterization

where $L^g(\Sigma_{z+u})$ is the mean curvature of Σ_{z+u} with respect to the metric g. Here we also fix a direction along the z-axes in the cylinder by assuming that the inner unit normal η of Σ_{z+u} with respect to g satisfies

$$g(\eta, \frac{\partial}{\partial z}) \geq 0.$$

Let M be the Banach space of all $C^{2,\beta}$ symmetric 2-tensors on $\mathbb{S}^{n-1} \times [a,b]$ with respect to the standard metric g_0 and let us denote by M_0 the open cone of all positive tensors in M. Also we need to fix the vector spaces

$$\mathcal{V}((a_0,b_0)) := C^{0,\beta}(\mathbb{S}^{n-1};(a_0,b_0)) \cap \left\{u \,:\, \int_{\mathbb{S}^{n-1}} u(x)\,\mathrm{d}\mu_0(x) = 0\right\}$$

for any $a_0 < b_0 \in \mathbb{R}$. Then we define the map

$$\mathcal{B} : M_0 \times (a+\delta_0, b-\delta_0) \times C^{2,\beta}(\mathbb{S}^{n-1};(-\delta_0,\delta_0)) \to \mathcal{V}(((-\delta_0,\delta_0)) \times \mathbb{R}$$

via

$$(g,z,u) \mapsto \mathcal{B}(g,z,u) := \left(L^g(\Sigma_{z+u}) - \fint_{\Sigma_{z+u}} L^g(\Sigma_{z+u})\,\mathrm{d}\mu_\Sigma, V(z)\right)$$

where $V(z)$ is the volume between some reference cross section Σ_{z_0} and Σ_{z+u} with respect to g. This situation is indicated in Figure 3.1.

This defines a map between products of open sets in Banach spaces and Banach spaces. Also by our assumption that all components are at least $C^{2,\beta}$ regular it is a continuously differentiable map. The next step is to compute the linearization. The perturbation parameters t and the values $z \in [a,b]$ are given, so we wanna look at the linearization with respect to u at the point $(g_0, z_0, 0)$. First of all recall that for a normal variation D_h of Σ_z with speed h along η the stability operator $\mathcal{J}$ comes into play. That is

$$D_h(L^g(\Sigma_z)) = \mathcal{J}h = \Delta^{g,\Sigma_z} h + h\left(|l_z|_g^2 + \mathrm{Ric}^g(\eta,\eta)\right)$$

where $\Delta^{g,\Sigma_{z_0}}$ is the Laplacian on Σ_z induced by g and η the corresponding normal. In particular, if $t=0$ the term $|l_z|_g^2 + \mathrm{Ric}^g(\eta,\eta)$ vanishes identically. Hence, the linearization $D_u\mathcal{B} : C^{2,\beta}(\mathbb{S}^{n-1}) \to V(\mathbb{R}) \times \mathbb{R}$ is just

$$\mathcal{D}(w) := D_u\mathcal{B}(g_0, z_0, 0)(w, K) = \left(\Delta^{\Sigma_{z_0}} w, \int_{\mathbb{S}^{n-1}} w \,\mathrm{d}\mu_0\right)$$

for $w \in V(\mathbb{R})$. By the definition of V this map is injective. Namely, if w_0 is such that

$$\Delta^{\Sigma_{z_0}} w_0 = 0 = \int_{\mathbb{S}^{n-1}} w_0 \,\mathrm{d}\mu_0,$$

then $w_0 \equiv 0$ because the Eigenfunctions to the 0 eigenvalue of the Laplacian on the sphere are the constant functions. This is why we included the volume into the definition of the operator B which is one way to parameterize the CMC surfaces along the z-axes. It is well known that the Laplace operator $\Delta : C^{2,\beta}(\mathbb{S}^{n-1}) \to V(\mathbb{R})$ is an isomorphism by standard Schauder-theory. Thus for any $f \in C^{0,\beta}(\mathbb{S}^{n-1})$ with vanishing integral we can find a unique $w_f \in V(\mathbb{R})$ with

$$\Delta^{\Sigma_{z_0}} w_f = f$$

which implies the surjectivity of $\mathcal{D}$. Furthermore, the standard Schauder-theory a-priori estimate yields

$$\|w_f\|_{2,\beta} \le \|f\|_{0,\beta} = \|\mathcal{D}(w_f)\|_{0,\beta}$$

implying that also $\mathcal{D}^{-1}$ is a continuous operator. The implicit function theorem now guarantees that for any metric g close to the standard metric and a threshold δ_0, we can find a family of functions $u_z \in C^{2,\beta}(\mathbb{S}^{n-1})$, $z \in (a+\delta_0, b-\delta_0)$, such that the resulting surfaces Σ_{z+u_z} have constant mean curvature, namely

$$L^g(\Sigma_{z+u_z}) = \fint_{\Sigma_{z+u_z}} L^g(\Sigma_{z+u_z}) \,\mathrm{d}\mu$$

and are very close to horizontal. This means that whenever $|g-\bar{g}|_{C^k} \le \varepsilon$ then

$$\|u_z\|_{C^{2,\beta}} \le C(n)\varepsilon \quad \forall z \in [a+2\delta_0, b-2\delta_0].$$

A priori how close the metric needs to be at least to the standard metric depends on z, too, but since z ranges in an interval with compact closure, we can forget about this dependence, as long as the metric is close enough, which we can a priori assume by choosing $\varepsilon = \varepsilon(n) > 0$ small enough. From this we can also conclude that for all $z_1, z_2 \in (a+\delta, a-\delta)$ we have $r(z_1) = r(z_2) + \mathcal{O}(\varepsilon)$. The proof now follows from the following Lemma.

Lemma 3.11.
The resulting surfaces

$$\{\Sigma_{z+u_z} \,:\, z \in (a+\delta_0, b-\delta_0)\}$$

are smooth constant mean curvature hypersurfaces in $\mathbb{S}^{n-1} \times [a,b]$ that are distinct and cover the whole stripe $[a+2\delta_0, b-2\delta_0]$. Furthermore they form a foliation.

Proof. The equation on u to form a constant mean curvature graph is a quasilinear elliptic equation with coefficients depending on u and ∇u and on the metric g. Thus by a standard PDE theory bootstrap argument we can conclude that each of the surfaces is smooth . Also by assumption
$$\int_{\mathbb{S}^{n-1}} u_z(x)\, \mathrm{d}\mu_0(x) = 0$$
for all z in the range such that if for some $z_1, z_2 \in (a+\delta_0, b-\delta_0)$ we have $z_1 - z_2 = u_{z_1}(x) - u_{z_2}(x)$ for all $x \in \mathbb{S}^{n-1}$, then
$$\mathrm{Vol}(\mathbb{S}^{n-1})(z_1 - z_2) = \int_{\mathbb{S}^{n-1}} (u_{z_1}(x) - u_{z_2}(x))\, \mathrm{d}\mu_0(x) = 0.$$
So the surfaces are distinct. Now if z is very close to $a+\delta_0$, then $z + u_z(x) \le a + 2\delta_0$ for any $x \in \mathbb{S}^{n-1}$. Similarly for z close to $b - \delta_0$, we have $z + u_z(x) \ge b - 2\delta_0$. In this way the function
$$(a+\delta_0, b-\delta_0) \ni z \mapsto z + u_z(x) \in [a + 2\delta_0, b - 2\delta_0]$$
is surjective for any $x \in \mathbb{S}^{n-1}$ which shows that the family of hypersurfaces cover the stripe as claimed. By our parameterization of the foliation via the volume compared to a reference sphere Σ_{z_0} we have $\partial_z V(z) \neq 0$ such that by the inverse function theorem the map $\tilde{\mathcal{B}} := \mathcal{B}(g, \cdot, \cdot)$ is invertible locally around z_0. This means the surfaces cannot intersect locally. But z_0 can be chosen arbitrarily as long as there is enough distance to the boundaries of the neck. □

□

Remark 3.12. In (c) Λ_0 is a threshold and plays the same role as the parameter δ in (a) and (b). It means that the necks have to overlap in a region large enough, so that there is still enough distance to the ends of both necks.

If we now combine all definitions and results from above we can find for any (ε', k') a set of parameters $\varepsilon > 0$ and $k \ge 1$ and $\Lambda \ge 10$ such that if the curvature around a point p is $(\varepsilon, k, \Lambda)$-cylindrical then we can also find a neighborhood which can be written as the graph of a function
$$u : \mathbb{S}^{n-1} \times [-(\Lambda - 1), \Lambda - 1] \to \mathbb{R}.$$
This graph is $c(n,\Lambda)\varepsilon$ close to the standard cylinder in the C^{k+2}-norm. We can apply Theorem 3.10 (a) to find a normal parametrization which is a normal (ε', k')-hypersurface neck. The statement also guarantees that this neck is contained in a maximal normal hypersurface neck. In this way we have essentially two different ways to look at a neck. One is the parameterization which is very close in a C^k norm to the one of the standard cylinder embedding. The other is the cylindrical graph representation.
In the following we need a time dependent notion of necks in order to cope with the movement of the surface. We say that a point (p,t) lies in the center of a (geometric or hypersurface) neck if $p \in \mathcal{M}$ lies at the center of a (geometric or hypersurface) neck w.r.t the immersion $F(.,t)$. For $s \le 0$ we define
$$\rho(r,s) := \sqrt{r^2 - 2(n-1)s}$$
which is the radius of a standard cylinder with radius r evolving by mean curvature flow. We will also introduce the following notation for so called parabolic neighborhoods
$$\mathcal{P}(p,t,r,\theta) = \Big\{(q,s)\ :\ q \in \mathcal{B}_{g(t)}(p,r),\ s \in [t-\theta, t]\Big\}$$
where $\mathcal{B}_{g(t)}(p,r)$ is the closed ball of radius r around p with respect to the metric $g(t)$. Now we introduce shrinking curvature necks.

Definition 3.13 (Centers of shrinking curvature necks)**.**
A point (p_0, t_0) lies in the center of an $(\epsilon, k, \Lambda, \theta)$-**shrinking curvature neck**, if for $r_0 := \hat{r}_0(p_0, t_0) = \frac{n-1}{H(p_0,t_0)}$,$\mathcal{B}_0 := \mathcal{B}_{g(t_0)}(p_0, r_0\Lambda)$ and $\hat{\mathcal{P}}(p, t, \Lambda, \theta) := \mathcal{P}(p, t, \hat{r}\Lambda, \hat{r}^2\theta)$ the following two properties hold

1. the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, \Lambda, \theta)$ does not contain surgeries
2. For all $t \in [t_0 - r_0^2\theta+, t_0]$ the region $\mathcal{B}_0$ w.r.t. the immersion $F(., t)$ scaled by $\rho(r_0, t - t_0)$, is ϵ-cylindrical and (ϵ, k)-parallel at all points.

We will also define centers of shrinking hypersurface necks according to Definition 3.5

Definition 3.14 (Centers of shrinking hypersurface necks)**.**
We say the point (p_0, t_0) lies in a $(\varepsilon, k, \Lambda, \theta)$ **shrinking hypersurface neck**, if for all $t \in [t_0 - r_0^2\theta+, t_0]$ the point (p_0, t) lies in a (ε, k) hypersurface neck $\mathcal{N}_t \subset \mathcal{B}_0$ such that

1. the mean radius of each crosssection $\Sigma_z \subset \mathcal{N}_t$ satisfies $r(z) = \rho(r_0, t - t_0)(1 + \mathcal{O}(\varepsilon))$
2. the length of $\mathcal{N}_t$ is at least Λ.
3. there exists a unit vector $\omega \in \mathbb{R}^{n+1}$ such that
$$|\langle\nu(p, t), \omega\rangle| \leq \varepsilon$$
for each $p \in \mathcal{N}_t$.

Remark 3.15. From Proposition 3.5 we see that if we choose $\varepsilon > 0$ sufficiently small, we can always arrange that a point that lies a shrinking curvature neck also lies in a shrinking hypersurface neck such that we will use the notion of a shrinking neck in the sense of Definition 3.14 in the following chapters.

Here the notation "+" means that if $t_0 - r_0^2\theta$ is a surgery time, we consider the altered manifold right after the surgery procedure has been performed.

Lemma 3.16.
The mean radius function $r(z)$ on a normal neck normal (ε, k)-hypersurface neck $\mathcal{N}$: $\mathbb{S}^{n-1} \times [0, \Lambda] \to \mathcal{M}^n$ of length Λ satisfies
$$r(z_1) - r(z_2) \leq C(n)\Lambda\varepsilon$$
for any $z_1, z_2 \in (0, \Lambda)$.

Proof. By assumption (3.4) we have $r'(z) \leq |r'(z)| \leq \varepsilon r(z)$ such that
$$\begin{aligned} r(z_1) - r(z_2) = \int_{z_2}^{z_1} r'(z)\,\mathrm{d}z &\leq \varepsilon(z_1 - z_2)^{1-\frac{1}{n}} \left(\int_{z_2}^{z_1} r(z)^n\,\mathrm{d}z\right)^{\frac{1}{n}} \\ &= \varepsilon(z_1 - z_2)^{\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}} \mathrm{vol}(\mathbb{S}^{n-1} \times [z_2, z_1])^{\frac{1}{n}} \\ &\leq C(n)(z_1 - z_2)\varepsilon \end{aligned}$$
□

Remark 3.17. There are a couple of easy consequences from this definition. One is the following. If $\varepsilon > 0$ is small enough then at any point in space and time (p, t) within the shrinking curvature neck we can find universal constants $0 < c_1 < c_2$ close to 1 such that
$$c_1\frac{n-1}{\rho(r_0, t - t_0)} \leq H(p, t) \leq c_2\frac{n-1}{\rho(r_0, t - t_0)}.$$

This follows from the compactness of the neck region and the fact that by (3.1)

$$|\lambda_1| \le \frac{\varepsilon}{R(t)} \quad \text{and} \quad |\lambda_i - \frac{1}{R(t)}| \le \frac{\varepsilon}{R(t)}$$

for $i = 2, \dots, n$ for all $t \in [t_0 - r_0^2\theta+, t_0]$ since $W = \frac{1}{R(t)}\mathrm{Id}$ on the standard sphere of radius $R(t)$.

The next result is crucial and guarantees the existence of necks of arbitrary length in space and time.

Theorem 3.18 (Neck Detection Theorem, [HS09]).
Let $\mathcal{M}_t$, $t \in [0, T)$ be a mean curvature flow with surgeries in $\mathbb{R}^{n+1}$ for $n \ge 3$ with $\mathcal{M}_0 \in C(R, \gamma)$. Let $\epsilon, \theta, \Lambda$ and $k \ge k_0$ be given. Then we can find $\eta_0 = \eta_0(\gamma, \epsilon, k, \Lambda, \theta)$ and $H_0 = h_0(\gamma, \epsilon, k, \Lambda, \theta)R^{-1}$ s.t. if $p_0 \in \mathcal{M}$ and $t_0 \in [0, T)$ satisfy

$$\textbf{(ND1)} \quad H(p_0, t_0) \ge H_0, \qquad \frac{\lambda_1(p_0, t_0)}{H(p_0, t_0)} \le \eta_0$$

(ND2) the neighbourhood $\hat{\mathcal{P}}(p_0, t_0, \Lambda, \theta)$ does not contain surgeries.

Then

1. the neighborhood $\hat{\mathcal{P}}(p_0, t_0, \Lambda, \theta)$ is an $(\epsilon, k_0 - 1, \Lambda, \theta)$-shrinking curvature neck,
2. the neighborhood $\hat{\mathcal{P}}(p_0, t_0, \Lambda - 1, \frac{\theta}{2})$ is an $(\epsilon, k, \Lambda - 2, \frac{\theta}{2})$-shrinking curvature neck.

For the sake of completeness we will give a proof of this result which is taken from [HS09, Lemma 7.4]. We will use some of the estimates that we will cover extensively in Chapter 4.

Proof. Assume the statement is wrong, that is we can find a sequence $\{\mathcal{M}_t^j\}_{j\ge1}$ of solutions to the flow, a sequence of time steps t_j a sequence of points $p_j \in \mathcal{M}^j$ s.t. for values of H and λ_1 evaluated at $F_j(p_j, t_j) \in \mathcal{M}_{t_j}^j$, say H_j and $\lambda_{1,j}$ we have the following properties:

1. Each flow starts with an initial surface in same class $C(R, \alpha)$ satisfying all estimates with the same constants.
2. The parabolic neighborhood $\mathcal{P}^j(p_j, t_j, \hat{r}_j\Lambda, \hat{r}_j^2\theta)$ is not changed by surgeries ($\mathcal{P}^j$ is the neighborhood belonging to $\mathcal{M}^j$).
3. $H_j \to \infty$ and $\frac{\lambda_{1,j}}{H_j} \to 0$ as $j \to \infty$.
4. (p_j, t_j) does not lie in the center of an $(\epsilon, k_0 - 1, \Lambda, \theta)$-shrinking neck.

Since H_j tends to infinity and at $t = 0$ the mean curvature is uniformly bounded, the sequence t_j must be bounded away from 0. Therefore, for large j we can guarantee that $t_j > \theta(n-1)^2 H_j^{-2} = \theta\hat{r}_j^2$ which implies $t_j - \hat{r}_j^2\theta > 0$. Thus $\mathcal{P}^j(p_j, t_j, \hat{r}_j\Lambda, \hat{r}_j^2\theta)$ is well defined for j large enough.
In the next step we perform a parabolic rescaling on each flow M_t^j to get $H(p_j, t_j) = n-1$. Additionally, we translate the flow in space and time to map (p_j, t_j) to the origin and the time 0. If F_j is the parametrisation of the original M_t^j, then we denote the rescaled flow by $\bar{\mathcal{M}}_\tau^j$ defined through

$$\bar{F}_j(p, \tau) := \frac{1}{\hat{r}_j}\left(F_j\left(p, \hat{r}_j^2\tau + t_j\right) - F_j(p_j, t_j)\right).$$

Note that the mean curvature satisfies $H(\lambda p) = \lambda^{-1}H(p)$. Thus, if we choose coordinates centered at p_j for every flow then we can write 0 instead of p_j and the neighbourhood $\mathcal{P}^j(p_j, t_j, \hat{r}_j\Lambda, \hat{r}_j^2\theta)$ becomes $\bar{\mathcal{P}}^j(0,0,\Lambda,\theta)$ belonging to $\bar{\mathcal{M}}_t^j$. Also by construction we have $\bar{F}_j(0,0) = 0$ and $\bar{H}_j(0,0) = n-1$. Because the initial surface has normalised curvature and lies in $C(R,\alpha)$ for every flow j, the gradient estimate in Theorem 4.23 which we will cover in Chapter 4 guarantees estimates on $|A|$ and all derivatives up to order k_0 at least in a small neighbourhood $\bar{P}_j(0,0,d,d)$ independent from j. This implies in turn uniform estimates on the immersions $\bar{F}_j$ in the C^{k_0+2}-norm. Therefore, we can extract a subsequence converging to some limit flow $\tilde{\mathcal{M}}_\tau^\infty$ in the C^{k_0+1}-norm. Since for every flow j the convexity estimates from Theorem 4.1 hold, we obtain for the rescaled surfaces

$$\bar{S}_m^j \geq -\eta\bar{H}_j^m - H_j^{-m}(n-1)^m C_{\eta,\mathcal{M}_0}, \tag{3.7}$$

where S_k is the k-th symmetric polynomial in the principle curvatures $\lambda = (\lambda_1,\ldots,\lambda_n)$. Thus, by 3., the limit flow satisfies $\tilde{S}_m \geq -\eta\tilde{H}^m$ for all $\eta > 0$ which implies $S_m \geq 0$ for all $m = 1,\ldots,n$. This means, however, that all principle curvatures are non negative $\tilde{\lambda}_i \geq 0$ and since we have the preserved inequality $\tilde{\lambda}_1 + \tilde{\lambda}_2 \geq \alpha_0\tilde{H}$ we know that $\tilde{\lambda}_i > 0$ for all $i = 2,\ldots,m$ and so $\tilde{S}_m > 0$ for all $m = 2,\ldots,n$. Additionally, by the convergence in property 3. from above we get $\tilde{\lambda}_1(0,0) = 0$. If the quotient $\tilde{Q}_n := \frac{\tilde{S}_n}{\tilde{S}_{n-1}}$ is positive somewhere in $\bar{\mathcal{P}}^\infty(0,0,d,d)$, then it remains so by the strong parabolic maximum principle. But $\tilde{Q}_n(0,0) = 0$, therefore $\tilde{Q} \equiv 0$ on $\bar{\mathcal{P}}^\infty(0,0,d,d)$. Consequently, we have $\tilde{\lambda}_1 \equiv 0$ in this neighborhood. Since the principle curvatures satisfy $\tilde{\lambda}_1 = 0$, $\tilde{\lambda}_i > 0$, $i = 2,\ldots,n$ we have $|\tilde{A}|^2 - \frac{1}{n-1}\tilde{H} \geq 0$. Similar to the convexity estimate we can pass to the limit in the cylindrical estimates taken from Theorem 4.19 to get the other inequality $|\tilde{A}|^2 - \frac{1}{n-1}\tilde{H} \leq 0$ and we can conclude that the quantity $|\tilde{A}|^2 - \frac{1}{n-1}\tilde{H} =: f_{2,0}$ vanishes identically on the same neighborhood. By the evolution equation for $f_{2,0}$ (4.22) in the proof of the cylindrical estimates, we also obtain that $|\tilde{H}\nabla_i\tilde{h}_{kl} - \nabla_i\tilde{H}\tilde{h}_{kl}|^2$ vanishes identically. Considering only the anti symmetric part of this tensor, we obtain

$$|\nabla_i\tilde{H}h_{jk} - \nabla_j\tilde{H}h_{ik}|^2 \equiv 0$$

on $\bar{\mathcal{P}}^\infty(0,0,d,d)$. If we choose an orthonormal frame such that $e_1 = \nabla\tilde{H}/|\nabla\tilde{H}|$ this implies

$$|\nabla\tilde{H}|^2\left(|\tilde{A}|^2 - \sum_{i=1}^n \tilde{h}_{1i}^2\right) = 0.$$

