On the Effective Dynamics of Interacting Fermionic Many-Body Systems

Dissertation

der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades
eines Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
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Tübingen
2025

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation:	22.04.2025
Dekan:	Prof. Dr. Thilo Stehle
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Zusammenfassung

In der vorliegenden Dissertation untersuchen wir die Dynamik von quantenmechanischen Vielteilchensystemen für N wechselwirkende, nicht-relativistische Fermionen. Wir entwickeln effektive Beschreibungen der N-Teilchensysteme und konstruieren eine rigorose Näherung im Grenzfall $N \to \infty$. Ein bemerkenswertes Ergebnis dieser Arbeit ist die Behandlung von Hamilton-Operatoren mit Kopplungsparameter der Größenordnung 1 in physikalisch relevanten Situationen, wodurch Regime adressiert werden, die bisher von bestehenden Methoden noch nicht erforscht wurden.

Im ersten Teil leiten wir die zeitabhängigen Hartree-Fock-Gleichungen (TDHF) für singuläre Paarwechselwirkungen $|\cdot|^{-s}$ mit $s \in (0, 2/3)$ in einem neu erforschten, stark wechselwirkenden Regime ab. Im Gegensatz zu anderen Regimen enthält der Hamilton-Operator in unserer Situation keine N-abhängigen Wechselwirkungsparameter, wenn er auf Volumina der Größenordnung 1 und Zeitskalen der Größenordnung $N^{-2/3}$ betrachtet wird. Äquivalent dazu skaliert der Wechselwirkungsparameter des Hamiltonian-Operator auf Volumina der Größenordnung N mit $g_N = N^{-\frac{2-s}{3}}$ für makroskopische Zeitskalen. Wir verwenden und erweitern die Zählfunktionalmethode, die für Fermionen erstmals in [Pet14, PP16] entwickelt wurde, um dieses Regime zu behandeln. Dazu führen wir eine Eichtransformation ein, die das große Wechselwirkungspotenzial extrahiert und den Hamilton-Operator in eine magnetartige Struktur transformiert. Diese umfasst nun Dreikörperterme sowie Zweikörperterme, die Differentialoperatoren anstelle einfacher Multiplikationsoperatoren beinhalten. Eine zentrale Herausforderung besteht darin, diese Differentialoperatoren effektiv in das Framework zu integrieren. Um dies zu erreichen, entwickeln wir eine Strategie, die das Zählfunktional mit einer Normnäherung relativ zu einem Hilfs-Hamilton-Operator kombiniert. Dieser Hilfs-Hamilton-Operator dient als vereinfachtes Referenzsystem, das es uns ermöglicht, die schlechten kinetischen Energieterme rigoros zu kontrollieren.

Im zweiten Teil untersuchen wir die Quantendynamik eines homogenen idealen Fermi-Gases, das an ein Fremdteilchen im Hochdichtegrenzfall gekoppelt ist. Wir betrachten eine dreidimensionale Box mit periodischen Randbedingungen und nehmen an, dass die Anfangswellenfunktion ein Produktzustand zwischen dem Fremdteilchen und einer gefüllten Fermi-Kugel ist. Im Grenzfall großer Fermi-Impulse k_F beweisen wir, dass die effektive Dynamik durch einen Fröhlich-artigen Polaron-Hamilton-Operator erzeugt wird, der das Fremdteilchen linear an ein nahezu bosonisches Anregungsfeld koppelt. Unsere Methode integriert kollektive Anregungen in die effektive Beschreibung, indem sie nahezu bosonische Operatoren verwendet, die kürzlich zur Analyse der Korrelationsenergie entwickelt wurden [BNP+19, BNP+21a, BNP+21b, BPSS23]. Dies ermöglicht es uns, die Bildung kollektiver Anregungen zu beschreiben, indem wir die effektive Dynamik mit einem expliziten gekoppelten kohärenten Zustand approximieren. Unser Ansatz deckt eine breite Palette von Wechselwirkungsstärken ab, einschließlich Kopplungen der Größenordnung 1 und Zeitskalen der Größenordnung k_F^{-1}. Als Anwendung berechnen wir das Loschmidt-Echo als Antwortfunktion des interagierenden Systems und zeigen, dass unsere theoretischen Vorhersagen mit den universellen Merkmalen übereinstimmen, die in jüngsten Experimenten mit ultrakalten Atomen beobachtet wurden. Diese Ergebnisse liefern neue Einblicke in die Bildung von Polaron-Quasiteilchen in fermionischen Systemen und unterstreichen die Bedeutung unserer Methoden zur Beschreibung der Hauptmerkmale stark wechselwirkender Quantensysteme im Hochdichtegrenzfall.

Abstract

In this thesis, we study the dynamics of quantum many-body systems for N interacting non-relativistic fermions. We establish effective descriptions of the N-body systems and construct a rigorous approximation in the limit of $N \to \infty$. A notable achievement of this work is the treatment of Hamiltonians with coupling parameters of order 1 in physically relevant settings, addressing regimes previously unexplored by existing methods.

In the first part, we will derive the time-dependent Hartree-Fock equations (TDHF) for singular pair interactions $|\cdot|^{-s}$ for $s \in (0, 2/3)$ in a newly explored strongly interacting regime. In contrast to other regimes, the Hamiltonian in our setting does not involve N-dependent interaction parameters when considered on volumes of order 1 and time scales of order $N^{-2/3}$. Equivalently, on volumes of order N the interaction parameter scales as $g_N = N^{-\frac{2-s}{3}}$ for macroscopic time scales. We use and extend the counting functional method, which was first developed in [Pet14, PP16], to treat this regime by introducing a gauge transformation that extracts the large interaction potential, transforming the Hamiltonian into a magnetic-type structure which involves three-body terms and two-body terms that involve differential operators rather than simple multiplication operators. A central challenge is effectively incorporating these differential operators into the framework. To overcome this, we develop a strategy combining the counting functional with a norm approximation relative to an auxiliary Hamiltonian. This auxiliary Hamiltonian will serve as a simplified reference system, allowing us to rigorously control the bad kinetic energy terms.

In the second part, we study the quantum dynamics of a homogeneous ideal Fermi gas coupled to an impurity particle in the high density limit. We consider a three-dimensional box with periodic boundary conditions and assume that the initial wave function is a product state between the impurity and a filled Fermi ball. In the limit of large Fermi momentum k_{F}, we prove that the effective dynamics is generated by a Fröhlich-type polaron Hamiltonian, which linearly couples the impurity particle to an almost-bosonic excitation field. Our method incorporates collective excitations into the effective description by employing almost-bosonic operators recently developed for analyzing the correlation energy [BNP+19, BNP+21a, BNP+21b, BPSS23]. This allows us to describe the formation of collective excitations by approximating the effective dynamics with an explicit coupled coherent state. Our approach covers a broad range of interaction strengths, including couplings of order 1 and time scales of the order k_{F}^{-1}. As an application, we compute response functions of the interacting system such as the Loschmidt echo and demonstrate that our theoretical predictions align with the universal features observed in recent ultracold atom experiments. These results provide new insights into the formation of polaron quasi-particles in fermionic systems and highlight the effectiveness of our methods for describing the main features of a strongly interacting quantum system in the high density limit.

Acknowledgment

I am deeply grateful to Prof Dr. Peter Pickl for his unwavering support and enlightening guidance. His intuition and enthusiasm have been a constant source of inspiration, and his openness to discuss mathematical ideas while seeking the broader picture has been invaluable. Peter's initial trust in me allowed for great personal and professional development and provided me the freedom to take responsibility and grow. Navigating both moments of struggle and excitement has strengthened my appreciation for the process and allowed me to eventually fully enjoy every aspect of my research—something for which I am deeply grateful!

I am very honored that Prof. Dr. Sören Petrat agreed to co-supervise my thesis. His excellent pioneering work on the counting functional for fermionic settings provided great motivation to work on this topic.

I would also like to express my gratitude to Dr. David Mitrouskas for his collaboration on the project concerning the fermionic Hartree equations and for many insightful discussions on various topics on quantum many-body systems. His careful proofreading and numerous helpful suggestions have been invaluable.

Great thanks to the entire mathematical physics group in Tübingen. I felt very comfortable, supported, and inspired by the conversations we shared and the wonderful time we spent together. My particular gratitude goes to my long-time office mates, Manuela Feistl, Cedric Igelspacher and Siegfried Spruck, as well as Karla Schön and Sahand Tokasi from our working group, for the vivid discussions on math and our perspectives on life. You truly brought inspiration, motivation and laughter to our collaborative environment. Also, I extend my thanks to Dr. Stefan Keppeler and Prof. Dr. Roderich Tumulka for their support while I worked as a teaching assistant. Our collaboration in presenting and discussing math with our esteemed students was both beneficial and rewarding.

Special thanks for proofreading go to Manuela Feistl, Viktoria Kabel and Karla Schön for taking the effort to read parts of my thesis and pointing out mistakes.

Last but not least, I would like to extend my gratitude to my friends and family for their ever-renewing interest and warm support. In particular, I thank Viktoria for her constant love and encouragement throughout this journey.

Contents

Chapter 1

Introduction

> Mind: Apparently, color is appearance, sweetness is appearance, bitterness is appearance; in truth, only atoms and empty space exist.
>
> Senses: Poor mind, you take your evidence from us and wish to defeat us with it? Your victory is your downfall.
>
> — Democritus, c. 450 BC[1]

The perspective of macroscopic physical systems—such as a gas in a cylinder, a liquid in a glass or a solid of definite dimension—as entities emerging from their constituent parts and their interaction is both enlightening and challenging. It is enlightening, since it enables the explanation of macroscopic phenomena on the more fundamental level of the constituents and thus a deeper and profound understanding of nature. The success of this *reductionist perspective* in modern physics owes much to the seminal contribution of Ludwig Boltzmann and James C. Maxwell in the 19th century, who provided a microscopic explanation of thermodynamics from the level of individual gas particles. Since then the reductionist perspective has become a guiding principle to explain the macroscopic behavior of complex systems in terms of the underlying components in many different sub-disciplines of physics, ranging from quantum gases and condensed matter physics to biophysics and astrophysics.

However, this perspective is also challenging due to the sheer complexity on the microscopic level: a comprehensive analysis on the level of constituents requires accounting for an enormous number of degrees of freedom, arising from the typically large number of constituents and their intricate interactions. This calls for the necessity to establish *effective equations* while capture the essential behavior of the microscopic system and its time evolution and explaining some characteristic macroscopic features in a limiting regime with reduced complexity. More concretely, this typically involves three key steps: Firstly, select an appropriate microscopic picture which is typically characterized by a large number of degrees of

[1] Own translation from [Sch12]. Here, "apparent" has the meaning of "according to conventional perception as being true".

freedom. Secondly, define an order parameter which is associated with a coarse-graining of the microscopy. The order parameter is usually a macroscopic quantity such as the number of particles or the temperature. Varying the order parameter towards the asymptotic setting (in which we have for instance a large number of particles or low temperature) corresponds to "zooming out" from the microscopic details towards the limiting regime. The third step is to identify which processes are dominant and which are sub-dominant as the order parameter varies. The dominant processes are usually associated with collective phenomena where microscopic interactions combine in a coherent way whereas the sub-dominant processes contribute only marginally and therefore become irrelevant as the order parameter changes.

The main subject of this thesis is to establish such effective equations for the dynamics of non-relativistic fermionic many-body systems with mathematical rigor in physically relevant regimes. In cases where the effective description is unknown this corresponds to the rigorous development of a new equation capturing the effective behavior of the system. Quite often, an effective candidate equation is already known and commonly used in the physics community. Establishing the effective equation in such settings therefore corresponds to a justification or a derivation thereof.
Regardless of whether the effective description is known or unknown, a rigorous mathematical approach contributes to a deeper understanding of nature on many levels: It allows to specify the topology in which the microscopic description is close to the effective description. In the context of coarse-graining where the order parameter is sent to a limiting regime, this corresponds to a specification of the sense of convergence of the microscopic to the candidate description. Moreover, one is able to state the conditions of convergence in the limiting regime and therefore identifies situations where the effective description holds and where it breaks down, which is pivotal for advancing theoretical understanding. By estimating the sub-dominant processes by explicit error bounds one obtains a rate of convergence which depends on the order parameter. These results not only to accurately distinguish the most dominant from the sub-dominant effects but also to offer a reliable and robust understanding of the governing principles of the system.

In this thesis, we will consider a large number of non-relativistic indistinguishable fermions in the spatial region $\Lambda \subseteq \mathbb{R}^3$ at zero temperature neglecting their spins. The relevant order parameter will therefore be the number N of fermionic particles which parameterizes the N-body description. The dynamics of an individual particle is governed by the microscopic N-body Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Psi_N(t) = H_N \Psi_N(t), \qquad \Psi_N(0) = \Psi_{N,0} \in L^2_{\mathrm{as}}(\Lambda^N) \tag{1.0.1}$$

where $\hbar$ is the reduced Planck constant and $\Psi_N \in L^2_{\mathrm{as}}(\Lambda^N)$ is the N-body wave function describing our system. $L^2_{\mathrm{as}}(\Lambda^N)$ denotes the antisymmetric subspace of the Hilbert space

$L^2(\Lambda^N)$ whose elements are square integrable and antisymmetric under pairwise exchange of coordinates, i.e. if $\{x_i\}_{i=1,\dots,N} \subset \Lambda$ are the coordinates of the fermions than for all $i, j \in \{1, \dots, N\}$ it shall hold

$$\Psi_N(x_1, \dots, x_i, \dots, x_j, \dots, x_N) = -\Psi_N(x_1, \dots, x_j, \dots, x_i, \dots, x_N). \tag{1.0.2}$$

The N-body Hamiltonian for the fermionic system is given by

$$H_N = \frac{1}{2m} \sum_{i=1}^{N} (-\Delta_{x_i}) + g_N \sum_{1 \le i < j \le N} W(x_i - x_j) \tag{1.0.3}$$

where m denotes the mass of the fermionic particles, $-\Delta_{x_i}$ denotes the Laplace operator with respect to the i-th particle with coordinate x_i and $W : \Lambda \to \mathbb{R}$ denotes the two-body interaction between the fermions. In all models under consideration, W is such that H_N is bounded from below and self-adjoint on an appropriate dense subset. The parameter g_N takes the role of a coupling parameter of the interaction and usually depends on N. Its role is twofold: an N-dependent g_N reflects the physical scaling regime of our interest. It is very often possible to physically justify such choices by re-scaling space and time and to connect them with a suitable N-dependent choice. We will explain in more detail the coupling parameter in Subsection 1.2.2. Moreover, a choice of g_N inherently encodes the coarse-graining procedure we are aiming for, and makes the rigorous analysis more convenient in practice and allows to state the results more clearly. We note that in most cases, we will use the convention $\hbar = 1$ and $m = 1/2$ to simplify the form of each equation unless explicitly stated otherwise.

A substantial part of the thesis will be concerned with a slight variation of the Hamiltonian (1.0.3) to describe an additional interacting particle as an "impurity" immersed in the fermionic system. The fermionic system takes the role of a medium whose properties are tested by the impurity particle. The corresponding Hamiltonian is given as

$$H_N = \frac{-\Delta_y}{2m_{\text{imp}}} + \frac{1}{2m} \sum_{i=1}^{N} (-\Delta_{x_i}) + g_N \sum_{1 \le i < j \le N} W(x_i - x_j) + \lambda_N \sum_{i=1}^{N} V(x_i - y) \tag{1.0.4}$$

where m_{imp} is the mass of the impurity particle, $V : \Lambda \to \mathbb{R}$ denotes the two-body interaction between the fermions and the impurity particle and λ_N is the coupling parameter. Since in this case the fermions act as a surrounding medium which will be probed by the impurity particle, we will typically refer to these fermions as "gas particles". More details will be given in Section 1.3.

In this thesis we will establish two effective dynamical theories from the microscopic many-body Hamiltonians (1.0.3) and (1.0.4):

Firstly, we investigate Hartree-Fock dynamics for interacting fermions in a regime characterized by significant potential energy, featuring singular long-range potentials of the form $|\cdot|^{-s}$ for $s \in (0, 2/3)$ and a weak coupling of $g_N = N^{\frac{s-2}{3}}$. We adapt and extend the counting functional method which were developed in [Pic11] and in [Pet14, PP16] for fermions to address this regime. To exploit the large potential energy, we introduce a gauge transformation, leading to the treatment of a time-dependent, magnetic-type Hamiltonian. Main steps in the proof involve the introduction an auxiliary dynamics, a separate propagation of a gradient term—referred to as the bad kinetic energy—and a norm approximation utilizing weight functions. The regime is believed to more closely approximate the physical conditions under which such mean-field equations are applied (see Subsection 1.2.2 for an overview of scaling regimes). Most notably, our setting turns out to be equivalent to a system on a volume of order 1 described by a Hamiltonian with $g_N = 1$ on short time scales, where particles can traverse the inter-particle distance. Thus, the treated regime can be seen as a strongly interacting one. We will also explore the connections to other regimes. The central motivation for this research is the generality of the microscopic Hamiltonian (1.0.3) covering a wide range of physical systems such as orbitals of molecules and atoms in chemistry, the electron gas in a solid metal or fermionic neutrons in a star. Understanding how this fundamental description gives rise to effective models allows for broader applicability across different settings. The time-dependent Hartree-Fock equations are widely used in simulations and serves as a foundation for more refined methods [Fie92, LSM+05, Mey11, SU18]. Since many higher-level approximations build upon Hartree-Fock, it is essential to rigorously analyze the conditions under which it remains valid.

Secondly, we examine the dynamics of an impurity particle interacting with an ideal Fermi gas of high density described by (1.0.4) with $g_N = 0$ and $\lambda_N = 1$. To handle the strong coupling constant, we employ collective, almost-bosonic operators, which were developed in the study of correlation energy, to model a linear coupling of the impurity with the excitations. We show that the effective dynamics is approximately governed by a Fröhlich-type Hamiltonian which is associated with a coherent dressing of the impurity which is usually interpreted as the formation of a polaron as quasi-particle. The main method involves the inclusion of coherent excitations in the form of collective particle-hole excitations, which were recently developed for analyzing the correlation energy [BNP+19, BNP+21a, BNP+21b, BPSS23]. Crucial steps involve controlling the error in terms of the number of excitations via a Grönwall argument and introducing an almost-bosonic Weyl operator. Additionally, our approach enables an effective approximation at the level of states through an explicit coupled coherent state. Subsequently, we will approximate the Loschmidt echo as overlap between the free dynamics and the interacting dynamics, a quantity of particular interest in experimental contexts. We emphasize that the impurity problem plays a crucial role in various physical contexts, including ultracold atomic gases (see Subsection 1.3.3), semiconductors [SBC+16] and plasmas [Rit59, DGGP+74]. In particular, polaron formation is of great interest in condensed matter physics, as polarons are believed to influence macroscopic fermionic properties such as superconductivity[Mot90, AK92]. Theoretical considerations suggest that polarons contribute to pairing mechanisms and collective behaviors in strongly correlated systems, but their precise role remains an open question [JWB+07, LMI+23, MS24].

The ultimate goal of this PhD thesis, to which all the above topics point, is to achieve a deeper understanding of fermionic systems by rigorously investigating the conditions and validity of effective descriptions. By clarifying when and why these approximations work, this research contributes to the broader effort of linking microscopic models to physically relevant phenomena.

The thesis is structured as follows: The remaining subsections of this chapter provide a basic introduction to the mathematical formulation of the problems, review the results that form the foundation for the main results of this thesis, summarize open questions in the literature and discuss how our work fits within the broader research community. More specifically, we provide an overview of effective dynamical theories for interacting fermions and discuss and compare corresponding scaling regimes. Additionally, we offer intuition and results on the behavior of an impurity particle immersed in a dense Fermi gas on the torus, with a particular focus on experimental observations that our effective theories aim to describe. With the foundational concepts established, we proceed to Chapter 2, where we present our work on the derivation of the time-dependent Hartree-Fock equation in the strongly interacting regime. In Chapter 3, we present the work on the effective polaron dynamics of an impurity interacting with a dense Fermi gas. Finally, we conclude with a summary of our findings, their implications and open questions in Chapter 4.

1.1 Mathematical formalism of quantum many-body systems

This section provides an overview of the fundamental mathematical concepts used to describe quantum many-body systems. These concepts form the foundation of our analysis and are essential for both the static and dynamical perspectives of quantum systems. We will introduce the key quantities and concepts necessary to rigorously study the evolution of many-body wave functions and effective equations.

1.1.1 Dynamical and static quantities of interest

The mathematical description of our N-body system is based on the framework of quantum mechanics, where states are represented by elements of a Hilbert space. Specifically, we consider wave functions $\Psi_N \in L^2(\mathbb{R}^{3N}) \simeq L^2(\mathbb{R}^3)^{\otimes N}$ equipped with the scalar product $\langle \Psi_N, \Phi_N \rangle = \int_{\mathbb{R}^{3N}} \mathrm{d}x \overline{\Psi_N(x)} \Phi_N(x)$. We will always consider normalized wave functions

$\sqrt{\langle \Psi_N, \Psi_N \rangle} = 1$. For convenience, we will not explicitly distinguish between the N-body and one-body scalar products when the context allows. Since we will primarily describe fermions on a spatial subset $\Lambda \subseteq \mathbb{R}^3$, we restrict to the subspace of antisymmetric wave functions, given by

$$L^2_{\mathrm{as}}(\Lambda^N) = \Big\{ \Psi_N \in L^2(\Lambda^N) \mid \forall i \neq j : \\ \Psi_N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = -\Psi_N(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N) \Big\}. \tag{1.1.1}$$

Quantum mechanics provides a framework for predicting the average outcome of measuring an observable over a large ensemble of identical experimental repetitions. This average outcome is computed by the expectation value $\langle O \rangle_{\Psi_N} = \langle \Psi_N, O \Psi_N \rangle$ where O is a self-adjoint operator on $L^2(\mathbb{R}^{3N})$ modeling an observable. Examples of typical observables of interest include the position operator, acting as multiplication by the coordinate x_k on the k-th particle, the corresponding momentum operator $P_k = -i\nabla_{x_k}$ and the Hamilton operator H_N, which not only generates the time evolution by (1.0.1) but also models the total energy of the system.

A crucial distinction in quantum many-body theory is between static and dynamical settings. The *static setting* concerns expectation values with respect to specific wave functions, typically those that minimize the energy, i.e. $\Psi_N \in L^2(\mathbb{R}^{3N})$ such that $\langle \psi, H_N \psi \rangle = \inf \sigma(H_N)$ where $\sigma(H_N)$ denotes the spectrum of H_N, or other relevant states, such as low-lying energy states. This is of particular interest since according to fundamental physical principles, a system naturally tends to minimize its energy. These theoretical predictions can be compared with experimental measurements, for example, through various spectroscopic methods that provide insights into energy levels. The mathematical study of static properties is often based on a detailed spectral analysis of the Hamiltonian H_N. In some cases, stability properties of the system can be inferred via inequalities involving order parameters, with one of the most notable examples being the proof of the stability of matter of the second kind [Lie76].

In contrast, the *dynamical setting*—which is the primary focus of our work—concerns the time evolution of a given initial state under a prescribed Hamiltonian. Here, one aims to approximate either the full time-dependent wave function $\Psi_{N,t}$, which is the solution of (1.0.1) with appropriate initial state and whose existence and uniqueness is governed by the self-adjointness of the Hamiltonian by Stone's theorem, or its marginals in the form of correlation functions or *reduced k-body density matrices*

$$\gamma^{(k)}_{\Psi_{N,t}}(x_1, \ldots, x_k; y_1, \ldots, y_k) \\ := \frac{N!}{(N-k)!} \int_{\Lambda^{N-k}} \Psi_{N,t}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_N) \overline{\Psi_{N,t}(y_1, \ldots, y_k, x_{k+1}, \ldots, x_N)} \, \mathrm{d}x_{k+1} \ldots \mathrm{d}x_N. \tag{1.1.2}$$

Dynamical properties are of great interest in many physical systems, as real-world conditions often prevent the system from reaching the ground state of a theoretical Hamiltonian due to external perturbations, such as finite temperature or interactions. Instead of studying equilibrium states, explicitly tracking the time evolution of the system reveals how potentially new states emerge and transitions occur—insights that are often inaccessible through spectral methods. This time-dependent perspective is particularly suited for understanding the formation processes of quasi-particles, collective averaging effects, and the adiabatic behavior of many-body systems, all of which might be encoded in an effective equation. Advances in quantum gas experiments and modern measurement techniques now allow direct observations of these dynamical effects, providing an experimental benchmark for mathematical predictions (see (1.3.3)).

The typical approach in this setting is to start from an initial state $\Psi_{N,0}$ that approximates a ground state of a simplified Hamiltonian—one that neglects certain interactions—and analyze its evolution under a more complex Hamiltonian that includes these interactions. Our central objective is to derive effective equations that accurately capture the essential features of the microscopic system while reducing the complexity in a limiting regime. A rigorous analysis of the quantum dynamics often involves distinguishing between different time scales, as distinct physical processes become relevant in different regimes.

In order to quantify the approximation of solutions of the Schrödinger equation (1.0.1), we can employ different notions of distance. The most straightforward is given by the L^2-norm $\|\Psi_N\| \coloneqq \sqrt{\langle \Psi_N, \Psi_N \rangle}$ induced by scalar product of our Hilbert space. Note that this norm is sensitive to deviations of single particles of the many-body system.

In cases where approximations are expected to hold in the average sense, we will work at the level of reduced density matrices (1.1.2). Although this provides a weaker notion of approximation, it is still sufficient for estimating expectation values of bounded k-body operators, provided that we can control the trace norm. This follows from the following fact: For any $k = 1, \dots, N$, let O^k be a bounded k-body operator with $O^k = O^k \otimes \mathrm{id}^{\otimes(N-k)}$, consider the norms

$$\|O^k\|_{\mathrm{op}} \coloneqq \sup_{\Psi \in \mathcal{H}, \|\Psi\|=1} \|O^k \Psi\|, \tag{1.1.3}$$

$$\|O^k\|_{\mathrm{tr}} \coloneqq \mathrm{tr}|O^k| = \sum_i \langle \psi_i, |O^k| \psi_i \rangle, \tag{1.1.4}$$

$$\|O^k\|_{\mathrm{HS}} \coloneqq \sqrt{\mathrm{tr}\left((O^k)^* O^k\right)}, \tag{1.1.5}$$

where $\{\psi_i\}_{i \in \mathbb{N}}$ denotes a orthonormal basis and $(O^k)^*$ the adjoint of O^k. Then it is well-known that $\|O^k\|_{\mathrm{op}} \leq \|O^k\|_{\mathrm{HS}} \leq \|O^k\|_{\mathrm{tr}}$ and $\|AO^k\|_{\mathrm{tr}} \leq \|A\|_{\mathrm{op}} \|O^k\|_{\mathrm{tr}}$. Thus, by using that the expectation value can be written as

$$\langle \Psi_N, O^k \Psi_N \rangle = \mathrm{tr}\left(O^k \gamma^{(k)}_{\Psi_N}\right), \qquad \text{with } \gamma^{(k)}_{\Psi_N} \equiv \mathrm{tr}_{k+1,\dots,N} |\Psi_N\rangle\langle\Psi_N|, \tag{1.1.6}$$

the difference of expectation values can be bounded by the trace norm of the corresponding reduced densities

$$\langle \Psi_N, O^k \Psi_N \rangle - \langle \Phi_N, O^k \Phi_N \rangle = \operatorname{tr}\left(\gamma^{(k)}_{\Psi_N} O^k - \gamma^{(k)}_{\Phi_N} O^k \right) \leq \| O^k \|_{\mathrm{op}} \| \gamma^{(k)}_{\Psi_N} - \gamma^{(k)}_{\Phi_N} \|_{\mathrm{tr}}. \tag{1.1.7}$$

We will see later in Subsection 1.2.3.1 that the trace norm difference can be related to counting the relative number of particles described by Ψ_N which are close to Φ_N.

1.1.2 Second quantized formalism

To describe fermionic states of variable particle numbers, one introduces the *fermionic Fock space* over the one-body Hilbert space $\mathscr{H} = L^2(\Lambda)$ for a subspace $\Lambda \subseteq \mathbb{R}^d$:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n^-, \quad \mathscr{H}_n^- := L^2_{\mathrm{as}}(\Lambda^n). \tag{1.1.8}$$

The elements are denoted by $\psi = \left(\psi^{(n)}\right)_{n\geq 0}$ and we equip $\mathcal{F}$ with the inner product $\langle \psi, \phi \rangle = \sum_{n\geq 0} \langle \psi^{(n)}, \phi^{(n)} \rangle_{\mathscr{H}_n^-}$ and the norm $\|\psi\| = \sqrt{\langle \psi, \psi \rangle} = \sum_{n \geq 0} \|\psi^{(n)}\|^2_{\mathscr{H}_n^-}$. We consider solely normalized vectors $\psi \in \mathcal{F}$ such that $\|\psi^{(n)}\|^2_{\mathscr{H}_n^-}$ determines the probability that the state ψ has exactly n particles. In the case of a fixed number of particles, we expect to find an eigenvector of a number operator. This is achieved by defining the *number operator* by

$$(\mathcal{N}\psi)^{(n)} := n\psi^{(n)}.$$

The *vacuum state* describing the absence of any particles is defined as $\Omega := (1, 0, 0, \ldots)$.

For a function $f \in \mathscr{H}$, we define the fermionic *creation and annihilation operators* $a^*(f)$ and $a(f)$ as

$$\left(a^*(f)\psi\right)^{(n)}(x_1, \ldots, x_n) = \sum_{j=1}^{n} \frac{(-1)^j}{\sqrt{n}} f(x_j) \psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), \tag{1.1.9}$$

$$\left(a(f)\psi\right)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int_{\Lambda} \mathrm{d}x \; \overline{f(x)} \psi^{(n+1)}(x_1, \ldots, x_n, x). \tag{1.1.10}$$

In order to write them in a more convenient form, one commonly uses the operator-valued distributions a_x^* and a_x to write

$$a^*(f) = \int_{\Lambda} \mathrm{d}x \; f(x) \; a_x^*, \qquad a^*(f) = \int_{\Lambda} \mathrm{d}x \; \overline{f(x)} \; a_x. \tag{1.1.11}$$

For any $f, g \in \mathscr{H}$, the creation and annihilation operators satisfy the fermionic *canonical anti-commutation relation* (CAR)

$$\{a(f), a^*(g)\} = \langle f, g \rangle_{\mathscr{H}}, \qquad \{a(f), a(g)\} = 0 = \{a^*(f), a^*(g)\}. \tag{1.1.12}$$

Correspondingly, it holds for the operator-valued distributions that

$$\{a_x, a_y^*\} = \delta(x-y), \qquad \{a_x, a_y\} = 0 = \{a_x^*, a_y^*\}. \tag{1.1.13}$$

The operators $a(f)$ and $a^*(f)$ satisfy the bound for all $\psi \in \mathcal{F}$

$$\|a(f)\psi\|^2 = \langle \psi, a^*(f)a(f)\psi\rangle = \|f\|^2_{\mathscr{H}} - \langle \psi, a(f)a^*(f)\psi\rangle \le \|f\|^2_{\mathscr{H}} \tag{1.1.14}$$

and are therefore bounded operators (unlike their bosonic analogs). This can be interpreted as a consequence of Pauli's exclusion principle since $a^*(f)a^*(f) = 0$. It is easy to check that, in fact, it holds that $\|a(f)\| = \|a^*(f)\| = \|f\|_{\mathscr{H}}$.

The second quantization $\mathrm{d}\Gamma\left(O^{(1)}\right)$ of a one-body operator $O^{(1)}$ acting on $\mathscr{H}$ is defined by

$$\left(\mathrm{d}\Gamma\left(O^{(1)}\right)\psi\right)^{(n)} \coloneqq \sum_{i=1}^{n} O_i^{(1)}\psi^{(n)} \tag{1.1.15}$$

where $O_i^{(n)} = \mathrm{id}^{\otimes(i-1)} \otimes O^{(1)} \otimes \mathrm{id}^{\otimes(n-j)}$. In the case that $O^{(1)}$ has an integral kernel $O^{(1)}(x;y)$ it holds

$$\mathrm{d}\Gamma\left(O^{(1)}\right) = \int_{\Lambda\times\Lambda} \mathrm{d}x\ \mathrm{d}y\ O^{(1)}(x;y)a_x^*a_y. \tag{1.1.16}$$

The generalization for k-body operators is given by

$$\mathrm{d}\Gamma\left(O^{(k)}\right) \coloneqq \int_{\Lambda^k\times\Lambda^k} \mathrm{d}x_1\cdots\mathrm{d}x_k\mathrm{d}y_1\cdots\mathrm{d}y_k\ O^{(k)}(x_1,\dots,x_k;y_1,\dots,y_k)a_{x_1}^*\cdots a_{x_k}^*a_{y_1}^*\cdots a_{y_k}. \tag{1.1.17}$$

The most important examples are

$$\mathcal{N} = \int_\Lambda \mathrm{d}x\ a_x^*a_x = \mathrm{d}\Gamma(\mathrm{id}) \tag{1.1.18}$$

and the Hamiltonian

$$\mathcal{H} = \underbrace{\int_\Lambda \mathrm{d}x\ \nabla_x a_x^*\nabla_x a_x}_{=\mathrm{d}\Gamma(-\Delta)} + \frac{g_N}{2}\int_{\Lambda\times\Lambda} \mathrm{d}x\mathrm{d}y\ W(x-y)a_x^*a_y^*a_ya_x \tag{1.1.19}$$

which coincides with H_N if restricted to the N-particle sector $L^2(\Lambda^N) \subset \mathcal{F}$. Furthermore, one can write the reduced one-body density $\gamma_\psi^{(1)}$ and the *pairing density* a_ψ with respect to $\psi \in \mathcal{F}$ as

$$\gamma_\psi^{(1)}(x;y) = \langle \psi, a_y^*a_x\psi\rangle, \tag{1.1.20}$$

$$a_\psi(x;y) = \langle \psi, a_ya_x\psi\rangle \tag{1.1.21}$$

with $\mathrm{tr}\gamma_\psi^{(1)} = \langle \psi, \mathcal{N}\psi\rangle$. The standard Fock space bounds (see for example [BPS14b, BBP+16]) for a bounded operator O on $\mathscr{H}$ are

$$\|\mathrm{d}\Gamma(O)\psi\| \leq \|O\|_{\mathrm{op}}\|\mathcal{N}\psi\|. \tag{1.1.22}$$

and for O Hilbert-Schmidt operator on $\mathscr{H}$

$$\|\mathrm{d}\Gamma(O)\psi\| \leq \|O\|_{\mathrm{HS}}\|\mathcal{N}^{1/2}\psi\|, \tag{1.1.23}$$

$$\|\int_{\Lambda\times\Lambda} \mathrm{d}x\mathrm{d}y\; O(x;y)a_x a_y\psi\| \leq \|O\|_{\mathrm{HS}}\|\mathcal{N}^{1/2}\psi\|, \tag{1.1.24}$$

$$\|\int_{\Lambda\times\Lambda} \mathrm{d}x\mathrm{d}y\; O(x;y)a_x^* a_y^*\psi\| \leq 2\|O\|_{\mathrm{HS}}\|(\mathcal{N}+1)^{1/2}\psi\|. \tag{1.1.25}$$

For O trace-class on $\mathscr{H}$ (i.e. $\|O\|_{\mathrm{tr}} \leq C$ for $C > 0$) it holds

$$\|\mathrm{d}\Gamma(O)\psi\|, \|\int \mathrm{d}x\mathrm{d}y\; O(x;y)a_x a_y\psi\|, \|\int \mathrm{d}x\mathrm{d}y\; O(x;y)a_x^* a_y^*\psi\| \leq 2\|O\|_{\mathrm{tr}}. \tag{1.1.26}$$

As a common example, we consider the three-dimensional box $\Lambda = \mathbb{R}^3/(L\mathbb{Z}^3)$ with periodic boundary conditions (also referred to as *torus*). Most conveniently, one chooses $\{f_k\}_{k\in(2\pi/L)\mathbb{Z}^3}$ with $f_k(x) = L^{-\frac{3}{2}e^{ik\cdot x}}$ as orthonormal basis of $L^2(\Lambda)$ such that one obtains the momentum representation

$$a_k^* \coloneqq a^*(f_k), \qquad a_k \coloneqq a(f_k) \tag{1.1.27}$$

satisfying $\{a_k, a_l^*\} = \delta_{k,l}$ and $\{a_k, a_l\} = 0 = \{a_k^*, a_l^*\}$. In this representation, one finds

$$\mathcal{N} = \sum_{k\in(2\pi/L)\mathbb{Z}^3} a_k^* a_k, \tag{1.1.28}$$

$$\mathbb{H} = \sum_{k\in(2\pi/L)\mathbb{Z}^3} |k|^2 a_k^* a_k + \frac{g_N}{2L^3}\sum_{k\in(2\pi/L)\mathbb{Z}^3}\sum_{p,q\in(2\pi/L)\mathbb{Z}^3} \hat{W}(k)a_{p+k}^* a_{q-k}^* a_q a_p \tag{1.1.29}$$

with the Fourier transform $\hat{W}(k) = \int_\Lambda W(x)e^{-ik\cdot x}\,\mathrm{d}x$.

1.1.3 Notation

Throughout the thesis, the notations are defined in the respective chapters. Unless specified otherwise, we use the following implicit notation for unspecified norms and scalar products, depending on the objects involved:

- $|\cdot|$ denotes the standard euclidean norm on $\mathbb{C}^d$ for dimension $d = 1, 2, 3$.

- If $\psi, \varphi \in L^2(\Lambda^k)$ for $\Lambda \subset \mathbb{R}^d$ and $k \in \mathbb{N}$, then $\langle \psi, \varphi \rangle \coloneqq \langle \cdot, \cdot \rangle_{L^2(\Lambda^k)}$ denotes the corresponding inner product and $\|\psi\| \coloneqq \sqrt{\langle \psi, \psi \rangle_{L^2(\Lambda^k)}}$ the corresponding norm.
- If $A : L^2(\Lambda^k) \to L^2(\Lambda^k)$ is a linear operator, we denote $\|A\| \coloneqq \|A\|_{\mathrm{op}}$.
- If $\xi \in \mathscr{H}_A$ and $\psi \in \mathscr{H}_B$ with Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$, then $\|\xi \otimes \psi\| \coloneqq \|\xi \otimes \psi\|_{\mathscr{H}_A \otimes \mathscr{H}_B}$ and similarly for the scalar product.
- The letter C is commonly used to denote a positive constant whose numerical value may vary from line to line in the estimates. It is independent of the order parameters or time unless explicitly stated otherwise.

1.2 The interacting Fermi gas

In this section we present an overview of effective dynamical theories for the interacting Fermi gas described by (1.0.3) with different coupling parameters $g_N > 0$. We will start with a mean-field description of the interacting Fermi gas where correlations are neglected.

1.2.1 The time-dependent Hartree-Fock equations

In order to develop an effective dynamical theory that reduces complexity, the fundamental idea behind the mean-field approach is to assume that particles are as independent as Fermi statistics permits and evolve in an effective, self-consistent one-particle potential. Suppose that at the time $t = 0$ the N fermions of our system are described in the most uncorrelated state given by a antisymmetric product

$$\Psi_{N,0} \coloneqq \bigwedge_{k=1}^{N} \varphi_k^0 \tag{1.2.1}$$

where $\{\varphi_k^0\}_{k=1}^N \subset L^2(\mathbb{R}^3)$ are orthonormal one-body wave functions which are usually referred to as *orbitals*. This state is also often known as *Slater determinant* since we can write $\Psi_{N,0}(x_1, \ldots, x_N) = (N!)^{-1/2} \det \big(\varphi_k(x_j)\big)_{1 \leq j,k \leq N}$. It has the nice property that the reduced one-particle density is a rank-N projector

$$\gamma^{(1)}_{\wedge\varphi^0} \coloneqq N \mathrm{tr}_{2,\ldots,N} |\Psi_{N,0}\rangle\langle\Psi_{N,0}| = N \sum_{k=1}^{N} |\varphi_k^0\rangle\langle\varphi_k^0| \tag{1.2.2}$$

with integrating kernel $\gamma^{(1)}_{\wedge\varphi^0}(x;y) = N \sum_{k=1}^N \overline{\varphi_k^0(x)} \varphi_k^0(y)$. In particular this means that the spectral decomposition is of the form

$$\gamma^{(1)}_{\wedge\varphi^0} = \sum_{i=1}^{\infty} \lambda_i |\chi_i\rangle\langle\chi_i| \tag{1.2.3}$$

where exactly N of the coefficients λ_i equal 1 and the remaining coefficients equal 0. Furthermore, the expectation values of N-particle observables with respect to such a Slater determinant can be calculated with respect to the reduced one-particle density by Wick's rule. In particular, we find that the energy can be written in the form

$$\langle \Psi_{N,0}, H_N \Psi_{N,0} \rangle = \operatorname{tr}(-\Delta)\,\gamma^{(1)}_{\wedge\varphi^0} + \frac{1}{2} g_N \int \mathrm{d}x\,\mathrm{d}y\, W(x-y) \left(\gamma^{(1)}_{\wedge\varphi^0}(x,x)\gamma^{(1)}_{\wedge\varphi^0}(y,y) - |\gamma^{(1)}_{\wedge\varphi^0}(x,y)|^2 \right). \tag{1.2.4}$$

Since these properties have great benefits for many calculations it is natural to ask whether these are applicable in general. More concretely, one might ask if the antisymmetric product structure, which carries these properties, is preserved in the time evolution generated by the Hamiltonian (1.0.3). Or even more concretely, one might ask: Is there an appropriate law governing the dynamics of the orbitals $\{\varphi_k^t\}_{k=1}^N$ such that it holds

$$\Psi_{N,t} = e^{-iH_N t} \bigwedge_{k=1}^N \varphi_k^0 \approx \bigwedge_{k=1}^N \varphi_k^t \tag{1.2.5}$$

where H_N is given by (1.0.3). By inserting the ansatz $\Psi_{N,t} = \bigwedge_{k=1}^N \varphi_k^t$ into the Schrödinger equation (1.0.1) one obtains, analogously to (1.2.4), the coupled one-body equation

$$i\partial_t \varphi_k^t = -\Delta \varphi_k^t + g_N \sum_{j=1}^N \int \mathrm{d}y\; W(\cdot - y) \left(|\varphi_j^t|^2\; \varphi_k^t - \overline{\varphi_j^t(y)} \varphi_k^t(y)\; \varphi_j^t \right) \tag{1.2.6}$$

for each $k = 1, \dots, N$ from which we identify the **Hartree-Fock Hamiltonian**

$$h^{\mathrm{HF}}(t) \coloneqq -\Delta + g_N W * \rho^t - g_N W * N^{-1} \gamma^{(1)}_{\wedge\varphi^t} \tag{1.2.7}$$

with $*$ denoting the convolution and

$$\rho^t \coloneqq \sum_{k=1}^N |\varphi_k^t|^2, \qquad N^{-1}\gamma^{(1)}_{\wedge\varphi^t} \coloneqq \sum_{k=1}^N |\varphi_k^0\rangle\langle\varphi_k^0|. \tag{1.2.8}$$

The coupled non-linear equations

$$i\partial_t \varphi_k^t = h^{\mathrm{HF}}(t)\varphi_k^t \tag{1.2.9}$$

for each $k = 1, \dots, N$ are called **time-dependent Hartree-Fock equations (TDHF)**. The so-called *direct term* $g_N W * \rho^t$ describes an interaction over the mean-field given by the convolution with the density of all orbitals whereas the *exchange term* $g_N W * N^{-1}\gamma^{(1)}_{\wedge\varphi^t}$ takes into account the antisymmetric nature of the fermion system. We emphasize that despite assuming the most uncorrelated structure of our N-body wave function (1.2.5), the evolution equation for each one-body wave function is inherently coupled. This is because both the

direct and exchange terms involve all other orbitals which is a direct consequence of the antisymmetric property involving all N fermions.
However, we observe a drastic reduction in complexity with this approximation, as the dimensionality of the solution space is reduced. Specifically, we only need the functions $\varphi_1^t, \dots \varphi_N^t$ on $\mathbb{R}^3$ instead of $\Psi_{N,t}$ on $\mathbb{R}^{3N}$. This results in a linear scaling of $N \cdot \dim(L^2(\mathbb{R}^3))$ instead of an exponential scaling $\dim(L^2(\mathbb{R}^3))^N$. Therefore, this effective theory is valuable in numerous contexts within theoretical physics and theoretical chemistry. Most notably, it is employed for the numerical analysis of the electronic structure of large atoms and molecules [Sza12] and in solid-state physics, atomic theory, and nuclear physics. In addition, the TDHF serves as a fundamental prototype model and an important stepping stone for more involved theories, such as Density Functional Theory (DFT).

Alternatively, one can formulate the time-dependent Hartree-Fock equations for the reduced one-particle density matrix $p^t := \sum_{j=1}^N |\varphi_j^t\rangle\langle\varphi_j^t|$ with

$$i\partial_t p^t = [h^{\mathrm{HF}}(t), p^t]. \tag{1.2.10}$$

In many regimes, it is well-established that the Hartree-Fock description can be further simplified while retaining a substantial degree of accuracy. More specifically, it is often possible to neglect the exchange term in (1.2.7) in many mathematical proofs of the validity of the mean-field approximation without significantly altering the error bounds and the convergence rates. This gives rise to the **time-dependent fermionic Hartree equations** (also referred to as time-dependent reduced Hartree-Fock equations)

$$i\partial_t \varphi_k^t = h^{\mathrm{mf}}(t)\varphi_k^t \tag{1.2.11}$$

with the **fermionic Hartree Hamiltonian** (or reduced Hartree-Fock Hamiltonian)

$$h^{\mathrm{mf}}(t) := -\Delta + g_N W * \rho^t. \tag{1.2.12}$$

The global-in-time well-posedness of (2.1.4), (1.2.9) and(1.2.10) has been covered for singular interactions including the repulsive Coulomb case in [CG75, BDPF76]. Both equations are usually referred to as *mean-field description* of the microscopic system since one considers averaged interactions and neglects correlations by assuming that the antisymmetric product structure is preserved. It is therefore natural to expect that such an approximation holds only on the average level or more concretely on the level of reduced one particle density matrices instead of wave functions. A derivation of the mean-field equations can be formulated as

$$\lim_{N\to\infty} \operatorname{tr}\left|\gamma^{(1)}_{\Psi_{N,0}} - \gamma^{(1)}_{\wedge\varphi^0}\right| = 0 \quad \Longrightarrow \quad \lim_{N\to\infty} \operatorname{tr}\left|\gamma^{(1)}_{\Psi_{N,t}} - \gamma^{(1)}_{\wedge\varphi^t}\right| = 0. \tag{1.2.13}$$

In Chapter 2, we will focus on the rigorous derivation of the time-dependent fermionic Hartree equations (2.1.4) from the microscopic dynamics leveraging the fact that it is straightforward to deduce that also the Hartree-Fock equations (1.2.9) provide a valid approximation in our regime of interest. This approach allows for a rigorous derivation of both equations while maintaining the tractability of the mathematical framework since we can focus on the simpler Hartree equations (2.1.4).

1.2.2 Scaling regimes

There are different scenarios where the approximation (1.2.5) is expected to hold, which are explained in the following. We note that non-linearity in (1.2.7) arises from the interaction terms incorporating averaging processes. Thus, we expect that the limiting regime is of large particle number N. Moreover, since we assume in the approximation (1.2.5) that the antisymmetric product structure is preserved, it is implicitly assumed that no other correlations form despite those from the antisymmetrization. Therefore, it is expected that the approximation is restricted to short times, such that correlations do not form, or to weak interaction, which can be realized by choosing a N-dependent coupling of the form $g_N = N^{-\gamma}$ for a $\gamma > 0$. Typically, mathematical results are formulated under appropriate assumptions on the solutions $\{\varphi_k^t\}_{k=1}^N$ for N-dependent coupling parameters and macroscopic time scales which translate to short time scales with N-independent coupling parameters. We focus on the case of singular long-range pair interactions of the form

$$W(x) = |x|^{-s}, \qquad s \in (0, 1]$$

in our review of the relevant scaling regimes of interest and subsequently discuss the corresponding literature in Subsection 1.2.3.

1.2.2.1 Mean-field regime

Note that in the Hamiltonian (1.0.3) each fermion interacts with $(N-1)$ other particles. Since our objective is to develop a mean-field description, a natural first educated guess is $g_N = N^{-1}$ as averaging factor to balance the large number of interaction partners. This regime serves as a first toy model in which the effect of the interaction can be treated as a small perturbation in the limit of $N \to \infty$ and in which therefore the free dynamics dominates.

1.2.2.2 High density and semiclassical regime

An alternative motivation arises from considering a setting in which the kinetic energy and the interaction energy are of the same order in the number N of particles. In such a setting, it is reasonable to expect that the mean-field description remains valid up to macroscopic times, which are independent of the particle number. If the volume is of order 1 with respect to N, then the kinetic energy is high due to the Pauli exclusion principle and can be estimated by the Lieb-Thirring inequality

$$\langle \psi, \sum_{i=1}^{N} (-\Delta_{x_i}) \psi \rangle \sim N^{5/3}. \tag{1.2.14}$$

Since the interaction term in the Hamiltonian (1.0.3) contains $N(N-1)$ summands and the expected interaction is of order 1 with respect to N, one considers a coupling parameter of $g_N = N^{-1/3}$. The fact that the expected interaction per particle is of order N follows from the high spatial density in this regime, i.e., $\|\rho\|_\infty \sim N$. This can be seen by considering the corresponding mean-field interaction:

$$|\cdot|^{-s} * \rho \sim \|\rho\|_\infty \int_0^L \mathrm{d}r \; r^{-s} r^2 \sim N. \tag{1.2.15}$$

The highest momenta of the fermions are of order $N^{1/3}$. Consequently, it is natural to look at times of order $N^{-1/3}$ and re-scale the time variable accordingly:

$$iN^{\frac{1}{3}} \partial_t \Psi_N(t) = \sum_{i=1}^N (-\Delta_{x_i}) + N^{-\frac{1}{3}} \sum_{1 \le i < j \le N} W(x_i - x_j) \tag{1.2.16}$$

By multiplying the equation with $N^{-2/3}$ and identifying $\hbar_N := N^{-1/3}$ as semiclassical parameter which takes formally the role of the physical constant $\hbar$ one obtains

$$i\hbar_N \partial_t \Psi_N(t) = \hbar_N^2 \sum_{i=1}^N (-\Delta_{x_i}) + N^{-1} \sum_{1 \le i < j \le N} W(x_i - x_j). \tag{1.2.17}$$

In the limit $N \to \infty$, it holds $\hbar_N \to 0$ and thus, this regime is expected to exhibit classical behavior under suitable assumptions on the mean-field solutions.

1.2.2.3 Extensive volume regime

The same motivation of a balanced kinetic and interaction energy, leads to a different situation when considering a volume of order N. This assumption arises naturally when describing a system with an extensive particle number while maintaining a fixed average density, as in the thermodynamic limit of gas systems.

By a scaling argument, the kinetic energy per particle remains of order 1, necessitating a comparable interaction energy per particle. Despite density of order 1, i.e. $\|\rho\|_\infty \sim 1$, the long-range behavior becomes significant due to the large volume

$$g_N |\cdot|^{-s} * \rho \sim g_N \|\rho\|_\infty \int_0^{N^{1/3}} \mathrm{d}r \; r^{-s} r^2 \sim g_N N^{\frac{3-s}{3}}. \tag{1.2.18}$$

To ensure a balanced contribution of the interaction energy, we choose the coupling parameter as $g_N = N^{\frac{s}{3}-1}$. This regime is usually called *extensive volume regime*. In this regime, it is expected that the system experiences more quantum features since it is not associated with a small semiclassical parameter such as in the semiclassical regime. It is also a regime where the long-range character of the interaction potential is visible for the particles of the system. Notably, while the mean-field potential remains of order 1, it is nearly constant across the inter-particle distance of order $N^{1/3}$ which can be seen by considering the spatial variation

$$g_N \nabla |\cdot|^{-s} * \rho \sim \|\rho\|_\infty \int_0^{N^{1/3}} \mathrm{d}r \; r^{-(s+1)} r^2 \sim g_N N^{\frac{2-s}{3}} = N^{-\frac{1}{3}}. \tag{1.2.19}$$

As a result, it is expected that the mean-field interaction term $g_N W * \rho^t$ can be approximated by a constant real-valued number. Thus, the leading-order dynamics is expected to be dominated by the free evolution.

1.2.2.4 Strongly interacting regime

In order to overcome the dominance of the free evolution in the case of an extensive volume of order N, one has to abandon the initial motivation of balancing interaction and kinetic energy. Instead, one requires that the spatial variation of the mean-field interaction is of order 1, i.e.

$$1 \overset{!}{=} g_N \nabla |\cdot|^{-s} * \rho \sim \|\rho\|_\infty \int_0^{N^{1/3}} \mathrm{d}r \; r^{-(s+1)} r^2 \sim g_N N^{\frac{2-s}{3}} \tag{1.2.20}$$

which would imply that the mean-field interaction is not expected to be constant across the system such as in the extensive volume regime. This leads to the novel choice of $g_N = N^{\frac{s-2}{3}}$. This regime inherits the quantum and long-range features from the extensive volume regime but is considerably more difficult since the particles interact with a factor of $N^{1/3}$ stronger than in the extensive volume regime. Consequently, the long-range interaction dominates the kinetic energy which is still of order 1 per particle whereas the expected interaction energy per particle is of larger order,

$$g_N |\cdot|^{-s} * \rho \sim N^{\frac{1}{3}}. \tag{1.2.21}$$

Note that it is exactly the spatial variation of the mean-field interaction in the form of the mean-field force which is the main reason why the system exhibits correlation over time. This can be seen by investigating the change of the kinetic energy of the solutions of the Hartree equation (2.1.4)

$$\begin{aligned}
\left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \varphi_k^t, (-\Delta) \varphi_k^t \rangle \right| &= \left| \langle \varphi_k^t, \left[-\Delta, g_N W * \rho^t \right] \varphi_k^t \rangle \right| \\
&= \left| \langle \varphi_k^t, (g_N \Delta W * \rho_t) \varphi_k^t + 2 \left(g_N \nabla W * \rho_t \right) \cdot \nabla \varphi_k^t \rangle \right|
\end{aligned} \tag{1.2.22}$$

which only depends on the mean-field force $g_N \nabla W * \rho^t$ and not on the mean-field interaction itself. This is natural since a strong interaction between the particles is expected to change the average kinetic energy of an orbital over time, thereby inducing correlations among particles. In Proposition 16 of Chapter 2, we will show rigorously that the order of the mean-field force dictates the time scale in which we can expect the kinetic energy to change significantly and therefore correlations to form due to the interaction. In such a case, we do not expect that the mean-field description remains valid. Thus, by requiring (1.2.20), we ensure that the approximation is expected to hold for macroscopic time scales.

1.2.2.5 Short time regime

Finally, let us discuss how to transform a Hamiltonian in the weak-coupling limit, that is of the form (1.0.3) with $g_N = N^{-\beta}$ and singular interaction $W = |\cdot|^{-s}$, to an unscaled Hamiltonian. We assume that in the beginning the system is described by variables $(t, x) \in \mathbb{R} \times \mathbb{R}^{3N}$ where $t \in \mathcal{O}(1)$ with respect to the particle number N. Consider the Schrödinger equation

$$i\partial_t \Psi_{N,t}(x_1, \ldots, x_N) = \left(\sum_{j=1}^{N} (-\Delta_{x_j}) + N^{-\beta} \sum_{i<j} |x_i - x_j|^{-s} \right) \Psi_{N,t}(x_1, \ldots, x_N). \tag{1.2.23}$$

Now consider re-scaled coordinates $(\tilde{t}, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{3N}$

$$\begin{cases} \tilde{t} &= N^{-\frac{2\beta}{2-s}} t, \\ \tilde{x} &= N^{-\frac{\beta}{2-s}} x \end{cases}$$

and the re-scaled many-body wave function

$$\tilde{\Psi}_{N,\tilde{t}}(\tilde{x}_1, \ldots, \tilde{x}_N) = \Psi_{N, N^{-\frac{2\beta}{2+s}} \tilde{t}} \left(N^{-\frac{\beta}{2+s}} \tilde{x}_1, \ldots, N^{-\frac{\beta}{2+s}} \tilde{x}_N \right) \tag{1.2.24}$$

which satisfies

$$\begin{aligned} &iN^{-\frac{2\beta}{2+s}} \partial_{\tilde{t}} \tilde{\Psi}_{N,\tilde{t}}(\tilde{x}_1, \ldots, \tilde{x}_N) \\ &= \left(N^{-\frac{2\beta}{2-s}} \sum_{j=1}^{N} (-\Delta_{\tilde{x}_j}) + N^{-\frac{\beta}{2-s}s} N^{-\beta} \sum_{i<j} |\tilde{x}_i - \tilde{x}_j|^{-s} \right) \partial_{\tilde{t}} \tilde{\Psi}_{N,\tilde{t}}(\tilde{x}_1, \ldots, \tilde{x}_N) \end{aligned} \tag{1.2.25}$$

and thus the unscaled Schrödinger equation. Given that the time variable now operates on shorter time scales, specifically $\tilde{t} \in \mathcal{O}(N^{-\frac{2\beta}{2-s}})$, and the spatial variable $\tilde{x}$ is correspondingly smaller, this can be interpreted as a formal "zooming in" process.
For the specific choice of the strongly interacting regime, one obtains the unscaled Schrödinger equation for $\tilde{t} \in \mathcal{O}(N^{-2/3})$ and on a re-scaled volume $\tilde{V} \in \mathcal{O}(1)$. Given that the kinetic energy per particle is also re-scaled, it is observed that particles possess momenta of the order

$N^{1/3}$. Consequently, particles can traverse the inter-particle distance of $N^{-1/3}$ within this time scale. Also in this case, we can show with Proposition 16 of Chapter 2 that this is the optimal time scale before we can expect the kinetic energy to change significantly, leading to the formation of correlations due to the interaction. We remark that in the extensive volume regime and the semiclassical regime, the same considerations lead to a much smaller volume and extremely short time scale.

1.2.3 Rigorous derivations and open questions

In the following, we provide a brief overview over the mathematical results in the derivation of the time-dependent mean-field description of many-body fermionic systems within which our work is situated.

The first results were formulated in the mean-field regime by [BGGM03, BGGM04]. The authors focused on bounded symmetric interaction potentials and initial states close to Slater determinants and extended later Gibbs equilibrium states for macroscopic time scales. Shortly later, first results were obtained in the semiclassical regime for real-analytic interactions [EESY04] for macroscopic time scales using the Wigner transform and the Husimi function to exploit the semiclassical character. In [FK11] the validity of the time-dependent fermionic Hartree equations was proved in the mean-field regime for the Coulomb interaction.

All results up to this point have been derived with the *Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy method* (or shortly the *BBGKY method*) which inherently does not provide explicit convergence rates. To explain this limitation, we briefly outline the core steps of the BBGKY approach: The method begins by deriving hierarchical equations of motion for the reduced k-particle density $\gamma^{(k)}$ which depend explicitly on the higher-order reduced density matrix $\gamma^{(k+1)}$. This coupling reflects the intrinsic many-body nature of the system, as the dynamics of $\gamma^{(k)}$ cannot be closed without incorporating higher correlations. To circumvent the infinite hierarchy, one establishes compactness for the sequence of reduced densities $\{\gamma^{(k)}\}_k$ within an appropriate (weak) topology. This compactness ensures the existence of a limit point as $N \to \infty$ which formally satisfies the infinite hierarchy of equations. In a critical subsequent step, one proves the uniqueness of solutions to the infinite hierarchy. This is typically achieved by leveraging the hierarchical structure and employing combinatorial arguments to bound differences between potential solutions. Finally, one observes that the antisymmetric product state satisfies the infinite hierarchy. By the uniqueness of solutions, this implies that the limiting hierarchy is solved by the mean-field ansatz, thereby establishing convergence.

In order to obtain convergence rates, new methods were developed based on Grönwall's lemma for the semiclassical regime [BPS14b] and for the extensive volume regime [PP16]. We will review both approaches in the following subsections. In the semiclassical regime, the *fluctuation dynamics method* was applied to the case of bounded interactions and non-relativistic

dispersion relation [BPS14b] and to pseudo-relativistic dispersion [BPS14a]. Later on, quasi-free mixed states as initial states were included [BJP+15].
In the extensive volume regime, singular interactions including the Coulomb interaction could be included under appropriate assumptions of the Hartree solutions [PP16]. Within the *counting functional method*, results in the semiclassical regime were reproduced. Subsequently, the Coulomb case could be covered solely under assumptions on the kinetic energy of the initial states of the Hartee solutions [BBP+16] by using the Fefferman decomposition of radial singular potentials. Its was proved in [Pet17] that the Hartree dynamics provides a correction to the free dynamics in this setting. In the semiclassical regime, the Coulomb case was treated in [PRSS17] using a smooth version of the Fefferman decomposition but under the assumption that the solutions of the Hartree equation preserve a semiclassical structure. To our best knowledge, the proof of the propagation of the semiclassical structure from a general class of initial states for singular interaction potentials remains an open problem in the semiclassical regime. The derivation for mixed initial states in semiclassical regime for singular potentials including the Coulomb interaction and gravitational attraction was achieved for short time scales in [CLS24]. Recently, the semiclassical regime was extended to large volumes Λ of high density $\varrho = N/|\Lambda| \gg 1$ for rapidly decaying bounded pair potentials. The fermionic Hartree equations were derived in this setting for non-relativistic dispersion and short time scales [FPS23] and for pseudo-relativistic dispersion for macroscopic times [FPS24].

We also mention that there is a geometrical approach to the fermionic Hartree equations from the so-called Dirac-Frenkel principle [Lub08, BSS18]. This is a variational method used to approximate the dynamics of quantum systems by projecting the time evolution described by the Schrödinger equation onto a restricted manifold of trial wave functions (in this case the Slater determinants). The projection ensures that the time derivative of the trial wave function remains as close as possible to the exact evolution within the tangent space of this manifold.

The mentioned classical behavior in the limit of $N \to \infty$ in the semiclassical regime, was rigorously established with different notions of convergence for singular potentials. More concretely, one can demonstrate the convergence in the weak sense of the solutions of the fermionic Hartree equations [LP93, MM93] or the fermionic Hartree-Fock equations [GIMS98] to the classical Vlasov equation, which describes the dynamics of the phase-space density. Explicit error bounds were subsequently obtained in [Laf19, Laf21] for singular potentials. Convergence in the strong sense was established for bounded potentials in [APPP11, BPSS16] and for the Coulomb case and initial mixed states in [Saf19, Saf20, LS23, CLS23]. Note that it is also possible to derive the Vlasov equations from the microscopic many-body dynamics. This is discussed in more detail in Pragraph 1.2.4.1.

1.2.3.1 The counting functional method

We give a short introduction of the counting functional method which was first established and applied to bosonic systems in [Pic11] and later on extended to fermionic systems in [PP16]. In Chapter 2 we develop and refine the techniques of this method to cover the strongly interacting regime.

The main idea here is to find a precise measure of closeness of a N-body state $\Psi \in L^2_{\text{as}}(\mathbb{R}^{3N})$ to a specific Slater determinant $\bigwedge_{k=1}^N \varphi_k$ where $\{\varphi_k\}_{k=1}^N$ is an orthonormal system. This is achieved by constructing the so-called counting functional or alpha functional $\alpha_f(\Psi, \{\varphi_k\}_{k=1}^N)$. One starts by defining a natural notion to detect the parts which are inside and outside the specific Slater determinant: Consider for each $k \in \{1, \dots, N\}$ the one-body projector $p^{\varphi_k} := |\varphi_k\rangle\langle\varphi_k|$ and lift it to the tensor space $L^2(\mathbb{R}^{3N}) \simeq L^2(\mathbb{R}^3) \otimes \dots \otimes L^2(\mathbb{R}^3)$ by defining its action on the m-th particle component as $p_m^{\varphi_k} : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ with

$$\left(p_m^{\varphi_k}\Psi\right)(x_1, \dots, x_N) := \left(|\varphi_k\rangle\langle\varphi_k|_m\Psi\right)(x_1, \dots, x_N) := \varphi_k(x_m) \int_{\mathbb{R}^3} \mathrm{d}x_m \; \varphi_k(x_m)^* \; \Psi(x_1, \dots x_N). \tag{1.2.26}$$

Now note that the sum

$$p_m := p_m^{\varphi_1, \dots \varphi_N} := \sum_{k=1}^N p_m^{\varphi_k} = \sum_{k=1}^N |\varphi_k\rangle\langle\varphi_k|_m \tag{1.2.27}$$

also defines a projector (since $\{\varphi_k\}_{k=1}^N$ are pairwise orthogonal) and detects the overlap of the m-th particle component with the subspace spanned by the orbitals $\{\varphi_k\}_{k=1}^N$. Similarly, one defines the orthogonal projector as

$$q_m = 1 - p_m \tag{1.2.28}$$

to detect the part which lies in the orthogonal complement. In this context, it is common to refer to the operator p_m as projecting onto the *good* part of the m-th particle, while q_m projects onto the *bad* part.
In order to lift to the level of the fermionic many-body wave function, we have to take into account the antisymmetric structure. Thus, one defines for $k \in \{0, \dots, N\}$

$$P_{N,k} := P_{N,k}^{\varphi_1, \dots, \varphi_N} := \left(\prod_{m=1}^k q_m \prod_{m=k+1}^N p_m\right)_{\text{sym}} \tag{1.2.29}$$

as the symmetrized tensor product of k q-operators and $(N-k)$ p-operators. In other words, the operator $P_{N,k}$ projects onto a many-body state where we find exactly k-times the bad part of a particle and exactly $(N-k)$-times the good part of a particle. Alternatively, in a more heuristic but practical manner, one may say the projected state contains k *bad particles* and $(N-k)$ *good particles*, as is common in this framework. One can easily check, that $P_{N,k}$ can be written with a $0-1$-sequence $\boldsymbol{a}=(a_1,\dots,a_N)$ satisfying $k=|\boldsymbol{a}|=a_1+\dots+a_N$ as

$$P_{N,k}=\sum_{\substack{\boldsymbol{a}\in\{0,1\}^N,\\ |a|=k}}\prod_{m=1}^{N}(p_m)^{1-a_m}(q_m)^{a_m} \tag{1.2.30}$$

and is again a projector, in the sense that $P_{N,k}^2=P_{N,k}=P_{N,k}^*$. As an easy example one might check

$$\begin{aligned}
P_{3,0}&=p_1p_2p_3,\\
P_{3,1}&=p_1p_2q_3+p_1q_2p_3+q_1p_2p_3,\\
P_{3,2}&=p_1q_2q_3+q_1p_2q_3+q_1q_2p_3,\\
P_{3,3}&=q_1q_2q_3.
\end{aligned}$$

Since the sum over all $P_{N,k}$ covers all possibilities, we have the property $\sum_{k=0}^N P_{N,k}=1$.
In order to obtain a full picture of the closeness of our many-body wave function $\Psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$ to the Slater determinant $\bigwedge_{k=1}^N\varphi_k$, we consider the sum over all $P_{N,k}$ weighted with $f(k)$ with respect to the number k of bad particles:

Let $f:\{0,\dots,N\}\to[0,1]$ with $f(0)=0$ and $f(N)=1$ and define

$$\hat{f}\coloneqq\hat{f}^{\varphi_1,\dots,\varphi_N}\coloneqq\sum_{k=0}^{N}f(k)P_{N,k}. \tag{1.2.31}$$

Then we define for any $\Psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$ the *counting functional (or alpha functional) with weight f* as

$$\alpha_f\coloneqq\alpha_f(\Psi,\varphi_1,\dots,\varphi_N)\coloneqq\langle\Psi,\hat{f}\Psi\rangle=\sum_{k=0}^{N}f(k)\langle\Psi,P_{N,k}^{\varphi_1,\dots,\varphi_N}\Psi\rangle \tag{1.2.32}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L^2(\mathbb{R}^{3N})$.

