Deriving ancient foot units from building dimensions: a statistical approach employing cosine quantogram analysis

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Abstract: It is generally held by scholars of Greek metrology that the length of a foot standard can be derived from the dimensions of building elements. The Erechtheion at Athens is used as a case study to demonstrate that besides architectural measurements and inscriptional evidence, statistical analysis should also be used in the process. The method is based on D. G. Kendall's cosine quantogram analysis, and the result validity is assessed by Monte Carlo simulation. In the second part of the study an artificial data set based on the Arsenal in Piraeus is used to study the effects of noise on quantogram analysis. The main results of the study are that based on building block dimensions the length of the Erechtheion foot unit can be established as c. *324 mm and that a uniform noise-level of ±10 mm does not cause difficulties in identifying units of quarter-foot or larger in quantogram analysis.*

Key words: Greek architecture, cosine quantogram analysis, Monte Carlo simulation, kernel density estimates, metrology, foot unit, Erechtheion, Arsenal in Piraeus

Introduction

It is very often illuminating to get an external evaluation of a problem, and ancient metrology is no exception. N. R. J. Fieller has recently written a survey on some statistical questions in archaeology, and on metrology he writes as follows: "[T]here is a substantial literature on the subject which is largely contained in the more informal publications of local archaeological societies and it is in the main by 'amateur archaeologists'. Measuring archaeological constructions is an enjoyable holiday activity. Sadly, although many of these apply some version of the quantal techniques [...], most fall into obvious pitfalls in selecting data" (Fieller 1993:286). However, in the context of Greek architecture, it is possible in some cases to solve the question of data selection on the basis of inscriptional evidence: this is not a new idea, German archaeologist W. Dörpfeld used it to calculate the length Greek foot units at the end of the 19th century (Dörpfeld 1890), but it should be combined with a thorough statistical evaluation. In general, metrological analyses of Greek architecture have not employed statistics to any great depth[1] and the whole procedure of deriving foot unit lengths from architectural dimensions is in need of further study.

One of the buildings Dörpfeld used in his metrological studies was the late fifth-century BC Erechtheion on the Athenian Acropolis (Dörpfeld 1890:168–170). The first part of this paper presents a restudy of Dörpfeld's definition of the Erechtheion foot standard based on a cosine quantogram analysis of the blocks named in the inventory carried out on the incomplete building in 409/8 BC (*Inscriptiones Graecae* I[3] 474). The quantogram results are compared against the inscription dimensions, and different data models can be used to test the reliability of the analysis. Also, the Erechtheion data provides an opportunity to study the effect of small sample size on quantogram analysis: as often in archaeological contexts, the number of relevant measurements is limited.

In the second part of the paper I will use another Attic inscription to produce a simulated data set (*Inscriptiones Graecae* II[2] 1668). Parts of the fourth-century BC Arsenal in Piraeus, designed by Philon, were recently discovered near the ancient harbour installations, but the main features of the building had already previously been reconstructed based solely on the inscription.[2] The reason for using an artificial data set is that the *exact* length of the foot unit behind the measurements is then known: the effect of different levels of noise on the reliability of the quantogram analysis can therefore be evaluated.

Method

The method of this paper is based on D. G. Kendall's cosine quantogram analysis for detecting a quantum of an unknown size in a data set; after the initial analysis the result validity is evaluated by using Monte Carlo simulations (Kendall 1974). The effect of simulation distributions in the second stage of the study has been questioned by P. R. Freeman (1976),[3] which is why I will use several different data models to test whether the results are dependent on modelling. I will also introduce kernel density estimate (KDE) distributions as possible models to be used in the Monte Carlo simulations.[4]

In the case of Greek architecture the quantum hypothesis is that a building dimension X can be expressed in terms of an integral multiple M times a quantum q plus a small error component e:

$$X = Mq + e. \tag{1}$$

The error may equally well be due to ancient building execution or modern measurement, and it should be significantly smaller than q.

To find out whether dimension X can usefully be given in terms of quantum q, X is divided by q and the remainder (or error) e is analysed: the closer to 0 or q it is, the better the fit. The formula for calculating the amount of clustering around q is

$$f(q) = \sqrt{2/N}\sum_{i=1}^{n}\cos(2\pi\varepsilon_i/q), \qquad (2)$$

where N is the number of building dimensions. The higher the score $f(q)$, the higher is the probability of q being the right candidate for the quantum (fig. 1); Monte Carlo simulations can then be used to assess whether the function peak is high enough to be considered a 'true quantum'. In the simulations random data sets are created from non-quantal distributions, and these are then analysed in the same way as the original data to determine whether peaks as high as or higher than the original regularly arise from these distributions.