If $|\nabla\tilde{H}|^2 = 0$ then also $|\nabla\tilde{A}|^2 = 0$ and we deduce by Lawson's theorem [Law69, Proposition 1] that the limit flow $\tilde{\mathcal{M}}_\tau^\infty$ is a portion of a shrinking cylinder in the parabolic neighborhood $\bar{\mathcal{P}}^\infty(0,0,d,d)$. On the other hand if $|\tilde{A}|^2 = \sum\tilde{h}_{1i}^2$ then this contradicts $\tilde{\lambda}_i > 0$. So far we have shown that if j is large enough, the mean curvature is close to the one of a unite cylinder in the small neighborhood $\bar{\mathcal{P}}^j(0,0,d,d)$. Therefore, we have the estimate $\bar{H}_j \leq 2(n-1)$. The gradient estimates yield uniform estimates on a slightly larger neighborhood, say $\bar{\mathcal{P}}^j(0,0,2d,2d)$. By repeating the previous argument, we know the convergence to a cylinder also in this larger neighborhood. In this way, after a finite number of steps we can conclude the convergence in the C^{k_0+1}-norm on the whole neighborhood $\bar{\mathcal{P}}^j(0,0,\Lambda,\theta)$. This implies that for large j these neighborhoods are $(\epsilon, k_0-1, \Lambda, \theta)$-shrinking necks which contradicts assumption 4.. The second assertion can be proven similarly. Again we have C^2-bounds on the Shape-operator. If we take a slightly smaller parabolic neighborhood $\bar{\mathcal{P}}^j(0,0,\Lambda-1,\theta/2)$ we are able to employ the interior regularity results for surfaces evolving by mean curvature flow to find uniform bounds also in the C^{k+1}-norm. From this point on, we can repeat all steps from above and obtain the desired conclusion. □

Remark 3.19. The proof can be adapted such that the point (p_0, t_0) lies in a $(\varepsilon, k-1, \Lambda - 2, \theta)$ shrinking hypersurface neck. This is essentially an application of Proposition 3.5 at each time t. We refer to [HS09, Lemma 7.2] and [HS09, Lemma 7.9].

Chapter 4

The central three estimates

One of the main efforts in the surgery construction for mean curvature flow is to derive estimates that control all relevant geometric quantities and are not affected by the surgery procedure. In this chapter we want to briefly state those estimates and then later aim to improve the estimates by localization on necks. To avoid issues when we want to integrate partially we assume that the solution called $\mathcal{C}_t$ for $t \in (-\theta R_0^2, 0]$ is either closed or a periodic and smooth solution to mean curvature flow with final time 0 and final radius R_0 whose periodic components are also $(\varepsilon, k, \Lambda, \theta)$ shrinking curvature necks. By periodic we mean that it has periodic boundary data, i.e. it is periodic in space and time with period $\Lambda \geq 10$. In other words the normal derivative along the boundary of the parameterization is 0 such that if $v \in \mathbb{R}^{n+1}$ is a vector normal to the boundary of a component then we have

$$\frac{\partial \mathcal{C}}{\partial v}(p) = 0 \quad \forall p \in \partial \mathcal{C}.$$

Before we state the estimates and show how to improve them we want to make a brief statement about this assumption. In view of possible applications to 2-convex mean curvature flow with surgery or even weaker convexity assumptions the present situation is unrealistic since necks naturally appear as singularity formations and there is no reason to believe that they should be periodic. Usually when one aims to localize an estimate the standard procedure is to look at cutoff functions. The naive approach to just take a linear cutoff function that for example involves the z-coordinate along the neck and thus isolates a part away from the boundaries of the neck region has little hope for success. The main reason is that when we compute the evolution equation for the test function f under consideration multiplied with the square of such a cutoff function φ we will get upon partial integration extra terms of the form

$$\int_{\mathcal{M}} |\nabla f| |\nabla \varphi| \varphi \, \mathrm{d}\mu.$$

However, the good gradient terms have a factor of φ^2 such that there is no obvious way to absorb these new terms. Surely, such an approach is way to simple, as the behavior of the necks also largely depends on the behavior of the regions nearby. In particular, this is a very delicate part of the proof of the existence of 2-convex mean curvature flow with surgeries in [HS09]. Intuitively, for the time evolution of the neck it should make a difference weather it is closing up to a convex cap or opening up whilst significantly reducing the maximal curvature. That is why we could think of a cut off function which is time dependent. This method has been successfully employed for mean curvature flow in the past. For example, there are local versions of Huisken's Monotonicity formula. One of which was proved by J. Buckland in [Buc05]. His clever choice of cutoff function involves the heat kernel and a distance to the end of the necks related cutoff. However, this a whole new project on its one as there are several parameters involving regularity of the neck and surrounding regions to be chosen properly. For now we will continue with the assumptions we have made above in order to avoid extra boundary terms coming from the partial integration.

4.1 Convexity Estimate

The first estimate is the so called convexity estimate. The convexity estimate measures how far from being positive the first principle curvature is, i.e. how far the surface is from being convex. The following estimate even holds if we only assume mean convexity for which we refer to [HS99a].

Theorem 4.1 (Convexity Estimate)**.**
Let $\mathcal{M}_t$ be a smooth solution to mean curvature flow in the class $C(R;\gamma)$. Then for any $\delta > 0$ there exists a constant $K_\delta(n, \alpha)$ such that

$$\lambda_1 \geq -\delta H - K_\delta$$

on $\mathcal{M}_t$ for all $t > 0$.

For small $\delta > 0$ the constant K_δ can be really large. This is the reason why we want to eliminate this constant by exploiting the neck assumptions. In order to do so we will carefully repeat the steps in the proof of Theorem 4.1. To prove such a result we will need a way to compare the different principle curvatures and use that some are larger than the others. For this we define the symmetric elementary polynomials.

Definition 4.2 (Symmetric polynomials)**.**
For any $k = 0, \dots, n$ we define the k-th symmeytric polynomial as

$$S_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_n} \tag{4.1}$$

for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Also $s_0 \equiv 1$ and $s_k \equiv 0$ for $k > n$. Further we define the open cones

$$\Gamma_k := \{\mu \in \mathbb{R}^n : S_1(\mu) > 0, \dots S_k(\mu) > 0\}. \tag{4.2}$$

In this way the convexity assumptions can be reformulated.

Proposition 4.3.
For any smooth closed hypersurface with principle curvatures $\lambda_1 \leq \cdots \leq \lambda_n$ we have

$$\lambda_1 \geq 0, \dots, \lambda_n \geq 0 \quad \Leftrightarrow \quad S_1(\lambda) \geq 0, \dots, S_n(\lambda) \geq 0$$

where $\lambda := (\lambda_1, \dots, \lambda_n)$.

Proof. The first implication is trivial. To prove the converse, we recall that if we denote by $S_{k;i}$ the sum over all terms with λ_i left out, then $S_1 \geq 0, \dots, S_k \geq 0$ implies

$$S_{h;i} \geq 0$$

for any $1 \leq h \leq k-1$ and $i \in \{1, \dots, n\}$ which is a consequence of the Newton McLaurin inequality, see Chapter 2 in [HS99a]. The case $n = 3$ we have $\lambda_1 + \lambda_2 = S_{1;3} \geq 0$ and

$S_1 = H \geq 0$ implies that $\lambda_2, \lambda_3 \geq 0$. Also $S_3 = \lambda_1\lambda_2\lambda_3 \geq 0$ such that $\lambda_1 \geq 0$. By induction we argue that for general $n > 3$ we have

$$S_{1,n} \geq 0, \dots, S_{n-1,n} \geq 0$$

such that $\lambda_1, \dots, \lambda_{n-1}$ satisfy the induction hypothesis and the result is proven. □

It is thus quiet natural to look at these polynomials. In fact, starting from $S_1 = H > 0$, we can use an induction procedure to prove estimates on how positive the remaining S_k are all the way up to S_n from which we can then deduce the result above.

Theorem 4.4 (Estimates on S_k)**.**
Let $\mathcal{M}_t$ be a smooth solution to mean curvature flow with non negative mean curvature. Then for any $\delta > 0$ and $2 \leq k \leq n$, there exists a constant $C_{\delta,k} := C(\delta, k, n, \mathcal{M}_0) > 0$ such that

$$S_k \geq -\delta H^k - C_{\delta,k}.$$

In the following we will return to the particular solution C_t we defined in the introduction of this chapter. We will show now that in the neck situation all symmetric polynomials of the curvature but the last are positive.

Proposition 4.5.
If $\varepsilon = \varepsilon(n)$ is small enough, then any point in a (ε, k) hypersurface neck $\mathcal{N}$ satisfies

$$S_l \geq C(n,l)H^l$$

for each $l = 1, \dots, n-1$ uniformly on $\mathcal{N}$.

Proof. We recall that since $\mathcal{N}$ is ε close to the standard cylinder in the C^k topology, we have that if ε is small enough there are two constants c_1, c_2 close to 1, $c_1 < c_2$ such that

$$c_1 \frac{H}{n-1} \leq \lambda_i \leq c_2 \frac{H}{n-1}$$

for $i > 1$ and $\lambda_1 \geq -\varepsilon H$. Then we compute for $l \leq n-1$

$$\begin{aligned} S_l(\lambda) &= \sum_{1=i_1<i_2<\dots<i_l\leq n} \lambda_{i_1} \cdots \lambda_{i_l} + \sum_{1<i_1<i_2<\dots<i_l\leq n} \lambda_{i_1} \cdots \lambda_{i_l} \\ &\geq -\varepsilon c(n,l) c_2^l \frac{H^l}{(n-1)^{l-1}} + c_1^l c(n,l) \frac{H^l}{(n-1)^l} \\ &= c(n)(c_1^l - \varepsilon c_2^l(l)) \frac{H^{n-1}}{(n-1)^{n-1}} \geq c_0(n,l)H^l, \end{aligned}$$

if $\varepsilon = \varepsilon(n)$ is small enough. This proofs the proposition. □

In particular, $\mathcal{C}_t$ satisfies $S_l \geq cH^l$ for all times. The last symmetric polynomial, however, is not positive everywhere on $\mathcal{N}$. In general, we will assume that for some $k \in \{1, \dots, n\}$ we have the uniform bound $S_k > cH^k$ for some $c > 0$. For all partial integrations to be well defined we will either assume that we have a shrinking neck with periodic boundary data or that the solution is closed and call it $\mathcal{C}_t$. By [HS99a, Proposition 3.4] such a uniform estimate is also preserved under the flow. For now we will keep the calculations as general as possible. As in the proof of Theorem 4.1 we look at the test function

$$f_{\sigma,\delta} := \frac{-Q_{k+1} - \delta H}{H^{1-\sigma}}$$

where $Q_{k+1} := \frac{S_{k+1}}{S_k}$. By Proposition 4.5 this function is well defined. Before we proceed with the usual analysis we need a couple of algebraic identities involving the symmetric polynomials. We remark that this is also a first simplification compared to the proof in [HS99a] as we do not need to regularize S_k to guarantee that f is well defined.

Lemma 4.6. [HS99a, Proposition 2.2]
Let k be any positive integer from 0 to n and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be a vector. We denote by $\sigma_{k,i}(\lambda)$ the sum over the terms of σ_k, which do not contain λ_i. Then the following identities hold

$$\frac{\partial \sigma_{k+1}}{\partial \lambda_i}(\lambda) = \sigma_{k,i}(\lambda) \tag{4.3}$$

$$\sigma_{k+1}(\lambda) = \sigma_{k+1,i}(\lambda) + \lambda_i \sigma_{k,i}(\lambda) \tag{4.4}$$

$$\sum_{i=1}^{n} \sigma_{k,i}(\lambda) = (n-k)\,\sigma_k(\lambda) \tag{4.5}$$

$$\sum_{i=1}^{n} \lambda_i \sigma_{k,i}(\lambda) = (k+1)\,\sigma_{k+1}(\lambda) \tag{4.6}$$

$$\sum_{i=1}^{n} \lambda_i^2 \sigma_{k,i}(\lambda) = \sigma_{k+1}(\lambda)\sigma_1(\lambda) - (k+1)\,\sigma_{k+2}(\lambda) \tag{4.7}$$

Proof. If we take the derivative of $\sum_{i_1<\cdots<i_{k+1}} \lambda_{i_1} \cdots \lambda_{i_{k+1}}$ with respect to λ_i, then all terms not containing λ_i disappear and in the others we just have to leave out λ_i. So, the first claim is obvious from the definition. The second equation comes from the fact, that $\lambda_i \sigma_{k,i}(\lambda)$ is exactly the sum over the terms, that are left out in $\sigma_{k+1,i}$. We recall that for a homogeneous function f of order k we have Euler's theorem telling us

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf(x).$$

Applying this to the homogeneous function σ_{k+1} of order $k+1$ we get

$$\sum_{i=1}^{n} \lambda_i \sigma_{k,i}(\lambda) = \sum_{i=1}^{n} \lambda_i \frac{\partial \sigma_{k+1}}{\partial \lambda_i}(\lambda) = (k+1)\sigma_{k+1}(\lambda),$$

which is exactly equation (4.6). We now take the sum in the second equation and apply (4.6) to get

$$\begin{aligned} n\sigma_{k+1}(\lambda) &= \sum_{i=1}^{n} \sigma_{k+1,i}(\lambda) + \sum_{i=1}^{n} \lambda_i \sigma_{k,i}(\lambda) \\ &= \sum_{i=1}^{n} \sigma_{k+1,i}(\lambda) + (k+1)\sigma_{k+1}(\lambda). \end{aligned}$$

This is equivalent to (4.5). Furthermore, by applying equation (4.4) twice we get

$$\sigma_{k+2}(\lambda) - \sigma_{k+2,i}(\lambda) = \lambda_i \sigma_{k+1,i}(\lambda) = \lambda_i \sigma_{k+1}(\lambda) - \lambda_i^2 \sigma_{k,i}(\lambda).$$

We take the sum from $i = 1$ to n on both sides and arrive at

$$\begin{aligned} n\sigma_{k+2}(\lambda) - (n-(k+2))\,\sigma_{k+1}(\lambda) &= \sigma_{k+1}(\lambda)\sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n}\lambda_i^2\sigma_{k,i}(\lambda) \\ &= \sigma_{k+1}(\lambda)\sigma_1(\lambda) - \sum_{i=1}^{n}\lambda_i^2\sigma_{k,i}(\lambda) \end{aligned}$$

by equation (4.6). The left-hand side equals $(k+1)\sigma_{k+2}(\lambda)$ which finishes the proof for (4.7). $\square$

In the following, we will consider S_k as a function acting on the eigenvalues of the symmetric matrix h^i_j. With an abuse of notation we can also let S_k act on symmetric matrices as the sum of $k \times k$ sub determinants, such that we will write

$$\nabla_l S_k = \frac{\partial S_k}{\partial h^i_j}\nabla_l h^i_j.$$

We will now cite some important properties of S_k. Especially the concavity of Q_{k+1} will be crucial.

Theorem 4.7. [HS99a, Theorem 2.14]
Let $c > \delta > 0$ and choose orthonormal coordinates around the point under consideration. Then there exists a constant $C = C(c, \delta, n)$ such that at any point p with $(\lambda_1(p), \dots, \lambda_n(p)) \in \Gamma_k$ and

$$-cH(p) \leq Q_{k+1}(p) \leq -\delta H(p)$$

we have

$$\sum_{i,j,p,q,l}\frac{\partial^2 Q_{k+1}}{\partial h_{ij}\partial h_{pq}}\nabla_l h_{ij}\nabla_l h_{pq} \leq -\frac{1}{C}\frac{|\nabla A|^2}{|A|}. \tag{4.8}$$

Furthermore, we have the algebraic identity [HS99a, Lemma 2.15]

$$\begin{aligned} \sum_{ij}\frac{\partial S_k}{\partial h_{ij}}\nabla_i\nabla_j S_{k+1} &= \sum_{i,j,l,m}\frac{\partial S_k}{\partial h_{ij}}\frac{\partial S_{k+1}}{\partial h_{lm}}\nabla_l\nabla_m h_{ij} && (4.9) \\ &\quad + \sum_{i,j,p,q,l,m}\frac{\partial S_k}{\partial h_{ij}}\frac{\partial^2 S_{k+1}}{\partial h_{lm}\partial h_{pq}}\nabla_i h_{lm}\nabla_j h_{pq} \\ &\quad - HS_kS_{k+1} + (k+1)S_{k+1}^2 + k\Big((k+1)S_{k+1}^2 - (k+2)S_kS_{k+2}\Big) && (4.10) \end{aligned}$$

and the estimates

$$\begin{aligned} \left|\frac{\partial S_k}{\partial h_{ij}}\xi^i\zeta^j\right| &\leq c(n,k)H^{k-1}|\xi||\zeta| \\ \left|\nabla_l\frac{\partial S_k}{\partial h_{ij}}\right| &\leq c(n,k)H^{k-2}|\nabla A|. \end{aligned}$$

Remark 4.8.

1. We remark that (4.10) is a form of commutator identity in the same spirit as the identity in Proposition 1.2.

2. If on a closed manifold $\mathcal{M}^n$ we have $S_k > cH^k$ as in our assumption we also have $S_l > 0$ for all $l = 1, \ldots, k$. This is true since the cones Γ_k are connected and satisfy $\Gamma_k \subset \Gamma_l$ for $l < k$. Then $S_l > 0$ follows from the fact that there is at least one point $p \in \mathcal{M}^n$ in each connected component such that $\lambda_1(p) > 0$.