On a heuristic level, the *weight function f* can be seen as k-dependent observable where k is the number of bad particles. The counting functional α_f takes the role of an expectation with respect to the operator-valued counting measure $P_{N,\cdot}$ on the set of antisymmetric wave functions. For the natural choice of $n(k)\coloneqq\frac{k}{N}$, the functional α_n corresponds to the expected relative number of bad particles and counts in this sense the number of bad particles. The freedom of the weights of the sum corresponds to adjusting the sensitivity of counting in regimes with different number of bad particles. The most important choices of weight functions are defined by

$$n^2(k) := \frac{k^2}{N^2}, \tag{1.2.33}$$

$$m(k) := m_\gamma(k) := \min\{1, \frac{k}{N^\gamma}\} \qquad \text{with } \gamma \in (0,1], \tag{1.2.34}$$

$$l(k) := \sqrt{\frac{k}{N}} \tag{1.2.35}$$

for all $k \in \{0, \dots, N\}$ (see Figure 1.2.1 on page 25). We stress that in all applications, a discrete derivative of the weight functions appears. Thus, the heuristic interpretation is that l corresponds to a higher sensitivity and n^2 to a lower sensitivity in the regime of small numbers of bad particles, whereas the cut-off weight m_γ is only sensitive in the regime of $k < N^\gamma$.
In this framework, it is not surprising that the q-operators, as projectors onto the bad particles, can be related to the counting functional, which counts the bad particles, via

$$\langle \Psi, q_1^{\varphi_1, \dots \varphi_N} \Psi \rangle = \alpha_n(\Psi, \varphi_1, \dots \varphi_N) \tag{1.2.36}$$

and that it measures the distance between Ψ and $\bigwedge \varphi := \varphi_1 \wedge \dots \wedge \varphi_N$. More specifically, the following inequality holds (see [PP16, Lemma 3.2.])

$$\|\gamma_\Psi^{(1)} - \gamma_{\wedge\varphi}^{(1)}\|_{\mathrm{tr}}^2 \le 8\alpha_n \left(\Psi, \{\varphi_k\}_{k=1}^N\right) \le 4\|\gamma_\Psi^{(1)} - \gamma_{\wedge\varphi}^{(1)}\|_{\mathrm{tr}} \tag{1.2.37}$$

where $\gamma_\Psi^{(1)}$ and $\gamma_{\wedge\varphi}^{(1)}$ are the reduced one-body density matrices with respect to Ψ and $\varphi_1 \wedge \dots \wedge \varphi_k$ respectively.

The main strategy is now to insert time-dependent states $\Psi_{N,t}$ as solutions of the Schrödinger equation with Hamiltonian (1.0.3) and $\{\varphi_k^t\}_{k=1}^N$ as solutions of the Hartree equation (1.2.9) into the definition of the projectors $p^t = p^{\varphi_1^t, \dots \varphi_N^t}, q^t = 1 - p^t$ and the counting functional

$$\alpha_n(t) := \alpha_n \left(\Psi_{N,t}, \varphi_1^t, \dots, \varphi_N^t\right) = \sum_{k=0}^N f(k) \langle \Psi_{N,t}, P_{N,k}^{\varphi_1^t, \dots, \varphi_N^t} \Psi_{N,t} \rangle \tag{1.2.38}$$

and to show that α_f is bounded by a Grönwall argument. I.e. we want to find an upper bound for the derivative of α_f by α_f itself:

$$\partial_t \alpha_n(t) \le C(\alpha_n(t) + N^{-\delta})$$

for a $C, \delta > 0$.

The derivative of α_f can be decomposed into three distinct transition processes, each associated to changes of the number of bad particles:

$$\partial_t \alpha_n(t) = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) \tag{1.2.39}$$

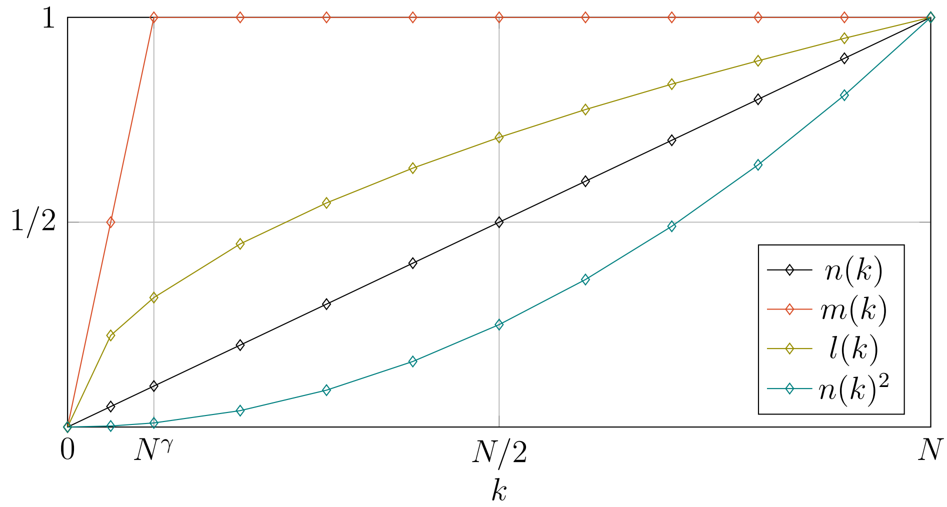

Figure 1.2.1: Plot of the most relevant weight functions.

with

$$\text{(I)} = g_N 2\text{Im}\langle q_1^t \left((N-1)p_2^t W(x_1-x_2)p_2^t - W * \rho^t(x_1)\right) p_1^t\rangle_{\Psi_{N,t}}, \tag{1.2.40}$$

$$\text{(II)} = g_N 2\text{Im}\langle q_1^t q_2^t (N-1)W(x_1-x_2)p_2^t p_1^t\rangle_{\Psi_{N,t}}, \tag{1.2.41}$$

$$\text{(III)} = g_N 2\text{Im}\langle q_1^t q_2^t (N-1)W(x_1-x_2)q_2^t p_1^t\rangle_{\Psi_{N,t}}. \tag{1.2.42}$$

The overarching strategy is to utilize each q-operator to close the Grönwall argument and to evaluate each p-operator with the interaction term. A remarkable outcome of this approach is the emergence of the mean-field interaction in the (I)-term with $p_2^t W(x_1-x_2)p_2^t$ which can be rigorously seen by a diagonalization argument. From the perspective of the microscopic derivation of an effective theory, one may interpret the term $p_2^t W(x_1-x_2)p_2^t$ as determining the choice of the mean-field interactions of the effective theory. Another key aspect of this approach is that the conditions required to close the Grönwall argument reduce to controlling $W*\rho^t$ and $W^2*\rho^t$ which represent the mean and variance[2] of the interaction $\sum_{k=1}^N W(x_k-\cdot)$ with respect to the density $\rho^t = \sum_{k=1}^N |\varphi_k^t|^2$. This nicely supports the heuristics that the mean-field approximation is valid only when the variance of the interaction is small.

We will apply and extend the presented method in Chapter 2 to address the strongly interacting regime. By introducing a suitable gauging phase, the transformed Hamiltonian will exhibit both three-body interaction terms and two-body terms of the form $f_{12}\cdot\nabla_1$ involving differential operators instead of simple multiplication operators. The primary challenge lies

[2]If one defines $X := \sum_{k=1}^N W(x_k - \cdot)$ and denote mean of X as $\mathbb{E}[X] := \langle X\rangle_{\wedge\varphi}$, then a simple calculation yields $\mathbb{E}[X] = W*\rho$ and $\text{Var}[X] = W^2*\rho - \sum_{i,j=1}^N |\langle\varphi_i, W(\cdot - y)\varphi_j\rangle|^2$.

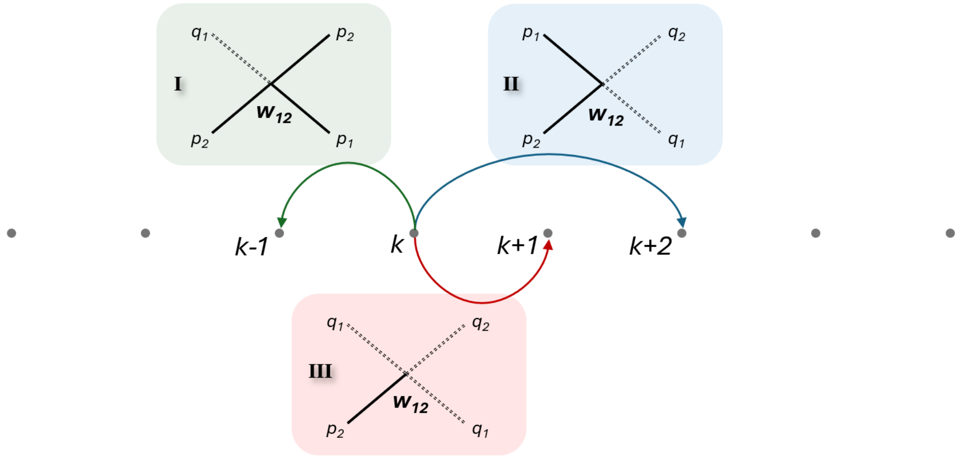

Figure 1.2.2: Exemplary processes which are included by (I), (II) and (III).

in effectively incorporating these differential operators into the framework of our analysis. This requires a generalization of the diagonalization argument for the terms of the form $p_2^t w_{12} p_2^t$ and a careful treatment of the bad kinetic energy, which cannot be directly handled using standard methods. To overcome this, we will develop a strategy that combines the counting functional with a norm approximation relative to an auxiliary Hamiltonian. This auxiliary Hamiltonian will serve as a simplified reference system, allowing us to rigorously control the bad kinetic energy terms.

1.2.3.2 The fluctuation dynamics method

In the following we give a short summary of the closely related fluctuation dynamics method which is embedded in the framework of second quantization and was first established in [BPS14b]

This method tracks excitations relative to the Slater determinant $\bigwedge \varphi^t := \varphi_1^t \wedge \ldots \wedge \varphi_N^t$ of interest by introducing a *fermionic Bogoliubov transformation*. Given the solutions $\{\varphi_k^t\}_{k=1}^N$ of the Hartree equation (2.1.4), set $p^t = \sum_{k=1}^N |\varphi_k^t\rangle\langle\varphi_k^t|$. Define the unitary map $R_{p^t} : \mathcal{F} \to \mathcal{F}$ satisfying

$$R_{p^t}\Omega = a^*(\varphi_1^t)\ldots a^*(\varphi_N^t)\Omega \tag{1.2.43}$$

and for all $g \in L^2(\mathbb{R}^3)$

$$R_{p^t}^* a^*(g) R_{p^t} = a^*\left((1-p^t)g\right) + a(p^t g). \tag{1.2.44}$$

The first condition (1.2.43) can be restated as: R maps the vacuum to the Slater determinant $\bigwedge \varphi^t$. The condition (1.2.44) can be interpreted as a basis transformation in Fock space which maps the creation of a particle with wave function g to the creation of a particle outside the Slater determinant $\bigwedge_{k=1}^{N} \varphi_k^t$, while annihilating a particle with the same wave function g inside the Slater determinant. In this manner, one obtains a Fock space basis that focuses on deviations from $\bigwedge_{k=1}^{N} \varphi_k^t$ while preserving the total number of particles. With a slight abuse of notation, we can translate (1.2.44) to the operator-valued distribution a_x as

$$R_{p^t}^* a_x R_{p^t} = a^* \left((1-p^t)\right) + a(p^t)$$

where $a(p^t) \equiv \sum_{k=1}^{N} a\left(\overline{\varphi_k^t(x)}\varphi_k^t(\cdot)\right)$.

Next, one re-writes the reduced one-body density with respect to the solution $\Psi_{N,t}$ of the Schrödinger equation as

$$\begin{aligned}\gamma^{(1)}_{\Psi_{N,t}}(x_1;x_2) &= \langle e^{-i\mathcal{H}t/\hbar_N} R_{p^0}\Omega, a^*_{x_1} a_{x_2} e^{-i\mathcal{H}t/\hbar_N} R_{p^0}\Omega\rangle \\ &= \langle R_{p^t} \underbrace{R^*_{p^t} e^{-i\mathcal{H}t/\hbar_N} R_{p^0}}_{=:\mathcal{U}(t)} \Omega, a^*_{x_1} a_{x_2} R_{p^t} R^*_{p^t} e^{-i\mathcal{H}t/\hbar_N} R_{p^0}\Omega\rangle \end{aligned} \tag{1.2.45}$$

where the unitary operator $\mathcal{U}(t)$ denotes the *fluctuation dynamics*. By the CAR for

$$R_{p^t}^* a_x^* R_{p^t} R_{p^t}^* a_y R_{p^t} = \left(a^*\left((1-p^t)\right) + a(p^t)\right)\left(a\left((1-p^t)\right) + a^*(p^t)\right) \tag{1.2.46}$$

one obtains

$$\gamma^{(1)}_{\Psi_{N,t}}(x_1;x_2) = p^t(x_1;x_2) + \text{"other terms in normal order"} \tag{1.2.47}$$

which allows to bound for a $C > 0$

$$\mathrm{tr}|\gamma^{(1)}_{\Psi_{N,t}} - p^t| \leq C\langle \mathcal{U}(t)\Omega, (\mathcal{N}+1)\mathcal{U}(t)\Omega\rangle. \tag{1.2.48}$$

The goal is now to control the expected number of fluctuations by a Grönwall argument:

$$i\hbar_N \partial_t \langle \mathcal{U}(t)\Omega, \mathcal{N}\mathcal{U}(t)\Omega\rangle = (1) + (2) + (3) \tag{1.2.49}$$

with the decomposition

$$(1) = \frac{4i}{N}\mathrm{Im}\int \mathrm{d}x\ \mathrm{d}x_2\ V(x_1-x_2)\langle \mathcal{U}(t)\Omega, a^*(p_1^t)a(p_2^t)a(1-p_1^t)a(1-p_2^t)\ \mathcal{U}(t)\Omega\rangle, \tag{1.2.50}$$

$$(2) = \frac{4i}{N}\mathrm{Im}\int \mathrm{d}x\ \mathrm{d}x_2\ V(x_1-x_2)\langle \mathcal{U}(t)\Omega, a(p_1^t)a(p_2^t)a(1-p_1^t)a(1-p_2^t)\ \mathcal{U}(t)\Omega\rangle, \tag{1.2.51}$$

$$(3) = \frac{4i}{N}\mathrm{Im}\int \mathrm{d}x\ \mathrm{d}x_2\ V(x_1-x_2)\langle \mathcal{U}(t)\Omega, a^*(p_1^t)a^*(p_2^t)a^*(1-p_1^t)a(1-p_2^t)\ \mathcal{U}(t)\Omega\rangle. \tag{1.2.52}$$

The most difficult term is typically (2) as it changes the number of excitations the most and therefore is associated with the largest deviations from the Slater determinant. For bounded interactions V one can make use of the Fourier transform and the fermionic Fock space bounds to obtain

$$|(2)| \leq \frac{4}{N} \int \mathrm{d}k \; \hat{V}(k) \| p^t e^{ikx} (1-p^t) \|_{\mathrm{HS}}^2 \| (\mathcal{N}+1)^{1/2} \mathcal{U}(t) \Omega \|^2 \tag{1.2.53}$$

with the desired bound

$$\| p^t e^{ikx} (1-p^t) \|_{\mathrm{HS}}^2 \leq \| [e^{ikx}, p^t] \|_{\mathrm{HS}}^2 \overset{!}{\leq} \hbar_N N \tag{1.2.54}$$

to close Grönwall. In order to obtain the desired bound, one has to make use of the initial conditions satisfying the *semiclassical structure*

$$\| [x, p^0] \|_{\mathrm{tr}} \leq C \hbar_N N, \qquad \| [\hbar_N \nabla, p^0] \|_{\mathrm{tr}} \leq C \hbar N \tag{1.2.55}$$

and propagate these inequalities in time by a separate Grönwall argument.

In [PRSS17] this approach has been extended to cover the Coulomb interaction by using the Fefferman-de la Llave decomposition for singular interactions of the form

$$\frac{1}{|x_1 - x_2|} = \frac{4}{\pi^2} \int_0^\infty \frac{\mathrm{d}r}{r^5} \int \mathrm{d}z \; e^{-\frac{(x-z)^2}{r^2}} e^{-\frac{(y-z)^2}{r^2}}. \tag{1.2.56}$$

However in this case, the preservation of the semiclassical structure for all times is part of the assumptions.

To translate this into the counting functional framework as discussed in [BBP+16], one can establish an equivalence between the expectation value of the number operator with respect to the fluctuation dynamics and the *degree of evaporation*, which serves as a second-quantized analogue of the counting functional. This is expressed as:

$$S_1(\gamma^{(1)}_{\Psi_{N,t}}, p^t) \coloneqq \langle \Psi_{N,t}, d\Gamma(1-p^t) \Psi_{N,t} \rangle. \tag{1.2.57}$$

Consider an initial state of the form

$$\xi_N \equiv R^*_{p^0} \; (0, ..., 0, \underbrace{\Psi_{N,0}}_{n\text{-th entry}}, 0, ...), \tag{1.2.58}$$

then it holds $\mathcal{U}(t)\xi_N = R^*_{p^t} e^{-i\mathcal{H}t/\hbar_N} \Psi_{N,0}$ .Thus, by utilizing the identity

$$R_{p^t} \mathcal{N} R^*_{p^t} = \mathcal{N} - 2\mathrm{d}\Gamma(p^t) + N \tag{1.2.59}$$

with the fact that the Hamiltonian $\mathcal{H}$ preservers the particle number, one finds

$$\begin{aligned} \langle \mathcal{N} \rangle_{\mathcal{U}(t)\xi_N} &= \langle e^{-i\mathcal{H}t/\hbar_N} \Psi_{N,0}, R_{p^t} \mathcal{N} R^*_{p^t} e^{-i\mathcal{H}t/\hbar_N} \Psi_{N,0} \rangle \\ &= 2 \langle \Psi_{N,t}, d\Gamma(1-p^t) \Psi_{N,t} \rangle \\ &= 2 S_1(\gamma^{(1)}_{\Psi_{N,t}}, p^t). \end{aligned} \tag{1.2.60}$$

1.2.4 Related effective mean-field models

Besides the described mean-field-model there are different effective equations of the microscopic many-body system available in specific regimes. We shall give a brief discussion in the following.

1.2.4.1 The Vlasov equation

Let us consider for a moment the classical phase-space density $k_t : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}, (x,p) \mapsto k_t(x,p)$ satisfying $\int_{\mathbb{R}^3\times\mathbb{R}^3} k_t(x,p)\mathrm{d}x\mathrm{d}p = 1$ for all times $t \geq 0$. The **Vlasov equation** governing its evolution is given by

$$\partial_t k + p \cdot \nabla_x k + \left(f * \tilde{k}\right) \cdot \nabla_p k = 0 \tag{1.2.61}$$

where $f = -\nabla W$ denotes the force as gradient of the interaction potential W and $\tilde{k}_t = \int_{\mathbb{R}^3} k_t(x,p)\mathrm{d}p$ denotes the spatial density. The term $p \cdot \nabla_x k$ denotes the free transport term whereas $\left(f * \tilde{k}\right) \cdot \nabla_p k$ denotes the acceleration term by the mean-force $f * \tilde{k}$. Unlike the Boltzmann equation, the Vlasov equation neglects short-range collisions (collisionless approximation) and instead focuses on a mean-field interaction generated by all particles, leading to a nonlinear equation as in the case of the Hartree equation. The Vlasov equation plays a pivotal role in plasma physics, astrophysical dynamics of galaxies and charged-particle modeling.

It can be interpreted as a coarse-grained description of the time evolution of a many-particle system governed by Newtonian dynamics. This has been rigorously established in different classical settings with the mean-field scaling. Early results date back to the 1970s [NW74], with further developments addressing singular potentials [HJ06, HJ15] including Coulomb interactions with small cut-off parameters [LP17, Gra19] and delta-like interactions [GPI18, Fei24, FP24]. However, a mathematical derivation for a classical system with (the full) Coulomb interaction remains an open problem.

It is expected that also in the semiclassical regime the dynamics of interacting fermions governed by the Schrödinger equation (1.0.1) is well approximated by the Vlasov dynamics (1.2.61). For this purpose, one typically considers the *Wigner transform* W_ψ to associate the reduced one-body density with a phase-space pseudo-density. It is defined by $W_\psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ and for all $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$

$$W_\psi(x,p) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}y \; e^{ip\cdot y/\hbar_N} \gamma_\psi^{(1)} \left(x + y/2, x - y/2\right) \tag{1.2.62}$$

with the semiclassical parameter $\hbar_N$ which tends to zero. Conversely, one can also associate a phase-space density k with a reduced one-particle density via the inverse transformation of the Wigner transform which, given by the *Weyl quantization*

$$\omega_k(x, y) = N \int_{\mathbb{R}^3} \mathrm{d}p \; k\left(\frac{x+y}{2}, p\right) e^{ip\cdot(x-y)/\hbar_N}. \tag{1.2.63}$$

The appropriate notion of convergence can then be formulated either in terms of reduced densities using the trace norm or in terms of phase-space densities with respect to a suitable norm.

Early results in this direction were obtained for analytic potentials [NS81] and C^2-potentials [Spo81]. The direct convergence from the many-body system to the Vlasov description was recently established for large system by using the Husimi measure [CLL21]. A related approach, which employs the Hartree equation as an intermediate step, was discussed in Subsection 1.2.3.

1.2.4.2 Hartree-Fock-Bogoliubov equation

While the Hartree-Fock equation (1.2.9) solely involves the reduced one-particle density, the mean-field approach can be extended to account for two-body correlations through the pairing density a^t as defined in (1.1.21). This generalization is achieved by introducing an additional equation for a^t, analog to (1.2.10), yielding the **Hartree-Fock-Bogoliubov equation** (or **Bogoliubov-de Gennes equation**)

$$\begin{cases} i\partial_t p^t &= [h^{\mathrm{HF}}(t), p^t] + \Pi^t(a^t)^* + a^t(\Pi^t)^*, \\ i\partial_t a^t &= h^{\mathrm{HF}}(t)a^t + a^t\overline{h^{\mathrm{HF}}(t)} + \Pi^t(1-\overline{p^t}) - p^t\Pi^t, \end{cases} \tag{1.2.64}$$

where Π^t denotes an integral operator with kernel

$$\Pi^t(x, y) \coloneqq g_N W(x-y)a^t(x, y). \tag{1.2.65}$$

This system is expected to describe the dynamics of attractive fermionic many-body systems, such as electrons in superconductors, by incorporating pairing correlations. Such correlations are crucial for modeling superconducting and superfluid systems.

A mathematically rigorous derivation of this system was recently established in [MPS24] for bounded and regular interaction potentials in the semiclassical regime for macroscopic time scales. Further results include the well-posedness and a derivation from the Dirac-Frenkel principle in [BSS18]. Additionally, connections to Ginzburg-Landau dynamics have been discussed in [FHSS16], while the classical limit of $\hbar_N \to 0$ has been analyzed in [CLS25].

1.2.5 Bosonic results

In the bosonic setting, there are significantly more results regarding effective dynamics compared to the fermionic case. The main reason is that due to the symmetric property of the many-body wave function all particles might occupy the same state, leading to a *Bose-Einstein condensate*. This allows for an effective single-particle approximation that captures the leading-order behavior in many settings.

The mean-field description is analogous to (2.1.4) but instead of a Slater determinant one assumes a symmetric product structure $\Psi_{N,t} \approx (\varphi^t)^{\otimes N}$ where

$$i\partial_t \varphi^t = (-\Delta + g_N W * |\varphi^t|^2)\varphi^t \tag{1.2.66}$$

and φ^t describes a one-body *condensate wave function.*

A rigorous derivation of the mean-field equation 1.2.66 corresponds to showing

$$\lim_{N\to\infty} \operatorname{tr} \left| \gamma^{(1)}_{\Psi_{N,0}} - |\varphi^0\rangle\langle\varphi^0| \right| = 0 \quad \Longrightarrow \quad \lim_{N\to\infty} \operatorname{tr} \left| \gamma^{(1)}_{\Psi_{N,t}} - |\varphi^t\rangle\langle\varphi^t| \right| = 0. \tag{1.2.67}$$

The Hartree equation was first developed in the 1970s. Early findings were made for smooth interaction potentials [Hep74]. Later, these findings were extended to singular potentials [GV79a, GV79b, Spo80]. These early studies showed the statement (1.2.67). Subsequent research has expanded these findings in numerous directions, as comprehensively reviewed in [Lew15]. Notably, many of the fermionic methods, which were discussed in Subsection 1.2.3, also apply to bosonic systems and were in fact first developed in the bosonic analogue (see e.g. [Pic11]).

Moreover, significant attention has been placed to the fluctuations around the condensate, known as *excitations*, allowing to find an effective approximation on the level of the L^2-norm. Bogoliubov provided the first systematic description of these excitations in his seminal work [Bog47] by introducing a quadratic Hamiltonian to describe the excitations. This bosonic Hamiltonian (commonly known as *Bogoliubov Hamiltonian*) can be diagonalized, yielding an effective dispersion relation for the excitations. This relationship between the excitations and their corresponding energy implies that excitations, describable within the Bogoliubov framework, are widely interpreted as *quasi-particles*. The first rigorous results were obtained in [GMM10, GMM11], providing a L^2-norm convergence of the N-body wave function Ψ_t generated by the many-body Hamiltonian (1.0.3) to the solution of the Bogoliubov dynamics Ψ_t^{Bog} generated by the quadratic Bogoliubov Hamiltonian as $N \to \infty$. Further advancements in this field have explored the dynamics of the excitations [LNS15, NN17, MPP19, COS24]. Additionally, recent studies have extended beyond the Bogoliubov description in appropriate regimes [PP19, BPPS20, BPPS22].

1.2.6 Norm approximations by effective dynamical theories

In order to obtain an effective description for the microscopic many-body dynamics generated by (1.0.3) at the level of an L^2-norm approximation, it is essential to take correlations into account that were neglected in the previous subsections on effective mean-field models. The necessity of including correlations arises from the fundamental nature of position-dependent pair interactions. Heuristically, interactions induce correlations between specific pairs of fermions over time, meaning that the assumption of a purely mean-field interaction, where each fermion experiences only an averaged potential, becomes inaccurate. Note that since the norm measures differences at the level of the entire wave function, even the deviation of a single particle from the mean-field evolution can lead to a large norm difference between the microscopic many-body and the effective time evolution. Thus, a more refined effective theory beyond mean-field models is required to accurately approximate the microscopic dynamics in a norm sense.

The key idea comes from a bosonic analogue where correlations can be described by a Bogoliubov transformation. For interacting bosons, the next-order corrections are exactly pair excitations of the symmetric product state. This can be seen in the alpha functional description of the transition terms (1.2.39) as the (III)-term is dominated by the (II)-term (see [MPP19]).

For interacting fermions, however, the situation is more complex. Unlike bosons, single excitations from the antisymmetric product state are not inherently improbable due to the Pauli exclusion principle. Nevertheless, these single-particle excitations may still combine coherently to form collective excitations that are approximately bosonic and can be effectively described by a Bogoliubov transformation.

This perspective has proven to be highly fruitful and has led to the rigorous calculation of the *correlation energy*, a strongly related static problem, on the torus [BNP+19, CHN23b, CHN23a, BNP+21b, BPSS23, FRS24]. More concretely, one calculates

$$E^{\mathrm{cor}}(H_N) := \inf \sigma(H_N) - \langle \psi^{\mathrm{Slater}}, H_N, \psi^{\mathrm{Slater}} \rangle \tag{1.2.68}$$

where H_N denotes the microscopic Hamiltonian (1.0.3) with an appropriate choice of g_N and ψ^{Slater} is the Slater determinant minimizing the Hartree-Fock energy functional 1.2.4. An expression for 1.2.68 was already predicted by Bohm and Pines [BP53] using the *random phase approximation* (RPA) and later on by Gell-Mann and Brueckner [GMB57] whose studies paved the study of so-called *plasmons*, the quasi-particles describing collective excitations of an interacting Fermi gas.

The main idea is to consider the momentum generated by an interaction along with the hole left in the Fermi ball as a unified entity known as a *particle hole pair*. To rigorously describe the excitations of the Fermi gas one can utilize individual particle hole-pairs on the torus

$$b_{k,p} = a_p^* a_{p-k}, \qquad p \in B_{\mathrm{F}}^{\mathrm{c}}, k \in \mathbb{Z}^3, \tag{1.2.69}$$

which are on average almost bosonic [CHN23b, CHN23a] or collective particle-hole pairs patched over the Fermi ball surface

$$b^*_{\alpha,k} = \frac{1}{n_{\alpha,k}} \sum_{p \in B_{\mathrm{F}}^c \cap (B_{\mathrm{F}}+k) \cap B_\alpha} a^*_p a_{p-k}, \qquad k \in \mathbb{Z}^3, \tag{1.2.70}$$

with patch B_α and normalization $n_{\alpha,k}$ which are almost bosonic themselves, i.e. the operators satisfy the bosonic commutation relation up to a small error [BNP$^+$19, BNP$^+$21b, BPSS23].

Based on this concept, a recent study [BNP$^+$21a] established a norm approximation for the many-body dynamics in the semiclassical regime on $\Lambda = \mathbb{R}^3/(2\pi\mathbb{Z}^3)$, considering interactions with compact support in momentum space and initial states of the form:

$$\xi_0 = \frac{1}{Z_m} c^*(\eta_1) \cdots c^*(\eta_m)\Omega. \tag{1.2.71}$$

Here c^* and c denote almost-bosonic creation and annihilation operators, respectively, η represents a one-body bosonic wave function, and Ω the fermionic vacuum state. These operators, c^* and c, are identical to those employed in [BNP$^+$19, BNP$^+$21b, BPSS23] to describe collective particle-hole excitations and are explicitly constructed from the fermionic creation and annihilation operators.

In the main theorem of [BNP$^+$21a], the time evolution operator is conjugated with two unitary maps: one to track excitations relative to the filled Fermi ball via a particle-hole transformation R, and another to diagonalize the Hamiltonian using an almost-bosonic Bogoliubov transformation T. This results in the inequality

$$\| T^* R^* e^{-H_N t/\hbar_N} R T \xi_0 - e^{-(E_N^{\mathrm{HF}} + E_N^{\mathrm{RPA}})t/\hbar_N} \xi_t \| \leq C_{\xi_0} \hbar_N^{1/15} |t| \tag{1.2.72}$$

where $\hbar_N = N^{-1/3}$ is the semiclassical parameter from 1.2.17 and $E_N^{\mathrm{HF}}, E_N^{\mathrm{RPA}}$ are the energy constants from the Hartree-Fock and random phase approximation theory. Since it holds

$$\xi_t = \frac{1}{Z_m} c^*(\eta_1^t) \cdots c^*(\eta_m^t)\Omega \tag{1.2.73}$$

with each η satisfying $i\hbar_N \eta_k^t = H_{\mathrm{B}} \eta_k^t$, the initially assumed product structure of the almost-bosonic excitations is effectively preserved in the limit $N \to \infty$.

In Chapter 3, we will adapt this idea to incorporate almost-bosonic excitations of the Fermi gas. However, instead of initially assuming such excitations, we will demonstrate that they are formed through interactions with an impurity particle.

1.3 Probing the ideal Fermi gas

In this section, we will give an introduction to the dynamics of an impurity particle immersed in a Fermi gas. The concepts and literature discussed here will serve as the foundation for Chapter 3.

To build intuition, we first examine the case of non-interacting fermions, commonly referred to as the *ideal Fermi gas*, instead of the full Hamiltonian (1.0.4). Despite the absence of interactions, this system already exhibits interesting quantum features arising from the Pauli exclusion principle and the structure of the Fermi sea. Understanding these fundamental properties is essential before incorporating impurity interactions, which lead to more complex phenomena such as polaron formation and collective excitations.

The free Hamiltonian of an ideal Fermi gas is relatively simple to solve, as it consists solely of the Laplacian operator describing the kinetic energy of non-interacting fermions

$$H_0 \coloneqq \sum_{i=1}^{N} (-\Delta_{x_i}). \tag{1.3.1}$$

To make use of a discrete spectrum with proper eigenfunctions, we restrict the spatial domain to a finite region Λ and impose appropriate boundary conditions. A natural choice is a cubic box of site length $L > 0$ with periodic boundary conditions, i.e. $\Lambda \equiv \mathbb{R}^3/(L\mathbb{Z}^3)$. This choice simplifies the analysis, as it allows for a straightforward expansion of the eigenfunctions in plane waves $\{f_k\}_k$ with

$$f_k(x) \coloneqq \frac{e^{ikx}}{L^{3/2}} \in L^2(\Lambda) \qquad \text{with momenta } k \in (2\pi/L)\mathbb{Z}^3. \tag{1.3.2}$$

Note that with this spatial restriction, we have a *dense* Fermi gas in the limit of $N \to \infty$. Due to the antisymmetry of the wave function Ψ_N, each fermion must occupy a distinct quantum state since otherwise

$$\Psi_N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = -\Psi_N(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N) = 0. \tag{1.3.3}$$

The ground state Ω_0 of the system is then constructed by filling up all the N available single-particle momentum states up to a characteristic momentum, known as *Fermi momentum* k_F:

$$\Omega_0 \coloneqq \bigwedge_{k\in B_\mathrm{F}} f_k, \qquad B_\mathrm{F} \coloneqq \left\{k \in (2\pi/L)\mathbb{Z}^3 : |k| \le k_\mathrm{F}\right\}, \tag{1.3.4}$$

$$N \equiv N(k_\mathrm{F}, L) = |B_\mathrm{F}|. \tag{1.3.5}$$

This results in a *(filled) Fermi ball* Ω_0 in momentum space (or a *Fermi sea*) which defines the fundamental structure of the ideal Fermi gas. Note that in second quantization we can also write $\Omega_0 = \prod_{k\in B_\mathrm{F}} a_k^*\Omega$ where Ω denotes the vacuum state. The ground state energy is then given by

$$E_N(k_\mathrm{F}, L) = \langle \Omega_0, H_0\Omega_0\rangle = \sum_{k\in B_\mathrm{F}} |k|^2. \tag{1.3.6}$$

Note that for finite $L > 0$ the number of particles $N \equiv N(k_F, L) = |B_F|$ described in this framework cannot be any integer due to the discreteness of the momenta on the lattice $(2\pi/L)\mathbb{Z}^3$. This can be easily seen for $L = 2\pi$. As k_F increases, $N = N(k_F, 2\pi)$ increases only when k_F crosses radii where new lattice points enter the sphere. These jumps occur at radii $\sqrt{r}$ which can be written as sum of three squares. The number of integer points at radius $\sqrt{r}$, denoted by $r_3(r)$, counts how many possibilities r can be written as sum of three squares.[3] It then holds

$$N = N(k_F, 2\pi) = \sum_{r=0}^{\lfloor k_F^2 \rfloor} r_3(r), \tag{1.3.7}$$

i.e. N can only take values from the cumulative sum not arbitrary integers.

Via Gauss' counting argument, one can relate the number of particles with the momentum as

$$k_F = \left(\frac{6\pi^2}{L^3}\right)^{1/3} N^{\frac{1}{3}} + \mathcal{O}(1) \tag{1.3.8}$$

and therefore it holds for the density

$$\rho = \frac{N(k_F, L)}{L^3} = \frac{1}{6\pi^2} k_F^3 + o(1). \tag{1.3.9}$$

It is therefore convenient to parameterize the system with the Fermi momentum k_F, which scales with the density of the fermionic system.

1.3.1 Free time evolution of an impurity particle

In a high-density regime, the Fermi momentum is large, leading to a bulk-dominated Fermi ball with a relatively small surface-to-volume ratio. To assess the impact of interactions, we consider (1.0.4) with $g_N \equiv 0$, i.e.

$$H_N = \frac{-\Delta_y}{2m_{\mathrm{imp}}} + \sum_{i=1}^{N} (-\Delta_{x_i}) + \lambda_N \sum_{i=1}^{N} V(x_i - y) \tag{1.3.10}$$

and assume that the impurity-fermion interaction is compactly supported in momentum space, i.e. $\hat{V}$ is compactly supported.[4]

Under the assumption, that the effect of the interaction can be understood as shifts in momentum space, fermions deep inside the Fermi ball cannot undergo momentum shifts

[3] By Legendre's three-square theorem, $r_3(r) = 0$ for $r = 4^k(8n + 7)$ where $n, k \in \mathbb{N}$, meaning no points are added at these radii.

[4] By the Paley-Wiener theorem it holds that V is rapidly decreasing in position space.

because all momenta nearby are already occupied. Only fermions near the surface of the Fermi ball have accessible unoccupied states to scatter into. This implies that interactions predominantly affect particles at the Fermi surface, while the bulk remains largely unaffected. Since only a small fraction of fermions (those near the surface) can participate in interaction-driven momentum shifts, the overall response of the Fermi sea is heavily constrained. This suggests that the Fermi ball remains rigid in the presence of weak impurity interactions.

This heuristic argument indicates that at high densities, an impurity is unlikely to significantly change the bulk structure of the Fermi sea. This phenomenon of *free decoupled time evolution* of an impurity particle will be explored in more detail in the following.

1.3.1.1 Rigorous results and open questions

The first rigorous result in this context was established in [JMPP17] which provided a L^2-norm approximation between a free decoupled time evolution and the microscopic many-body evolution in dimension $d = 2$ in the thermodynamic limit of $N, L \to \infty$ while keeping k_F constant. More specifically, assuming an initial state of the form $\Psi_{N,0} = \xi_0 \otimes \Omega_0$, where Ω_0 denotes the filled Fermi ball and ξ_0 is sufficiently regular, and evolving the state with respect to (1.3.10) with coupling $\lambda_N = 1$, they established for all $t \geq 0$:

$$\lim_{L\to\infty} \|e^{-iH_N t}(\xi_0 \otimes \Omega_0) - e^{-iE(k_\mathrm{F},L)t} e^{i\Delta_y t/(2m_\mathrm{imp})} \xi_0 \otimes e^{-iH_0 t}\Omega_0\| \leq C(1+t)k_\mathrm{F}^{-b} \tag{1.3.11}$$

where $b < 1/2$. This result demonstrates that the effective Hamiltonian governing the system takes a decoupled form

$$H^\mathrm{eff} = \frac{-\Delta_y}{2m_\mathrm{imp}} + E_N + E_V \tag{1.3.12}$$

which acts separately on each tensor factor of the Hilbert space. The constants $E_V := \frac{N}{L}\hat{V}(0) = \frac{N}{L}\int_\Lambda \mathrm{d}x\, V(x)$ and $E_N := E_N(k_\mathrm{F}, L)$ coinciding with (1.3.6) capture the leading-order effect of the medium. Importantly, this approximation remains valid for macroscopic times $t \ll k_\mathrm{F}^b$. Furthermore, as k_F increases—corresponding to a higher particle density—the approximation improves.

This result can be interpreted as a form of mean-field approximation, where the interaction between the impurity and the Fermi gas can be effectively replaced by a constant energy shift. The underlying mechanism relies on the suppression of fluctuations: in a large Fermi ball, momentum transfers induced by interactions are close to the surface of the Fermi ball and are therefore momentum differences are typically large, leading to destructive interference of excitations. Consequently, the initial state remains stable under perturbations, as significant excitations are suppressed.

This behavior differs fundamentally from bosonic and classical systems, where no Fermi ball structure exists. In those cases, the mean free path of an impurity decreases with increasing density, leading to frequent scattering events of the impurity with the medium [DFPP14]. In contrast, in a fermionic system, the Pauli exclusion principle prevents such frequent scattering, preserving the motion of the impurity.

The result was extended in [JMP18] to the case of two impurity particles, revealing an additional effective interaction mediated by the Fermi gas. This interaction is a nontrivial consequence of the underlying fermionic structure. Further generalizations were carried out in [MP21], where the analysis was extended to include dimension $d = 3$ to long-range interaction potentials and multiple impurity particles, demonstrating that the Fermi gas can mediate an effective pair interaction potential among the impurities. A crucial distinction arises in $d = 3$, where the previous result does not hold under the same interaction strength. Instead, a weaker coupling regime is required, specifically $\lambda_N = k_{\mathrm{F}}^{-1/2} \in \mathcal{O}(N^{-1/6})$. Alternatively, one can retain $\lambda_N = 1$ in $d = 3$ but then result holds only for short time of order $k_{\mathrm{F}}^{-1-\delta}$ for $\delta > 0$. However, this restriction is significant: on such short time scales, the fermions in the gas remain almost static, since the fastest particles in the filled Fermi ball have momenta on the order of k_{F}, implying that their characteristic timescale is at least k_{F}^{-1}.

The mentioned results leave several important questions open, including:

1. Can the effective decoupling persist in $d = 3$ under stronger coupling parameters λ_N? Alternatively, can one extend the validity of the approximation to longer timescales beyond $k_{\mathrm{F}}^{-1-\delta}$?

2. The current models assume a non-interacting Fermi gas. How does the introduction of weak or strong interactions among the fermions affect the effective description? In particular, can one still justify the mean-field-like approximation, or do interaction-induced correlations qualitatively alter the impurity dynamics?

3. In the present setting, the impurity remains effectively free in leading order. However, does the impurity eventually lose energy due to residual interactions with the Fermi gas? If so, how does this dissipation depend on the interaction strength and the density of the gas?

We will give partial answers to the questions in our work in Chapter 2. Moreover, we will give a brief explanation of the polaron concept in Subsection 1.3.2. The method used in the discussed works for $d = 3$ will be reviewed briefly in the following section.

1.3.1.2 Perturbative resolvent method

The method is based on the fact that the kinetic energy transfer of fermions dominates over the potential energy transfer arising from interactions. This observation is leveraged by constructing a perturbation series that capitalizes on the significant kinetic energy transfer.

Specifically, when an interaction excites fermions near the Fermi surface, the kinetic energy transfer becomes substantial as part of the system's time evolution post-interaction. This results in high phase oscillations over time for the excited fermions. Consequently, the time integral over these excited states is dampened due to destructive interference, as the large phase factor effectively reduces their contributions after integration. As a result, the contributions from these excitations become sub-leading.

Instead of rigorously presenting every calculation, we will give a heuristic explanation of the results of the key estimates. In order to treat this mathematically, one considers the Hamiltonian (1.3.10) in a second quantization

$$\mathbb{H} = \underbrace{-\frac{\Delta_y}{2m_{\text{imp}}}}_{=:h_0} + \underbrace{\sum_{k\in(2\pi/L)\mathbb{Z}^3} |k|^2 a_k^* a_k}_{=:\mathbb{T}} + \underbrace{\lambda_N \sum_{k,p\in(2\pi/L)\mathbb{Z}^3} \hat{V}(k) e^{iky} a_p^* a_{p-k}}_{=:\mathbb{V}} \tag{1.3.13}$$

with $\hat{V} \geq 0$ compactly supported and the initial state of a product between Fermi sea Ω_0 and a general tracer wave function ξ_0 with bounded kinetic energy. The difference of the time evolutions can be written by Duhamel's formula as

$$\begin{aligned} &\| \left(e^{-i\mathbb{H}t} - e^{-iH^{\text{eff}}t}\right) \xi_0 \otimes \Omega_0 \| \\ &= \| \int_0^t \mathrm{d}s \; e^{i\left(\mathbb{T}-E_N+\mathbb{V}_{\neq 0}+h_0\right)s} \mathbb{V} e^{-ih_0 s} \left(\xi_0 \otimes \Omega_0\right) \| \end{aligned} \tag{1.3.14}$$

with $\mathbb{V}_{\neq 0} \coloneqq \mathbb{V} - E_V = \mathbb{V} - \frac{N}{L}\hat{V}(0)$. The idea is now to introduce a resolvent operator $R \coloneqq (\mathbb{T} - E_N)^{-1}$, which is a bounded operator on the subspace $\{\Omega_0\}^\perp \subset L^2_{\text{as}}(\Lambda^N)$, by the identity

$$e^{-iR^{-1}s} \left(\frac{\mathrm{d}}{\mathrm{d}s} e^{iR^{-1}s}\right) R = \text{id}. \tag{1.3.15}$$

and integrate by parts yielding

$$\begin{aligned} &\| \left(e^{-i\mathbb{H}t} - e^{-iH^{\text{eff}}t}\right) \xi_0 \otimes \Omega_0 \| \\ &= \| \int_0^t \mathrm{d}s \; e^{i\left(\mathbb{T}-E_N+\mathbb{V}_{\neq 0}+h_0\right)s} \mathbb{V}_{\neq 0} R \mathbb{V} e^{-ih_0 s} \left(\xi_0 \otimes \Omega_0\right) + \text{"boundary terms"} \|. \end{aligned} \tag{1.3.16}$$

By inserting $P_{\Omega_0} = 1 \otimes |\Omega_0\rangle\langle\Omega_0|$ and $Q_{\Omega_0} = \text{id} - P_0$, we can distinguish between *bubble processes*, which return the system to its initial state and therefore effectively leaving the Fermi ball invariant, and *excitation processes*, where we can again insert the resolvent and integrate by parts. In this way, the bubble processes contribute as expectation values with respect to Ω_0 and the excitation process can be estimated by exploiting the resolvent: One uses that supp$\hat{V} \subset B_{r_0}(0)$, the fact that h_0 only acts on the impurity space and the orthogonality property

$$\langle \prod_{k\in S} a_k^* \Omega_0, \prod_{k'\in S'} a_{k'}^* \Omega_0 \rangle = 0 \iff S' = S \tag{1.3.17}$$

to bound

$$\|\mathbb{V} e^{-ih_0 s} (\xi_0 \otimes \Omega_0) \|^2 \leq C r_0 \lambda_N^2 k_{\mathrm{F}}^2. \tag{1.3.18}$$

Since $\mathbb{V}$ can be interpreted as momentum shift, $C r_0 k_{\mathrm{F}}^2$ corresponds to the volume of admissible momenta, which is located in the Fermi ball surface due to the Pauli exclusion principle and the filled Fermi ball structure of Ω_0. The resolvent can be evaluated by utilizing $(\mathbb{T} - E_N) a_p^* a_{p-k} \Omega_0 = (p^2 - (p-k)^2) a_p^* a_{p-k} \Omega_0$ where $k \in \mathrm{supp}\hat{V}$ and $p \in (2\pi/L)\mathbb{Z}^3$ is close to the Fermi surface. Hence, the kinetic energy difference is of order k_{F} and the resolvent is small. One obtains for higher order terms the bounds

$$\|Q_{\Omega_0} \mathbb{V}_{\neq 0} R \mathbb{V} e^{-ih_0 s} (\xi_0 \otimes \Omega_0) \| \leq C r_0 \lambda_N^2 k_{\mathrm{F}}^{1+\delta}, \tag{1.3.19}$$

$$\|\mathbb{V}_{\neq 0} R Q_{\Omega_0} \mathbb{V}_{\neq 0} R \mathbb{V} e^{-ih_0 s} (\xi_0 \otimes \Omega_0) \| \leq C r_0^{3/2} \lambda_N^3 k_{\mathrm{F}}^{1+\delta} \tag{1.3.20}$$

with a $\delta > 0$. This heuristic explanation can be made rigorous by employing Wick's rule to evaluate the creation and annihilation operators, along with a careful analysis of all possible excitation histories. The bounds are derived by using the limit $L \to \infty$ to convert the sums to integrals. Due to the integration of singular terms arising from the resolvent a logarithmic factor of k_{F}^{δ} has to be taken into account.

For the choice of $\lambda_N = k_{\mathrm{F}}^{-1/2}$ the perturbation series can be truncated after two iterations. The leading error is given by a boundary term after the first iteration, of the form

$$\|R \mathbb{V} e^{-ih_0 s} (\xi_0 \otimes \Omega_0) \| \leq C \lambda_N \leq C k_{\mathrm{F}}^{-1/2+\delta}. \tag{1.3.21}$$

Alternatively, for the choice of $\lambda_N = 1$ one can require short times of order $k_{\mathrm{F}}^{-1-2\delta}$ in (1.3.16) to obtain a meaningful error bound. Applying this method to the two-dimensional case, reveals that $\lambda_N = 1$ can be treated for macroscopic times since the volume of admissible momenta for a momentum shift is only of order $C r_0 k_{\mathrm{F}}$.

The method can be generalized to situations where the Hamiltonian is of the form $H = A \otimes \mathrm{id} + \mathrm{id} \otimes B + \lambda V$ and initial state is given of the form $\Psi_{N,0} = \phi \otimes \psi$, with ψ being an eigenstate of A, i.e. $A\psi = a\psi$. In addition, if $\|(\mathrm{id} \otimes B + \lambda V)\Psi_{N,0}\| \ll \|(A \otimes \mathrm{id})\Psi_{N,0}\|$ in the sense that the right-hand side if of larger order in the order parameter, then one can exploit the destructive interference, as $(\mathrm{id} \otimes B + \lambda V)$ can be treated as a perturbation.

Recent applications of this approach include the derivation of an effective interaction between multiple impurity particles mediated by a dense Fermi gas [MP21], radiative corrections to the dynamics of an impurity particle interacting with a bosonic scalar field [CM24a], the derivation of the two-body Coulomb potential from the renormalized Nelson model [CM24b] and the stability analysis of Bose-Fermi mixtures [CMMP25].

In principle, one expects the result to hold for $\lambda_N = k_{\mathrm{F}}^{-\beta}$ for any $\beta > 0$ and error bounds $k_{\mathrm{F}}^{-\beta/2+\delta}$ for times of order 1, as the perturbation series can be extended to higher orders. However, the number of distinct excitation processes grows exponentially with the order of the series, making estimates in the current form highly tedious.

Additionally, one anticipates that weak internal interactions can be incorporated to generalize this result for Hamiltonians of the form given in (1.0.4) where g_N, the coupling parameter of the gas interaction, must be chosen appropriately. One expects $g_N = k_{\mathrm{F}}^{-\frac{4}{3}-\delta}$ to be sufficient such that the leading error is small. This can be understood as follows: an internal interaction W would correspond to the second quantized two-body operator $\mathbb{W}$ with four creation and annihilation operators. Hence, $\mathbb{W}$ can be thought as shifting two momenta simultaneously associated with square of the volume of admissible momenta. Adapting the heuristic explanation for (1.3.16), one expects the leading error to be given by

$$\|Q_{\Omega_0}\mathbb{W}_{\neq 0}RQ_{\Omega_0}\mathbb{W}_{\neq 0}R\mathbb{W}e^{-ih_0 s}\left(\xi_0 \otimes \Omega_0\right)\| \leq Cr_0^3 g_N^3 k_{\mathrm{F}}^{4+\delta} \leq Cr_0^3 k_{\mathrm{F}}^{-2\delta}. \tag{1.3.22}$$

1.3.2 Polaron dynamics of an impurity particle

The concept of a *polaron* was originally introduced by Lev Landau to study the motion of an electron in a dielectric crystal [Lan65]. His idea was that the impurity is "dressed" by excitations of the medium, forming a new quasi-particle with renormalized properties (e.g., effective mass and mobility). This later led to the well-known *Fröhlich Hamiltonian* which describes electron-phonon interactions:

$$\mathbb{H}^{\mathrm{F}} = (-\Delta_y) \otimes 1 + 1 \otimes \mathbb{D}_{\mathrm{B}} + \Phi(h_y) \tag{1.3.23}$$

with $(-\Delta_y)$ describing the kinetic energy of the impurity particle, $\mathbb{D}_{\mathrm{B}}$ describing the kinetic energy of the excitation field, and $\Phi(h_y) := c^*(h_y) + c(h_y)$ the linear coupling between impurity particle and bosonic excitations. In the case of the original Fröhlich Hamiltonian the dispersion relation for the excitation field is assumed to be constant to model acoustic phonons as lattice excitations of a crystal [Frö54].

Subsequently, the polaron concept was extended to all kind of surrounding media including Bose and Fermi gases with the overarching feature that the impurity couples linearly to a bosonic background field (e.g. density excitations in a Bose-Einstein condensate) describing excitations of the medium. In the case of a Fermi polaron, a natural analog exists where the impurity couples to particle-hole excitations of a Fermi sea. Central questions in polaron physics concern the formation conditions and stability conditions of this quasi-particle, i.e. whether the impurity retains its individuality or becomes strongly correlated with the surrounding medium.

The first rigorous derivations of a polaron Hamiltonian from the underlying many-body Hamiltonian were carried out for the Bose polaron [MS20, LP22]. In this context, stability properties have been established [HL24] and the excitation spectrum has been analyzed [LT25].

Extensive work has been devoted to studying the Fröhlich polaron as an effective model (see, for example, early works [LY58, Spo88] and a comprehensive review [Sei20]). However, a rigorous derivation of this model from a microscopic many-body Hamiltonian remains an open problem. Starting from the Fröhlich polaron model, it has been shown in the strong-coupling regime that the impurity dynamics are governed by the so-called *Landau-Pekar equation*, which describes the macroscopic evolution of the impurity coupled to a classical polarization field [FZ17, LMR⁺21, LRSS21, LMS21]. These equations model the situation in which there is a separation of time scales between a fast moving impurity particle (usually associated to an electron) and a slow varying bosonic field with constant dispersion relation.

For the fermionic setting, much less is known. The static case has been explored in two dimensions and in the dilute regime with an attractive contact interactions [GL19, LM19] where rigorous upper bounds have been established for the ground-state energy of the many-body Hamiltonian, aligning with the polaron energy predictions from the physics literature. To our best knowledge, there are no general dynamical results in this setting where one starts from the many-body dynamics generated by (1.3.10) and derive an effective polaron dynamics described by (3.1.1). We will address this gap in Chapter 3, where we demonstrate that polaron dynamics can be derived for the described high-density case.

1.3.3 Experiments

Over the past two decades, advancements in ultracold atom experiments [SWSZ09, KPV⁺12, NSF⁺20, FBD⁺21, BHF⁺23] and ultrafast spectroscopy techniques [CJL⁺16] have given access to isolated fermionic systems, allowing for detailed investigations of quantum many-body phenomena in the laboratory. In the following, we give a non-technical overview of common experimental set-ups, observables and measurements. In Section 3.8, we will demonstrate how to rigorously calculate such an observable.

1.3.3.1 Experimental set-ups

In ultracold atom experiments, ^{6}Li atoms are commonly used to model Fermi gases due to their favorable properties: As one of the lightest fermionic alkali atoms, ^{6}Li has a relatively high *Fermi temperature* $T_F \sim \frac{1}{m}$, that is the temperature at which thermal effects become significant compared to the quantum effects. Additionally, ^{6}Li has two accessible hyperfine states which possess high tunability properties. Typical experiments involve approximately 10^5 to 10^6 fermionic atoms confined in an optical dipole trap with cylindrical symmetry of waist of $\sim 10^2 \mu$m . The gas is cooled to temperatures on the order of nanokelvins. To ensure a well-defined fermionic system, the ^{6}Li atoms are typically prepared in their lowest hyperfine state, which prevents unwanted spin-changing collisions and allows for precise control of interactions. The fermions in such experiments are usually assumed to be approximately free and to exhibit a Fermi ball structure with Fermi momentum k_{F}.

As impurity particles, several options are available: A common choice is to use ^{6}Li atoms in a different spin state than the majority species, effectively creating a *spin impurity* within the Fermi gas. In addition, heavy ^{40}K atoms are frequently used as impurities, allowing for a *mass-imbalanced impurity*, which introduces additional complexity into the polaron problem. Other choices include ^{41}K, ^{173}Yb or ^{87}Rb impurities, which enable studies of mixed Fermi-Bose systems. In all cases, the impurities are immersed with a much lower concentration in the gas.

The interaction between the impurity and the surrounding Fermi gas particles can be tuned via the *magnetic Feshbach resonance*. This is achieved by applying an external magnetic field that shifts the energy levels of the ^{6}Li atoms due to the Zeeman effect: A Feshbach resonance occurs when the energy of a bound molecular state (a two-body bound state of the impurity and a Fermi gas particle) becomes equal to the energy of the scattering continuum (the energy of the unbound, free particles). At this point, the *scattering length a* diverges, indicating that the interaction strength between the impurities and the gas becomes maximally strong. By adjusting the external magnetic field, the interaction can be tuned from attractive to repulsive and from small values $|a| \ll k_{\mathrm{F}}^{-1}$ (weakly interacting regime) to arbitrarily large values $|a| \gg k_{\mathrm{F}}^{-1}$ (strongly interacting or unitary regime). We remark that while the interaction radius of the Feshbach resonance is typically much smaller than the inter-particle distance, the strong-interaction regime is characterized by a large scattering length that can significantly exceed the inter-particle distance. On the theoretical side, this scenario is often modeled using a delta potential with an ultraviolet cut-off in momentum space to align with the scattering length [CM10, KSN+12].

1.3.3.2 Measurements and theoretical interpretations

In the context of measurements, a radio-frequency (rf) pulse is typically applied to transfer impurities from an initial spin state to a final spin state [SWSZ09]. This transition occurs at a frequency determined by the energy difference between the two states, which is influenced by the interactions between the impurity and the surrounding Fermi gas. In the experiment, the rf frequency ω is swept, and the number of impurities transferred to the final state is recorded. This produces an rf absorption spectrum $I(\omega)$, which reveals how the impurity responds at different frequencies. The absorption spectrum provides crucial insights into the quasi-particle formation:
For weak interactions, the spectrum exhibits a single, sharp peak, indicating that the impurity retains its individuality with minimal dressing by the surrounding medium. As the interaction strength increases, a broader background appears in the spectrum (usually referred to as *quasi-particle residue*), reflecting enhanced dressing effects as the impurity becomes more

correlated with its environment. In the strong-interaction regime, the impurity binds to a fermion from the medium and forms a bound molecule rather than a well-defined polaron. This leads to a distinct spectral shift and the emergence of a separate molecular peak in the absorption spectrum. The transition between the two states is observed to be smooth instead of discontinuous [NSF+20]. It is observed that also under repulsive interactions the polaron forms, but it is metastable with finite lifetime and eventually decays [KPV+12, SVM+17].

A precise time-resolved measurements of dynamical quantities and the formation process of polarons is possible using *Ramsey interferometry*.
In this method, impurity particles are first prepared in a specific spin state. A $\pi/2$-*pulse*, which is a rf pulse that rotates the spin by 90 degrees, is then applied to bring the impurity into a superposition of two spin states. The two branches of this superposition evolve differently: One branch remains non-interacting with the surrounding Fermi gas and evolves freely, whereas the other branch interacts with the medium via the Feshbach resonance with a tunable interaction strength. After a controlled evolution period $t > 0$, during which the interaction is present, the impurity's interaction is suddenly turned off and a second $\pi/2$ pulse is then applied, bringing the two spin states back to interference. The overlap between the two wave functions of the corresponding branches is measured via the *contrast* (also referred to as *Loschmidt echo*)

$$S(t) = \langle e^{-iH^{\text{free}}t}\psi_0, e^{-iH^{\text{int}}t}\psi_0\rangle = |S(t)|e^{-i\phi(t)}, \tag{1.3.24}$$

which quantifies the degree of coherence of the two branches. If $|S(t)| \approx 1$ then the impurity remained mostly unperturbed, whereas if $|S(t)| < 1$ suggests a polaron formation, as the interacting branch of the superposition experiences significant modifications due to the surrounding Fermi gas. In Section 3.8, we will find a rigorous approximation for the contrast (1.3.24), revealing the characteristic features described below.

For the repulsive case, the measured signal $S(t)$ exhibits distinctive features that offer insights into the underlying quasi-particle dynamics [CJL+16]: Initially, $|S(t)|$ follows a parabolic transient phase, which is followed by a power-law decay. The phase $\phi(t)$, on the other hand, exhibits a linear increase over time. As the interaction strength increases, pronounced oscillations appear in $|S(t)|$, which can even decay to almost zero before recovering. For longer timescales, these oscillations become damped.

These observations can be interpreted as follows [CJL+16, KSN+12]:

- The parabolic transient phase is a well-documented feature in complex quantum systems [JP01] and provides direct insight into the ultrafast real-time formation of the quasi-particle.
- The power-law decay is a characteristic feature of the so-called *Anderson's orthogonality catastrophe*: Turning on an interaction between the Fermi gas and an impurity particle

at $t = 0$ can be interpreted as perturbing the non-interacting Fermi gas. The power-law decay indicates two key aspects: first, that the perturbed system becomes orthogonal to the free system, and second, that the perturbed system retains a strong memory of the initial perturbation, contributing to a scale-invariant response.

- The oscillatory behavior in $|S(t)|$ can be understood as a collective response of the Fermi gas, where coherent interactions between the impurity and the medium induce transient many-body effects.
- The linear evolution of the phase $\phi(t)$ corresponds to the energy shift of the quasi-particle state, which arises due to interactions with the surrounding medium.

The lifetime of a polaron is typically associated with the width of its spectral peak, in accordance with Heisenberg's uncertainty relation between energy and time. This lifetime is often considered crucial in understanding pairing instabilities in repulsively interacting Fermi gases, as discussed in [CZ10, SBC+16, ALS+20]. Furthermore, it plays a significant role in the approach to observing the Stoner transition in itinerant ferromagnetism, as explored in reference [MZB14]. We remark that the Fermi polaron setting has even been extended beyond cold atoms, which is realized in charge-tunable atomically thin semiconductors [SBC+16].

Chapter 2

Derivation of the Time-Dependent Hartree Equations for Strongly Interacting Fermions

This chapter presents results that form the basis for the collaborative preprint [HMP25] with my supervisor Peter Pickl and David Mitrouskas. The article focuses on the case of a dense Fermi gas with regular interactions, while this chapter addresses the more technically demanding setting of singular interactions and extended gases. The key techniques and strategy were first developed during my thesis research and parts of the structure and exposition, especially in the overlapping technical sections, were subsequently adapted from my thesis manuscript for the dense case in collaboration with the co-authors. Conversely, some formulations and structural elements from the preprint are also reflected in this chapter.

I contributed substantially to the conceptual development and all central parts of the argument. In particular, I am grateful for the valuable collaboration with David Mitrouskas and the support throughout the project.

2.1 Introduction

The analysis of interacting many body quantum systems is indispensable in many areas of quantum physics or chemistry, but trying to give a direct analytic or numeric solution of systems of many particles is practically impossible. Fortunately there is a large number of regimes where effective descriptions could be found that significantly reduce the complexity of these systems - and thus make them accessible to analytical and numerical methods - that give a rather accurate approximation of the main features of the systems.

One example for such an effective description is the Hartree-Fock approximation used to approximate systems of many interacting fermions. Since many decades, Hartree-Fock is widely used, both in atomic [Foc30] and nuclear physics [GL86]. In recent years there was quite some progress in the mathematical understanding of the connection between the many-body Schrödinger description and the Hartree-Fock formalism. Most of these results are based in a semiclassical setting where the rigidity of a filled Fermi-sea of high density is used to obtain the respective estimates [BPS14a, PRSS17, FPS23, FPS24]. In this setting the leading order description is classical, i.e. given by the Vlasov dynamics [LP93, MM93, Saf19, Saf20, CLL21, CLS23]. However, Hartree-Fock theory is also widely used in settings, where the quantum mechanical nature of the system plays an important role to describe leading order effects of the dynamics, for example when simulating chemical reactions of large molecules or other problems that arise from quantum chemistry [Sza12].

The goal of the present paper is to prove that the Hartee-Fock equation approximates the dynamics of an N-body system in non-semiclassical situations where the effective interaction has an effect on the dynamics. We assume that the particles interact via long-range singular interactions, i.e. interaction potentials of the form $\frac{1}{|x|^s}$ for $0 < s$. While the Coulomb-case, i.e. $s = 1$, is physically the most relevant choice for s, we have to restrict ourselves to smaller $s < 2/3$ for technical reasons. Further we assume that the density of fermions is bounded, due to the Pauli exclusion principle, and occupies an area of volume of the same order as the particle number N. Note, that in this setting the spatial variation of the potential in the form of the force is sub-leading compared to the potential itself. Choosing a coupling constant which is such that the total interaction potential per particle is of order 1 with respect to the particle number [PP16, Pet17] leads to a force term which is sub-leading for any choice of $s > 0$. It follows that in this setting the influence of the interaction is mostly in the phase of the wave-function not in the profile of the probability density. Thus, coupling constants, that are such that the force per particle is of leading order and therefore the total interaction potential per particle is predominant, are of great interest.