The Erechtheion

The block dimensions of the Erechtheion inscription are listed in table 1. Instead of using Dörpfeld's measurements I have used a more recent monograph on the building (Dörpfeld 1890:169–170; Paton & Stevens 1927).[5] There are some discrepancies in the data, but I have intentionally retained them: the most obvious is the length of the frieze block which is given in the inscription as 4 feet while the measured average length for the east frieze block is 1.94 m, which must be 6 feet. As we see from column 4 of table 1, the reported block dimensions in the inscription are not overly precise. There are several possible explanations: if the inventory was only needed for recognising the blocks which required further work, taking measurements to the closest quarter-feet or in some cases to the nearest round number of feet could have been thought sufficient; or, if the surveyors were working with the original building specification which also listed the block dimensions, these might be quite different from the final sizes of the blocks finished only at the building site. The length of the Erechtheion foot unit is usually given as 326–328 mm, but this is calculated on the basis of only a few dimensions (Dörpfeld 1890:171; Dinsmoor 1961:358).

In fig. 1 is presented the cosine quantogram of the Erechtheion data with the quantum score $f(q)$ is plotted against q. There are two clear peaks, the first at 162.0 mm and the second almost exactly twice the first at 324.2 mm. Since the analysis makes no *a priori* assumption on the quantum length, or actually even its existence, it is very interesting that the correspondence with the inscription data is as good as it is. The height of the peak at 324 mm is 3.67 and at the half-foot mark 162 mm only a fraction less, 3.65.

The results of Monte Carlo simulations employing several different data models are collected in table 2; plots of four distribution models are presented in fig. 2. For each distribution 1000 simulations were run. The results are quite similar except for the first simulation employing the bootstrap method. The reason why bootstrap should not be used in quantum analysis is that multiple replication of an observation in the resampled data set very often emphasizes the height of the maximum quantum peak, which is evident when the bootstrap score is compared with other simulations (cols. 2–3 in table 2): the 5% significance level for the peak is very high at 5.12 and 85% of bootstrap simulations produced a higher peak than the Erechtheion score of 3.67. The significance level of 3.50 of the uniform distribution is a fraction more than the distribution models on lines 3–7. This is most likely due to a slight over-representation of long measurements produced by the model. The F distribution, the histogram distribution and the kernel estimate distributions all produce very similar results with the 5% significance level range as 3.34–3.41.[6] Therefore, the choice of simulation distribution does not have an effect on the interpretation of the results *in this case*. However, more extensive tests with other data sets should be carried out to determine if they lead to the same conclusions and whether kernel density estimation provides a general solution for the choice of distribution model. Since the results of the five last models are as close as they are, the simulations can be combined to obtain more accurate values of 5000 runs (line 8. in table 2): the score for 5% significance level is 3.36, and the Erechtheion peak height of 3.67 is significant at the level of 1.7%. It is very important that even without inscriptional evidence on the length of the unit, we would have clearly accepted the quantum at 5% significance level.

Philon's Arsenal in Piraeus

The data set used in the computer simulations to study the effect of noise on quantogram analysis is based on 18 Arsenal inscription measurements which could very likely be discovered in a modern excavation of a Greek building, even if it was not well preserved: these are all dimensions of the building plan or of individual blocks. They, and their length in feet given in the inscription, are listed in table 3. The calculated size in millimetres (col. 3) is based on the above determined Erechtheion average foot length of 0.3245 m; actually, any other length within the range of supposed Greek foot units would have served the analysis equally well. One of the measurements, the width of the wall orthostate block, is given in the inscription with the accuracy of one-sixteenth of a foot,[7] all others in terms of quarter-, half- or a round number of feet.