3. An inductive argument using the maximum principle repeatedly then yields the same estimates
$$S_k > cH^k, \quad S_l > 0$$
for all times and $l = 1, \ldots, k$, if the solution is closed and satisfies the estimates initially. We refer to [HS99a, Proposition 3.3 and 3.4]

Also on the set where $f_\sigma > 0$ the assumption of Theorem 4.7 is satisfied and we also have by (4.10) together with the Newton-McLaurin inequality $(k+1)S_{k-1}S_{k+1} \leq kS_k^2$

$$\begin{aligned}\sum_{ij} \frac{\partial S_k}{\partial h_{ij}} \nabla_i \nabla_j S_{n+1} > &\sum_{i,j,p,q,l,m} \frac{\partial S_k}{\partial h_{ij}} \frac{\partial^2 S_{k+1}}{\partial h_{lm} \partial h_{pq}} \nabla_i h_{lm} \nabla_j h_{pq} \\ &+ \sum_{i,j,l,m} \frac{\partial S_k}{\partial h_{ij}} \frac{\partial S_{k+1}}{\partial h_{lm}} \nabla_l \nabla_m h_{ij} \\ &+ \delta H^2 S_k^2. \end{aligned} \tag{4.11}$$

Here we used $-HS_kS_{k+1} \geq -\delta H^2 S_k^2$ by assumption. Now we compute the evolution equation of f_σ.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f = \Delta f &+ \frac{2(1-\sigma)}{H} \langle \nabla H, \nabla f \rangle - \frac{\sigma(1-\sigma)}{H^2} f |\nabla H|^2 \\ &+ \frac{1}{H^{1-\sigma}} \frac{\partial^2 Q_{k+1}}{\partial h_{ij} \partial h_{pq}} \nabla_m h_{ij} \nabla_m h_{pq} + \sigma |A|^2 f. \end{aligned} \tag{4.12}$$

As usual we cannot argue with the standard maximum principle due to the positive absolute term on the right hand side. We follow the standard proof scheme and derive an L^p-estimate for f. We denote by f_+ the positive part of f. Then by multiplying (4.10) with pf^{p-1} and integrating over $\mathcal{C}_t$ we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu = &-p(p-1) \int_{\mathcal{C}_t} f_+^{p-2} |\nabla f|^2 \,\mathrm{d}\mu + 2p(1-\sigma) \int_{\mathcal{C}_t} \frac{f_+^{p-1}}{H} \langle \nabla H, \nabla f \rangle \,\mathrm{d}\mu \\ &+ p \int_{\mathcal{C}_t} \frac{f_+^{p-1}}{H^{1-\sigma}} \frac{\partial^2 Q_{k+1}}{\partial h_{ij} \partial h_{pq}} \nabla_m h_{ij} \nabla_m h_{pq} \,\mathrm{d}\mu + p\sigma \int_{\mathcal{C}_t} |A|^2 f_+^p \,\mathrm{d}\mu. \end{aligned}$$

Since we are only looking at the positive part of f_σ, we are in the situation of Theorem 4.7, so we can use (4.8) for the third term on the right hand side which leads to

$$\frac{\partial^2 Q_{k+1}}{\partial h_{ij} \partial h_{pq}} \nabla_m h_{ij} \nabla_m h_{pq} \leq -\frac{1}{C\sqrt{n}} \frac{|\nabla A|^2}{H}.$$

Also for any $\gamma > 0$, by the Peter Paul inequality we get

$$\begin{aligned}\frac{f_+^{p-1}}{H} \langle \nabla H, \nabla f \rangle \leq \frac{f_+^{p-1}}{H} |\nabla H| \nabla f| \leq \frac{f_+^{p-1}}{H} \left(\frac{n+2}{3} \gamma \frac{|\nabla A|^2}{H^{1-\sigma}} + \frac{1}{\gamma} H^{1-\sigma} |\nabla f|^2 \right) \\ \leq \frac{(n+2)\gamma}{3} \frac{f^{p-1}}{H^{2-\sigma}} |\nabla A|^2 + \frac{1}{\gamma} f^{p-2} \underbrace{H^{-\sigma} f_+}_{\leq c_0(\delta,n,k)} |\nabla f|^2. \end{aligned}$$

Then for $\gamma = \frac{(1-\sigma)c_0}{4(p-1)}$ and $p \geq 1 + c_0 C\sqrt{n}(n+2)$ we derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu \leq &-\frac{p(p-1)}{2}\int_{\mathcal{C}_t} f_+^{p-2}|\nabla f|^2\,\mathrm{d}\mu - \frac{p}{2C\sqrt{n}}\int_{\mathcal{C}_t}\frac{f_+^{p-1}}{H^{2-\sigma}}|\nabla A|^2\,\mathrm{d}\mu \\ &+ p\sigma n\int_{\mathcal{C}_t} H^2 f_+^p\,\mathrm{d}\mu\end{aligned}$$

where $\mathcal{C}_t$ is one periodic component of the flow by abuse of notation. The next step is to use a Poincaré-type inequality in order to absorb the absolute term into the negative gradient terms. Similar to [HS15] we can not only absorb this term once but twice, so we end up with an entirely negative right hand side.

Proposition 4.9 (Poincaré-Inequality)**.**
There is a constant $c_3 = c_3(n,\delta,k) > 0$ such that for any $p > 2$ and $\kappa > 0$

$$\int_{\mathcal{C}_t} H^2 f_+^p\,\mathrm{d}\mu \leq c_3\left((p+p\kappa^{-1})\int_{\mathcal{C}_t} f_+^{p-2}|\nabla f|^2\,\mathrm{d}\mu + (1+\kappa p)\int_{\mathcal{C}_t}\frac{f_+^{p-1}}{H^{2-\sigma}}|\nabla A|^2\,\mathrm{d}\mu\right). \tag{4.13}$$

Proof. As in [HS99a, Proposition 3.6] we compute the derivatives of f.

$$\nabla_i f = -H^{\sigma-1}S_k^{-1}\nabla_j S_{k+1} + H^{\sigma-1}S_k^{-2}S_{k+1}\nabla_j S_k - \delta H^{\sigma-1}\nabla_j H + (\sigma-1)H^{-1}f\nabla_j H$$

and

$$\begin{aligned}\nabla_i\nabla_j f = &-H^{\sigma-1}S_k^{-1}\nabla_i\nabla_j S_n - (\sigma-1)H^{\sigma-2}S_k^{-1}\nabla_i H\nabla_j S_n + H^{\sigma-1}S_k^{-2}\nabla_i S_k\nabla_j S_{k+1} \\ &+ (\sigma-1)H^{2-\sigma}S_k^{-2}S_{k+1}\nabla_i H\nabla_j S_k - 2H^{\sigma-1}S_k^{-3}S_{k+1}\nabla_i S_k\nabla_j S_k \\ &+ H^{\sigma-1}S_k^{-2}\nabla_i S_{k+1}\nabla_j S_k \\ &+ H^{\sigma-1}S_k^{-2}S_{k+1}\nabla_i\nabla_j S_k - \delta(\sigma-1)H^{\sigma-2}\nabla_i H\nabla_j H - \delta H^{\sigma-1}\nabla_i\nabla_j H \\ &+ (\sigma-1)H^{-1}\nabla_i f\nabla_j H - (\sigma-1)H^{-2}f\nabla_i H\nabla_j H + (\sigma-1)H^{-1}\nabla_i\nabla_j H.\end{aligned}$$

We now take the trace with respect to $\frac{\partial S_k}{\partial h_{ij}}$ and collect all terms involving second derivatives of H. Then we use the estimates from Theorem 4.7 to get

$$\begin{aligned}\frac{\partial S_k}{\partial h_{ij}}\nabla_i\nabla_j f \leq &-H^{\sigma-1}S_k^{-1}\frac{\partial S_k}{\partial h_{ij}}\nabla_i\nabla_j S_{k+1} + C(n)H^{k+\sigma-3}|\nabla A|^2 + C(n)H^{k-2}|\nabla f||\nabla A| \\ &+ \frac{\partial S_k}{\partial h_{ij}}\Big(H^{\sigma-1}S_k^{-2}S_{k+1}\nabla_i\nabla_j S_k - (\delta H^{\sigma-1} - (\sigma-1)H^{-1}f)\nabla_i\nabla_j H\Big).\end{aligned}$$

In order to obtain the right powers of H after integrating in view of the claim we need to multiply this inequality with $f_+^p H^{1-k-\sigma}$. Then integrating together with (4.11) yields

$$\begin{aligned}\int_{\mathcal{C}_t}\frac{\partial S_k}{\partial h_{ij}}\nabla_i\nabla_j f H^{1-k-\sigma}f_+^p\,\mathrm{d}\mu \leq &-\int_{\mathcal{C}_t} f_+^p H^{-k}S_k^{-1}\frac{\partial S_k}{\partial h_{ij}}\frac{\partial^2 S_{k+1}}{\partial h_{lm}\partial h_{pq}}\nabla_i h_{lm}\nabla_j h_{pq}\,\mathrm{d}\mu \\ &-\int_{\mathcal{C}_t} f_+^p H^{-k}S_k^{-1}\frac{\partial S_k}{\partial h_{ij}}\frac{\partial S_{k+1}}{\partial h_{lm}}\nabla_l\nabla_m h_{ij}\,\mathrm{d}\mu - \int_{\mathcal{C}_t}\delta f_+^p H^{2-k}S_k\,\mathrm{d}\mu \\ &+ C(n)\int_{\mathcal{C}_t} f_+^p\Big(H^{-2}|A|^2 + H^{-1-\sigma}|\nabla f||\nabla A|\Big)\,\mathrm{d}\mu \\ &+\int_{\mathcal{C}_t}\frac{\partial S_k}{\partial h_{ij}}f_+^p H^{1-n}S_k^{-2}S_{k+1}\nabla_i\nabla_j S_k\,\mathrm{d}\mu \\ &-\int_{\mathcal{C}_t} f_+^p\frac{\partial S_k}{\partial h_{ij}}H^{1-n}\Big(\delta - (\sigma-1)H^{1-\sigma}f_+\Big)\nabla_i\nabla_j H\,\mathrm{d}\mu.\end{aligned}$$

By our assumptions on the solution $\mathcal{C}_t$ we have no extra boundary integrals. Now we need to partially integrate all terms involving second derivatives of curvature. Also by our initial assumption we have $\delta f_+^p H^{2-k} S_k \geq c(n,k)\delta H^2 f_+^p$ such that we can bring this term to the other side.

$$\begin{aligned}\delta \int_{\mathcal{C}_t} H^2 f_+^p \,\mathrm{d}\mu \leq C(n,k) \int_{\mathcal{C}_t} f_+^p \left(H^{-2}|\nabla A|^2 + H^{-1-\sigma}|\nabla f||\nabla A| \right) \mathrm{d}\mu \\ + pC(n,k) \int_{\mathcal{C}_t} f_+^{p-2} \left(|\nabla f|^2 + f_+ H^{-2}|\nabla f||\nabla A| \right) \mathrm{d}\mu.\end{aligned}$$

The conclusion follows by the Peter-Paul inequality

$$f_+^p H^{-1-\sigma}|\nabla f||\nabla A| \leq f^{p-2}\kappa^{-1}|\nabla f|^2 + \kappa f_+^{p-1}\frac{|\nabla A|^2}{H^{2-\sigma}}$$

and

$$f_+^{p-1} H^{-2}|\nabla f||\nabla A| \leq f^{p-2}\kappa^{-1}|\nabla f|^2 + \kappa f_+^{p-1}\frac{|\nabla A|^2}{H^{2-\sigma}}$$

for any $\kappa > 0$. □

Now if we set $\kappa = p^{-\frac{1}{2}}$ and

$$\sigma \leq \frac{1}{16 c_3 \sqrt{p} n \sqrt{n} C} \qquad p \geq \max\{4, C, c_3\}$$

then

$$\begin{aligned}2p^2 n c_3 \sigma (1+\sqrt{p}) &\leq \frac{p^2}{4\sqrt{n}C} \leq \frac{p(p-1)}{2} \\ 2\sigma n p c_3 (1+\sqrt{p}) &\leq \frac{1}{2C\sqrt{n}} p\end{aligned}$$

such that we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu \leq -np\sigma \int_{\mathcal{C}_t} H^2 f_+^p \,\mathrm{d}\mu. \tag{4.14}$$

We use Hölders inequality with $q = \frac{2}{p\sigma+1}$ such that for $p\sigma \geq 2n+1 \geq \frac{1}{n}$ we have $q < \frac{1}{n+1} < 1$ and therefore,

$$\begin{aligned}\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu \leq \int_{\mathcal{C}_t} f_+^{p(1-q)} H^{p\sigma q} \,\mathrm{d}\mu = \int_{\mathcal{C}_t} \left(f_+^p H^2 \right)^{1-q} H^q \,\mathrm{d}\mu \\ \leq \left(\int_{\mathcal{C}_t} f_+^p H^2 \,\mathrm{d}\mu \right)^{1-q} \left(\int_{\mathcal{C}_t} H \,\mathrm{d}\mu \right)^q.\end{aligned}$$

Here we used that $f_+^{-pq} H^{\sigma pq} \geq 1$. We remark that so far we have only used the uniform bound from below on S_k and that no boundary integrals appear. No other neck assumptions have been involved so far. We suppose for now that we have a certain volume growth rate namely

$$|\Omega_t| = \int_t^0 \left(\int_{C_\tau} H \,\mathrm{d}\mu \right) \mathrm{d}\tau \leq c(1+|t|)^{n+1}.$$

Here Ω_t is the area enclosed by the solution C_t. The relation to the integral of H is given by the evolution equation of the surface element

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_t| = \int_{C_t} H \,\mathrm{d}\mu_t.$$

This assumption is of course true, if we are in a shrinking curvature neck when the constant c also depends on Λ.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu &\leq -\int_{\mathcal{C}_t} f_+^p H^2 \,\mathrm{d}\mu \\ &\leq -\left(\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu\right)^{\frac{1}{1-q}} \left(\int_{\mathcal{C}_t} H \,\mathrm{d}\mu\right)^{-\frac{q}{q-1}} \\ &= -\left(\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu\right)^{1+\frac{2}{p\sigma-1}} \left(\int_{\mathcal{C}_t} H \,\mathrm{d}\mu\right)^{-\frac{2}{p\sigma-1}}.\end{aligned}$$

Now we are in position to apply the same ODE argument as in [HS15] with the notations

$$\phi(t) := \int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu, \quad \psi(t) = \int_{\mathcal{C}_t} H \,\mathrm{d}\mu, \quad r = \frac{2}{p\sigma - 1}.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi^{-r} = r\phi^{-r-1} \geq r\psi^{-r}.$$

Suppose there is a time t_1 such that for all $t \leq t_1$ $\phi(t) \neq 0$. This together with the inequality above implies that for any $t_0 < t_1$ we arrive at

$$\frac{1}{\phi(t_1)^r} - \frac{1}{\phi(t_0)^r} \geq r\int_{t_0}^{t_1} \psi^{-r}\,\mathrm{d}t \geq r(t_1 - t_0)^{1+r}\left(\int_{t_0}^{t_1} \psi \,\mathrm{d}t\right)^{-r} \geq r(t_1 - t_0)^{1+r} c(-t_0)^{-r(n+1)}.$$

Therefore,

$$\frac{1}{\phi(t_1)^r} \geq Cr(t_1 - t_0)^{1-rn}$$

for all $t_0 < t_1 < 0$. Now if we let $t_0 \to -\infty$ this yields a contradiction since

$$1 - rn > 0.$$

In this way we can find arbitrarily negative t with $|t|$ large such that $\phi(t) = 0$. But this can only be true if $S_{k+1} \geq -\delta H S_k$. Since we can make δ as small as we like this in turn implies that $S_{k+1} \geq 0$. This inequality is preserved by the flow such that it has to hold for all times $t < 0$. By the parabolic maximum principle used on one periodic component of the surface there are two possibilities either $S_{k+1} \equiv 0$ or $S_{k+1} > 0$ for all times. In this way we can classify the following type of ancient solutions.

Theorem 4.10.
We suppose $\mathcal{C}_t$ is an ancient closed solution which satisfies the following volume growth

$$|\Omega_t| \leq c(1 + |t|)^{n+1}.$$

and whose principle curvatures satisfy the uniform estimate $S_k > cH^k$ for some $k = 1, \ldots, n-1$. Then either $S_{k+1} \equiv 0$ or $S_{k+1} > 0$ for all points in space and time. In particular if $k = n-1$, we have $S_n > 0$ for all points in space and time.

Proof. The first statement was shown in the computations above. If $k = n-1$ by the same ODE argument we can now conclude that either $S_n \equiv 0$ for all times which implies that $\lambda_1 \equiv 0$ or $S_n > 0$ which implies $\lambda_1 > 0$. In the first case we have $\lambda_1 \equiv 0$ everywhere on $\mathcal{C}_t$ for all $t < 0$. This is impossible since a smooth closed surface has to have a point where $\lambda_1 > 0$ otherwise $\mathcal{C}_t$ splits of a flat direction $\mathcal{C}_t = \mathbb{R} \times \Sigma_t^{n-1}$ for some strictly convex and closed Σ_t^{n-1} solving mean curvature flow in $\mathbb{R}^n$ which is impossible by the closedness. To see this we define the set

$$\mathcal{D} := \{v \in T_p\mathcal{C}_t : p \in \mathcal{C}_t, A(v,v) = 0\}.$$

Because $\lambda_i > 0$ for all $i \geq 2$ this a one dimensional distribution. In particular, it is integrable and defines a vectorfield $X : \mathcal{C}_t \to T\mathcal{C}_t$. For any $p \in \mathcal{C}_t$ on the inside of the periodic component we consider a geodesic $\gamma : [0,S) \to \mathcal{C}_t$ with $\gamma(0) = p$ and $\dot\gamma(0) = X_p$. Due to the fact that A satisfies a parabolic equation we can cite [Ham86, Lemma 8.2, Lemma 8.2] applied to the tensor $A = (h_{ij})$ and its null-space $\mathcal{D}$ to conclude that X is parallel, meaning $\nabla X \equiv 0$. Therefore, we can compute

$$\nabla_{\dot\gamma}\langle\dot\gamma, X\rangle = \langle\nabla_{\dot\gamma}\dot\gamma, X\rangle + \langle\dot\gamma, \nabla_{\dot\gamma}X\rangle = 0$$

such that the geodesics are the integral lines of X. Now it follows that γ must be a piece of a straight line in $\mathbb{R}^{n+1}$ as long as it stays in the component because

$$\nabla^{\mathbb{R}^{n+1}}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + A(\dot\gamma,\dot\gamma) = 0.$$

□

As a consequence we can conclude that there is only one possibility for an ancient shrinking neck.

Corollary 4.11.
Let $0 < \Lambda < \infty$ and $k \geq 2$ be fixed. Let further C_t be an ancient periodic $(\varepsilon, k, \Lambda)$ shrinking curvature neck for some $\varepsilon = \varepsilon(n)$ so small such that $S_{n-1} \geq cH^{n-1}$ is valid on C_t in view of Proposition 4.5. Then $C_t = \mathbb{S}^{n-1}_{R(t)} \times \mathbb{R}$ for all $t < 0$ which means that C_t is the standard shrinking cylinder solution.

Proof. By the same argument from above we can conclude that $\lambda_1 \equiv 0$ or $\lambda_1 > 0$ for all times and on each periodic component. We note that the maximum principle can be applied on all of the components since the normal derivative at the boundaries is 0 by the periodicity of the solution such that we have Neumann boundary data. Now the case $\lambda_1 > 0$ is ruled out because by the periodicity we would have $\lambda_1 \geq c > 0$ for some constant which is impossible because the neck does not close up. In the other case we have $\mathcal{C}_t = \mathbb{R} \times \Sigma_t^{n-1}$ for some strictly convex and closed Σ_t^{n-1} solving mean curvature flow in $\mathbb{R}^n$. But since $\mathcal{C}_t$ is a shrinking neck, we have

$$H(\Sigma^{n-1})\sqrt{-t} \leq C(c_1, n)$$

such that Σ_t^{n-1} is off Type I. A result by Huisken and Sinestrari [HS15, Proposition 4.6] then says $\Sigma_t^{n-1} = \mathbb{S}^{n-1}_{R(t)}$. We can do this on each component of the periodic solution such that they must combine to the shrinking cylinder solution since they agree on the boundaries.