We study the time-dependent Schrödinger equation

$$i\partial_t \Phi_t = H\Phi_t. \tag{2.1.1}$$

for an antisymmetric wave function $\Phi_t \in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$ describing N fermions at zero temperature. The Hamiltonian is of the form

$$H = \sum_{i=1}^{N}(-\Delta_i) + \sum_{1\leq i<j\leq N} u^{(N)}(x_i - x_j) \tag{2.1.2}$$

where we assume a long-range two-body potential of the form

$$u^{(N)}(x) := \lambda_N(s)\frac{1}{|x|^s}, \qquad \lambda_N(s) = N^{\frac{s-2}{3}} \tag{2.1.3}$$

with $0 < s \leq 1$.

Starting from an initial state Φ_0, which is approximately an antisymmetric product of N one-body orbitals $\{\varphi_0^k\}_{k=1}^N$, our goal is to show that this property is conserved during time-evolution, i.e. $\Phi_0 \approx \bigwedge_{k=1}^N \varphi_k^0 \Longrightarrow \Phi_t \approx \bigwedge_{k=1}^N \varphi_k^t$. The one-body orbitals φ_k^t solve the coupled Hartree equations

$$i\partial_t \varphi_k^t = h(t)\varphi_k^t \tag{2.1.4}$$

with mean-field Hamiltonian

$$h(t) = -\Delta + v^{(N)} * \rho_t, \qquad \rho_t \coloneqq \rho_t^N \coloneqq \sum_{k=1}^N |\varphi_k^t|^2 \tag{2.1.5}$$

where $v^{(N)} * \rho_t$ denotes the convolution.

Note that it is well-known that adding the exchange term to $h(t)$ such that the evolution of the orbitals is governed by the fermionic Hartree-Fock equation given by the Hamiltonian

$$h^{\mathrm{HF}}(t) = -\Delta + v^{(N)} * \rho_t - \sum_{j=1}^N v^{(N)} * (\overline{\varphi_j^t} \varphi_k^t), \tag{2.1.6}$$

would not improve the error estimates since the exchange term is negligible small [PP16, BPS14a].

In contrast to previous works on the derivation of the fermionic Hartree equation in the semiclassical scaling [BPS14a, PRSS17] and in the extensive volume scaling [PP16, Pet17, BBP+16], we are focusing on a regime where the average interaction is large in a volume of order N. This can be seen as follows:

We consider initial data which is given in a volume of order N. Due to the Pauli exclusion principle, the kinetic energy of a system of N fermions confined in a volume of order N is at least of the order N and the kinetic energy per particle of order 1. The interaction energy per particle is in our setting much larger and is expected to decrease from $\lambda_N N$ due to the long range nature of the interaction. This can be heuristically seen by estimating the average interaction energy by the mean-field potential

$$\lambda_N \left(|\cdot|^{-s} * \rho_t \right) \sim C\lambda_N \int_{B_{N^{1/3}}(0)} |x|^{-s} \,\mathrm{d}x \sim \lambda_N N^{1-\frac{s}{3}} = N^{\frac{1}{3}}, \tag{2.1.7}$$

whereas the average spatial variation of the mean-field potential is expected to be of order 1:

$$\lambda_N \nabla \left(|\cdot|^{-s} * \rho_t \right) \sim C\lambda_N \int_{B_{N^{1/3}}(0)} |x|^{-s-1} \,\mathrm{d}x \sim \lambda_N N^{\frac{2}{3}-\frac{s}{3}} = 1. \tag{2.1.8}$$

In comparison to [PP16, Pet17, BBP+16] where the scaling parameter is chosen to be $\tilde{\lambda}_N(s) = N^{\frac{s-3}{3}}$, our interaction energy is not only of much large order but the mean-field interaction potential does also significantly vary over typical distances of the system. This is not the case for the scaling of $\tilde{\lambda}_N(s) = N^{\frac{s-3}{3}}$ as one easily estimates

$$\tilde{\lambda}_N \left(|\cdot|^{-s} * \rho_t \right) \sim 1, \quad \tilde{\lambda}_N \nabla \left(|\cdot|^{-s} * \rho_t \right) \sim N^{-\frac{1}{3}}. \tag{2.1.9}$$

Thus, we do not expect that the leading order of our regime is free dynamics of the fermionic system as in the case for $\tilde{\lambda}_N(s)$ [Pet17].

In [BPS14a, PRSS17] a much smaller volume of order 1 is considered with a scaling parameter of order $N^{\frac{s-2}{3}}$ and re-scaled time-scales of order $N^{-1/3}$. Since the small spatial volume the kinetic energy per particle is of order $N^{2/3}$ and therefore large. This setting is naturally coupled to a joint mean-field and semiclassical scaling where one identifies $N^{-1/3} = \hbar$ as small semiclassical parameter. It is well known [NS81, Spo81] that in this setting the Vlasov equation describing the classical phase space density is a good approximation of the system. To compare with our setting, we re-scale our Hamiltonian (2.1.2) in its spatial and temporal component by $N^{-1/3}$ yielding

$$iN^{-\frac{1}{3}} \partial_t \Phi_t = \left(-N^{-\frac{2}{3}} \sum_{i=1}^{N} \Delta_i + N^{\frac{2s-2}{3}} \sum_{1 \leq i < j \leq N} v(x_i - x_j) \right) \Phi_t. \tag{2.1.10}$$

The scaled-interaction parameter $\lambda_N^{\mathrm{sc}}(s) = N^{\frac{2s-2}{3}}$ is in our case much larger than the mean-field scaling of N^{-1} in [BPS14a, PRSS17], and thus we are not expecting a semiclassical behavior such as convergence to the Vlasov dynamics in our setting.

To provide an early overview, we state the main result below. While the detailed notations appearing in the assumptions will be introduced later in Section 2.2, they are briefly summarized in the following main theorem:

Theorem 1. *Consider the interaction parameter $s \in (0, 2/3)$ as introduced in (2.1.3). Let Φ_t be the solution of (3.1.3) with normalized initial state $\Phi_0 \in L^2_{as}(\mathbb{R}^{3N})$ and let $\{\varphi_k^t\}_{k=1}^N$ be the solutions of (2.1.4) with normalized initial data $\{\varphi_k^0\}_{k=1}^N \in L^2(\mathbb{R}^3)$ satisfying for a $C_0 > 0$*

$$\langle \Phi_0, (1 - p_1)\Phi_0 \rangle \leq C_0 N^{-1},$$

$$|\langle \Psi_0, (-\Delta_1)\Psi_0 \rangle - \frac{1}{N} \sum_{i=1}^{N} \langle \varphi_i^0, (-\Delta)\varphi_i^0 \rangle| \leq C_0 N^{-\frac{1}{2}},$$

$$\sum_{k=1}^{N} \left(\|\Delta \varphi_k^0\|^2 + \|\nabla \varphi_k^0\|^2 \right) \leq C_0 N$$

where $p_1 \coloneqq \sum_{k=1}^N |\varphi_k^0\rangle\langle\varphi_k^0|_1$ *denotes the projector onto the orbitals acting on the first particle variable* x_1*. Additionally, consider for* $t \geq 0$ *the quantities* $\rho_{\varphi^t}^{\nabla} \coloneqq \sum_{k=1}^N |\nabla\varphi_k^t|^2$ *and* $\rho_{\varphi^t}^{\Delta} \coloneqq \sum_{k=1}^N |\Delta\varphi_k^t|^2$ *and for all* $t \geq 0$ *the quantity*

$$S(t) \coloneqq \max\{\|\rho_t\|_\infty, \|\rho_{\varphi^t}^{\nabla}\|_\infty, \|\rho_{\varphi^t}^{\Delta}\|_\infty, 1\}.$$

Then there exists a universal constant $C > 0$ *such that for all bounded multiplication operators* $M : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$

$$\left| \langle \Phi_t, M\Phi_t\rangle - \langle \bigwedge_{k=1}^N \varphi_k^t, M \bigwedge_{k=1}^N \varphi_k^t\rangle \right| \leq Ce^{C(t)}\|M\|_{\mathrm{op}} \max\left\{N^{-\frac{1}{32}}, N^{-\frac{2-3s}{32}}\right\}$$

with $C(t) = \int_0^t d\tau\ \exp\left(C\ln\left((2+t)S(t)\right)\int_0^\tau d\sigma\ S(\sigma)\right)$ *.eq*

Remark 2. The right hand side of the inequality of Theorem 51 is small for $N \to \infty$ for macroscopic times $t \in \mathcal{O}(1)$ with respect to the particle number N as long as $S(t) \in \mathcal{O}(1)$. The first two assumptions on the initial conditions address the fraction of particles outside the condensate $\langle\Phi_0, (1-p_1)\Phi_0\rangle$ and the difference between the kinetic energy of the macroscopic and effective initial states. Both assumptions may be slightly relaxed at the expense of a deterioration in the error bounds; we refer the reader to Subsection 2.3.2 and Lemma 27 for more details. For the reader's convenience, we have stated the slightly stricter version of the assumptions here. The third initial condition of $\sum_{k=1}^N (\|\Delta\varphi_k^0\|^2 + \|\nabla\varphi_k^0\|^2) \leq C_0 N$ corresponds physically to the setting where the particles occupy a volume of order N. Note that the setting of the extensive volume of order N, is also reflected in $S(t) \in \mathcal{O}(1)$ as the density ρ_t and the pointwise kinetic energy density ρ_t^{∇} are not pathologically concentrated in space.

Remark 3. Instead of considering this setting of an extensive volume and macroscopic times, we can equivalently treat the case of a volume of order 1 and short times of order $N^{-2/3}$. This is a direct consequence of the homogeneous interaction potential $|\cdot|^{-s}$ and can be seen as follows: For any $s > 0$, we can introduce re-scaled time and space variables $(\tilde{t}, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{3N}$ via

$$\begin{cases} \tilde{t} &= N^{-\frac{2}{3}}t, \\ \tilde{x} &= N^{-\frac{1}{3}}x \end{cases} \tag{2.1.11}$$

and the re-scaled many-body wave function via

$$\tilde{\Psi}_{N,\tilde{t}}(\tilde{x}_1, \ldots, \tilde{x}_N) \coloneqq \Psi_{N,N^{-\frac{2}{3}}\tilde{t}}\left(N^{-\frac{1}{3}}\tilde{x}_1, \ldots, N^{-\frac{1}{3}}\tilde{x}_N\right) \tag{2.1.12}$$

which now solves the Schrödinger equation in the new variables with coupling parameter 1. Thus, by this re-scaling argument we can treat the Hamiltonian (2.1.2) with $\lambda_N(s) = 1$

for all $s \in (0, 2/3)$ for shorter times $t \in \mathcal{O}(N^{-2/3})$ and conditions on the Hartree solutions adjusted to the volume of order 1, that are:

$$\max\left\{N^{-7/3}\sum_{k=1}^{N}\|\Delta\varphi_k^0\|^2, N^{-5/3}\sum_{k=1}^{N}\|\nabla\varphi_k^0\|^2\right\} \leq C_0 \tag{2.1.13}$$

for a $C_0 > 0$. We emphasize that the time scale of $\mathcal{O}(N^{-2/3})$ is expected to be optimal for the mean-field description since it neglects correlations arising from the interaction. Thus, as soon as the expected the kinetic energy per particle changes significantly due to the interaction, the mean-field description is not expected to be valid anymore. Due to Proposition 16 we can bound the change of the expected kinetic energy per particle by the product of mean-field force and momentum per particle. It follows, that for $t \in \mathcal{O}(N^{-2/3})$ the expected kinetic energy per particle cannot be bounded of the same order anymore. Thus, the time scale is expected to be optimal.

Remark 4. In [HMP25], we replaced the singular interaction by a regular one and studied the re-scaled dense case with a volume of order 1 and short times of order $N^{-2/3}$. Due to the restriction to regular interactions, the analysis simplifies making for example Lemma 10 obsolete as the long-range behavior of the interaction is sub-dominant. In this case, one circumvents the quantity $S(t) = \max\{\|\rho_t\|_\infty, \|\rho^\nabla_{\varphi^t}\|_\infty, \|\rho^\Delta_{\varphi^t}\|_\infty, 1\}$ and all assumptions can be fully reduced to initial conditions on the orbitals and microscopic state.

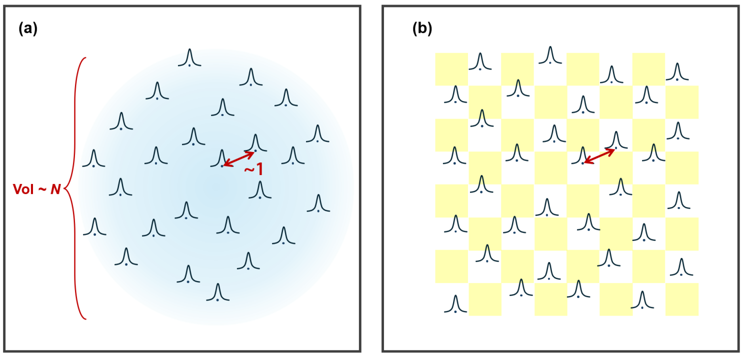

Figure 2.1.1: (a) Illustration of a possible setting satisfying the conditions of the main theorem. The total occupied volume is of order N and hence extensive, the distance between the cells is of order 1. (b) Illustration of the alternating occupation number using a checkerboard pattern as a representative example. The system is divided into a regular array of cells, with fermions initially confined to alternating occupied cells (white) while neighboring cells remain unoccupied (yellow). In the time scale of interest, the particles can traverse the nearest-cell distance.

Remark 5. Given that we assume the particles have an initial kinetic energy of order 1 with respect to N, and we can prove that this property persists over time due to Proposition 16, the particles can traverse the typical inter-particle distance within the time scale for which our proof is valid. Thus, similar to the extensive volume setting in [PP16, BBP+16], observables M of interest should be sensitive for local properties on small scales. As an example, one considers M as an alternating occupation number: The system's volume is partitioned into $2N$ cells of order-one volume. As an initial state, we consider smooth wave packets with disjoint supports, arranged such that occupied and unoccupied cells alternate. The alternating occupation number M is then defined as the number of fermions found in the initially unoccupied cells at a given time. This observable is particularly sensitive to small-scale spatial transport and captures local spatial redistribution in an interacting fermionic system.

Strategy of the proof

We briefly explain the strategy of the derivation. The central idea is to extract the large potential by introducing a gauging phase and consider the gauged Schrödinger and Hartree equations. We start by decomposing the potential (2.1.3) into a singular part $u_{\leq}^{(N)}$ and a long-range part $v^{(N)}$:

$$u^{(N)} = u_{\leq}^{(N)} + v^{(N)}, \tag{2.1.14}$$

$$u_{\leq}^{(N)}(x) := u^{(N)}\chi(|x| \leq 1) := \begin{cases} \lambda_N |x|^{-s} & , |x| \leq 1 \\ 0 & \text{else} \end{cases}, \tag{2.1.15}$$

$$v^{(N)}(x) := u^{(N)}\chi(|x| > 1). \tag{2.1.16}$$

The gauged wave functions

$$\Psi_t = e^{i\left(\sum_{1\leq k<l\leq N} v^{(N)}(x_k - x_l)\right)t}\Phi_t, \qquad \psi_k^t = e^{i(v^{(N)} * \rho_t)t}\varphi_k^t \tag{2.1.17}$$

satisfy the equations

$$i\partial_t \Psi_t = H^{\mathrm{g}}(t)\Psi_t, \qquad H^{\mathrm{g}}(t) = \sum_{i=1}^{N}\left(i\nabla_i + tF_i\right)^2 + \sum_{1\leq k<l\leq N} u_{\leq}^{(N)}(x_k - x_l), \tag{2.1.18}$$

$$i\partial_t \psi_k^t = h^{\mathrm{g}}(t)\psi_k^t, \qquad h^{\mathrm{g}}(t) = \left(i\nabla + t\overline{f}\right)^2 - t\partial_t\left(v^{(N)} * \rho_t\right) + \overline{u_{\leq}} \tag{2.1.19}$$

with

$$F_i \coloneqq F_i^N \coloneqq \sum_{j=1; j\neq i}^{N} f_{ij}^{(N)}, \quad f_{ij}^{(N)} \coloneqq -\nabla_i v^{(N)}(x_i - x_j), \tag{2.1.20}$$

$$\overline{f}(x) \coloneqq \overline{f^{(N)}}(x) \coloneqq -(\nabla v^{(N)} * \rho_t)(x), \tag{2.1.21}$$

$$\overline{u_{\leq}}(x) \coloneqq (u_{\leq}^{(N)} * \rho_t)(x) \tag{2.1.22}$$

Let us note that

$$-\partial_t (v^{(N)} * \rho_t) = \overline{(i\nabla \cdot f)} + \overline{(f \cdot i\nabla)} + 2t\overline{f \cdot \overline{f}} \tag{2.1.23}$$

with

$$\overline{(f \cdot i\nabla)}(x_1) \coloneqq \int \mathrm{d}y \, \sum_{j=1}^{N} \psi_j^*(y) \nabla_1 v^{(N)}(x_1 - y) \cdot i\nabla\psi_j(y), \tag{2.1.24}$$

$$\overline{(i\nabla \cdot f)}(x_1) \coloneqq \int \mathrm{d}y \, \sum_{j=1}^{N} (i\nabla\psi_j)^*(y) \cdot \nabla_1 v^{(N)}(x_1 - y) \; \psi_j(y), \tag{2.1.25}$$

$$\overline{f \cdot \overline{f}}(x_1) \coloneqq \int \mathrm{d}y \; \nabla_1 v^{(N)}(x_1 - y) \cdot \overline{f}(y) \; \rho_t(y). \tag{2.1.26}$$

Note that (2.1.23) is a direct consequence of the continuity equation $\partial_t \rho_t = -\nabla \cdot J_t$ with $J_t : \mathbb{R}^3 \to \mathbb{R}^3$ and

$$J_t(y) \coloneqq 2\sum_{j=1}^{N} \mathrm{Im}\left\{\psi_j^*(y)(\nabla\psi_j)(y)\right\} + 2t\overline{f}(y)\rho_N^t(y) = 2\sum_{j=1}^{N} \mathrm{Im}\left\{\varphi_j^*(y)(\nabla\varphi_j)(y)\right\}. \tag{2.1.27}$$

We note that well-posedness of (2.1.18) and (2.1.19) are inherited from (3.1.3) and (2.1.4) since the solutions only differ by a gauging phase. More specifically, we can write

$$\Psi_t = U_t \Psi_0, \qquad U_t \coloneqq e^{i\left(\sum_{1\leq k<l\leq N} v^{(N)}(x_k - x_l)\right)t} e^{-iHt}, \tag{2.1.28}$$

$$i\partial_t \Psi_t = H^{\mathrm{g}}(t)\Psi_t. \tag{2.1.29}$$

The existence and uniqueness of solutions of (2.1.4) is given by [BDPF76] and our regularity assumptions on our initial conditions (see also [BBP+16, Theorem A.1.]).

In order to simplify H^{g} we make use of the projectors $p^t \coloneqq p^{\psi_1^t,\ldots,\psi_N^t} \coloneqq \sum_{k=1}^{N} |\psi_k^t\rangle\langle\psi_k^t|$ and $q^t \coloneqq 1 - p_t$ with the orthonormal solutions $\{\psi_k^t\}_{k=1}^N$ of (2.1.19). The new equations will then be studied with the aid of an auxiliary Hamiltonian that is denoted by $\widetilde{H}^{\mathrm{g}}(t)$ and obtained by writing

$$H^{\mathrm{g}}(t) = \sum_{m=1}^{N} (p_m^t + q_m^t) \left(i\nabla_m + t \sum_{k=1, k\neq m}^{N} (p_k^t + q_k^t) f_{mk} (p_k^t + q_k^t) \right) \times \\ \times \left(i\nabla_m + t \sum_{l=1, l\neq m}^{N} (p_l^t + q_l^t) f_{ml} (p_l^t + q_l^t) \right) (p_m^t + q_m^t) + \sum_{1\leq k<l\leq N} u_{\leq}^{(N)}(x_k - x_l). \tag{2.1.30}$$

and discarding all terms that contain more than three q operators except for the singular part $u_{\leq}^{(N)}$. The reason for introducing this auxiliary Hamiltonian is that although the gauging procedure eliminated the large potential $u^{(N)}$ from the generator of our gauged dynamics, we are still facing a magnetic-type explicitly time-dependent Hamiltonian $H^{\mathrm{g}}(t)$. The main difficulty is to treat the gradient terms where $\nabla q_1^t \Psi_t$-terms appear making direct Grönwall estimates in terms of $q_1^t \Psi_t$ or $\nabla q_1^t \Psi_t$ impossible. Due to the restricted number of q^t-operators in the auxiliary Hamiltonian, we can avoid such gradient terms in the Grönwall estimates. Note also that this problem does not apply to the singular part since no gradient operators occur.

For the auxiliary dynamics generated by $\widetilde{H^{\mathrm{g}}}(t)$, we will show convergence in the sense that there exists a $C(t) > 0$ only depending on t and a $\eta > 0$ such that for all $t \in [0, T]$ for a $T > 0$

$$\tilde{\alpha}(t) := \left\langle \tilde{\Psi}_t, q_1^t \tilde{\Psi}_t \right\rangle \leq C(t) N^{-\eta} \tag{2.1.31}$$

where $q_1 := 1 - \sum_{k=1}^{N} |\psi_k^t\rangle\langle\psi_k^t|_1$ and $\tilde{\Psi}_t := \tilde{U}_{t,0}\Psi_0$ with $\tilde{U}_{t,s}$ being the two-parameter groups generated by $\widetilde{H^{\mathrm{g}}}(t)$. The rigorous result is stated in Proposition 21 and will be shown using ideas and techniques similar to the ones introduced in [BBP+16, Pet14, PP16]. Moreover, due to the magnetic-type structure of (2.1.18), it is necessary to find kinetic bounds of the form $\|\nabla_1 q_1^t \tilde{\Psi}_t\| \leq C N^{-\nu}$ for a $\nu > 0$. Since the quantity $\|\nabla_1 q_1^t \tilde{\Psi}_t\|^2$ corresponds to the kinetic energy of the particles outside of the orbitals, we need to control the time evolution of the *bad kinetic energy*. Such a bound is provided by propagating an appropriate energy difference (see Proposition 23).

In a second step, we will show that the auxiliary dynamics allows a norm approximation for the dynamics generated by $H^g(t)$, that is for all $t \in [0, T]$ for a $T > 0$

$$\|\tilde{\Psi}_t - \Psi_t\| \xrightarrow{N\to\infty} 0 \tag{2.1.32}$$

where $\Psi_t = U_t \Psi_0$ as defined in (2.1.28). The main challenge lies in establishing sufficiently good bounds for q-operators acting on Ψ_t which cannot be achieved by directly applying a Duhamel argument. Instead, we will use a cut-off weight function in the Duhamel argument that seamlessly integrates with the techniques employed to prove Proposition 21. Note that

in this step all the difference $(H^{\mathrm{g}} - \widetilde{H^{\mathrm{g}}})$ appears and necessarily the gradient term $\nabla_1 q_1^t \tilde{\Psi}_t$ appears which has to be estimated with the previously derived kinetic bound. Further details on this approach are explained in Section 2.4.
Note that due to the fact that it holds

$$\begin{aligned}
\alpha(t) - \tilde{\alpha}(t) &:= \langle \Psi_t, q_1^t \Psi_t \rangle - \langle \tilde{\Psi}_t, q_1^t \tilde{\Psi}_t \rangle \\
&= \langle \Psi_t, q_1^t \Psi_t \rangle - \langle \Psi_t, q_1^t \tilde{\Psi}_t \rangle + \langle \Psi_t, q_1^t \tilde{\Psi}_t \rangle - \langle \tilde{\Psi}_t, q_1^t \tilde{\Psi}_t \rangle \\
&= \langle \Psi_t, q_1^t (\Psi_t - \tilde{\Psi}_t) \rangle + \langle \Psi_t - \tilde{\Psi}_t, q_1^t \tilde{\Psi}_t \rangle \\
&\leq \tilde{\alpha}(t)^{1/2} \| \Psi_t - \tilde{\Psi}_t \| + \alpha(t)^{1/2} \| \Psi_t - \tilde{\Psi}_t \| \qquad (2.1.33) \\
&\leq \tilde{\alpha}(t) + 2 \| \Psi_t - \tilde{\Psi}_t \|^2 + \frac{1}{4} \alpha(t) \qquad (2.1.34)
\end{aligned}$$

we find

$$\alpha(t) \leq \frac{8}{3} \left(\| \tilde{\Psi}_t - \Psi_t \|^2 + \tilde{\alpha}(t) \right) \xrightarrow{N \to \infty} 0. \qquad (2.1.35)$$

This is already sufficient to conclude the desired result due to the following known results:

Lemma 6 ([PP16, Lemma 3.2])**.** *Let μ_1^Ψ be the reduced one-particle density matrix with respect to $\Psi \in L^2_{as}(\mathbb{R}^{3N})$ and $\mu_1^{\mathrm{Slater}} = \frac{1}{N} \sum_{k=1}^N |\psi^k\rangle\langle\psi^k|_1$ with $\{\psi^k\}_{k=1}^N$ being a orthonormal system. For any bounded operator $A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^2)$ it holds*

$$\begin{aligned}
\left| \mathrm{tr} A \mu_1^\Psi - \mathrm{tr} A \mu_1^{\mathrm{Slater}} \right| \leq \|A\|_{\mathrm{op}} \| \mu_1^\Psi - \mu_1^{\mathrm{Slater}} \|_{tr} \leq 2\sqrt{2} \|A\|_{\mathrm{op}} \langle \Psi, q_1 \Psi \rangle^{1/2}, \\
\langle \Psi, q_1 \Psi \rangle \leq \frac{1}{2} \| \mu_1^\Psi - \mu_1^{\mathrm{Slater}} \|_{tr}.
\end{aligned}$$

From that we can conclude:

Proposition 7. *Let μ_1^Φ be the reduced one-particle density matrix with respect to Φ_t being the solution of (2.1.4) and $\mu_1^{Slater} = \frac{1}{N} \sum_{k=1}^N |\varphi_k^t\rangle\langle\varphi_k^t|_1$ with $\{\varphi_k^t\}_{k=1}^N$ being the solution of (2.1.19). For any bounded multiplication operator $M : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^2)$ it holds for any $t \geq 0$*

$$\left| \mathrm{tr} M \mu_1^{\Phi_t} - \mathrm{tr} M \mu_1^{\mathrm{Slater}} \right| \leq \|M\|_{\mathrm{op}} \| \mu_1^{\Phi_t} - \mu_1^{\mathrm{Slater}} \|_{tr} \leq 2\sqrt{2} \|M\|_{\mathrm{op}} \alpha(t)^{1/2}$$

with $\alpha(0) \leq \| \mu_1^{\Phi_0} - \mu_1^{\mathrm{Slater}} \|_{tr} / 2$.

2.2 Preliminaries

2.2.1 The counting functional formalism

We use the following counting functional as introduced in [PP16] for the fermionic setting:

Definition 8. Let $\{\varphi_j\}_{j=1}^N \in L^2(\mathbb{R}^3)$ be orthonormal. Define the projectors

$$p := p^{\varphi_1,\ldots,\varphi_N} := \sum_{j=1}^N |\varphi_j\rangle\langle\varphi_j| = N\gamma_{\wedge\varphi_j}, \qquad q = 1 - p.$$

For any one-particle operator $A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ and $i \in \{1, \ldots, N\}$, we define the notation A_i to indicate that the operator acts only on the first variable, i.e.

$$A_i := 1 \otimes \ldots \otimes \underbrace{A}_{i\text{-th position}} \otimes \ldots \otimes 1.$$

Definition 9. Let $\{\varphi_j\}_{j=1}^N \in L^2(\mathbb{R}^3)$ be orthonormal. We define a weight function as $f : \{0, \ldots, N\} \to [0, 1]$ satisfying $f(0) = 0$, $f(N) = 1$. The weight operator is defined by

$$\hat{f} := \hat{f}^{\varphi_1,\ldots,\varphi_N} := \sum_{k=0}^N f(k)P_{N,k} := \sum_{k=0}^N f(k) \left(\prod_{m=1}^k q_m \prod_{m=k+1}^N p_m \right)_{\text{sym}}.$$

For $\psi \in L^2(\mathbb{R}^{3N})$ be normalized define the functional

$$\alpha_f := \alpha_f(\psi, \varphi_1, \ldots, \varphi_N) := \langle\psi, \hat{f}\psi\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L^2(\mathbb{R}^{3N})$.

Note that in general $P_{N,k}$ satisfies for each $k \in \{1, \ldots, N\}$ that $P_{N,k}P_{N,l} = \delta_{k,l}P_{N,k}$ and

$$P_{N,k} := \sum_{\{b_m\}_m \in \mathcal{A}_k} \prod_{m \in 1}^N p_m^{1-b_m}(q_m)^{b_m}$$

where

$$\mathcal{A}_k := \left\{ \overrightarrow{b} = \{b_m\}_{m=1}^N \in \{0,1\}^N : \sum_{m=1}^N b_m = k \right\}.$$

The most important choices of weight functions which appear in Theorem 51 are defined by

$$n(k) := \frac{k}{N}, \tag{2.2.1}$$

$$m(k) := m_\gamma(k) := \min\{1, \frac{k}{N^\gamma}\} \qquad \text{with } \gamma \in (0, 1], \tag{2.2.2}$$

$$l(k) := \sqrt{\frac{k}{N}} \tag{2.2.3}$$

for all $k \in \{0, \ldots, N\}$. Note that the definition for the functional is consistent with (2.1.31) since it holds for $\psi \in L^2_{\text{as}}(\mathbb{R}^{3N})$

$$\alpha_n = \alpha_{m_{\gamma=1}} = \sum_{k=0}^N \frac{k}{N}\langle\psi, P_{N,k}\psi\rangle = \sum_{k=0}^N \frac{1}{N}\langle\psi, \sum_{m=1}^N q_m P_{N,k}\psi\rangle = \langle\psi, q_1\psi\rangle \tag{2.2.4}$$

due to the antisymmetry of ψ.

2.2.2 Estimates of the Hartree solution

Throughout the entire proofs, we have to rely on bounds of mean-field quantities depending on $\{\psi_k^t\}_{k=1}^N$ as solutions of (2.1.19) and the corresponding spatial density $\rho_t = \sum_{i=1}^N |\psi_i^t|^2 = \sum_{i=1}^N |\varphi_i^t|$ where $\{\varphi_i^t\}$ are the solutions of (2.1.4).

2.2.2.1 Estimates of mean-field quantities

In the following, we summarize the relevant bounds for the mean-field quantities of interest in dependence of $\|\rho_t\|_\infty$ and $\|\rho_t\|_1$. We collect the estimates under the following statement:

Lemma 10 (Estimates of the mean-field terms). *There exists a constant $C > 0$ such that*

1. $N^{-\frac{1}{3}}\|v^{(N)} * \rho_t\|_\infty, \||f^{(N)}| * \rho_t\|_\infty \leq (\|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1})$,
2. $\||f^{(N)}|^2 * \rho_t\|_\infty \leq C_\varepsilon N^{-a}(\|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1})$ *where*
$$a := a(s) := \begin{cases} 1 & , s < 1/2 \\ 1-\varepsilon & , s = 1/2 \\ \frac{2}{3}(2-s) & , s > 1/2 \end{cases}$$
for $\varepsilon > 0$ arbitrarily small,
3. $\||\nabla \cdot f^{(N)}| * \rho_t\|_\infty \leq C(\|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1}) \begin{cases} N^{-\frac{1}{3}} & , s < 1 \\ N^{-\frac{1}{3}} \ln N & , s = 1 \end{cases}$,
4. $\||\Delta f^{(N)}| * \rho_t\|_\infty, \||\Delta\nabla \cdot f^{(N)}| * \rho_t\|_\infty \leq C\lambda_N(\|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1})$,
5. $\||\nabla \cdot f^{(N)}|^2 * \rho_t\|_\infty, \||\Delta f^{(N)}|^2 * \rho_t\|_\infty, \||f^{(N)}|^4 * \rho_t\|_\infty, \||u_\leq^{(N)}|^2 * \rho_t\|_\infty \leq C\lambda_N^2(\|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1})$,
6. $\|u_\leq^{(N)} * \rho_t\|_\infty, \||\nabla u_\leq^{(N)}| * \rho_t\|_\infty \||\Delta u_\leq^{(N)}| * \rho_t\|_\infty \leq C\lambda_N(\|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1})$ *for $s < 1$.*

Remark 11. The condition $\|\rho_t\|_1 \leq cN$ is trivially satisfied since $\{\psi_k^t\}_{k=1}^N$ are normalized. Later on, we will apply the bounds to quantities convoluted with $\rho_t^A := \sum_{k=1}^N |A\psi_k^t|^2$ for operators A on $L^2(\mathbb{R}^3)$.

Proof. In the following, we demonstrate explicit bounds for a representative subset of the terms. The remaining terms can be bounded analogously. Note that it holds $\|\rho_t\|_1 = N$

since the orbitals are normed such that we can estimate by the Hölder inequality

$$
\begin{aligned}
\left(v^{(N)} * \rho_t\right)(y) &\leq \lambda_N \left(\int_{B_{N^{1/3}}(y)} v^{(N)}(x-y)\rho_t(x)\,\mathrm{d}x + \int_{B^c_{N^{1/3}}(y)} |x-y|^{-s}\rho_t(x)\,\mathrm{d}x \right) \\
&\leq \lambda_N \left(\|\rho_t\|_\infty \int_{B_{N^{1/3}}(0)} v^{(N)}(x-y)\,\mathrm{d}x + \|\rho_t\|_1 \sup_{x\in B^c_{N^{1/3}}(0)} |x|^{-s} \right) \\
&\leq \lambda_N \left(\|\rho_t\|_\infty \int_1^{N^{1/3}} |x|^{-s+2}\,\mathrm{d}|x| + \|\rho_t\|_1 N^{-\frac{s}{3}} \right) \\
&\leq \lambda_N N^{1-\frac{s}{3}} \left(\frac{1}{2}\|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1} \right) \leq N^{\frac{1}{3}} \left(\frac{1}{2}\|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1} \right) \qquad (2.2.5)
\end{aligned}
$$

and analogously

$$
\begin{aligned}
\left(u_{\leq}^{(N)} * \rho_t\right)(y) &\leq \|\rho_t\|_\infty \lambda_N \left(\int_0^1 |x|^{-s+2}\,\mathrm{d}|x| \right) \\
&\leq \frac{1}{2}\lambda_N \left(\|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1} \right), \qquad (2.2.6) \\
\left(|f^{(N)}| * \rho_t\right)(y) &\leq \|\rho_t\|_\infty + \|\rho_t\|_1 N^{-1}. \qquad (2.2.7)
\end{aligned}
$$

■

In order to bound the mean-field quantities with respect to variations of ρ_t we make the following definition:

Definition 12. Let $\rho_t^O := \sum_{k=1}^N |O\psi_k^t|^2$ where $\{\psi_k^t\}$ are orthornomal solutions of (2.1.4) and O is an operator on $L^2(\mathbb{R}^3)$. We define

$$D(t) := \max\{\|\rho_t\|_\infty, \|\rho_t^\nabla\|_\infty, \|\rho_t^\Delta\|_\infty, N^{-1}\|\rho_t^\nabla\|_1, N^{-1}\|\rho_t^\Delta\|_1, 1\}.$$

Lemma 13. *Let $\rho_t^h := \sum_{k=1}^N |h^{\mathrm{g}}(t)\psi_k^t|^2$ where $\{\psi_k^t\}$ are orthornomal normalized solutions of the gauged Schrödinger equation generated by $h^{\mathrm{g}}(t)$ as defined in (2.1.19). There exists a $C > 0$ such that for $t \geq 0$ it holds*

$$
\begin{aligned}
\|\rho_t^h\|_\infty &\leq C(1+t)^4 D(t)^5, \\
\|\rho_t^h\|_1 &\leq C(1+t)^4 D(t)^4 N, \\
\|h_1^{\mathrm{g}} p_1\|_{\mathrm{op}} &\leq C(1+t)^2 D(t)^2 N^{\frac{1}{2}}, \\
\|h_1^{\mathrm{g}} p_1\|_{\mathrm{op,as}} &\leq C(1+t)^2 D(t)^2.
\end{aligned}
$$

Remark 14. In order to control the kinetic energy of particles outside the orbitals (the so-called *bad kinetic energy*), we will later consider the slight deviation $\widetilde{h}^{\mathrm{g}}(t)$ of the gauged Hamiltonian $h^{\mathrm{g}}(t)$ as

$$\widetilde{h}^{\mathrm{g}}(t) = h^{\mathrm{g}}(t) - \frac{t}{2}R - \frac{1}{2}\overline{u_{\leq}} - \frac{2t^2}{3}W \tag{2.2.8}$$

with R and W as defined in (2.3.20) and (2.3.21) respectively. It is easy to check that $\widetilde{h}^{\mathrm{g}}(t)$ satisfies the same estimates as $h^{\mathrm{g}}(t)$ in Lemma 13.

Proof. Recall that $h^{\mathrm{g}} = (i\nabla + t\bar{f})^2 - t\partial_t\bar{v} + \overline{u_{\leq}}$ thus by direct computation we obtain

$$\begin{aligned}
\rho_t^h &= \sum_{k=1}^N \left|i\partial_t\psi_k^t\right|^2 \\
&\leq C\left(t^2\|\partial_t\bar{v}\|_\infty^2\rho_t + \|\overline{u_{\leq}}\|_\infty\rho_t + t^2\|\bar{f}\|_\infty^2\rho_t^\nabla + \rho_t^\Delta\right) \\
&\leq C\left((1+t)^4D(t)^4\rho_t + t^2D(t)^2\rho_t^\nabla + \rho_t^\Delta\right). \qquad (2.2.9)
\end{aligned}$$

For antisymmetric wave functions we find

$$\|h_1^{\mathrm{g}}p_1\|_{\mathrm{op,as}}^2 = \sup_{\substack{\psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N}),\\ \|\psi_{\mathrm{as}}\|=1}} \langle p_1^h\psi_{\mathrm{as}}, p_1^h\psi_{\mathrm{as}}\rangle \tag{2.2.10}$$

with $p_1^h = \sum_{i=1}^N |h^{\mathrm{g}}(t)\psi_i^t\rangle\langle\psi_i^t|$. Thus, by using Lemma 38 it follows immediately

$$\|h_1^{\mathrm{g}}p_1\|_{\mathrm{op,as}}^2 \leq \frac{1}{N}\|\rho_t^h\|_1. \tag{2.2.11}$$

As a simple consequence of Lemma 38, we obtain analogously $\|h_1^{\mathrm{g}}p_1\|_{\mathrm{op}}^2 \leq \|\rho_t^h\|_1$. ■

2.2.2.2 Conversion of gauged to ungauged quantities

Let us note that the properties on the L^p-norms for $p \in [1,\infty]$ of the solutions $\{\varphi_k^t\}_{k=1}^N$ of the original mean-field equation (2.1.4) translate to the solutions $\{\psi_k^t\}_{k=1}^N$ of the gauged mean-field equation (2.1.19) as both differ only via a phase factor. In particular we find the following statement by direct calculation:

Lemma 15. *Consider* $D(t)$ *as defined in Definition 12 and* $d(t) := \max\{\|\rho_t\|_\infty, 1\}$ *for* $t \geq 0$. *There exists a* $C > 0$ *such that for* $p \in [1,\infty]$ *and* $t \geq 0$ *it holds*

$$\begin{aligned}
\|i\partial_t\psi_k^t\|_p &\leq C\left(\left((1+t)D(t) + t^2D(t)^2\right)\|\varphi_k^t\|_p + \|\Delta\varphi_k^t\|_p\right) \\
\|\nabla\psi_k^t\|_p &\leq C\left(td(t)\|\varphi_k^t\|_p + \|\nabla\varphi_k^t\|_p\right), \\
\|\Delta\psi_k^t\|_p &\leq C\left(\left(td(t) + t^2d(t)^2\right)\|\varphi_k^t\|_p + td(t)\|\nabla\varphi_k^t\|_p + \|\Delta\varphi_k^t\|_p\right).
\end{aligned}$$

Moreover, with $\rho^{\nabla}_{\varphi^t} := \sum_{k=1}^N |\nabla\varphi_k^t|^2, \rho^{\Delta}_{\varphi^t} := \sum_{k=1}^N |\Delta\varphi_k^t|^2$*, there exists a* $C > 0$ *such that for* $p \in [1,\infty]$ *and* $t \geq 0$ *it holds*

$$\begin{aligned}\|\rho_t^{\nabla}\|_p &\leq C\left(\|\rho^{\nabla}_{\varphi_t}\|_p + t^2 d(t)^2 \|\rho_t\|_p\right),\\ \|\rho_t^{\Delta}\|_p &\leq C\left(\|\rho^{\Delta}_{\varphi_t}\|_p + t^2 d(t)^2 \|\rho^{\nabla}_{\varphi_t}\|_p + (t^2 d(t)^2 + t^4 d(t)^4)\|\rho_t\|_p\right).\end{aligned}$$

Proof. This is an immediate consequence of the back transform $\varphi_k^t = e^{-i\overline{v}t}\psi_k^t$ yielding

$$i\partial_t \psi_k^t = (-t\partial_t \overline{v} - \Delta + \overline{u_{\leq}})\varphi_k^t. \tag{2.2.12}$$

Recall that $-\partial_t\overline{v} = \overline{(i\nabla\cdot f)} + \overline{(f\cdot i\nabla)} + 2t\overline{f\cdot\overline{f}}$ which can be estimated by Cauchy-Schwarz and Lemma 10

$$\begin{aligned}&|\overline{(f\cdot i\nabla)}(x_1)|\\ &\leq \sum_{j=1}^N \int \mathrm{d}y\ |\psi_j^*(y)|\ |\nabla_1 v^{(N)}(x_1-y)|\ |\nabla\psi_j(y)|\\ &\leq \left(\int \mathrm{d}y\ |\nabla_1 v^{(N)}(x_1-y)| \sum_{j=1}^N |\psi_j(y)|^2\right)^{1/2}\left(\int \mathrm{d}y\ |\nabla_1 v^{(N)}(x_1-y)| \sum_{j=1}^N |\nabla\psi_j(y)|^2\right)^{1/2}\\ &\leq |\ |f^{(N)}| * \rho_t(x_1)|^{1/2}\ |\ |f^{(N)}| * \rho_t^{\nabla}(x_1)|^{1/2} \qquad (2.2.13)\end{aligned}$$

and similarly

$$|\overline{(i\nabla\cdot f)}(x_1)| \leq |\ |f^{(N)}| * \rho_t(x_1)|^{1/2}\ |\ |f^{(N)}| * \rho_t^{\nabla}(x_1)|^{1/2}, \tag{2.2.14}$$

$$\begin{aligned}|\overline{f\cdot\overline{f}}(x_1)| &\leq \int \mathrm{d}y\ |\nabla_1 v^{(N)}(x_1-y)|\ |\overline{f}(y)|\ \rho_t(y)\\ &\leq \|\,|f^{(N)}| * \rho_t\|_\infty|\ |f^{(N)}| * \rho_t(x_1)|.\end{aligned} \tag{2.2.15}$$

Furthermore, it holds

$$\nabla\psi_k^t = e^{i\overline{v}t}\left(-it\overline{f}\varphi_k^t + \nabla\varphi_k^t\right), \tag{2.2.16}$$

$$\Delta\psi_k^t = e^{i\overline{v}t}\left(-t^2\overline{f}^2\varphi_k^t - it(\nabla\cdot\overline{f})\varphi_k^t - it\overline{f}\cdot\nabla\varphi_k^t + \Delta\varphi_k^t\right). \tag{2.2.17}$$

We find the desired estimates using Lemma 10 with $\|\rho_t\|_\infty, \|\rho_t^{\nabla}\|_\infty, \|\rho_t^{\Delta}\|_\infty \leq D(t)$ by definition. In addition, it holds by direct computation

$$\|\rho_t^{\nabla}\|_p \leq C\left(\|\rho^{\nabla}_{\varphi_t}\|_\infty + t^2(\|\rho_t\|_\infty + 1)^2\|\rho_t\|_p\right), \tag{2.2.18}$$

$$\begin{aligned}\|\rho_t^{\Delta}\|_p \leq C\Big(&\|\rho^{\Delta}_{\varphi_t}\|_\infty + t^2(\|\rho_t\|_\infty+1)^2\|\rho^{\nabla}_{\varphi_t}\|_p\\ &+ t^2(\|\rho_t\|_\infty+1)^2(1+t^2(\|\rho_t\|_\infty+1)^2)\|\rho_t\|_p\Big).\end{aligned} \tag{2.2.19}$$

∎

2.2.2.3 Kinetic energy estimates

Since the normed (ungauged) solutions $\{\varphi_k^t\}_{k=1}^N$ of (2.1.4) satisfy energy conservation in the sense that $E(t) := \langle \varphi_k^t, h(t)\varphi_k^t \rangle$ is conserved in time, we obtain under the assumption that $\|\rho_t\|_\infty \leq D(t)$ for all $t \geq 0$ the simple bound

$$\sum_{k=1}^N \|\nabla \varphi_k^t\|^2 \leq E(t) + \sum_{k=1}^N |\langle \varphi_k^t, \overline{v}\varphi_k^t \rangle| \leq \sum_{k=1}^N \|\nabla \varphi_k^0\|^2 + 2\left(D(0) + D(t)\right) N^{\frac{4}{3}} \tag{2.2.20}$$

where we used Lemma 10 to estimate $\|\overline{v}\|_\infty$ and $\|\overline{f}\|_\infty$. Using $\nabla \varphi_k = e^{-i\overline{v}t}\left(it\overline{f}\psi_k + \nabla \psi_k\right)$ we obtain for the gauged solutions $\{\psi_k^t\}_{k=1}^N$ satisfying $\sum_{k=1}^N \|\nabla \psi_k^0\|^2 \leq CN$ for a $C > 0$ that

$$\sum_{k=1}^N \|\nabla \psi_k^t\|^2 \leq C(t) N^{\frac{4}{3}} \tag{2.2.21}$$

where $C(t)$ depends quadratically on $D(t)$ and t.

In the following we will show that the N-dependence of the kinetic energy (2.2.21) can be improved by a Grönwall argument.

Proposition 16. *Let $\{\varphi_k^t\}_{k=1}^N$ be orthonormal solutions of (2.1.4) satisfying for each $k \in \{1, \dots, N\}$ and for a $C > 0$ it holds*

$$\|\nabla \varphi_k^0\|^2 \leq C.$$

There is a constant $C > 0$ such that for all $k \in \{1, \dots, N\}$ and for all $t \geq 0$

$$\|\nabla \varphi_k^t\|^2 \leq C e^{C \int_0^t d\tau (\|\rho_t\|_\infty + 1)}.$$

Under the additional assumption that for each $k \in \{1, \dots, N\}$ and for a $C > 0$

$$\|\Delta \varphi_k^0\|^2 \leq C,$$

it follows that there exists a non-negative function $t \mapsto C(t)$ which is affine linear in $D(t)$ such that for all $k \in \{1, \dots, N\}$ and for all $t \geq 0$

$$\|\Delta \varphi_k^0\|^2 + \|\nabla \varphi_k^t\|^2 \leq C e^{C \int_0^t d\tau (\|\rho_t\|_\infty + 1)}.$$

Proof. We neglect the implicit t-dependencies in the notation of our proof. Consider

$$\begin{aligned}
\left| \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \varphi_k^t\|^2 \right| &= \left| \langle \varphi_k^t, \left[-\Delta, u^{(N)} * \rho\right] \varphi_k^t \rangle \right| \\
&= \left| \langle \varphi_k^t, (\Delta u^{(N)} * \rho_t)\varphi_k^t + 2\left(\nabla u^{(N)} * \rho_t\right) \cdot \nabla \varphi_k^t \rangle \right| \\
&\leq 2\|\nabla u^{(N)} * \rho_t\|_\infty \|\nabla \varphi_k^t\| + \|\Delta u^{(N)} * \rho_t\|_\infty \|\varphi_k^t\|
\end{aligned} \tag{2.2.22}$$

Due to the assumption that $\|\rho\|_\infty \leq D(t)$ and Lemma 10 it holds $N^{1/3}\|\Delta u^{(N)} * \rho_t\|_\infty, \|\nabla u^{(N)} * \rho_t\|_\infty \leq C\left(\|\rho_t\|_\infty + 1\right)$. Thus, it holds by Grönwall's inequality

$$\|\nabla\varphi_k^t\|^2 \leq e^{C\int_0^t \mathrm{d}\tau(\|\rho_\tau\|_\infty+1)}\left(\|\nabla\varphi_k^0\|^2+1\right). \tag{2.2.23}$$

Similarly, it holds

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\varphi_k^t\|^2\right| &= \left|\langle\varphi_k^t, \left[(i\nabla)^4, u^{(N)} * \rho\right]\varphi_k^t\rangle\right| \\ &\leq 2\left|\langle-\Delta\varphi_k^t, (\nabla u^{(N)} * \rho_t)\cdot\nabla\varphi_k^t\rangle\right| + 3\left|\langle\varphi_k^t, (\Delta u^{(N)} * \rho_t)(-\Delta)\varphi_k^t\rangle\right| \\ &\quad + 2\left|\langle(\nabla u^{(N)} * \rho_t)\cdot\nabla\varphi_k^t, (-\Delta)\varphi_k^t\rangle\right| \\ &\leq 3\|\nabla u^{(N)} * \rho_t\|_\infty\left(\|\Delta\varphi_k^t\|^2 + \|\nabla\varphi_k^t\|^2\right) + \|\Delta u^{(N)} * \rho_t\|_\infty\left(1+\|\Delta\varphi_k^t\|^2\right) \\ &\leq C\left(\|\rho_t\|_\infty+1\right)\|\Delta\varphi_k^t\|^2 + C\left(\|\rho_t\|_\infty+1\right)\|\nabla\varphi_k^t\|^2 \end{aligned} \tag{2.2.24}$$

and therefore

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta\varphi_k^t\|^2 + \|\nabla\varphi_k^t\|^2\right)\right| \leq C\left(\|\rho_t\|_\infty+1\right)\left(\|\Delta\varphi_k^t\|^2 + \|\nabla\varphi_k^t\|^2 + 1\right). \tag{2.2.25}$$

■

2.3 Properties of the auxiliary dynamics

In this section, we define the auxiliary dynamics $\tilde{\Psi}_t$ generated by the simplified Hamiltonian $\tilde{H}^{\mathrm{g}}$. Subsequently, we show that the counting functional with respect to the auxiliary dynamics $\tilde{\Psi}_t$ can be bounded with a Grönwall estimate. In order to bound the norm difference between the gauged dynamics Ψ_t, generated by H^{g}, and the auxiliary dynamics $\tilde{\Psi}_t$, we need to establish a bound for the gradient term $\|\nabla_1 q_1\tilde{\Psi}_t\|$. This will be done in Subsection 2.3.2.

Let $\{\psi_k^t\}_{k=1}^N$ be orthonormal normalized solutions of the gauged mean-field equation (2.1.19). We define the projectors p and q with respect to orthonormal system $\{\psi_k^t\}_{k=1}^N$ as

$$p := p^t := p^{\psi_1^t,\ldots,\psi_N^t} := \sum_{j=1}^N |\psi_j^t\rangle\langle\psi_j^t|, \tag{2.3.1}$$

$$q := q^t := 1 - p^t. \tag{2.3.2}$$

It is convenient to define the projector $P_a^{\mathcal{C}}$ only acting on particles with indices in $\mathcal{C} \subseteq \{1,2,\ldots,N\}$ to for all $a = 0,1,\ldots|\mathcal{C}|$ as the sum over all products of projectors where in each summand q-projectors occur exactly a times (and p-operators $(|\mathcal{C}|-a)$ times) as

$$P_a^{\mathcal{C}} := \sum_{\{b_m\}_{m\in\mathcal{A}_a}} \prod_{m\in\mathcal{C}} p_m^{1-b_m}(q_m)^{b_m} \tag{2.3.3}$$

with

$$\mathcal{A}_a := \left\{ \overrightarrow{b} = \{b_m\}_{m \in \mathcal{C}} \in \{0,1\}^{|\mathcal{C}|} : \sum_{m \in \mathcal{C}} b_m = a \right\}. \tag{2.3.4}$$

We introduce the following notation

$$w_{mk}^{(a)} := \sum_{b=0}^{a} P_b^{\{m,k\}}(t) w_{mk} P_{a-b}^{\{m,k\}}(t), \tag{2.3.5}$$

$$w_{mkl}^{(a)} := \sum_{b=0}^{a} P_b^{\{m,k,l\}}(t) w_{mkl} P_{a-b}^{\{m,k,l\}}(t) \tag{2.3.6}$$

for any two-body multiplication operator of the form $w_{mk} = w(x_m - x_k)$ for $m, k \in \{1, \ldots, N\}$ and three-body multiplication operator w_{mkl} for $m, k, l \in \{1, \ldots, N\}$. Now, we consider the two-body terms

$$w_{mk}(t) := t(w_{\nabla f})_{mk} + t^2 (w_f)_{mk} + (u_{\leq})_{mk} \tag{2.3.7}$$

$$\text{with} \quad (w_{\nabla f})_{mk} := (i\nabla_m) \cdot f_{mk} + f_{mk} \cdot (i\nabla_m) + (i\nabla_k) \cdot f_{km} + f_{km} \cdot (i\nabla_k), \tag{2.3.8}$$

$$(w_f)_{mk} := f_{mk} \cdot f_{mk} + f_{km} \cdot f_{km}, \tag{2.3.9}$$

$$(u_{\leq})_{mk} := u_{\leq}^{(N)}(x_m - x_k) \tag{2.3.10}$$

and the three-body term

$$w_{mkl}(t) := t^2 (w_{ff})_{mkl} \tag{2.3.11}$$

$$\text{with} \quad (w_{ff})_{mkl} := 2f_{mk} \cdot f_{ml} + 2f_{km} \cdot f_{kl} + 2f_{lm} \cdot f_{lk} \tag{2.3.12}$$

for $m, k, l \in \{1, \ldots, N\}$.

The auxiliary Hamiltonian can be decomposed into its one-body, two-body and three-body part the following way

$$\widetilde{H^{\mathrm{g}}}(t) = \sum_{m=1}^{N} (-\Delta_m) + \sum_{1 \leq m < k \leq N} \tilde{w}_{mk}(t) + \sum_{1 \leq m < k < l \leq N} \tilde{w}_{mkl}(t) \tag{2.3.13}$$

with

$$\tilde{w}_{mk}(t) = w_{mk}^{(0)}(t) + w_{mk}^{(1)}(t) + w_{mk}^{(2)}(t) + (u_{\leq})_{mk}^{(3)}(t) + (u_{\leq})_{mk}^{(4)}(t), \tag{2.3.14}$$

$$\tilde{w}_{mkl}(t) = w_{mkl}^{(0)}(t) + w_{mkl}^{(1)}(t) + w_{mkl}^{(2)}(t). \tag{2.3.15}$$

The well-posedness of the auxiliary time evolution generated by $\widetilde{H^{\mathrm{g}}}(t)$ is given by the following statement:

Lemma 17. *Let $T > 0$ and assume that there is a $C > 0$ such that $D(\tau) = \max\{\|\rho_\tau\|_\infty, \|\rho_\tau^\nabla\|_\infty, \|\rho_\tau^\Delta\|_\infty, N^{-}$ C for all $\tau \in [0, T]$. Then there exists a unique two-parameter family of unitary operators $\tilde{U}(\tau, \sigma) : L^2_{as}(\mathbb{R}^{3N}) \to L^2_{as}(\mathbb{R}^{3N})$ such that*

1. *the $\tau \mapsto \tilde{U}(\tau, \sigma)\Psi$ is continuously differentiable for each $\sigma \in [0, T]$ and $\Psi \in L^2_{as}(\mathbb{R}^{3N})$ with $i\partial_\tau \tilde{U}(\tau, \sigma)\Psi_0 = \widetilde{H^{\mathrm{g}}}(\tau)\tilde{U}(\tau, \sigma)\Psi_0$,*
2. *$\tilde{U}(\tau, \tau) = 1$ and $\tilde{U}(\tau, \rho)\tilde{U}(\rho, \sigma) = \tilde{U}(\tau, \sigma)$ for all $\rho, \sigma, \tau \in [0, T]$,*
3. *$(\tau, \sigma) \mapsto \tilde{U}(\tau, \sigma)\Psi$ is strongly continuous on $[0, T]^2$ for all $\Psi \in L^2_{as}(\mathbb{R}^{3N})$.*
4. *The solution map $\Psi_0 \mapsto \tilde{U}(\tau, 0)\Psi_0$ is unitary.*

Proof. The proof is based on [SG16, Theorem 2.5] and analyzing the time-dependent part of $\widetilde{H^{\mathrm{g}}}(t)$

$$B(t) := \widetilde{H^{\mathrm{g}}}(t) - \sum_{i=1}^{N}(-\Delta_i) - \sum_{1\le k<l\le N} u_{\le}^{(N)}(x_k - x_l). \tag{2.3.16}$$

It is sufficient to show that $B(t)$ is a bounded operator and that $t \mapsto B(t)$ is Lipschitz continuous in operator norm.

Boundedness can be seen by $\|q_1^t\|_{\mathrm{op}}, \|p_1^t\|_{\mathrm{op}} \le C$, (36) and 10 since for all $m, k = 1, \ldots, N$ and $t \in [0, T]$

$$\|f_{mk}p_m^t\|_{\mathrm{op}}^2 \le \sup_{\|\psi\|=1} \langle \psi, p^t|f|^2 p^t\psi\rangle \lesssim \||f|^2 * \rho_t\|_\infty \lesssim D(t). \tag{2.3.17}$$

In addition, note that by definition of $\widetilde{H^{\mathrm{g}}}$ contains the full singular part $u_{\le}^{(N)}$ of the interaction and at most two q-operators per term such that gradient terms can always be evaluated with a p-operator, for instance it holds for all $\Psi \in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$ that

$$\begin{aligned}&\|p_1^t p_2^t f_{12} \cdot \nabla_1 q_1^t q_2^t \Psi\| \\ &\lesssim \|\nabla_1 p_1^t\|_{\mathrm{op}} \||f|^2 * \rho_t\|_\infty^{1/2} + \|p_1\|_{\mathrm{op}} \||\nabla_1 \cdot f_{12}|^2 * \rho_t\|_\infty^{1/2} \lesssim D(t)^{3/2}.\end{aligned} \tag{2.3.18}$$

where we used $\|\nabla_1 p_1\|_{\mathrm{op}} \lesssim \|\rho_t^\Delta\|_1$ in the last step.

The Lipschitz continuity of $t \mapsto B(t)$ follows from the Lipschitz continuity of $t \mapsto p^t = p^{\psi_1^t, \ldots, \psi_N^t}$. This is achieved by considering the time-derivative using $i\partial_t p^t = [h^{\mathrm{g}}(t), p^t]$ and $\|h_1^{\mathrm{g}}(t)p_1^t\|_{\mathrm{op}} \lesssim D(t)^2$ by Lemma 13. The last item is a direct consequence that $\widetilde{H^{\mathrm{g}}}(t)$ is self-adjoint. ■

We will use the notation $\tilde{U}_t := \tilde{U}(t, 0)$ and $\tilde{\Psi}_t := U_t\Psi_0$ with respect to an initial state $\Psi_0 \in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$ from now on for the two-parameter group form Lemma 17.

2.3.1 Convergence of the counting functional

Let $\tilde{\Psi}_t = \tilde{U}_t \Psi_0$ be generated by (2.3.13) for some $\Psi_0 \in L^2_{\mathrm{a}}(\mathbb{R}^{3N})$ with $\|\Psi_0\| = 1$ and $\{\psi_k^t\}_{k=1}^N$ be the orthonormal solutions of the gauged mean-field equation (2.1.19). In the following, we study the time-dependent counting functional

$$\tilde{\alpha}_f(t) := \alpha_f\left(\tilde{\Psi}_t, \psi_1^t, \dots, \psi_N^t\right)$$

for specific weight functions f. We define for $\gamma \in (0, 1]$

$$\begin{aligned} \tilde{\alpha}_m :=& \tilde{\alpha}_m(t) := \tilde{\alpha}_{m_{\gamma,t}} := \left\langle \tilde{\Psi}_t, \widehat{m_{\gamma,t}} \tilde{\Psi}_t \right\rangle, \\ \tilde{\alpha}_n :=& \tilde{\alpha}_n(t) := \tilde{\alpha}_{n_t} := \left\langle \tilde{\Psi}_t, \widehat{m_{1,t}} \tilde{\Psi}_t \right\rangle. \end{aligned}$$

It is convenient to decompose the gauged mean-field Hamiltonian as

$$h_1^{\mathrm{g}}(t) := -\Delta_1 + tR_1 + t^2 W_1 + \overline{u_{\leq}}(x_1) \tag{2.3.19}$$

with

$$R_j := R_j(t) := i\nabla_j \cdot \overline{f}(x_j) + \overline{f}(x_j) \cdot i\nabla_j + \overline{(i\nabla \cdot f)}(x_j) + \overline{(f \cdot i\nabla)}(x_j), \tag{2.3.20}$$

$$W_j := W_j(t) := 2\overline{f \cdot \overline{f}}(x_j) + \overline{f} \cdot \overline{f}(x_j). \tag{2.3.21}$$

Lemma 18 (Derivative of $\tilde{\alpha}_f$)**.** *It holds for all $t \geq 0$ that $\partial_t \tilde{\alpha}_f(t) = (\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+(\mathrm{V})$ with*

$$\begin{aligned} (\mathrm{I}) &:= \mathrm{Im}\left\langle \tilde{\Psi}_t, N\big(\hat{f} - \hat{f}_{-1}\big) \left((N-1) P_1^{\{1,2\}} w_{12}(t) P_0^{\{1,2\}} - 2q_1 \left(tR_1 + \overline{u_{\leq}}(x_1)\right) p_1 \right) \tilde{\Psi}_t \right\rangle, \\ (\mathrm{II}) &:= \mathrm{Im}\left\langle \tilde{\Psi}_t, N\big(\hat{f} - \hat{f}_{-2}\big) P_2^{\{1,2\}} (N-1) w_{12}(t) P_0^{\{1,2\}} \tilde{\Psi}_t \right\rangle, \\ (\mathrm{III}) &:= \mathrm{Im}\left\langle \tilde{\Psi}_t, N\big(\hat{f} - \hat{f}_{-1}\big) P_2^{\{1,2\}} (N-1) (u_{\leq})_{12} P_1^{\{1,2\}} \tilde{\Psi}_t \right\rangle \\ (\mathrm{IV}) &:= \frac{1}{3} \mathrm{Im}\langle \tilde{\Psi}_t, N\left(\hat{f} - \hat{f}_{-1}\right) \left((N-1)(N-2) P_1^{\{1,2,3\}} w_{123}(t) P_0^{\{1,2,3\}} - 6t^2 q_1 W_1 p_1 \right) \tilde{\Psi}_t \rangle, \\ (\mathrm{V}) &:= \frac{1}{3} \mathrm{Im}\langle \tilde{\Psi}_t, N\left(\hat{f} - \hat{f}_{-2}\right) P_2^{\{1,2,3\}} (N-1)(N-2) w_{123}(t) P_0^{\{1,2,3\}} \tilde{\Psi}_t \rangle. \end{aligned}$$

Proof of Lemma 18. Due to the antisymmetry of $\tilde{\Psi}_t$ it holds

$$
\begin{aligned}
\partial_t \tilde{\alpha}_f(t) &= \partial_t \langle \tilde{\Psi}_t, \hat{f} \tilde{\Psi}_t \rangle \\
&= i \langle \tilde{\Psi}_t, \left[\widetilde{H}^{\mathrm{g}}(t) - \sum_{j=1}^{N} h_j^{\mathrm{g}}(t), \hat{f} \right] \tilde{\Psi}_t \rangle \\
&= i \langle \tilde{\Psi}_t, \left[\sum_{1 \le m < k \le N} \tilde{w}_{mk}(t) + \sum_{1 \le m < k < l \le N} \tilde{w}_{mkl}(t) - \sum_{j=1}^{N} \left(tR_j + t^2 W_j + \overline{u_{\le}}(x_j) \right), \hat{f} \right] \tilde{\Psi}_t \rangle \\
&= i \langle \tilde{\Psi}_t, \left[\frac{N(N-1)}{2} \tilde{w}_{12}(t) - \frac{N}{2} \left(tR_1 + \overline{u_{\le}}(x_1) + tR_2 + \overline{u_{\le}}(x_2) \right), \hat{f} \right] \tilde{\Psi}_t \rangle \\
&\quad + i \langle \tilde{\Psi}_t, \left[\frac{N(N-1)(N-2)}{6} \tilde{w}_{123}(t) - t^2 \frac{N}{6} \left(2W_1 + 2W_2 + 2W_3 \right), \hat{f} \right] \tilde{\Psi}_t \rangle \\
&= i \langle \tilde{\Psi}_t, \left[A_{12}(t), \hat{f} \right] \tilde{\Psi}_t \rangle + i \langle \tilde{\Psi}_t, \left[B_{123}(t), \hat{f} \right] \tilde{\Psi}_t \rangle .
\end{aligned}
\tag{2.3.22}
$$

Analogously for as in [PP16, Lemma 6.5] one obtains for the two-body part by combining that $P_a^{\{1,2\}}$ consists of exactly two terms for $a = 1, 2$ with $w_{12} = w_{21}$ and $(u_{\le})_{12} = (u_{\le})_{21}$ and using that R_1 and R_2 give exactly the same contribution on antisymmetric states

$$
\begin{aligned}
&- \langle \tilde{\Psi}_t, \left[A_{12}(t), \hat{f} \right] \tilde{\Psi}_t \rangle \\
&= \frac{N}{2} \langle \tilde{\Psi}_t, (\hat{f} - \hat{f}_{-1}) \left((N-1) P_1^{\{1,2\}} w_{12}(t) P_0^{\{1,2\}} - 2 q_1 \left(tR_1 + \overline{u_{\le}}(x_1) \right) p_1 \right) \tilde{\Psi}_t \rangle - \text{h.c.} \\
&\quad + \frac{N(N-1)}{2} \langle \tilde{\Psi}_t, (\hat{f} - \hat{f}_{-2}) P_2^{\{1,2\}} w_{12}(t) P_0^{\{1,2\}} \tilde{\Psi}_t \rangle - \text{h.c.} \\
&\quad + \frac{N(N-1)}{2} \langle \tilde{\Psi}_t, (\hat{f} - \hat{f}_{-2}) P_2^{\{1,2\}} (u_{\le})_{12} P_1^{\{1,2\}} \tilde{\Psi}_t \rangle - \text{h.c.}
\end{aligned}
\tag{2.3.23}
$$

Similarly we obtain by combining that $P_a^{\{1,2,3\}}$ consists of exactly three terms for $a = 1, 2$ with $w_{123} = w_{132} = w_{213} = w_{321}$ and using that W_1, W_2 and W_3 give exactly the same contribution on antisymmetric states

$$
\begin{aligned}
&- \langle \tilde{\Psi}_t, \left[B_{123}(t), \hat{f} \right] \tilde{\Psi}_t \rangle \\
&= \frac{N}{6} \langle \Psi_t, \left(\hat{f} - \hat{f}_{-1} \right) \left((N-1)(N-2) P_1^{\{1,2,3\}} \tilde{w}_{123}(t) P_0^{\{1,2,3\}} \right) \tilde{\Psi}_t \rangle - \text{h.c.} \\
&\quad - \frac{N}{6} \langle \tilde{\Psi}_t, \left(\hat{f} - \hat{f}_{-1} \right) 6t^2 q_1 W_1 p_1 \tilde{\Psi}_t \rangle + \text{h.c.} \\
&\quad + \frac{N(N-1)(N-2)}{6} \langle \Psi_t, \left(\hat{f} - \hat{f}_{-2} \right) P_2^{\{1,2,3\}} w_{123}(t) P_0^{\{1,2,3\}} \tilde{\Psi}_t \rangle - \text{h.c.}
\end{aligned}
\tag{2.3.24}
$$

■

Lemma 19 (Estimates on $\partial_t \tilde{\alpha}_m$)**.** *Consider $D(t)$ as defined in Definition 12 as and $a = a(s)$ as defined in Lemma 10. In the setting of Lemma 18 with $\hat{f} = \hat{m}$ it holds for all $t \in \mathbb{R}$ that*

$$
\begin{aligned}
|(\mathrm{I})| \leq{}& CtD(t)\tilde{\alpha}_m^{1/2}\left(\tilde{\alpha}_m + N^{-\gamma}\right)^{1/2} + Ct^2D(t)\tilde{\alpha}_m^{1/2}N^{\frac{1}{2}-\frac{\gamma}{2}-a} \\
&+ CD(t)\tilde{\alpha}_m^{1/2}\left(N^{-a}\tilde{\alpha}_m + N^{-a-\gamma}\right)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
|(\mathrm{II})| \leq{}& CtD(t)\tilde{\alpha}_m^{1/2}\left(\left(\tilde{\alpha}_m + N^{1-\gamma-a}\right)^{1/2} + \left(\lambda_N^2\tilde{\alpha}_m + \lambda_N N^{1-\gamma}\right)^{1/2}\right) \\
&+ Ct^2D(t)\tilde{\alpha}_m^{1/2}\left(N^{-a}\tilde{\alpha}_m + \lambda_N^2 N^{-\gamma}\right)^{1/2} \\
&+ CD(t)\tilde{\alpha}_m^{1/2}\left(N^{-a}\tilde{\alpha}_m + N^{1-\gamma-a}\right)^{1/2},
\end{aligned}
$$

$$|(\mathrm{III})| \leq CD(t)N^{\frac{\gamma}{2}-\frac{a}{2}}\tilde{\alpha}_m,$$

$$|(\mathrm{IV})| \leq Ct^2D(t)^3\tilde{\alpha}_m^{1/2}\left(\tilde{\alpha}_m + N^{-\gamma}\right)^{1/2},$$

$$|(\mathrm{V})| \leq Ct^2D(t)^3\tilde{\alpha}_m^{1/2}\left(\tilde{\alpha}_m + N^{1-\gamma-a}\right)^{1/2}.$$

Remark 20. We remark that in all estimates it is essential to utilize the q-operators via the bounds Lemma 34 to close the Grönwall argument with the right order of N. By construction of the auxiliary Hamiltonian (2.3.13), we can avoid the occurrence of $\nabla_1 q_1$-terms, which would otherwise worsen the estimates.