Fig. 3 presents a summary of the Arsenal simulations. Adding a *uniform* noise of ±10 mm to the building dimensions does not have any effect on the quantogram analysis: the top two KDE plots in fig. 3 show that all 50 simulations picked the quarter-foot mark of *c.* 81 mm as the quantum (top left) and all had a very high maximum peak score of 4 or more (top right). The second set of simulations demonstrates that a noise of ±20 mm has very little effect on the length of detected quantum, but peak scores are now clearly lower than in the first set. The vertical line at 3.5 indicates a peak score which most often will be recognized as significant in Monte Carlo simulations, depending of course on the data: only 22 of 50 simulations produced a peak of 3.5 or higher. A noise-level of ±30 mm is enough to collapse the peak at quarter-foot and make the half-foot mark of c. 162 mm the mode of the distribution; 29 simulations peak in the region of 81 and 162 mm, and only a few produce significantly high peaks. Addition of a noise of ±40 mm and ±50 mm gradually diminishes the proportion of correct quanta being detected while there is very little change in the height of maximum peaks produced in the simulations.

The next obvious question is that what kind of 'noise' can be expected in measurements of Greek buildings. J. J. Coulton has made the following observation: "It would appear that in many buildings a variation of about 0.01 m. between 'identical' elements was considered acceptable for the smaller elements, while variations of several centimetres might occur in the larger ones. [...] We may [...] presume that discrepancies between the model and the design were at least as great as, and perhaps greater than, the variations of 'identical' parts from the mean sizes"(Coulton 1975:94). The published dimensions of the Arsenal give some indication of the unit length and variation of the building execution, but they are problematic, as is evident from Table 4.[8] In the case of the pillar stylobate width there is quite likely a quarter-foot discrepancy between the design and the execution. Luckily, such a discrepancy would not effect quantogram analysis since the modified length would be recognized as 3.5 feet instead of 3.25 feet. However, it might be advisable to avoid the longer measurements where larger errors are likely to occur, since as we see in fig. 3, a noise-level of ±10 mm does not create any problems for detecting the correct quantum.

Conclusions

The cosine quantogram analysis of the building block dimensions suggests that the length of the Erechtheion foot standard was *c.* 324 mm. The difference in length compared to the previous proposals of 326–328 mm suggests that a more thorough metrological analysis of the building blocks should be carried out. However, it is quite encouraging that even though the Erechtheion sample size is small with $n = 19$, quantogram analysis can produce significant results when the building inscription is used to select the appropriate dimensions. In general, provided that a sufficient number of blocks were executed relatively accurately in multiples of quarter-feet, using block dimensions rather than larger measurements, such as the width and length of the building, will more likely result in discovery of an architectural foot unit.

Software

I have implemented the computer programs used in the cosine quantogram analyses, Monte Carlo simulations, and kernel density estimations on top of Survo MM, the Windows version of the statistical program; very warm thanks are due to S. Mustonen for providing a copy of the program. C. C. Beardah's MATLAB routines were used for calculating the optimal window widths of the kernel density estimates (see Beardah & Baxter 1999 for a recent bibliography on KDE).

Endnotes

[1] R. C. A. Rottländer's work should be mentioned here as an exception; see e.g. Rottländer 1996.

[2] The inscription is dated to 347/6 BC. For earlier reconstructions of the building, see e.g. Jeppesen 1958, 69–101. On the remains of the building, see Steinhauer 1994, 1996; Rottländer 1997.

[3] Freeman's method of calculating posterior distributions is very closely connected to Kendall's quantogram analysis, as has been demonstrated by B. W. Silverman 1976.

[4] For a recent bibliography on KDE in archaeology, see Beardah & Baxter 1999, 168–169.

[5] When several measurements are reported, the average is given in Table 1. I have omitted Dörpfeld's 'Eckquader' because his identification of the block cannot be correct: it was only discovered in later excavations and it is very fragmentarily preserved; see Paton & Stevens 1927:51, 330 n. 2. The length of the Kariatid porch roof block is given for the visible part: only the upper surfaces needed redressing and these blocks were in place when the inventory was carried out; Paton & Stevens 1927:pl. 27.

[6] The step used in the quantogram analysis of the simulations was 0.25 mm; to test that the step is adequately small, two control simulations of 500 runs using a step of 0.05 mm and STE and DPI-2 KDE distributions were executed: there are no discrepancies between results of the larger and smaller step values.

[7] The reason for this is that the width is one dactyl or fingerbreath wider than the normal wall block width of $2^1/_2$ feet.