□

Remark 4.12. We have carried out the computations that lead us to the classification in Corollary 4.11 not just for $k = n-1$ such that this result generalizes to perodic solutions with more than one flat direction. A k-periodic solutions can be seen as solution that lives in a Torus $\mathbb{R}^{n+1-k} \times \mathbb{T}^k$ where $\mathbb{T}^k = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{k-\text{times}}$. We could then define a notion of a periodic neck with k almost flat directions as a parameterizations whose metric is close to the standard metric on the standard shrinking cylinder $\mathbb{S}^k_{R(t)} \times \mathbb{R}^{n+1-k}$ but lives in $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$. The computations from above together with Proposition 4.5 then imply that for all points in space and time we have either $S_{k+1} \equiv 0$ or $S_{k+1} > 0$. The latter case being impossible since by Remark 4.8 we have $S_1 > 0, \ldots, S_{k+1} > 0$ which gives us $\lambda_1 + \cdots + \lambda_{n-k} > 0$ everywhere and for all times, see [HS09, Lemma 2.3]. By the periodicity we would again obtain $\lambda_1 + \cdots + \lambda_{n-k} \geq c > 0$ which is excluded. Thus by a similar computation as in Proposition 4.5 we get $\lambda_i = 0$ for all $i = 1, \ldots, n-k$ and $\lambda_j > 0$ for $j > n-k$. Then we already know that the solution is weakly convex in this situation and so the gauss equations imply that the sectional curvatures and hence also the ricci curvature must be non negative. Therefore, the Cheeger-Gromoll splitting theorem [CG71, Theorem 2] tells us that $\mathcal{C}_t$ splits isometrically as a product $\Sigma^k \times \mathbb{R}^{n+1-k}$ where Σ_k is a convex solution to mean curvature flow in $\mathbb{R}^{k+1}$. At this point we can conclude exactly as in the proof of Corollary 4.11.

While the previous results where concerned with a qualitative behavior of ancient solutions, we are now interested in a quantitative way to describe the convexity improvement. For this reason let (p_0, t_0) be a point in the center of $\mathcal{P}(p_0, t_0, R_0\Lambda, R_0^2\theta)$ being a $(\varepsilon, k-1, \Lambda, \theta)$ shrinking curvature neck which in our case is the periodic shrinking solution $\mathcal{C}_t$. On the shrinking neck we have

$$H(p,t) \geq c\frac{n-1}{\rho(R_0, t-t_0)}$$

for any $t \in [t_0 - R_0^2\theta, t_0]$, where

$$\rho(r,s) := \sqrt{r^2 - 2(n-1)s}$$

is the radius of the standard shrinking cylinder of starting radius r. We define $R(t) := \rho(R_0, t)$. From inequality (4.14), by using the Gronwall's inequality and the fact that

$$H \geq c_1 \frac{n-1}{R(t)}$$

for $\frac{1}{2} \leq c_1 \leq 1$ if ε is small enough, we can also obtain a direct time dependend bound for the L^p integral. We know that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu &\leq -n(n-1)^2\frac{1}{2}p\sigma\frac{1}{R(t)^2}\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu \\ &\leq -p\sigma(n-1)\frac{1}{R(t)^2}\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu\end{aligned}$$

such that by Gronwall's inequality we have

$$\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu \leq (\int_{C_0} f_+^p \,\mathrm{d}\mu)\exp\left(-p\sigma(n-1)\int_{t_0-R_0^2\theta}^{t_0-t}\frac{1}{R_0^2 - 2(n-1)(s-t_0)}\,\mathrm{d}s\right)$$

where $C_0 := \mathcal{C}_{t_0 - R_0^2\theta}$.

$$\begin{aligned}\int_{t_0-R_0^2\theta}^{t_0-t}\frac{1}{R_0^2 - 2(n-1)(s-t_0)}\,\mathrm{d}s &= -\frac{1}{2(n-1)}\left[\ln\left(R_0^2 - 2(n-1)(s-t_0)\right)\right]_{t_0-R_0^2\theta}^{t_0-t} \\ &= -\frac{1}{2(n-1)}\ln\left(\frac{R_0^2 - 2(n-1)t}{R_0^2(1+2(n-1)\theta)}\right)\end{aligned}$$

such that we have

$$\int_{\mathcal{C}_t} f_+^p \,\mathrm{d}\mu \leq \int_{C_0} f_+^p \,\mathrm{d}\mu \left(\frac{R_0^2 - 2(n-1)t}{R_0^2(1+2(n-1)\theta)}\right)^{\frac{\sigma p}{2}}.$$

W.l.o.g. we will assume $t_0 = 0$. At this point we have established an L^p estimate for $\sigma = o(p^{-\frac{1}{2}})$ of the function

$$g := f_+ \cdot \left(\frac{R_0^2 - 2(n-1)t}{R_0^2(1+2(n-1)\theta)}\right)^{\frac{-\sigma}{2}}.$$

Namely

$$\int_{\mathcal{C}_t} g^p \,\mathrm{d}\mu \leq \int_{C_0} f_+^p \,\mathrm{d}\mu = \int_{C_0} g^p \,\mathrm{d}\mu,$$

since $g(-R_0^2\theta, p) = f(-R_0^2\theta, p)$ for all $p \in C_0$. In the following we want to choose $\sigma = o(p^{-\frac{1}{2}})$ small enough such that not only the L^p bound from above is valid, but also the integrals

$$\int_{\mathcal{C}_t} H^n g^p \,\mathrm{d}\mu, \quad \int_{\mathcal{C}_t} H^{2r} g^{pr} \,\mathrm{d}\mu$$

are non increasing in time where $r > 1$ is a suitable constant which depends on the dimension. Starting from here we want to employ the standard Stamppachia iteration trick that leads us to a sup estimate of g. For this purpose we set $g_k := \max\{g - k, 0\}$ and $A(k,t) = \mathrm{Supp}(g_k) \subset \mathcal{C}_t$ for $k \geq k_0 := \sup_{\mathcal{C}_t} g_\sigma$. The evolution equation of g contains an extra term coming from the additional time derivative. We denote by $c(t) := \left(\frac{R_0^2-2(n-1)t}{R_0^2(1+2(n-1)\theta)}\right) = (R_0(1+2(n-1)\theta)^{-2}R(t)^2$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} g = \Delta g &+ \frac{2(1-\sigma)}{H}\langle \nabla H, \nabla g\rangle - \frac{\sigma(1-\sigma)}{H^2} g |\nabla H|^2 \\ &+ \frac{1}{c(t)^{\frac{\sigma}{2}} H^{1-\sigma}} \frac{\partial^2 Q_n}{\partial h_{ij}\partial h_{pq}} \nabla_m h_{ij} \nabla_m h_{pq} + \sigma |A|^2 g - \frac{\sigma}{2} c'(t) c(t)^{-\frac{\sigma}{2}-1} f. \end{aligned} \tag{4.15}$$

Now we use the closeness to a standard cylinder in the following way

$$-\frac{\sigma}{2} c'(t) c(t)^{-\frac{\sigma}{2}-1} f = \sigma(n-1) R(t)^{-2} g \leq \frac{c\sigma}{n-1} H^2 g.$$

Further

$$\frac{1}{c(t)^{\frac{\sigma}{2}p}} \frac{\partial^2 Q_n}{\partial h_{ij}\partial h_{pq}} \nabla_m h_{ij} \nabla_m h_{pq} \leq -\frac{1}{c(t)^{\frac{\sigma}{2}p}} \frac{1}{C(n,\delta)\sqrt{n}} \frac{|\nabla A|^2}{H} \leq 0.$$

If we now multiply (4.15) with $p g_k^{p-1}$ and integrate then we obtain in the same way as above

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{C}_t} g_k^p \,\mathrm{d}\mu &+ \frac{p(p-1)}{2} \int_{A(k,t)} g_k^{p-2} |\nabla g|^2 \,\mathrm{d}\mu \\ &\leq (np\sigma + \frac{c\sigma p}{n-1}) \int_{A(k,t)} H^2 g^p \,\mathrm{d}\mu\end{aligned}$$

for p large enough and σ small such that the L^p estimate which we derived on the previous pages is still true. To exploit the good gradient term on the right hand side we will need the Michael-Simon Sobolev inequality

Proposition 4.13. [Michael-Simon Sobolev Inequality]
For any $u \in C^{0,1}(C)$ we have

$$\left(\int_C |u|^{\frac{2n}{n-2}}\,\mathrm{d}\mu\right)^{\frac{n-2}{2n}} \le c(n)\left(\int_C |\nabla u|^2\,\mathrm{d}\mu + \int_C H^2|u|^2\,\mathrm{d}\mu\right). \tag{4.16}$$

We remark that since $p > 2$ we can estimate on the set $A(k)$

$$\left|\nabla g_k^{p/2}\right|^2 \le \frac{1}{2}p(p-1)g_k^{p-2}|\nabla g|^2.$$

Now we set $v = g_k^{p/2}$ and use Proposition 4.13 to get

$$\begin{aligned}\left(\int_C v^{2q}\,\mathrm{d}\mu\right)^{\frac{1}{q}} &\le c_n\int_{\mathcal{C}_t}|\nabla v|^2\,\mathrm{d}\mu + c_n\int_{\mathcal{C}_t}H^2v^2\,\mathrm{d}\mu\\ &\le c_n\int_{\mathcal{C}_t}|\nabla v|^2\,\mathrm{d}\mu + c_n\left(\int_{A(k)}H^n\,\mathrm{d}\mu\right)^{\frac{2}{n}}\left(\int_{\mathcal{C}_t}v^{2q}\,\mathrm{d}\mu\right)^{\frac{1}{q}}\end{aligned}$$

where $q = \frac{n}{n-2}$. Here we used the obvious inclusion $\mathrm{Supp}(v) \subset A(k)$. Now by the L^p estimate from above we can further conclude

$$\int_{A(k)}H^n\,\mathrm{d}\mu \le \int_{A(k)}H^n\frac{g^p}{k^p}\,\mathrm{d}\mu \le k^{-p}\int_{C_0}H^ng^p\,\mathrm{d}\mu.$$

Here we also used that $\int H^ng^p\,\mathrm{d}\mu$ is non increasing in time if we make σ slightly smaller compared to above. Thus if we take $k \ge k_2 := \max\{k_0, k_1\}$ where $k_1 := 2c_n^{\frac{n}{2p}}(\int_{C_0}H^ng^p\,\mathrm{d}\mu)^{\frac{1}{p}}$, then we can absorb this term into the LHS above to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_t}v^2 + \frac{1}{2c_n}\left(\int_{\mathcal{C}_t}v^{2q}\,\mathrm{d}\mu\right)^{\frac{1}{q}} \le c\sigma p\int_{A(k)}H^2g^p\,\mathrm{d}\mu.$$

If we integrate in time from $-R_0^2\theta$ to 0, there is no contribution of the initial data and we get

$$\begin{aligned}\sup_{[-R_0^2\theta,0]}\int_{A(k)}v^2\,\mathrm{d}\mu + c_n\int_{-R_0^2\theta}^0\left(\int_{A(k)}v^{2q}\,\mathrm{d}\mu\right)^{\frac{1}{q}}\mathrm{d}t&\\ \le c\sigma p\int_{-R_0^2\theta}^0\int_{A(k)}H^2g^p\,\mathrm{d}\mu&\end{aligned}$$

for any $k \ge k_2$. Now we can argue as in [HS09, Theorem 4.5] and set $q_0 = 2 - 1/q$ and find $r = r(n) > 1$ such that $\gamma := 1 - 1/q_0 - 1/r > 0$ to get

$$\int_{-R_0^2\theta}^0\int_{A(k)}v^2\,\mathrm{d}\mu\,\mathrm{d}t \le c\sigma p\|A(k)\|^{\gamma-1}\left(\int_{-R_0^2\theta}^0\int_{A(k)}H^{2r}g^{pr}\,\mathrm{d}\mu\,\mathrm{d}t\right)^{\frac{1}{r}}$$

where

$$\|A(k)\| := \int_{-R_0^2}^0\int_{A(k)}\mathrm{d}\mu\,\mathrm{d}t.$$

The integral on the right hand side is again non-increasing in time if we make σ sufficiently small such that if we set

$$k_3 := \int_{C_0}H^{2r}f^{pr}\,\mathrm{d}\mu$$

we get

$$\int_{-R_0^2\theta}^{0}\int_{A(k)} v^2 \,\mathrm{d}\mu\,\mathrm{d}t \le c\sigma p\|A(k)\|^{1+\gamma}(R_0^2\theta)^{\frac{1}{r}}k_3^{\frac{1}{r}}.$$

Since $\gamma > 0$ we can use a well known Lemma by Stampacchia in order to conclude

$$g \le k_2 + k_4$$

with

$$k_4^p := c\sigma p(R_0^2\theta)^{\frac{1}{r}}2^{\frac{p(\gamma+1)}{\gamma}}\|A(k_2)\|.$$

We need to control the constants k_i explicitly. Therefore, we set $R_\theta := (R_0^2+2(n-1)\theta R_0^2)^{\frac{1}{2}}$.

$$\begin{aligned}
k_0 &= \sup_{C_0} g \le c(\sup_{C_0} H)^\sigma \le c(n)R_\theta^{-\sigma}\\
k_1 &\le c(n,\Lambda)(R_\theta)^{n/p}(\sup_{C_0} H)^{n/p}k_0 \le c(n,\Lambda)(R_\theta)^{-\sigma}\\
k_3 &\le C(n,\Lambda)(R_\theta)^{n-2r-\sigma pr}\\
k_4^p &\le C(R_\theta)^{\frac{n}{r}-2-\sigma p+n\gamma}(R_0^2\theta)^{\frac{1}{r}+\gamma}\\
&\le CR_\theta^{-\sigma p},
\end{aligned}$$

since by the choice of our exponents we have

$$(n+2)(\gamma+\frac{1}{r}) = (n+2)(1-\frac{1}{q_0}) = 2$$

and

$$R_0^2\theta \le R_\theta^2.$$

We sum up our previous computations in the following theorem

Theorem 4.14 (Convexity Improvement Theorem)**.**
Let $0 < \Lambda < \infty$ and $k \ge 2$ be fixed and let $\varepsilon = \varepsilon(n)$ be so small such that the conclusions of Proposition 4.5 hold. Then for any $\beta > 0$ large and $\delta > 0$ small there exists a large constant $\theta = \theta(\delta, \beta, n, \Lambda)$ such that if $\mathcal{C}_t$ is a $(\varepsilon, k, \Lambda, \theta)$ periodic shrinking curvature neck, then

$$S_n \ge -2\delta H^n \tag{4.17}$$

everywhere on $\mathcal{C}_t$ for all $t \in [-\beta R_0^2, 0]$.

Proof. Let $\varepsilon = \varepsilon(n) > 0$ be given by Proposition 4.5 and let $\beta > 0$ be a constant. By the previous computation we have

$$-Q_n - \delta H \le C(n,\delta,\Lambda)R_\theta^{-\sigma}H^{1-\sigma}\left(\frac{R_0^2-2(n-1)t}{R_0^2(1+2(n-1)\theta)}\right)^{\frac{\sigma}{2}}$$

for some small σ depending on δ and n. We recall that we have the estimate $S_{n-1} \le C(n)H^{n-1}$ such that after rearranging terms we end up with

$$\begin{aligned}
S_n &\ge -C(n)\delta H^n - C(n,\delta,\Lambda)R_\theta^{-\sigma}H^{n-\sigma}\left(\frac{R_0^2-2(n-1)t}{R_0^2(1+2(n-1)\theta)}\right)^{\frac{\sigma}{2}}\\
&\ge -\tilde{\delta}H^n - \frac{C}{\tilde{\delta}}\left(\frac{R_0^2-2(n-1)t}{R_0^2(1+2(n-1)\theta)}\right)^{\frac{n}{2}}R_\theta^{-n}
\end{aligned}$$

where we used Youngs inequality and $\tilde{\delta} = 2\delta C(n,\varepsilon)^{-1}$ can be made as small as we like.The estimate above is improving as t approaches 0 such that we only look at the time $t = -R_0^2\beta$. There we have

$$S_n \geq -\delta H^n - C(n,\delta,\Lambda)\delta^{-1}\left(\frac{1+2(n-1)\beta}{(1+2(n-1)\theta)^2}\right)^{\frac{n}{2}} R_0^{-n}.$$

Furthermore, as the neck shrinks the mean curvature H grows, meaning that $H(p,t) \geq c_2\frac{n-1}{R_0}\frac{1}{\sqrt{1+2(n-1)\theta}}$ for all (p,t) which implies

$$\geq -\delta H^n - C\delta^{-1}H^n\left(\frac{1+2(n-1)\beta}{1+2(n-1)\theta}\right)^{\frac{n}{2}}.$$

Hence choosing

$$\theta = \frac{1}{2(n-1)}\left(\left(C\frac{1+2(n-1)\beta}{\delta^2}\right)^{\frac{2}{n}} - 1\right)$$

yields the desired estimate. □

Remark 4.15. We notice that making δ tend to zero results in θ growing to infinity. Also if we make β larger we have to increase θ as well. With Theorem 4.14 we found a way to measure how "old" solutions become more and more convex.

4.2 Estimates on λ_1

Now that we have shown the improvement of λ_1 becoming more and more convex, we also want to see if the initial bound $\lambda_1 \leq \varepsilon H$ coming from the neck assumptions improves as time passes. Naturally we would define the following test function

$$g_{\sigma,\delta} = \frac{\lambda_1 - \delta H}{H^{1-\sigma}}$$

for $\delta,\sigma \in (0,\frac{1}{4})$ and carry out a similar approach as above. The second variation of the first principle curvature λ_1 however, yields an extra correction term with the wrong sign.

Proposition 4.16.
We have the following relation

$$\Delta\lambda_1 = \Delta h_{11} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=2}^{n}\frac{1}{\lambda_j - \lambda_1}\left(\nabla_i h_{1j}\right)^2. \tag{4.18}$$

Moreover, we have $\nabla_i h_{1j} = \nabla_i \lambda_1\, \delta_{1j}$.

Proof. The formula is a consequence of the variational formula on parameter dependent eigenvalues in [Lan64, Theorem 9] by observing that $(h_i^j)_{ij}$ is a symmetric matrix and λ_1 is smooth and distinct from the other eigenvalues. For any $p \in \mathcal{C}_t$ and $i = 1,\dots,n$ we simply choose a geodesic γ such that $\gamma(0) = p$ and $\dot\gamma(0) = e_i$ and use $\partial_s\lambda_1(\gamma(s))|_{s=0} = \nabla_i\lambda_1$. □

In order to overcome this difficulty we will use a different approach based on the argument of Corollary 4.11. This will give us the following estimate on λ_1 from above

Theorem 4.17. Let $0 < \Lambda < \infty$ and $k \geq 2$ be fixed and let $\varepsilon = \varepsilon(n)$ be so small such that the conclusions of Proposition 4.5 hold. Then for any $\beta > 0$ large and $\delta > 0$ small there exists a large constant $\theta = \theta(\delta, \beta, n, \Lambda)$ such that if $\mathcal{C}_t$ is a $(\varepsilon, k, \Lambda, \theta)$ periodic shrinking curvature neck, then

$$\lambda_1 \leq \delta H \tag{4.19}$$

everywhere on $\mathcal{C}_t$ for all $t \in [-\beta R_0^2, 0]$.

Proof. If the statement above was wrong, we could find a $\delta > 0$, a time parameter $\beta > 0$ and a sequence $\mathcal{C}^j$ of periodic $(\varepsilon, k, \Lambda, \Theta_j)$ shrinking curvature necks all having final radius $R_0 > 0$ such that

$$\lambda_1^j > \delta H^j$$

somewhere and at a time in $[-\beta R_0^2, 0]$ where $\Theta_j \to \infty$ for $j \to \infty$. Since by the neck assumptions we have a uniform gradient estimate in the form $|\nabla A^j| \leq C(\varepsilon, n) R_0^{-2}$ (and also for higher order derivatives) we can extract a subsequence converging smoothly to a smooth limit flow $\mathcal{C}^\infty$ existing on $[-\infty, 0]$ which is still a periodic $(\varepsilon, k, \Lambda)$ shrinking curvature neck and satisfies

$$\lambda_1 \geq \delta H > 0$$

for at least some point and some time in the interval $[-\beta R_0^2, 0]$ since $\mathcal{C}^\infty$ is a smooth limit. This, however, is impossible, since by Corollary 4.11 $\mathcal{C}^\infty$ is a round shrinking cylinder solution and therefore has to satisfy $\lambda_1 = 0$ everywhere and for all times. □

Now we can combine this result with the previous convexity estimate.