Proof. We estimate each composite of $w_{12} = t(w_{\nabla f})_{12} + t^2(w_f)_{12}$ and $w_{123} = t^2(w_{ff})_{123}$ separately. The general strategy for $\hat{f} = \hat{m}$ is that we split

$$\left(\hat{m} - \hat{m}_{-d}\right) = \left(\hat{m} - \hat{m}_{-d}\right)^{1/2}\left(\hat{m} - \hat{m}_{-d}\right)^{1/2} \equiv \hat{D}_{-d}\hat{D}_{-d} \tag{2.3.25}$$

and push $\hat{D}_{-d}$ via Lemma 30 to the right. Subsequently, the q-operators are estimated with the weight functions via Lemma 34 and the p-operators are estimated with the interaction terms via Lemma 38.

Error terms: First we show that the w_f-term contributes only to the error. We estimate with Lemma 43

$$
\begin{aligned}
&|\mathrm{Im}\langle\tilde{\Psi}_t, N\left(\hat{m} - \hat{m}_{-1}\right)(N-1)P_1^{\{1,2\}}t^2(w_f)_{12}P_0^{\{1,2\}}\tilde{\Psi}_t\rangle| \\
&\leq t^2N^2\langle\hat{D}_{-1}\tilde{\Psi}_t, P_1^{\{1,2\}}(w_f)_{12}P_0^{\{1,2\}}\hat{E}_{-1}\tilde{\Psi}_t\rangle \\
&\leq 2t^2N\||f^{(N)}|^2 * \rho_t\|_\infty\|q_1\hat{D}_{-1}\tilde{\Psi}_t\|\|\hat{E}_{-1}\tilde{\Psi}_t\| \\
&\leq Ct^2D(t)N^{\frac{1}{2}-\frac{\gamma}{2}-a}\tilde{\alpha}_m^{1/2}
\end{aligned}
\tag{2.3.26}
$$

where we used the weight estimates for $\hat{D}_{-1}$ and $\hat{E}_{-1}$ from Lemma 34 and the bounds for the mean-field quantity from Lemma 10 in the last inequality.

Furthermore, it holds by applying Lemma 45 and Lemma 34

$$\begin{aligned}
&|\mathrm{Im}\big\langle\tilde{\Psi}_t, N\big(\hat{m}-\hat{m}_{-2}\big)P_2^{\{1,2\}}(N-1)t^2(w_f)_{12}P_0^{\{1,2\}}\tilde{\Psi}_t\big\rangle| \\
&\leq t^2N^2|\big\langle P_2^{\{1,2\}}\hat{D}_{-2}\tilde{\Psi}_t,(w_f)_{12}P_0^{\{1,2\}}\hat{E}_{-2}\tilde{\Psi}_t\big\rangle| \\
&\leq Ct^2\tilde{\alpha}_m^{1/2}\left(D(t)^2N^{-a}\tilde{\alpha}_m+D(t)\lambda_N^4N^{-\gamma}\right)^{1/2}.
\end{aligned} \tag{2.3.27}$$

<u>1q-terms</u>: The $1q$-term for $w_{\nabla f}$ is estimated by applying Lemma 41 after using $\hat{l}\widehat{l^{-1}}q_1\tilde{\Psi}_t$ and pushing $\hat{l}$ to the left using Lemma 30 and the estimate

<u>2q-terms</u>: For $w_{\nabla f}$-term we estimate with Lemma 45

$$\begin{aligned}
&|\mathrm{Im}\big\langle\tilde{\Psi}_t, N\big(\hat{m}-\hat{m}_{-2}\big)P_2^{\{1,2\}}(N-1)t(w_{\nabla f})_{12}P_0^{\{1,2\}}\tilde{\Psi}_t\big\rangle| \\
&\leq tN^2|\big\langle\hat{D}_{-2}\tilde{\Psi}_t,P_2^{\{1,2\}}(w_{\nabla f})_{12}P_0^{\{1,2\}}\hat{E}_{-2}\tilde{\Psi}_t\big\rangle| \\
&\leq Ct\tilde{\alpha}_m^{1/2}\Bigg(\left(D(t)^2N^{1-\gamma-a}+\left(D(t)^2+D(t)\|\rho_t^{\nabla}\|_1N^{-1}\right)\tilde{\alpha}_m\right)^{1/2} \\
&\qquad\qquad+\left(D(t)^2N^{-\frac{2}{3}}\tilde{\alpha}_m+D(t)^2N^{\frac{1}{3}-\gamma}\right)^{1/2}\Bigg)
\end{aligned} \tag{2.3.28}$$

and

$$\begin{aligned}
&|\mathrm{Im}\langle\tilde{\Psi}_t, N\left(\hat{m}-\hat{m}_{-2}\right)P_2^{\{1,2,3\}}(N-1)(N-2)w_{123}(t)P_0^{\{1,2,3\}}\tilde{\Psi}_t\rangle| \\
&\leq t^2N^3|\langle\hat{D}_{-2}\tilde{\Psi}_t,P_2^{\{1,2,3\}}(w_{ff})_{123}P_0^{\{1,2,3\}}\hat{E}_{-2}\tilde{\Psi}_t\rangle| \\
&\leq Ct^2D(t)^3\tilde{\alpha}_m^{1/2}\left(\tilde{\alpha}_m+N^{1-\gamma-a}\right)^{1/2}
\end{aligned} \tag{2.3.29}$$

Similarly, it holds for the $u_{\leq}$-term

$$\begin{aligned}
&|\mathrm{Im}\big\langle\tilde{\Psi}_t, N\big(\hat{m}-\hat{m}_{-2}\big)(N-1)P_2^{\{1,2\}}(u_{\leq})_{12}P_0^{\{1,2\}}\tilde{\Psi}_t\big\rangle| \\
&\leq N^2|\big\langle\hat{D}_{-2}\tilde{\Psi}_t,P_2^{\{1,2\}}(u_{\leq})_{12}P_0^{\{1,2\}}\hat{E}_{-2}\tilde{\Psi}_t\big\rangle| \\
&\leq CD(t)\lambda_N\tilde{\alpha}_m^{1/2}\left(\tilde{\alpha}_m+N^{1-\gamma}\right)^{1/2}
\end{aligned} \tag{2.3.30}$$

with $\lambda_N^2\leq N^{-a}$.

<u>3q-terms</u>: We estimate with Cauchy-Schwarz and Lemma 46

$$\begin{aligned}
&|\mathrm{Im}\big\langle\tilde{\Psi}_t, N\big(\hat{m}-\hat{m}_{-1}\big)q_1q_2(N-1)(u_{\leq})_{12}q_2p_1\tilde{\Psi}_t\big\rangle| \\
&\leq N(N-1)|\big\langle\hat{D}_{-1}\tilde{\Psi}_t,q_1q_2(u_{\leq})_{12}p_1q_2\hat{E}_{-1}\tilde{\Psi}_t\big\rangle| \\
&\leq CD(t)\lambda_NN^{\frac{\gamma}{2}}\tilde{\alpha}_m.
\end{aligned} \tag{2.3.31}$$

■

Proposition 21. *Let $D(t)$ as defined in Definition 12 and $a = a(s)$ as defined in Lemma 10. There exists a $C > 0$ such that for all $t \geq 0$ and for $\gamma \in (0,1]$*

$$\begin{aligned}\tilde{\alpha}_{m_\gamma}(t) &\leq e^{C(t)}\left(\tilde{\alpha}_{m_\gamma}(0) + N^{-(a+\gamma-1)}\right),\\ \tilde{\alpha}_n(t) &\leq e^{C(t)}\left(\tilde{\alpha}_l(0) + N^{-a}\right)\end{aligned}$$

with $C(t) = C\int_0^t d\tau\,((1+\tau)^2 D(\tau)^3)$ for a $C > 0$.

Remark 22. Recall that we we decompose the interaction potential into a singular part, as described in 2.1.15, and a residual part, which is incorporated in the gauge phase. The radius at which the decomposition occurs is chosen to be $N^{-\kappa}$ with $\kappa = 0$. For $\kappa \neq 0$, one would find a deteriorated convergence rate in Proposition 21 arising from the estimate on the singular part and Lemma 10.

Proof. Note that $\hat{m}_{\gamma=1} = \hat{n}$. The statements follow from Grönwall's inequality[1] ■

2.3.2 Estimates of the bad kinetic energy

In this section we want to show that the kinetic energy of of particles outside the orbitals is bounded in the following sense:

Proposition 23. *Let $\tilde{\Psi}_t$ be the solution of auxiliary dynamics as given in Lemma 17 with normalized initial state $\Phi_0 \in L^2_{as}(\mathbb{R}^{3N})$ and let $\{\psi_k^t\}_{k=1}^N$ be the solutions of (2.1.19) with normalized initial data $\{\varphi_k^0\}_{k=1}^N \in L^2(\mathbb{R}^3)$ satisfying for a $C_0 > 0$*

$$\begin{aligned}\|\nabla_1\Phi_0\|^2 - \frac{1}{N}\sum_{k=1}^N \|\nabla\varphi_k^0\|^2 &\leq C_0 N^{-\frac{a}{2}},\\ \tilde{\alpha}_n(0) &\leq C_0 N^{-a}\end{aligned}$$

with $a = a(s)$ as defined in Lemma 10. There exists a non-negative map $t \mapsto C(t)$ such that for all $t \geq 0$

$$\|\nabla_1 q_1 \tilde{\Psi}_t\| \leq e^{C(t)} N^{-\frac{a}{4}}$$

with $C(t) := C\int_0^t d\tau\,((1+\tau)^4 D(\tau)^4)$ for a $C > 0$.

[1] Grönwall's inequality states that for a differentiable function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\partial_t\alpha(t) \leq C(t)\,(\alpha(t) + \varepsilon)$ for $t \geq 0$ and for a continuous function $C : \mathbb{R} \to \mathbb{R}$ it follows for $t \geq 0$

$$\alpha(t) \leq e^{\int_0^t C(s)\mathrm{d}s}\alpha(0) + (e^{\int_0^t C(s)\mathrm{d}s} - 1)\varepsilon.$$

Remark 24. In fact, we find a more general estimate of the form

$$\|\nabla_1 q_1 \tilde{\Psi}_t\| \leq e^{C(t)/2} \left(\|\nabla_1 \Phi_0\|^2 - \frac{1}{N} \sum_{k=1}^{N} \|\nabla \varphi_k^0\|^2 + \tilde{\alpha}_n(0)^{1/2} + N^{1-a}\tilde{\alpha}_n(0) + N^{-\frac{a}{2}} \right)^{1/2} \tag{2.3.32}$$

with the same $C(t)$ as in the proposition. For the ease of the reader, we restrict here to the assumptions of the main theorem under which we obtain a more compact bound.

For the kinetic energy estimate, we define an adjusted energy functional with the following mean-field Hamiltonian

$$\widetilde{h^{\mathrm{g}}}_m(t) := -\Delta_m + \frac{t}{2} R_m + \frac{1}{2} \overline{u_{\leq}}(x_m) + \frac{t^2}{3} W_m \tag{2.3.33}$$

with R_m and W_m as defined in (2.3.20) and (2.3.21) respectively and for $m \in \{1, \dots, N\}$. Note the bounds in Lemma 13 for h^{g} can be applied to $\widetilde{h^{\mathrm{g}}}$ with the same dependencies in $D(t)$ and t since they are obtained by estimating $R_m, \overline{u_{\leq}}(x_m)$ and W_m separately.

We introduce the gauged fermionic Hartree energy

$$E^{\mathrm{g}}(t) := \sum_{k=1}^{N} \langle \psi_k^t, \widetilde{h^{\mathrm{g}}}(t) \psi_k^t \rangle \tag{2.3.34}$$

and the energy difference between the microscopic system and the fermionic Hartree system

$$\beta(t) := \langle \tilde{\Psi}_t \Big(\widetilde{H^{\mathrm{g}}}(t) - E^{\mathrm{g}}(t) \Big) \tilde{\Psi}_t \rangle. \tag{2.3.35}$$

With the following statement we are able to control the bad kinetic energy in terms of the energy difference:

Lemma 25. *Under the assumptions of Proposition 23, there exists a constant $C > 0$ such that for all $t \geq 0$ it holds*

$$\sum_{k=1}^{N} \|\nabla_k q_k \tilde{\Psi}_t\|^2 \leq C(1+t) D(t) \beta(t) + C(1+t)^2 D(t)^2 N \left(\tilde{\alpha}_n^{1/2}(t) + N^{-\frac{a}{2}} + N^{\frac{1-a}{2}} \tilde{\alpha}_n(t) \right).$$

Proof. We rewrite

$$\widetilde{H^{\mathrm{g}}} \equiv \sum_{m=1}^{N} (-\Delta_m) + W^{(0)} + W^{(1)} + W^{(2)} + U^{(3)} + U^{(4)} \tag{2.3.36}$$

where $W^{(j)}$ only consists of summands with exactly j q-operators and correspondingly for $U^{(j)} := \sum_{1\le m\le k\le N}(u_\le)^{(j)}_{mk}$ (see notation (2.3.5)). It then holds

$$\begin{aligned}\sum_{m=1}^N q_m(-\Delta_m)q_m &= \sum_{m=1}^N \{(-\Delta_m) - p_m(-\Delta_m)p_m - p_m(-\Delta_m)q_m - q_m(-\Delta_m)p_m\}\\ &= \widetilde{H}^{\mathrm{g}} - \left\{W^{(0)} + W^{(1)} + W^{(2)} + U^{(3)} + U^{(4)}\right\}\\ &\qquad - \sum_{m=1}^N \{p_m(-\Delta_m)p_m + p_m(-\Delta_m)q_m + q_m(-\Delta_m)p_m\}\,. \end{aligned} \tag{2.3.37}$$

Now we add and subtract $E^{\mathrm{g}}(t)$ such that

$$\sum_{m=1}^N \|\nabla_m q_m \Psi_t\|^2 = \beta(t) + D_0 + D_1 - D_2 - D_3 \tag{2.3.38}$$

with

$$D_0 = \left\langle \tilde{\Psi}_t, \left(E^{\mathrm{g}}(t) - \sum_{m=1}^N p_m(-\Delta_m)p_m - W^{(0)}\right) \tilde{\Psi}_t \right\rangle, \tag{2.3.39}$$

$$D_1 = \left\langle \tilde{\Psi}_t, \left(\sum_{m=1}^N (p_m(-\Delta_m)q_m + q_m(-\Delta_m)p_m) - W^{(1)}\right) \tilde{\Psi}_t \right\rangle, \tag{2.3.40}$$

$$D_2 = \left\langle \tilde{\Psi}_t, W^{(2)}\tilde{\Psi}_t \right\rangle, \tag{2.3.41}$$

$$D_3 = \left\langle \tilde{\Psi}_t, \left(U^{(3)} + U^{(4)}\right) \tilde{\Psi}_t \right\rangle. \tag{2.3.42}$$

The goal is now to show that D_0, D_1 and D_2 can be bounded by the alpha functionals. We emphasize that the main purpose of working with $\widetilde{h}^{\mathrm{g}}$ instead of h^{g} is to bound D_0 with an appropriate diagonalization estimate.

***D*₀-term**: To bound this term, we add and subtract $p_1\widetilde{h}^{\mathrm{g}}{}_1 p_1$ such that

$$D_0 = \left\langle \tilde{\Psi}_t \left(, \sum_{m=1}^N p_m \left(\widetilde{h}^{\mathrm{g}}{}_m + \Delta_m\right) p_m - W^{(0)} \right) \tilde{\Psi}_t \right\rangle + \left\langle \tilde{\Psi}_t, \left(E^{\mathrm{g}} - \sum_{m=1}^N p_m \widetilde{h}^{\mathrm{g}}{}_m p_m \right) \tilde{\Psi}_t \right\rangle. \tag{2.3.43}$$

Recall that $w_{12}(t) = t(w_{\nabla f})_{12} + t^2(w_f)_{12} + (u_\le)_{12}$ and $w_{123}(t) = t^2(w_{ff})_{123}$ and $\widetilde{h}^{\mathrm{g}}{}_m = -\Delta_m + \frac{t}{2}R_m + \frac{1}{2}\overline{u_\le}(x_m) + t^2 W_m$. We employ Lemma 40:

$$|\left\langle \tilde{\Psi}_t, \left(W^{(0)} - \sum_{m=1}^N p_m \left(\widetilde{h}^{\mathrm{g}}{}_m + \Delta_m\right) p_m \right) \tilde{\Psi}_t \right\rangle|$$

$$\le |\langle \tilde{\Psi}_t, \left(\sum_{1 \le m < k \le N} w^{(0)}_{mk}(t) - \frac{1}{2} \sum_{m=1}^{N} p_m \left(tR_m + \overline{u_{\le}}(x_m) \right) p_m \right) \tilde{\Psi}_t \rangle|$$

$$+ |\langle \tilde{\Psi}_t, \left(\sum_{1 \le m < k < l \le N} w^{(0)}_{mkl}(t) - \frac{t^2}{3} \sum_{m=1}^{N} p_m W_m p_m \right) \tilde{\Psi}_t \rangle|$$

$$
\begin{aligned}
&\leq t\frac{N}{2}|\langle\tilde{\Psi}_t, \left((N-1)P_0^{\{1,2\}}(w_{\nabla f})_{12}P_0^{\{1,2\}} - p_1 R_1 p_1\right)\tilde{\Psi}_t\rangle| \\
&\quad + \frac{N}{2}|\langle\tilde{\Psi}_t, \left((N-1)P_0^{\{1,2\}}(u_{\leq})_{12}P_0^{\{1,2\}} - p_1\overline{u_{\leq}}(x_1)p_1\right)\tilde{\Psi}_t\rangle| \\
&\quad + t^2\frac{N}{2}|\langle\tilde{\Psi}_t, (N-1)P_0^{\{1,2\}}(w_f)_{12}P_0^{\{1,2\}}\tilde{\Psi}_t\rangle| \\
&\quad + t^2\frac{N}{6}|\langle\tilde{\Psi}_t, \left((N-1)(N-2)P_0^{\{1,2,3\}}(w_{ff})_{123}P_0^{\{1,2,3\}} - 2t^2 p_1 W_1 p_1\right)\tilde{\Psi}_t\rangle| \\
&\leq C(1+t)^2 D(t) N^{1-a} + C(1+t)^2 D(t) N\tilde{\alpha}_n^{1/2}
\end{aligned}
\tag{2.3.44}
$$

with $D(t)$ from Definition 12. Additionally, we use Lemma 40 with $\tilde{h^{\mathrm{g}}}$ instead of h^{g} to estimate

$$
\begin{aligned}
|\langle\tilde{\Psi}_t, \left(E^{\mathrm{g}} - \sum_{m=1}^{N} p_m \widetilde{h^{\mathrm{g}}}_m p_m\right)\tilde{\Psi}_t\rangle| &\leq \|\rho_t^{\tilde{h}}\|_1^{1/2}\left(N\tilde{\alpha}_n + 1\right)^{1/2} \\
&\leq C(1+t)^2 D(t)^2 N\tilde{\alpha}_n^{1/2}.
\end{aligned}
\tag{2.3.45}
$$

<u>***D_1*-term**</u>: In contrast to the approach in the proof of Lemma 19 where we had to use the cancellation with the mean-field term, we estimate this term directly in terms of $\tilde{\alpha}_n^{1/2}$:

$$
\begin{aligned}
\langle\tilde{\Psi}_t, W^{(1)}\tilde{\Psi}_t\rangle &= \sum_{1\leq m<k\leq N}\langle\tilde{\Psi}_t, w_{mk}^{(1)}\tilde{\Psi}_t\rangle + \sum_{1\leq m<k<l\leq N}\langle\tilde{\Psi}_t, w_{mkl}^{(1)}\tilde{\Psi}_t\rangle \\
&= N(N-1)\mathrm{Re}\langle\tilde{\Psi}_t, P_0^{\{1,2\}} w_{12}(t) P_1^{\{1,2\}}\tilde{\Psi}_t\rangle \\
&\quad + \frac{N(N-1)(N-2)}{3}\mathrm{Re}\langle\tilde{\Psi}_t, P_0^{\{1,2,3\}} w_{123}(t) P_1^{\{1,2,3\}}\tilde{\Psi}_t\rangle
\end{aligned}
$$

with

$$
\begin{aligned}
&N(N-1)\mathrm{Re}\langle\tilde{\Psi}_t, P_0^{\{1,2\}}(w_{\nabla f})_{12}P_1^{\{1,2\}}\tilde{\Psi}_t\rangle \\
&\leq CtD(t)N(1+N^{-\frac{1}{3}})\tilde{\alpha}_n^{1/2},
\end{aligned}
\tag{2.3.46}
$$

$$
\begin{aligned}
&N(N-1)\mathrm{Re}\langle\tilde{\Psi}_t, P_0^{\{1,2\}}(w_f)_{12}P_1^{\{1,2\}}\tilde{\Psi}_t\rangle \\
&\leq Ct^2 D(t) N^{1-a}\tilde{\alpha}_n^{1/2},
\end{aligned}
\tag{2.3.47}
$$

$$
\begin{aligned}
&N(N-1)\mathrm{Re}\langle\tilde{\Psi}_t, P_0^{\{1,2\}}(u_{\leq})_{12}P_1^{\{1,2\}}\tilde{\Psi}_t\rangle \\
&\leq CD(t)\lambda_N N\tilde{\alpha}_n^{1/2}
\end{aligned}
\tag{2.3.48}
$$

with $\lambda_N \leq N^{-a/2}$ and

$$
\begin{aligned}
&\frac{N(N-1)(N-2)}{3}\mathrm{Re}\langle\tilde{\Psi}_t, P_0^{\{1,2,3\}} w_{123}(t) P_1^{\{1,2,3\}}\tilde{\Psi}_t\rangle \\
&= Ct^2 D(t)^2 N\tilde{\alpha}_n^{1/2}.
\end{aligned}
\tag{2.3.49}
$$

Moreover, the kinetic terms are estimated by Cauchy-Schwarz

$$\begin{aligned}
&|\langle\tilde{\Psi}_t, \sum_{m=1}^{N}\{p_m(-\Delta_m)q_m + q_m(-\Delta_m)p_m\}\,\tilde{\Psi}_t\rangle| \\
&\leq 2N|\langle\tilde{\Psi}_t, q_1(-\Delta_1)p_1\tilde{\Psi}_t\rangle| \\
&\leq 2N\|q_1\tilde{\Psi}_t\|\;\|\Delta_1 p_1\tilde{\Psi}_t\| \\
&\leq 2N\tilde{\alpha}_n^{1/2}\|\Delta_1 p_1\tilde{\Psi}_t\| \leq 2D(t)^{1/2}N\tilde{\alpha}_n^{1/2} \qquad (2.3.50)
\end{aligned}$$

where we used Lemma 38 in the last line to estimate

$$\|\Delta_1 p_1\tilde{\Psi}_t\|^2 \leq \frac{1}{N}\|\tilde{\Psi}_t\|^2\sum_{k=1}^{N}\|\Delta\psi_k^t\|^2 \leq D(t) \qquad (2.3.51)$$

with $D(t)$ from Definition 12.

<u>***D_2*-term**</u>: We decompose

$$\begin{aligned}
\left\langle\tilde{\Psi}_t, W^{(2)}\tilde{\Psi}_t\right\rangle &= \sum_{1\leq m<k\leq N}\left\langle\tilde{\Psi}_t, w_{mk}^{(2)}(t)\tilde{\Psi}_t\right\rangle + \sum_{1\leq m<k<l\leq N}\left\langle\tilde{\Psi}_t, w_{mkl}^{(2)}(t)\tilde{\Psi}_t\right\rangle \\
&= \frac{N(N-1)}{2}\left\langle\tilde{\Psi}_t, P_1^{\{1,2\}}w_{12}(t)P_1^{\{1,2\}}\tilde{\Psi}_t\right\rangle \qquad (2.3.52a)\\
&\quad + N(N-1)\mathrm{Re}\left\langle\tilde{\Psi}_t, P_0^{\{1,2\}}w_{12}(t)P_2^{\{1,2\}}\tilde{\Psi}_t\right\rangle \qquad (2.3.52b)\\
&\quad + \frac{N(N-1)(N-2)}{6}\left\langle\tilde{\Psi}_t, P_1^{\{1,2,3\}}w_{123}(t)P_1^{\{1,2,3\}}\tilde{\Psi}_t\right\rangle \qquad (2.3.52c)\\
&\quad + \frac{N(N-1)(N-2)}{3}\mathrm{Re}\left\langle\tilde{\Psi}_t, P_0^{\{1,2,3\}}w_{123}(t)P_2^{\{1,2,3\}}\tilde{\Psi}_t\right\rangle \qquad (2.3.52d)
\end{aligned}$$

where we recall that $w_{mk}(t) = t(w_{\nabla f})_{mk} + t^2(w_f)_{mk} + (u_\leq)_{mk}$ and $w_{mkl}(t) = t^2(w_{ff})_{mkl}$. We estimate using Lemma 44 and Lemma 45 for the symmetric $2q$-terms

$$\begin{aligned}
|(2.3.52a)| &\leq C(1+t)D(t)N\tilde{\alpha}_n + CtD(t)N\tilde{\alpha}_n^{1/2}\|\nabla_1 q_1\tilde{\Psi}_t\| + Ct^2D(t)N^{1-a}\tilde{\alpha}_n, \qquad (2.3.53)\\
|(2.3.52c)| &\leq Ct^2D(t)^2N\tilde{\alpha}_n \qquad (2.3.54)
\end{aligned}$$

and for the antisymmetric $2q$-terms

$$\begin{aligned}
|(2.3.52b)| \leq\; & CtD(t)N\tilde{\alpha}_n^{1/2}\left(\tilde{\alpha}_n + N^{-a}\right)^{1/2} \\
& + CtD(t)N\tilde{\alpha}_n^{1/2}\left(\tilde{\alpha}_n N^{-\frac{2}{3}} + \lambda_N^2\right)^{1/2} \\
& + D(t)\tilde{\alpha}_n^{1/2}\left(\tilde{\alpha}_n\lambda_N^2 + \lambda_N^2\right)^{1/2} \\
& + Ct^2D(t)N\tilde{\alpha}_n, \qquad (2.3.55)\\
|(2.3.52d)| \leq\; & Ct^2D(t)^2N\tilde{\alpha}_n^{1/2}\left(\tilde{\alpha}_n + N^{-a}\right)^{1/2}. \qquad (2.3.56)
\end{aligned}$$

Note that it holds due to the antisymmetry of $\tilde{\Psi}_t$

$$tD(t)N\tilde{\alpha}_n^{1/2}\|\nabla_1 q_1\tilde{\Psi}_t\| \leq Ct^2 D(t)^2 N\tilde{\alpha}_n + \frac{1}{2}\sum_{m=1}^{N}\|\nabla_m q_m\tilde{\Psi}_t\|^2. \tag{2.3.57}$$

<u>***D*₃-term**</u>: Note that since $0 \leq u_{\leq}$ we can neglect the $U^{(4)}$-term and it holds with Lemma 46

$$\begin{aligned}
-D_3 &\leq \sum_{1\leq k<l\leq N} |\langle\tilde{\Psi}_t, (u_{\leq})^{(3)}_{kl}\tilde{\Psi}_t\rangle| \\
&\leq N^2 \mathrm{Re}\langle\tilde{\Psi}_t, P_1^{\{1,2\}}(u_{\leq})_{12}P_2^{\{1,2\}}\tilde{\Psi}_t\rangle \\
&\leq CD(t)^{1/2}\lambda_N N^{\frac{3}{2}}\tilde{\alpha}_n \leq CD(t)^{1/2}N^{\frac{3}{2}-\frac{a}{2}}\tilde{\alpha}_n.
\end{aligned} \tag{2.3.58}$$

■

Lemma 26. *Consider the energy differences $\beta(t)$ as defined in (2.3.35) for $t \geq 0$. Under the assumptions of Proposition 23, there exists a constant $C > 0$ such that for all $t \geq 0$ it holds*

$$N^{-1}|\partial_t\beta(t)| \leq C(1+t)^4 D(t)^4\left(N^{-1}\beta(t) + \tilde{\alpha}_n(t)^{1/2} + N^{1-a}\tilde{\alpha}_n(t) + N^{-\frac{a}{2}}\right).$$

Proof. We use the time evolutions from Lemma 17 in the form of

$$i\partial_t\tilde{\Psi}_t = \widetilde{H}^{\mathrm{g}}\tilde{\Psi}_t, \tag{2.3.59}$$

$$i\partial_t p^t_m = [h^{\mathrm{g}}_m, p^t_m] = q^t_m h^{\mathrm{g}}_m p^t_m - p^t_m h^{\mathrm{g}}_m q^t_m \tag{2.3.60}$$

and will neglect the t-dependence of the operators p^t and q^t in our further notation. We calculate

$$\begin{aligned}
i\partial_t\beta(t) &= \langle\tilde{\Psi}_t, i\partial_t\left(\widetilde{H}^{\mathrm{g}} - E^{\mathrm{g}}\right)\tilde{\Psi}_t\rangle \\
&= \langle\tilde{\Psi}_t, \left((i\partial_t\widetilde{H}^{\mathrm{g}}) - \sum_{k=1}^{N}\langle\psi^t_k, (i\partial_t\widetilde{h}^{\mathrm{g}})\psi^t_k\rangle\right)\tilde{\Psi}_t\rangle + \sum_{k=1}^{N}\langle\psi^t_k, \left[h^{\mathrm{g}}, \widetilde{h}^{\mathrm{g}}\right]\psi^t_k\rangle
\end{aligned} \tag{2.3.61}$$

Recall that the Hamiltonians are time-dependent

$$\begin{aligned}
h^{\mathrm{g}}_m &= -\Delta_m + tR_m + t^2 W_m + \overline{u_{\leq}}(x_m) = \widetilde{h}^{\mathrm{g}}{}_m + \frac{t}{2}R_m + \frac{2}{3}t^2 W_m + \frac{1}{2}\overline{u_{\leq}}(x_m), \\
\widetilde{H}^{\mathrm{g}} &= \sum_{m=1}^{N}(-\Delta_m) + t\sum_{1\leq m<k\leq N}\left((w_{\nabla f})^{(0)}_{mk} + (w_{\nabla f})^{(1)}_{mk} + (w_{\nabla f})^{(2)}_{mk}\right) \\
&\quad + t^2\sum_{1\leq m<k\leq N}\left((w_f)^{(0)}_{mk} + (w_f)^{(1)}_{mk} + (w_f)^{(2)}_{mk}\right) \\
&\quad + t^2\sum_{1\leq m<k<l\leq N}\left((w_{ff})^{(0)}_{mkl} + (w_{ff})^{(1)}_{mkl} + (w_{ff})^{(2)}_{mkl}\right) \\
&\quad + \sum_{1\leq m<k\leq N}(u_{\leq})^{(1)}_{mk} + (u_{\leq})^{(2)}_{mk} + (u_{\leq})^{(3)}_{mk} + (u_{\leq})^{(4)}_{mk}
\end{aligned}$$

with R_m and W_m defined in (2.3.20) and (2.3.21), respectively, and the shorthand notation (2.3.5).

There are three sources of t-dependence in $\widetilde{h}^{\mathrm{g}}$ and $\widetilde{H}^{\mathrm{g}}$:

(a) the derivative of R_m, W_m and $\overline{u_{\leq}}(x_m)$ for each $m \in \{1, \ldots, N\}$ will cancel with the commutator term $[h^{\mathrm{g}}, \widetilde{h}^{\mathrm{g}}]$,

(b) the derivative of the explicit t-dependence will be treated as in the proof of Lemma 25,

(c) the derivative of $w^{(a)}$ will only contribute if the derivative hits a p-operator in $W^{(2)}$ due to cancellations of the following form

$$(\partial_t q_1) q_2 p_3 h_{123} p_1 p_2 p_3 + (\partial_t p_1) q_2 p_3 h_{123} p_1 p_2 p_3 = 0,$$

that is each derivative of a q-operator in a $W^{(a)}$-term has a canceling derivative of a p-operator in the corresponding $W^{(a-1)}$-term for $a = 1, 2$. Note that in (c), the singular part $u_{\leq}$ does not contribute since we included the singular interaction without any p, q-decomposition in the auxiliary Hamiltonian.

More explicitly, to estimate the time derivative (2.3.61) we decompose

$$\begin{aligned}
&(i\partial_t \widetilde{H}^{\mathrm{g}}) - \sum_{k=1}^{N} \langle \psi_k^t, (i\partial_t \widetilde{h}^{\mathrm{g}}) \psi_k^t \rangle + \sum_{k=1}^{N} \langle \psi_k^t, \left[h^{\mathrm{g}}, \widetilde{h}^{\mathrm{g}} \right] \psi_k^t \rangle \\
&= (\mathrm{a}) + (\mathrm{b}) + (\mathrm{c})
\end{aligned} \tag{2.3.62}$$

with

$$
\begin{aligned}
(\mathrm{a}) &:= \sum_{k=1}^{N} \langle \psi_k^t, \left[h^{\mathrm{g}}, \widetilde{h^{\mathrm{g}}}\right] \psi_k^t \rangle \\
&\quad - \sum_{k=1}^{N} \langle \psi_k^t, \left(\frac{t}{2}(i\partial_t R) + \frac{t^2}{3}(i\partial_t W) + \frac{1}{2}(i\partial_t \overline{u_{\leq}})\right) \psi_k^t \rangle, \qquad (2.3.63)\\
(\mathrm{b}) &:= i \sum_{1 \leq m < k \leq N} \left((w_{\nabla f})_{mk}^{(0)} + (w_f)_{mk}^{(0)}\right) - i\frac{1}{2} \sum_{m=1}^{N} p_m R_m p_m \\
&\quad + i2t \sum_{1 \leq m < k < l \leq N} (w_{ff})_{mkl}^{(0)} - i2\frac{t}{3} \sum_{m=1}^{N} p_m W_m p_m \\
&\quad + i \sum_{1 \leq m < k \leq N} \left((w_{\nabla f})_{mk}^{(1)} + (w_f)_{mk}^{(1)}\right) + i2t \sum_{1 \leq m < k < l \leq N} (w_{ff})_{mkl}^{(1)} \\
&\quad + i \sum_{1 \leq m < k \leq N} \left((w_{\nabla f})_{mk}^{(2)} + (w_f)_{mk}^{(2)}\right) + i2t \sum_{1 \leq m < k < l \leq N} (w_{ff})_{mkl}^{(2)}, \qquad (2.3.64)\\
(\mathrm{c}) &:= \sum_{1 \leq m < k \leq N} \left\langle \tilde{\Psi}_t, w_{mk}^{(2,\partial)} \tilde{\Psi}_t \right\rangle + \sum_{1 \leq m < k < l \leq N} \left\langle \tilde{\Psi}_t, w_{mkl}^{(2,\partial)} \tilde{\Psi}_t \right\rangle \qquad (2.3.65)
\end{aligned}
$$

where we introduced the notation $w^{(2,\partial)}$ corresponding to the cases where the derivative hits a p-operator in (c).

Now, we explain how to estimate each part of the time derivative.

(a): We first note that the following trace identities hold

$$
\begin{aligned}
E^{\mathrm{g}}(t) &= \mathrm{tr}_1\left(p_1 \widetilde{h^{\mathrm{g}}}_1(t) p_1\right), \qquad (2.3.66)\\
R_1 &= \mathrm{tr}_2\left(p_2 (w_{\nabla f})_{12} p_2\right), \qquad (2.3.67)\\
W_1 &= \frac{1}{2}\mathrm{tr}_{2,3}\left(p_2 p_3 (w_{ff})_{123} p_3 p_2\right), \qquad (2.3.68)\\
\overline{u_{\leq}}(x_1) &= \mathrm{tr}_2\left(p_2 u_{\leq}(x_1 - x_2) p_2\right) \qquad (2.3.69)
\end{aligned}
$$

where $\mathrm{tr}_2(A_{12}) = \sum_{n\in\mathbb{N}} \langle u_n, A_{12} u_n \rangle_2$ denotes the partial trace over the second variable of a two-body operator A_{12} and $\{u_n\}_{n\in\mathbb{N}}$ denotes an orthonormal system of $L^2(\mathbb{R}^3)$ and similarly for a three body operator A_{123}. Moreover, we can re-write (a) as the following trace terms

$$
\begin{aligned}
(\mathrm{a}) &= \mathrm{tr}_1\left(p_1 \left[h_1^{\mathrm{g}}, \left(\frac{t}{2} R_1 + \frac{t^2}{3} W_1 + \frac{1}{2}\overline{u_{\leq}}(x_1)\right)\right] p_1\right) \\
&\quad - \mathrm{tr}_1\left(p_1 \left(\frac{t}{2}(i\partial_t R_1) + \frac{t^2}{3}(i\partial_t W_1) + \frac{1}{2}(i\partial_t \overline{u_{\leq}}(x_1))\right) p_1\right). \qquad (2.3.70)
\end{aligned}
$$

Now, using the symmetry $(w_{\nabla f})_{12} = (w_{\nabla f})_{21}$ and $u_{\leq}(x_1 - x_2) = u_{\leq}(x_2 - x_1)$ it follows

$$\begin{aligned} \mathrm{tr}_1\left(p_1(i\partial_t R_1)p_1\right) &= -\mathrm{tr}_1\left(p_1 \mathrm{tr}_2\left(p_2\left[h_2^{\mathrm{g}}, (w_{\nabla f})_{12}\right] p_2\right) p_1\right) \\ &= -\mathrm{tr}_{1,2}\left(p_1 p_2\left[h_2^{\mathrm{g}}, (w_{\nabla f})_{12}\right] p_2 p_1\right) \\ &= -\mathrm{tr}_2\left(p_2\left[h_2^{\mathrm{g}}, (w_{\nabla f})_{12}\right] p_2\right) \\ &= -\mathrm{tr}_2\left(p_2\left[h_2^{\mathrm{g}}, R_2\right] p_2\right) \end{aligned} \tag{2.3.71}$$

and completely analogously

$$\mathrm{tr}_1\left(p_1(i\partial_t \overline{u_{\leq}}(x_1))p_1\right) = -\mathrm{tr}_2\left(p_2\left[h_2^{\mathrm{g}}, \overline{u_{\leq}}(x_2)\right] p_2\right). \tag{2.3.72}$$

Similarly, using the symmetry of $(w_{ff})_{123}$ we obtain

$$\begin{aligned} \mathrm{tr}_1\left(p_1(i\partial_t W_1)p_1\right) &= -\frac{1}{2}\mathrm{tr}_1\left(p_1 \mathrm{tr}_{2,3}\left(p_2 p_3\left[h_2^{\mathrm{g}} + h_3^{\mathrm{g}}, (w_{ff})_{123}\right] p_3 p_2\right) p_1\right) \\ &= -\mathrm{tr}_{1,2,3}\left(p_1 p_2 p_3\left[h_2^{\mathrm{g}}, (w_{ff})_{123}\right] p_3 p_2 p_1\right) \\ &= -2\mathrm{tr}_2\left(p_2\left[h_2^{\mathrm{g}}, W_2\right] p_2\right). \end{aligned} \tag{2.3.73}$$

Therefore, we obtain the identity

$$(\mathrm{a}) = \mathrm{tr}_1\left(p_1\left[h_1^{\mathrm{g}}, \left(tR_1 + t^2 W_1 + \overline{u_{\leq}}(x_1)\right)\right] p_1\right) = \mathrm{tr}_1\left(p_1\left[h_1^{\mathrm{g}}, h_1^{\mathrm{g}}\right] p_1\right) = 0. \tag{2.3.74}$$

(b): This term corresponds to the D_0-term, D_1-term and D_2-term from the proof of Lemma 25 where the time derivative acted on the explicit time dependence. Thus, we find the estimate

$$|\langle \tilde{\Psi}_t, (\mathrm{b})\tilde{\Psi}_t\rangle| \leq C(1+t)D(t)N\left(N^{-1}\beta + \tilde{\alpha}_n^{1/2} + N^{\frac{1-a}{2}}\tilde{\alpha}_n + N^{-\frac{a}{2}}\right). \tag{2.3.75}$$

(c): In this part, we are only estimating the non-singular part

$$\breve{w}_{mk}(t) := w_{mk}(t) - (u_{\leq})_{mk} = t(w_{\nabla f})_{mk} + t^2(w_f)_{mk} \tag{2.3.76}$$

It holds due to the antisymmetry

$$\begin{aligned} &\left\langle \tilde{\Psi}_t, (\mathrm{c})\tilde{\Psi}_t\right\rangle \\ &= \sum_{1\leq m<k\leq N}\left\langle \tilde{\Psi}_t, w_{mk}^{(2,\partial)}(t)\tilde{\Psi}_t\right\rangle + \sum_{1\leq m<k<l\leq N}\left\langle \tilde{\Psi}_t, w_{mkl}^{(2,\partial)}(t)\tilde{\Psi}_t\right\rangle \\ &= \frac{N(N-1)}{2}\left\langle \tilde{\Psi}_t, w_{12}^{(2,\partial)}(t)\tilde{\Psi}_t\right\rangle + \frac{N(N-1)(N-2)}{6}\left\langle \tilde{\Psi}_t, w_{123}^{(2,\partial)}(t)\tilde{\Psi}_t\right\rangle \end{aligned}$$

with

$$\begin{aligned}
w_{12}^{(2,\partial)}(t) &:= \left(i\partial_t P_0^{\{1,2\}}\right) \breve{w}_{12}(t) P_2^{\{1,2\}} + P_2^{\{1,2\}} \breve{w}_{12}(t) \left(i\partial_t P_0^{\{1,2\}}\right) \\
&\quad + \left((i\partial_t p_1) q_2 + q_1 (i\partial_t p_2)\right) \breve{w}_{12}(t) P_1^{\{1,2\}} + P_1^{\{1,2\}} \breve{w}_{12}(t) \left((i\partial_t p_1) q_2 + q_1 (i\partial_t p_2)\right),
\end{aligned} \tag{2.3.77}$$

$$\begin{aligned}
w_{123}^{(2,\partial)}(t) &:= \left(i\partial_t P_0^{\{1,2,3\}}\right) w_{123}(t) P_2^{\{1,2,3\}} + P_2^{\{1,2,3\}} w_{123}(t) \left(i\partial_t P_0^{\{1,2,3\}}\right) \\
&\quad + \left(q_3 (i\partial_t p_1 p_2) + q_1 (i\partial_t p_2 p_3) + q_2 (i\partial_t p_1 p_3)\right) w_{123}(t) P_1^{\{1,2,3\}} \\
&\quad + P_1^{\{1,2,3\}} w_{123}(t) \left(q_3 (i\partial_t p_1 p_2) + q_1 (i\partial_t p_2 p_3) + q_2 (i\partial_t p_1 p_3)\right).
\end{aligned} \tag{2.3.78}$$

Using $w_{12}(t) = w_{21}(t)$ we further simplify

$$\begin{aligned}
&\left\langle \tilde{\Psi}_t, \left(\partial_t P_0^{\{1,2\}}\right) \breve{w}_{12}(t) P_2^{\{1,2\}} \tilde{\Psi}_t \right\rangle \\
&= 2\langle \tilde{\Psi}_t, \left(h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}}\right) p_2 \breve{w}_{12}(t) P_2^{\{1,2\}} \tilde{\Psi}_t \rangle,
\end{aligned} \tag{2.3.79}$$

$$\begin{aligned}
&\langle \tilde{\Psi}_t, \left((\partial_t p_1) q_2 + q_1 (\partial_t p_2)\right) \breve{w}_{12}(t) P_1^{\{1,2\}} \tilde{\Psi}_t \rangle \\
&= 2\langle \tilde{\Psi}_t, \left(h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}}\right) q_2 \breve{w}_{12}(t) P_1^{\{1,2\}} \tilde{\Psi}_t \rangle,
\end{aligned} \tag{2.3.80}$$

and using $w_{123} = w_{213} = w_{321}$

$$\begin{aligned}
&\left\langle \tilde{\Psi}_t, \left(\partial_t P_0^{\{1,2,3\}}\right) w_{123}(t) P_2^{\{1,2,3\}} \tilde{\Psi}_t \right\rangle \\
&= 3\langle \tilde{\Psi}_t, \left(h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}}\right) p_2 p_3 w_{123}(t) P_2^{\{1,2,3\}} \tilde{\Psi}_t \rangle,
\end{aligned} \tag{2.3.81}$$

and

$$\begin{aligned}
&\langle \tilde{\Psi}_t, \left(q_3 (\partial_t p_1 p_2) + q_1 (\partial_t p_2 p_3) + q_2 (\partial_t p_1 p_3)\right) w_{123}(t) P_1^{\{1,2,3\}} \tilde{\Psi}_t \rangle \\
&= 6\langle \tilde{\Psi}_t, \left(h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}}\right) \left(p_2 q_3 w_{123}(t) P_1^{\{1,2,3\}}\right) \tilde{\Psi}_t \rangle.
\end{aligned} \tag{2.3.82}$$

Thus, it holds

$$\left\langle \tilde{\Psi}_t, (\mathrm{c}) \tilde{\Psi}_t \right\rangle$$

$$= i2N(N-1)\mathrm{Re}\langle \tilde{\Psi}_t, \left(h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}}\right) p_2 \breve{w}_{12}(t) P_2^{\{1,2\}} \tilde{\Psi}_t \rangle \tag{2.3.83a}$$

$$+ iN(N-1)(N-2)\mathrm{Re}\langle \tilde{\Psi}_t, \left(h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}}\right) p_2 p_3 w_{123}(t) P_2^{\{1,2,3\}} \tilde{\Psi}_t \rangle \tag{2.3.83b}$$

$$+ i2N(N-1)\mathrm{Re}\langle \tilde{\Psi}_t, \left(h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}}\right) q_2 \breve{w}_{12}(t) P_1^{\{1,2\}} \tilde{\Psi}_t \rangle \tag{2.3.83c}$$

$$+ i2N(N-1)(N-2)\mathrm{Re}\langle \tilde{\Psi}_t, \left(h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}}\right) \left(p_2 q_3 w_{123}(t) P_1^{\{1,2,3\}}\right) \tilde{\Psi}_t \rangle \tag{2.3.83d}$$

which is estimated by using Lemma 49 with $\psi \equiv \tilde{\Psi}_t$ and the bounds for ρ_t^h by Lemma 13

$$\begin{aligned}
&|(2.3.83a)| \\
&\leq C(1+t)^3 D(t)^3 N \|\nabla_1 q_1 \tilde{\Psi}_t\| \left(N^{1-a}\tilde{\alpha}_n + N^{-a}\right)^{1/2} \\
&\quad + C(1+t)^3 D(t)^3 N \|\nabla_1 q_1 \tilde{\Psi}_t\| \left(N^{-a} + \tilde{\alpha}_n\right)^{1/2} \\
&\quad + C(1+t)^3 D(t)^3 N^{2-a}\tilde{\alpha}_n \\
&\quad + C(1+t)^4 D(t)^3 N^{1-a} \tilde{\alpha}_n^{1/2} \left(1+\tilde{\alpha}_n\right)^{1/2},
\end{aligned} \tag{2.3.84}$$

$$\begin{aligned}
&|(2.3.83b)| \\
&\leq C(1+t)^4 D(t)^4 N \tilde{\alpha}_n^{1/2} \left(N^{-a} + \tilde{\alpha}_n\right)^{1/2} \\
&\quad + Ct^2 (1+t)^4 D(t)^4 N^{1+\frac{1-a}{2}} \tilde{\alpha}_n,
\end{aligned} \tag{2.3.85}$$

and by using Lemma 50

$$\begin{aligned}
&|(2.3.83c)| \\
&\leq C(1+t)^3 D(t)^3 N^{1+\frac{1-a}{2}} \tilde{\alpha}_n^{1/2} \left(\tilde{\alpha}_n^{1/2} + \|\nabla_1 q_1 \tilde{\Psi}_t\|\right) \\
&\quad + C(1+t)^3 D(t)^3 N \left(\tilde{\alpha}_n^{1/2} \|\nabla_1 q_1 \tilde{\Psi}_t\| + N^{-\frac{1}{3}} \tilde{\alpha}_n\right) \\
&\quad + C(1+t)^4 D(t)^4 N^{\frac{1}{2}-a} \tilde{\alpha}_n,
\end{aligned} \tag{2.3.86}$$

$$\begin{aligned}
&|(2.3.83d)| \\
&\leq C(1+t)^4 D(t)^4 N^{1+\frac{1-a}{2}} \tilde{\alpha}_n \\
&\quad + C(1+t)^4 D(t)^4 N \tilde{\alpha}_n.
\end{aligned} \tag{2.3.87}$$

In total, by using Lemma 25 we arrive at

$$|\langle \tilde{\Psi}_t, (\mathrm{c}) \tilde{\Psi}_t \rangle| \leq C(1+t)^4 D(t)^4 N \left(N^{-1}\beta + \tilde{\alpha}_n^{1/2} + N^{1-a}\tilde{\alpha}_n + N^{-\frac{a}{2}}\right). \tag{2.3.88}$$

■

Proof of Proposition 23. From Lemma 26, we infer by Grönwall's inequality

$$\begin{aligned}
&N^{-1}\beta(t) \\
&\leq e^{C\int_0^t \mathrm{d}\tau\ (1+\tau)^4 D(\tau)^4} \left(N^{-1}\beta(0) + \int_0^t \mathrm{d}s \left(\tilde{\alpha}_n(t)^{1/2} + N^{1-a}\tilde{\alpha}_n(t) + N^{-\frac{a}{2}}\right)\right) \\
&\leq e^{C\int_0^t \mathrm{d}\tau\ (1+\tau)^4 D(\tau)^4} \left(N^{-1}\beta(0) + t\left(\tilde{\alpha}_n(0)^{1/2} + N^{1-a}\tilde{\alpha}_n(0) + N^{-\frac{a}{2}}\right)\right)
\end{aligned} \tag{2.3.89}$$

where we inserted Proposition 21 and used $a \geq 2/3$ for $s \leq 1$. Using the antisymmetry of $\tilde{\Psi}_t$ it holds with Lemma 25

$$\begin{aligned}\|\nabla_1 q_1 \tilde{\Psi}_t\|^2 &= \frac{1}{N}\sum_{k=1}^{N} \|\nabla_k q_k \tilde{\Psi}_t\|^2 \\ &\leq C(1+t)D(t)N^{-1}\beta(t) \\ &\quad + C(1+t)^2 D(t)^2 \left(\tilde{\alpha}_n^{1/2}(t) + N^{-\frac{a}{2}} + N^{\frac{1-a}{2}}\tilde{\alpha}_n(t)\right) \\ &\leq e^{C\int_0^t \mathrm{d}\tau\ (1+\tau)^4 D(\tau)^4}\left(N^{-1}\beta(0) + \tilde{\alpha}_n(0)^{1/2} + N^{1-a}\tilde{\alpha}_n(0) + N^{-\frac{a}{2}}\right) \end{aligned} \tag{2.3.90}$$

where we used that $(1+t)^n D(t)^n \geq 1$ for all $n \in \mathbb{N}$ and thus the prefactor can be absorbed in the exponent. Note that by definition (2.3.35) it holds

$$N^{-1}\beta(0) = \langle \tilde{\Psi}_0, (-\Delta_1)\tilde{\Psi}_0\rangle - \frac{1}{N}\sum_{k=1}^{N}\langle \psi_k^0, (-\Delta)\psi_k^0\rangle = \|\nabla_1 \Phi_0\|^2 - \frac{1}{N}\sum_{k=1}^{N}\|\nabla \varphi_k^0\|^2.$$

We assumed initial conditions such that $N^a\tilde{\alpha}_n(0) \leq C_0$ for a universal constant $C_0 > 0$ to obtain the desired result. ■

2.4 Norm approximation of the gauged dynamics

It remains to show the norm approximation between the gauged dynamics and its auxiliary dynamics. However, a direct application of Duhamel's formula in the form of

$$\|\Psi_t - \tilde{\Psi}_t\|^2 \leq 2\int_0^t \mathrm{d}s\ \langle \Psi_s, \left(H^{\mathrm{g}}(s) - \widetilde{H^{\mathrm{g}}}(s)\right), \tilde{\Psi}_s\rangle \tag{2.4.1}$$

is not sufficient, as one encounters terms of the form

$$\begin{aligned}&N(N-1)t|\langle \Psi_s, q_1 p_2 \nabla_1 \cdot f_{12}^{(N)} q_2 q_1 \tilde{\Psi}_s\rangle| \\ &\leq N^{\frac{3}{2}} \||f^{(N)}|^2 * \rho_t\|_\infty^{1/2} \|q_1 q_2 \tilde{\Psi}_s\|\ \|\nabla_1 q_1 \Psi_s\|\end{aligned} \tag{2.4.2}$$

Even if it would hold $\|\nabla_1 q_1 \Psi_s\|, N^a\|q_1 q_2 \tilde{\Psi}_s\| \leq C(s)$ for a $C(s) > 0$ under appropriate assumptions, we still can only find an upper bound for (2.4.2) of the $\mathcal{O}(N^{3/2}N^{-3a/2})$ which is even at best not small with the estimates of Lemma 10.

In order to make the most of the properties of the auxiliary dynamics we introduce a weight function

$$\hat{w}_t := \hat{w}_{\gamma,t} = 1 - \hat{m}_{\gamma,t} \qquad \text{for } \gamma \in (0,1] \tag{2.4.3}$$

and prove the following lemma:

Lemma 27. *Let $U_t\Psi_0 = \Psi_t$ as defined in (2.1.28) and $\tilde{U}_t\Psi_0 = \tilde{\Psi}_t$ be the solutions of (3.1.3) and (2.1.18), respectively, with normalized initial state $\Psi_0 \in L^2_{as}(\mathbb{R}^{3N})$ and interaction parameter $s \in (0, 2/3)$. Consider*

$$A_{\gamma(s)}(t) := \left\langle \Psi_t, \left(H^{\mathrm{g}}(t) - \widetilde{H^{\mathrm{g}}}(t)\right) \hat{w}_{\gamma(s),t}\tilde{\Psi}_t \right\rangle,$$

$$B_{\gamma(s)}(t) := \left\langle \Psi_t, \left[\widetilde{H^{\mathrm{g}}}(t) - \sum_{j=1}^{N} h_j^{\mathrm{g}}(t), \hat{m}_{\gamma(s),t}\right] \tilde{\Psi}_t \right\rangle$$

with $\gamma(s) = \max\{1/8, (2-s)/12\}$.
Under the assumption of Proposition 23 such that for all $t \geq 0$

$$|A_{\gamma(s)}(t)| + |B_{\gamma(s)}(t)| \leq e^{C(t)} \max\left\{N^{-\frac{1}{16}}, N^{-\frac{2-3s}{8}}\right\}$$

with $C(t) = C\int_0^t d\tau\ (1+\tau)^4 D(\tau)^2$ for a constant $C > 0$.

Remark 28. The term $B_\gamma(t)$ corresponds to an analogue of the difference appearing in the Grönwall argument Lemma 18 as derivative of the counting functional. The term $A_\gamma(t)$ contains the remainder terms neglected in the auxiliary Hamiltonian. Since $(H^{\mathrm{g}} - \widetilde{H^{\mathrm{g}}})$ include terms with many q-operators, it is unavoidable to encounter a gradient term $\nabla_1 q_1$. As to be seen, it is crucial that we can control the $\|\nabla_1 q_1 \tilde{\Psi}_t\|$ via Proposition 23. In comparison to (2.4.1) we have an additional weight function $\hat{w}_{\gamma,t}$ to utilize q-operators via Lemma 35.

Before turning to the proof, we show how to obtain as an immediate consequence of Lemma 27 the desired norm approximation:

Proposition 29. *Let $U_t\Psi_0 = \Psi_t$ as defined in (2.1.28) and $\tilde{U}_t\Psi_0 = \tilde{\Psi}_t$ be the solutions of (3.1.3) and (2.1.18) respectively with normalized initial state $\Psi_0 \in L^2_{as}(\mathbb{R}^{3N})$. Under the assumptions of Proposition 23, there exists a $C > 0$ such that for all $t \geq 0$ and $s \in (0, 2/3)$*

$$\|\tilde{\Psi}_t - \Psi_t\| \leq e^{C(t)} \max\left\{N^{-\frac{1}{32}}, N^{-\frac{2-3s}{16}}\right\}$$

with $C(t) = C\int_0^t d\tau\ (1+\tau)^4 D(\tau)^2$.

Proof of Proposition 29. By triangle inequality it holds for any $\gamma \in [0, 1]$

$$\begin{aligned}
\|\tilde{\Psi}_t - \Psi_t\| &\leq \|\tilde{\Psi}_t - \hat{w}_{\gamma,t}\tilde{\Psi}_t\| + \|\hat{w}_{\gamma,t}\tilde{\Psi}_t - \Psi_t\| \\
&\leq \sqrt{2\mathrm{Re}\langle \tilde{\Psi}_t, \tilde{\Psi}_t - \hat{w}_{\gamma,t}\tilde{\Psi}_t\rangle} + \sqrt{2\mathrm{Re}\langle \Psi_t, \Psi_t - \hat{w}_{\gamma,t}\tilde{\Psi}_t\rangle} \\
&= \sqrt{2}\tilde{\alpha}_{m_\gamma}^{1/2} + \sqrt{2\mathrm{Re}\left\{\langle \Psi_0, \hat{m}_{\gamma,0}\Psi_0\rangle + \langle \Psi_0, (\hat{w}_{\gamma,0} - U_t^*\hat{w}_{\gamma,t}\tilde{U}_t)\Psi_0\rangle\right\}}
\end{aligned} \tag{2.4.4}$$

using $U_{t,0}\Psi_0 = \Psi_t$ and $\tilde{U}_{t,0}\Psi_0 = \tilde{\Psi}_t$. Note that $\tilde{\alpha}_{m_\gamma}$ is bounded by Proposition 21 and

$$-\langle \Psi_0, (\hat{w}_{\gamma,0} - U_t^* \hat{w}_{\gamma,t} \tilde{U}_t)\Psi_0\rangle = \left\langle \Psi_0, \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} U_s^* \hat{w}_{\gamma,s} \tilde{U}_s \Psi_0 \ \mathrm{d}s \right\rangle = i \int_0^t \mathrm{d}s \ \{A_\gamma(s) + B_\gamma(s)\}\,. \tag{2.4.5}$$

Note that for any $\gamma \in [0,1]$ it holds $kN^{-\gamma} \leq N^{1-\gamma} kN^{-1}$ it follows $\tilde{\alpha}_{m_\gamma} \leq N^{1-\gamma}\tilde{\alpha}_n$ and therefore $\tilde{\alpha}_n(0) \leq C_0 N^{-a(s)}$ implies $\tilde{\alpha}_{m_{\gamma(s)}}(0) \leq N^{1-\gamma-a(s)}$. Recall that $a(s) \in [2/3, 1]$ for any $s \in (0,1]$. Thus, by taking the choice of Lemma 27, which can be written as $\gamma(s) = a(s)/8$, it follows

$$|A_{\gamma(s)}(t)| + |B_{\gamma(s)}(t)| \leq Ce^{C(t)} N^{-\frac{9a(s)-8}{32}} \tag{2.4.6}$$

and we obtain the desired result. ■

Proof of Lemma 27. Note that the bound for $|B_\gamma(s)|$ is similar to the Grönwall argument Lemma 18 but with two different states on the left and right side of the inner product. Similarly to Lemma 18, we can use of $\|q_1 \hat{m}_\gamma^{1/2} \tilde{\Psi}_t\|^2 \leq CN^{-1}\tilde{\alpha}_{m_\gamma}$ and $\tilde{\alpha}_{m_\gamma} \leq C(t)\left(N^{-(a+\gamma-1)}\right)$ from Proposition 21.