[8] The published width of 18 m for the building is not without problems since it results in a reconstruction where the side walls and pillars are not placed at the centres of discovered foundations; see Steinhauer 1994:fig. 40. The centre- and sidenave widths are also dependent on the building width.

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Tables

1.	2. *IG* I³ 474 (feet)	3. Measured dimension (mm)	4. Length of foot unit (mm)
Wall block L	4	1300	325
Wall block W	2	664	332
Wall block H	1 1/2	490	327
Epikranitis L	4	1300	325
Epikranitis W	3	912	304
Epikranitis H	1 1/2	491	327
Architrave L	8	2608	326
Architrave W	2 1/4	770	342
Architrave H	2	633	317
Kariatid porch roof block L	13	3890	299
Kariatid porch roof block W	5	1648	330
East frieze block L	4	1940	485
East frieze block W	1	285	285
East frieze block H	2	617	309
North frieze block W	1	298	298
North frieze block H	2	683	342
Geison block L	4	1297	324
Geison block W	3	960	320
Geison block H	1 1/4	284	227
			$x \approx 323.3$

Table 1. Erechtheion, Athens. Building dimensions.

1.	2. $\phi(q)$, $\alpha = 5\%$	3. α, $\phi(q) = 3.67$
1. Bootstrap	5.12	85%
2. Uniform distribution	3.50	2%
3. *F* distribution	3.34	2%
4. Histogram distribution	3.37	2%
5. KDE, STE, $h = 286.9$	3.35	1%
6. KDE, DPI-2, $h = 320.3$	3.36	2%
7. KDE, Normal scale, $h = 403.6$	3.41	1%
Combined models 3.–7., $n = 5000$	3.36	1.7%

*Table 2. Erechtheion, Athens. Cosine quantogram analysis of building blocks; results of the Monte Carlo simulations (*n = *1000 for each run).*

1.	2. *IG* II² 1668 (feet)	3. Calculated dimension (mm)
Building W	55	17848
Doorpost L	10	3245
Doorpost W	2	649
Door W	9	2921
Centre-nave W	20	6490
Side-nave W	12 1/4	3975
Euthynteria block L	4	1298
Euthynteria block W	3	974
Euthynteria block H	1 1/2	487
Corner euthynteria block L	4 3/4	1541
Orthostate block L	4	1298
Orthostate block W	2 9/16	831
Orthostate block H	3	974
Wall block W	2 1/2	811
Wall block H	1 1/2	487
Pillar W	2 3/4	892
Pillar stylobate L	4	1298
Pillar stylobate W	3 1/4	1055

Table 3. Arsenal, Piraeus. Simulated inscription data (1 foot = 0.3245 m).

1.	2. (feet)	3. Measured dimension (m)	4. Length of foot unit (mm)
Building W		18	
Steinhauer (between walls)	55		327.3
Rottländer (at euthynteria)	55 1/2		324.3
Axial distance of pillars		3.50	
Steinhauer	10 3/4		325.6
Rottländer	11		318.2
Jeppesen	11 1/4		311.1
Centre-nave W	20	6.55	327.5
Side-nave W with pillar	15	4.90	326.7
Pillar stylobate L	4	1.30	325.0
Pillar stylobate W	3 1/4	1.15	353.9

Table 4. Arsenal, Piraeus. Building dimensions and foot unit lengths (based on Jeppesen 1958, Steinhauer 1994 and Rottländer 1997).

Figures

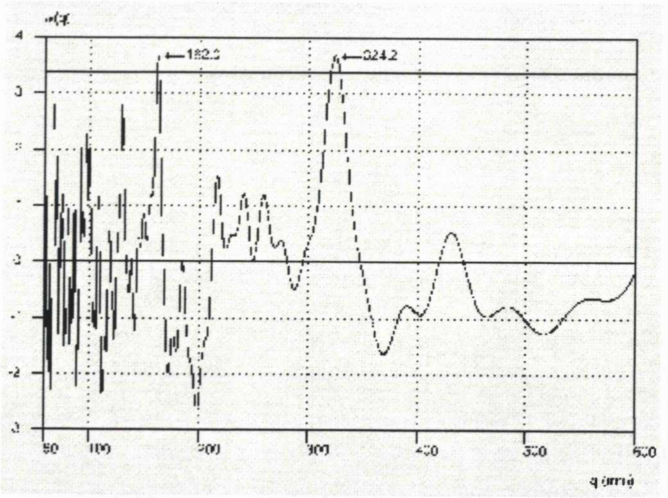

Figure 1. Erechtheion, Athens. Cosine quantogram of building block measurements.