Theorem 4.18 (Improvement of λ_1)**.**
Let $0 < \Lambda < \infty$ and $k \geq 2$ be fixed and let $\varepsilon = \varepsilon(n)$ be so small such that the conclusions of Proposition 4.5 hold. Then for any $\beta > 0$ large and $\delta > 0$ small there exists a large constant $\theta = \theta(\delta, \beta, n, \Lambda)$ such that if $\mathcal{C}_t$ is a $(\varepsilon, k, \Lambda, \theta)$ periodic shrinking curvature neck, then

$$|\lambda_1| \leq 2\delta H \tag{4.20}$$

everywhere on $\mathcal{C}_t$ for all $t \in [-\beta R_0^2, 0]$.

Proof. For $\beta > 0$ and δ we get the existence of θ_1 from Theorem 4.14 and likewise the existence of a θ_2 such that we have

$$S_n = \lambda_1 \cdots \lambda_n \geq -\frac{\delta}{(n-1)^{n-1}} H^n$$

and

$$\lambda_1 \leq \delta H$$

for all $t \in -[\beta R_0^2, 0]$ if the neck existed for on $[-\bar{\theta} R_0^2, 0]$ where $\bar{\theta} = \max\{\theta_1, \theta_2\}$. If $\lambda_1 \geq 0$ we are done so we only consider the case where $\lambda_1 < 0$. Then we compute

$$\lambda_1 \geq -\frac{\delta}{(n-1)^{n-1}} H^n \frac{1}{\lambda_2 \cdots \lambda_n} \geq -\frac{\delta}{(n-1)^{n-1}} H^n \frac{(n-1)^{n-1}}{H^{n-1}}.$$

□

4.3 Cylindrical Estimate

A characterizing feature of the standard cylinder is that it has a flat direction corresponding to the first principle curvature λ_1 vanishing identically everywhere while all other principle curvatures are equal to $\frac{H}{n-1}$ which corresponds to the spherical part of the cylinder. Therefore, once we have established control on the first principle curvature it is somewhat natural to examine how far the other eigenvalues of the Weingarten operator are apart from each other. This analysis is provided by the crucial cylindrical estimate.

Theorem 4.19 (Cylindrical Estimate)**.**
Let $\mathcal{M}_t^n$ be a smooth 2-convex solution to mean curvature flow in $\mathbb{R}^{n+1}$. Then for every $\eta > 0$ we can find a constant $C_\delta = C(\eta, \mathcal{M}_0^n)$ such that

$$|A|^2 - \frac{1}{n-1}H^2 \leq \eta H^2 + C_\eta \tag{4.21}$$

everywhere on $\mathcal{M}_t^n$ for all $t > 0$.

Proof. We refer to [HS09, Theorem 5.3]. □

The proof of Theorem 4.19 has a very similar structure to the one of the convexity estimate 4.1 but it relies on the convexity estimate. From Theorem 4.14 we can deduce that for a suitable time interval a (ε, k) shrinking curvature neck has an improved estimate of the form

$$\lambda_1 \geq -\delta H.$$

Now we hope to be able to eliminate the constant C_η which a priori can be really large. For this reason we go through the main steps of the proof of the original result in [HS09, Theorem 5.3] and look at the test function

$$f_{\sigma,\eta} := \frac{|A|^2 - \left(\frac{1}{n-1} + \eta\right) H^2}{H^{2-\sigma}}$$

where η is a small parameter and $\sigma \in (0,1)$ breaks the scaling of the quotient. For $\eta = 0$ this function vanishes on the standard cylinder. In this section we want to examine this estimate more closely in the case we already know that we are on an (ε, k) hypersurface neck. Similar to the original proof of Theorem 4.19 we can thus estimate on the set where $f_{\sigma,\eta} \geq 0$ the quantity $Z := H\mathrm{tr}(A^3) - |A|^4$ in the following way. First of all we have

$$|A| - \frac{1}{n-1}H^2 = \frac{1}{n-1}\left(\sum_{1<i<j}(\lambda_i - \lambda_j)^2 + \lambda_1(n\lambda_1 - 2H)\right).$$

$$
\begin{aligned}
Z &= \sum_{j=2}^{n} \lambda_1\lambda_j(\lambda_1-\lambda_j)^2 + \sum_{1<i<j} \lambda_i\lambda_j(\lambda_i-\lambda_j)^2 \\
&\geq \sum_{j=2}^{n} \lambda_1\lambda_j(\lambda_1-\lambda_j)^2 + c_1^2\frac{H^2}{(n-1)^2}\sum_{1<i<j}(\lambda_i-\lambda_j)^2 \\
&= c_1^2\frac{H^2}{(n-1)^2}((n-1)|A|^2-H^2) \\
&+ \lambda_1\left(\sum_{j=2}^{n}\lambda_j(\lambda_1-\lambda_j)^2 + c_1^2\frac{H^2}{(n-1)^2}(2H-n\lambda_1)\right) \\
&\geq c_1^2\frac{H^2}{(n-1)^2}((n-1)|A|^2-H^2) \\
&- \delta H\left((n-1)H(2H)^2 + c_1^2\frac{H^2}{(n-1)^2}(n+2\delta)H\right) \\
&= C(n,c_1)H^2\left(|A|^2 - \frac{1}{n-1}H^2 - C(n)\delta H^2\right)
\end{aligned}
$$

for $\lambda_1 < 0$. For $\lambda_1 > 0$ we can just drop the λ_1 term entirely in the second step and get the same estimate. and thus

$$
\frac{2Z}{H^2}f_+^p \geq C\eta\frac{1}{n-1}|A|^2 f_+^p
$$

for any $\eta > C(n)\delta$ where $f_+ := (f_{\sigma,\eta})_+$ is the positive part of the test function. We will denote by $\mathcal{C}_t$ a component of the periodic solution. As in the proof of the cylindrical estimate we obtain then via partial integration

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_t} f_+^p\,\mathrm{d}\mu \leq{}& \frac{-p(p-1)}{2}\int_{\mathcal{C}_t} f_+^{p-2}|\nabla f|^2\,\mathrm{d}\mu - \frac{p}{c_1}\int_{\mathcal{C}_t}\frac{f_+^p}{H^2}|\nabla H|^2\,\mathrm{d}\mu \\
&- p\int_{\mathcal{C}_t}\frac{f_+^{p-1}}{H^{4-\sigma}}|H\cdot\nabla A - \nabla H\cdot A|^2\,\mathrm{d}\mu + p\sigma\int_{\mathcal{C}_t}|A|^2 f_+^p\,\mathrm{d}\mu
\end{aligned}
$$

for any $p \geq c_2$ where $c_1, c_2 > 1$ are positive constants depending on the class parameters n, ε, see [HS09, Lemma 5.4]. Then we have to absorb the absolute term on the right hand side into the good negative terms. It turns out that we can actually absorb the bad term twice such that we are left over with an entirely negative right hand side. To do this we need a Poincare type inequality. For this we use the Simons' identity which implies

$$
\begin{aligned}
\Delta f ={}& \frac{2}{H^m}\langle h_{ij}, \nabla_i\nabla_j H\rangle + \frac{2}{H^m}Z + \frac{2}{H^{m+2}}|H\cdot\nabla A - \nabla H\cdot A|^2 \\
&- \frac{2(m-1)}{H}\langle\nabla H, \nabla f\rangle - \left(\frac{mf}{H} + 2\left(\frac{1}{n-1}+\eta\right)H^{1-m}\right)\Delta H \\
&+ \frac{(2-m)(m-1)}{H^2}f|\nabla H|^2
\end{aligned}
$$

with $m = 2 - \sigma$. Multiplying this inequality with $f_+^p H^{m-2}$ and integrating partially, we obtain

$$\begin{aligned}\int_{\mathcal{C}_t}\frac{2Z}{H^2}f_+^p\,\mathrm{d}\mu &= -p\int_{\mathcal{C}_t}\frac{f_+^p|\nabla f|^2}{H^{2-m}}\,\mathrm{d}\mu + 2p\int_{\mathcal{C}_t}\frac{f_+^{p-1}\langle h_{ij},\nabla_i f\nabla_j H\rangle}{H^2}\,\mathrm{d}\mu\\
&- 4\int_{\mathcal{C}_t}\frac{f_+^p\langle h_{ij},\nabla_i H\nabla_j H\rangle}{H^3}\,\mathrm{d}\mu\\
&- 2\int_{\mathcal{C}_t}\frac{f_+^p|H\cdot\nabla A-\nabla H\cdot A|^2}{H^4}\,\mathrm{d}\mu - \int_{\mathcal{C}_t}\left(\frac{mpf_+^p}{H^{3-m}}+2p\left(\frac{1}{n-1}+\eta\right)\frac{f_+^p}{H}\right)\langle\nabla f,\nabla H\rangle\,\mathrm{d}\mu\\
&+ 2\int_{\mathcal{C}_t}\left(\frac{f_+^{p+1}}{H^{4-m}}+\left(2+\frac{1}{n-1}+\eta\right)\frac{f_+^p}{H^2}\right)|\nabla H|^2\,\mathrm{d}\mu\end{aligned}$$

Now since $f_+^p \le nH^\sigma$ by the 2-convexity and since $0<\eta<1$ we can further estimate

$$\le -p\int_{\mathcal{C}_t}\frac{f_+^{p-1}}{H^\sigma}|\nabla f|^2\,\mathrm{d}\mu + (4np+3p)\int_{\mathcal{C}_t}\frac{f_+^{p-1}}{H}|\nabla H||\nabla f|\,\mathrm{d}\mu + (6n+5)\int_{\mathcal{C}_t}\frac{f_+^p}{H^2}|\nabla H|^2\,\mathrm{d}\mu.$$

By using the estimate from below on Z and the Peter Paul inequality for the second term on the right hand side we can now obtain a Poincare type inequality

$$C\eta\frac{1}{n-1}\int_{\mathcal{C}_t}|A|^2 f_+^p\,\mathrm{d}\mu \le (6n+5+2p\gamma(n+1))\int_{\mathcal{C}_t}\frac{f_+^p}{H^2}|\nabla H|^2\,\mathrm{d}\mu \tag{4.22}$$

$$+\frac{2p(n+1)}{\gamma}\int_{\mathcal{C}_t}f_+^{p-2}|\nabla f|^2\,\mathrm{d}\mu \tag{4.23}$$

which we can write as

$$\eta\int_{\mathcal{C}_t}|A|^2 f_+^p\,\mathrm{d}\mu \le c_3\left(\frac{p}{\gamma}\int_{\mathcal{C}_t}f_+^{p-1}|\nabla f|^2\,\mathrm{d}\mu + (1+\gamma p)\int_{\mathcal{C}_t}\frac{f_+^p}{H^2}|\nabla H|^2\,\mathrm{d}\mu\right)$$

for any $p>2$ and $\gamma>0$ and $c_3 = c_3(n) > 0$. In this way we want σ and p to be chosen such that

$$\begin{aligned}4n^2\frac{p^2\sigma c_3}{\eta\gamma} &\le \frac{p(p-1)}{2}\\
2(1+\gamma p)p\frac{\sigma c_3}{\eta} &\le \frac{p}{c_1}\end{aligned}$$

and $\gamma := \frac{1}{\sqrt{p}}$. This is satisfied as long as $p \ge \max\{2, c_1\}$ and $\sigma \le \frac{\eta}{16n^2 c_1 c_3\gamma}$ such that

$$2\sigma pn^2\int_{\mathcal{C}_t}|A|^2 f_+^p\,\mathrm{d}\mu \le \frac{p(p-1)}{2}\int_{\mathcal{C}_t}f_+^{p-2}|\nabla f|^2\,\mathrm{d}\mu + \frac{p}{2c_1}\int_{\mathcal{C}_t}\frac{f_+^p}{H^2}|\nabla H|^2\,\mathrm{d}\mu$$

Additionally, we finally obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_t}f_+^p\,\mathrm{d}\mu &\le -n^2\sigma p\int_{\mathcal{C}_t}|A|^2 f_+^p\,\mathrm{d}\mu\\
&\le -n\sigma p\int_{\mathcal{C}_t}H^2 f_+^p\,\mathrm{d}\mu\end{aligned}$$

since $\frac{1}{n}H^2 \le |A|^2 \le nH^2$. We can write $H(p) \ge \frac{1}{2}\frac{n-1}{R(t)}$ such that

$$\int_{\mathcal{C}_t}H^2 f_+^p\,\mathrm{d}\mu \ge \frac{1}{2}(n-1)^2 R(t)^{-2}\int_{\mathcal{C}_t}f_+^p\,\mathrm{d}\mu.$$

Then we can proceed exactly as in the proof of the convexity estimate in order to obtain the following improved cylindrical estimate

$$|A|^2 - \left(\frac{1}{n-1} + \eta\right) H^2 \leq C(n, \Lambda, \eta) R_\beta^{-\sigma} H^{2-\sigma} \left(\frac{R_0^2 - 2(n-1)t}{R_0^2(1+2(n-1)\beta)}\right)^{\frac{\sigma}{2}} \tag{4.24}$$

which takes the form:

Theorem 4.20 (Cylindrical Improvement)**.**
Let $0 < \Lambda < \infty$ and $k \geq 2$ be fixed and let $\varepsilon = \varepsilon(n)$ be so small such that the conclusions of Proposition 4.5 hold. Then for any $\zeta > 0$ and $\eta > \delta$ there exists $\beta(\zeta, \eta) > 0$ and a constant $\theta = \theta(\delta, \eta, \beta, \zeta, n, \Lambda)$ such that if $\mathcal{C}_t$ is a $(\varepsilon, k, \Lambda, \theta)$ periodic shrinking curvature neck with $\lambda_1 > \delta H$ for all times $[-R_0^2\beta, 0]$, then

$$|A|^2 - \frac{1}{n-1} H^2 \leq \eta H^2$$

on $[-R_0^2\zeta, 0]$.

Proof. The parameters ε, k, Λ are given. Then we fix $\zeta > 0$. By the time estimate for the cylindrical estimate, we need $\lambda_1 > -\delta H$ for a large time interval $[-R_0^2\beta, 0]$ which we can achieve by choosing θ large in Theorem 4.14. Both the choices of θ and β depend on ζ. Then we proceed exactly as in the proof of Theorem 4.14. □

In particular, in the special case where we can bound λ_1 also from above we get a two sided bound. This is in particular interesting for the upcoming section where we will use the quantity on the right hand side of the cylindrical estimate.

Proposition 4.21.
Suppose at a point p we have $|\lambda_1| \leq \delta H$, then we also have

$$\frac{1}{n-1} H^2 - |A|^2 \leq \delta H^2$$

at that point.

Proof. We can write the quantity on the R.H.S. in the following way

$$\begin{aligned}
\frac{1}{n-1} H^2 - |A|^2 &= -\frac{1}{n-1}\left(\sum_{i<j}(\lambda_i - \lambda_j)^2 + \lambda_1(n\lambda_1 - 2H)\right) \\
&\leq \frac{|\lambda_1|}{n-1}(2H - n\lambda_1) \leq \frac{\delta}{n-1} H^2
\end{aligned}$$

□

Remark 4.22. We remark that by Theorem 4.18 we will be able to make use of Proposition 4.21 on a large time interval which will be crucial to get an improved gradient control.

4.4 Gradient Estimates

The classical gradient estimate for mean curvature flow of 2-convex surfaces in $\mathbb{R}^{n+1}$ is the following.

Theorem 4.23.
Let $\mathcal{M}_t$ in the class $C(R,\gamma)$ be a solution to mean curvature flow with surgery and normalized initial data. Then there exists a constant $c_2 = c_2(n)$ and a constant $c_3 = c_3(n,\alpha)$ such that for suitable surgery parameters the flow satisfies the uniform estimate

$$|\nabla A|^2 \leq c_2|A|^4 + c_3 R^{-4}. \tag{4.25}$$

for all $t \geq \frac{1}{4}R^2$.

In this section we are looking to improve this estimate using the improved cylindrical estimate. We will follow the ideas in [HS09, Chapter 6]. Heuristically, the change of the curvature is determined by the change of the principle curvatures. Thus, if the principle curvatures are almost equal, we also expect the gradient of the curvature to be very small. In this way we want to find a way to use our cylindrical estimate to obtain the gradient estimate. We fix $\eta > 0$ coming from Theorem 4.20, such that we have the improved cylindrical estimate

$$((\frac{1}{n-1} + \tau)H^2 - |A|^2 \geq (-\eta + \tau)H^2 > 0$$

for any $\tau > \eta$ and all times $t \in [-R_0^2\beta, 0]$. Then simillar to the proof of Theorem 4.23 in [HS09, Theorem 6.1] we define a test function

$$f := (R_0^2\beta + t)|\nabla A|^2 \cdot g^{-2}$$

where $g := (\frac{1}{n-1} + \tau)H^2 - |A|^2$. We note that as we have just argued f is a well-defined function. The difference to the classical gradient estimate is the extra time factor which we introduced to eliminate the dependence on the initial values. This factor also depends on the parameter β which determines the length of the time interval. In order to be able to compute the evolution equation of f we need the evolution equation of $|\nabla A|^2$. By [Hui84, Theorem 7.1] we have the general formula

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla^m A|^2 = \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A \tag{4.26}$$

where for general tensors T, S the notation $T * S$ stands for any contraction of the tensors S and T with the metric g. From this formula we can deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla A|^2 - \Delta|\nabla A|^2 \leq -2|\nabla^2 A|^2 + c_n|A|^2|\nabla A|^2, \tag{4.27}$$

where $c_n > 0$ is just a dimensional constant.