Using the antisymmetry of Ψ_t and $\tilde{\Psi}_t$, we obtain similarly to (2.3.23) and (2.3.24)

$$\begin{aligned}
&\left\langle \Psi_t, \left[\widetilde{H}^{\mathrm{g}} - \sum_{j=1}^N h_j^{\mathrm{g}}(t), \hat{m} \right] \tilde{\Psi}_t \right\rangle \\
&= \left\langle \Psi_t, \left[\frac{N(N-1)}{2} \tilde{w}_{12}(t) - \frac{N}{2}\left(tR_1 + \overline{u_\leq}(x_1) + tR_2 + \overline{u_\leq}(x_2)\right), \hat{m} \right] \tilde{\Psi}_t \right\rangle \\
&\quad + \left\langle \Psi_t, \left[\frac{N(N-1)(N-2)}{6} \tilde{w}_{123}(t) - t^2 \frac{N}{3}(W_1 + W_2 + W_3), \hat{m} \right] \tilde{\Psi}_t \right\rangle \\
&= \frac{1}{2} \left\langle \Psi_t, [A_{12}(t), \hat{m}]\, \tilde{\Psi}_t \right\rangle + \frac{1}{6} \left\langle \Psi_t, [B_{123}(t), \hat{m}]\, \tilde{\Psi}_t \right\rangle
\end{aligned} \tag{2.4.7}$$

with

$$\begin{aligned}
&\left\langle \Psi_t, [A_{12}(t), \hat{m}]\, \tilde{\Psi}_t \right\rangle \\
&= N\left\langle \Psi_t, \left((N-1) P_0^{\{1,2\}} w_{12}(t) P_1^{\{1,2\}} - 2p_1 \left(tR_1 + \overline{u_\leq}(x_1)\right) q_1 \right) \left(\hat{m} - \hat{m}_{-1}\right) \tilde{\Psi}_t \right\rangle && \text{(2.4.8a)} \\
&\quad - N\left\langle \Psi_t, \left(\hat{m} - \hat{m}_{-1}\right) \left((N-1) P_1^{\{1,2\}} w_{12}(t) P_0^{\{1,2\}} - 2q_1 \left(tR_1 + \overline{u_\leq}(x_1)\right) p_1 \right) \tilde{\Psi}_t \right\rangle && \text{(2.4.8b)} \\
&\quad + N(N-1)\left\langle \Psi_t, P_0^{\{1,2\}} w_{12}(t) P_2^{\{1,2\}} \left(\hat{m} - \hat{m}_{-2}\right) \tilde{\Psi}_t \right\rangle && \text{(2.4.8c)} \\
&\quad - N(N-1)\left\langle \Psi_t, \left(\hat{m} - \hat{m}_{-2}\right) P_2^{\{1,2\}} w_{12}(t) P_0^{\{1,2\}} \tilde{\Psi}_t \right\rangle && \text{(2.4.8d)} \\
&\quad + N(N-1)\left\langle \Psi_t, P_1^{\{1,2\}} (u_\leq)_{12} P_2^{\{1,2\}} \left(\hat{m} - \hat{m}_{-1}\right) \tilde{\Psi}_t \right\rangle && \text{(2.4.8e)} \\
&\quad - N(N-1)\left\langle \Psi_t, \left(\hat{m} - \hat{m}_{-1}\right) P_2^{\{1,2\}} (u_\leq)_{12} P_1^{\{1,2\}} \tilde{\Psi}_t \right\rangle && \text{(2.4.8f)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle \Psi_t, [B_{123}(t), \hat{m}]\, \tilde{\Psi}_t \right\rangle \\
&= N\langle \Psi_t, \Big((N-1)(N-2) P_0^{\{1,2,3\}} w_{123}(t) P_1^{\{1,2,3\}} - 6t^2 p_1 W_1 q_1 \Big) \left(\hat{m} - \hat{m}_{-1}\right) \tilde{\Psi}_t \rangle && \text{(2.4.9a)} \\
&\quad - N\langle \Psi_t, \left(\hat{m} - \hat{m}_{-1}\right) \Big((N-1)(N-2) P_1^{\{1,2,3\}} w_{123}(t) P_0^{\{1,2,3\}} - 2t^2 q_1 W_1 p_1 \Big) \tilde{\Psi}_t \rangle && \text{(2.4.9b)} \\
&\quad + N(N-1)(N-2)\langle \Psi_t, P_0^{\{1,2,3\}} w_{123}(t) P_1^{\{1,2,3\}} \left(\hat{m} - \hat{m}_{-2}\right) \tilde{\Psi}_t \rangle && \text{(2.4.9c)} \\
&\quad - N(N-1)(N-2)\langle \Psi_t, \left(\hat{m} - \hat{m}_{-2}\right) P_1^{\{1,2,3\}} w_{123}(t) P_0^{\{1,2,3\}} \tilde{\Psi}_t \rangle. && \text{(2.4.9d)}
\end{aligned}$$

Now, we estimate each term:
Using Lemma 41 with $\psi \equiv \hat{D}_{-1}\tilde{\Psi}_t$ and $\varphi \equiv \hat{E}_{-1}\Psi_t$ and Lemma 34 yields

$$\begin{aligned}
&|(3.1.4)| \\
&\leq tN^{\frac{1}{2}} \| |\nabla \cdot f^{(N)}| * \rho_t \|_\infty \|\hat{E}_{-1}\Psi_t\| \Big((N-1)\|q_2 q_1 \hat{D}_{-1}\tilde{\Psi}_t\|^2 + \|q_1 \hat{D}_{-1}\tilde{\Psi}_t\|^2 \Big)^{1/2} \\
&\quad + CtN^{\frac{1}{2}} \| |f^{(N)}| * \rho_t^\nabla \|_\infty^{1/2} \| |f^{(N)}| * \rho_t \|_\infty^{1/2} \|\hat{E}_{-1}\Psi_t\| \Big((N-1)\|q_2 q_1 \hat{D}_{-1}\tilde{\Psi}_t\|^2 + \|q_1 \hat{D}_{-1}\tilde{\Psi}_t\|^2 \Big)^{1/2} \\
&\quad + Ct^2 N \| |f^{(N)}|^2 * \rho_t \|_\infty \|\hat{E}_{-1}\Psi_t\| \; \|q_2 \hat{D}_{-1}\tilde{\Psi}_t\| \\
&\leq C(1+t)^2 D(t) \tilde{\alpha}_m^{1/2} N^{-\frac{\gamma}{2}} && \text{(2.4.10)}
\end{aligned}$$

where the last line of each inequality is given by the error term w_f estimated by Lemma 43.

With ψ and φ switched we obtain

$$\begin{aligned}
&|(2.4.8b)| \\
&\leq tN^{\frac{1}{2}} \| |\nabla \cdot f^{(N)}| * \rho_t \|_\infty \|\hat{n}_{+1}^{1/2} \hat{E}_{-1}\tilde{\Psi}_t\| \Big((N-1)\|\hat{n}^{-1/2} q_2 q_1 \hat{D}_{-1}\Psi_t\|^2 + \|\hat{n}^{-1/2} q_1 \hat{D}_{-1}\Psi_t\|^2 \Big)^{1/2} \\
&\quad + CtN^{\frac{1}{2}} \| |f^{(N)}| * \rho_t^\nabla \|_\infty^{1/2} \| |f^{(N)}| * \rho_t \|_\infty^{1/2} \|\hat{n}_{+1}^{1/2} \hat{E}_{-1}\tilde{\Psi}_t\| \\
&\qquad\qquad \times \Big((N-1)\|\hat{n}^{-1/2} q_2 q_1 \hat{D}_{-1}\Psi_t\|^2 + \|\hat{n}^{-1/2} q_1 \hat{D}_{-1}\Psi_t\|^2 \Big)^{1/2} \\
&\quad + Ct^2 N \| |f^{(N)}|^2 * \rho_t \|_\infty \|\hat{n}_{+1}^{1/2} \hat{E}_{-1}\tilde{\Psi}_t\| \; \|\hat{n}^{-1/2} q_2 \hat{D}_{-1}\Psi_t\| \\
&\leq C(1+t)^2 D(t) \tilde{\alpha}_m^{1/2} N^{-\frac{\gamma}{2}} && \text{(2.4.11)}
\end{aligned}$$

Similarly, using Lemma 45 with $\psi \equiv \hat{D}_{-2}\tilde{\Psi}_t$ and $\varphi \equiv \hat{E}_{-2}\Psi_t$ and Lemma 34 yields

$$\begin{aligned}
&|(2.4.8c)| \\
&\leq tN\|q_1\hat{E}_{-2}\Psi_t\| \Bigg(\left(6\| |f^{(N)}|^2 * \rho_t^\nabla\|_\infty \|\hat{D}_{-2}\tilde{\Psi}_t\|^2 + 3\||f^{(N)}| * \rho_t\|_\infty \||f^{(N)}| * \rho_t^\nabla\|_\infty \|q_1\hat{D}_{-2}\tilde{\Psi}_t\|^2\right)^{1/2} \\
&\qquad + \left(\|(\nabla\cdot f^{(N)})^2 * \rho_t\|_\infty \|\hat{D}_{-2}\tilde{\Psi}_t\|^2 + \|(\nabla\cdot f^{(N)}) * \rho_t\|_\infty^2 \|q_1\hat{D}_{-2}\tilde{\Psi}_t\|^2\right)^{1/2} \Bigg) \\
&\qquad + t^2N\| \| |f^{(N)}|^2 * \rho_t\|_\infty \|q_1\hat{E}_{-2}\Psi_t\| \; \|q_1\hat{D}_{-2}\tilde{\Psi}_t\| \\
&\Bigg(\left(12\||f^{(N)}|^2 * \rho_t^\nabla\|_\infty NN^{-\gamma} + 18\||f^{(N)}| * \rho_t\|_\infty \||f^{(N)}| * \rho_t^\nabla\|_\infty\right)^{1/2} \\
&\qquad\qquad + \left(2\|(\nabla\cdot f^{(N)})^2 * \rho_t\|_\infty NN^{-\gamma} + 6\||\nabla\cdot f^{(N)}| * \rho_t\|_\infty^2\right)^{1/2} \Bigg) \\
&\leq C(1+t)^2 D(t)\tilde{\alpha}_m^{1/2}\left(1 + N^{\frac{1-a-\gamma}{2}}\right)
\end{aligned} \tag{2.4.12}$$

and with ψ and φ switched

$$\begin{aligned}
&|(2.4.8d)| \\
&\leq tN\|q_1\hat{D}_{-2}\Psi_t\| \Bigg(\left(6\| |f^{(N)}|^2 * \rho_t^\nabla\|_\infty \|\hat{E}_{-2}\tilde{\Psi}_t\|^2 + 3\||f^{(N)}| * \rho_t\|_\infty \||f^{(N)}| * \rho_t^\nabla\|_\infty \|q_1\hat{E}_{-2}\tilde{\Psi}_t\|^2\right)^{1/2} \\
&\qquad + \left(\|(\nabla\cdot f^{(N)})^2 * \rho_t\|_\infty \|\hat{E}_{-2}\tilde{\Psi}_t\|^2 + \|(\nabla\cdot f^{(N)}) * \rho_t\|_\infty^2 \|q_1\hat{E}_{-2}\tilde{\Psi}_t\|^2\right)^{1/2} \Bigg) \\
&\qquad + t^2N\| \| |f^{(N)}|^2 * \rho_t\|_\infty \|q_1\hat{E}_{-2}\tilde{\Psi}_t\| \; \|q_1\hat{D}_{-2}\Psi_t\| \\
&\leq C(1+t)^2 D(t)\tilde{\alpha}_m^{1/2}\left(1 + N^{\frac{1-a-\gamma}{2}}\right).
\end{aligned} \tag{2.4.13}$$

Furthermore, it using Lemma 46 with $\psi \equiv \hat{D}_{-1}\tilde{\Psi}_t$ and $\varphi \equiv \hat{E}_{-1}\Psi_t$ and Lemma 34 yields

$$\begin{aligned}
&|(2.4.8e)|, |(2.4.8f)| \\
&\leq CN^{\frac{3}{2}}\||u_{\leq}^{(N)}|^2 * \rho_t\|_\infty^{1/2} \|\hat{l}_{+1}q_1\hat{E}_{-1}\Psi_t\| \; \|\hat{l}^{-1}q_1q_2\hat{D}_{-1}\tilde{\Psi}_t\| \\
&\leq CD(t)N^{\frac{\gamma}{2}-\frac{a}{2}}.
\end{aligned} \tag{2.4.14}$$

For the three-body term we use Lemma 41 with $\psi \equiv \hat{D}_{-1}\tilde{\Psi}_t$ and $\varphi \equiv \hat{E}_{-1}\Psi_t$ and Lemma 34

$$\begin{aligned}
&|(2.4.9a)| \\
&\leq Ct^2N^{\frac{1}{2}}\||f^{(N)}| * \rho_t\|_\infty^2 \|\hat{E}_{-1}\Psi_t\| \left((N-1)\|q_2q_1\hat{D}_{-1}\tilde{\Psi}_t\|^2 + \|q_1\hat{D}_{-1}\tilde{\Psi}_t\|^2\right)^{1/2} \\
&\leq Ct^2D(t)^2\tilde{\alpha}_m N^{-\frac{\gamma}{2}}
\end{aligned} \tag{2.4.15}$$

and with ψ and φ switched

$$\begin{aligned}
&|(2.4.9b)| \\
&\leq Ct^2 N^{\frac{1}{2}} \| |f^{(N)}| * \rho_t \|_\infty^2 \| \hat{l}_{+1} \hat{E}_{-1} \tilde{\Psi}_t \| \left((N-1) \| \hat{n}^{-1/2} q_2 q_1 \hat{D}_{-1} \Psi_t \|^2 + \| \hat{l}^{-1} q_1 \hat{D}_{-1} \Psi_t \|^2 \right)^{1/2} \\
&\leq Ct^2 D(t)^2 \tilde{\alpha}_m N^{-\frac{\gamma}{2}}.
\end{aligned} \tag{2.4.16}$$

Similarly, using Lemma 41 with $\psi \equiv \hat{D}_{-2}\tilde{\Psi}_t$ and $\varphi \equiv \hat{E}_{-2}\Psi_t$ and Lemma 34 yields

$$\begin{aligned}
&|(2.4.9c)| \\
&\leq Ct^2 N \| |f^{(N)}| * \rho_t \|_\infty \| \hat{E}_{-2} \Psi_t \| \times \\
&\qquad \times \left(\| |f^{(N)}| * \rho_t \|_\infty^3 \| q_1 q_3 \hat{D}_{-2} \tilde{\Psi}_t \|^2 + \| |f^{(N)}|^2 * \rho_t \|_\infty \| |f^{(N)}| * \rho_t \|_\infty \| q_1 \hat{D}_{-2} \tilde{\Psi}_t \|^2 \right)^{1/2} \\
&\leq Ct^2 D(t)^3 \tilde{\alpha}_m^{1/2} \left(1 + N^{\frac{1-a-\gamma}{2}} \right)
\end{aligned} \tag{2.4.17}$$

and with ψ and φ switched

$$\begin{aligned}
&|(2.4.9d)| \\
&\leq Ct^2 N \| |f^{(N)}| * \rho_t \|_\infty \| q_1 \hat{E}_{-2} \tilde{\Psi}_t \| \times \\
&\qquad \times \left(\| |f^{(N)}| * \rho_t \|_\infty^3 \| q_3 \hat{D}_{-2} \Psi_t \|^2 + \| |f^{(N)}|^2 * \rho_t \|_\infty \| |f^{(N)}| * \rho_t \|_\infty \| \hat{D}_{-2} \Psi_t \|^2 \right)^{1/2} \\
&\leq Ct^2 D(t)^3 \tilde{\alpha}_m^{1/2} \left(1 + N^{\frac{1-a-\gamma}{2}} \right).
\end{aligned} \tag{2.4.18}$$

In total one obtains with Proposition 21 a $C > 0$ such that

$$|B_\gamma(t)| \leq C e^{C_1(t)} \left(N^{\frac{1-a-\gamma}{2}} + \tilde{\alpha}_m(0)^{1/2} \right) \tag{2.4.19}$$

with $C_1(t) = C \int_0^t \mathrm{d}\tau \ (1+\tau)^2 D(\tau)^3$.

In order to estimate $A_\gamma(s)$ first note that the difference between the gauged Hamiltonian and the auxiliary can be written as

$$H^{\mathrm{g}}(s) - \widetilde{H^{\mathrm{g}}}(s) = W^{(3)} + W^{(4)} + W^{(5)} + W^{(6)} \tag{2.4.20}$$

with $j = 3, 4, 5, 6$

$$W^{(j)} := \sum_{1 \leq m < k \leq N} \breve{w}_{mk}^{(j)}(t) + \sum_{1 \leq m < k < l \leq N} w_{mkl}^{(j)}(t), \tag{2.4.21}$$

$$\breve{w}_{mk}(t) := w_{mk}(t) - (u_{\leq})_{mk}, \qquad \text{for } m, k \in \{1, \ldots, N\} \tag{2.4.22}$$

and the projector notation from (2.3.5) and (2.3.6). Note that since $\widetilde{H^{\mathrm{g}}}$ was defined to include all $u_{\leq}$-terms, we need to subtract them here from the two-body terms $w_{mk}(t)$.

Every term is in the following estimated by applying Lemma 46, Lemma 47 and Lemma 48. The general strategy is shift the q-operators in form of $\hat{n}$ to $\hat{w}_d\Psi_t$ and to make use of the weight estimates from Lemma 35. When possible we shift all ∇_i-operators to $q_i\tilde{\Psi}_t$ to use .

$\underline{W^{(3)}}$: We first decompose

$$\begin{aligned}
&\left\langle \Psi_t, W^{(3)}\hat{w}\tilde{\Psi}_t \right\rangle \\
&= \sum_{1\le m<k\le N} \left\langle \hat{w}_t\Psi_t, \breve{w}^{(3)}_{mk}(t)\tilde{\Psi} \right\rangle + \sum_{1\le m<k<l\le N} \left\langle \hat{w}_t\Psi_t, w^{(3)}_{mkl}(t)\tilde{\Psi}_t \right\rangle && (2.4.23) \\
&= \frac{N(N-1)}{2}\left\langle \Psi_t, \left(\hat{w}_{-1}P_1^{\{1,2\}}\breve{w}_{12}(t)P_2^{\{1,2\}} + \hat{w}_{+1}P_2^{\{1,2\}}\breve{w}_{12}(t)P_1^{\{1,2\}}\right)\tilde{\Psi}_t \right\rangle \\
&\quad + \frac{N(N-1)(N-2)}{6}\left\langle \Psi_t, \left(\hat{w}_{-1}P_1^{\{1,2,3\}}w_{123}(t)P_2^{\{1,2,3\}} + \hat{w}_{+1}P_2^{\{1,2,3\}}w_{123}(t)P_1^{\{1,2,3\}}\right)\tilde{\Psi}_t \right\rangle \\
&\quad + \frac{N(N-1)(N-2)}{6}\left\langle \hat{w}_t\Psi_t, \left(\hat{w}_{-3}P_0^{\{1,2,3\}}w_{123}(t)P_3^{\{1,2,3\}} + \hat{w}_{+3}P_3^{\{1,2,3\}}w_{123}(t)P_0^{\{1,2,3\}}\right)\tilde{\Psi}_t \right\rangle \\
& && (2.4.24)
\end{aligned}$$

and use Lemma 46 with $\varphi \equiv \hat{w}_{\pm 1}\Psi_t$ and $\psi \equiv \tilde{\Psi}_t$ in combination with the weight estimates Lemma 35 and Lemma 32 to obtain

$$\begin{aligned}
&\frac{N(N-1)}{2}\left|\left\langle \Psi_t, \left(\hat{w}_{+1}P_1^{\{1,2\}}\breve{w}_{12}(t)P_2^{\{1,2\}} + \hat{w}_{-1}P_2^{\{1,2\}}\breve{w}_{12}(t)P_1^{\{1,2\}}\right)\tilde{\Psi}_t \right\rangle\right| \\
&\le Ct\ N^{\frac{1}{2}+\gamma}\||f^{(N)}|^2 * \rho_t\|_\infty^{1/2}\ \|\nabla_1 q_1\tilde{\Psi}_t\| + 2N^{\frac{3}{2}\gamma}\||\nabla\cdot f^{(N)}|^2 * \rho_t\|_\infty^{1/2} \\
&\quad + Ct\ N^{\frac{3}{2}\gamma}\left(\||f^{(N)}|^2 * \rho_t^{\nabla}\|_\infty^{1/2} + \||\nabla\cdot f^{(N)}|^2 * \rho_t\|_\infty^{1/2}\right) \\
&\quad + Ct\ N^{\frac{3}{2}\gamma}\||f^{(N)}|^4 * \rho_t\|_\infty^{1/2} \\
&\le CtD(t)\left(e^{C_2(t)}N^{\frac{1-a}{2}+\gamma-\frac{a}{4}} + N^{\frac{3}{2}\gamma-\frac{a}{2}}\right) && (2.4.25)
\end{aligned}$$

where in the last line Proposition 23 was used to obtain $C_2(t) = C\int_0^t \mathrm{d}\tau\ (1+\tau)^4 D(\tau)^4$. Furthermore, it holds

$$\begin{aligned}
&\frac{N(N-1)(N-2)}{6}\left|\left\langle \Psi_t, \left(\hat{w}_{+1}P_1^{\{1,2,3\}}w_{123}(t)P_2^{\{1,2,3\}} + \hat{w}_{-1}P_2^{\{1,2,3\}}w_{123}(t)P_1^{\{1,2,3\}}\right)\tilde{\Psi}_t \right\rangle\right| \\
&\le Ct^2\ \tilde{\alpha}_m^{1/2}N^{\frac{3}{2}\gamma}\||f^{(N)}|^2 * \rho_t\|_\infty^{1/2}\ \||f^{(N)}| * \rho_t\|_\infty \\
&\le Ct^2 D(t)^2 N^{\frac{1}{2}-a+\gamma}. && (2.4.26)
\end{aligned}$$

Similarly, we use Lemma 46 with $\varphi \equiv \hat{w}_{-3}\Psi_t$ and $\psi \equiv \tilde{\Psi}_t$ and Lemma 35 to estimate

$$\begin{aligned}
&\frac{N(N-1)(N-2)}{6}\left|\left\langle \hat{w}_t\Psi_t, \left(\hat{w}_{+3}P_0^{\{1,2,3\}}w_{123}(t)P_3^{\{1,2,3\}} + \hat{w}_{-3}P_3^{\{1,2,3\}}w_{123}(t)P_0^{\{1,2,3\}}\right)\tilde{\Psi}_t \right\rangle\right| \\
&\le Ct^2\ \tilde{\alpha}_m^{1/2}N^{\frac{3}{2}\gamma}\||f^{(N)}|^2 * \rho_t\|_\infty^{1/2}\left(\||f^{(N)}|^2 * \rho_t\|_\infty + \||f^{(N)}| * \rho_t\|_\infty^2\right)^{1/2} \\
&\le Ct^2 D(t)^2 N^{\frac{1}{2}-a+\gamma}. && (2.4.27)
\end{aligned}$$

$\underline{W^{(4)}}$: We first decompose

$$
\begin{aligned}
&\left\langle \Psi_t, W^{(4)} \hat{w} \tilde{\Psi}_t \right\rangle \\
&= \sum_{1\le m<k\le N} \left\langle \Psi_t, \breve{w}^{(4)}_{mk}(t) \hat{w} \tilde{\Psi}_t \right\rangle + \sum_{1\le m<k<l\le N} \left\langle \Psi_t, w^{(4)}_{mkl}(t) \hat{w} \tilde{\Psi}_t \right\rangle \\
&= \frac{N(N-1)}{2} \left\langle \Psi_t, \hat{w} P_2^{\{1,2\}} \breve{w}_{12} P_2^{\{1,2\}} \tilde{\Psi}_t \right\rangle \\
&\quad + \frac{N(N-1)(N-2)}{6} \left\langle \Psi_t, \left(\hat{w}_{+2} P_1^{\{1,2,3\}} w_{123}(t) P_3^{\{1,2,3\}} + \hat{w}_{-2} P_3^{\{1,2,3\}} w_{123}(t) P_1^{\{1,2,3\}} \right) \tilde{\Psi}_t \right\rangle \\
&\quad + \frac{N(N-1)(N-2)}{6} \left\langle \Psi_t, \hat{w} P_2^{\{1,2,3\}} w_{123}(t) P_2^{\{1,2,3\}} \tilde{\Psi}_t \right\rangle
\end{aligned}
\tag{2.4.28}
$$

and use Lemma 47 with $\varphi \equiv \hat{w}\Psi_t$ and $\psi \equiv \tilde{\Psi}_t$ in combination with the weight estimates Lemma 35 and Lemma 32 to estimate

$$
\begin{aligned}
&\frac{N(N-1)}{2} \left| \left\langle \Psi_t, \hat{w} P_2^{\{1,2\}} \breve{w}_{12} P_2^{\{1,2\}} \tilde{\Psi}_t \right\rangle \right| \\
&\le Ct\; N^{\frac{1}{2}+\frac{3}{2}\gamma} \|f^{(N)}\|_\infty \; \|\nabla_1 q_1 \tilde{\Psi}_t\| + N^{2\gamma} \left(\|\nabla \cdot f^{(N)}\|_\infty + \|f^{(N)}\|_\infty^2 \right) \\
&\le Ct \left(e^{C_2(t)} N^{\frac{1-a}{2}+\frac{3}{2}\gamma-\frac{a}{4}} + N^{2\gamma-\frac{a}{2}} \right)
\end{aligned}
\tag{2.4.29}
$$

where we used $\|\nabla \cdot f^{(N)}\|_\infty, \|f^{(N)}\|_\infty \le \lambda_N \le N^{-a/2}$ due to the cut-off and Proposition 23. Furthermore, using Lemma 47 with $\varphi \equiv \hat{w}\Psi_t$ and $\psi \equiv \tilde{\Psi}_t$ and Lemma 35 we estimate

$$
\begin{aligned}
&\frac{N(N-1)(N-2)}{6} \left| \left\langle \Psi_t, \hat{w} P_2^{\{1,2,3\}} w_{123}(t) P_2^{\{1,2,3\}} \tilde{\Psi}_t \right\rangle \right| \\
&\le t^2\; \tilde{\alpha}_m^{1/2} N^{2\gamma} \left(\| |f^{(N)}|^2 * \rho_t \|_\infty + \| |f^{(N)}| * \rho_t \|_\infty \|f^{(N)}\|_\infty \right) \\
&\le Ct^2 D(t) N^{\frac{1}{2}-a+\frac{3}{2}\gamma}
\end{aligned}
\tag{2.4.30}
$$

where we used $\|f^{(N)}\|_\infty \le \lambda_N \le N^{-a/2}$ and

$$
\begin{aligned}
&\frac{N(N-1)(N-2)}{6} \left| \left\langle \Psi_t, \left(\hat{w}_{+2} P_1^{\{1,2,3\}} w_{123}(t) P_3^{\{1,2,3\}} + \hat{w}_{-2} P_3^{\{1,2,3\}} w_{123}(t) P_1^{\{1,2,3\}} \right) \tilde{\Psi}_t \right\rangle \right| \\
&\le Ct^2\; \tilde{\alpha}_m^{1/2} N^{2\gamma} \| |f^{(N)}|^2 * \rho_t \|_\infty \le Ct^2 D(t) N^{\frac{1}{2}-\frac{3}{2}a+2\gamma}
\end{aligned}
\tag{2.4.31}
$$

using Lemma 48 with $\varphi \equiv \hat{w}_{\pm 2}\Psi_t$ and $\psi \equiv \tilde{\Psi}_t$ and Lemma 35.

$\underline{W^{(5)}}$: We apply Lemma 48 with $\varphi \equiv \hat{w}_{\pm 1}\Psi_t$ and $\psi \equiv \tilde{\Psi}_t$ in combination with the weight estimates Lemma 35 and Lemma 32

$$\begin{aligned}
&|\langle \Psi_t, W^{(5)} \hat{w} \tilde{\Psi}_t \rangle| \\
&\leq \sum_{1 \leq m<k<l \leq N} |\langle \Psi_t, w^{(5)}_{mkl}(t) \hat{w} \tilde{\Psi}_t \rangle| \\
&= \frac{N(N-1)(N-2)}{6} |\langle \Psi_t, \left(\hat{w}_{+1} P_2^{\{1,2,3\}} w_{123}(t) P_3^{\{1,2,3\}} + \hat{w}_{-1} P_3^{\{1,2,3\}} w_{123}(t) P_2^{\{1,2,3\}} \right) \tilde{\Psi}_t \rangle| \\
&\leq Ct^2 \, \tilde{\alpha}_m^{1/2} N^{\frac{5}{2}\gamma} \| |f^{(N)}|^2 * \rho_t \|_\infty^{1/2} \, \|f^{(N)}\|_\infty \leq Ct^2 D(t) N^{\frac{1}{2}+2\gamma-\frac{3}{2}a} \qquad (2.4.32)
\end{aligned}$$

where we used $\|f^{(N)}\|_\infty \leq \lambda_N \leq N^{-a/2}$.

$\underline{W^{(6)}}$: We apply Lemma 48 with $\varphi \equiv \hat{w}\Psi_t$ and $\psi \equiv \tilde{\Psi}_t$ in combination with the weight estimates Lemma 35 and Lemma 32

$$\begin{aligned}
|\langle \Psi_t, W^{(6)} \hat{w} \tilde{\Psi}_t \rangle| &\leq \sum_{1 \leq m<k<l \leq N} |\langle \Psi_t, w^{(6)}_{mkl}(t) \hat{w} \tilde{\Psi}_t \rangle| \\
&= \frac{N(N-1)(N-2)}{6} |\langle \hat{w}\Psi_t, P_3^{\{1,2,3\}} w_{123}(t) P_3^{\{1,2,3\}} \tilde{\Psi}_t \rangle| \\
&\leq t^2 \, \tilde{\alpha}_m^{1/2} N^{3\gamma} \|f^{(N)}\|_\infty^2 \leq Ct^2 N^{\frac{1}{2}+3\gamma-a}. \qquad (2.4.33)
\end{aligned}$$

It follows that there is a constant $C_1(t), C_2(t) > 0$ only depending on t such that for sufficiently large N it holds

$$\begin{aligned}
|A_\gamma(t)| &= |\langle \Psi_t, \left(H^{\mathrm{g}}(t) - \widetilde{H^{\mathrm{g}}}(t) \right) \hat{w}_{\gamma,t} \tilde{\Psi}_t \rangle| \\
&\leq C(1+t)^2 D(t)^2 \left(N^{\frac{1}{2}+3\gamma-a} + N^{\frac{1}{2}+2\gamma-\frac{3}{2}a} + e^{C_2(t)} N^{\frac{1}{2}+\frac{3}{2}\gamma-\frac{1}{2}a-\frac{a}{4}} \right) \qquad (2.4.34)
\end{aligned}$$

In total, by combining (2.4.34) and (2.4.34), we obtain

$$\begin{aligned}
&|A_\gamma(t)| + |B_\gamma(t)| \\
&\leq C(1+t)^2 D(t)^3 \left(N^{\frac{1}{2}+3\gamma-a} + N^{\frac{1}{2}+2\gamma-\frac{3}{2}a} + e^{C_2(t)} N^{\frac{1}{2}+\frac{3}{2}\gamma-\frac{3}{4}a} + e^{C_1(t)} N^{\frac{1}{2}-\frac{\gamma}{2}-\frac{a}{2}} \right). \qquad (2.4.35)
\end{aligned}$$

For $s \in (0, 1/2)$, we find with Lemma 10 and Proposition 23 $a = a(s) = 1$ and thus, the optimal choice of $\gamma = 1/8$ such that

$$|A_\gamma(t)| + |B_\gamma(t)| \leq C(1+t)^2 D(t)^3 e^{\max\{C_2(t), C_1(t)\}} N^{-\frac{1}{16}}. \qquad (2.4.36)$$

For $s \in [\frac{1}{2}, \frac{2}{3})$, we find with Lemma 10 and Proposition 23 $a = a(s) = \frac{2}{3}(2-s)$ and the optimal choice of $\gamma(s) = (2-s)/12$

$$|A_\gamma(t)| + |B_\gamma(t)| \leq C(1+t)^2 D(t)^3 e^{C_2(t)} N^{\frac{3}{8}s-\frac{1}{4}}.$$

■

Proof of Theorem 51. Recall that due to Proposition 7 it holds

$$\left| \langle \Phi_t, M\Phi_t \rangle - \langle \bigwedge_{k=1}^{N} \varphi_k^t, M \bigwedge_{k=1}^{N} \varphi_k^t \rangle \right| \leq 2\sqrt{2} \|M\|_{\mathrm{op}} \alpha_n(t)^{1/2} \tag{2.4.37}$$

with $\alpha_n(t) = \langle \Psi_t, q_1, \Psi_t \rangle$ which can be bounded by $\tilde{\alpha}_n(t) = \langle \tilde{\Psi}_t, q_1, \tilde{\Psi}_t \rangle$ due to

$$\alpha_n(t) \leq \frac{8}{3} \left(\tilde{\alpha}_n(t) + \|\tilde{\Psi}_t - \Psi_t\|^2 \right) \tag{2.4.38}$$

as shown in (2.1.35).

We apply the norm approximation given by Proposition 29 and the bound for the counting functional given by (21) to obtain the desired result

$$\begin{aligned} \left| \langle \Phi_t, M\Phi_t \rangle - \langle \bigwedge_{k=1}^{N} \varphi_k^t, M \bigwedge_{k=1}^{N} \varphi_k^t \rangle \right| &\leq \frac{8}{\sqrt{3}} \|M\|_{\mathrm{op}} \left(\tilde{\alpha}_n(t) + \|\tilde{\Psi}_t - \Psi_t\|^2 \right)^{1/2} \\ &\leq C e^{C(t)} \|M\|_{\mathrm{op}} \left(\tilde{\alpha}_n(0)^{1/2} + \max\left\{ N^{-\frac{1}{32}}, N^{-\frac{2-3s}{32}} \right\} \right) \end{aligned} \tag{2.4.39}$$

with $C(t) = C \int_0^t \mathrm{d}\tau \; (1+\tau)^4 D(\tau)^4$. The desired constant follows from the initial condition $\sum_{k=1}^N (\|\Delta \varphi_k^0\|^2 + \|\nabla \varphi_k^0\|^2) \leq C_0 N$, Proposition 16 and Lemma 15 such that we can estimate the term $D(t) = \max\{\|\rho_t\|_\infty, \|\rho_t^\nabla\|_\infty, \|\rho_t^\Delta\|_\infty, N^{-1}\|\rho_t^\nabla\|_1, N^{-1}\|\rho_t^\Delta\|_1, 1\}$ with respect to $S(t) = \max\{\|\rho_t\|_\infty, \|\rho_{\varphi^t}^\nabla\|_\infty, \|\rho_{\varphi^t}^\Delta\|_\infty, 1\}$ in the following form:

$$\begin{aligned} D(t) &\leq C \left((1+t)^4 S(t)^5 + (1+t)^2 S(t) e^{C \int_0^t \mathrm{d}\sigma \; S(\sigma)} \right) \\ &\leq e^{C \ln((2+t)S(t)) \int_0^t \mathrm{d}\sigma \; S(\sigma)}. \end{aligned} \tag{2.4.40}$$

■

2.5 Toolbox and general estimates

We consider in this subsection the projectors $p = p^{\varphi_1, \ldots, \varphi_N}$ and $q = 1 - p$ with respect to a general orthonormal system $\{\varphi_i\}_{i=1}^N$. Moreover, we consider for $m = 1, 2, \ldots, N$ and self-adjoint operators A on $L^2(\mathbb{R}^3)$

$$p_m^A := \sum_{i=1}^{N} |A\varphi_i\rangle\langle\varphi_i|_m, \tag{2.5.1}$$

$$\rho^A := \sum_{i=1}^{N} |A\varphi_i|^2 \tag{2.5.2}$$

with $p_m^1 \equiv p_m$ and $\rho^1 \equiv \rho$.

2.5.1 Properties of the counting functional

It is convenient to use the operator $P_a^{\mathcal{C}}$ only acting on particles with indices in $\mathcal{C} \subseteq \{1, 2, \ldots, N\}$ to define for all $a = 0, 1, \ldots |\mathcal{C}|$ as the sum over all products of projectors where in each summand q-projectors occur exactly a times as defined in (2.3.3). Due to the antisymmetry of the wave function we simplify our presentation for the case $\mathcal{C} = \{1, 2, 3\}$ and $\{1, 2\}$ in the following:

$$\begin{aligned}
P_0^{\{1,2,3\}} &:= p_1 p_2 p_3, \\
P_1^{\{1,2,3\}} &:= p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 p_3, \\
P_2^{\{1,2,3\}} &:= q_1 q_2 p_3 + q_1 p_2 q_3 + p_1 q_2 q_3, \\
P_3^{\{1,2,3\}} &:= q_1 q_2 q_3,
\end{aligned}$$

$$\begin{aligned}
P_0^{\{1,2\}} &:= p_1 p_2, \\
P_1^{\{1,2\}} &:= p_1 q_2 + q_1 p_2, \\
P_2^{\{1,2\}} &:= q_1 q_2.
\end{aligned}$$

Lemma 30. *Let h be a self-adjoint operator only acting on the particles with indices $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2 \subseteq \{1, 2, \ldots, N\}$ with $|\mathcal{C}|, |\mathcal{C}_1|, |\mathcal{C}_2| \leq 3$ and define for any $d = 0, \pm 1, \ldots, \pm N$*

$$\hat{f}_d := \sum_{k=0}^{N} \chi_{[0,N]}(k+d)\ f(k+d) P_{N,k}.$$

Then it holds for all $a, b = 0, 1, \ldots, 3$ and $k = 1, \ldots, N$ that

$$\hat{f}\left(P_a^{\mathcal{C}} h P_b^{\mathcal{C}}\right) = \left(P_a^{\mathcal{C}} h P_b^{\mathcal{C}}\right) \hat{f}_{a-b}$$

and for $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ and $a, b = 0, 1, \ldots, 3$ and $a', b' = 0, 1, \ldots, 3$

$$\hat{f} P_a^{\mathcal{C}_1} \left(P_{a'}^{\mathcal{C}_2} h P_{b'}^{\mathcal{C}_2}\right) P_b^{\mathcal{C}_1} = P_a^{\mathcal{C}_1} \left(P_{a'}^{\mathcal{C}_2} h P_{b'}^{\mathcal{C}_2}\right) P_{N,k} P_b^{\mathcal{C}_1} \hat{f}_{a+a'-b-b'}.$$

It also holds that

$$\sum_{a=0}^{|\mathcal{C}|} P_a^{\mathcal{C}} = \prod_{k=1}^{|\mathcal{C}|} (p_k + q_k) = 1.$$

Proof. The first set of equation was shown in [PP16, Lemma 6.4], the second set can be proved by using the decomposition

$$P_{N,k} = \sum_{d=0}^{|\mathcal{C}_1|} \sum_{d'=0}^{|\mathcal{C}_2|} P_d^{\mathcal{C}_1} P_{d'}^{\mathcal{C}_2} P_{k-d-d'}^{\{1,\ldots,N\} \setminus (\mathcal{C}_1 \sqcup \mathcal{C}_2)}$$

and shifting the $\{1,\dots,N\}\backslash(\mathcal{C}_1\sqcup\mathcal{C}_2)$-part to see that for all $a'=0,1,\dots,|\mathcal{C}_1|$ and $b'=0,1,\dots,|\mathcal{C}_2|$ it holds

$$P_{b'}^{\mathcal{C}_2}P_{N,k}P_{a'}^{\mathcal{C}_1}=P_{b'}^{\mathcal{C}_2}P_{a'}^{\mathcal{C}_1}P_{N,k}=P_{N,k}P_{b'}^{\mathcal{C}_2}P_{a'}^{\mathcal{C}_1}.$$

■

Lemma 31 (Inversion of $\hat{f}$, [Pet14, Lemma 6.5])**.** *Let $f(k)>0$ for all $0<k\le N$ and $f(0)=0$. Let $0<s\in\mathbb{Q}$, then then it holds for $\widehat{f^{-s}}:=\sum_{k=1}^N f(k)^{-s}P_{N,k}$ that*

$$\widehat{f^{-1}}=\hat{f}^{-1}(id-P_{N,0}).$$

Furthermore, we need the following statements for concrete choices of weights $\hat{f}$.

Lemma 32 (q-conversion to $\hat{n}$)**.** *Let $\psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$. It holds for $n_0=1,\dots,6$ and sufficiently large N*

$$\|\prod_{i=1}^{n_0+1}q_i\psi\|^2\le 2\langle\psi,(\hat{n})^{n_0+1}\psi\rangle.$$

Proof. Since $P_{N,k}$ contains exactly k q-operators it holds $\sum_{m=1}^N q_mP_{N,k}=kP_{N,k}$ which implies

$$\hat{n}=N^{-1}\sum_{k=0}^N kP_{N,k}=N^{-1}\sum_{m=1}^N q_m\sum_{k=0}^N P_{N,k}=N^{-1}\sum_{m=1}^N q_m. \tag{2.5.3}$$

Thus, the statement is true for $n_0=1$. Moreover, it holds, recursively by the projector property of q_i

$$\begin{aligned}
0\le\langle\psi,\prod_{i=1}^{n_0}q_iq_{n_0+1}\psi\rangle&=\frac{1}{N-n_0}\sum_{m=n_0+1}^N\langle\psi,\prod_{i=1}^{n_0}q_iq_m\psi\rangle\\
&=\frac{1}{N-n_0}\left(\langle\psi,\prod_{i=1}^{n_0}q_i\sum_{m=1}^N q_m\psi\rangle-n_0\langle\psi,\prod_{i=1}^{n_0}q_i\psi\rangle\right)\\
&\le\frac{N}{N-n_0}\langle\psi,\prod_{i=1}^{n_0}q_iN^{-1}\sum_{m=1}^N q_m\psi\rangle\\
&\le\frac{(N-n_0-1)!N^{n_0+1}}{N!}\langle\psi,\left(N^{-1}\sum_{m=1}^N q_m\right)^{n_0+1}\psi\rangle\\
&=\frac{(N-n_0-1)!N^{n_0}}{(N-1)!}\langle\psi,(\hat{n})^{n_0+1}\psi\rangle.
\end{aligned} \tag{2.5.4}$$

The desired result holds for sufficiently large N and in particular in the case of $n_0\le 6$ the result holds for $N\ge 33$. ■

Lemma 33 (l-conversion)**.** *Let $\psi \in L^2_{\text{as}}(\mathbb{R}^{3N})$ and N sufficiently large.*
It holds for $s, n_0 \in \mathbb{N}$ with $n_0 < N$, $s/2 \leq n_0$ and $d = 0,1,2,3$

$$\begin{aligned}
\|\widehat{l^{-s/2}} \prod_{i=1}^{n_0} q_i \psi\|^2 &\leq 2\langle \psi, (\hat{n})^{n_0 - s/2}\psi\rangle, \\
\|\hat{l}_{+d} \prod_{i=1}^{n_0} q_i \psi\|^2 &\leq 2\left(\langle \psi, (\hat{n})^{n_0+1}\psi\rangle + \frac{d}{N}\langle \psi, (\hat{n})^{n_0}\psi\rangle\right).
\end{aligned}$$

Proof. Note that we can commute $P_{N,k}$ and $\prod_{i=1}^{n_0+1} q_i$ with the same argument as used in the proof of Lemma 30. Since $P_{N,k}$ preserves the antisymmetry we can apply Lemma 32:

$$\begin{aligned}
\|\widehat{l^{-s/2}} \prod_{i=1}^{n_0} q_i \psi\|^2 &= \sum_{k=1}^{N} \left(\frac{N}{k}\right)^{-s/2} \langle \psi, P_{N,k} \prod_{i=1}^{n_0} q_i \psi\rangle \\
&\leq 2\langle \psi, (\hat{n})^{n_0 - s/2}\psi\rangle
\end{aligned}$$

and using $(\hat{l}_{+d})^2 = \hat{n}_{+d}$ it follows

$$\begin{aligned}
\|\hat{l}_{+d} \prod_{i=1}^{n_0} q_i \psi\|^2 &= \sum_{k=0}^{N-d} \left(\frac{k+d}{N}\right) \langle \psi, P_{N,k} \prod_{i=1}^{n_0} q_i \psi\rangle \\
&\leq 2\langle \psi, (\hat{n})^{n_0+1}\psi\rangle + \frac{2d}{N}\langle \psi, (\hat{n})^{n_0}\psi\rangle.
\end{aligned}$$

∎

Lemma 34 (Weight estimate for $\hat{m}_\gamma$, [PP16, Lemma 7.1])**.** *Let $\hat{D}_{-d} \coloneqq (\hat{m}_\gamma - \hat{m}_{\gamma,-d})^{1/2}$ and $\hat{E}_{-d} \coloneqq (\hat{m}_{\gamma,+d} - \sum_{k=0}^{N-d} m_\gamma(k) P_{N,k})^{1/2}$ for $d = 0,1,2,3$. Then it holds for all $\chi \in L^2_{\text{as}}(\mathbb{R}^{3N})$, h acting on particles with indices $\mathcal{C} \subseteq \{1,2,\ldots,N\}$ and $a = 0,1,\ldots,|\mathcal{C}| - d$ that*

$$(\hat{m}_\gamma - \hat{m}_{\gamma,-d}) P^{\mathcal{C}}_{a+d} h P^{\mathcal{C}}_a = \hat{D}_{-d}\left(P^{\mathcal{C}}_{a+d} h P^{\mathcal{C}}_a\right)\hat{E}_{-d}.$$

Furthermore, it holds for all $\chi \in L^2_{\text{as}}(\mathbb{R}^{3N})$ the following estimates

$$\begin{aligned}
\|\hat{D}_{-d}\chi\|^2 &\leq d N^{-\gamma}, \\
\|q_1 \hat{D}_{-d}\chi\|^2 &\leq d(d+1) N^{-1}\ \alpha_m, \\
\|q_1 q_2 \hat{D}_{-d}\chi\|^2 &\leq d(d+1)^2 N^{\gamma-2}\ \alpha_m
\end{aligned}$$

and

$$\begin{aligned}
\|\hat{E}_{-d}\chi\|^2 &\leq d N^{-\gamma}, \\
\|q_1 \hat{E}_{-d}\chi\|^2 &\leq d N^{-1}\ \alpha_m, \\
\|q_1 q_2 \hat{E}_{-d}\chi\|^2 &\leq d N^{\gamma-2}\ \alpha_m.
\end{aligned}$$

Lemma 35 (Weight estimate for $\hat{w}_\gamma$)**.** *It holds for all $\psi \in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$, sufficiently large N, $n = 1, \dots, 6$ and $d = 0, 1, 2, 3$*

$$\begin{aligned}
\|\prod_{i=1}^{n} q_i \hat{w}_{\gamma,-d}\psi\|^2 &\leq 2N^{n(\gamma-1)}\alpha_m, \\
\|\prod_{i=1}^{n} q_i \hat{w}_{\gamma,+d}\psi\|^2 &\leq 2N^{n(\gamma-1)}\left(\alpha_m + dN^{-\gamma}\right).
\end{aligned}$$

Proof. It holds

$$\hat{w}(k) = \sum_{k=0}^{N^\gamma}\left(1 - \frac{k}{N^\gamma}\right) P_{N,k} \leq \sum_{k=0}^{N^\gamma} P_{N,k} \tag{2.5.5}$$

and furthermore

$$\begin{aligned}
\|\prod_{i=1}^{n} q_i \hat{w}_{-d}\psi\|^2 &= \sum_{k=d}^{N} w(k-d)^2 \langle \psi, \prod_{i=1}^{n} q_i P_{N,k}\psi\rangle \\
&\leq 2\sum_{k=d}^{N^\gamma}\left(\frac{k}{N}\right)^n \left(1 - \frac{k-d}{N^\gamma}\right)^2 \langle\psi, P_{N,k}\psi\rangle \\
&= 2N^{n(\gamma-1)}\sum_{k=d}^{N^\gamma}\left(\frac{k}{N^\gamma}\right)^n \left(1 - \frac{k-d}{N^\gamma}\right)^2 \langle\psi, P_{N,k}\psi\rangle \\
&\leq 2N^{n(\gamma-1)}\sum_{k=d}^{N^\gamma}\frac{k}{N^\gamma}\langle\psi, P_{N,k}\psi\rangle \\
&\leq 2N^{n(\gamma-1)}\alpha_m
\end{aligned} \tag{2.5.6}$$

where we used $\max\{x^{n-1}(1-x-c)^2 \mid x \in [c,1], 0 < c < 1\} \leq 1$. In addition, it holds

$$\begin{aligned}
\|\prod_{i=1}^{n} q_i \hat{w}_{+d}\psi\|^2 &= \sum_{k=0}^{N} w(k+d)^2 \langle \psi, \prod_{i=1}^{n} q_i P_{N,k}\psi\rangle \\
&\leq 2N^{n(\gamma-1)}\sum_{k=0}^{N^\gamma}\left(\frac{k+d}{N^\gamma}\right)^n \left(1 - \frac{k+d}{N^\gamma}\right)^2 \langle\psi, P_{N,k}\psi\rangle \\
&\leq 2N^{n(\gamma-1)}\left(\alpha_m + dN^{-\gamma}\right)
\end{aligned} \tag{2.5.7}$$

where we used $\max\{x^n(1-x)^2 \mid x \in [0,1]\} \leq 1$. ■

2.5.2 Diagonalization estimates

In the following we state a generalized version of the diagonalization lemma [Pet14, Lemma 6.9] which is needed to study our magnetic-type Hamiltonians. We emphasize that the two-body diagonalization is sufficient for our purpose as the three-body terms are estimated by a simple triangle inequality argument.

Lemma 36 (Diagonalization of $(p_2^A)^* h_{12} p_2^A$)**.** *Consider a two-body operator h_{12} on $L^2(\mathbb{R}^3, dx_1) \otimes L^2(\mathbb{R}^3, dx_2)$ acting on the first and second particles and satisfying $h_{12} = B_2 g_{12} + g_{12} B_2^* :=$ $(\mathrm{id} \otimes B_2) g_{12} + g_{12}(\mathrm{id} \otimes B_2^*)$ where $g_{12} = g(x_1 - x_2)$ is a multiplication operator for some measurable function $g : \mathbb{R}^3 \to \mathbb{R}$ and $B_2 = B_2^*$ is a symmetric operator acting on the second particle. Let A be an operator such that*

$$\sup_{x_1 \in \mathbb{R}^3} |\langle A\varphi_i, h_{12} A\varphi_j \rangle_2(x_1)| < \infty \qquad \text{for all } 1 \le i, j \le N.$$

For fixed x_1 there exists an orthonormal set $\{\chi_i^1\}_{i=1,\ldots,N} := \{\chi_i^{x_1}\}_{i=1,\ldots,N} \subset \mathrm{span}(\varphi_1, \ldots, \varphi_N)$ with eigenvalues $\{\lambda_i(x_1)\}_{i=1,\ldots,N}$ such that

$$\left(p_2^A\right)^* h_{12} p_2^A = \sum_{i=1}^N \lambda_i(x_1) |\chi_i^1\rangle\langle\chi_i^1|_2$$

with $\langle A\chi_i^1, h_{12} A\chi_j^1 \rangle_2 = 0$ for $i \neq j$ and

$$\sum_{i=1}^N \lambda_i(x_1) = \sum_{i=1}^N \langle A\varphi_i, h_{12} A\varphi_i \rangle_2(x_1),$$
$$\lambda_i(x_1) = \langle A\chi_i^1, h_{12} A\chi_i^1 \rangle_2(x_1).$$

Remark 37. In the special case of $h_{12} = h(x_1 - x_2) = h(x_2 - x_1)$ for a measurable function $h : \mathbb{R}^3 \to \mathbb{R}$, then it holds $\lambda_i(x_1) = h * |A\chi_i^1|^2(x_1)$ and in particular

$$\sum_{i=1}^N \lambda_i(x_1) = h * \rho^A(x_1). \tag{2.5.8}$$

Proof. The proof works similar as in [Pet14, Lemma 6.9] where $A \equiv 1 \equiv B$, h is a multiplication operator and $\left(p_2^A\right)^* \equiv p_2^* = p_2$. First note that

$$\left(p_2^A\right)^* h_{12} p_2^A = \sum_{i,j=1}^N \langle A\varphi_i, h_{12} A\varphi_j \rangle_2(x_1) |\varphi_i\rangle\langle\varphi_j|_2 = \sum_{i,j=1}^N \langle A\varphi_i, \left(B_2 g_{12} + g_{12} B_2^*\right) A\varphi_j \rangle_2(x_1) |\varphi_i\rangle\langle\varphi_j|_2 \tag{2.5.9}$$

is a self-adjoint $N\times N$ matrix in the second variable and can be diagonalized with eigenvalues, i.e. there exists a unitary $N \times N$ matrix $U(x_1)$ depending on the first variable x_1 such that for all $i = 1, \dots, N$

$$|\chi_i^1\rangle_2 = \sum_{k=1}^{N} U_{ik}(x_1)|\varphi_k\rangle_2 \tag{2.5.10}$$

and

$$\left(p_2^A\right)^* h_{12} p_2^A = \sum_{i=1}^{N} \lambda_i(x_1)|\chi_i^1\rangle\langle\chi_i^1|_2. \tag{2.5.11}$$

Thus, it holds

$$\begin{aligned}
\lambda_i(x_1) &= \langle\chi_i^1, \left(p_2^A\right)^* h_{12} p_2^A \chi_i^1\rangle_2(x_1) \\
&= \sum_{j,k=1}^{N} U_{ij}^*(x_1)U_{ik}(x_1)\langle\varphi_j^1, \left(p_2^A\right)^* h_{12} p_2^A \varphi_k^1\rangle_2(x_1) \\
&= \sum_{j,k=1}^{N} U_{ij}^*(x_1)U_{ik}(x_1)\langle A\varphi_j^1, h_{12} A\varphi_k^1\rangle_2(x_1) \\
&= \langle A\chi_i^1, h_{12} A\chi_i^1\rangle_2(x_1).
\end{aligned} \tag{2.5.12}$$

Moreover, for all $k, l = 1, 2, \dots, N$ it holds

$$\begin{aligned}
\langle A\chi_k^1, h_{12} A\chi_l^1\rangle_2(x_1) &= \langle\chi_k^1, \left(p_2^A\right)^* h_{12} p_2^A \chi_l^1\rangle_2(x_1) \\
&= \sum_{i=1}^{N} \lambda_i(x_1)\langle\chi_k^1, \chi_i^1\rangle\langle\chi_i^1, \chi_l^1\rangle_2 \\
&= \lambda_k(x_1)\delta_{k,l}.
\end{aligned} \tag{2.5.13}$$

Thus, it holds

$$\sum_{i=1}^{N} \lambda_i(x_1) = \sum_{i=1}^{N}\langle A\chi_i^1, h_{12} A\chi_i^1\rangle_2(x_1) = \sum_{j=1}^{N}\langle A\varphi_j, h_{12} A\varphi_j\rangle_2(x_1) \tag{2.5.14}$$

where we used $\sum_i U_{ij}^*(x_1)U_{ik}(x_1) = \delta_{jk}$. ■

Lemma 38 (Diagonalization estimates)**.** *Let $h_{mk} = h(x_m - x_k)$ for a non-negative measurable function $h \geq 0$ for $m, k \in \{1, \dots, N\}$ and A be an operator such that*

$$\|h * \rho^A\|_\infty < \infty \qquad \textit{with } \rho^A = \sum_{i=1}^{N} |A\varphi_i|^2.$$

Let ψ be antisymmetric in $x_2, \dots, x_N$. Then it holds

1.

$$\langle\psi, \left(p_2^A\right)^* h_{12} p_2^A \psi\rangle \leq \frac{1}{N-1} \|h * \rho^A\|_\infty \langle\psi, \psi\rangle,$$

2.

$$\langle\psi, p_1 \left(p_2^A\right)^* h_{12} p_2^A p_1 \psi\rangle \leq \frac{1}{N(N-1)} \|\left(h * \rho^A\right) \rho\|_1 \langle\psi, \psi\rangle,$$

3.

$$\langle\psi, q_3 \left(p_1^A\right)^* p_2 h_{12} h_{13} p_1^A p_3 q_2 \psi\rangle \leq \frac{1}{(N-1)(N-2)} \|h * \rho\|_\infty \|h * \rho^A\|_\infty \langle\psi, q_1 \psi\rangle.$$

In particular, it holds for $h_2 = h(x_2)$ the estimate

$$\langle\psi, \left(p_2^A\right)^* h_2 p_2^A \psi\rangle \leq \frac{1}{N} \|h \rho^A\|_1 \langle\psi, \psi\rangle.$$

Remark 39. We emphasize that Lemma 36 applies to a broader class of two-body operators h_{12}, whereas Lemma 38 is valid only for non-negative multiplication operators. This is consistent, as in practice, we always take the absolute modulus before estimating with Lemma 38. The diagonalization in Lemma 36 will be needed for cancellations with the mean-field quantities in Lemma 40 and Lemma 41.

Proof. The proofs work similar to [Pet14, Lemma 6.10, 6.11, 6.12] where $A \equiv \mathrm{id}$. Note that due to the antisymmetry of ψ we can lift a projector of the second particle variable to all antisymmetric variables in ψ from which the projector does not depend on. In particular for the case that ψ is antisymmetric in $x_2, \ldots, x_N$ it holds for any $i = 1, \ldots, N$ that

$$\begin{aligned}\langle\psi, |\chi_i^1\rangle\langle\chi_i^1|_2 \psi\rangle &= \int \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N \; \psi(x_1, x_2, x_3, \ldots, x_N) \chi_i^{x_1}(x_2) \int \mathrm{d}y \; \overline{\chi_i^{x_1}(y)} \psi(x_1, y, x_3, \ldots, x_N) \\ &= \frac{1}{N-1} \langle\psi, \sum_{m=2}^{N} |\chi_i^1\rangle\langle\chi_i^1|_m \psi\rangle \leq \frac{1}{N-1} \|\psi\|^2 \qquad (2.5.15)\end{aligned}$$

since $\sum_{m=2}^N |\chi_i^1\rangle\langle\chi_i^1|_m$ is a projector (the mixed terms vanish since $\{\chi_i^1\}_{i=1}^N$ is an orthonormal system). Thus the first statement follows immediately from $\lambda_i(x_1) \geq 0$ for all $i = 1, \ldots, N$ and

$$\langle\psi, p_2^A h_{12} p_2^A \psi\rangle = \sum_{i=1}^N \lambda_i(x_1) \langle\psi, |\chi_i^1\rangle\langle\chi_i^1|_2 \psi\rangle. \qquad (2.5.16)$$

The second inequality follows applying Lemma 36 a second time with an orthonormal system $\{\xi_j\}_{j=1}^N$ satisfying

$$\langle\psi, p_1 \sum_{i=1}^N \lambda_i(x_1) p_1\rangle = \sum_{j=1}^N \mu_j |\xi_j\rangle\langle\xi_j|_2 \qquad (2.5.17)$$

with $\mu_j \geq 0$ for any $j = 1, \ldots, N$ and

$$\sum_{j=1}^{N} \mu_j = \int \mathrm{d}x \sum_{i=1}^{N} \lambda_i(x)\rho(x) = \int \mathrm{d}x \, \left(h * \rho^A\right)(x)\rho(x). \tag{2.5.18}$$

Additionally, with $A^j \subset L^2_{\mathrm{as}}(\mathbb{R}^{3N})$ denoting the subset of wave functions which are anti-symmetric except for the j-th variable and $A^{j,k} := A^j \cap A^k$ for any $j, k = 1, \ldots, N$ it holds that

$$\begin{aligned}
\langle \psi, q_3 \left(p_1^A\right)^* p_2 h_{12} h_{13} p_1^A p_3 q_2 \psi \rangle &= \langle \sqrt{h_{12}} p_2 \sqrt{h_{13}} p_1^A q_3 \psi, \sqrt{h_{13}} p_3 \sqrt{h_{12}} p_1^A q_2 \psi \rangle \\
&\leq \|q_3\psi\| \left(\sup_{\phi \in A^3} \|\sqrt{h_{13}} p_1^A \phi\| \right) \left(\sup_{\phi \in A^{1,3}} \|\sqrt{h_{12}} p_2 \phi\| \right) \times \\
&\quad \times \left(\sup_{\phi \in A^{1,2}} \|\sqrt{h_{13}} p_3 \phi\| \right) \left(\sup_{\phi \in A^2} \|\sqrt{h_{12}} p_1^A \phi\| \right) \|q_2\psi\| \\
&\leq \|q_1\psi\|^2 \left(\sup_{\phi \in A^3} \|\sqrt{h_{13}} p_1 \phi\| \right)^2 \left(\sup_{\phi \in A^{1,3}} \|\sqrt{h_{12}} p_2 \phi\| \right)^2 \\
&\leq \frac{1}{(N-1)(N-2)} \|h * \rho\|_\infty \|h * \rho^A\|_\infty \langle \psi, q_1 \psi \rangle.
\end{aligned} \tag{2.5.19}$$

Note the proof works analogously with a different constant if we replace q_3 with q_4. ■

2.5.3 Estimates of terms in H^{g} and h^{g}

We give a collection of estimates depending on the number of q-operators occurring. Recall that $\rho^h = \sum_{k=1}^{N} |h^{\mathrm{g}} \varphi_k|^2$.