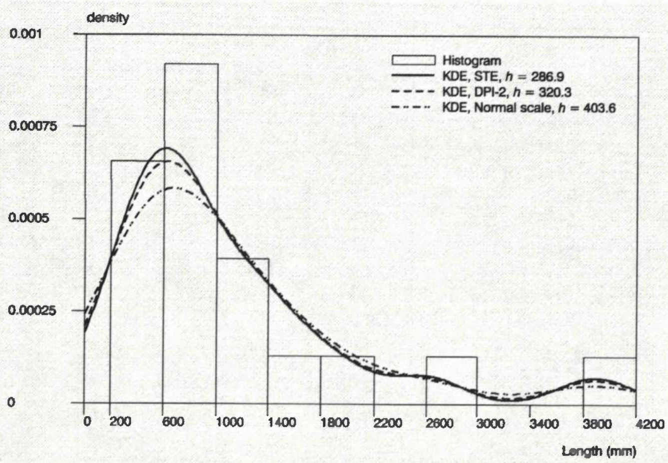

Figure 2. Erechtheion, Athens. Four of the distribution models used in the cosine quantogram Monte Carlo simulations of building block dimensions.

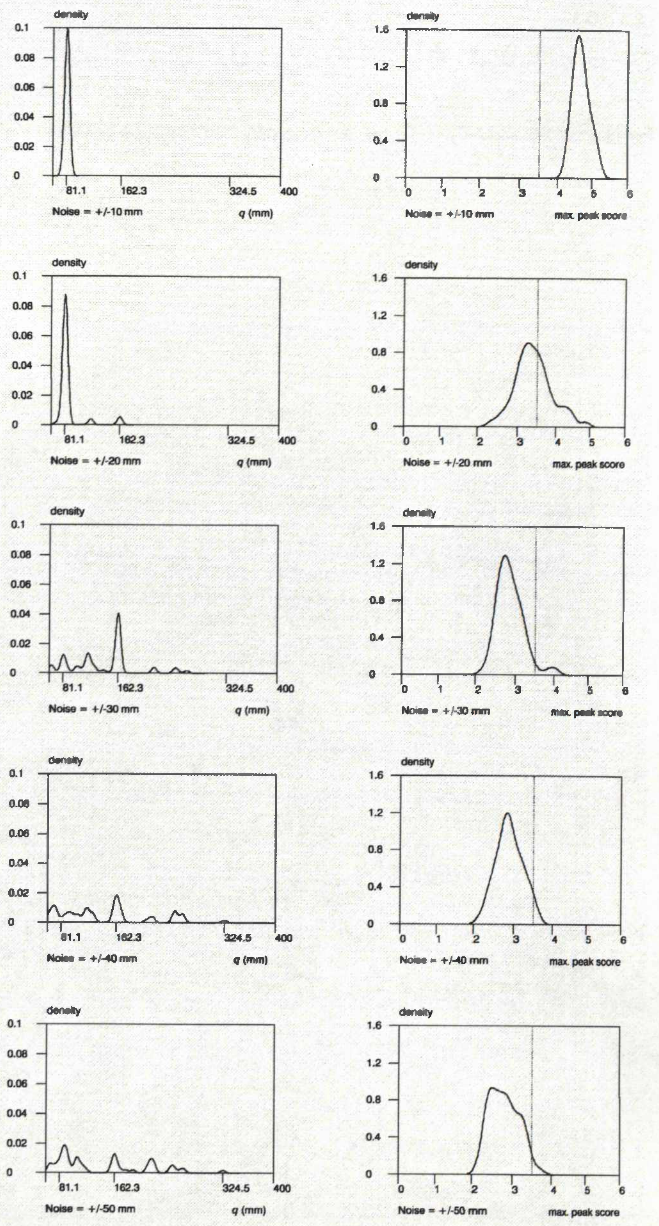

Figure 3. Arsenal, Piraeus. Kernel density estimates of the effect of noise on the reliability of the quantogram analyses of simulated inscription data (n = 50 for each run). Length of maximum quantum (on the left, band-width h = 4) and maximum peak score (on the right, h = 0.14).