Lemma 4.24.
Let $\kappa_n = \frac{1}{2}\left(\frac{3}{n+2} - \frac{1}{n-1}\right) > 0$, then following evolutionary inequality holds for any $\tau \leq \kappa_n$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(R_0^2\beta + t)|\nabla A|^2}{g^2}\right) - \Delta\left(\frac{(R_0^2\beta + t)|\nabla A|^2}{g^2}\right) &\leq \frac{2(R_0^2\beta + t)}{g}\left\langle \nabla g, \nabla\left(\frac{|\nabla A|^2}{g^2}\right)\right\rangle - \frac{|\nabla A|^2}{g^2} \\ &+ \frac{(R_0^2\beta + t)}{g}\left(c_n|A|^2\left(\frac{|\nabla A|^2}{g}\right) - 2\kappa_n\frac{n+2}{3}\frac{|\nabla A|^4}{g^2}\right). \end{aligned} \tag{4.28}$$

Proof. We will compute the evolution equation step by step. First of all we recall that H and $|A|^2$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}H^2 = \Delta H^2 + |A|^2H^2 - 2|\nabla H|^2, \quad \frac{\mathrm{d}}{\mathrm{d}t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + |A|^4.$$

Using this and the well known estimate $|\nabla A|^2 \geq \frac{3}{n+2}|\nabla H|^2$ we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}g - \Delta g &= -2\left(\left(\frac{1}{n-1}+\tau\right)|\nabla H|^2 - |\nabla A|^2\right) + 2|A|^2 g \\
&\geq -2\left(\left(\frac{1}{n-1}+\tau\right)\frac{n+2}{3}+1\right)|\nabla A|^2 \\
&= 2\frac{n+2}{3}(2\kappa_n - \tau)|\nabla A|^2 \geq 2\frac{n+2}{3}\kappa_n|\nabla A|^2
\end{aligned}$$

Now we compute the evolution equation for $|\nabla A|^2 g^{-1}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|\nabla A|^2}{g}\right) - \Delta\left(\frac{|\nabla A|^2}{g}\right) \leq \frac{2}{g}\left\langle \nabla g, \nabla\left(\frac{|\nabla A|^2}{g}\right)\right\rangle + \frac{1}{g}\left(-2|\nabla^2 A|^2 + c_n|A|^2|\nabla A|^2\right) \\
- 2\kappa_n\frac{n+2}{3}\left(\frac{|\nabla A|^4}{g^2}\right)
\end{aligned}$$

Here we can take Kato's inequality to estimate further, namely

$$\begin{aligned}
\left\langle \nabla g, \nabla\left(\frac{|\nabla A|^2}{g}\right)\right\rangle &= \langle \nabla g, \nabla|\nabla A^2|\rangle - \frac{1}{g}|\nabla g|^2|\nabla A|^2 \\
&\leq 2|\nabla g|^2|\nabla^2 A|^2|\nabla A| - \frac{1}{g}|\nabla g|^2|\nabla A|^2 \\
&\leq \frac{1}{g}|\nabla g|^2|\nabla A|^2 - \frac{1}{g}|\nabla g|^2\nabla A|^2 + g|\nabla^2 A|^2.
\end{aligned}$$

This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|\nabla A|^2}{g}\right) - \Delta\left(\frac{|\nabla A|^2}{g}\right) \leq c_n|A|^2\left(\frac{|\nabla A|^2}{g}\right) - 2\kappa_n\frac{n+2}{3}\left(\frac{|\nabla A|^4}{g^2}\right).$$

Finally, if we recall that $-g \leq 0$, we obtain our desired result

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|\nabla A|^2}{g^2}\right) - \Delta\left(\frac{|\nabla A|^2}{g^2}\right) \leq \left\langle \nabla g, \nabla\left(\frac{|\nabla A|^2}{g^2}\right)\right\rangle \\
+ \frac{1}{g}\left(c_n|A|^2\left(\frac{|\nabla A|^2}{g}\right) - 2\kappa_n\frac{n+2}{3}\left(\frac{|\nabla A|^4}{g^2}\right)\right).
\end{aligned}$$

□

Let (p_0, t_0) be the first event such that $\left(\frac{(R_0^2\beta+t_0)|\nabla A|^2}{g^2}\right) = K$ obtains a new maximum. Then we have $\nabla(\left((R_0^2\beta + t_0)\frac{|\nabla A|^2}{g^2}\right)) = 0$, $\frac{\mathrm{d}}{\mathrm{d}t}\left((R_0^2\beta + t_0)\frac{|\nabla A|^2}{g^2}\right) \geq 0$ and $\Delta\left((R_0^2\beta + t_0)\frac{|\nabla A|^2}{g^2}\right) \leq 0$ such that by Lemma 4.24 we obtain

$$0 \leq (R_0^2\beta + t_0)g^{-1}\left(c_n|A|^2\left(\frac{|\nabla A|^2}{g}\right) - 2\kappa_n\frac{n+2}{3}\frac{|\nabla A|^4}{g^2}\right) + \frac{|\nabla A|^2}{g^2}$$

which implies

$$0 \le c_n K|A|^2 - 2\kappa_n K^2 \frac{g}{R_0^2\beta + t_0} + \frac{K}{(R_0^2\beta + t_0)}$$

Since $g = \left(\frac{1}{n-1} + \tau\right)|H|^2 - |A|^2 \ge (\tau - \eta)\frac{1}{n}|A|^2$ we can further conclude

$$0 \le c_n K - 2(\tau - \eta)\frac{1}{n}\kappa_n \frac{K^2}{R_0^2\beta + t_0} + \frac{K}{(R_0^2\beta + t_0)|A|^2}.$$

We rearrange the terms to get

$$\begin{aligned} K &\le \frac{n}{2\kappa_n(\tau - \eta)}\left(n(R_0^2\beta + t_0) + \frac{10}{11(n-1)}(R_0^2 - 2(n-1)t_0)\right) \\ &\le C(n)\frac{1}{\tau - \eta}\left(R_0^2\beta + 2nR_0^2\beta\right) \\ &\le C(n)\frac{1}{\tau - \eta}R_0^2(1 + \beta). \end{aligned}$$

As there is no contribution of the initial data we thus obtain

$$|\nabla A|^2 \le C(n)\frac{1}{\tau - \eta}g^2\frac{R_0^2(1 + \beta)}{R_0^2\beta + t} \tag{4.29}$$

for all $t > -R_0^2\beta$.

Theorem 4.25 (Gradient Estimate)**.**
Let $\varepsilon = \varepsilon(n) > 0$, $k \ge 2$ and $0 < \Lambda < \infty$ be fixed such that on a (ε, k) cylindrical region we have $S_{n-1} \ge C(n)H^{n-1}$. Then for any $\beta > 0$ large and $\tau > 0$ small and there exists a constant $\theta = \theta(\varepsilon, \beta, n, \Lambda, \tau)$ such that every point $p \in \mathcal{C}_t$ at the center of an $(\varepsilon, k, \Lambda, \theta)$ periodic shrinking curvature neck satisfies

$$|\nabla A| \le C(n)\tau H^2 \tag{4.30}$$

for all times $t \in [\frac{-R_0^2\beta}{2}, 0]$.

Proof. We fix a small $\tau << \varepsilon$ and a constant β. By choosing $\delta = \frac{\tau^2}{4}$ and $\eta = \frac{\tau^2}{2}$ in the convexity estimate Theorem 4.14 and cylindrical estimate Theorem 4.20, we get the existence of a large constant and θ only depending on $\varepsilon, \beta, \Lambda, n$ and τ such that

$$|A| - \frac{1}{n-1}H^2 \le \frac{\tau}{2}H^2$$

for all times in $[-\beta R_0^2, 0]$ if the neck exists on $[-\theta R_0^2, 0]$. Then by (4.29) and Remark 4.22 we get

$$|\nabla A|^2 \le 2C(n)\frac{1}{\tau^2}\frac{1}{4}\tau^4 H^4 = \frac{1}{2}C(n)\tau^2 H^4$$

for all times $t > -\frac{1}{2}R_0^2\beta$. $\square$

By interior parabolic regularity we further obtain an estimate for higher derivatives of A.

Corollary 4.26.
In the situation of Theorem 4.25 for any $\tau > 0$ and $\beta > 0$ there also exist θ like above such that

$$|\partial_t^h \nabla^m A|^2 \leq C(n,k)\tau^2 H^{4h+2m+2} \tag{4.31}$$

for all $h \geq 0$ and $m \geq 1$ such that $2h + m \leq k$ on $[-\frac{1}{2}R_0^2\beta, 0]$.

Proof. Let us agree that all constants that are only dimensional will be denoted by $c(n)$. By [Hui84, Theorem 7.1] we have the general formula

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla^m A|^2 = \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A.$$

We will focus on the case $m = 2$. For this purpose we suppose that in view of Theorem 4.25 we have chosen $\theta > 0$ such that $|\nabla A|^2 \leq \delta^2 H^4$ on $[-R_0^2\beta, 0]$. By formula (4.26) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla^2 A|^2 &\leq \Delta|\nabla^2 A|^2 - 2|\nabla^3 A|^2 + c(n)|A||\nabla A|^2|\nabla^2 A| + c(n)|A|^2|\nabla^2 A|^2 \\ &\leq \Delta|\nabla^2 A|^2 - 2|\nabla^3 A|^2 + c(n)\delta^4 H^8 + c(n)H^2|\nabla^2 A|^2\end{aligned}$$

by the estimate from Theorem 4.25. Now we define the following function for $t \geq -R_0^2\beta$

$$f := (R_0^2\beta + t)|\nabla^2 A|^2 + N|\nabla A|^2 - \tau H^4$$

for $N > 0$ to be chosen later. We recall from the proof of the gradient estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla A|^2 \leq \Delta|\nabla A|^2 - |\nabla^2 A|^2 + c(n)\delta^2 H^6$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}H^4 = \Delta H^4 - 12|\nabla H|^2 H^2 + 4|A|^2 H^4.$$

Computing the time derivative we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f - \Delta f &\leq +c(n)\delta^2 H^8(R_0^2\beta + t) + |\nabla^2 A|^2 + c(n)H^2|\nabla^2 A|^2(R_0^2\beta + t) \\ &\quad - 2N|\nabla^2 A|^2 + Nc_2(n)\delta^2 H^6 + 12\tau|\nabla H|^2 H^2 - 4c(n)\tau H^6 \\ &\leq 0\end{aligned}$$

for $N \geq 2c(n)$ and $\tau \geq c(n)\delta^2$. Here we used that by the neck assumptions we can estimate

$$H^2(R_0^2\beta + t) \leq c(n-1)\frac{(R_0^2\beta + t)}{R_0^2 - 2(n-1)t} \leq c(n)\frac{\beta}{1 + 2(n-1)\beta} \leq c(n).$$

$\square$

Now the maximum principle implies that

$$\max_{\mathcal{C}_t} g \leq \max_{\mathcal{C}_{-R_0^2\beta}}$$

such that we get

$$\begin{aligned}|\nabla^2 A|^2 &\leq \frac{1}{(R_0^2\beta + t)}\left(c(n)|\nabla A|^2 + \tau H^4\right) \\ &\leq 2c(n)\tau H^6 \end{aligned} \tag{4.32}$$

for all $\tau > \delta$ small and $t \geq -\frac{R_0^2\beta}{2}$ because the contribution of the initial values of $\nabla^2 A$ vanishes due to the extra t factor in f. The last line follows from $H^2 \geq C(n)R(t)^{-2}$ by neck assumption and

$$\frac{c(n)}{(R_0^2\beta + t)} \leq c(n)H^2\frac{R_0^2 - 2(n-1)t}{R_0^2\beta + t} \leq c(n)H^2\frac{1+\beta}{\beta}$$

for all $t > -\frac{1}{2}R_0^2\beta$. From which we conclude the estimate for $m = 2$. For $m > 2$ we can repeat the same argument by induction. However, in each step we have to choose τ slightly larger and make the time interval smaller as (4.35). Since the regularity level is the fixed number $k \geq 2$ this procedure ends after a finite number of steps such that we have to choose $\theta > 0$ in dependence of k to get the estimate on the desired time interval. The estimates on the time derivatives then immediately follow from the equation (4.26).

It is well known that gradient estimates give us control over the change of the curvature in a neighborhood.

Corollary 4.27.
We have for all $p \in \mathcal{C}_t$, for $t \in [-\frac{R_0^2\beta}{2}, 0]$

$$H(q) \geq \frac{H(p)}{1 + \tau d(p,q)H(p)}$$

for all $q \in \mathcal{C}_t$ such that p and q lie in the same periodic component.

Proof. This is the same argument like in [HS09, Lemma 6.6]. □

4.5 Neck Improvement

So far we have seen that the only possibility for an ancient shrinking periodic neck is the standard cylinder. Therefore, if the parameter θ which determines the existence time of the shrinking periodic neck tends to infinity we expect the neck to become closer and closer to the standard cylinder. In the following section we want to quantify this phenomena by turning the local estimate into a global statement on the parameterization. Before we start, we want to collect all estimates we have obtained in the previous chapter in an for us relevant manner. We fix $\varepsilon = \varepsilon(n) > 0, k \geq 2$ and $\Lambda > 0$ such that the theorems of chapter 4 can be applied. Furthermore, we fix a $\beta > 0$ which determines the length of the time interval we want to examine. Before we continue we want to recall the following result from [HS09, Theorem 3.14] which is a consequence of Proposition 3.5.

Theorem 4.28.
Let $n \geq 3$. For any set of neck parameters $(\varepsilon, k, \Lambda)$ with $\Lambda \geq 10$ there exists (ε', k') such that if the extrinsic curvature is $(\varepsilon', k', \Lambda)$ around a point p, then p lies in the center of a normal maximal (ε, k)-hypersurface neck $\mathcal{N} : [-\Lambda + 1, \Lambda - 1] \times \mathbb{S}^{n-1} \to \mathcal{C}_t$.

We also note that the curvature is measured after rescaling it to radius one. First of all as in the definition we denote by $\overline{W}$ the shape operator corresponding to the standard cylinder $[-\Lambda, \Lambda] \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$. It is well known that the Eigenvalues are given by $\lambda_1 = 0$ and $\lambda_i = 1 = \frac{H}{n-1}$. Similarly the rescaled cylinder of radius R with Shape operator $\overline{W}_R$ has eigenvalues $\lambda_1 = 0$ and $\lambda_i = \frac{1}{R} = \frac{H}{n-1}$ for $i = 2, \ldots, n$. We want to apply Theorem 4.28 to (δ, k) for a small $0 < \delta < \varepsilon$, which yields the existence of parameters (ε', k'). Now if we denote by $R(t)$ the radius of the standard shrinking cylinder, then by the convexity and cylindrical estimates we get existence of a parameter $\theta(n, \varepsilon', k', \beta)$ such that if $\mathcal{C}_t$ is a $(\varepsilon, k', \Lambda, \theta)$ shrinking curvature neck we have

$$|\lambda_1| \leq \varepsilon' R(t)^{-1}, \quad |\lambda_i - \lambda_j| \leq \varepsilon' R(t)^{-1}$$

for any $p \in \mathcal{C}_t$ and $t \in [-\beta R_0^2, 0]$. Furthermore, the gradient estimate implies

$$|\nabla^l W(p)| \leq \varepsilon' R(t)^{-1-l}$$

for any $1 \leq l \leq k'$ and all times $t \in [-\beta R_0^2, 0]$. In other words, every point in $\mathcal{C}_t$ lies in the center of (ε', k', L)-extrinsic curvature neck in the sense of Definition 3.3 and the conclusion of Theorem 4.28 applies. This means for any $t \in [-R_0^2\beta, 0]$ we have a maximal normal (δ, k) hypersurface neck $\mathcal{N} : [-\Lambda + 1, \Lambda - 1] \times \mathbb{S}^{n-1} \to \mathcal{C}_t$. In particular, since the scaling parameters are time dependent we can say that any point p lies in the center of $(\delta, k, \Lambda, \beta)$ shrinking curvature neck. We recall that the neck assumptions are measured after a suitable rescaling to radius 1. Here we note that we can do the same analysis for any point in the periodic component such that the whole component can be parameterized as a normal (δ, k) hypersurface neck. We sum this up in the following Theorem.

Theorem 4.29 (Neck improvement)**.**
Let $\varepsilon = \varepsilon(n) > 0$, $0 < \Lambda < \infty$ and $k \geq 2$ be given so that the conclusions of the previous sections are true. Then for any large $\beta > 0$ and any small $0 < \delta < \varepsilon$ there exists a constant $\theta = \theta(n, \Lambda, k, \beta, \varepsilon, \delta) \geq \beta$ such that every point $(p, 0)$ being at the center of a periodic $(\varepsilon, k, \Lambda, \theta)$ shrinking curvature neck lies in the center of a periodic $(\delta, k, \Lambda, \beta)$ shrinking curvature neck.

Chapter 5

Roundness of Crosssections

The shape of the cross sections of a neck region is key as they are the starting point of the surgery construction. Due to the neck being close to the standard cylinder within ε range in some C^k topology, we expect that also the cross sections are round up to an error of order ε. Here round means that the induced second fundamental form l almost vanishes meaning that there is almost no change in the interior normal when moving along the cross section. In the previous chapter we have established various estimates that control the geometry of the neck. The change of the induced geometry of the cross section during the flow is determined by these ambient geometric factors. Thus we also expect an improvement here induced by the improvement of the ambient geometry. In order to quantify this heuristic thoughts we need to analyze the cross sections and their geometry further. Let $\mathcal{N} : [0, \Lambda] \times \mathbb{S}^{n-1} \to \mathcal{M}_t^n$ be a $(\varepsilon, k, \Lambda)$- hypersurface neck in normal parameterization and denote by Σ_z the corresponding foliation with constant mean curvature coming from the normal parameterization in chapter 3. We will assume that $\varepsilon > 0$ is sufficiently small. Let us denote by ω the unique Eigen-vector to λ_1 in $\mathcal{N}$ which satisfies $|\omega - \partial_z| = \mathcal{O}(\varepsilon)$ since it is almost parallel to the approximate axis by [HS09, Proposition 7.18]. Throughout this section, whenever we speak of the standard cylinder, we assume it to be in standard regular coordinates. We consider local coordinates x^i around a point $p \in \mathcal{N}$ such that $e_a := \frac{\partial}{\partial x^a}, \eta$ form a basis of T_pM, where $a = 1, \ldots, n-1$. Since $\mathcal{N}$ after rescaling to radius 1 is ε close in the C^k-norm to the standard cylinder in $\mathbb{R}^{n+1}$ we have that

$$|A(\omega, \omega)| \leq \frac{\varepsilon}{r(z)}; \quad |h_{ab} - \frac{1}{r(z)} g_{ab}| \leq \frac{\varepsilon}{r(z)} \tag{5.1}$$

$$|A(e_a, \omega)| \leq \frac{\varepsilon}{r(z)}; \quad |\nabla^m A|_{\bar{g}} \leq \frac{\varepsilon C_m}{r(z)^{m+1}} \tag{5.2}$$

for any $m \leq k$ and any $z \in [0, \Lambda]$ where e_a are tangential directions to Σ_z running from 1 to $n-1$ and $\bar{g}$ is the standard metric on the cylinder.

Remark 5.1. The estimates (5.1) and (5.2) also hold true with $r(z)$ replaced by $CR(t)$ with a constant C because by Definition 3.14 we have

$$r(z) = R(t)(1 + \mathcal{O}(\varepsilon)),$$

whenever $\Sigma_z \subset \mathcal{C}_t$. We will use this fact frequently and replace $r(z)$ by $R(t)$ whenever we are in the time dependent case.

The foliation by the CMC surfaces is very close to the standard foliations $\mathbb{S}^{n-1} \times \{z\} \subset \mathbb{S}^{n-1} \times [0, \Lambda]$ in the standard cylinder. That is why an interior normal direction is almost parallel to ω.

Lemma 5.2.
Let η be a choice of an interior normal to Σ_z in $\mathcal{N}$. Then there exists some constant C such that

$$|\eta - \omega| \leq C\varepsilon. \tag{5.3}$$

Furthermore, the following estimates hold

$$|h_{nn}| \leq C\frac{\varepsilon}{r(z)} \quad \text{and} \quad |h_{an}| \leq C\frac{\varepsilon}{r(z)} \tag{5.4}$$

where $h_{nn} = h(\eta, \eta)$ and $h_{na} = h(\eta, e_a)$ and therefore

$$|\mathrm{Ric}^{\mathcal{M}^n}(\eta, \eta)| \leq C\varepsilon r(z)^{-2}. \tag{5.5}$$

Here by choosing $\varepsilon = \varepsilon(n)$ sufficiently small we can always arrange that $C \leq \frac{11}{10}$.

Proof. On a standard cylinder we have that $\eta = \omega$ since ω is a unit vector. We recall that by the implicit function theorem Σ_z can be written as a graph of a function u on $\mathbb{S}^{n-1}$ close to 0, for example with C^2-norm less than $C\varepsilon$ for some constant C. Thus on Σ_z compared to the corresponding sphere in standard cylinder we have

$$|\eta - \partial_z| \leq C\varepsilon.$$

This together with the property that $|\omega - \partial_z| = \mathcal{O}(\varepsilon)$ gives the desired conclusion. The estimates (5.4) then follow from the corresponding assumptions on the neck in (5.1),(5.2) and the gradient estimate. Then (5.5) follows with $|H| \leq \frac{n-1+o(\varepsilon)}{r(z)} \leq C(n)\frac{1}{r(z)}$

$$\begin{aligned}
|\mathrm{Ric}(\eta, \eta)| &= |Hh_{nn} - h_{nk}h^k_n| \\
&\leq C\frac{\varepsilon}{r(z)^2} + C\frac{\varepsilon^2}{r(z)^2} \\
&\leq C\varepsilon(1+\varepsilon)r(z)^{-2}.
\end{aligned}$$

□

As a consequence we also get estimates on the second fundamental form and mean curvature of Σ_z.