Lemma 40 (0q-estimates using diagonalization)**.** *It holds for $\varphi, \psi \in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$*

1.

$$\begin{aligned}
&N \left| \left\langle \varphi, \left((N-1) P_0^{\{1,2\}} (w_{\nabla f})_{12} P_0^{\{1,2\}} - p_1 R_1 p_1 \right) \psi \right\rangle \right| \\
&\leq 3N^{\frac{1}{2}} \| |f^{(N)}| * \rho \|_\infty^{1/2} \| |f^{(N)}| * \rho^\nabla \|_\infty^{1/2} \|\varphi\| \left((N-1) \|p_1 q_2 \psi\|^2 + \|p_1 \psi\|^2 \right)^{1/2} \\
&\quad + N^{\frac{1}{2}} \| |f^{(N)}| * \rho \|_\infty^{1/2} \| |f^{(N)}| * \rho^\nabla \|_\infty^{1/2} \|\psi\| \left((N-1) \|p_1 q_2 \varphi\|^2 + \|p_1 \varphi\|^2 \right)^{1/2},
\end{aligned}$$

2.

$$N(N-1) \left| \left\langle \varphi, P_0^{\{1,2\}} (w_f)_{12} P_0^{\{1,2\}} \psi \right\rangle \right| \leq 2N \| |f^{(N)}|^2 * \rho \|_\infty \|\varphi\| \, \|\psi\|,$$

3.

$$
\begin{aligned}
&N|\langle\varphi,\left((N-1)P_0^{\{1,2\}}(u_{\le})_{12}P_0^{\{1,2\}}-p_1\overline{u_{\le}}(x_1)p_1\right)\psi\rangle| \\
&\le N^{\frac{1}{2}}\|u_{\le}^{(N)}*\rho\|_\infty\|\varphi\|\left((N-1)\|p_1q_2\psi\|^2+\|p_1\psi\|^2\right)^{1/2},
\end{aligned}
$$

4.

$$
\begin{aligned}
&N|\langle\varphi,\left((N-1)(N-2)P_0^{\{1,2,3\}}(w_{ff})_{123}P_0^{\{1,2,3\}}-2p_1W_1p_1\right)\psi\rangle| \\
&\le 12N^{\frac{1}{2}}\||f^{(N)}|*\rho\|_\infty^2\|\varphi\|\left((N-1)\|p_1q_2\psi\|^2+\|p_1\psi\|^2\right)^{1/2}.
\end{aligned}
$$

5.

$$
\begin{aligned}
&|\langle\varphi,\left(\sum_{m=1}^N p_m h_m^{\mathrm{g}} p_m-\sum_{k=1}^N\langle\varphi_k,h^{\mathrm{g}}\varphi_k\rangle\right)\psi\rangle| \\
&\le \|\rho^h\|_1^{1/2}\|\varphi\|\left(N\|q_2\psi\|^2+\|\psi\|^2\right)^{1/2}
\end{aligned}
$$

Proof. We estimate

$$
\begin{aligned}
&N|\langle\varphi,\left((N-1)P_0^{\{1,2\}}(w_{\nabla f})_{12}P_0^{\{1,2\}}-p_1R_1p_1\right)\psi\rangle| \\
&\le N|\langle\varphi,p_1\left((N-1)p_2(i\nabla_1)\cdot f_{12}p_2-i\nabla_1\cdot\overline{f}(x_1)\right)p_1\psi\rangle| \\
&\quad+N|\langle\varphi,p_1\left((N-1)p_2f_{12}\cdot(i\nabla_1)p_2-\overline{f}(x_1)\cdot i\nabla_1\right)p_1\psi\rangle| \\
&\quad+N|\langle\varphi,p_1\left((N-1)p_2\left((i\nabla_2)\cdot f_{21}+f_{21}\cdot(i\nabla_2)\right)p_2-\overline{(i\nabla\cdot f)}(x_1)-\overline{(f\cdot i\nabla)}(x_1)\right)p_1\psi\rangle|.
\end{aligned}
\tag{2.5.20}
$$

Now, we apply Lemma 36 for each of the three differences. More specifically, for the first term $(i\nabla_1)\cdot f_{12}$ we use Lemma 36 for each component of f_{12} to obtain

$$
p_2f_{12}p_2=\sum_{i=1}^N\lambda_i(x_1)|\chi_i^1\rangle\langle\chi_i^1|_2 \tag{2.5.21}
$$

with $\lambda_i(x_1)=\left(f^{(N)}*|\chi_i^1|^2\right)(x_1)$. The strategy is now to utilize that the mean-field term is equal to the trace $\sum_{i=1}^N\lambda_i(x_1)$. The difference of (2.5.21) and the mean-field term thus effectively corresponds to a q-operator. We separate the the projector part the trace and the projector part by applying the Cauchy-Schwarz inequality:

$$
\begin{aligned}
& N\left|\left\langle\varphi, p_1\left((N-1) p_2(i \nabla_1) \cdot f_{12} p_2-i \nabla_1 \cdot \overline{f}(x_1)\right) p_1 \psi\right\rangle\right| \\
& =N\left|\left\langle\varphi, p_1(i \nabla_1) \cdot\left(\sum_{m=2}^N p_m f_{1 m} p_m-\overline{f}(x_1)\right) p_1 \psi\right\rangle\right| \\
& =N\left|\left\langle\varphi, p_1(i \nabla_1) \cdot\left(\sum_{i=1}^N \lambda_i(x_1)\Big(\sum_{m=2}^N |\chi_i^1\rangle\langle\chi_i^1|_m-1\Big)\right) p_1 \psi\right\rangle\right| \\
& \leq N\left(\sum_{i=1}^N \||\lambda_i(x_1)| p_1^{\nabla} \varphi\|^2\right)^{1/2}\left(\sum_{i=1}^N \Big\|\Big(\sum_{m=2}^N |\chi_i^1\rangle\langle\chi_i^1|_m-1\Big) p_1 \psi\Big\|^2\right)^{1/2} \\
& \leq N N^{-\frac{1}{2}}\||f^{(N)}| * \rho\|_{\infty}^{1/2}\Big\| \max_{i \in\{1, \ldots, N\}}\left(|f^{(N)}| *|\chi_i^1|^2\right) \rho^{\nabla}\Big\|_1^{1/2}\|\varphi\|\left((N-1)\|p_1 q_2 \psi\|^2+\|p_1 \psi\|^2\right)^{1/2}
\end{aligned} \tag{2.5.22}
$$

where we used two ingredients in the last line: Firstly, the operator $\sum_{i=1}^N \Big(1-\sum_{m=2}^N |\chi_i^1\rangle\langle\chi_i^1|_m\Big)$ acts as a projector on ψ^1 that are antisymmetric in all variables except x_1 since $\{\chi_i\}_{i=1}^N$ is an orthonormal system and mixed terms vanish

$$
\begin{aligned}
\Big\langle\psi^1, \sum_{i=1}^N\Big(1-\sum_{m=2}^N |\chi_i^1\rangle\langle\chi_i^1|_m\Big) \psi^1\Big\rangle & =\Big\langle\psi^1,\Big(N-\sum_{m=2}^N p_m\Big) \psi^1\Big\rangle \\
& =\Big\langle\psi^1,\Big(N-(N-1) p_2\Big) \psi^1\Big\rangle \\
& =(N-1)\langle\psi^1, q_2 \psi^1\rangle+\langle\psi^1, \psi^1\rangle .
\end{aligned} \tag{2.5.23}
$$

Secondly, it holds

$$
\begin{aligned}
\sum_{i=1}^N \||\lambda_i(x_1)| p_1^{\nabla} \varphi\|^2 & \leq \sum_{i=1}^N|\lambda_i(x_1)| \max_{i \in\{1, \ldots, N\}}\langle\varphi,(p_1^{\nabla})^*|\lambda_i(x_1)| p_1^{\nabla} \varphi\rangle \\
& \leq \||f^{(N)}| * \rho\|_{\infty}\Big\| \max_{i \in\{1, \ldots, N\}}\left(|f^{(N)}| *|\chi_i^1|^2\right) \rho^{\nabla}\Big\|_1 \\
& \leq \||f^{(N)}| * \rho\|_{\infty}\||f^{(N)}| * \rho^{\nabla}\|_{\infty}
\end{aligned} \tag{2.5.24}
$$

since $\||\chi_i^1|^2\|_1=1$ for all $i \in\{1, \ldots, N\}$. Thus, by combining both ingredients, it holds

$$
\begin{aligned}
& N\left|\left\langle\varphi, p_1\left((N-1) p_2(i \nabla_1) \cdot f_{12} p_2-i \nabla_1 \cdot \overline{f}(x_1)\right) p_1 \psi\right\rangle\right| \\
& \leq N^{\frac{1}{2}}\||f^{(N)}| * \rho\|_{\infty}^{1/2}\||f^{(N)}| * \rho^{\nabla}\|_{\infty}^{1/2}\|\varphi\|\left((N-1)\|q_2 \psi\|^2+\|\psi\|^2\right)^{1/2} .
\end{aligned} \tag{2.5.25}
$$

Analogously to (2.5.22), it follows

$$
\begin{aligned}
& N\left|\left\langle\varphi, p_1\left((N-1) p_2 f_{12} \cdot(i \nabla_1) p_2-\overline{f}(x_1) \cdot(i \nabla_1)\right) p_1 \psi\right\rangle\right| \\
& \leq N^{\frac{1}{2}}\||f^{(N)}| * \rho\|_{\infty}^{1/2}\||f^{(N)}| * \rho^{\nabla}\|_{\infty}^{1/2}\|\psi\|\left((N-1)\|q_2 \varphi\|^2+\|\varphi\|^2\right)^{1/2} .
\end{aligned} \tag{2.5.26}
$$

Since $((i\nabla_2)\cdot f_{21}+f_{21}\cdot(i\nabla_2))$ is symmetric in x_2, we can apply Lemma 36 to obtain

$$p_2\sum_{m=1}^{N}p_m h^{\mathrm{g}}_m p_m-\sum_{k=1}^{N}\langle\psi^t_k,h^{\mathrm{g}}\psi^t_k\rangle\left((i\nabla_2)\cdot f_{21}+f_{21}\cdot(i\nabla_2)\right)p_2=\sum_{j=1}^{N}\kappa_j(x_1)|\chi^1_j\rangle\langle\chi^1_j|_2$$

with $\kappa_j(x_1)=\left(f^{(N)}*\chi^1_j\overline{(i\nabla\chi^1_j)}+f^{(N)}*(i\nabla\chi^1_j)\overline{\chi^1_j}\right)(x_1)$ and estimate by Cauchy-Schwarz

$$|\kappa_i(x_1)|\le 2\left(|f^{(N)}|*|\chi^1_i|^2(x_1)\right)^{1/2}\left(|f^{(N)}|*|\nabla\chi^1_i|^2(x_1)\right)^{1/2}, \tag{2.5.27}$$
$$\sum_{i=1}^{N}|\kappa_i(x_1)|^2\le 2\||f^{(N)}|*\rho^\nabla\|_\infty\max_{i\in\{1,\dots,N\}}|f^{(N)}|*|\chi^1_i|^2. \tag{2.5.28}$$

With the same approach as in (2.5.22) it follows

$$\begin{aligned}&N|\langle\varphi,p_1\left((N-1)p_2\left((i\nabla_2)\cdot f_{21}+f_{21}\cdot(i\nabla_2)\right)p_2-\overline{(i\nabla\cdot f)}(x_1)-\overline{(f\cdot i\nabla)}(x_1)\right)p_1\psi\rangle|\\&\le 2N^{\frac12}\||f^{(N)}|*\rho^\nabla\|_\infty^{1/2}\ \||f^{(N)}|*\rho\|_\infty^{1/2}\ \|\varphi\|\left((N-1)\|q_2\psi\|^2+\|\psi\|^2\right)^{1/2}\end{aligned} \tag{2.5.29}$$

and

$$\begin{aligned}&N|\langle\varphi,p_1\left((N-1)p_2(u_\le)_{12}p_2-\overline{u_\le}(x_1)\right)p_1\psi\rangle|\\&\le N^{\frac12}\|u^{(N)}_\le*\rho\|_\infty\ \|\varphi\|\left((N-1)\|q_2\psi\|^2+\|\psi\|^2\right)^{1/2}.\end{aligned} \tag{2.5.30}$$

Now for $(w_{ff})_{123}$ we estimate with the triangle inequality

$$\begin{aligned}&N|\langle\varphi,\left((N-1)(N-2)P_0^{\{1,2,3\}}(w_{ff})_{123}P_0^{\{1,2,3\}}-2p_1W_1p_1\right)\psi\rangle|\\&\le 4N|\langle\varphi,p_1\left((N-1)(N-2)p_2p_3\left(f_{21}\cdot f_{23}\right)p_3p_2-\overline{f\cdot\overline{f}}(x_1)\right)p_1\psi\rangle|\\&\quad+2N|\langle\varphi,p_1\left((N-1)(N-2)p_2p_3\left(f_{12}\cdot f_{13}\right)p_3p_2-\overline{f}(x_1)\cdot\overline{f}(x_1)\right)p_1\psi\rangle|\\&\le 4N|\langle\varphi,p_1p_2\left((N-1)(N-2)p_3f_{21}\cdot f_{23}p_3-(N-1)f_{21}\cdot\overline{f}(x_2)\right)p_2p_1\psi\rangle|\\&\quad+4N|\langle\varphi,p_1\left((N-1)p_2f_{21}\cdot\overline{f}(x_2)p_2-\overline{f\cdot\overline{f}}(x_1)\right)p_1\psi\rangle|\\&\quad+2N|\langle\varphi,p_1p_2\left((N-1)(N-2)p_3\left(f_{12}\cdot f_{13}\right)p_3-(N-1)f_{12}\cdot\overline{f}(x_1)\right)p_2p_1\psi\rangle|\\&\quad+2N|\langle\varphi,p_1\left((N-1)(N-2)p_2(N-1)f_{12}p_2-\overline{f}(x_1)\cdot\overline{f}(x_1)\right)p_1\psi\rangle|\end{aligned} \tag{2.5.31}$$

The first line can be estimated similarly to (2.5.22) with $(f_{21})^{1/2} = |f_{21}|^{1/2}\hat{e}_{f_{12}}$

$$
\begin{aligned}
&N\langle\varphi, p_1\left((N-1)(N-2)p_2p_3f_{21}\cdot f_{23}p_3p_2 - (N-1)p_2f_{21}\cdot\overline{f}(x_2)p_2\right)p_1\psi\rangle \\
&\leq N(N-1)|\langle\varphi, p_1p_2(f_{21})^{1/2}\left(\sum_{i=1}^{N}\lambda_i(x_2)\Big(\sum_{m=3}^{N}|\chi_i^2\rangle\langle\chi_i^2|_m - 1\Big)\right)|f_{21}|^{1/2}p_2p_1\psi\rangle| \\
&\leq N(N-1)\left(\sum_{i=1}^{N}\|\lambda_i(x_2)|f_{21}|^{1/2}p_2p_1\varphi\|^2\right)^{1/2}\left(\sum_{i=1}^{N}\|\Big(\sum_{m=3}^{N}|\chi_i^2\rangle\langle\chi_i^2|_m - 1\Big)|f_{21}|^{1/2}p_2p_1\psi\|^2\right)^{1/2} \\
&\leq N\||f^{(N)}|*\rho\|_\infty^{3/2}\langle\varphi, p_2\max_{i\in\{1,\ldots,N\}}|\lambda_i(x_2)|p_2\varphi\rangle^{1/2}\|\sum_{i=1}^{N}\Big(\sum_{m=3}^{N}|\chi_i^2\rangle\langle\chi_i^2|_m - 1\Big)p_2\psi\| \\
&\leq N^{\frac{1}{2}}\||f^{(N)}|*\rho\|_\infty^2\|\varphi\|\left((N-1)\|p_2q_3\psi\|^2 + \|p_2\psi\|^2\right)^{1/2}. \qquad (2.5.32)
\end{aligned}
$$

Furthermore, for the second line of (2.5.31) we write

$$
p_2f_{21}\cdot\overline{f}(x_2)p_2 = \sum_{i=1}^{N}\mu_i(x_1)|\chi_i^1\rangle\langle\chi_i^1|_2
$$

with $\mu_i(x_1) = \left(f^{(N)} * \cdot\overline{f}|\chi_i^1|^2\right)(x_1)$ and estimate

$$
\begin{aligned}
\mu_i(x_1) &\equiv \int \mathrm{d}y\ \chi_i^1(y)^* f^{(N)}(x_1-y)\cdot\overline{f}(y)\chi_i^1(y) \\
&\leq \||f^{(N)}|*\rho\|_\infty\int \mathrm{d}y\ |f^{(N)}(x_1-y)|\ |\chi_i^1(y)|^2. \qquad (2.5.33)
\end{aligned}
$$

Thus by taking the same approach as in (2.5.22) we find

$$
\begin{aligned}
&N\langle\varphi, p_1\left((N-1)p_2f_{21}\cdot\overline{f}(x_2)p_2 - \overline{f\cdot\overline{f}}(x_1)\right)p_1\psi\rangle \\
&\leq N\langle\varphi, p_1\left(\sum_{i=1}^{N}\mu_i(x_1)\Big((N-1)|\chi_i^1\rangle\langle\chi_i^1|_2 - 1\Big)\right)p_1\psi\rangle \\
&\leq N^{\frac{1}{2}}\||f^{(N)}|*\rho\|_\infty^2\|\varphi\|\left((N-1)\|q_2\psi\|^2 + \|\psi\|^2\right)^{1/2}. \qquad (2.5.34)
\end{aligned}
$$

Similarly, we obtain for the last two lines of (2.5.31)

$$
\begin{aligned}
&N\langle\varphi, p_1\left((N-1)(N-2)p_2p_3f_{12}\cdot f_{13}p_3p_2 - (N-1)p_2f_{12}\cdot\overline{f}(x_1)p_2\right)p_1\psi\rangle \\
&\leq N^{\frac{1}{2}}\||f^{(N)}|*\rho\|_\infty^2\|\varphi\|\left((N-1)\|p_1q_2\psi\|^2 + \|p_1\psi\|^2\right)^{1/2}, \qquad (2.5.35)
\end{aligned}
$$

and additionally

$$
\begin{aligned}
&N\langle\varphi, p_1\left((N-1)p_2 f_{12}\cdot\overline{f}(x_1)p_2 - \overline{f}\cdot\overline{f}(x_1)\right)p_1\psi\rangle\\
&\leq N\langle\varphi, p_1\left(\sum_{i=1}^N \lambda_i(x_1)\cdot\overline{f}(x_1)\Big(\sum_{m=3}^N |\chi_i^1\rangle\langle\chi_i^1|_m - 1\Big)\right)p_1\psi\rangle\\
&\leq N^{\frac{1}{2}}\||f^{(N)}|*\rho\|_\infty^2\|\varphi\|\left((N-1)\|p_1q_2\psi\|^2+\|p_1\psi\|^2\right)^{1/2}.
\end{aligned}
\tag{2.5.36}
$$

The w_f-term is estimated with Lemma 38 as an error term without mean-field pendant:

$$
\begin{aligned}
&N(N-1)|\langle\varphi, P_0^{\{1,2\}}(w_f)_{12}P_0^{\{1,2\}}\psi\rangle|\\
&\leq N(N-1)|\langle\varphi, p_1p_2 f_{12}\cdot f_{12}p_2p_1\psi\rangle\\
&\leq N\||f^{(N)}|^2*\rho\|_\infty\|\varphi\|\,\|\psi\|.
\end{aligned}
\tag{2.5.37}
$$

The last item follows again from Lemma 36 which yields

$$
\sum_{m=1}^N p_m h_m^{\mathrm{g}} p_m - \sum_{k=1}^N\langle\psi_k, h^{\mathrm{g}}\psi_k\rangle = \sum_{i=1}^N e_i\left(\sum_{m=1}^N|\xi_i\rangle\langle\xi_i|_m - 1\right)
$$

with $e_i = \langle\xi_i, h^{\mathrm{g}}\xi_i\rangle$, where $\{\xi_i\}_{i=1}^N\subset\mathrm{span}(\varphi_1,\ldots,\varphi_N)$ is an orthonormal set, defined as in 36, that satisfies $\sum_{i=1}^N|\xi_i\rangle\langle\xi_i|_m = p_m$. Thus, by Cauchy-Schwarz it follows

$$
\begin{aligned}
&|\langle\varphi,\left(\sum_{m=1}^N p_m h_m^{\mathrm{g}}p_m - \sum_{k=1}^N\langle\varphi_k, h^{\mathrm{g}}\varphi_k\rangle\right)\psi\rangle|\\
&\leq\left(\sum_{i=1}^N\||e_i|\varphi\|^2\right)^{1/2}\left(\sum_{i=1}^N\|\left(\sum_{m=1}^N|\xi_i\rangle\langle\xi_i|_m - 1\right)\psi\|^2\right)^{1/2}\\
&\leq\|\rho^h\|_1^{1/2}\|\varphi\|\left(N\|q_2\psi\|^2+\|\psi\|^2\right)^{1/2}
\end{aligned}
\tag{2.5.38}
$$

where in the last line we used (2.5.23) and

$$
\sum_{i=1}^N|e_i|^2\leq\sum_{i=1}^N\langle h^{\mathrm{g}}\xi_i, h^{\mathrm{g}}\xi_i\rangle = \sum_{k=1}^N\langle h^{\mathrm{g}}\varphi_k, h^{\mathrm{g}}\varphi_k\rangle = \|\rho^h\|_1.
\tag{2.5.39}
$$

■

Lemma 41 (1q-estimates using diagonalization)**.** *It holds for* $\varphi,\psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$

1.

$$
\begin{aligned}
&N|\langle\varphi,\left((N-1)P_0^{\{1,2\}}(w_{\nabla f})_{12}P_1^{\{1,2\}} - 2p_1R_1q_1\right)\psi\rangle|\\
&\leq 2N^{\frac{1}{2}}\||\nabla\cdot f^{(N)}|*\rho\|_\infty\|\varphi\|\left((N-1)\|q_2q_1\psi\|^2+\|q_1\psi\|^2\right)^{1/2}\\
&\quad+6N^{\frac{1}{2}}\||f^{(N)}|*\rho^\nabla\|_\infty^{1/2}\||f^{(N)}|*\rho\|_\infty^{1/2}\|\varphi\|\left((N-1)\|q_2q_1\psi\|^2+\|q_1\psi\|^2\right)^{1/2}
\end{aligned}
$$

2.

$$N|\langle\varphi,\Big((N-1)(N-2)P_0^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}}-6p_1W_1q_1\Big)\,\psi\rangle|$$
$$\leq 36N^{\frac{1}{2}}\||f^{(N)}|*\rho\|_\infty^2\|\varphi\|\left((N-1)\|q_2q_1\psi\|^2+\|q_1\psi\|^2\right)^{1/2},$$

3.

$$N|\langle\varphi,\Big((N-1)P_0^{\{1,2\}}(u_\leq)_{12}P_1^{\{1,2\}}-2p_1\overline{u_\leq}(x_1)q_1\Big)\,\psi\rangle|$$
$$\leq 2N^{\frac{1}{2}}\|u_\leq^{(N)}*\rho\|_\infty\|\varphi\|\left((N-1)\|q_1q_2\psi\|^2+\|q_1\psi\|^2\right)^{1/2}$$

Remark 42. The estimates in Lemma 40 and Lemma 41 alter from the diagonalization estimates in [Pet14, PP16] where one finds estimates in terms of $\||u^{(N)}|^2*\rho\|_\infty$. Since in our scaling regime the corresponding quantity is not sufficiently small $\||f^{(N)}|^2*\rho\|_\infty\gg N^{-1}$ (see Lemma 10), we rather want to utilize all p-operators via Lemma 38 and

$$\||f_{12}|^{1/2}p_2\psi\|\leq(N-1)^{-1/2}\||f^{(N)}|*\rho\|_\infty \tag{2.5.40}$$

where we can use Lemma 10 to estimate the right hand side in our applications.

Proof. The estimates work similarly to Lemma 40 with the only difference that we want to avoid that the ∇-operator acts on a q-operator. Therefore, all expressions with a ∇-operator are acting on the left φ-side here. We decompose

$$\begin{aligned}
&N|\langle\varphi,\Big((N-1)P_0^{\{1,2\}}(w_{\nabla f})_{12}P_1^{\{1,2\}}-2p_1R_1q_1\Big)\,\psi\rangle|\\
&\leq 2N|\langle\varphi,p_1\left((N-1)p_2(i\nabla_1)\cdot f_{12}p_2-i\nabla_1\cdot\overline{f}(x_1)\right)q_1\psi\rangle|\\
&\quad+2N|\langle\varphi,p_1\left((N-1)p_2f_{12}\cdot(i\nabla_1)p_2-\overline{f}(x_1)\cdot i\nabla_1\right)q_1\psi\rangle|\\
&\quad+2N|\langle\varphi,p_1\Big((N-1)p_2\left((i\nabla_2)\cdot f_{21}+f_{21}\cdot(i\nabla_2)\right)p_2-\overline{(i\nabla\cdot f)}(x_1)-\overline{(f\cdot i\nabla)}(x_1)\Big)\,q_1\psi\rangle|.
\end{aligned} \tag{2.5.41}$$

In comparison to the proof of Lemma 40 we estimate

$$\begin{aligned}
&N|\langle\varphi,p_1\left((N-1)p_2f_{12}\cdot(i\nabla_1)p_2-\overline{f}(x_1)\cdot i\nabla_1\right)q_1\psi\rangle|\\
&\leq N|\langle\varphi,p_1\left(\sum_{m=2}^N p_m(\nabla_1\cdot f_{1m})p_m-(\nabla_1\cdot\overline{f})(x_1)\right)q_1\psi\rangle|\\
&\quad+N|\langle\varphi,p_1^\nabla\cdot\left((N-1)p_2f_{12}p_2-\overline{f}(x_1)\right)q_1\psi\rangle|\\
&=N|\langle\varphi,p_1^\nabla\left(\sum_{i=1}^N\nu_i(x_1)\Big((N-1)|\sigma_i^1\rangle\langle\sigma_i^1|_2-1\Big)\right)q_1\psi\rangle|\\
&\quad+N|\langle\varphi,p_1^\nabla\left(\sum_{i=1}^N\lambda_i(x_1)\Big((N-1)|\chi_i^1\rangle\langle\chi_i^1|_2-1\Big)\right)q_1\psi\rangle|
\end{aligned} \tag{2.5.42}$$

with $\lambda_i(x_1) := f^{(N)} * |\chi_i^1|^2(x_1)$ and $\nu_i(x_1) = \nabla \cdot f^{(N)} * |\chi_i^1|^2(x_1)$. Thus, with the same approach as in (2.5.22) it holds

$$\begin{aligned}
&N|\langle\varphi, p_1\left((N-1)p_2 f_{12}\cdot(i\nabla_1)p_2 - \overline{f}(x_1)\cdot i\nabla_1\right) q_1\psi\rangle| \\
&\leq N^{\frac{1}{2}}\||\nabla\cdot f^{(N)}| * \rho\|_\infty \|\varphi\|\left((N-1)\|q_2q_1\psi\|^2 + \|q_1\psi\|^2\right)^{1/2} \\
&\quad + N^{\frac{1}{2}}\||f^{(N)}| * \rho^\nabla\|_\infty^{1/2}\||f^{(N)}| * \rho\|_\infty^{1/2}\|\varphi\|\left((N-1)\|q_2q_1\psi\|^2 + \|q_1\psi\|^2\right)^{1/2}. \qquad (2.5.43)
\end{aligned}$$

The other terms are estimated in the same manner as in the the proof of Lemma 40. For instance, write again

$$p_2\left((i\nabla_2)\cdot f_{21} + f_{21}\cdot(i\nabla_2)\right)p_2 = \sum_{j=1}^N \kappa_j(x_1)|\chi_j^1\rangle\langle\chi_j^1|_2$$

with $\kappa_j(x_1) = \left(f^{(N)} * \chi_j^1\overline{(i\nabla\chi_j^1)} + f^{(N)} * (i\nabla\chi_j^1)\overline{\chi_j^1}\right)(x_1)$. With the same approach as in (2.5.22) it follows

$$\begin{aligned}
&N|\langle\varphi, p_1\left((N-1)p_2(i\nabla_2)\cdot f_{21}p_2 - \overline{(i\nabla\cdot f)}(x_1)\right) q_1\psi\rangle| \\
&\leq N^{\frac{1}{2}}\||f^{(N)}| * \rho^\nabla\|_\infty^{1/2}\ \||f^{(N)}| * \rho\|_\infty^{1/2}\ \|\varphi\|\left((N-1)\|q_2q_1\psi\|^2 + \|q_1\psi\|^2\right)^{1/2}. \qquad (2.5.44)
\end{aligned}$$

Note that it holds

$$\begin{aligned}
&N|\langle\varphi, \left((N-1)(N-2)P_0^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}} - 6p_1W_1q_1\right)\psi\rangle| \\
&= 6N|\langle\varphi, ((N-1)(N-2)p_1p_2p_3\left(f_{12}\cdot f_{13}\right)\left(q_1p_2p_3 + 2p_1q_2p_3\right) - p_1W_1q_1)\psi\rangle| \\
&\leq 6N|\langle\varphi, p_1\left((N-1)(N-2)p_2p_3\left(f_{12}\cdot f_{13}\right)p_2p_3 - \overline{f}(x_1)\cdot\overline{f}(x_1)\right) q_1\psi\rangle| \\
&\quad + 12N|\langle\varphi, p_2\left((N-1)(N-2)p_1p_3\left(f_{12}\cdot f_{13}\right)p_3p_1 - \overline{f\cdot\overline{f}}(x_2)\right) q_2\psi\rangle| \qquad (2.5.45)
\end{aligned}$$

which allows us to use the same triangle inequalities as in the proof of Lemma 40, that is (2.5.31) with

$$\begin{aligned}
&N\langle\varphi, p_1\left((N-1)(N-2)p_2p_3f_{12}\cdot f_{13}p_3p_2 - (N-1)p_2f_{12}\cdot\overline{f}(x_1)p_2\right) q_1\psi\rangle \\
&\qquad \leq N^{\frac{1}{2}}\||f^{(N)}| * \rho\|_\infty^2\|\varphi\|\left((N-1)\|q_1q_2\psi\|^2 + \|q_1\psi\|^2\right)^{1/2}, \qquad (2.5.46) \\
&N\langle\varphi, p_1\left((N-1)p_2f_{21}\cdot\overline{f}(x_2)p_2 - \overline{f\cdot\overline{f}}(x_1)\right) q_1\psi\rangle \\
&\qquad \leq N^{\frac{1}{2}}\||f^{(N)}| * \rho\|_\infty^2\|\varphi\|\left((N-1)\|q_1q_2\psi\|^2 + \|q_1\psi\|^2\right)^{1/2}. \qquad (2.5.47)
\end{aligned}$$

and additionally

$$\begin{aligned}
&N\langle\varphi, p_1\left((N-1)p_2f_{12}\cdot\overline{f}(x_1)p_2 - \overline{f}\cdot\overline{f}(x_1)\right) q_1\psi\rangle \\
&\leq N\langle\varphi, p_1\left(\sum_{i=1}^N \lambda_i(x_1)\cdot\overline{f}(x_1)\Big(\sum_{m=3}^N |\chi_i^1\rangle\langle\chi_i^1|_m - 1\Big)\right) q_1\psi\rangle \\
&\leq N^{\frac{1}{2}}\||f^{(N)}| * \rho\|_\infty^2\|\varphi\|\left((N-1)\|q_1q_2\psi\|^2 + \|q_1\psi\|^2\right)^{1/2}. \qquad (2.5.48)
\end{aligned}$$

Furthermore, it holds

$$
\begin{aligned}
&N|\langle\varphi, \left((N-1)P_0^{\{1,2\}}(u_\leq)_{12}P_1^{\{1,2\}} - 2p_1\overline{u_\leq}q_1\right)\psi\rangle| \\
&\leq 2N|\langle\varphi, p_1\left((N-1)p_2(u_\leq)_{12}p_2 - \overline{u_\leq}\right)q_1\psi\rangle| \\
&\leq 2N|\langle\varphi, p_1\sum_{i=1}^N \mu_i(x_1)\left((N-1)|\chi_i^1\rangle\langle\chi_i^1|_2 - 1\right)q_1\psi\rangle| \\
&\leq 2N\left(\sum_{i=1}^N \|\mu_i(x_1)p_1\varphi\|^2\right)^{1/2}\left(\sum_{i=1}^N \|\left((N-1)|\chi_i^1\rangle\langle\chi_i^1|_2 - 1\right)q_1\psi\|^2\right)^{1/2} \\
&\leq 2N^{\frac{1}{2}}\|u_\leq^{(N)} * \rho\|_\infty\|\varphi\|\left((N-1)\|q_1q_2\psi\|^2 + \|q_1\psi\|^2\right)^{1/2}
\end{aligned}
\tag{2.5.49}
$$

with $\mu_i(x_1) = \left(u_\leq^{(N)} * |\chi_i^1|^2\right)(x_1)$. ■

Lemma 43 (1q-estimates)**.** *It holds for $\varphi, \psi \in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$*

1.

$$
\begin{aligned}
\frac{N(N-1)}{2}|\langle\varphi, P_0^{\{1,2\}}(w_{\nabla f})_{12}P_1^{\{1,2\}}\psi\rangle| \leq\ & 12N\||f^{(N)}| * \rho^\nabla\|_\infty^{1/2}\||f^{(N)}| * \rho\|_\infty^{1/2}\|\varphi\|\ \|q_2\psi\| \\
&+ 2N\||\nabla\cdot f^{(N)}| * \rho\|_\infty\|\varphi\|\ \|q_2\psi\|,
\end{aligned}
$$

2.

$$
\frac{N(N-1)}{2}|\langle\varphi, P_0^{\{1,2\}}(w_f)_{12}P_1^{\{1,2\}}\psi\rangle| \leq 2N\||f^{(N)}|^2 * \rho\|_\infty\|\varphi\|\ \|q_2\psi\|,
$$

3.

$$
\frac{N(N-1)}{2}|\langle\varphi, P_0^{\{1,2\}}(u_\leq)_{12}P_1^{\{1,2\}}\psi\rangle| \leq N\|u_\leq^{(N)} * \rho\|_\infty\|\varphi\|\ \|q_2\psi\|,
$$

4.

$$
\begin{aligned}
&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_0^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}}\psi\rangle| \\
&\leq 3N\||f^{(N)}| * \rho\|_\infty^2\|\varphi\|\ \|q_1\psi\|.
\end{aligned}
$$

Proof. The estimates are straightforward by exploiting the p-operators with Lemma 38 and splitting $f_{12} = (f_{12})^{1/2}|f_{12}|^{1/2}$ with $(f_{12})^{1/2} = |f_{12}|^{1/2}\hat{e}_{f_{12}}$ if possible. Furthermore, we prioritize $\nabla_1 p_1$ over $\nabla_1 q_1$. We estimate

$$
\begin{aligned}
&N(N-1)|\langle\varphi, P_0^{\{1,2\}}(w_{\nabla f})_{12}P_1^{\{1,2\}}\psi\rangle| \\
&\leq 2N(N-1)|\langle\varphi, p_1p_2\left((i\nabla_1)\cdot f_{12} + f_{12}\cdot(i\nabla_1)\right)(q_2p_1+p_2q_1)\psi\rangle| \\
&\leq 4N(N-1)\|(f_{12})^{1/2}\cdot p_1^\nabla p_2\varphi\|\ \||f_{12}|^{1/2}(q_2p_1+p_2q_1)\psi\| \\
&\quad + 2N(N-1)\||\nabla_1\cdot f_{12}|^{1/2}p_2p_1\varphi\|\ \||\nabla_1\cdot f_{12}|^{1/2}(q_2p_1+p_2q_1)\psi\| \\
&\leq 24N\||f^{(N)}|*\rho^\nabla\|_\infty^{1/2}\||f^{(N)}|*\rho\|_\infty^{1/2}\|\varphi\|\ \|q_2\psi\| \\
&\quad + 4N\||\nabla\cdot f^{(N)}|*\rho\|_\infty\|\varphi\|\ \|q_2\psi\|
\end{aligned}
\tag{2.5.50}
$$

and similarly

$$
\begin{aligned}
&N(N-1)|\langle\varphi, P_1^{\{1,2\}}(w_{\nabla f})_{12}P_0^{\{1,2\}}\psi\rangle| \\
&\leq 24N\||f^{(N)}|*\rho^\nabla\|_\infty^{1/2}\||f^{(N)}|*\rho\|_\infty^{1/2}\|\psi\|\ \|q_2\varphi\| + 4N\||\nabla\cdot f^{(N)}|*\rho\|_\infty\|\psi\|\ \|q_2\varphi\|.
\end{aligned}
\tag{2.5.51}
$$

In addition, we estimate

$$
\begin{aligned}
&N(N-1)|\langle\varphi, P_0^{\{1,2\}}(w_f)_{12}P_1^{\{1,2\}}\psi\rangle| \\
&\leq 2N(N-1)|\langle\varphi, p_1p_2\left(f_{12}\cdot f_{12}\right)(q_2p_1+p_2q_1)\psi\rangle| \\
&\leq 2N(N-1)\||f_{12}|p_2p_1\varphi\|\ \||f_{12}|(q_2p_1+p_2q_1)\psi\| \\
&\leq 4N\||f^{(N)}|^2*\rho\|_\infty\|\varphi\|\ \|q_2\psi\|,
\end{aligned}
\tag{2.5.52}
$$

$$
\begin{aligned}
&N(N-1)|\langle\varphi, P_0^{\{1,2\}}(u_\leq)_{12}P_1^{\{1,2\}}\psi\rangle| \\
&\leq 2N(N-1)\||(u_\leq)_{12}|^{1/2}p_2p_1\varphi\|\ \||(u_\leq)_{12}|^{1/2}(q_2p_1+p_2q_1)\psi\| \\
&\leq 2N\|u_\leq^{(N)}*\rho\|_\infty\|\varphi\|\ \|q_2\psi\|,
\end{aligned}
\tag{2.5.53}
$$

and is

$$
\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_0^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}}\psi\rangle| \\
&\leq 6N(N-1)(N-3)|\langle\varphi, p_1p_2p_3\left(f_{12}\cdot f_{13}\right)(q_1p_2p_3+2p_1q_2p_3)\psi\rangle| \\
&\leq 18N\||f^{(N)}|*\rho\|_\infty^2\|\varphi\|\ \|q_1\psi\|.
\end{aligned}
\tag{2.5.54}
$$

■

Lemma 44 (2q-estimates, symmetric terms)**.** *It holds for $\varphi,\psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$*

1.
$$
\begin{aligned}
&\frac{N(N-1)}{2}|\langle\varphi, P_1^{\{1,2\}}(w_{\nabla f})_{12}P_1^{\{1,2\}}\psi\rangle| \\
&\leq 12N\||f^{(N)}|*\rho^\nabla\|_\infty^{1/2}\||f^{(N)}|*\rho\|_\infty^{1/2}\|q_2\varphi\|\ \|q_2\psi\| \\
&\quad + 2N\||f^{(N)}|*\rho\|_\infty\left(\|\nabla_1q_1\varphi\|\ \|q_2\psi\| + \|q_2\varphi\|\ \|\nabla_1q_1\psi\|\right),
\end{aligned}
$$

2.
$$\frac{N(N-1)}{2}|\langle\varphi, P_1^{\{1,2\}}(w_f)_{12}P_1^{\{1,2\}}\psi\rangle| \leq N\||f^{(N)}|^2 * \rho\|_\infty \|q_1\varphi\| \ \|q_2\psi\|,$$

3.
$$\frac{N(N-1)}{2}|\langle\varphi, P_1^{\{1,2\}}(u_{\leq})_{12}P_1^{\{1,2\}}\psi\rangle| \leq 2N\|u_{\leq}^{(N)} * \rho\|_\infty \|q_1\varphi\| \ \|q_2\psi\|,$$

4.
$$\begin{aligned}&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_1^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}}\psi\rangle \\ &\leq 9N\||f^{(N)}| * \rho\|_\infty^2 \|q_1\varphi\| \ \|q_2\psi\|.\end{aligned}$$

Proof. We estimate

$$\begin{aligned}&N(N-1)|\langle\varphi, P_1^{\{1,2\}}(w_{\nabla f})_{12}P_1^{\{1,2\}}\psi\rangle| \\ &\leq 2N(N-1)|\langle\varphi, (q_1p_2+p_1q_2)\left((i\nabla_1)\cdot f_{12} + f_{12}\cdot(i\nabla_1)\right)(q_2p_1+p_2q_1)\psi\rangle| \\ &\leq 2N(N-1)\|(f_{12})^{1/2}\cdot(p_1^{\nabla}q_2 + p_2\nabla_1 q_1)\varphi\| \ \||f_{12}|^{1/2}(q_2p_1+p_2q_1)\psi\| \\ &\quad + 2N(N-1)\||f_{12}|^{1/2}(q_1p_2+p_1q_2)\varphi\| \ \|(f_{12})^{1/2}\cdot(p_1^{\nabla}q_2 + p_2\nabla_1 q_1)\psi\| \\ &\leq 4N\||f^{(N)}| * \rho\|_\infty \left(\|\nabla_1 q_1\varphi\| \ \|q_2\psi\| + \|q_2\varphi\| \ \|\nabla_1 q_1\psi\|\right) \\ &\quad + 24N\||f^{(N)}| * \rho^{\nabla}\|_\infty^{1/2}\||f^{(N)}| * \rho\|_\infty^{1/2}\|q_2\varphi\| \ \|q_2\psi\|.\end{aligned} \tag{2.5.55}$$

We estimate with Lemma 38

$$\begin{aligned}&\frac{N(N-1)}{2}|\langle\varphi, P_1^{\{1,2\}}(w_f)_{12}P_1^{\{1,2\}}\psi\rangle| \\ &\leq 2N(N-1)\langle\varphi, q_1p_2f_{12}\cdot f_{12}(p_2q_1+q_2p_1)\psi\rangle \\ &\leq 2N\|\||f^{(N)}|^2 * \rho\|_\infty \|q_1\varphi\| \ \|q_1\psi\|.\end{aligned} \tag{2.5.56}$$

Furthermore, it holds

$$\begin{aligned}&\frac{N(N-1)(N-2)}{6}\langle\varphi, P_1^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}}\psi\rangle \\ &= N(N-1)(N-2)\langle\varphi, q_1p_2p_3f_{12}\cdot f_{13}q_1p_2p_3\psi\rangle \\ &\quad + 2N(N-1)(N-2)\langle\varphi, q_1p_2p_3f_{12}\cdot f_{13}p_1q_2p_3\psi\rangle \\ &\quad + 2N(N-1)(N-2)\langle\varphi, p_1q_2p_3f_{12}\cdot f_{13}q_1p_2p_3\psi\rangle \\ &\quad + 2N(N-1)(N-2)\langle\varphi, p_1q_2p_3f_{12}\cdot f_{13}(p_1q_2p_3+p_1p_2q_3)\psi\rangle\end{aligned} \tag{2.5.57}$$

We estimate with Lemma 38

$$\begin{aligned}&N(N-1)(N-2)|\langle\varphi, q_1p_2p_3f_{12}\cdot f_{13}q_1p_2p_3\psi\rangle| \\ &\leq N\||f^{(N)}| * \rho\|_\infty^2 \|q_1\varphi\| \ \|q_1\psi\|\end{aligned} \tag{2.5.58}$$

and

$$\begin{aligned}&4N(N-1)(N-2)|\mathrm{Re}\langle\varphi, q_1p_2p_3f_{12}\cdot f_{13}p_1q_2p_3\psi\rangle| \\ &\leq 4N\||f^{(N)}|*\rho\|_\infty^2\|q_1\varphi\|\ \|q_2\psi\| \end{aligned} \tag{2.5.59}$$

and

$$\begin{aligned}&2N(N-1)(N-2)\langle\tilde{\Psi}_t, p_1q_2p_3f_{12}\cdot f_{13}(p_1q_2p_3+p_1p_2q_3)\tilde{\Psi}_t\rangle \\ &\leq 4N\||f^{(N)}|*\rho\|_\infty^2\|q_2\varphi\|\ \|q_2\psi\|. \end{aligned} \tag{2.5.60}$$

$\blacksquare$

Lemma 45 (2q-estimates, asymmetric terms)**.** *It holds for* $\varphi,\psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$

1.
$$\begin{aligned}&\frac{N(N-1)}{2}|\langle\varphi, P_0^{\{1,2\}}(w_{\nabla f})_{12}P_2^{\{1,2\}}\psi\rangle| \\ &\leq N\|q_1\varphi\|\Bigg(\left(6N^{-1}\|(f^{(N)})^2*\rho\,\rho^{\nabla}\|_1\|\psi\|^2+3\||f^{(N)}|*\rho\|_\infty\||f^{(N)}|*\rho^{\nabla}\|_\infty\|q_1\psi\|^2\right)^{1/2} \\ &\quad+\left(N^{-1}\|(\nabla\cdot f^{(N)})^2*\rho\,\rho\|_1\|\psi\|^2+\|(\nabla\cdot f^{(N)})*\rho\|_\infty^2\|q_1\psi\|^2\right)^{1/2}\Bigg).\end{aligned}$$

2.
$$\begin{aligned}&\frac{N(N-1)}{2}|\langle\varphi, P_0^{\{1,2\}}(w_f)_{12}P_2^{\{1,2\}}\psi\rangle| \\ &\leq N\|q_1\varphi\|\left(\||f^{(N)}|^4*\rho\|_\infty\|\psi\|^2+\||f^{(N)}|^2*\rho\|_\infty^2\|q_2\psi\|^2\right)^{1/2},\end{aligned}$$

3.
$$\begin{aligned}&\frac{N(N-1)}{2}|\langle\varphi, P_0^{\{1,2\}}(u_\leq)_{12}P_2^{\{1,2\}}\psi\rangle| \\ &\leq \frac{N}{2}\|q_1\varphi\|\left(\||u_\leq^{(N)}|^2*\rho\|_\infty\|\psi\|^2+\|u_\leq^{(N)}*\rho\|_\infty^2\|q_2\psi\|^2\right)^{1/2}\end{aligned}$$

4.
$$\begin{aligned}&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_0^{\{1,2,3\}}(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle| \\ &\leq 3N\||f^{(N)}|*\rho\|_\infty\|q_1\varphi\|\times \\ &\qquad\times\left(\||f^{(N)}|*\rho\|_\infty^2\|q_3\psi\|^2+\||f^{(N)}|^2*\rho\|_\infty\|\psi\|^2\right)^{1/2}.\end{aligned}$$

Proof. Similar to before, we exploit the p-operators with Lemma 38, split $|f_{12}| = |f_{12}|^{1/2}|f_{12}|^{1/2}$ if possible and avoid $\nabla_1 q_1$ by applying ∇ to the corresponding p-operator. We estimate by utilizing the antisymmetry of φ, ψ to introduce a sum and apply Cauchy-Schwarz to estimate

$$\begin{aligned}
&N(N-1)|\langle\varphi, P_0^{\{1,2\}}(w_{\nabla f})_{12}P_2^{\{1,2\}}\psi\rangle| \\
&\le 2N(N-1)|\langle\varphi, p_1p_2\left((i\nabla_1)\cdot f_{12} + f_{12}\cdot(i\nabla_1)\right)q_2q_1\psi\rangle| \\
&\le 2N(N-1)|\langle\varphi, p_1p_2\left(2(i\nabla_1)\cdot f_{12} + (i\nabla_1\cdot f_{12})\right)q_2q_1\psi\rangle| \\
&\le 2N\|q_1\psi\|\Big(2\|\sum_{m=2}^{N} q_m f_{1m}p_m\cdot p_1^{\nabla}\varphi\| + \|\sum_{m=2}^{N} q_m\left(\nabla_1\cdot f_{1m}\right)p_mp_1\varphi\|\Big) \qquad (2.5.61)
\end{aligned}$$

and furthermore, we employ the symmetrization method with estimate 3 of Lemma 38

$$\begin{aligned}
&\|\sum_{m=2}^{N} q_m f_{1m}p_m\cdot p_1^{\nabla}\varphi\|^2 \\
&= (N-1)\langle\varphi, p_2p_1^{\nabla}\cdot f_{12}q_2f_{12}\cdot p_1^{\nabla}p_2\varphi\rangle \\
&\quad + (N-1)(N-2)\langle q_2\varphi, p_3p_1^{\nabla}\cdot f_{13}f_{12}\cdot p_1^{\nabla}p_2q_3\varphi\rangle \\
&\le 3N^{-1}\|(f^{(N)})^2 * \rho\ \rho^{\nabla}\|_1\|\varphi\|^2 + 3\||f^{(N)}| * \rho\|_\infty\||f^{(N)}| * \rho^{\nabla}\|_\infty\|q_1\varphi\|^2, \qquad (2.5.62)
\end{aligned}$$

$$\begin{aligned}
&\|\sum_{m=2}^{N} q_m\nabla_1\cdot f_{1m}p_mp_1\varphi\|^2 \\
&\le N^{-1}\|(\nabla\cdot f^{(N)})^2 * \rho\ \rho\|_1\|\varphi\|^2 + \||\nabla\cdot f^{(N)}| * \rho\|_\infty^2\|q_1\varphi\|^2. \qquad (2.5.63)
\end{aligned}$$

In addition, it holds

$$\begin{aligned}
&N(N-1)|\langle\varphi, P_0^{\{1,2\}}(w_f)_{12}P_2^{\{1,2\}}\psi\rangle| \\
&\le 2N(N-1)|\langle q_1\varphi, q_2f_{12}\cdot f_{12}p_2p_1\psi\rangle| \\
&\le 2N\|q_1\varphi\|\ \|\sum_{m=2}^{N} q_mf_{1m}\cdot f_{1m}p_mp_1\psi\| \qquad (2.5.64)
\end{aligned}$$

with

$$\begin{aligned}
&\|\sum_{m=2}^{N} q_mf_{1m}\cdot f_{1m}p_mp_1\psi\| \\
&\le (N-1)\langle\psi, p_1p_2f_{12}\cdot f_{12}q_2f_{12}\cdot f_{12}p_2p_1\psi\rangle \\
&\quad + (N-1)(N-2)\langle\psi, p_1p_2f_{12}\cdot f_{12}q_2q_3f_{13}\cdot f_{13}p_3p_1\psi\rangle \\
&\le \||f^{(N)}|^4 * \rho\|_\infty\|\psi\|^2 + \||f^{(N)}|^2 * \rho\|_\infty^2\|q_2\psi\|^2. \qquad (2.5.65)
\end{aligned}$$

Similarly, it holds

$$N(N-1)|\langle\varphi, P_0^{\{1,2\}}(u_{\le})_{12}P_2^{\{1,2\}}\psi\rangle| \le N\|q_1\varphi\| \left(\||u_{\le}^{(N)}|^2 * \rho\|_\infty \|\psi\|^2 + \|u_{\le}^{(N)} * \rho\|_\infty^2 \|q_2\psi\|^2\right)^{1/2} \tag{2.5.66}$$

In addition, we estimate

$$\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_0^{\{1,2,3\}}(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle| \\
&= 6N(N-1)(N-2)|\langle\varphi, p_1p_2p_3f_{12}\cdot f_{13}(p_1q_2q_3 + 2q_1q_2p_3)\psi\rangle| \\
&\le 6N(N-1)\||f_{12}|^{1/2}p_1q_2\varphi\| \, \|\sum_{m=3}^N q_m|f_{1m}| \; |f_{12}|^{1/2}p_1p_2p_m\psi\| \\
&\quad + 12N(N-1)\||f_{13}|^{1/2}p_3q_1\varphi\| \, \|\sum_{m=2,4,\dots,N} q_m|f_{1m}| \; |f_{13}|^{1/2}p_1p_mp_3\psi\| \\
&\le 18N\||f^{(N)}| * \rho\|_\infty \|q_1\varphi\| \times \\
&\qquad \times \left(\||f^{(N)}| * \rho\|_\infty^2 \|q_3\psi\|^2 + \||f^{(N)}|^2 * \rho\|_\infty \|\psi\|^2\right)^{1/2}
\end{aligned} \tag{2.5.67}$$

where we used

$$\begin{aligned}
&\|\sum_{m=3}^N q_m|f_{1m}| \; |f_{12}|^{1/2}p_1p_2p_m\psi\|^2 \\
&\le (N-2)\langle\psi, p_1p_2|f_{12}|^{1/2}p_3|f_{13}|^2p_3|f_{12}|^{1/2}p_2p_1\psi\rangle \\
&\quad + (N-2)(N-3)\langle\psi, p_3p_2p_1|f_{12}|^{1/2}|f_{13}|q_4q_3|f_{14}| \; |f_{12}|^{1/2}p_1p_2p_4\psi\rangle \\
&\le \||f^{(N)}|^2 * \rho\|_\infty \langle\psi, p_1p_2|f_{12}|p_2p_1\psi\rangle \\
&\quad + (N-2)(N-3)\langle q_4\psi, p_1|f_{14}|^{1/2}p_2|f_{12}|^{1/2}p_3|f_{13}|^{1/2} \; |f_{14}|^{1/2}p_4|f_{12}|^{1/2}p_2|f_{13}|^{1/2}p_1q_3\psi\rangle \\
&\le N^{-1}\||f^{(N)}|^2 * \rho\|_\infty \||f^{(N)}| * \rho\|_\infty \langle\psi,\psi\rangle \\
&\quad + N^{-1}\||f^{(N)}| * \rho\|_\infty^3 \|q_4\psi\| \; \|q_3\psi\|
\end{aligned}$$

which gives us the desired result. ■

Lemma 46 (3q-estimates)**.** *It holds for* $\varphi, \psi \in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$

1.

$$\begin{aligned}
&\frac{N(N-1)}{2}|\langle\varphi, P_1^{\{1,2\}}(w_{\nabla f})_{12}P_2^{\{1,2\}}\psi\rangle| \\
&\leq 4N(N-1)^{1/2}\||f^{(N)}|^2 * \rho\|_\infty^{1/2} \ \|q_1\hat{n}_{+1}^{1/2}\varphi\| \ \|\hat{n}^{-1/2}q_2\nabla_1 q_1\psi\| \\
&\quad + 2N(N-1)^{1/2}\||\nabla\cdot f^{(N)}|^2 * \rho\|_\infty^{1/2} \ \|\hat{n}_{+1}q_1\varphi\| \ \|\hat{n}^{-1}q_2q_1\psi\|, \\
&\frac{N(N-1)}{2}|\langle\varphi, P_2^{\{1,2\}}(w_{\nabla f})_{12}P_1^{\{1,2\}}\psi\rangle| \\
&\leq 4N(N-1)^{1/2}\||f^{(N)}|^2 * \rho\|_\infty^{1/2} \ \|q_1q_2\varphi\| \ \|\nabla_1 q_1\psi\| \\
&\quad + 12N(N-1)^{1/2}\||f^{(N)}|^2 * \rho^{\nabla}\|_\infty^{1/2} \ \|\hat{n}_{-1}^{1/2}q_1q_2\varphi\| \ \|\hat{n}^{-1/2}q_2\psi\| \\
&\quad + 4N(N-1)^{1/2}\||\nabla\cdot f^{(N)}|^2 * \rho\|_\infty^{1/2} \ \|\hat{n}_{-1}^{1/2}q_1q_2\varphi\| \ \|\hat{n}^{-1/2}q_2\psi\|,
\end{aligned}$$

2.

$$\begin{aligned}
&\frac{N(N-1)}{2}|\langle\varphi, P_1^{\{1,2\}}(w_f)_{12}P_2^{\{1,2\}}\psi\rangle| \leq 2N(N-1)^{1/2}\||f^{(N)}|^4 * \rho\|_\infty^{1/2} \ \|\hat{n}_{+1}q_1\varphi\| \ \|\hat{n}^{-1}q_1q_2\psi\|, \\
&\frac{N(N-1)}{2}|\langle\varphi, P_2^{\{1,2\}}(w_f)_{12}P_1^{\{1,2\}}\psi\rangle| \leq 2N(N-1)^{1/2}\||f^{(N)}|^4 * \rho\|_\infty^{1/2} \ \|\hat{n}_{-1}^{1/2}q_1q_2\varphi\| \ \|\hat{n}^{-1/2}q_1\psi\|,
\end{aligned}$$

3.

$$\begin{aligned}
&\frac{N(N-1)}{2}|\langle\varphi, P_1^{\{1,2\}}(u_{\leq})_{12}P_2^{\{1,2\}}\psi\rangle| \leq N(N-1)^{1/2}\||u_{\leq}^{(N)}|^2 * \rho\|_\infty^{1/2} \ \|\hat{n}_{+1}q_1\varphi\| \ \|\hat{n}^{-1}q_1q_2\psi\|, \\
&\frac{N(N-1)}{2}|\langle\varphi, P_2^{\{1,2\}}(u_{\leq})_{12}P_1^{\{1,2\}}\psi\rangle| \leq N(N-1)^{1/2}\||u_{\leq}^{(N)}|^2 * \rho\|_\infty^{1/2} \ \|\hat{n}_{-1}^{1/2}q_1q_2\varphi\| \ \|\hat{n}^{-1/2}q_1\psi\|,
\end{aligned}$$

4.

$$\begin{aligned}
&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_1^{\{1,2,3\}}(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle| \\
&\leq 9N(N-1)^{1/2}\||f^{(N)}|^2 * \rho\|_\infty^{1/2} \ \||f^{(N)}| * \rho\|_\infty \ \|q_1\hat{n}_{+1}\varphi\| \ \|\hat{n}^{-1}q_2q_3\psi\|, \\
&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_2^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}}\psi\rangle| \\
&\leq 9N(N-1)^{1/2}\||f^{(N)}|^2 * \rho\|_\infty^{1/2} \ \||f^{(N)}| * \rho\|_\infty \ \|\hat{n}_{-1}^{1/2}q_2q_3\varphi\| \ \|\hat{n}^{-1/2}q_1\psi\|,
\end{aligned}$$

5.

$$
\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_0^{\{1,2,3\}}(w_{ff})_{123}P_3^{\{1,2,3\}}\psi\rangle| \\
&\leq 6N^{\frac{1}{2}}(N-1)\|\hat{n}^{-3/2}q_1q_2q_3\psi\|\ \|\hat{n}_{+3}^{3/2}\varphi\|\times \\
&\qquad\times\left(\||f^{(N)}|^2*\rho\|_\infty^2+\||f^{(N)}|^2*\rho\|_\infty\||f^{(N)}|*\rho\|_\infty^2\right)^{1/2}, \\
&N(N-1)(N-2)|\langle\varphi, P_3^{\{1,2,3\}}(w_{ff})_{123}P_0^{\{1,2,3\}}\psi\rangle| \\
&\leq 6N^{\frac{1}{2}}(N-1)\|q_1q_2q_3\varphi\|\ \|\psi\|\times \\
&\qquad\times\left(\||f^{(N)}|^2*\rho\|_\infty^2+\||f^{(N)}|^2*\rho\|_\infty\||f^{(N)}|*\rho\|_\infty^2\right)^{1/2}.
\end{aligned}
$$

Proof. Similar to before, we exploit the p-operators with Lemma 38. Note that here we cannot avoid $\nabla_1 q_1$ anymore. We estimate

$$
\begin{aligned}
&N(N-1)|\langle\varphi, P_1^{\{1,2\}}(w_{\nabla f})_{12}P_2^{\{1,2\}}\psi\rangle| \\
&\leq 2N(N-1)|\langle\varphi, (q_1p_2+p_1q_2)\left((i\nabla_1)\cdot f_{12}+f_{12}\cdot(i\nabla_1)\right)q_2q_1\psi\rangle| \\
&\leq 4N(N-1)|\langle\varphi, \hat{n}_{+1}^{1/2}(q_1p_2+p_1q_2)\left(f_{12}\cdot i\nabla_1\right)q_1\hat{n}^{-1/2}q_2\psi\rangle| \\
&\quad+2N(N-1)|\langle\varphi, \hat{n}_{+1}(q_1p_2+p_1q_2)(i\nabla_1\cdot f_{12})\hat{n}^{-1}q_2q_1\psi\rangle| \\
&\leq 8N(N-1)^{1/2}\||f^{(N)}|^2*\rho\|_\infty^{1/2}\ \|q_1\hat{n}_{+1}^{1/2}\varphi\|\ \|\hat{n}^{-1/2}q_2\nabla_1q_1\psi\| \\
&\quad+4N(N-1)^{1/2}\||\nabla\cdot f^{(N)}|^2*\rho\|_\infty^{1/2}\ \|q_1\hat{n}_{+1}\varphi\|\ \|\hat{n}^{-1}q_2q_1\psi\|
\end{aligned}
\tag{2.5.68}
$$

and similarly,

$$
\begin{aligned}
&N(N-1)|\langle\varphi, P_2^{\{1,2\}}(w_{\nabla f})_{12}P_1^{\{1,2\}}\psi\rangle| \\
&\leq 2N(N-1)|\langle\varphi, q_1q_2\left((i\nabla_1)\cdot f_{12}+f_{12}\cdot(i\nabla_1)\right)(q_1p_2+p_1q_2)\psi\rangle| \\
&\leq 4N(N-1)|\langle\varphi, q_1q_2f_{12}\cdot(i\nabla_1q_1p_2+p_1^\nabla\hat{n}^{1/2}\hat{n}^{-1/2}q_2)\psi\rangle| \\
&\quad+2N(N-1)|\langle\varphi, \hat{n}_{-1}^{1/2}q_1q_2(i\nabla_1\cdot f_{12})\hat{n}^{-1/2}(q_1p_2+p_1q_2)\psi\rangle| \\
&\leq 4N(N-1)^{1/2}\||f^{(N)}|^2*\rho\|_\infty^{1/2}\ \|q_1q_2\varphi\|\ \|\nabla_1q_1\psi\| \\
&\quad+12N(N-1)^{1/2}\||f^{(N)}|^2*\rho^\nabla\|_\infty^{1/2}\ \|\hat{n}_{-1}^{1/2}q_1q_2\varphi\|\ \|\hat{n}^{-1/2}q_2\psi\| \\
&\quad+4N(N-1)^{1/2}\||\nabla\cdot f^{(N)}|^2*\rho\|_\infty^{1/2}\ \|\hat{n}_{-1}^{1/2}q_1q_2\varphi\|\ \|\hat{n}^{-1/2}q_2\psi\|.
\end{aligned}
\tag{2.5.69}
$$

In addition, it holds

$$
\begin{aligned}
&N(N-1)|\langle\varphi, P_1^{\{1,2\}}(w_f)_{12}P_2^{\{1,2\}}\psi\rangle| \\
&\leq 2N(N-1)|\langle\varphi, (p_1q_2+q_1p_2)f_{12}\cdot f_{12}q_1q_2\psi\rangle| \\
&\leq 4N(N-1)^{1/2}\||f^{(N)}|^4*\rho\|_\infty^{1/2}\ \|\hat{n}_{+1}q_1\varphi\|\ \|\hat{n}^{-1}q_1q_2\psi\|
\end{aligned}
\tag{2.5.70}
$$

and

$$
\begin{aligned}
&N(N-1)|\langle\varphi, P_2^{\{1,2\}}(w_f)_{12}P_1^{\{1,2\}}\psi\rangle| \\
&\leq 2N(N-1)|\langle\varphi, q_1q_2f_{12}\cdot f_{12}(p_1q_2+q_1p_2)\psi\rangle| \\
&\leq 2N(N-1)^{1/2}\||f^{(N)}|^4 * \rho\|_\infty^{1/2}\ \|\hat{n}_{-1}^{1/2}q_1q_2\varphi\|\ \|\hat{n}^{-1/2}q_1\psi\|.
\end{aligned}
\tag{2.5.71}
$$

In addition, we estimate

$$
\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_1^{\{1,2,3\}}(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle| \\
&= 6N(N-1)(N-2)|\langle\varphi, (q_1p_2p_3+p_1q_2p_3+p_1p_2q_3)f_{12}\cdot f_{13}(p_1q_2q_3+2q_1q_2p_3)\psi\rangle| \\
&\leq 54N(N-1)(N-2)\||f_{12}|p_2|f_{13}|^{1/2}p_3q_1\hat{n}_{+1}\varphi\|\ \||f_{13}|^{1/2}p_1\hat{n}^{-1}q_2q_3\psi\| \\
&\leq 54N(N-1)^{1/2}\||f^{(N)}|^2 * \rho\|_\infty^{1/2}\ \||f^{(N)}| * \rho\|_\infty\ \|q_1\hat{n}_{+1}\varphi\|\ \|\hat{n}^{-1}q_2q_3\psi\|
\end{aligned}
\tag{2.5.72}
$$

and

$$
\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_2^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}}\psi\rangle| \\
&= 6N(N-1)(N-2)|\langle\varphi, (p_1q_2q_3+2q_1q_2p_3)f_{12}\cdot f_{13}(q_1p_2p_3+p_1q_2p_3+p_1p_2q_3)\psi\rangle| \\
&\leq 54N(N-1)(N-2)\||f_{13}|^{1/2}p_3q_1\hat{n}_{-1}^{1/2}\varphi\|\ \||f_{12}|p_2|f_{13}|^{1/2}p_1\hat{n}^{-1/2}q_3\psi\| \\
&\leq 54N(N-1)^{1/2}\||f^{(N)}|^2 * \rho\|_\infty^{1/2}\ \||f^{(N)}| * \rho\|_\infty\ \|q_1\hat{n}_{-1}^{1/2}\varphi\|\ \|\hat{n}^{-1/2}q_3\psi\|.
\end{aligned}
\tag{2.5.73}
$$

Moreover, we estimate

$$
\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_0^{\{1,2,3\}}(w_{ff})_{123}P_3^{\{1,2,3\}}\psi\rangle| \\
&= 6N(N-1)(N-2)|\langle\hat{n}_{+3}^{3/2}\varphi, p_1p_2p_3f_{12}\cdot f_{13}\hat{n}^{-3/2}q_1q_2q_3\psi\rangle| \\
&\leq 6N(N-1)\|\hat{n}^{-3/2}q_1q_2q_3\psi\|\ \|\sum_{m=3}^{N}|f_{1m}|\ |f_{12}|p_1p_2p_m\hat{n}_{+3}^{3/2}\varphi\| \\
&\leq 6N^{\frac{1}{2}}(N-1)\|\hat{n}^{-3/2}q_1q_2q_3\psi\|\ \|\hat{n}_{+3}^{3/2}\varphi\|\times \\
&\qquad\times\left(\||f^{(N)}|^2 * \rho\|_\infty^2 + \||f^{(N)}|^2 * \rho\|_\infty\||f^{(N)}| * \rho\|_\infty^2\right)^{1/2}.
\end{aligned}
\tag{2.5.74}
$$

where we used

$$
\begin{aligned}
&\|\sum_{m=3}^{N}|f_{1m}|\ |f_{12}|p_1p_2p_m\varphi\|^2 \\
&\leq (N-2)\langle\varphi, p_1p_2|f_{12}|p_3|f_{13}|^2p_3|f_{12}|p_2p_1\varphi\rangle \\
&\quad + (N-2)(N-3)\langle\varphi, p_3p_2p_1|f_{12}|\ |f_{13}|\ |f_{14}|\ |f_{12}|p_1p_2p_4\varphi\rangle \\
&\leq \||f^{(N)}|^2 * \rho\|_\infty\langle\varphi, p_1p_2|f_{12}|^2p_2p_1\varphi\rangle \\
&\quad + (N-2)(N-3)\langle\varphi, p_1|f_{14}|^{1/2}p_2|f_{12}|p_3|f_{13}|^{1/2}\ |f_{14}|^{1/2}p_4|f_{12}|p_2|f_{13}|^{1/2}p_1\varphi\rangle \\
&\leq N^{-1}\||f^{(N)}|^2 * \rho\|_\infty^2\|\varphi\|^2 \\
&\quad + N^{-1}\||f^{(N)}|^2 * \rho\|_\infty\||f^{(N)}| * \rho\|_\infty^2\|\varphi\|^2
\end{aligned}
\tag{2.5.75}
$$

and similarly,

$$\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_3^{\{1,2,3\}}(w_{ff})_{123}P_0^{\{1,2,3\}}\psi\rangle| \\
&\le 6N^{\frac{1}{2}}(N-1)\|q_1q_2q_3\varphi\|\ \|\psi\|\times \\
&\quad\times\left(\||f^{(N)}|^2*\rho\|_\infty^2+\||f^{(N)}|^2*\rho\|_\infty\||f^{(N)}|*\rho\|_\infty^2\right)^{1/2}
\end{aligned} \tag{2.5.76}$$

which gives us the desired result. ∎

Lemma 47 (4q-estimates)**.** *It holds for* $\varphi,\psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$

1.
$$\begin{aligned}
&\frac{N(N-1)}{2}|\langle\varphi, P_2^{\{1,2\}}(w_{\nabla f})_{12}P_2^{\{1,2\}}\psi\rangle| \\
&\le 2N(N-1)\|f^{(N)}\|_\infty\ \|q_1q_2\hat{n}^{1/2}\varphi\|\ \|\hat{n}^{-1/2}q_2\nabla_1q_1\psi\| \\
&\quad+N(N-1)\|\nabla\cdot f^{(N)}\|_\infty\ \|q_1q_2\hat{n}\varphi\|\ \|\hat{n}^{-1}q_2q_1\psi\|,
\end{aligned}$$

2.
$$\frac{N(N-1)}{2}|\langle\varphi, P_2^{\{1,2\}}(w_f)_{12}P_2^{\{1,2\}}\psi\rangle|\le N(N-1)\|f^{(N)}\|_\infty^2\ \|\hat{n}q_1q_2\varphi\|\ \|\hat{n}^{-1}q_1q_2\psi\|,$$

3.
$$\begin{aligned}
&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_2^{\{1,2,3\}}(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle| \\
&\le N(N-2)\left(7\||f^{(N)}|^2*\rho\|_\infty+2\||f^{(N)}|*\rho\|_\infty\ \|f^{(N)}\|_\infty\right)\|\hat{n}q_2q_3\varphi\|\ \|\hat{n}^{-1}q_1q_2\psi\|,
\end{aligned}$$

4.
$$\begin{aligned}
&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_1^{\{1,2,3\}}(w_{ff})_{123}P_3^{\{1,2,3\}}\psi\rangle| \\
&\le 3N(N-1)\||f^{(N)}|^2*\rho\|_\infty\ \|q_1\hat{n}_{+2}^{3/2}\varphi\|\ \|\hat{n}^{-3/2}q_1q_2q_3\psi\|, \\
&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_3^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}}\psi\rangle| \\
&\le 3N(N-1)\||f^{(N)}|^2*\rho\|_\infty\ \|\hat{n}_{-2}^{1/2}q_1q_2q_3\varphi\|\ \|\hat{n}^{-1/2}q_1\psi\|.
\end{aligned}$$

Proof. We estimate

$$\begin{aligned}
&N(N-1)|\langle\varphi, P_2^{\{1,2\}}(w_{\nabla f})_{12}P_2^{\{1,2\}}\psi\rangle| \\
&\le 2N(N-1)|\langle\varphi, q_1q_2\left((i\nabla_1)\cdot f_{12}+f_{12}\cdot(i\nabla_1)\right)q_2q_1\psi\rangle| \\
&\le 4N(N-1)|\langle\varphi, \hat{n}^{1/2}q_1q_2\left(f_{12}\cdot i\nabla_1\right)q_1\hat{n}^{-1/2}q_2\psi\rangle| \\
&\quad+2N(N-1)|\langle\varphi, \hat{n}q_1q_2(i\nabla_1\cdot f_{12})\hat{n}^{-1}q_2q_1\psi\rangle| \\
&\le 4N(N-1)\|f^{(N)}\|_\infty\ \|q_1q_2\hat{n}^{1/2}\varphi\|\ \|\hat{n}^{-1/2}q_2\nabla_1q_1\psi\| \\
&\quad+2N(N-1)\|\nabla\cdot f^{(N)}\|_\infty\ \|q_1q_2\hat{n}\varphi\|\ \|\hat{n}^{-1}q_2q_1\psi\|.
\end{aligned} \tag{2.5.77}$$

In addition, it holds

$$\begin{aligned}
&N(N-1)|\langle\varphi, P_2^{\{1,2\}}(w_f)_{12}P_2^{\{1,2\}}\psi\rangle| \\
&\leq 2N(N-1)|\langle\varphi, \hat{n}q_1q_2f_{12}\cdot f_{12}\hat{n}^{-1}q_1q_2\psi\rangle| \\
&\leq 2N(N-1)\|f^{(N)}\|_\infty^2\ \|\hat{n}q_1q_2\varphi\|\ \|\hat{n}^{-1}q_1q_2\psi\|. \qquad (2.5.78)
\end{aligned}$$

In addition, we estimate

$$\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_1^{\{1,2,3\}}(w_{ff})_{123}P_3^{\{1,2,3\}}\psi\rangle| \\
&= 6N(N-1)(N-2)|\langle\varphi, (q_1p_2p_3+p_1q_2p_3+p_1p_2q_3)f_{12}\cdot f_{13}q_1q_2q_3\psi\rangle| \\
&\leq 18N(N-1)(N-2)\||f_{12}|p_2|f_{13}|p_3q_1\hat{n}_{+2}^{3/2}\varphi\|\ \|\hat{n}^{-3/2}q_1q_2q_3\psi\| \\
&\leq 18N(N-1)\||f^{(N)}|^2*\rho\|_\infty\ \|q_1\hat{n}_{+2}^{3/2}\varphi\|\ \|\hat{n}^{-3/2}q_1q_2q_3\psi\| \qquad (2.5.79)
\end{aligned}$$

and similarly

$$\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_3^{\{1,2,3\}}(w_{ff})_{123}P_1^{\{1,2,3\}}\psi\rangle| \\
&= 6N(N-1)(N-2)|\langle\varphi, q_1q_2q_3f_{12}\cdot f_{13}(q_1p_2p_3+p_1q_2p_3+p_1p_2q_3)\psi\rangle| \\
&\leq 18N(N-1)(N-2)\|\hat{n}_{-2}^{1/2}q_1q_2q_3\varphi\|\ \||f_{12}|p_2|f_{13}|p_3\hat{n}^{-1/2}q_1\psi\| \\
&\leq 18N(N-1)\||f^{(N)}|^2*\rho\|_\infty\ \|\hat{n}_{-2}^{1/2}q_1q_2q_3\varphi\|\ \|\hat{n}^{-1/2}q_1\psi\|. \qquad (2.5.80)
\end{aligned}$$

Moreover, we estimate

$$\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_2^{\{1,2,3\}}(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle| \\
&= 6N(N-1)(N-2)|\langle\varphi, (p_1q_2q_3+q_1p_2q_3+q_1q_2p_3)f_{12}\cdot f_{13}(p_1q_2q_3+2q_1q_2p_3)\psi\rangle| \\
&\leq 42N(N-2)\||f^{(N)}|^2*\rho\|_\infty\|\hat{n}^{-1}q_1q_2\psi\|\ \|\hat{n}q_2q_3\varphi\| \\
&\quad + 12N(N-2)\||f^{(N)}|*\rho\|_\infty\|f^{(N)}\|_\infty\|\hat{n}^{-1}q_1q_2\psi\|\ \|\hat{n}q_2q_3\varphi\| \qquad (2.5.81)
\end{aligned}$$

which yields the desired result. ■

Lemma 48 (5q- & 6q-estimates). *It holds for $\varphi,\psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$*

1.

$$\begin{aligned}
&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_2^{\{1,2,3\}}(w_{ff})_{123}P_3^{\{1,2,3\}}\psi\rangle| \\
&\leq 3N(N-1)(N-2)^{\frac{1}{2}}\||f^{(N)}|^2*\rho\|_\infty^{1/2}\|f^{(N)}\|_\infty\ \|q_1q_2\hat{n}_{+1}^{3/2}\varphi\|\ \|\hat{n}^{-3/2}q_1q_2q_3\psi\|, \\
&\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_3^{\{1,2,3\}}(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle| \\
&\leq 3N(N-1)(N-2)^{\frac{1}{2}}\||f^{(N)}|^2*\rho\|_\infty^{1/2}\|f^{(N)}\|_\infty\ \|q_1q_2q_3\hat{n}_{-1}\varphi\|\ \|\hat{n}^{-1}q_1q_2\psi\|,
\end{aligned}$$

2.

$$\frac{N(N-1)(N-2)}{6}|\langle\varphi, P_3^{\{1,2,3\}}(w_{ff})_{123}P_3^{\{1,2,3\}}\psi\rangle|$$
$$\leq N(N-1)(N-2)\|f^{(N)}\|_\infty^2\|\hat{n}^{3/2}q_1q_2q_3\varphi\|\ \|\hat{n}^{-3/2}q_1q_2q_3\psi\|.$$

Proof. In addition, we estimate

$$\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_2^{\{1,2,3\}}(w_{ff})_{123}P_3^{\{1,2,3\}}\psi\rangle|\\
&= 6N(N-1)(N-2)|\langle\varphi, (q_1q_2p_3 + p_1q_2q_3 + q_1p_2q_3)f_{12}\cdot f_{13}q_1q_2q_3\psi\rangle|\\
&\leq 18N(N-1)(N-2)\|f_{12}\cdot f_{13}p_3q_1q_2\hat{n}_{+1}^{3/2}\varphi\|\ \|\hat{n}^{-3/2}q_1q_2q_3\psi\|\\
&\leq 18N(N-1)(N-2)^{\frac{1}{2}}\||f^{(N)}|^2*\rho\|_\infty^{1/2}\|f^{(N)}\|_\infty\ \|q_1q_2\hat{n}_{+1}^{3/2}\varphi\|\ \|\hat{n}^{-3/2}q_1q_2q_3\psi\|
\end{aligned} \tag{2.5.82}$$

and similarly

$$\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_3^{\{1,2,3\}}(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle|\\
&= 6N(N-1)(N-2)|\langle\varphi, q_1q_2q_3f_{12}\cdot f_{13}(q_1q_2p_3 + p_1q_2q_3 + q_1p_2q_3)\psi\rangle|\\
&\leq 18N(N-1)(N-2)^{\frac{1}{2}}\||f^{(N)}|^2*\rho\|_\infty^{1/2}\|f^{(N)}\|_\infty\ \|q_1q_2q_3\hat{n}_{-1}\varphi\|\ \|\hat{n}^{-1}q_1q_2\psi\|.
\end{aligned} \tag{2.5.83}$$

Moreover, we estimate

$$\begin{aligned}
&N(N-1)(N-2)|\langle\varphi, P_3^{\{1,2,3\}}(w_{ff})_{123}P_3^{\{1,2,3\}}\psi\rangle|\\
&= 6N(N-1)(N-2)|\langle\varphi, q_1q_2q_3f_{12}\cdot f_{13}q_1q_2q_3\psi\rangle|\\
&\leq 6N(N-1)(N-2)\|f^{(N)}\|_\infty^2\|\hat{n}^{-3/2}q_1q_2q_3\psi\|\ \|\hat{n}^{3/2}q_1q_2q_3\varphi\|
\end{aligned}$$

which gives us the desired result. ■

2.5.4 Estimates of terms in $\partial_t\beta(t)$

The terms appearing in $\partial_t\beta(t)$ typically involve $h^{\mathrm{g}}(t)$. We show the following statements:

Lemma 49 (2q-estimates of kinetic terms, asymmetric terms)**.** *Let $\rho^h := \sum_{j=1}^N |h^{\mathrm{g}}\psi_j|^2 = \sum_{j=1}^N |\partial_t\psi_j|^2$ and assume $h_1^{\mathrm{g}}p_1$ is a bounded operator. It holds for $\psi \in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$*

1.

$$\begin{aligned}
&(N-1)|\langle\psi, (h_1^{\mathrm{g}}p_1 - p_1h_1^{\mathrm{g}})\, p_2(w_{\nabla f})_{12}P_2^{\{1,2\}}\psi\rangle|\\
&\leq 2\|\nabla_1q_1\psi\|\||f^{(N)}|^2*\rho\|_\infty^{1/2}\|h_1^{\mathrm{g}}p_1\|_{\mathrm{op,as}}\left((N-2)\|q_2\psi\|^2+1\right)^{1/2}\\
&\quad + \|q_1\psi\|\ \||\nabla\cdot f^{(N)}|^2*\rho\|_\infty^{1/2}\|h_1^{\mathrm{g}}p_1\|_{\mathrm{op,as}}\left((N-2)\|q_2\psi\|^2+1\right)^{1/2}\\
&\quad + 2\|\nabla_1q_1\psi\|\left(N^{-1}\||f^{(N)}|^2*\rho\ \rho^h\|_1 + \|q_1\psi\|^2\||f^{(N)}|*\rho\|_\infty\||f^{(N)}|*\rho^h\|_\infty\right)^{1/2}\\
&\quad + \|q_1\psi\|\left(N^{-1}\||\nabla\cdot f^{(N)}|^2*\rho\ \rho^h\|_1 + \|q_1\psi\|^2\||\nabla\cdot f^{(N)}|*\rho\|_\infty\||\nabla\cdot f^{(N)}|*\rho^h\|_\infty\right)^{1/2},
\end{aligned}$$

2.

$$
\begin{aligned}
&(N-1)|\langle\psi,(h_1^{\mathrm{g}}p_1-p_1h_1^{\mathrm{g}})\,p_2(w_f)_{12}P_2^{\{1,2\}}\psi\rangle| \\
&\leq 2\|q_1\psi\|\ \||f^{(N)}|^4*\rho\|_\infty^{1/2}\|h^{\mathrm{g}}p_1\|_{\mathrm{op,as}}\left((N-2)\|q_2\psi\|^2+1\right)^{1/2} \\
&\quad+2\|q_1\psi\|\left(N^{-1}\||f^{(N)}|^4*\rho\ \rho^h\|_1+\|q_1\psi\|^2\||f^{(N)}|^2*\rho\|_\infty^2\right)^{1/2},
\end{aligned}
$$

3.

$$
\begin{aligned}
&(N-1)(N-2)|\langle\psi,(h_1^{\mathrm{g}}p_1-p_1h_1^{\mathrm{g}})\,p_2p_3(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle| \\
&\leq 6\||f^{(N)}|*\rho\|_\infty^{1/2}\||f^{(N)}|*\rho^h\|_\infty^{1/2}\ \|q_1\psi\|\left(\||f^{(N)}|^2*\rho\|_\infty+\|q_3\psi\|^2\||f^{(N)}|*\rho\|_\infty^2\right)^{1/2} \\
&\quad+9(N-1)^{\frac{1}{2}}\|h_1^{\mathrm{g}}p_1\|_{\mathrm{op,as}}\||f^{(N)}|*\rho\|_\infty\||f^{(N)}|^2*\rho\|_\infty^{1/2}\|q_1\psi\|\left(\|q_3\psi\|^2+(N-2)^{-1}\right)^{1/2}.
\end{aligned}
$$

Proof. It holds due to the antisymmetry of ψ

$$
\begin{aligned}
&(N-1)|\langle\psi,(h_1^{\mathrm{g}}p_1-p_1h_1^{\mathrm{g}})\,p_2(w_{\nabla f})_{12}P_2^{\{1,2\}}\psi\rangle| \\
&=(N-1)|\langle\psi,(h_1^{\mathrm{g}}p_1-p_1h_1^{\mathrm{g}})\,p_2\,(\nabla_1\cdot f_{12}+f_{12}\cdot\nabla_1)\,q_2q_1\psi\rangle| \\
&\quad+(N-1)|\langle\psi,(h_2^{\mathrm{g}}p_2-p_2h_2^{\mathrm{g}})\,p_1\,(\nabla_1\cdot f_{12}+f_{12}\cdot\nabla_1)\,q_2q_1\psi\rangle|.
\end{aligned}
$$

We first focus on the first line and estimate the following terms separately

$$
\begin{aligned}
&(N-1)|\langle\psi,(h_1^{\mathrm{g}}p_1-p_1h_1^{\mathrm{g}})\,p_2\,(\nabla_1\cdot f_{12}+f_{12}\cdot\nabla_1)\,P_2^{\{1,2\}}\psi\rangle| \\
&\leq(N-1)|\langle\psi,h_1^{\mathrm{g}}p_1p_2\,((\nabla_1\cdot f_{12})+2f_{12}\cdot\nabla_1)\,q_2q_1\psi\rangle| \qquad (2.5.84a)\\
&\quad+(N-1)|\langle\psi,p_1h_1^{\mathrm{g}}p_2\,((\nabla_1\cdot f_{12})+2f_{12}\cdot\nabla_1)\,q_2q_1\psi\rangle|. \qquad (2.5.84b)
\end{aligned}
$$

It holds for the first line of the right hand side using $h_1^{\mathrm{g}}p_1$ is a bounded operator

$$
\begin{aligned}
&(2.5.84a) \\
&\leq 2\|\nabla_1q_1\psi\|\ \|\sum_{m=2}^{N}q_mf_{1m}p_mp_1h_1^{\mathrm{g}}\psi\|+\|q_1\psi\|\ \|\sum_{m=2}^{N}q_m(\nabla_1\cdot f_{1m})p_mp_1h_1^{\mathrm{g}}q_1\psi\| \\
&\leq\left(2\|\nabla_1q_1\psi\|\ \||f^{(N)}|^2*\rho\|_\infty^{1/2}+\|q_1\psi\|\ \||\nabla\cdot f^{(N)}|^2*\rho\|_\infty^{1/2}\right) \\
&\quad\times\|h_1^{\mathrm{g}}p_1\|_{\mathrm{op,as}}\left((N-2)\|q_2\psi\|^2+\|\psi\|^2\right)^{1/2} \qquad (2.5.85)
\end{aligned}
$$

where we used with $\|\cdot\|_{\mathrm{op,as}}$ denoting the operator norm on antisymmetric wave functions

$$\|p_1h_1^{\mathrm{g}}\psi\|\leq\|h_1^{\mathrm{g}}p_1\|_{\mathrm{op,as}}\|\psi\|.$$

Furthermore, it holds with including $\rho^h = \sum_{i=1}^N |h_i^{\mathrm{g}} \psi_i|^2$ and Lemma 38

$$
\begin{aligned}
&(2.5.84\mathrm{b})\\
&\leq 2\|\nabla_1 q_1 \psi\| \; \|\sum_{m=2}^N q_m f_{1m} p_m h_1^{\mathrm{g}} p_1 \psi\| + \|q_1\psi\| \; \|\sum_{m=2}^N q_m (\nabla_1 \cdot f_{1m}) p_m h_1^{\mathrm{g}} p_1 \psi\| \\
&\leq 2\|\nabla_1 q_1 \psi\| \left(N^{-1} \| |f^{(N)}|^2 * \rho \; \rho^h \|_1 + \|q_1\psi\|^2 \| |f^{(N)}| * \rho \|_\infty \| |f^{(N)}| * \rho^h \|_\infty \right)^{1/2} \\
&\qquad + \|q_1\psi\| \left(N^{-1} \| |\nabla \cdot f^{(N)}|^2 * \rho \; \rho^h \|_1 + \|q_1\psi\|^2 \| |\nabla \cdot f^{(N)}| * \rho \|_\infty \| |\nabla \cdot f^{(N)}| * \rho^h \|_\infty \right)^{1/2} .
\end{aligned}
\tag{2.5.86}
$$

Similarly, we estimate

$$
\begin{aligned}
&(N-1) |\langle \psi, (h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}}) \, p_2 (w_f)_{12} P_2^{\{1,2\}} \psi \rangle| \\
&\leq 2(N-1) |\langle \psi, h_1^{\mathrm{g}} p_1 p_2 f_{12} \cdot f_{12} q_1 q_2 \psi \rangle| && (2.5.87\mathrm{a})\\
&\qquad + 2(N-1) |\langle \psi, p_1 h_1^{\mathrm{g}} p_2 f_{12} \cdot f_{12} q_1 q_2 \psi \rangle| . && (2.5.87\mathrm{b})
\end{aligned}
$$

with

$$
\begin{aligned}
&(2.5.87\mathrm{a})\\
&\leq 2 \|\sum_{m=2}^N q_m f_{1m} \cdot f_{1m} p_m p_1 h_1^{\mathrm{g}} \psi\| \; \|q_1 \psi\| \\
&\leq 2 \|q_1\psi\| \; \| |f^{(N)}|^4 * \rho \|_\infty^{1/2} \|h_1^{\mathrm{g}} p_1\|_{\mathrm{op,as}} \left((N-2) \|q_2\psi\|^2 + \|\psi\|^2 \right)^{1/2}
\end{aligned}
\tag{2.5.88}
$$

and

$$
\begin{aligned}
&(2.5.87\mathrm{b})\\
&\leq 2\|q_1\psi\| \; \|\sum_{m=2}^N q_m f_{1m} \cdot f_{1m} p_m h_1^{\mathrm{g}} p_1 \psi\| \\
&\leq 2\|q_1\psi\| \left(N^{-1} \| |f^{(N)}|^4 * \rho \; \rho^h \|_1 + \|q_1\psi\|^2 \| |f^{(N)}|^2 * \rho \|_\infty^2 \right)^{1/2} .
\end{aligned}
\tag{2.5.89}
$$

Finally, we estimate

$$\begin{aligned}
&|\langle\psi,(h_1^{\mathrm{g}}p_1-p_1h_1^{\mathrm{g}})\,p_2p_3(w_{ff})_{123}P_2^{\{1,2,3\}}\psi\rangle| \\
&\leq |\langle\psi,(h_1^{\mathrm{g}}p_1-p_1h_1^{\mathrm{g}})\,p_2p_3\,(f_{12}\cdot f_{13}+2f_{21}\cdot f_{23})\,(p_1q_2q_3+q_1p_2q_3+q_1q_2p_3)\psi\rangle| \\
&\leq |\langle\psi,(h_1^{\mathrm{g}}p_1-p_1h_1^{\mathrm{g}})\,p_2p_3\,(f_{12}\cdot f_{13})\,(p_1q_2q_3+2q_1p_2q_3)\psi\rangle| \\
&\qquad +2|\langle\psi,(h_1^{\mathrm{g}}p_1-p_1h_1^{\mathrm{g}})\,p_2p_3\,(f_{21}\cdot f_{23})\,(p_1q_2q_3+q_1p_2q_3+q_1q_2p_3)\psi\rangle| \\
&\leq 2|\langle\psi,(h_1^{\mathrm{g}}p_1)\,p_2p_3\,(f_{12}\cdot f_{13})\,(q_1p_2q_3)\psi\rangle| \\
&\qquad +2|\langle\psi,(h_1^{\mathrm{g}}p_1)\,p_2p_3\,(f_{21}\cdot f_{23})\,(q_1q_2p_3)\psi\rangle| \\
&\qquad +2|\langle\psi,(h_1^{\mathrm{g}}p_1)\,p_2p_3\,(f_{21}\cdot f_{23})\,(p_1q_2q_3)\psi\rangle| \\
&\qquad +2|\langle\psi,(h_1^{\mathrm{g}}p_1)\,p_2p_3\,(f_{21}\cdot f_{23})\,(q_1p_2q_3)\psi\rangle| \\
&\qquad +|\langle\psi,(h_1^{\mathrm{g}}p_1)\,p_2p_3\,(f_{12}\cdot f_{13})\,(p_1q_2q_3)\psi\rangle| \\
&\quad +|\langle\psi,(p_1h_1^{\mathrm{g}})\,p_2p_3\,(f_{12}\cdot f_{13})\,(2q_1p_2q_3+p_1q_2q_3)\psi\rangle| \\
&\qquad +2|\langle\psi,(p_1h_1^{\mathrm{g}})\,p_2p_3\,(f_{21}\cdot f_{23})\,(q_1q_2p_3+q_1p_2q_3+p_1q_2q_3)\psi\rangle|.
\end{aligned}$$

We introduce $p_1^h=\sum_{j=1}^N|h^{\mathrm{g}}\psi_j\rangle\langle\psi_j|_1$ and estimate analogously to Lemma 45

$$\begin{aligned}
&|\langle\psi,(p_1h_1^{\mathrm{g}})\,p_2p_3\,(f_{12}\cdot f_{13})\,(2q_1p_2q_3+p_1q_2q_3)\psi\rangle| \\
&\leq (N-2)^{-1}\left(2\||f_{12}|^{1/2}p_2q_1\psi\|+\||f_{12}|^{1/2}p_1q_2\psi\|\right)\ \|\sum_{m=3}^N q_m|f_{12}|^{1/2}\ |f_{1m}|p_mp_2p_1^h\psi\| \\
&\leq 3(N-1)^{-1}(N-2)^{-1}\||f^{(N)}|*\rho\|_\infty^{1/2}\ \|q_1\psi\|\times \\
&\quad\times\left(\||f^{(N)}|^2*\rho\|_\infty\||f^{(N)}|*\rho^h\|_\infty+\|q_3\psi\|^2\||f^{(N)}|*\rho\|_\infty^2\||f^{(N)}|*\rho^h\|_\infty\right)^{1/2}
\end{aligned}\tag{2.5.90}$$

and

$$\begin{aligned}
&|\langle\psi,(p_1h_1^{\mathrm{g}})\,p_2p_3\,(f_{21}\cdot f_{23})\,(q_1q_2p_3+q_1p_2q_3+p_1q_2q_3)\psi\rangle| \\
&\leq (N-2)^{-1}\left(\||f_{21}|^{1/2}p_2q_1\psi\|+\||f_{21}|^{1/2}p_1q_2\psi\|\right)\ \|\sum_{m=3}^N q_m|f_{21}|^{1/2}\ |f_{2m}|p_mp_2p_1^h\psi\| \\
&\quad +(N-2)^{-1}\||f_{23}|^{1/2}p_3q_2\psi\|\ \|\sum_{m=1,4,\dots,N} q_m|f_{23}|^{1/2}\ |f_{2m}|p_3p_2p_m^h\psi\| \\
&\leq 3(N-1)^{-1}(N-2)^{-1}\||f^{(N)}|*\rho\|_\infty^{1/2}\ \||f^{(N)}|*\rho^h\|_\infty^{1/2}\ \|q_1\psi\|\times \\
&\quad\times\left(\||f^{(N)}|^2*\rho\|_\infty+\|q_3\psi\|^2\||f^{(N)}|*\rho\|_\infty^2\right)^{1/2}.
\end{aligned}\tag{2.5.91}$$

Next, we estimate

$$
\begin{aligned}
&|\langle\psi,(h^{\mathrm{g}}_1 p_1)\,p_2p_3\,(f_{12}\cdot f_{13})\,(q_1p_2q_3)\psi\rangle| \\
&\le (N-1)^{-\frac{1}{2}}(N-2)^{-1}\|\sum_{m=3}^{N} q_m|f_{12}|^{1/2}|f_{1m}|p_2p_mp_1h^{\mathrm{g}}_1\psi\|\ \||f^{(N)}|*\rho\|_\infty^{1/2}\|q_1\psi\| &(2.5.92)\\
&\le (N-1)^{-\frac{1}{2}}(N-2)^{-1}\|q_1\psi\|\ \|h^{\mathrm{g}}_1p_1\|_{\mathrm{op,as}} &(2.5.93)\\
&\qquad\times \||f^{(N)}|*\rho\|_\infty\||f^{(N)}|^2*\rho\|_\infty^{1/2}\left(\|q_3\psi\|^2+(N-2)^{-1}\right)^{1/2} &(2.5.94)
\end{aligned}
$$

■

Lemma 50 (2q-estimates of kinetic terms, symmetric terms)**.** *Let $\rho^h := \sum_{j=1}^N|h^{\mathrm{g}}\psi_j|^2 = \sum_{j=1}^N|\partial_t\psi_j|^2$ and assume $h^{\mathrm{g}}_1p_1$ is a bounded operator. It holds for $\psi\in L^2_{\mathrm{as}}(\mathbb{R}^{3N})$*

1.
$$
\begin{aligned}
&(N-1)|\langle\psi,(h^{\mathrm{g}}_1p_1-p_1h^{\mathrm{g}}_1)\,q_2(w_{\nabla f})_{12}P_1^{\{1,2\}}\psi\rangle| \\
&\le 4(N-1)^{\frac{1}{2}}\|q_2\psi\|\ \|h^{\mathrm{g}}_1p_1\|_{\mathrm{op,as}}\left(\||\nabla\cdot f^{(N)}|^2*\rho\|_\infty^{1/2}\|q_1\psi\|+2\||f^{(N)}|^2*\rho\|_\infty^{1/2}\|\nabla_1q_1\psi\|\right)\\
&\quad+8\|q_1\psi\|\ \|\nabla_1q_1\psi\|\ \||f^{(N)}|*\rho\|_\infty^{1/2}\||f^{(N)}|*\rho^h\|_\infty^{1/2}\\
&\quad+4\|q_1\psi\|^2\ \||\nabla\cdot f^{(N)}|*\rho\|_\infty^{1/2}\||\nabla\cdot f^{(N)}|*\rho^h\|_\infty^{1/2},
\end{aligned}
$$

2.
$$
\begin{aligned}
&(N-1)|\langle\psi,(h^{\mathrm{g}}_1p_1-p_1h^{\mathrm{g}}_1)\,q_2(w_f)_{12}P_1^{\{1,2\}}\psi\rangle|\\
&\le 4(N-1)^{\frac{1}{2}}\|q_2\psi\|\ \|h^{\mathrm{g}}_1p_1\|_{\mathrm{op,as}}\||f^{(N)}|^4*\rho\|_\infty^{1/2}\|q_1\psi\|\\
&\quad+4\|q_1\psi\|^2\||f^{(N)}|^2*\rho\|_\infty^{1/2}\||f^{(N)}|^2*\rho^h\|_\infty^{1/2},
\end{aligned}
$$

3.
$$
\begin{aligned}
&(N-1)(N-2)|\langle\psi,(h^{\mathrm{g}}_1p_1-p_1h^{\mathrm{g}}_1)\,q_2p_3(w_{ff})_{123}P_1^{\{1,2,3\}}\psi\rangle|\\
&\le 9(N-1)^{\frac{1}{2}}\|q_2\psi\|\ \|h^{\mathrm{g}}_1p_1\|_{\mathrm{op,as}}\||f^{(N)}|*\rho\|_\infty\||f^{(N)}|^2*\rho\|_\infty^{1/2}\|q_1\psi\|\\
&\quad+9\||f^{(N)}|*\rho\|_\infty^{3/2}\||f^{(N)}|*\rho^h\|_\infty^{1/2}\|q_1\psi\|^2.
\end{aligned}
$$

Proof. We give a sketch how to arrive at the stated inequalities starting with

$$
\begin{aligned}
&(N-1)|\langle\psi,(h^{\mathrm{g}}_1p_1-p_1h^{\mathrm{g}}_1)\,q_2(w_{\nabla f})_{12}P_1^{\{1,2\}}\psi\rangle|\\
&\le (N-1)|\langle\psi,(h^{\mathrm{g}}_1p_1q_2+h^{\mathrm{g}}_2p_2q_1)\left((\nabla_1\cdot f_{12})+2f_{12}\cdot(\nabla_1)\right)(p_1q_2+q_1p_2)\psi\rangle|\\
&\quad+(N-1)|\langle\psi,(p_1h^{\mathrm{g}}_1q_2+p_2h^{\mathrm{g}}_2q_1)\left((\nabla_1\cdot f_{12})+2f_{12}\cdot(\nabla_1)\right)(p_1q_2+q_1p_2)\psi\rangle|. &(2.5.95)
\end{aligned}
$$

The first line is estimated with $\|p_1 h_1^{\mathrm{g}} q_2\psi\| \leq \|h_1^{\mathrm{g}} p_1\|_{\mathrm{op,as}}\|q_2\psi\|$ on the left side and the second line is estimated similarly to (2.5.84b) with e.g. $\|(\nabla_1 \cdot f_{12})h_1^{\mathrm{g}} p_1 q_2\psi\| \leq (N-1)^{-1/2}\||\nabla \cdot f^{(N)}|^2 * \rho^h\|_\infty^{1/2}\|q_2\psi\|$. The right side is estimated similarly to (2.5.68).

Similarly, we estimate

$$\begin{aligned}
&(N-1)|\langle\psi, (h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}})\, q_2 (w_f)_{12} P_1^{\{1,2\}}\psi\rangle| \\
&\leq 2(N-1)|\langle\psi, (h_1^{\mathrm{g}} p_1)\, q_2 f_{12} \cdot f_{12}(p_1 q_2 + q_1 p_2)\psi\rangle| \\
&\quad + 2(N-1)|\langle\psi, (p_1 h_1^{\mathrm{g}})\, q_2 f_{12} \cdot f_{12}(p_1 q_2 + q_1 p_2)\psi\rangle|. \qquad (2.5.96)
\end{aligned}$$

In addition,

$$\begin{aligned}
&(N-1)(N-2)|\langle\psi, (h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}})\, q_2 p_3 (w_{ff})_{123} P_1^{\{1,2,3\}}\psi\rangle| \\
&= (N-1)(N-2)|\langle\psi, (h_1^{\mathrm{g}} p_1 - p_1 h_1^{\mathrm{g}})\, q_2 p_3 (w_{ff})_{123}(q_1 p_2 p_3 + p_1 q_2 p_3 + p_1 p_2 q_3)\psi\rangle| \\
&\leq 2(N-1)(N-2)|\langle\psi, (h_1^{\mathrm{g}} p_1)\, q_2 p_3 f_{21} \cdot f_{23}(q_1 p_2 p_3 + p_1 q_2 p_3 + p_1 p_2 q_3)\psi\rangle| \\
&\quad (N-1)(N-2)|\langle\psi, (h_1^{\mathrm{g}} p_1)\, q_2 p_3 f_{12} \cdot f_{13}(q_1 p_2 p_3 + p_1 q_2 p_3 + p_1 p_2 q_3)\psi\rangle| \\
&\qquad + (N-1)(N-2)|\langle\psi, (p_1 h_1^{\mathrm{g}})\, q_2 p_3 (w_{ff})_{123}(q_1 p_2 p_3 + p_1 q_2 p_3 + p_1 p_2 q_3)\psi\rangle|. \qquad (2.5.97)
\end{aligned}$$

The first two lines are estimated by using Cauchy-Schwarz and bringing two q-operators, one $(p_1 h_1^{\mathrm{g}})$-operator to the left side to obtain $\||f^{(N)}| * \rho\|_\infty^{1/2}\|p_1 h_1^{\mathrm{g}} q_2 q_1\psi\|$, in the third line we estimate similarly to Lemma 44. ■

Chapter 3

Effective Polaron Dynamics of an Impurity Particle Interacting with a Fermi Gas

This project is mostly a replication of the peer-reviewed paper [HP25] co-authored with my supervisor, Peter Pickl. I made substantial contributions to all results, particularly in developing the main idea and its proof. The final version of the paper was edited in close collaboration with Peter Pickl. New sections on the comparison with the Landau-Pekar equations and the derivation of the Loschmidt echo, which were not included in our publication, are added by me. We are grateful for helpful discussions with David Mitrouskas, Benjamin Schlein and Karla Schön.

3.1 Introduction

The study of impurities in quantum gases has garnered considerable attention due to its relevance in various physical contexts, ranging from solid-state physics to cold atom experiments. In this context quasi-particles such as polarons stand as intriguing entities emerging from the interaction of a single impurity particle with a surrounding medium. The concept of a polaron, originally introduced by Lev Landau to study the motion of an electron in a dielectric crystal [Lan65], most famously emerges from the celebrated Fröhlich Hamiltonian in second quantization formalism describing electron-phonon interactions [Frö54]. Subsequently, the polaron concept was extended to all kind of surrounding media including Bose and Fermi gases. The formation conditions and properties of polarons are believed to play a central role to understand the transport properties and the effective mass of impurities within the host material.

In this article, we study with mathematical rigor the dynamics of an impurity particle immersed in a dense gas of fermions as surrounding medium. Interactions between fermions are neglected, and we assume that the impurity particle interacts with the Fermi gas via a finite-range potential in momentum space. The initial state ψ of the system is a product state between the impurity state and a filled Fermi ball. This mathematical framework finds resonance with recent experimental and theoretical advancements in the study of ultracold atoms [SWSZ09, KSN+12, CJL+15]. We emphasize that our assumption on the interaction potential aligns with the growing interest among experimentalists and theorists in cases where the impurity is charged, and hence the interaction potential is long-ranged in position space [CCGB22, MJ24]. We show that the effective dynamics of the system is governed by a Fröhlich-type Hamiltonian, which linearly couples the impurity particle to an almost-bosonic excitation field. More specifically, the excitations relative to the filled Fermi ball are up to a constant described by the Hamiltonian

$$\mathbb{H}^{\mathrm{F}} = (-\Delta_y) \otimes 1 + 1 \otimes \mathbb{D}_{\mathrm{B}} + \Phi(h_y) \tag{3.1.1}$$

with $(-\Delta_y)$ describing the kinetic energy of the impurity particle, $\mathbb{D}_{\mathrm{B}}$ describing the kinetic energy of the excitation field and $\Phi(h_y) \coloneqq c^*(h_y)+c(h_y)$ the linear coupling between impurity particle and excitations. The operators c^* and c describing this excitation field coincide with those introduced in a series of pioneering studies on the correlation energy of interacting fermions [BNP+19, BNP+21a, BNP+21b]. We note that an effective Hamiltonian of a similar type to (3.1.1) has recently been derived in another microscopic setting involving a tracer particle interacting with excitations of a Bose Einstein condensate [LP22, MS20].
Subsequently, we show that the effective time evolved state can be up to a phase factor approximated by a time-dependent coupled coherent state $W(\eta_t)\phi \otimes \Omega$ where W is the Weyl operator of the excitation field which is simply parameterized by a function η_t and Ω is the Fock space vacuum. An explicit expression for η_t is derived which allows for determining the number of collective excitations over time. We believe that such quantities are particularly helpful to gain deeper insights into the formation process of quasi-particles as studied in experiments such as [CJL+16]. Eventually, we show that the linear coupling term $\Phi(h_y)$ cannot be omitted in the effective description but adds a leading order effect to the effective dynamics in the setting of interaction couplings of order 1.

Our results hold for a variety of time scales and couplings, describing different interaction strengths and mass ratios, which will be specified in the subsequent section. We note that the same microscopic model has been studied in [JMPP17, JMP18, MP21] but with very specific choices of couplings different from ours, leading to an effective decoupling of impurity and gas.

3.1.1 The microscopic model

We consider an impurity particle interacting with N spinless fermions on a 3-dimensional box with periodic boundaries described by $\Lambda := \mathbb{T}^3 := \mathbb{R}^3/(2\pi\mathbb{Z}^3)$. The system is described by a state in the Hilbert space $L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$ with $\mathcal{H}_N^- = L^2(\Lambda)^{\wedge N}$ where y is the coordinate of the impurity particle and $\{x_i\}_{i=1,\dots,N}$ are the coordinates of the fermions. The Hamiltonian for our main model of the system is given by

$$H_N = -\beta\Delta_y + \sum_{i=1}^{N}(-\Delta_{x_i}) + \lambda\sum_{i=1}^{N} V(x_i - y) \tag{3.1.2}$$

and parameterized by $\beta, \lambda > 0$. Note that the different parts of the Hamiltonian on different tensor components of our Hilbert space writing, i.e. we used the shorthand notation writing, for example, $-\Delta_y := -\Delta_y \otimes 1$ for the Laplacian acting on the impurity particle. The interaction V is assumed to have a Fourier transform $\hat{V}$ with compact support satisfying $\hat{V}(k) = \hat{V}(-k)$ and $\hat{V}(k) \geq 0$ for all $k \in \mathbb{Z}^3$. It is well-known that under this assumption the Hamiltonian (3.1.2) defines a self-adjoint operator which generates by Stone's theorem the unitary time evolution $e^{-H_N t}$.

We are interested in the dynamics of the system governed by the time-dependent Schrödinger equation of the form

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H_N\psi_t, \qquad \psi_0 \in L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-. \tag{3.1.3}$$

where $\hbar$ is the reduced Planck constant. For mathematical convenience we will always set $\hbar = 1$ unless explicitly stated otherwise.

Note that the filled Fermi ball is a ground state for the non-interacting Fermion system. It is non-degenerate and explicitly given by

$$\Omega_0 := \bigwedge_{k\in B_\mathrm{F}} f_k, \qquad f_k(x) := \frac{\exp(ikx)}{(2\pi)^{3/2}} \in L^2(\Lambda). \tag{3.1.4}$$

We choose the initial state to be of product form

$$\psi_0\big(y; x_1, \dots x_N\big) := \phi(y) \otimes \Omega_0(x_1, \dots x_N), \tag{3.1.5}$$

with a general state ϕ for the impurity particle i.e., the system is initially prepared in a state describing a ground state of the ideal Fermi gas which does not interact with the impurity particle.

Furthermore, we choose the Fermi momentum k_F to be our parameter of the system in the sense that the particle number is defined as

$$N \equiv N(k_{\mathrm{F}}) := |B_{\mathrm{F}}|, \qquad B_{\mathrm{F}} := \{k \in \mathbb{Z}^3 : |k| \leq k_{\mathrm{F}}\}, \tag{3.1.6}$$

i.e. the particle number N of the Fermi gas is chosen such that the Fermi ball is completely filled. Note that the average density is in this case proportional to the number N of gas particles due to the following relation

$$k_{\mathrm{F}} = \left(\frac{3}{4\pi}\right)^{1/3} N^{\frac{1}{3}} + \mathcal{O}(1) \tag{3.1.7}$$

which is a consequence of Gauss' counting argument.

3.1.2 Relevant parameters and time scales

In the following, we present the ranges of β and λ we are aiming for, and discuss the physical meaning of the parameters. In addition, it is important to discuss on which time scales our results hold.

- The parameter β determines the mass ratio of the impurity and Fermi gas particles such that $\beta \ll 1$ corresponds to a relatively heavy, $\beta \gg 1$ to a relatively light impurity particle. The case of equal masses corresponds to $\beta = 1$. Our work requires the restriction $\beta \in o(k_{\mathrm{F}})$ see Remark 57. Since k_{F} tends to infinity, this allows to include all three cases.

- The parameter $\lambda \in [k_{\mathrm{F}}^{-1/6}, 1]$ models the coupling strength between the Fermi gas and the impurity. For small λ we expect a decoupling between the gas and the impurity in the sense that the time evolution given by (3.1.3) does not entangle an initial state of product form $\phi \otimes \Omega_0$. Such a result was shown in [MP21] with $\beta = 1$ and $\lambda = k_{\mathrm{F}}^{-1/2}$ in three dimensions and [JMPP17, JMP18] with $\beta = \lambda = 1$ in two dimensions. Note that our interaction coupling λ is much larger than in [MP21] and in particular, we are able to include $\lambda = 1$. As will be discussed in Remark 60, it is necessary to include excitations of the Fermi gas into the effective description for $\lambda = 1$.

- Our approximations will apply for times $t \in \mathcal{O}(k_{\mathrm{F}}^{-1}\lambda^{-1})$, so that a weaker coupling strength λ corresponds to slightly larger times. The times $t \in \mathcal{O}(k_{\mathrm{F}}^{-1})$ are on the time scale of the fermions near the Fermi surface, which have an approximate momentum of k_{F} and thus travel a distance of order 1 in this time. We are therefore able to enter a time scale where the impurity particle can resolve the motion of the fermions. We remark that the results in [MP21] can be transferred to this setting of $\lambda = 1$, however, allowing only for shorter time scales of $t \in o(k_{\mathrm{F}}^{-1})$.

We remark that results obtained in the previously mentioned parameter range remain valid under re-scaling by multiplying the Hamiltonian by an overall factor and absorbing it by re-scaling the time variable. Although we will stick to the above choices, it might still be helpful for the reader to see some connections to other scaling regimes by re-scaling. We present two of them briefly:

- Time scales of order 1 are obtained by multiplying the Hamiltonian by an overall factor of k_{F}^{-1}. In this case, the factor k_{F}^{-1} appears as new parameters in front of the kinetic energy of the Fermi gas corresponding to a heavy fermion regime.
- In recent years the so-called semiclassical regime has been widely studied in the analysis of dense Fermi gases, as can be seen for example in [Ben22, Saf23]. This regime is associated with identifying $\hbar := k_{\mathrm{F}}^{-1}$ as small parameter instead of setting $\hbar = 1$. It is achieved by multiplying the Hamiltonian by an overall factor of k_{F}^{-2} and absorbing k_{F}^{-1} in the time variable. For this new time scale, our theorem makes a statement for times of order 1. The re-scaled Schrödinger equation takes the form of

$$i\hbar\partial_t\psi_t = \left(\hbar^2\beta(-\Delta_y) + \hbar^2\sum_{i=1}^{N}(-\Delta_{x_i}) + \lambda k_{\mathrm{F}}^{-2}\sum_{i=1}^{N} V(x_i - y)\right)\psi_t \tag{3.1.8}$$

  One identifies $\lambda k_{\mathrm{F}}^{-2} \in [k_{\mathrm{F}}^{-13/6}, k_{\mathrm{F}}^{-2}]$ as re-scaled interaction coupling.

3.2 Preliminaries

Second quantization It is convenient to consider $L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$ as the N-particle sector of $L^2(\Lambda) \otimes \mathcal{F}$ with the fermionic Fock space $\mathcal{F}$ constructed over $L^2(\Lambda)$. This way, we have access to the powerful formalism of second quantization with the fermionic creation operator a_p^* creating a particle with momentum $p \in \mathbb{Z}^3$ and the annihilation operator a_p annihilating a particle with momentum $p \in \mathbb{Z}^3$. Those operators satisfy the canonical anticommutation relations (CAR)

$$\forall p, q \in \mathbb{Z}^3 : \{a_p, a_q^*\} = \delta_{p,q}, \qquad \{a_p, a_q\} = 0 = \{a_p^*, a_q^*\}. \tag{3.2.1}$$

Furthermore we introduce the fermionic number operator $\mathcal{N} := \sum_{p\in\mathbb{Z}^3} a_p^* a_p$ and the vacuum Ω satisfying $a_p\Omega = 0$ for all $p \in \mathbb{Z}^3$.

We lift our N-particle Hamiltonian H_N to Fock space as

$$\mathbb{H} = \underbrace{-\beta\Delta_y}_{=:h_0} + \underbrace{\sum_{k\in\mathbb{Z}^3} |k|^2 a_k^* a_k}_{=:\mathbb{H}^{\mathrm{kin}}} + \underbrace{\lambda\sum_{k,p\in\mathbb{Z}^3} \hat{V}(k) e^{iky} a_p^* a_{p-k}}_{=:\mathbb{V}} \tag{3.2.2}$$

which agrees with H_N if restricted to $L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$ and by $\| \cdot \|$ the induced norm if not stated otherwise.

We will mostly use the abuse of notation $\mathbb{A} \equiv 1 \otimes \mathbb{A}$ as operator on $L^2(\Lambda) \otimes \mathcal{F}$ where $\mathbb{A}$ acts as an operator on the Fock space part.

Particle-hole transformation In our analysis, the primary objective is to focus on excitations relative to the non-interacting Fermi ball. In particular, we want to use a description of our fermionic system in which the non-interacting Fermi ball $\Omega_0 = \prod_{k \in B_{\mathrm{F}}} a_k^* \Omega$ is mapped to the vacuum. To achieve this, we employ the particle-hole transformation, which is a specific type of fermionic Bogoliubov transformation as creation operators are mapped to linear combinations of creation and annihilation operators while preserving the CAR. The *particle-hole transformation* is defined as the map $R : \mathcal{F} \to \mathcal{F}$ satisfying

$$R^* a_k^* R := \begin{cases} a_k^* & \text{if } k \in B_{\mathrm{F}}^{\mathrm{c}}, \\ a_k & \text{if } k \in B_{\mathrm{F}} \end{cases} \quad \text{and} \quad R\Omega := \Omega_0 \; . \tag{3.2.3}$$

It is easy to check that the map is well-defined, unitary and satisfies $R^{-1} = R^* = R$.

With this, we can re-write the initial state (3.1.5) representing a non-interacting impurity particle and a Fermi gas as

$$\psi_0 = \phi \otimes \Omega_0 = (1 \otimes R)\,(\phi \otimes \Omega) =: R\psi. \tag{3.2.4}$$

Later on, we will mostly use the product state $\psi = \phi \otimes \Omega$ of the impurity and the vacuum instead of ψ_0.
Furthermore, we define

$$E_N^{\mathrm{pw}} := \sum_{k \in B_{\mathrm{F}}} |k|^2 = \langle R\Omega, H_N, R\Omega \rangle \tag{3.2.5}$$

to be the energy of the non-interacting Fermi ball.

Of greatest interest is of course the action of the particle-hole transformation on the microscopic Hamiltonian as generator of the dynamics. The conjugation with R of $\mathbb{H} = -\beta\Delta_y + \mathbb{H}^{\mathrm{kin}} + \mathbb{V}$ yields

$$\mathbb{H}_0 := R^* \mathbb{H}^{\mathrm{kin}} R - E_N^{\mathrm{pw}} = \sum_{k \in \mathbb{Z}^3} |k|^2 R^* a_k^* R R^* a_k R - E_N^{\mathrm{pw}} \tag{3.2.6}$$

$$= \sum_{k \in B_{\mathrm{F}}} |k|^2 a_k a_k^* + \sum_{k \in B_{\mathrm{F}}^{\mathrm{c}}} |k|^2 a_k^* a_k - E_N^{\mathrm{pw}} \tag{3.2.7}$$

$$= \sum_{k \in \mathbb{Z}^3} e(k) a_k^* a_k \qquad \text{with } e(k) := \begin{cases} |k|^2 & \text{if } k \in B_{\mathrm{F}}^{\mathrm{c}}, \\ -|k|^2 & \text{if } k \in B_{\mathrm{F}}. \end{cases} \tag{3.2.8}$$

Similarly, we can see that

$$\begin{aligned}R^*\mathbb{V}R &= \lambda \sum_{k,p\in\mathbb{Z}^3} \hat{V}(k)e^{iky}R^*a_p^*RR^*a_{p-k}R \\ &= \lambda \sum_{k\in\mathbb{Z}^3} \sum_{\substack{p-k\in B_{\mathrm{F}}^c,\\ p\in B_{\mathrm{F}}}} \hat{V}(k)e^{iky}a_pa_{p-k} + \lambda \sum_{k\in\mathbb{Z}^3} \sum_{\substack{p\in B_{\mathrm{F}}^c,\\ p-k\in B_{\mathrm{F}}}} \hat{V}(k)e^{iky}a_p^*a_{p-k}^* \qquad (3.2.9a)\\ &\quad + \lambda \sum_{k\in\mathbb{Z}^3} \sum_{\substack{p\in B_{\mathrm{F}},\\ p-k\in B_{\mathrm{F}}}} \hat{V}(k)e^{iky}a_pa_{p-k}^* + \lambda \sum_{k\in\mathbb{Z}^3} \sum_{\substack{p\in B_{\mathrm{F}}^c,\\ p-k\in B_{\mathrm{F}}^c}} \hat{V}(k)e^{iky}a_p^*a_{p-k}. \qquad (3.2.9b)\end{aligned}$$

For later purposes we shall introduce for $\varphi \in l^2(\mathbb{Z}^3)$ the short-notation

$$b(\varphi) := \sum_{k\in\mathbb{Z}^3} \overline{\varphi(k)} \sum_{\substack{p\in B_{\mathrm{F}}^c,\\ p-k\in B_{\mathrm{F}}}} a_{p-k}a_p, \qquad (3.2.10)$$

$$b^*(\varphi) = \sum_{k\in\mathbb{Z}^3} \varphi(k) \sum_{\substack{p\in B_{\mathrm{F}}^c,\\ p-k\in B_{\mathrm{F}}}} a_p^*a_{p-k}^*. \qquad (3.2.11)$$

We can then write

$$R^*\mathbb{H}R = -\beta\Delta_y + \mathbb{H}_0 + b^*(\tilde{h}_y) + b(\tilde{h}_y) + \mathcal{E} \qquad (3.2.12)$$

with $\tilde{h}_y(k) := \lambda\hat{V}(k)e^{iky}$ and $\mathcal{E}$ is given by the terms of (3.2.9b) since

$$\begin{aligned}\sum_{k\in\mathbb{Z}^3} \sum_{\substack{p-k\in B_{\mathrm{F}}^c,\\ p\in B_{\mathrm{F}}}} \hat{V}(k)e^{iky}a_pa_{p-k} &= \sum_{k\in\mathbb{Z}^3} \sum_{\substack{\tilde{p}\in B_{\mathrm{F}}^c,\\ \tilde{p}+k\in B_{\mathrm{F}}}} \hat{V}(k)e^{iky}a_{\tilde{p}+k}a_{\tilde{p}} \\ &= \sum_{k\in\mathbb{Z}^3} \sum_{\substack{p\in B_{\mathrm{F}}^c,\\ p-k\in B_{\mathrm{F}}}} \hat{V}(k)e^{-iky}a_{p-k}a_p \qquad (3.2.13)\end{aligned}$$

where we used that $\hat{V}(-k) = \hat{V}(k)$.

Almost-bosonic operators and patch decomposition Our effective description of the microscopic system described by (3.1.2) will involve the emergence of almost-bosonic particles describing pair excitations of the Fermi ball. Those pair excitations will be delocalized over the Fermi surface in the sense that they correspond to a linear combination of pairs of fermionic operators. As mentioned before the almost-bosonic pair operators which occur in this article coincide with the ones introduced in the series of seminal works [BNP+19, BNP+21a, BNP+21b, BPSS23] on the correlation energy of a weakly interacting Fermi gas.

We give a brief introduction to the construction of those operators with the most relevant properties in this subsection and in Section A.1.

A key ingredient for the approximation of the microscopic fermionic system by almost-bosonic excitations is the decomposition of the Fermi surface into patches. This will allow to approximate the fermionic kinetic energy term by a term quadratic in the almost-bosonic pair operators.

Introduce the bisecting subset of $\mathbb{Z}^3 \cap \operatorname{supp}\hat{V}$

$$\Gamma := \Big\{(k_1, k_2, k_3) \in \mathbb{Z}^3 \cap \operatorname{supp}\hat{V} : (k_3 > 0 \vee k_3 = 0), (k_2 > 0 \vee k_2 = k_3 = 0), k_1 > 0\Big\} \tag{3.2.14}$$

allowing the decomposition $\Gamma \cup (-\Gamma) = \mathbb{Z}^3 \cap \operatorname{supp}\hat{V}$.

The construction works as follows:

(i). Choose the number M of patches satisfying

$$N^{2\delta} \ll M \ll N^{\frac{2}{3}-2\delta}, \qquad \delta \in (0, \frac{1}{6}). \tag{3.2.15}$$

The lower bound on M is needed to control the number of momenta inside each patch whereas the upper bound is needed to suppress Pauli's principle. The choice of δ and c will be taken later.

(ii). Define equal-area disjoint patches p_α as follows

- p_1 is spherical cap of area $4\pi/M$,
- decompose remaining semi-sphere into $\sqrt{M}/2$ collars,
- leave corridors of width $2R := 2 \operatorname{supp}\hat{V}$ between adjacent patches,
- define patches of southern semi-sphere by reflection $k \mapsto -k$.

Define $\omega_\alpha \in S^2$ as centers of the patch p_α. A useful graphical sketch of this patch construction is given in Figure 1 of [BNP+19].

(iii). For given $k \in \Gamma$ define the set of north and south patch indices

$$\mathcal{I}_k^+ := \{\alpha \in \{1, \dots, M\} \mid k \cdot \hat{\omega}_\alpha \geq N^{-\delta}\}, \tag{3.2.16}$$
$$\mathcal{I}_k^- := \{\alpha \in \{1, \dots, M\} \mid k \cdot \hat{\omega}_\alpha \leq -N^{-\delta}\} \tag{3.2.17}$$

and $\mathcal{I}_k := \mathcal{I}_k^+ \cup \mathcal{I}_k^-$. This has the effect of excluding a strip around the equator of the Fermi ball, where the number of momenta per patch may become too small. Note that $\delta > 0$ coincides with the parameter in step 1.

(iv). Define the collective almost-bosonic creation operator and its normalization factor as

$$c^*_\alpha(k) := \begin{cases} b^*_\alpha(+k) & \text{if } \alpha \in \mathcal{I}_k^+, \\ b^*_\alpha(-k) & \text{if } \alpha \in \mathcal{I}_k^-. \end{cases}, \quad n_\alpha(k) := \begin{cases} m_\alpha(k) & \text{if } \alpha \in \mathcal{I}_k^+, \\ m_\alpha(-k) & \text{if } \alpha \in \mathcal{I}_k^-. \end{cases} \tag{3.2.18}$$

with

$$b^*_\alpha(k) := \frac{1}{m_\alpha(k)} \sum_{\substack{p \in B_F^c \cap B_\alpha, \\ p-k \in B_F \cap B_\alpha}} a^*_p a^*_{p-k}, \quad m_\alpha(k)^2 := \sum_{\substack{p \in B_F^c \cap B_\alpha, \\ p-k \in B_F \cap B_\alpha}} 1 \tag{3.2.19}$$

being sensitive to being on the north or south hemisphere. The creation operator can be seen as collective in the sense that it involves a superposition of all possible fermion pairs with relative momentum k.

Similarly to (3.2.11) introduced in the previous subsection, we define as in [BNP+21a]

$$c^*(\eta) := \sum_{k \in \Gamma} \sum_{\alpha \in \mathcal{I}_k} \overline{\eta_\alpha(k)} c^*_\alpha(k) \tag{3.2.20}$$

with inner product $\langle \eta, \varphi \rangle := \sum_{k \in \Gamma} \sum_{\alpha \in \mathcal{I}_k} \overline{\eta_\alpha(k)} \varphi_\alpha(k)$ for all $\eta, \varphi \in \bigoplus_{k \in \Gamma} l^2(\mathcal{I}_k)$.

The following statements hold as a consequence of the above construction.

- The surface area of a patch satisfies $\sigma(p_\alpha) \in \mathcal{O}(1/M)$.
- The Canonical Commutation Relations (CCR) are satisfied up to an error term (see [BNP+19, Lemma 4.1]): It holds for all $k', k \in \Gamma$ and $\alpha \in \mathcal{I}_k, \beta \in \mathcal{I}_{k'}$

$$[c_\alpha(k), c_\beta(k')] = [c^*_\alpha(k), c^*_\beta(k')] = 0, \tag{3.2.21}$$

$$[c_\alpha(k), c^*_\beta(k')] = \delta_{\alpha,\beta}\big(\delta_{k,k'} + \mathcal{E}_\alpha(k,k')\big) \tag{3.2.22}$$

  satisfying $\mathcal{E}_\alpha(k,k) \leq 0$, $\mathcal{E}_\alpha(k,l) = \mathcal{E}_\alpha(l,k)^*$ and

$$\forall \psi \in \mathcal{F}: \qquad \|\mathcal{E}_\alpha(k,k')\psi\| \leq \frac{2}{n_\alpha(k) n_\alpha(k')} \|\mathcal{N}\psi\|. \tag{3.2.23}$$

- The almost-bosonic operators change the number operator by two (see [BNP+19, Lemma 2.3]) in the following sense

$$c_\alpha(k)\mathcal{N} = (\mathcal{N}+2)c_\alpha(k). \tag{3.2.24}$$

- The normalization constant satisfies (see [BNP+19, Proposition 3.1])

$$n_\alpha(k)^2 = \frac{4\pi k_{\mathrm{F}}^2}{M} |k \cdot \hat{\omega}_\alpha| \big(1 + o(1)\big). \tag{3.2.25}$$

Also note that the summation in the definition of the almost-bosonic operators $c_\alpha^*(k)$ and $c_\alpha(k)$ involves only finite sets. Unlike in the exactly bosonic case our almost-bosonic operators therefore inherit boundedness from the fermionic constituents which satisfy $\|a_k^*\| = \|a_k\| = 1$. Subtle questions about the mutual adjointness and domain of the almost-bosonic operators remain trivial in our case.

3.3 Main results

3.3.1 Effective time evolution

We are now focusing on the effective time evolution of the initial state $\psi_0 = R\psi \equiv \phi \otimes R\Omega \in L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$, i.e. a uncorrelated product state with the non-interacting Fermi gas prepared as its non-degenerate ground state with no initial excitations. This set-up corresponds to a system where the impurity does not interact with the cold Fermi gas at time $t = 0$. Over time, we expect that the influence of the impurity particle creates and annihilates excitations of the Fermi ball. Therefore, we will use the particle-hole transformation as defined in (3.2.3) to connect the microscopic description to the following effective description: Let

$$\begin{aligned}\mathbb{H}^{\mathrm{eff}} :=& -\beta\Delta_y + \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \epsilon_\alpha(k) c_\alpha^*(k) c_\alpha(k) + E_N^{\mathrm{pw}} \\ &+ \lambda \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \hat{V}(k) e^{iky} n_\alpha(k) \Big(c_\alpha^*(k) + c_\alpha(-k) \Big)\end{aligned} \tag{3.3.1}$$

be our effective Hamiltonian with $\epsilon_\alpha(k) = 2k_F|k \cdot \omega_\alpha|$ and E_N^{pw} as defined in (3.2.5). We introduce for all $k \in \Gamma$ and $\alpha \in \mathcal{I}_k$

$$(h_y)_\alpha(k) := \lambda \hat{V}(k) e^{iky} n_\alpha(k). \tag{3.3.2}$$

Note that the effective Hamiltonian acts on the components of the Hilbert space $L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$ in the sense that we can write

$$\mathbb{H}^{\mathrm{eff}} = (-\beta\Delta_y) \otimes 1 + 1 \otimes \mathbb{D}_{\mathrm{B}} + \Phi(h_y) + E_N^{\mathrm{pw}} \tag{3.3.3}$$

with $\Phi(h_y) := c^*(h_y) + c(h_y)$ and

$$c^*(h_y) := \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} (h_y)_\alpha(k)\ c_\alpha^*(k), \tag{3.3.4}$$

$$c(h_y) := \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \overline{(h_y)_\alpha(k)}\ c_\alpha(k), \tag{3.3.5}$$

$$\mathbb{D}_{\mathrm{B}} := \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \epsilon_\alpha(k) c_\alpha^*(k) c_\alpha(k) \quad \text{with } \epsilon_\alpha(k) = 2k_F|k \cdot \omega_\alpha| \tag{3.3.6}$$

describing the kinetic energy of the almost-bosonic pair excitations with linear dispersion relation. Note that by the Kato-Rellich theorem the effective Hamiltonian (3.3.1) is self-adjoint in its natural domain and generates a unitary time evolution.

To state an effective description of the time evolution of those excitations we compare the particle-hole transformed microscopic dynamics $R^* e^{-i\mathbb{H}t} R\psi$ with the effective time evolution $e^{-i\mathbb{H}^{\text{eff}}t}\psi$ in Hilbert space norm:

Theorem 51 (Effective dynamics of the system)**.** *Assume that $\hat{V} \geq 0$ is compactly supported and satisfies $\hat{V}(-k) = \hat{V}(k)$ for all $k \in \mathbb{Z}^3$. Let $\lambda \in [k_{\text{F}}^{-1/6}, 1]$ be the interaction parameter as introduced in (3.1.2) and take the number of patches to be $M = N^{\frac{16}{45}}$ with $\delta = \frac{2}{15}$ as introduced in (3.2.15). Then it holds for the initial state $\psi = \phi \otimes \Omega \in L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$ that there is a $C > 0$ depending only on the interaction V such that for all $k_{\text{F}} \geq 2$ and $t \geq 0$*

$$\|R^* e^{-i\mathbb{H}t} R\psi - e^{-i\mathbb{H}^{\text{eff}}t}\psi\| \leq C \left(e^{C\lambda k_{\text{F}} t} - 1\right) k_{\text{F}}^{-\frac{1}{5}}.$$

Remark 52. Note that the right hand side of the bound is indeed small as long as $t \in \mathcal{O}(k_{\text{F}}^{-1}\lambda^{-1})$. The error $k_{\text{F}}^{-1/5}$ is a result of the optimized choice of M and δ. The non-optimal error bound depends on the patch parameters and is given by

$$C\left((1+\lambda^{-1})\lambda^{-1}(M^{-\frac{1}{2}} + MN^{-\frac{2}{3}+\delta}) + (N^{-\frac{\delta}{2}} + N^{-\frac{1}{6}}M^{\frac{1}{4}})\right)\left(e^{C\lambda k_{\text{F}} t} - 1\right). \tag{3.3.7}$$

3.3.2 Effective coherent state

The effective time evolved state from Theorem 51 can be further simplified. Our second result shows how the dynamics can be approximated on the level of states. In order to state the second main result, we introduce almost-bosonic coherent states.

Note that since $B := c^*(\eta) - c(\eta)$ defines for all $\eta \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ a bounded operator and satisfies $B = -B^*$, the exponential operator e^B is well-defined and is unitary.

Definition 53. Define for $\eta \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ the Weyl operator

$$W(\eta) := e^B := e^{c^*(\eta) - c(\eta)}. \tag{3.3.8}$$

If $\eta \equiv \eta^y$ is additionally a bounded multiplication operator for each $y \in \mathbb{R}^3$, we call $W(\eta)\phi \otimes \Omega$ a coupled coherent state with $\phi \otimes \Omega \in L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$.

Remark 54. If c^*, c would satisfy the CCR without error, one could use the Baker-Campbell-

Hausdorff formula to formally write

$$
\begin{aligned}
W(\eta)\phi\otimes\Omega =& e^{-\|\eta\|^2/2}e^{c^*(\eta)}\phi\otimes\Big\{1,0,\dots,0,\dots\Big\}\\
=& e^{-\|\eta\|^2/2}e^{c^*(\eta)}\Big\{\phi,0,\dots,0,\dots\Big\}\\
=& e^{-\|\eta_s\|^2/2}\Big\{\phi,\eta\phi,\frac{\eta^{\otimes 2}\phi}{\sqrt{2!}},\dots,\frac{\eta^{\otimes n}\phi}{\sqrt{n!}},\dots\Big\}
\end{aligned}
\tag{3.3.9}
$$

i.e. the coupled coherent state corresponds to a superposition of different particle number.

Remark 55. The coupled coherent state from Definition 53 satisfies the following well-known properties of the Weyl operator (cf. for example [BPS16, Chapter 3] or [FZ17, Appendix A]) up to certain error terms:

- (shift property) For $\eta,\xi\in\bigoplus_{k\in\Gamma}l^2(\mathcal{I}_k)$ it holds
$$c(\xi)W(\eta)\phi\otimes\Omega\simeq\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\overline{\xi_\alpha(k)}\eta_\alpha(k)\phi\otimes\Omega, \tag{3.3.10}$$

- (expectation of the number operator) For $\eta\in\bigoplus_{k\in\Gamma}l^2(\mathcal{I}_k)$ it holds
$$\langle W(\eta)\phi\otimes\Omega,\mathcal{N}W(\eta)\phi\otimes\Omega\rangle\simeq 2\|\eta\|^2, \tag{3.3.11}$$

- (time derivative of the Weyl operator) For $\eta_t\in\bigoplus_{k\in\Gamma}l^2(\mathcal{I}_k)$ differentiable in t with derivative $\dot\eta_t\in\bigoplus_{k\in\Gamma}l^2(\mathcal{I}_k)$ it holds: for all $t\in\mathbb{R}$:
$$\partial_t W(\eta_t)\simeq\left(c^*(\dot\eta_t)-c(\dot\eta_t)+i\mathrm{Im}\langle\dot\eta_t,\eta_t\rangle\right)W(\eta_t), \tag{3.3.12}$$

We give rigorous statements on the error terms and proofs of the approximate properties in Lemma 65, Proposition 67, Lemma 69 of Section 3.5.

Consider now the following state for all times $t\in\mathbb{R}$

$$\psi_t := e^{iP(t)}W(\eta_t)\phi\otimes\Omega \tag{3.3.13}$$

$$\text{with } P(t)=2\mathrm{Im}(\nu_t)-E_N^{\mathrm{pw}}t-\mathrm{Im}\int_0^t \mathrm{d}s\langle\dot\eta_s,\eta_s\rangle_\Gamma \tag{3.3.14}$$

with the choices of

$$\big(\eta_s\big)_\alpha(k) :=\big(\eta_s^y\big)_\alpha(k) := \frac{e^{-is\epsilon_\alpha(k)}-1}{\epsilon_\alpha(k)}(h_y)_\alpha(k)=\frac{e^{-is\epsilon_\alpha(k)}-1}{\epsilon_\alpha(k)}\lambda\hat V(k)n_\alpha(k)e^{iky}, \tag{3.3.15}$$

$$\nu_s := \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\frac{e^{-is\epsilon_\alpha(k)}+is\epsilon_\alpha(k)-1}{\epsilon_\alpha(k)^2}|(h_y)_\alpha(k)|^2 \tag{3.3.16}$$

for all $k \in \Gamma, \alpha \in \mathcal{I}_k$. Due to $\left(\eta_s\right)_\alpha(k) = -ie^{-is\epsilon_\alpha(k)/2}\frac{\sin(\epsilon_\alpha(k)s/2)}{\epsilon_\alpha(k)/2}(h_y)_\alpha(k)$ and Lemma 82 the norm is bounded for all $s \in \mathbb{R}$

$$\|\eta_s\| \equiv \left(\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} |\left(\eta_s\right)_\alpha(k)|^2\right)^{1/2} \leq \min\left\{\sqrt{\pi}\|\hat{V}(\cdot)^{1/2}\|_2\lambda k_\mathrm{F}s, \sqrt{2\pi}\|\hat{V}\|_2\lambda\log(4k_\mathrm{F}s+2)\right\} \tag{3.3.17}$$

as shown later in Lemma 70 and similarly for $|\nu_s|$.

In particular, it holds $(\eta_0, \nu_0) = (0,0)$ and $\lim_{\epsilon_\alpha(k)\to 0}(\eta_s,\nu_s) = (-ish_y, 0)$ and therefore $\psi_0 = \phi\otimes\Omega$. Thus, as mentioned before, $\eta \equiv \eta^y$ as defined in (3.3.15) corresponds to an interaction term with a bounded multiplication operator acting on $L^2(\Lambda, \mathrm{d}y)$. The state $\psi_t = e^{iP(t)}W(\eta_t)\phi\otimes\Omega$ can therefore be seen as a time-dependent coupled coherent state.