Proposition 5.3.
As usual we denote by $(l^z_{ab})_{1\leq a,b\leq n-1}$ the second fundamental form and by L^z the corresponding mean curvature of the CMC slice Σ_z. Then we have

$$|l^z|_g \leq C\frac{\varepsilon}{r(z)} \tag{5.6}$$

$$|L^z| \leq C\frac{\varepsilon}{r(z)}. \tag{5.7}$$

Proof. As we have mentioned before each of the $\Sigma_z \subset \mathcal{N}$ is close to a sphere $\mathbb{S}^{n-1}_{r(z)}$ in the standard cylinder, since after rescaling to $r(z) = 1$ it can be written as a graph of a function $u : \mathbb{S}^{n-1} \to \mathbb{R}$ with $C^{2,\beta}$-norm less than some $C\varepsilon$. The foliation by the spheres $\mathbb{S}^{n-1}$ of the standard cylinder $\mathbb{S}^{n-1} \times [0, \Lambda]$ of radius 1 with the standard metric $\bar{g}$ satisfies $\bar{L} \equiv 0$ and $\bar{l} \equiv 0$ with respect to the induced metric by $\bar{g}$. Therefore, the corresponding quantities of the rescaled $\tilde{\Sigma}$ satisfy

$$|\tilde{l}| \leq C\varepsilon \quad |\tilde{L}| \leq C\varepsilon.$$

So, after the rescaling we get the desired estimates. □

Once we have detected a shrinking neck and fixed a normal parametrisation at a time t, we want to examine how we can follow the CMC surfaces $(\Sigma_z)_z$ in time such that the mean curvature stays constant. In this way, we can get a time dependent unique foliation via CMC surfaces such that the upcoming computations are well defined. A priori, if we just followed the points in one of the Σ_z under mean curvature flow the resulting surface could move away or even tilt. This is particulary the case when the neck closes up to a convex cap and becomes approximately close to a translating soliton. In Chapter 3 we have seen how a pertubation of the ambient metric influences the mean curvature of the hypersurfaces. We fix $z \in (0, \Lambda)$ with sufficient distance to the boundaries of the periodic component. In a next step we want to find a function $\alpha : \mathcal{M}^n \times [0, S_{\max}] \to \mathbb{R}$ such that if we evolve $G : \Sigma_z \times [0, S_{max}) \to \mathcal{M}^n$ via

$$\frac{\mathrm{d}}{\mathrm{d}s} G(p,s) = -\alpha \left((G(p,s)\,, s \right) \eta(G(p,s)) \tag{5.8}$$

the mean curvature of Σ stays constant. In other words, if the whole neck component moves with mean curvature flow, we want to find out how we have to deform the CMC foliation that has also moved along the flow in order to recover CMC surfaces. It is a well known result that for such a varation of the position within the Riemannian manifold $(\mathcal{M}^n, g)$ the mean curvature satisfies

$$\frac{\partial L}{\partial s}(G(p,s)) = \Delta^{\Sigma}\alpha + \alpha \left(|l|^2 + \mathrm{Ric}_g \left(\eta^t, \eta^t \right) \right), \tag{5.9}$$

where Ric_g is the Ricci-Tensor on $(\mathcal{M}^n, g)$, here at the point $G(p,s)$ and $\Delta^{\Sigma} = g^{ab}\nabla^{\Sigma}_a \nabla^{\Sigma}_b$ is the Laplace-Beltrami operator on Σ_z. The right hand side is also known as the stability operator $\mathcal{J}_{\Sigma_z}$ acting on smooth functions α. The next step is to consider a time dependent ambient metric g^t and consider an simoultaenous movement such that we need to combine the variations of the metric and of the location in order to obtain evolutions for the total variation. In this way we use t as the time parameter for both evolutions. For this purpose we think of l (similarly for all other relevant geometric quantities) as a quantity depending on a parameter τ coming from the variation of the metric (later $\tau = t$) and the location $q = G(p,t)$, i.e.

$$L = L(\tau, q)$$

where $q \in G(\Sigma_z \times \{t\})$ and $\tau \in [0, T_{\max})$. Therefore, all that is left to do is to compute the total derivative with respect to t. This is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} L(t, G(p,t)) &= (\frac{\partial}{\partial \tau} L)(t, G(p,t)) + (\frac{\partial}{\partial q} L)(t, G(p,t)) \\ &= -2g_t^{ab}\nabla_a (Hh)_{bn} - \nabla_n (H^2 - Hh_{nn}) + 2Hh^{ab}l_{ab} - LHh_{nn} \\ &\quad + \Delta\alpha + \alpha \left(|l|^2 + \mathrm{Ric}_{g^t} \left(\eta^t, \eta^t \right) \right). \end{aligned} \tag{5.10}$$

Thus, if we want the mean curvature of Σ_z to stay constant we need to solve

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(L(t, G(p,t)) - \fint_{\Sigma_t} L(t, G(p,t))\, \mathrm{d}\mu_t \right) = 0$$

which leads to

$$\begin{aligned} \mathcal{J}_{\Sigma_z}(\alpha) - \fint_{\Sigma_t} \frac{\mathrm{d}}{\mathrm{d}t} L_t(t, G(p,t))\, \mathrm{d}\mu_t &= -2g_t^{ab}\nabla_a (Hh)_{bn} - \nabla_n (H^2 - Hh_{nn}) \\ &\quad + 2Hh^{ab}l_{ab} - LHh_{nn} \end{aligned} \tag{5.11}$$

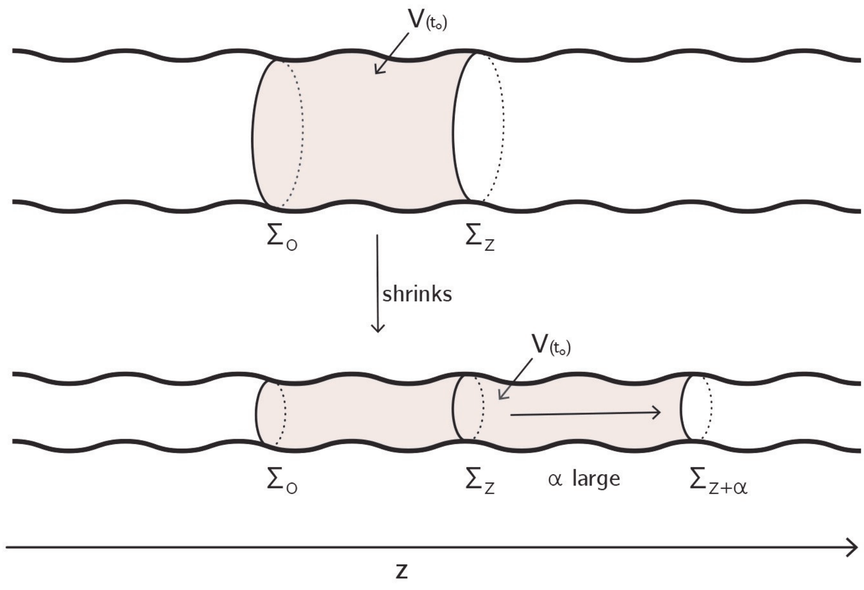

Figure 5.1: Problems with rescaling

being an elliptic equation in the speed α. We want to employ the same implicit function argument that we have used above to show existence of the CMC foliation. But now we do not linearize around the round standard metric but on a slightly perturbed metric such that we have to analyze the kernel of the full stability operator. In order to make sure the kernel is still trivial we choose a volume condition like in the standard metric case. Naturally, we would assume

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = 0$$

where $V(t)$ is the volume with respect to a reference sphere exactly like in the construction of the initial CMC foliation but now with respect to the metric g^t. However, since the neck will shrink very fast this would mean that the movement of the Σ_t in α direction would have to be very large in order to make the volume constant. In Figure **??** this problem is visualized. For this reason, we further choose to make this requirement scaling invariant. The standard shrinking cylinder solution to mean curvature flow with starting radius R_0 satisfies

$$R(t) = \sqrt{R_0^2 - 2(n-1)t}.$$

Thus it is quite natural to assume

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{V(t)}{(R_0^2 - 2(n-1)t)^{\frac{n-1}{2}}} = 0. \tag{5.12}$$

Remark 5.4. In general we could have prescribed any smooth function of t as the volume as long as the initial values fit. This might be necessary for an analysis of a neck that is closing up to a convex tip.

We recall that in our situation, $\mathcal{M}^n$ is only ε close to a standard cylinder in the C^k-norm. That means Δ^Σ is only ε close to the Laplacian on the sphere with respect to the standard metric $\bar{g}$ which implies the eigenvalues λ_i of Δ^Σ upon rescaling to radius 1 satisfy

$$\lambda_i = \bar{\lambda_i} + \mathcal{O}(\varepsilon)$$

where $\bar{\lambda_i}$ are those of the standard Laplacian. Because we have the metric g as close as we like to the standard metric $\bar{g}$ in the C^k norm, we can use Proposition 5.3 to ensure that the function

$$f_\Sigma := |l|^2 + \mathrm{Ric}_g(\eta, \eta)$$

can be estimated in the $C^{2,\beta}$ norm by lets say $\frac{n}{2}\varepsilon$ upon rescaling to radius 1. Then indeed the right hand side of (5.11) is an isomorphism. Before we proof this we need a Lemma which guarantees that the kernel is almost trivial.

Lemma 5.5.
Let $\mathcal{N}$ be an $(\varepsilon, k, \Lambda)$ hypersurface neck with foliation $(\Sigma_z)_z$. Then the Operator $A : C^{2,\beta}(\Sigma_z) \to C^{0,\beta}(\Sigma_z) \cap \{\int w \,\mathrm{d}\mu = 0\}$ given by

$$A(w) = \mathcal{J}_{\Sigma_z}(w) - \fint_{\Sigma_z} \mathcal{J}_{\Sigma_z}(w) \,\mathrm{d}\mu_z$$

has 1-dimensional kernel consisting of constant functions only.

Proof. We assume that we have found a function $w \in C^{2,\beta}(\Sigma_z)$ which also satisfies

$$\mathcal{J}_{\Sigma_z}(w) - \fint_{\Sigma_z} \mathcal{J}_{\Sigma_z}(w) \,\mathrm{d}\mu_z = 0.$$

We multiply this equation with $w - \bar{w}$ where $\bar{w} := \fint_{\Sigma_z} w \,\mathrm{d}\mu_z$ and integrate over Σ_z:

$$\begin{aligned}
\int_{\Sigma_z} |\nabla w|^2 \,\mathrm{d}\mu_{n-1} &= -\int_{\Sigma_z} \Delta^\Sigma w w \,\mathrm{d}\mu_z \\
&= \int_{\Sigma_z} (w - \bar{w})^2 f_\Sigma \,\mathrm{d}\mu_z \\
&\leq \frac{n}{2}\varepsilon \int_{\Sigma_z} (w - \bar{w})^2 \,\mathrm{d}\mu_z.
\end{aligned}$$

But on the other hand we have the variational characterization of the smallest eigenvalue via the Poincare inequality leading to

$$(n - 1 + \varepsilon) \int_{\Sigma_z} (w - \bar{w})^2 \,\mathrm{d}\mu_z \leq \int_{\Sigma_z} |\nabla w|^2 \,\mathrm{d}\mu_z,$$

which gives a contradiction. Thus $w = \bar{w}$ has to be constant. $\square$

Now we are in position to conclude the existence of a speed α.

Proposition 5.6.
Let $\mathcal{C}_t$ be a $(\varepsilon, k, \Lambda, \theta)$-shrinking curvature neck. We fix an initial folitation at time $t_0 = -R_0^2\theta$ by CMC surfaces Σ_z coming from a normal parameterization of the neck with initial Volume $V(z)$ like we constructed in Theorem 3.10. Then under the volume condition (5.12) there exists a unique evolution $\alpha_z : \Sigma_z \times [-R_0^2\theta, 0] \to \mathbb{R}$ such that the resulting time dependent family Σ_z^t given as graphs of $\alpha(t, \cdot)$ over Σ_z has constant mean curvature for all times $t \geq t_0$ and any $z \in [0, \Lambda]$. Furthermore, α depends smoothly on t and satisfies

$$R(t)|\alpha_z| + R(t)^2|\nabla\alpha_z| + R(t)^3|D^2\alpha_z| \leq C\varepsilon \quad \text{for all} \quad z \in [0, \Lambda]. \tag{5.13}$$

Proof. Again, since the Laplace Beltrami operator defines an isomorphism and further since the perturbation

$$F_\Sigma : C^{2,\beta}(\Sigma_z) \to C^{0,\beta}(\Sigma_z) \cap \{\int w = 0\}$$

given by $F_\Sigma(w) = f_\Sigma w - \fint_{\Sigma_z} w f_{\Sigma_z}\, \mathrm{d}\mu_z$ satisfies $\|F_\Sigma\|_{0,\beta} \leq Cw\varepsilon$ and is compact, we can conclude by standard Fredholm theory the operator $\mathcal{J}$ is again a Fredholm operator which has index 0. By Lemma 5.5 together with the volume condition (5.12) the constant functions are ruled out such that it has infact trivial kernel and we can conclude the surjectivity follows. Therefore, we can once again use the implicit function theorem exactly as is in the proof of Theorem 3.10 to obtain the existence of a family of functions α_z as we claimed above. Also since g depends smoothly on t the implicit function theorem guarantees that α depends smoothly on t, since it satisfies a linear PDE. This implies that for $z \in [0, \Lambda]$ we have found a smooth family of functions

$$\alpha_z : [-R_0^2\theta, 0] \times \Sigma_z \to \mathbb{R}$$

such that for every $t \in (-R_0^2\theta, 0]$ the function $\alpha_z(t, \cdot)$ is very close to 0 in the $C^{2,\beta}$-norm on Σ_z. All the arguments so far were carried out after rescaling to approximate radius 1, but since α acts like a speed of a movement of the surface within the neck, it scales like $R(t)^{-1}$. From this we conclude the estimate (5.13). □

The result of Proposition 5.6 allows us to look at the evolution equations we derived in the previous chapter and analyze them on one of the CMC surfaces as we have now found a controlled way how to follow them along the flow. In this way the normal direction towards the hypersurfaces Σ_z within the neck still corresponds to the almost flat direction of the neck up to an error of order ε. We recall that due to the results from the previous chapter we know that an a priori ε-neck starting at $-R_0^2\theta$ will in fact become a δ-neck at time $-R_0^2\beta$. In this way, if we fix an initial foliation of the neck component by Σ_z by CMC surfaces, we can use Proposition 5.3 to get a improved roundness estimate, namely

$$|l|^2 \leq C(n)\frac{\delta^2}{R(t)^2}$$

for all times $t \geq -R_0^2\beta$ and $z \in [0, \Lambda]$. This estimate comes from analyzing the elliptic PDE via the implicit function theorem at each individual time. In chapter 2 we have established a type of parabolic evolution equation for the quantity $|l|^2$. It will thus be interesting whether the same result can be obtained by pointwise parabolic argument involving the maximum principle. We will therefore ignore the result from Proposition 5.3. Of course we need an evolution equation as the basis of our analysis. Therefore, we have to combine these evolution equations with the movement coming from the speed α.

Proposition 5.7.
Suppose on we have a parameter dependent metric $\frac{\mathrm{d}}{\mathrm{d}t} g_{ij}^t = T_{ij}$ for some $(2, 0)$ Tensor T on a Riemannian manifold $\mathcal{M}^n$ and suppose we evolve a hypersurface $G_0 : \Sigma_0 \to \mathcal{M}^n$ via (5.8) then if we choose an interior normal η^t with respect to g^t, then we have the following

evolution equations on for the flow $\Sigma_t = G(t, \Sigma_0)$ in local coordinates

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta^t = \nabla^\Sigma \alpha - \frac{1}{2}T(\eta^t,\eta^t)\eta^t - T_(\eta, e_a)g_t^{ab}e_a \tag{5.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}l^t_{ab} = \nabla^\Sigma_a\nabla^\Sigma_b\alpha - \alpha(l_{ac}l^c_b + \mathrm{Rm}_{anbn}) - \frac{1}{2}\left(\nabla^t_a T_{bn} + \nabla^t_b T_{an} - \nabla^t_n T_{ab} - T_{nn}l^t_{ab}\right) \tag{5.15}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L &= \Delta^\Sigma\alpha + \alpha(|l|^2 - \mathrm{Ric}(\eta^t,\eta^t)) - g_t^{ab}\nabla^t_a T_{bn} + \frac{1}{2}\nabla_n \mathrm{Tr}_{\Sigma^{n-1}}(T) - g_t^{ap}g_t^{bq}T_{pq}l_{ab}\\ &\quad + \frac{1}{2}LT_{nn}\end{aligned} \tag{5.16}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|l|^2 &= 2\langle l_{ab}, \nabla^\Sigma_a\nabla^\Sigma_b\alpha\rangle + 2\alpha\mathrm{tr}_\Sigma(l^3) + 2\alpha l^{ab}\mathrm{Rm}_{anbn} - 2g_t^{ad}g_t^{be}T_{ab}\, l^c_e\, l_{cd}\\ &\quad - l^{ab}\left(\nabla^t_a T_{bn} + \nabla^t_b T_{an} - \nabla^t_n T_{ab}\right) + T_{nn}|l|^2\end{aligned} \tag{5.17}$$

Proof. The evolution equations for a flow with a general speed α in a fixed Riemmanian manifold can be found in [HP99, Lemma 7.6]. Then all we have left to do is take the evolution equations for the variation of the metric from Chapter 2 Lemma 2.2, 2.4 and Proposition 2.5 and combine them to the total variation exactly like in (5.10). □

We will continue with a small remark to related problems.

Remark 5.8. The reason why we computed the change of the submanifold geometry under general variations of the metric in Chapter 2 is that is approach is very flexible. In particular, we could think of other evolution equations not just Ricci-Flow and mean curvature flow. For example, in the ADM formalism [ADM08] where we consider a time slice in a space time which then is foliated by certain hypersurfaces and then the time evolution of such a time slice. A center of mass can then by defined as a CMC foliation of the time slice such that it could be interesting to reformulate this approach in the present setting. It could be useful to derive similar evolution equations for the Mean Curvature of those slices and see if they can be followed just like we did now. We refer to [HY96], [Ner15] and [CN15] for further material on that matter.

For our goal to prove a roundness estimate on the crosssection we need the evolution equation for our special case.

Proposition 5.9. We have the following evolution equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(|l|^2\right) &= \Delta^\Sigma|l|^2 - 2|\nabla^\Sigma l|^2 + 2l_{ab}\nabla^\Sigma_b\nabla^\Sigma_a L + 2|l|^2(\mathrm{Ric}_{nn} - Hh_{nn}) + 2(|l|^4 - L\mathrm{tr}_\Sigma(l^3))\\ &\quad + 4Hh_{ab}l_{bc}l_{ca} - 2l_{ab}l_{bc}\mathrm{Ric}_{ca} - 2Ll_{ab}\mathrm{Rm}_{nanb} - 2l_{ab}\mathrm{Rm}_{nacn} + 2l_{ab}l_{da}\mathrm{Rm}_{bccd}\\ &\quad + 4l_{ab}l_{dc}\mathrm{Rm}_{bcad} + 2l_{ab}\nabla_b A^2_{an} - 2\nabla_n H\cdot h_{ab}l_{ab} + 2l_{ab}\nabla_a H\cdot h_{nb} + 2l_{ab}\nabla_c\mathrm{Rm}_{nacb}\\ &\quad + 2l_{ab}\nabla^\Sigma_a\nabla^\Sigma_b\alpha + 2\alpha\mathrm{tr}_\Sigma(l^3) + 2\alpha_{ab}\mathrm{Rm}_{anbn}\end{aligned} \tag{5.18}$$

Proof. We combine the diffusion form equation from Theorem 2.7 with the general evolution equations from Proposition 5.7 applied to our speed α in the mean curvature flow case. □

We will assume that we have a time dependent CMC foliation in the sense of Proposition 5.6 and additionally we assume that for all times under consideration we have the estimate

$$|l|^2 \leq \frac{(n-1)}{4}R(t)^{-2} \tag{5.19}$$

which is much weaker than Proposition 5.3 since we have chosen $\varepsilon > 0$ small. Now we want to employ a parabolic argument to show that this estimate is actually much better. For this reason, similar to our arguments from above, we look at the function

$$f_\sigma := \left(|l|^2 \cdot R(t)^{2-\sigma} \right).$$

Note that again we introduce the parameter σ to break the scaling of f_σ, for $\sigma = 0$ f would be scaling invariant. First we make some elementary computations. We compute the time derivative of f_σ in orthogonal coordinates around the point under consideration.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} f_\sigma = -\frac{(2-\sigma)(n-1) f_\sigma}{R(t)^2} \\
&+ \Delta^\Sigma f_\sigma + R(t)^{2-\sigma}\big(-2|\nabla^\Sigma l|^2 + 2 l_{ab} \nabla^\Sigma_b \nabla^\Sigma_a L + 2|l|^2 (\mathrm{Ric}_{nn} - H h_{nn}) + 2(|l|^4 - L \mathrm{tr}_\Sigma(l^3))\big) \\
&+ R(t)^{2-\sigma} \left(4H h_{ab} l_{bc} l_{ca} - 2 l_{ab} l_{bc} \mathrm{Ric}_{ca} - 2L l_{ab} \mathrm{Rm}_{nanb} - 2 l_{ab} l_{cb} \mathrm{Rm}_{nacn} + 2 l_{ab} l_{da} \mathrm{Rm}_{bccd} \right) \\
&+ R(t)^{2-\sigma} \left(4 l_{ab} l_{dc} \mathrm{Rm}_{bcad} + 2 l_{ab} \nabla_b A^2_{an} - 2(\nabla_n H) h_{ab} l_{ab} + 2 l_{ab} (\nabla_a H) h_{nb} + 2 l_{ab} \nabla_c \mathrm{Rm}_{nacb} \right) \\
&+ R(t)^{2-\sigma} \left(2 l_{ab} \nabla^\Sigma_a \nabla^\Sigma_b \alpha + 2\alpha \mathrm{tr}_\Sigma(l^3) + 2\alpha l_{ab} \mathrm{Rm}_{anbn} \right)
\end{aligned}$$

Also since Σ stays CMC the term $2 l_{ab} \nabla^\Sigma_b \nabla^\Sigma_a L$ vanishes for all t. To analyze this parabolic equation we need to carefully look at each of the individual terms. By Lemma 5.2 we will identify some of the terms as lower order.

Lemma 5.10.
For every point p in an (ε, k)-hypersurface neck we have

$$\begin{aligned}
-2 l_{ab} l_{bc} \mathrm{Ric}_{ca} + 2 l_{ab} l_{da} \mathrm{Rm}_{bccd} + 4 l_{ab} l_{cd} \mathrm{Rm}_{bcad} + 4H h_{ab} l_{bc} l_{ca}& \\
= \frac{H^2 |l|^2}{(n-1)^2} \mathcal{O}(\varepsilon) + 4 \frac{H^2 L^2}{(n-1)^2} (1 \pm \mathcal{O}(\varepsilon))&
\end{aligned} \tag{5.20}$$

Proof. Due to the Gauss equations and since $h_{ab} = \frac{H(1 \pm \mathcal{O}(\varepsilon))}{n-1} g_{ab}$, we have

$$\begin{aligned}
\mathrm{Rm}_{abcd} &= h_{ac} h_{bd} - h_{ad} h_{bc} \\
&= \frac{H^2 (1 \pm \mathcal{O}(\varepsilon))}{(n-1)^2} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right).
\end{aligned}$$

Such that

$$l_{ab} l_{cd} (g_{ab} g_{cd} - g_{ad} g_{cb}) = L^2 - |l|^2$$

and

$$\begin{aligned}
l_{ab} l_{da} \mathrm{Rm}_{bccd} &= \frac{H^2 (1 \pm \mathcal{O}(\varepsilon))}{(n-1)^2} l_{ab} l_{da} (g_{bc} g_{cd} - g_{bd} g_{cc}) \\
&= \frac{H^2 (1 \pm \mathcal{O}(\varepsilon))}{(n-1)^2} l_{ab} l_{da} (g_{bd} - g_{bd} g_{cc}) = \frac{H^2 (1 \pm \mathcal{O}(\varepsilon))}{(n-1)^2} \left(|l|^2 - (n-1)|l|^2 \right)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ric}_{ac} = \mathrm{Rm}_{akck} = (h_{ac} h_{kk} - h_{ak} h_{ck}) &= \frac{H^2 (1 \pm \mathcal{O}(\varepsilon))}{n-1} g_{ac} - \frac{H^2 (1 \pm \mathcal{O}(\varepsilon))}{(n-1)^2} g_{ac} \\
= \frac{H^2 (n-2)(1 \pm \mathcal{O}(\varepsilon))}{(n-1)^2} g_{ac}.&
\end{aligned}$$

Then we obtain

$$
\begin{aligned}
l_{ab}l_{da}\mathrm{Ric}_{ac} &= \frac{n-2}{(n-1)^2}(H^2|l|^2(1\pm\mathcal{O}(\varepsilon)))\\
l_{ab}l_{da}\mathrm{Rm}_{bccd} &= -\frac{n-2}{(n-1)^2}H^2|l|^2(1\pm\mathcal{O}(\varepsilon))\\
l_{ab}l_{cd}\mathrm{Rm}_{bcad} &= (L^2-|l|^2)\frac{H^2}{(n-1)^2}(1\pm\mathcal{O}(\varepsilon))\\
Hh_{ab}l_{ad}l_{db} &= \frac{H^2}{n-1}|l|^2(1\pm\mathcal{O}(\varepsilon)),
\end{aligned}
$$

such that if we take into account the error terms of order ε we obtain

$$
\begin{aligned}
&-2l_{ab}l_{bc}\mathrm{Ric}_{ca}+2l_{ab}l_{da}\mathrm{Rm}_{bccd}+4l_{ab}l_{cd}\mathrm{Rm}_{bcad}+4Hh_{ab}l_{bc}l_{ca}\\
&\quad=\frac{H^2|l|^2}{(n-1)^2}(1\pm\mathcal{O}(\varepsilon))\left(-2(n-2)-2(n-2)-4+4(n-1)\right)\\
&\quad+4\frac{H^2L^2}{(n-1)^2}(1\pm\mathcal{O}(\varepsilon))\\
&\quad=\frac{H^2|l|^2}{(n-1)^2}\mathcal{O}(\varepsilon)+4\frac{H^2L^2}{(n-1)^2}(1\pm\mathcal{O}(\varepsilon))
\end{aligned}
$$

□

In order to be able to absorb the terms of the highest order that involve L we will need to split the norm of the second fundamental form in the following way

$$
|l|^2=|\mathring{l}|^2+\frac{1}{n-1}L^2. \tag{5.21}
$$

Now we can collect all terms of highest order including

$$
-\frac{(2-\sigma)(n-1)f_\sigma}{R(t)^2}=-(2-\sigma)(n-1)|l|^2R(t)^{-\sigma}
$$

together with the fact that we have $H^2(n-1)^{-2}=(\pm\mathcal{O}(\varepsilon))R(t)^{-2}$ to get

$$
\begin{aligned}
&R(t)^{-\sigma}\left(\left(-(2-\sigma)(n-1)+\mathcal{O}(\varepsilon)\right)|l|^2+(4+\mathcal{O}(\varepsilon))L^2\right)\\
&\quad=R(t)^{-\sigma}\left(\left(-(2-\sigma)(n-1)+\mathcal{O}(\varepsilon)\right)|\mathring{l}|^2+\left(4-(2-\sigma)+\mathcal{O}(\varepsilon)\right)L^2\right).
\end{aligned}
$$

At this point we have a good negative term in $|\mathring{l}|^2$ but the L^2 term has still a positive factor. Before we eliminate this factor we take care of the lower order terms. Here our improved estimates will come into play.

Proposition 5.11.
Let $0<\Lambda<\infty$ and $k\geq 2$ be fixed and $\varepsilon(n)>0$ be so small that the conclusions of the previous chapters hold. Then for any $\beta>0$ large and any small $0<\delta<\varepsilon$ we can find $\theta(\beta,n,\Lambda,\varepsilon,\delta)>0$ large such that if Σ_z^t is the evolution of the initial CMC surfaces in a periodic $(\varepsilon,k,\Lambda,\theta)$-shrinking curvature neck, then in the time interval $[-\frac{R_0^2\beta}{2},0]$ we have the following evolutionary estimate on the fest function f

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}f\leq{}&\Delta f+R(t)^{-\sigma}\left(\left(-(2-\sigma)(n-1)+\frac{3}{4}+C\varepsilon\right)|\mathring{l}|^2+\left(4-(2-\sigma)+\frac{1}{4}+C\varepsilon\right)L^2\right)\\
&+C(n)\delta^2R(t)^{-2-\sigma}.
\end{aligned}\tag{5.22}
$$

Proof. By [HS99b, Lemma 2.2] we have $\mathrm{tr}(l^3) \leq |l|^3$ such that we can use the neck assumptions together with Proposition 5.3 to estimate

$$\begin{aligned}2\left(|l|^4 - L\mathrm{tr}_\Sigma(l^3)\right) &- 2|l|^2 H h_{nn} + 2|l|^2 \mathrm{Ric}_{nn}\\ &\leq 2|l|^4 + 2L|l|^3 + C(n)\varepsilon|l|^2 R(t)^{-2}\\ &\leq \frac{3}{4}|l|^2 R(t)^{-2} + \frac{1}{4}L^2 R(t)^{-2} + C(n)\varepsilon|l|^2 R(t)^{-2}\end{aligned}$$

Here we used the additional bound on $|l|^2$ in (5.19) and Young's inequality on $|l|L$. Using the Gauss equations in a similar fashion as in the proof of Lemma 5.10 we can control the Riemann tensor contributions that involve η directions

$$-2Ll_{ab}\mathrm{Rm}_{nanb} - 2l_{ab}l_{cb}\mathrm{Rm}_{nacn} \leq C\left(\varepsilon L^2 R(t)^{-2} + \varepsilon|l|^2 R(t)^{-2}\right).$$

Furthermore, we can control the terms that have an α distribution by (5.13)

$$2l_{ab}\nabla_a^\Sigma\nabla_b^\Sigma\alpha + 2\alpha\mathrm{tr}_\Sigma(l^3) + 2\alpha l_{ab}\mathrm{Rm}_{anbn} \leq C\varepsilon|l|^2 R(t)^{-2} + 2|l|\left(|\alpha||l|^2 + |\nabla^2\alpha|\right)$$

By Theorem 4.29 there is a large constant θ such that in the smaller time interval $[-\frac{R_0^2\beta}{2}, 0]$ any point lies in the center of an (δ, k) curvature neck such that in particular we have $\|g^t - g_{\mathrm{std}}\|_{C^2} \leq C\frac{\delta}{R(t)^2}$, since the neck can be written as a graph of a function over the standard cylinder with C^2 norm of order δ. Therefore, Proposition 5.6 implies the estimate

$$|\nabla^2\alpha| \leq C\delta R(t)^{-3} \quad \text{and} \quad |\alpha| \leq C\delta R(t)^{-1}.$$

Last but not least we need to examine the terms involving derivatives of curvature. Here is where our improved gradient estimates come into play. Once again by Theorem 4.29 or Theorem 4.25 we obtain

$$|\nabla A| \leq \delta R(t)^{-2}$$

in the time interval $[\frac{-R_0^2}{2}\beta, 0]$ when the parameter θ is chosen accordingly. Furthermore, by the Gauß equations we can write

$$\mathrm{Rm}_{nacb} = h_{nc}h_{ab} - h_{nb}h_{ac}$$

such that all terms above are controlled by $c(n)|A||l||\nabla A|$. This together with

$$|A| \leq c(n)R(t)^{-1}$$

by the neck assumptions yields

$$\begin{aligned}2l_{ab}\nabla_b A_{an}^2 &- 2\nabla_n H \cdot h_{ab}l_{ab} + 2l_{ab}\nabla_a H \cdot h_{nb} + 2l_{ab}\nabla_c \mathrm{Rm}_{nacb}\\ &\leq C(n)\delta|l|R(t)^{-3}.\end{aligned}$$

Combining all these estimates with a simple application of the Peter-Paul inequality

$$\delta|l|R(t)^{-3} \leq \frac{1}{2}|l|^2 R(t)^{-2} + 4\delta^2 R(t)^{-4}$$

gives us the desired conclusion. □

From here on we fix $0 < \sigma < \sigma(n)$ so small such that the σn term can be absorbed into the good negative terms. We also replace $(n-1)$ by $(n-2)$ since we need to absorb the $\frac{3}{4}$. Furthermore, by Proposition 5.7, or more explicit formula 5.10, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L^2 &= 2L\left(2g^{ab}\nabla_a(Hh)_{bn} - g^{ab}\nabla_n(Hh)_{ab} - LHh_{nn}\right) + 4LHh^{ab}l_{ab}\\ &+ 2L\left(\Delta\alpha + \alpha(|l|^2 + \mathrm{Ric}(\eta,\eta))\right)\\ &= \frac{(4+\mathcal{O}(\varepsilon))H^2L^2}{n-1} + 2L\left(2g^{ab}\nabla_a(Hh)_{bn} - g^{ab}\nabla_n(Hh)_{ab} - LHh_{nn}\right)\\ &+ 2L\left(\Delta\alpha + \alpha(|l|^2 + \mathrm{Ric}(\eta,\eta))\right).\end{aligned}$$

We notice that in this evolution equation we find a term of the form $4(n-1)R(t)^{-2}(n-1)^{-2}L^2$ which is exactly what we need such that we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(|\mathring{l}|^2R(t)^{2-\sigma}) &\leq \Delta(|\mathring{l}|^2R(t)^{2-\sigma}) + -2(n-2)(|\mathring{l}|^2R(t)^{-\sigma})\\ &+ (\underbrace{4 + C\varepsilon - (2-\sigma) - 4 + 1}_{\leq 0})L^2R(t)^{-\sigma}\\ &+ C\delta^2R(t)^{-2-\sigma}.\end{aligned}$$

We note that here we have also estimated all terms coming from the evolution of L^2 in the same way as in Proposition 5.11 which gives us an extra "+1" term in the bracket before L^2. Now let us set $g := |\mathring{l}|^2R(t)^{2-\sigma}$ then for $n \geq 3$ because of $\Delta L = 0$ we have the following evolutionary inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}g \leq \Delta g - (n-2)\frac{g}{R(t)^2} + C\delta^2R(t)^{-2-\sigma}$$

for any $z \in [0,\Lambda]$ and all times $t \in [-\frac{R_0^2}{2}, 0]$. Thus, on each Σ_z^t the maximum principle implies

$$g(p,t) \leq \max\left\{g\left(p, \frac{-R_0^2\beta}{2}\right), C\frac{\delta^2}{n-2}R(t)^{-\sigma}\right\}$$

because $t = -\frac{\beta R_0^2}{2}$ is the initial time. Furthermore, the initial values can be estimated using our assumption (5.19)

$$g\left(p, \frac{-R_0^2\beta}{2}\right) \leq \frac{C}{4(R_0^\sigma(1+2(n-1)\beta_0))^{\frac{\sigma}{2}}} \leq \frac{C\delta^2}{(n-2)R_0^\sigma}$$

by choosing $\beta = \beta(n, R_0, \delta, \sigma)$ large enough. The corresponding result is the following

Theorem 5.12 (Roundness of Crosssections)**.**
Let $\varepsilon = \varepsilon(n) > 0$, $k \geq 10$ and the period $0 < \Lambda$ such that all conclusions of Chapter 4 hold true. Then for any small $0 < \delta < \varepsilon$ and any large $\beta > \beta_0$ we can find a constant $\theta(\beta, n, \Lambda, \varepsilon, \delta) > 0$ such that, if $\mathcal{C}_t$ is a periodic $(\varepsilon, k, \Lambda, \theta)$ shrinking curvature neck with finite Radius $R(0) = R_0$, then there is $\sigma(n) < 1$ such that if we denote by Σ_z^t the foliation of $[0,\Lambda]$ at time t coming from the original foliation at time $t = -R_0^2\theta$ by evolution with α_z, then we have

$$|\mathring{l}|^2R(t)^2 \leq C(n)\delta^2\left(\frac{R(t)}{R_0}\right)^\sigma \tag{5.23}$$

for all $t \in [-R_0^2\beta, 0]$ and $z \in [0,\Lambda]$.

Proof. We fix the corresponding CMC foliation of the neck at the starting time like in Chapter 3. Then by the computations from above and since $R(t) \geq R_0$ and Proposition 5.11, we get the existence of θ and β_0 as claimed such that

$$|\mathring{l}|^2 \leq C\delta^2 R(t)^{-2} \left(\frac{R(t)}{R_0}\right)^{\sigma}$$

for all $t \in [-R_0^2\beta, 0]$ and some $0 < \sigma(n) < 1$. □

Remark 5.13. This estimate agrees with what we expect from the elliptic estimate if β is large enough, σ is small and we are far away from the starting time. This is in line with the gradient estimate where we also had to make the time interval smaller since we used a maximum principle argument. The reason why we cannot estimate the full second fundamental form with the present pointwise estimate is that we have only used information on the gradient of A and the speed α within the neck. Also we do not use the information that the neck is periodic. A priori we could be in a situation where we have a cone with a very small opening angle. Away from the tip the cone is a regular surface and its structure is invariant under smooth mean curvature flow such that it will be a cone with small opening angle for a long time that is very close to the shape of a cylinder away from the tip. In this setting, however, depending on the choice of the interior normal η we would have $L_z \geq \delta_0 > 0$ for all z and for some small δ_0 which depends on the opening angle. Also with the present technique we cannot expect a further improvement due to the lower order terms depending on the neck improvement parameter δ. Another Ansatz towards a better estimate is to analyze the Laplacian and perform an eigenvalue decomposition of the speed α and of $\frac{\mathrm{d}}{\mathrm{d}t}\alpha$, both solutions of elliptic equations. The decomposition then yields ODEs for the different coefficients in the expansion which can be analyzed. In this way we could possibly prove an estimate on α and its change over time that does not require better neck regularity a priori. A further hint as to why this seems possible is that Brendle used a similar approach to prove his neck improvement result. We refer to [Bre19b] for more details.

We finish this Chapter with an auxiliary result on how the information on the change of curvature and the structure of the foliation gives us information on the curvature of each individual CMC surface.

Proposition 5.14.
Let $\mathcal{N} \subset \mathcal{M}^n$ be a $(\varepsilon, k, \Lambda)$ hypersurface neck with approximate Radius R_0 suppose that for some $0 < \delta << \varepsilon$ where $\varepsilon(n) > 0$ is small enough, we have $|\nabla A| \leq \delta H^2$. Denote by Σ_z the CMC foliation of the neck and suppose that for some z_0 and $p \in \Sigma_{z_0}$ we have $\lambda_1(p) > \frac{\delta H}{2}$ and $L_{z_0} \equiv L_0 R_0^{-1} > 0$. We further assume that the "lapse" u_z coming from the construction of this parametrization in Theorem 3.10 satisfies

$$R_0|u_z| + R_0^3|D^2 u_z| \leq \delta.$$

Then there is $D = D(\delta, \varepsilon, L_0)$ such that $L_z > 0$ for all $|z - z_0| \leq \min\{DR_0; |z - \Lambda|\}$.

Proof. First we observe that $|\nabla\lambda_1| \leq |\nabla A|$ by Proposition 4.18. Now if we consider a geodesic γ starting from p parametrized via arc length then

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_1(\gamma(t)) = \langle\nabla\lambda_1(\gamma(t)), \dot{\gamma}(t)\rangle \geq -\delta H^2(\gamma(t)) \geq -\frac{11}{10}\delta\frac{n-1}{R_0^2},$$

if $\varepsilon(n) > 0$ is small enough, such that if $t = d(p, \gamma(t)) \leq |z_0 - z|R_0 \leq \frac{11}{10}\frac{R_0}{4}$ we get

$$\lambda_1(\gamma(t)) \geq \lambda_1(p) - \delta\frac{n-1}{4R_0} \geq \frac{\delta H}{4}.$$

Let v be an eigenvector to the smooth eigenvalue λ_1 then we get

$$\mathrm{Ric}(v,v) = H\lambda_1 - \lambda_1^2 \geq (n-1)\lambda_1^2.$$

We pick orthonormal coordinates around a point q such that $v = \eta_z = e_n$ if $q \in \Sigma_z$.

$$\mathrm{Ric}(\eta_z, \eta_z)(q) \geq C(n,\varepsilon)\frac{\delta^2 H^2}{4}$$

for $|z - z_0| \leq c(n)\frac{1}{4}$. We need to compute the z derivative of L. The variation in this direction is given by the functions u_z coming from the proof of Theorem 3.10 where we proved the existence of the foliation.

$$\begin{aligned}
\partial_z L_z = \mathcal{J}(u_z) &= \Delta(u_z) + u_z\left(|l|^2 + \mathrm{Ric}(\eta,\eta)\right) \\
&\geq -\frac{\delta}{R_0^3} - C(n,\varepsilon)\frac{1}{R_0^3}\frac{\delta^3}{4} \geq -\frac{\delta}{2R_0^3}.
\end{aligned}$$

Integrating then leads us to

$$L_z \geq \frac{L_0}{R_0} - |z - z_0|\frac{\delta}{2R_0^2} > 0$$

for $|z - z_0| < 2R_0 L_0 \delta^{-1}$. $\square$

Statutory declaration

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

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date signature

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