The following theorem states that ψ_t approximately corresponds to the effective time evolution generated by $\mathbb{H}^{\mathrm{eff}}$.

Theorem 56 (Effective coherent dynamics)**.** *Consider the initial state $\psi = \phi\otimes\Omega \in L^2(\Lambda,\mathrm{d}y)\otimes\mathcal{H}_N^-$ with $\sum_{i=1}^3\left(\|\partial_{y_i}\phi\| + \|\partial_{y_i}^2\phi\|\right) \leq c < \infty$ for a constant $c > 0$. Under the assumptions of Theorem 51, there exists a constant $C > 0$ and a function $Q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ monotonically increasing with $Q(0) = 0$ such that for all $t \geq 0$*

$$\|e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi - e^{iP(t)}W(\eta_t)\psi\| \leq CQ(\lambda k_\mathrm{F}t)\max\{k_\mathrm{F}^{-\frac{2}{15}}, c\beta t\}$$

with $P(t)$ given by (3.3.14), η_t given by (3.5.30), C and Q depending only on the interaction V.

Remark 57. Note that the upper bound above is indeed meaningful in the sense that the bound is small for $t \in \mathcal{O}(k_\mathrm{F}^{-1}\lambda^{-1})$ and $\beta \in o(\lambda k_\mathrm{F})$. The latter condition together with the upper bound for $\|\Delta_y W(\eta_t)\phi\otimes\Omega\|$ from Lemma 73 ensures that the contribution from the kinetic energy term $h_0 = -\beta\Delta_y$ of the impurity particle remains negligible on the relevant time scale. This seems to be crucial in our approach since otherwise h_0 would generate non-trivial correlations and make the coherent state form inapplicable.

Remark 58. Due to the explicit formulation of the coupled coherent state provided in (3.3.15), we can quantify the number of collective excitations over time. More concretely, using (3.3.11) the term $\|\eta_t\|^2$, which is calculated in Lemma 70, represents the expected number of excitations generated by the interaction with the impurity. Assuming that the impurity, together with its excitations, can be interpreted as polaron-like quasi-particle, the quantity $\|\eta_t\|^2$ offers insight into the quasi-particle formation process. As displayed in Figure 3.3.1 on page 136 the graph shows a parabolic growth followed by a logarithmic increase as qualitative feature.

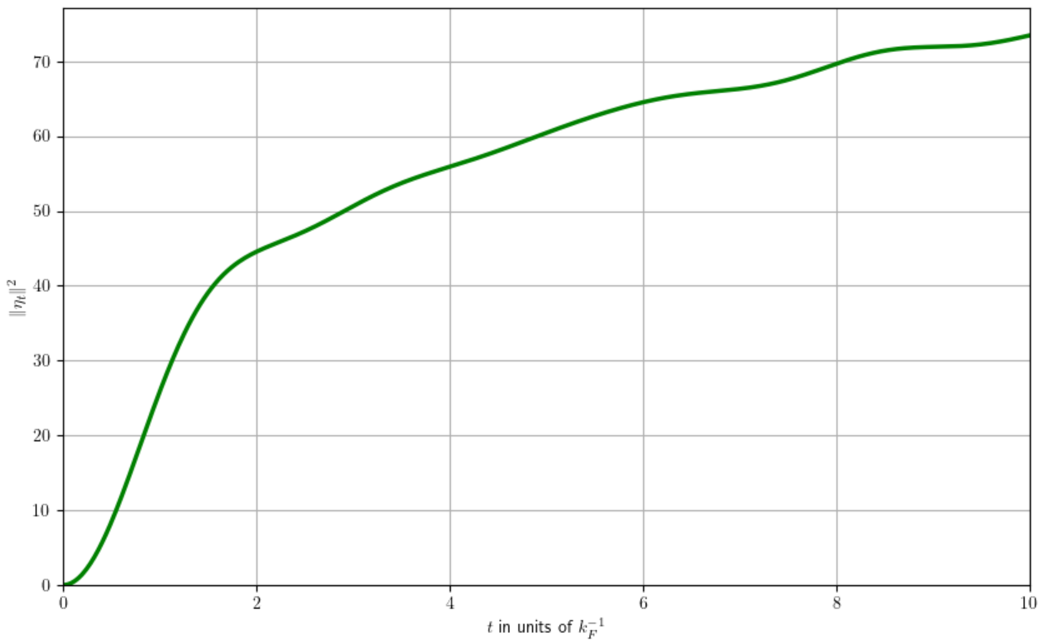

Figure 3.3.1: Plot of $\|\eta_t\|^2$ using Lemma 70 with constant $\hat{V}$ and $\lambda = 1$. As a qualitative feature one observes a parabolic growth followed by a logarithmic increase.

Remark 59. The proof is based on the following observation: It holds by virtue of Duhamel's formula

$$\begin{aligned}
&\|e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi - e^{iP(t)}W(\eta_t)\psi\| \\
&= \|\psi - e^{i\mathbb{H}^{\mathrm{eff}}t}e^{-iE_N^{\mathrm{pw}}t}e^{i2\mathrm{Im}(\nu_t)}e^{-i\mathrm{Im}\int_0^t \mathrm{d}s\langle\dot{\eta}_s,\eta_s\rangle}W(\eta_t)\psi\| \qquad (3.3.18)\\
&= \|\int_0^t \mathrm{d}s\; e^{i\mathbb{H}^{\mathrm{eff}}s}e^{iP(s)}\left\{\left(\mathbb{H}^{\mathrm{eff}} - E_N^{\mathrm{pw}} + 2\mathrm{Im}(\dot{\nu}_s) - \mathrm{Im}\langle\dot{\eta}_s,\eta_s\rangle\right)W(\eta_s)\psi - i\partial_s W(\eta_s)\psi\right\}\| \\
&\qquad (3.3.19)\\
&\leq \int_0^t \mathrm{d}s\|\left(\mathbb{H}^{\mathrm{eff}} - E_N^{\mathrm{pw}} + 2\mathrm{Im}(\dot{\nu}_s) - \mathrm{Im}\langle\dot{\eta}_s,\eta_s\rangle\right)W(\eta_s)\psi - i\partial_s W(\eta_s)\psi\|. \qquad (3.3.20)
\end{aligned}$$

If the collective operators $c^*_\alpha(k)$ and $c_\alpha(k)$ were exactly bosonic, i.e. the CCR held without error, we could use the shift property (3.3.10) of the Weyl operator to commute $c_\alpha(k)$ to the vacuum Ω with the cost of some inner product terms. In this case we would observe with the short-hand notation $c^*c(\epsilon) := \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\epsilon_\alpha(k)c^*_\alpha(k)c_\alpha(k)$ and by applying (3.3.12) that

$$\begin{aligned}
&\big[\mathbb{H}^{\text{eff}} - E_N^{\text{pw}} + 2\text{Im}(\dot\nu_s) - i\left\{c^*(\dot\eta_s) - c(\dot\eta_s)\right\}\big]W(\eta_s)\psi\\
=&\big[h_0 + c^*c(\epsilon) + c^*(h_y) + c(h_y) + 2\text{Im}(\dot\nu_s) + c^*(-i\dot\eta_s) - c(i\dot\eta_s)\big]W(\eta_s)\phi\otimes\Omega\\
=&\big[h_0 + c^*(\epsilon\eta_s) + c^*(h_y) + \langle h_y,\eta_s\rangle + 2\text{Im}(\dot\nu_s) + c^*(-i\dot\eta_s) - \langle i\dot\eta_s,\eta_s\rangle\big]W(\eta_s)\phi\otimes\Omega\\
=&h_0W(\eta_s)\phi\otimes\Omega
\end{aligned}\tag{3.3.21}$$

is exactly vanishing up to the kinetic energy term h_0 of the impurity particle since η_t as defined in (3.3.15) solves the ODE $\epsilon\eta_t + h_y = i\dot\eta_t$ and ν_t as defined in (3.3.16) absorbs all scalar terms.

Remark 60. Note that as long as $\|\eta_t\|$ is of order 1, the coupled coherent state is different from a free decoupled dynamics with $\psi_t^{\text{free}} \sim e^{i\beta\Delta_y t}\phi\otimes\Omega$. Vice versa, if $\|\eta_t\|\in o(1)$ in terms of k_F one can easily see with a Duhamel argument, analogue to the previous remark, that $\|\psi_t^{\text{free}} - e^{iP(t)}W(\eta_t)\psi\| \to 0$ for large k_F. This is because all the terms appearing in $\partial_t W(\eta_t)$ from (3.3.12) can be bounded in terms of $\|\eta_t\|$. Using (3.3.17) we can identify the following cases:

- $\lambda = 1, t\in o(k_\text{F}^{-1})$: $\|\eta_t\|\in o(1)$ with respect to k_F
  The time scale is too short for forming excitations thus ψ_t^{free} is a good approximation. Note that this is compatible with [MP21] as mentioned in Subsection 3.1.2.
- $\lambda = 1, t\in \mathcal{O}(k_\text{F}^{-1})$: $\|\eta_t\|\in \mathcal{O}(1)$ with respect to k_F
  On this time scale exitations of the Fermi gas become important and thus the free decoupled evolution is not a good approximation anymore. We refer to the proof of Corollary 61 on how to show rigorously that the derived effective description is relevant on this time scale.
- $\lambda\in o(1), 0\le t\in\mathcal{O}(k_\text{F}^{-1}\lambda^{-1})$: $\|\eta_t\|\in o(1)$ with respect to k_F
  Also here, ψ_t^{free} is a good description. Note that the number of excitations grows only logarithmically in time. Thus, for any $\lambda\in o(1)$ on gets $\|\eta_t\|\in o(1)$. In the case of $\lambda\in o(1)$ we expect this result of free decoupling to hold for even longer times, possibly of order 1. In order to show this expected behavior, we expect a method different from ours to be more suitable, for example a perturbative expansion in the spirit of [MP21] to higher orders to exploit the small coupling.

Now by virtue of the previous statement we are able to show that the linear coupling term of the effective Hamiltonian is essential for the effective description and cannot be neglected on an approximate level for $\lambda = 1$ and $t\in\mathcal{O}(k_\text{F}^{-1})$.

Let

$$\widetilde{\mathbb{H}^{\text{eff}}} := \mathbb{H}^{\text{eff}} - c^*(h_y) - c(h_y) \equiv \beta(-\Delta_y) + \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\epsilon_\alpha(k)c^*_\alpha(k)c_\alpha(k) + E_N^{\text{pw}} \tag{3.3.22}$$

be the effective Hamiltonian without the linear coupling.

Corollary 61. *Under the assumptions of Theorem 56 with $\lambda = 1$ and a $T > 0$, there exists a monotonically increasing function $C : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ only depending on V such that for all $t \in [0, T]$*

$$\|R^* e^{-i\mathbb{H}t} R\psi - e^{-i\widetilde{\mathbb{H}^{\mathrm{eff}}}t}\psi\| \geq C(t) - \mathcal{O}\left(\max\{k_{\mathrm{F}}^{-\frac{2}{15}}, \beta k_{\mathrm{F}}^{-1}\}\right).$$

In particular it holds $C(t) \in \mathcal{O}(1)$ with respect to k_{F} for all $t \in \mathcal{O}(k_{\mathrm{F}}^{-1})$.

3.4 Proof of Theorem 51

In all of our estimates, we need to control the number operator $\mathcal{N}$ acting on the time evolved state $\psi_t = e^{-i\mathbb{H}^{\mathrm{eff}}t}\phi \otimes \Omega$. The following statement shows that the effective time evolution preserves the order of magnitude of the number of excitations:

Proposition 62 (Effective time evolution of the number operator)**.** *There exists a constant $C > 0$ only depending on V such that it holds for all $t \in \mathbb{R}, n \in \mathbb{N}$ and $\psi \in L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$*

$$\langle e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi, (\mathcal{N}+1)^n e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\rangle \leq e^{nC\lambda k_{\mathrm{F}}t}\langle \psi, (\mathcal{N}+3)^n\psi\rangle.$$

Proof. We want to use Grönwall's lemma and therefore estimate the derivative

$$\begin{aligned}
&\left| i\partial_t \langle e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi, (\mathcal{N}+3)^n e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\rangle \right| \\
\leq &\left| \langle e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi, \sum_{j=0}^{n-1} (\mathcal{N}+3)^j [\mathcal{N}, \mathbb{H}^{\mathrm{eff}}](\mathcal{N}+3)^{n-j-1} e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\rangle \right| \\
= &\left| 2\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} (h_y)_\alpha(k) \sum_{j=0}^{n-1} \langle e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi, (\mathcal{N}+3)^j \left(c_\alpha^*(k) - c_\alpha(k)\right)(\mathcal{N}+3)^{n-j-1} e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\rangle \right| \qquad (3.4.1)
\end{aligned}$$

We split the difference of $\left(c_\alpha^*(k) - c_\alpha(k)\right)$ and consider the term with $c_\alpha^*(k)$ first. Insert here $\mathrm{id} = (\mathcal{N}+1)^{\frac{n}{2}-1-j}(\mathcal{N}+1)^{j+1-\frac{n}{2}}$ between $(\mathcal{N}+3)^j$ and $c_\alpha^*(k)$ and use the commutation $\mathcal{N}c_\alpha^*(k) = c_\alpha^*(k)(\mathcal{N}+2)$ to obtain

$$(\mathcal{N}+3)^j c_\alpha^*(k)(\mathcal{N}+3)^{n-j-1} = (\mathcal{N}+3)^j(\mathcal{N}+1)^{\frac{n}{2}-1-j} c_\alpha^*(k)(\mathcal{N}+3)^{\frac{n}{2}}. \qquad (3.4.2)$$

We introduce the notation $\xi_j := (\mathcal{N}+1)^{\frac{n}{2}-1-j}(\mathcal{N}+3)^j e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi$ and $\tilde{\xi} := (\mathcal{N}+3)^{\frac{n}{2}} e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi$ to estimate

$$\left| \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} (h_y)_\alpha(k) \sum_{j=0}^{n-1} \langle \xi_j, c_\alpha^*(k)\tilde{\xi}\rangle \right|$$

$$
\begin{aligned}
&\leq \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}|(h_y)_\alpha(k)|\sum_{j=0}^{n-1}\|c_\alpha(k)\xi_j\|\ \|\tilde{\xi}\|\\
&\leq C\lambda\sum_{k\in\Gamma}|\hat{V}(k)|\Big(\sum_{\alpha\in\mathcal{I}_k}n_\alpha(k)^2\Big)^{1/2}\sum_{j=0}^{n-1}\Big(\sum_{\alpha\in\mathcal{I}_k}\|c_\alpha(k)\xi_j\|^2\Big)^{1/2}\|\tilde{\xi}\|\\
&\leq C\lambda k_{\mathrm{F}}\sum_{k\in\Gamma}|\hat{V}(k)|\sum_{j=0}^{n-1}\|\mathcal{N}^{1/2}\xi_j\|\|\tilde{\xi}\|\\
&\leq C\lambda k_{\mathrm{F}}\|\hat{V}\|_1\sum_{j=0}^{n-1}\|(\mathcal{N}+1)\xi_j\|\ \|\tilde{\xi}\|\\
&\leq Cn\lambda k_{\mathrm{F}}\|\hat{V}\|_1\langle e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi,(\mathcal{N}+3)^n e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\rangle
\end{aligned}
\tag{3.4.3}
$$

where we used Lemma 84 and Lemma 81 in the fourth line and the operator inequality $\mathcal{N}^{1/2}\leq(\mathcal{N}+1)$ in the fifth line. Note that $|k|<C$ for all $k\in\Gamma$ since $\Gamma\subset\mathrm{supp}\hat{V}$ and $\hat{V}$ has bounded support by assumption.

The second term with $c_\alpha(k)$ can be treated by inserting $\mathrm{id}=(\mathcal{N}+1)^{\frac{n}{2}-j}(\mathcal{N}+1)^{j-\frac{n}{2}}$ and using the commutation $c_\alpha(k)\mathcal{N}=(\mathcal{N}+2)c_\alpha(k)$.

$$
(\mathcal{N}+3)^j c_\alpha(k)(\mathcal{N}+3)^{n-j-1}=(\mathcal{N}+3)^{\frac{n}{2}}c_\alpha(k)(\mathcal{N}+1)^{j-\frac{n}{2}}(\mathcal{N}+3)^{n-j-1}. \tag{3.4.4}
$$

We introduce the notation $\chi_j\coloneqq(\mathcal{N}+1)^{j-\frac{n}{2}}(\mathcal{N}+3)^{n-j-1}e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi$

$$
\begin{aligned}
&\Big|\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}(h_y)_\alpha(k)\sum_{j=0}^{n-1}\langle\tilde{\xi},c_\alpha(k)\chi_j\rangle\Big|\\
&\leq \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}|(h_y)_\alpha(k)|\sum_{j=0}^{n-1}\|c_\alpha(k)\chi_j\|\ \|\tilde{\xi}\|\\
&\leq C\lambda\sum_{k\in\Gamma}|\hat{V}(k)|\Big(\sum_{\alpha\in\mathcal{I}_k}n_\alpha(k)^2\Big)^{1/2}\sum_{j=0}^{n-1}\Big(\sum_{\alpha}\|c_\alpha(k)\chi_j\|^2\Big)^{1/2}\|\tilde{\xi}\|\\
&\leq C\lambda k_{\mathrm{F}}\sum_{k\in\Gamma}|\hat{V}(k)|\sum_{j=0}^{n-1}\|\mathcal{N}^{1/2}\chi_j\|\|\tilde{\xi}\|\\
&\leq C\lambda k_{\mathrm{F}}\|\hat{V}\|_1\sum_{j=0}^{n-1}\|(\mathcal{N}+1)\chi_j\|\ \|\tilde{\xi}\|\\
&\leq Cn\lambda k_{\mathrm{F}}\|\hat{V}\|_1\langle e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi,(\mathcal{N}+3)^n e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\rangle.
\end{aligned}
\tag{3.4.5}
$$

Altogether it follows with the Grönwall's lemma that

$$
\langle e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi,(\mathcal{N}+3)^n e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\rangle\leq\exp\big(Cn\lambda k_{\mathrm{F}}\|\hat{V}\|_1 t\big)\langle\psi,(\mathcal{N}+3)^n\psi\rangle \tag{3.4.6}
$$

which is the desired result since $\|\hat{V}\|_1 < C$. ■

The following statement shows that the fermionic kinetic energy term (3.2.8) can be approximated by the almost-bosonic kinetic energy term.

Proposition 63 (Approximation of the kinetic energy). *There exists a constant $C > 0$ only depending on V such that it holds for all $t \in \mathbb{R}$ and $\psi \in L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$*

$$\begin{aligned}
&\Big| \|(\mathbb{H}_0 - \mathbb{D}_B) e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi\| - \|(\mathbb{H}_0 - \mathbb{D}_B)\psi\| \Big| \\
&\leq C(1+\lambda^{-1}) N^{\frac{1}{3}} \left(e^{C\lambda k_{\mathrm{F}} t} - 1\right) \Big(M^{-\frac{1}{2}} \|(\mathcal{N}+3)\psi\| + k_{\mathrm{F}} M N^{-1+\delta} \|(\mathcal{N}+3)^2\psi\| \Big) \qquad (3.4.7)
\end{aligned}$$

using Proposition 62.

Remark 64. Note that in the case of $\psi \equiv \phi \otimes \Omega$ it holds $(\mathbb{H}_0 - \mathbb{D}_B)\psi = 0$ and $\|(\mathcal{N}+3)^2\psi\| = 9 \leq C$. Also note that $\left(e^{C\lambda k_{\mathrm{F}} t} - 1\right) = \lambda k_{\mathrm{F}} t + \mathcal{O}((\lambda k_{\mathrm{F}} t)^2)$.

Proof. It holds

$$\begin{aligned}
&\left| \partial_t \|(\mathbb{H}_0 - \mathbb{D}_B) e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi\|^2 \right| \\
&= \left| \partial_t \langle e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi, (\mathbb{H}_0 - \mathbb{D}_B)^2 e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi\rangle \right| \\
&= \left| \langle e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi, [(\mathbb{H}_0 - \mathbb{D}_B)^2, \mathbb{H}^{\mathrm{eff}}] e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi\rangle \right| \\
&= \left| \langle e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi, \big((\mathbb{H}_0 - \mathbb{D}_B)[(\mathbb{H}_0 - \mathbb{D}_B), \mathbb{H}^{\mathrm{eff}}] + [(\mathbb{H}_0 - \mathbb{D}_B), \mathbb{H}^{\mathrm{eff}}](\mathbb{H}_0 - \mathbb{D}_B)\big)\, e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi\rangle \right| \\
&\leq 2 \left| \langle e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi, (\mathbb{H}_0 - \mathbb{D}_B)[(\mathbb{H}_0 - \mathbb{D}_B), \mathbb{H}^{\mathrm{eff}}] e^{-i\mathbb{H}^{\mathrm{eff}} t}\psi\rangle \right| \qquad (3.4.8) \\
& \qquad (3.4.9)
\end{aligned}$$

and make use of

$$\begin{aligned}
[(\mathbb{H}_0 - \mathbb{D}_B), c_\alpha^*(k)] &=: \mathfrak{E}_\alpha^{\mathrm{lin}}(k)^* - \mathfrak{E}_\alpha^{\mathrm{B}}(k)^* =: \mathfrak{E}_\alpha(k)^*, &(3.4.10)\\
[(\mathbb{H}_0 - \mathbb{D}_B), c_\alpha(k)] &= -\mathfrak{E}_\alpha(k) &(3.4.11)
\end{aligned}$$

to arrive at

$$\begin{aligned}
[(\mathbb{H}_0 - \mathbb{D}_B), \mathbb{H}^{\mathrm{eff}}] &= [(\mathbb{H}_0 - \mathbb{D}_B), c^* c(\epsilon) + c^*(h_y) + c(h_y)] &(3.4.12)\\
&= \langle \mathfrak{E}, c(\epsilon)\rangle_\Gamma - \langle c(\epsilon), \mathfrak{E}\rangle_\Gamma + \langle \mathfrak{E}, h_y\rangle_\Gamma - \langle h_y, \mathfrak{E}\rangle_\Gamma &(3.4.13)
\end{aligned}$$

with the short notation $\langle A, B\rangle_\Gamma := \sum_{k\in\Gamma} \sum_{\alpha\in\mathcal{I}_k} A_\alpha^*(k) B_\alpha(k)$.

Bounds for the error terms are readily provided in Lemma 85 and Lemma 86 of the appendix:

$$\sum_{\alpha\in\mathcal{I}_k} \|\mathfrak{E}^{\text{lin}}_\alpha(k)\psi\|^2 \leq C\left(N^{\frac{1}{3}}M^{-\frac{1}{2}}\right)^2 \|(\mathcal{N}+1)^{\frac{1}{2}}\psi\|^2, \tag{3.4.14}$$

$$\sum_{\alpha\in\mathcal{I}_k} \|\mathfrak{E}^{\text{B}}_\alpha(k)\psi\|^2 \leq C\left(k_{\text{F}}MN^{-\frac{2}{3}+\delta}\right)^2 \|(\mathcal{N}+1)^{\frac{3}{2}}\psi\|^2. \tag{3.4.15}$$

Furthermore, use the bounds $\sum_{\alpha\in\mathcal{I}_k} c^*_\alpha(k)c_\alpha(k) \leq \mathcal{N}$, $\epsilon_\alpha(k) \leq Ck_{\text{F}}$ and that $\mathfrak{E}^{\text{B}}_\alpha(k), \mathfrak{E}^{\text{lin}}_\alpha(k), c_\alpha(k)$ all annihilate exactly two fermions to estimate

$$\begin{aligned}
&\langle e^{-i\mathbb{H}^{\text{eff}}t}\psi, (\mathbb{H}_0-\mathbb{D}_B)\langle c(\epsilon), \mathfrak{E}^{\text{lin}}-\mathfrak{E}^{\text{B}}\rangle_\Gamma e^{-i\mathbb{H}^{\text{eff}}t}\psi\rangle \\
\leq &\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\left|\langle\epsilon_\alpha(k)c_\alpha(k)(\mathcal{N}+1)^{-1/2}(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\text{eff}}t}\psi, \mathfrak{E}^{\text{lin}}_\alpha(k)(\mathcal{N}+1)^{1/2}e^{-i\mathbb{H}^{\text{eff}}t}\psi\rangle\right| \\
&+\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\left|\langle\epsilon_\alpha(k)c_\alpha(k)(\mathcal{N}+1)^{-1/2}(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\text{eff}}t}\psi, \mathfrak{E}^{\text{B}}_\alpha(k)(\mathcal{N}+1)^{1/2}e^{-i\mathbb{H}^{\text{eff}}t}\psi\rangle\right|
\end{aligned} \tag{3.4.16}$$

$$\begin{aligned}
\leq &\sum_{k\in\Gamma}\left(\sum_{\alpha\in\mathcal{I}_k}\|\epsilon_\alpha(k)c_\alpha(k)(\mathcal{N}+1)^{-1/2}(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\text{eff}}t}\psi\|^2\right)^{1/2} \\
&\times\left(\left(\sum_{\alpha\in\mathcal{I}_k}\|\mathfrak{E}^{\text{lin}}_\alpha(k)(\mathcal{N}+1)^{1/2}e^{-i\mathbb{H}^{\text{eff}}t}\psi\|^2\right)^{1/2}+\left(\sum_{\alpha\in\mathcal{I}_k}\|\mathfrak{E}^{\text{B}}_\alpha(k)(\mathcal{N}+1)^{1/2}e^{-i\mathbb{H}^{\text{eff}}t}\psi\|^2\right)^{1/2}\right)
\end{aligned} \tag{3.4.17}$$

$$\leq Ck_{\text{F}}\|(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\text{eff}}t}\psi\|\left(N^{\frac{1}{3}}M^{-\frac{1}{2}}\|(\mathcal{N}+1)e^{-i\mathbb{H}^{\text{eff}}t}\psi\|+k_{\text{F}}MN^{-\frac{2}{3}+\delta}\|(\mathcal{N}+1)^2e^{-i\mathbb{H}^{\text{eff}}t}\psi\|\right) \tag{3.4.18}$$

$$\leq C^2k_{\text{F}}N^{\frac{1}{3}}\|(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\text{eff}}t}\psi\|\left(M^{-\frac{1}{2}}\|(\mathcal{N}+1)e^{-i\mathbb{H}^{\text{eff}}t}\psi\|+k_{\text{F}}MN^{-1+\delta}\|(\mathcal{N}+1)^2e^{-i\mathbb{H}^{\text{eff}}t}\psi\|\right). \tag{3.4.19}$$

The term $\langle c(\epsilon), \mathfrak{E}\rangle_\Gamma$ can be treated analogously. The other terms can be estimated similarly using $|(h_y)_\alpha(k)| \leq C\lambda|\hat{V}(k)||n_\alpha(k)|$ and Cauchy-Schwarz inequality with (81)

$$\begin{aligned}
&\langle e^{-i\mathbb{H}^{\text{eff}}t}\psi, (\mathbb{H}_0-\mathbb{D}_B)\langle\mathfrak{E}^{\text{lin}}-\mathfrak{E}^{\text{B}}, h_y\rangle_\Gamma e^{-i\mathbb{H}^{\text{eff}}t}\psi\rangle \\
\leq &C\lambda\sum_{k\in\Gamma}|\hat{V}(k)|\sum_{\alpha\in\mathcal{I}_k}\left|\langle n_\alpha(k)(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\text{eff}}t}\psi, \mathfrak{E}^{\text{lin}}_\alpha(k)e^{-i\mathbb{H}^{\text{eff}}t}\psi\rangle\right| \\
&+C\lambda\sum_{k\in\Gamma}|\hat{V}(k)|\sum_{\alpha\in\mathcal{I}_k}\left|\langle n_\alpha(k)(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\text{eff}}t}\psi, \mathfrak{E}^{\text{B}}_\alpha(k)e^{-i\mathbb{H}^{\text{eff}}t}\psi\rangle\right|
\end{aligned} \tag{3.4.20}$$

$$\leq C\lambda k_{\text{F}}\|(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\text{eff}}t}\psi\|\sum_{k\in\Gamma}|\hat{V}(k)|\left\{\left(\sum_{\alpha\in\mathcal{I}_k}\|\mathfrak{E}^{\text{lin}}_\alpha(k)e^{-i\mathbb{H}^{\text{eff}}t}\psi\|^2\right)^{1/2}\right. \tag{3.4.21}$$

$$+\left(\sum_{\alpha\in\mathcal{I}_k}\|\mathfrak{E}^{\mathrm{B}}_\alpha(k)e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|^2\right)^{1/2}\Bigg\} \tag{3.4.22}$$

$$\leq C\lambda\|\hat{V}\|_1 k_{\mathrm{F}}\|(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|\Big\{M^{-\frac{1}{2}}N^{\frac{1}{3}}\|(\mathcal{N}+1)^{\frac{1}{2}}e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\| \tag{3.4.23}$$

$$+k_{\mathrm{F}}MN^{-\frac{2}{3}+\delta}\|(\mathcal{N}+1)^{\frac{3}{2}}e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|\Big\}$$

$$\leq C\lambda k_{\mathrm{F}}M^{-\frac{1}{2}}N^{\frac{1}{3}}\|(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|\Big\{\|(\mathcal{N}+1)^{\frac{1}{2}}e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\| \tag{3.4.24}$$

$$+k_{\mathrm{F}}M^{\frac{3}{2}}N^{-1+\delta}\|(\mathcal{N}+1)^{\frac{3}{2}}e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|\Big\}. \tag{3.4.25}$$

Thus, we derive

$$\begin{aligned}
&\partial_t\|(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|\\
\leq& C\lambda k_{\mathrm{F}}M^{-\frac{1}{2}}N^{\frac{1}{3}}\left(\|(\mathcal{N}+1)e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|+k_{\mathrm{F}}M^{\frac{3}{2}}N^{-1+\delta}\|(\mathcal{N}+1)^2e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|\right)\\
&+C\lambda k_{\mathrm{F}}N^{\frac{1}{3}}\left(\|(\mathcal{N}+1)^{\frac{1}{2}}e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|+k_{\mathrm{F}}M^{\frac{3}{2}}N^{-1+\delta}\|(\mathcal{N}+1)^{\frac{3}{2}}e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|\right) &(3.4.26)\\
\leq& C^2k_{\mathrm{F}}N^{\frac{1}{3}}\left(e^{C\lambda k_{\mathrm{F}}t}\|(\mathcal{N}+3)\psi\|+k_{\mathrm{F}}M^{\frac{3}{2}}N^{-1+\delta}e^{2C\lambda k_{\mathrm{F}}t}\|(\mathcal{N}+3)^2\psi\|\right)\\
&+C\lambda k_{\mathrm{F}}N^{\frac{1}{3}}\left(e^{\frac{1}{2}C\lambda k_{\mathrm{F}}t}\|(\mathcal{N}+3)^{\frac{1}{2}}\psi\|+k_{\mathrm{F}}M^{\frac{3}{2}}N^{-1+\delta}e^{\frac{3}{2}C\lambda k_{\mathrm{F}}t}\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\|\right) &(3.4.27)
\end{aligned}$$

where we used the Grönwall bound from Proposition 62 in the last inequality. Now integrating over t yields

$$\begin{aligned}
&\|(\mathbb{H}_0-\mathbb{D}_B)e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|-\|(\mathbb{H}_0-\mathbb{D}_B)\psi\|\\
\leq& C^2\lambda^{-1}M^{-\frac{1}{2}}N^{\frac{1}{3}}\left(e^{C\lambda k_{\mathrm{F}}t}-1\right)\left(\|(\mathcal{N}+3)\psi\|+k_{\mathrm{F}}M^{\frac{3}{2}}N^{-1+\delta}\|(\mathcal{N}+3)^2\psi\|\right)\\
&+CM^{-\frac{1}{2}}N^{\frac{1}{3}}\left(e^{C\lambda k_{\mathrm{F}}t}-1\right)\left(\|(\mathcal{N}+3)^{\frac{1}{2}}\psi\|+k_{\mathrm{F}}M^{\frac{3}{2}}N^{-1+\delta}\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\|\right) &(3.4.28)
\end{aligned}$$

which corresponds to the desired result. ∎

We are now ready to give the proof of the main theorem.

Proof of of Theorem 51. We employ Duhamel's formula for $t\mapsto e^{i\mathbb{H}t}Re^{-i\mathbb{H}^{\mathrm{eff}}t}$:

$$\|R^*e^{-i\mathbb{H}t}R\psi-e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\|=\|R\psi-e^{i\mathbb{H}t}Re^{-i\mathbb{H}^{\mathrm{eff}}t}\psi\| \tag{3.4.29}$$

$$=\|\int_0^t\mathrm{d}s\;e^{i\mathbb{H}s}\left(\mathbb{H}R-R\mathbb{H}^{\mathrm{eff}}\right)e^{-i\mathbb{H}^{\mathrm{eff}}s}\psi\| \tag{3.4.30}$$

$$\leq \int_0^t \mathrm{d}s \, \|(R^*\mathbb{H}R - \mathbb{H}^{\text{eff}})e^{-i\mathbb{H}^{\text{eff}}s}\psi\| \tag{3.4.31}$$

where we used that R is unitary.

From our considerations in (3.2.12) it follows that

$$\begin{aligned} R^*\mathbb{H}R - \mathbb{H}^{\text{eff}} &= -\beta\Delta_y + \mathbb{H}_0 + b^*(\tilde{h}_y) + b(\tilde{h}_y) + E_N^{\text{pw}} + \mathcal{E} - \mathbb{H}^{\text{eff}} \\ &= \mathbb{H}_0 - \mathbb{D}_B + b^*(\tilde{h}_y) - c^*(h_y) + b(\tilde{h}_y) - c(h_y) + \mathcal{E}. \end{aligned} \tag{3.4.32}$$

The interaction terms can be approximated in the sense of Lemma 87:

$$\begin{aligned} \| \left(b^*(\tilde{h}_y) - c^*(h_y)\right)\psi\| &\leq \lambda \sum_{k\in\Gamma} |\hat{V}(k)| \, \|\big(b(k) - \sum_{\alpha\in\mathcal{I}_k} n_\alpha(k)c_\alpha(k)\big)\psi\| \\ &\leq C\lambda N^{\frac{1}{3}}\|\hat{V}\|_1(N^{-\frac{\delta}{2}} + N^{-\frac{1}{6}}M^{\frac{1}{4}})\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\|. \end{aligned}$$

Therefore, by combining the bounds from Proposition 63 with $(\mathbb{H}_0 - \mathbb{D}_B)\psi = 0$, Lemma 88 with $\|\hat{V}\|_1 < C$ and Proposition 62 we obtain

$$\begin{aligned} &\|R^*e^{-i\mathbb{H}t}R\psi - e^{-i\mathbb{H}^{\text{eff}}t}\psi\| \\ &\leq \int_0^t \mathrm{d}s \, \|(\mathbb{H}_0 - \mathbb{D}_B)e^{-i\mathbb{H}^{\text{eff}}s}\psi\| + \int_0^t \mathrm{d}s \, \|\mathcal{E}e^{-i\mathbb{H}^{\text{eff}}s}\psi\| \\ &\quad + \int_0^t \mathrm{d}s \, \| \left(b^*(\tilde{h}_y) - c^*(h_y)\right) e^{-i\mathbb{H}^{\text{eff}}s}\psi\| + \int_0^t \mathrm{d}s \, \| \left(b(\tilde{h}_y) - c(h_y)\right) e^{-i\mathbb{H}^{\text{eff}}s}\psi\| \\ &\leq C(1+\lambda^{-1})N^{\frac{1}{3}} \left(M^{-\frac{1}{2}}\|(\mathcal{N}+3)\psi\| + k_{\mathrm{F}}MN^{-1+\delta}\|(\mathcal{N}+3)^2\psi\|\right) \int_0^t \mathrm{d}s \left(e^{C\lambda k_{\mathrm{F}}t} - 1\right) \\ &\quad + C\lambda \int_0^t \mathrm{d}s \, \|\mathcal{N}e^{-i\mathbb{H}^{\text{eff}}s}\psi\| \\ &\quad + C\lambda N^{\frac{1}{3}}(N^{-\frac{\delta}{2}} + N^{-\frac{1}{6}}M^{\frac{1}{4}}) \int_0^t \mathrm{d}s \, \|(\mathcal{N}+1)^{\frac{1}{2}}e^{-i\mathbb{H}^{\text{eff}}s}\psi\| \\ &\leq C(1+\lambda^{-1})\lambda^{-1}N^{\frac{1}{3}} \left(k_{\mathrm{F}}^{-1}M^{-\frac{1}{2}}\|(\mathcal{N}+3)\psi\| + MN^{-1+\delta}\|(\mathcal{N}+3)^2\psi\|\right) \left(e^{C\lambda k_{\mathrm{F}}t} - \lambda k_{\mathrm{F}}t - 1\right) \\ &\quad + Ck_{\mathrm{F}}^{-1}\|(\mathcal{N}+3)\psi\| \left(e^{C\lambda k_{\mathrm{F}}t} - 1\right) \\ &\quad + C(N^{-\frac{\delta}{2}} + N^{-\frac{1}{6}}M^{\frac{1}{4}})\|(\mathcal{N}+3)^{\frac{1}{2}}\psi\| \left(e^{C\lambda k_{\mathrm{F}}t} - 1\right) \\ &\leq C\tilde{C}\|(\mathcal{N}+3)^2\psi\| \left(e^{C\lambda k_{\mathrm{F}}t} - 1\right). \end{aligned} \tag{3.4.33}$$

The prefactor is given by

$$\tilde{C} = \max\left\{(1+\lambda^{-1})\lambda^{-1}k_{\mathrm{F}}^{-1}N^{\frac{1}{3}}\left(M^{-\frac{1}{2}} + k_{\mathrm{F}}MN^{-1+\delta}\right), k_{\mathrm{F}}^{-1}, (N^{-\frac{\delta}{2}} + N^{-\frac{1}{6}}M^{\frac{1}{4}})\right\} \tag{3.4.34}$$

where we optimize over M and δ with $\lambda \geq k_{\mathrm{F}}^{-\frac{1}{6}}$ to obtain the desired result. ∎

3.5 Proof of Theorem 56

Properties of almost-bosonic coherent states We first show the following rigorous properties of the Weyl operator which was introduced in Definition 53:

Lemma 65 (Approximate shift property). *Let $\eta, \xi \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ and $W_\sigma(\eta) := e^{\sigma B} := e^{\sigma c^*(\eta) - \sigma c(\eta)}$ for all $\sigma \in [0,1]$, then it holds*

$$W_\sigma(\eta)^* c(\xi) W_\sigma(\eta) = c(\xi) + \sigma\langle\xi,\eta\rangle + \langle\xi,\mathcal{R}^\sigma\rangle_\Gamma,$$
$$W_\sigma(\eta)^* c^*(\xi) W_\sigma(\eta) = c^*(\xi) + \sigma\langle\eta,\xi\rangle + \langle\mathcal{R}^\sigma,\xi\rangle_\Gamma,$$

with $\langle\cdot,\cdot\rangle_\Gamma : \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k) \times \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k) \to \mathbb{C}$ given by

$$\langle\xi,\mathcal{R}^\sigma\rangle_\Gamma := \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \xi_\alpha(k)^* \mathcal{R}^\sigma_\alpha(k) \tag{3.5.1}$$

and a σ-dependent error term $\mathcal{R}^\sigma_\alpha(k) := \int_0^\sigma \mathrm{d}\tau\, e^{-\tau B}\left(\sum_{l\in\Gamma}\eta_\alpha(l)\mathcal{E}_\alpha(l,k)\right)e^{\tau B}$.

Remark 66. We will give an estimate for $\mathcal{R}$ to show that this term corresponds indeed to a small error. Note that it holds $\mathcal{E}_\alpha(k,l) = \mathcal{E}_\alpha(l,k)^*$ for all $l,k \in \Gamma$ and $\alpha \in \mathcal{I}_k \cap \mathcal{I}_l$ from Lemma 83 and therefore

$$\mathcal{R}^\sigma_\gamma(l)^*\xi_\gamma(l) = \int_0^\sigma \mathrm{d}\tau\, e^{-\tau B}\left(\sum_{k\in\Gamma}\mathcal{E}_\gamma(l,k)\overline{\eta_\gamma(k)}\xi_\gamma(l)\right)e^{\tau B}, \tag{3.5.2}$$

$$\overline{\xi_\gamma(l)}\mathcal{R}^\sigma_\gamma(l) = \int_0^\sigma \mathrm{d}\tau\, e^{-\tau B}\left(\sum_{k\in\Gamma}\overline{\xi_\gamma(l)}\eta_\gamma(k)\mathcal{E}_\gamma(k,l)\right)e^{\tau B}. \tag{3.5.3}$$

For $\xi = \eta$ the above equations coincide, i.e.

$$\langle\eta,\mathcal{R}^\sigma\rangle_\Gamma = \langle\mathcal{R}^\sigma,\eta\rangle_\Gamma \tag{3.5.4}$$

from which it follows immediately that $\big(c^*(\eta) - c(\eta)\big)W_\sigma(\eta) = W_\sigma(\eta)\big(c^*(\eta) - c(\eta)\big)$, i.e. $[B, W_\sigma(\eta)] = 0$.

Proof. We observe that $W_\sigma(\eta) = e^{\sigma B}$ defines a strongly continuous one-parameter semigroup. Thus we can define a derivative and make use of Duhamel's formula of the form

$$e^{-\sigma B}c_\gamma(l)e^{\sigma B} = c_\gamma(l) + \int_0^\sigma \mathrm{d}\tau\, e^{-\tau B}[c_\gamma(l), B]e^{\tau B}. \tag{3.5.5}$$

The interested reader is referred to [Paz83, EN06] where definitions and properties of operator derivatives are discussed. The desired statement follows with the CCR as stated in (3.2.22)

$$[c_\gamma(l), e^{\sigma B}] = e^{\sigma B}\int_0^\sigma \mathrm{d}\tau\, e^{-\tau B}[c_\gamma(l), B]e^{\tau B}$$

$$= e^{\sigma B}\int_0^\sigma \mathrm{d}\tau\ e^{-\tau B}\left(\eta_\gamma(l) + \sum_{k\in\Gamma}\eta_\gamma(k)\mathcal{E}_\gamma(k,l)\right) e^{\tau B}$$
$$= \sigma\eta_\gamma(l)e^{\sigma B} + e^{\sigma B}\mathcal{R}^\sigma_\gamma(l) \tag{3.5.6}$$

Since $c(\xi) \equiv \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\overline{\xi_\alpha(k)}c_\alpha(k)$ is linear the result follows from the above identity. ■

The following statement shows that the number of particles in the state $W(\eta)\phi\otimes\Omega$ corresponds to a random variable with expectation approximately being $2\|\eta\|^2$.

Proposition 67 (Expectation of the number operator)**.** *Let $\zeta = \phi\otimes\Omega \in L^2(\Lambda, \mathrm{d}y)\otimes\mathcal{H}_N^-$, then it holds for all $\eta \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$*

$$\langle W(\eta)\zeta, \mathcal{N}W(\eta)\zeta\rangle = 2\|\eta\|^2 + 4\int_0^1 \mathrm{d}\sigma\langle\zeta, \langle\eta, \mathcal{R}^\sigma\rangle_\Gamma\zeta\rangle.$$

Proof. Using $W^*W = \mathrm{id}$ yields for all $\zeta \in L^2(\Lambda, \mathrm{d}y)\otimes\mathcal{H}_N^-$ with $\|\zeta\| = 1$

$$\langle W(\eta)\zeta, \mathcal{N}W(\eta)\zeta\rangle = \langle W(\eta)\zeta, [\mathcal{N}, W(\eta)]\zeta\rangle + \langle\zeta, \mathcal{N}\zeta\rangle. \tag{3.5.7}$$

We use Duhamel's formula to calculate

$$[\mathcal{N}, e^B] = e^B\int_0^1 \mathrm{d}\tau\ e^{-\tau B}[\mathcal{N}, B]e^{\tau B}$$
$$= 2e^B\int_0^1 \mathrm{d}\tau\ e^{-\tau B}\left(c^*(\eta) + c(\eta)\right)e^{\tau B}$$
$$= 2e^B\int_0^1 \mathrm{d}\tau\ e^{-\tau B}\left(B + 2c(\eta)\right)e^{\tau B}$$
$$= 2e^B B + 4e^B\int_0^1 \mathrm{d}\tau\ e^{-\tau B}c(\eta)e^{\tau B} \tag{3.5.8}$$

where we used (3.2.24). Therefore

$$\langle W(\eta)\zeta, [\mathcal{N}, W(\eta)]\zeta\rangle = 2\langle\zeta, B\zeta\rangle + 4\int_0^1 \mathrm{d}\tau\ \langle e^{\tau B}\zeta, c(\eta)e^{\tau B}\zeta\rangle$$
$$= 2\langle\zeta, B\zeta\rangle + 2\|\eta\|^2 + 4\langle\zeta, c(\eta)\zeta\rangle + 4\int_0^1 \mathrm{d}\tau\langle\zeta, \langle\eta, \mathcal{R}^\tau\rangle_\Gamma\zeta\rangle \tag{3.5.9}$$

where we used the shift property Lemma 65 and that $e^{\tau B}$ is unitary

$$\langle e^{\tau B}\zeta, c(\eta)e^{\tau B}\zeta\rangle = \langle e^{\tau B}\zeta, [c(\eta), e^{\tau B}]\zeta\rangle + \langle\zeta, c(\eta)\zeta\rangle$$
$$= \tau\|\eta\|^2 + \langle\zeta, \langle\eta, \mathcal{R}^1\rangle_\Gamma\zeta\rangle + \langle\zeta, c(\eta)\zeta\rangle. \tag{3.5.10}$$

Inserting (3.5.9) and (3.5.8) into (3.5.7) we obtain

$$\langle W(\eta)\zeta, \mathcal{N} W(\eta)\zeta\rangle = 2\|\eta\|^2 + 2\langle\zeta, (c^*(\eta) + c(\eta))\,\zeta\rangle + \langle\zeta, \mathcal{N}\zeta\rangle + 4\int_0^1 \mathrm{d}\tau \langle\zeta, \langle\eta, \mathcal{R}^\tau\rangle_\Gamma \zeta\rangle. \tag{3.5.11}$$

The desired result holds for $\zeta = \phi \otimes \Omega$ since $c(\eta)\phi \otimes \Omega = 0$.

■

For later purposes, we can bound the expectation of the number operator in the following way:

Proposition 68 (Stability of the number operator)**.** *Let $\eta \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$. There exists a constant $C > 0$ such that it holds for all $\tau \in [-1, 1], n \in \mathbb{N}$ and $\zeta \in L^2(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$*

$$\langle e^{\tau B}\zeta, (\mathcal{N} + 1)^n e^{\tau B}\zeta\rangle \leq e^{C\|\eta\|n|\tau|}\langle\zeta, (\mathcal{N} + 3)^n\zeta\rangle.$$

Proof. The proof works analogously to the proof of Proposition 62 with a Grönwall argument and B instead of $\mathbb{H}^{\mathrm{eff}}$ which is given later. Note that

$$[\mathcal{N}, B] = [\mathcal{N}, c^*(\eta) - c(\eta)] = 2\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \eta_\alpha(k)\big(c_\alpha^*(k) + c_\alpha(k)\big) \tag{3.5.12}$$

where we again used (3.2.24). The result is than obtained by using the same estimates with $\|\eta\|$ taking the role of $\|h_y\|$. ■

Lemma 69. *Let $\eta_t \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ be differentiable in t with derivative $\dot\eta_t \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ for all $t \in \mathbb{R}$. Then it holds for all $t \in \mathbb{R}$*

$$\partial_t W(\eta_t) = (c^*(\dot\eta_t) - c(\dot\eta_t) + i\mathrm{Im}\langle\dot\eta_t, \eta_t\rangle)\, W(\eta_t) + 2i\int_0^1 \mathrm{d}\tau\; W_{(1-\tau)}(\eta_t)\mathrm{Im}\langle\dot\eta_t, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_t)$$

with the shorthand notation $\mathrm{Im}\langle A, B\rangle_\Gamma \coloneqq -\frac{i}{2}\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}(A_\alpha^*(k)B_\alpha(k) - B_\alpha^*(k)A_\alpha(k))$.

Proof. For arbitrary $s \in \mathbb{R}$ it holds

$$\begin{aligned} W(\eta_s)^*\partial_s W(\eta_s) &= e^{-\tau B_s}\partial_s e^{\tau B_s}\big|_{\tau=0}^{\tau=1} = \int_0^1 \mathrm{d}\tau\; \partial_\tau\left(e^{-\tau B_s}\partial_s e^{\tau B_s}\right) \\ &= \int_0^1 \mathrm{d}\tau\left(-B_s e^{-\tau B_s}\partial_s e^{\tau B_s} + e^{-\tau B_s}\partial_s\partial_\tau e^{\tau B_s}\right) &\quad (3.5.13) \\ &= \int_0^1 \mathrm{d}\tau\left(-B_s e^{-\tau B_s}\partial_s e^{\tau B_s} + e^{-\tau B_s}\partial_s\left(B_s e^{\tau B_s}\right)\right) &\quad (3.5.14) \end{aligned}$$

$$= \int_0^1 \mathrm{d}\tau \left(-B_s e^{-\tau B_s}\partial_s e^{\tau B_s} + e^{-\tau B_s}\left(\partial_s B_s\right) e^{\tau B_s} + e^{-\tau B_s} B_s \partial_s e^{\tau B_s}\right) \quad (3.5.15)$$

$$= \int_0^1 \mathrm{d}\tau \ e^{-\tau B_s}\left(\partial_s B_s\right) e^{\tau B_s}. \quad (3.5.16)$$

Thus

$$\begin{aligned}\partial_s W(\eta_s) &= \int_0^1 \mathrm{d}\tau \ e^{(1-\tau)B_s}\left(\partial_s B_s\right) e^{\tau B_s} \\ &= \int_0^1 \mathrm{d}\tau \ \left(\partial_s B_s\right) e^{(1-\tau)B_s} e^{\tau B_s} + \int_0^1 \mathrm{d}\tau \ \left[e^{(1-\tau)B_s}, \partial_s B_s\right] e^{\tau B_s} &(3.5.17)\\ &= \left(\partial_s B_s\right) e^{B_s} + \int_0^1 \mathrm{d}\tau \ \left[e^{(1-\tau)B_s}, \partial_s B_s\right] e^{\tau B_s}. &(3.5.18)\end{aligned}$$

With

$$\partial_s B_s = \partial_s \left\{c^*(\eta_s) - c(\eta_s)\right\} = c^*(\dot\eta_s) - c(\dot\eta_s) \quad (3.5.19)$$

from the linearity of $c(\eta_s)$ it follows

$$\begin{aligned}&\left[e^{(1-\tau)B_s}, \partial_s B_s\right] e^{\tau B_s} \\ &= \left(\left[W_{(1-\tau)}(\eta_s), c^*(\dot\eta_s)\right] - \left[W_{(1-\tau)}(\eta_s), c(\dot\eta_s)\right]\right) W_\tau(\eta_s) &(3.5.20)\\ &= W_{(1-\tau)}(\eta_s)\Big((1-\tau)\langle\dot\eta_s, \eta_s\rangle - (1-\tau)\langle\eta_s, \dot\eta_s\rangle &(3.5.21)\\ &\qquad\qquad + \langle\dot\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma - \langle\mathcal{R}^{1-\tau}, \dot\eta_s\rangle_\Gamma\Big) W_\tau(\eta_s) &(3.5.22)\\ &= W(\eta_s)(1-\tau)2i\mathrm{Im}\langle\dot\eta_s, \eta_s\rangle + 2iW_{(1-\tau)}(\eta_s)\mathrm{Im}\langle\dot\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s). &(3.5.23)\end{aligned}$$

Inserting the above identity yields

$$\partial_s W(\eta_s) = \left(c^*(\dot\eta_s) - c(\dot\eta_s) + i\mathrm{Im}\langle\dot\eta_s, \eta_s\rangle\right) W(\eta_s) + 2i\int_0^1 \mathrm{d}\tau \ W_{(1-\tau)}(\eta_s)\mathrm{Im}\langle\dot\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s). \quad (3.5.24)$$

■

Proof of the main theorem First, we collect some useful observations on the function η_s in the form of the following two lemmata. We postpone the proofs to the end of the section in order to concentrate on presenting the proof of the main result.

Lemma 70. *Let η_s be defined as in (3.3.15) for $N^{2\delta} \ll M \ll N^{\frac{2}{3}-2\delta}$, then it holds for all $s \in \mathbb{R}$*

$$\|\eta_s\|^2 = \pi\lambda^2 \sum_{k\in\Gamma} \frac{\hat V(k)^2}{|k|}\left(\log(2k_{\mathrm{F}}|k|s) - \mathrm{Ci}(2k_{\mathrm{F}}|k|s) + \gamma\right)$$

$$\times \left\{1 + g(k_{\mathrm{F}}|k|s)\ \mathcal{O}\left(M^{\frac{1}{2}} N^{-\frac{1}{3}+\delta} + N^{-\delta}\right)\right\}.$$

where γ is the Euler-Mascheroni constant and $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a function independent of k_{F} and monotonically increasing.

Furthermore, define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$(y, x) \mapsto f_y(x) := \min\{e^{\sqrt{\pi}\|\hat{V}(\cdot)^{1/2}\|_2 yx}, e^{\sqrt{2\pi}\|\hat{V}\|_2(\log(18)+\frac{1}{9})y} e^{\frac{\sqrt{8\pi}}{9}\|\hat{V}\|_2 yx}\}.$$

Then there exists a $C > 0$ independent of k_{F} such that for $c_0 > 0$ and k_{F} sufficiently large

$$\|\eta_s\| \leq \log\left(f_1(\lambda k_{\mathrm{F}} s)\right), \tag{3.5.25}$$

$$e^{c_0\|\eta_s\|} \leq f_{c_0}(\lambda k_{\mathrm{F}} s). \tag{3.5.26}$$

Remark 71. Note that f_y is for all $y \geq 0$ monotonically increasing with $f_y(0) = 1$.

The previous statement is useful when combined with the following estimate:

Lemma 72. *There exists a constant $C > 0$ only depending on V such that it holds for all $s \in \mathbb{R}$, $n \in \mathbb{N}$, $\psi \in \mathcal{F}$*

(i). $\sum_{k\in\Gamma} \||k|^n \eta_s(k)\|_{l^2} \leq C\|\eta_s\|$,

(ii). $\langle \eta_s, |k|^n \eta_s\rangle \leq C\|\eta_s\|^2$,

(iii). $\|c^*(|k|^n \eta_s)\psi\| \leq C\|\eta_s\|\ \|(\mathcal{N}+1)^{1/2}\psi\|$.

We will now give the proof of the second main theorem.

Proof of Theorem 56. We use the approach as sketched in Remark 59. Since the bosonic property holds only with an error, the equality (3.3.21) holds only approximately:

$$\begin{aligned}
\|e^{-i\mathbb{H}^{\mathrm{eff}}t}\psi - e^{iP(t)}W(\eta_t)\psi\| &= \|\psi - e^{i\mathbb{H}^{\mathrm{eff}}t} e^{-iE_N^{\mathrm{pw}}t} e^{i2\mathrm{Im}(\nu_t)} e^{-i\mathrm{Im}\int_0^t \mathrm{d}s\langle\dot{\eta}_s,\eta_s\rangle} W(\eta_t)\psi\| \\
&\leq \int_0^t \mathrm{d}s \|\left(\mathbb{H}^{\mathrm{eff}} - E_N^{\mathrm{pw}} + 2\mathrm{Im}(\dot{\nu}_s) - \mathrm{Im}\langle\dot{\eta}_s,\eta_s\rangle\right)W(\eta_s)\psi - i\partial_s W(\eta_s)\psi\| \\
&\leq \int_0^t \mathrm{d}s \|h_0 W(\eta_s)\phi\otimes\Omega\| + \|\mathrm{Error}_1\| + \|\mathrm{Error}_2\|.
\end{aligned} \tag{3.5.27}$$

We will first estimate the error terms and then subsequently treat the h_0 term in a separate lemma. We give an explicit expression for the first error term using Lemma 65 on the approximate shift property applied to $c_\alpha(k)$ in the $c^*c(\epsilon), c(h_y)$ and $c(i\dot{\eta}_s)$ terms. Thus, the error of (3.3.21) is given by

$$\mathrm{Error}_1 := \int_0^t \mathrm{d}s \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \left(\epsilon_\alpha(k)c^*_\alpha(k) + (1-e^{is\epsilon_\alpha(k)})\overline{(h_y)_\alpha(k)}\right) W(\eta_s)\mathcal{R}^1_\alpha(k)\psi. \tag{3.5.28}$$

The second error term is given by Lemma 69 on the time derivative of the almost-bosonic Weyl operator and therefore

$$\begin{aligned}\mathrm{Error}_2 :=& -2i\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}\tau\; W_{1-\tau}(\eta_s)\mathrm{Im}\langle \dot{\eta}_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s)\psi \\ \equiv& 2\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}\tau e^{(1-\tau)B}\Bigg(\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} e^{is\epsilon_\alpha(k)}\overline{(h_y)_\alpha(k)}\mathcal{R}^{1-\tau}_\alpha(k) \\ &\qquad - \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\mathcal{R}^{1-\tau}_\alpha(k)^* e^{-is\epsilon_\alpha(k)}(h_y)_\alpha(k)\Bigg)e^{\tau B}\psi. \end{aligned}\tag{3.5.29}$$

Firstly, we show that the term $\mathcal{R}^\lambda_\alpha(k)\psi = \int_0^\sigma \mathrm{d}\tau\; e^{-\tau B}\left(\sum_{l\in\Gamma}\eta_\alpha(l)\mathcal{E}_\alpha(l,k)\right)e^{\tau B}\psi$ as defined in Lemma 65 constitutes indeed a small error. We estimate

$$\begin{aligned}&\sum_{l\in\Gamma}\sum_{\gamma\in\mathcal{I}_l}\|\mathcal{R}^1_\gamma(l)\psi\|^2 \\ &\le \sum_{l\in\Gamma}\sum_{\gamma\in\mathcal{I}_l\cap\mathcal{I}_k}\left(\sum_{k\in\Gamma}|\eta_\gamma(k)|\int_0^1 \mathrm{d}\tau\|\mathcal{E}_\gamma(k,l)e^{\tau B}\psi\|\right)^2 \\ &\le C\left(MN^{-\frac{2}{3}+\delta}(e^{C\|\eta_s\|}-1)\|(\mathcal{N}+3)\psi\|\right)^2 \end{aligned}\tag{3.5.30}$$

where we used $\sum_{\alpha\in\mathcal{I}_k}\left(\sum_{k\in\Gamma}|\eta_\alpha(k)|\right)^2 \le \sum_{k',k\in\Gamma}\|\eta(k)\|_{l^2}\|\eta(k')\|_{l^2} \le C\|\eta\|^2$ by Lemma 70 and

$$\int_0^1 \mathrm{d}\tau\|\mathcal{E}_\gamma(k,l)e^{\tau B}\psi\| \tag{3.5.31}$$

$$\le \int_0^1 \mathrm{d}\tau\langle e^{\tau B}\psi, |\mathcal{E}_\gamma(k,l)|^2 e^{\tau B}\psi\rangle^{1/2} \le CMN^{-\frac{2}{3}+\delta}\int_0^1 \mathrm{d}\tau\langle e^{\tau B}\psi, \mathcal{N}^2 e^{\tau B}\psi\rangle^{1/2} \tag{3.5.32}$$

$$\le CMN^{-\frac{2}{3}+\delta}\int_0^1 \mathrm{d}\tau e^{C\|\eta_s\|\tau}\|(\mathcal{N}+3)\psi\| \le C\|\eta_s\|^{-1}(e^{C\|\eta_s\|}-1)MN^{-\frac{2}{3}+\delta}\|(\mathcal{N}+3)\psi\| \tag{3.5.33}$$

which follows from $e^{\tau B}$ is unitary in the first inequality, Lemma 83 in the second inequality and Proposition 68 in the third inequality.

Secondly, we estimate

$$
\begin{aligned}
&\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\|c^*_\alpha(k)e^B\mathcal{R}^1_\alpha(k)\psi\| \\
&\leq \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\int_0^1 \mathrm{d}\tau\|c^*_\alpha(k)e^{(1-\tau)B}\left(\sum_{l\in\Gamma}\eta_\alpha(l)\mathcal{E}_\alpha(l,k)\right)e^{\tau B}\psi\| \\
&\leq \int_0^1 \mathrm{d}\tau\sum_{k,l\in\Gamma}\sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}|\eta_\alpha(l)|\ \|c^*_\alpha(k)\mathcal{E}_\alpha(l,k)e^{\tau B}\psi\| \\
&\quad + \int_0^1 \mathrm{d}\tau\sum_{k,l\in\Gamma}\sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}|\eta_\alpha(l)|\ \|[c^*_\alpha(k),e^{(1-\tau)B}]\mathcal{E}_\alpha(l,k)e^{\tau B}\psi\| \\
&\leq C\|\eta_s\|MN^{-\frac{2}{3}+\delta}\int_0^1 \mathrm{d}\tau\Big(\|(\mathcal{N}+1)^{\frac{3}{2}}e^{\tau B}\psi\|^2\Big)^{1/2} + C\|\eta_s\|^2MN^{-\frac{2}{3}+\delta}\int_0^1 \mathrm{d}\tau\ (1-\tau)\|\mathcal{N}e^{\tau B}\psi\| \\
&\quad + \sum_{k\in\Gamma}\|\eta_s\|\int_0^1 \mathrm{d}\tau\Big(\sum_{l\in\Gamma}\sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\|\mathcal{R}^1_\alpha(k)\mathcal{E}_\alpha(l,k)e^{\tau B}\psi\|^2\Big)^{1/2} \\
&\leq CMN^{-\frac{2}{3}+\delta}(e^{C\|\eta_s\|}-1)\|(\mathcal{N}+3)^{\frac{3}{2}}\psi\| + CMN^{-\frac{2}{3}+\delta}(e^{C\|\eta_s\|}-C\|\eta_s\|-1)\|(\mathcal{N}+3)\psi\| \\
&\quad + CM^{\frac{3}{2}}N^{-\frac{2}{3}+\delta}(e^{C\|\eta_s\|}-1)^2\|(\mathcal{N}+3)^2\psi\| \\
&\leq CM^{\frac{3}{2}}N^{-\frac{2}{3}+\delta}(e^{C\|\eta_s\|}-1)\|(\mathcal{N}+3)^2\psi\|.
\end{aligned}
\tag{3.5.34}
$$

where we used $[c^*_\alpha(k),e^{\sigma B}] = -\lambda\eta_\alpha(k)e^{\sigma B} - e^{\sigma B}\mathcal{R}^1_\alpha(k)$ from Lemma 65, the Cauchy-Schwarz inequality for the α-summation, Lemma 83 in the third inequality and in the fourth inequality we used Proposition 68 and (3.5.30).

In total by combining (3.5.30) and (3.5.34) we end up with the following estimate

$$
\begin{aligned}
\|\mathrm{Error}_1\| &\leq \int_0^t \mathrm{d}s\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\|\left\{\epsilon_\alpha(k)c^*_\alpha(k)+(1-e^{is\epsilon_\alpha(k)})\overline{(h_y)_\alpha(k)}\right\}e^B\mathcal{R}^1_\alpha(k)\psi\| \\
&\leq Ck_{\mathrm{F}}\int_0^t \mathrm{d}s\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\|c^*_\alpha(k)e^B\mathcal{R}^1_\alpha(k)\psi\| \\
&\quad + C\lambda\int_0^t \mathrm{d}s\sum_{k\in\Gamma}|\hat{V}(k)|\sum_{\alpha\in\mathcal{I}_k}\|n_\alpha(k)e^B\mathcal{R}^1_\alpha(k)\psi\| \\
&\leq Ck_{\mathrm{F}}\int_0^t \mathrm{d}s\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\|c^*_\alpha(k)e^B\mathcal{R}^1_\alpha(k)\psi\| \\
&\quad + C\lambda k_{\mathrm{F}}\int_0^t \mathrm{d}s\|\hat{V}\|_2\Big(\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\|\mathcal{R}^1_\alpha(k)\psi\|^2\Big)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C k_{\mathrm{F}} M N^{-\frac{2}{3}+\delta} \int_0^t \mathrm{d}s \; (e^{C\|\eta_s\|} - 1) \|(\mathcal{N}+3)^2 \psi\| \\
&\quad + C\lambda k_{\mathrm{F}} M N^{-\frac{2}{3}+\delta} \int_0^t \mathrm{d}s \; (e^{C\|\eta_s\|} - 1) \|(\mathcal{N}+3) \psi\| \\
&\leq C \; (f_C(\lambda k_{\mathrm{F}} t) - 1)(\lambda+1) k_{\mathrm{F}} t M N^{-\frac{2}{3}+\delta} \|(\mathcal{N}+3)^2 \psi\|. \qquad (3.5.35)
\end{aligned}
$$

where we used (82) and e^B unitary in the third inequality and Lemma 70 in the last line. Using $\psi = \phi \otimes \Omega$ we obtain the desired bound.

Similarly, we obtain an estimate for the second error term using Cauchy-Schwarz, (3.5.30) and Proposition 68

$$
\begin{aligned}
&\|\mathrm{Error}_2\| \\
&\leq 2 \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}\tau \sum_{k \in \Gamma} \sum_{\alpha \in \mathcal{I}_k} \Big(\|\overline{(h_y)_\alpha(k)} \mathcal{R}_\alpha^{1-\tau}(k) e^{\tau B} \psi\| + \|\mathcal{R}_\alpha^{1-\tau}(k)^* (h_y)_\alpha(k) e^{\tau B} \psi\| \Big) \\
&\leq 2\lambda \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}\tau \sum_{k \in \Gamma} \sum_{\alpha \in \mathcal{I}_k} |\hat{V}(k) n_\alpha(k)| \Big(\|\mathcal{R}_\alpha^{1-\tau}(k) e^{\tau B} \psi\| + \|\mathcal{R}_\alpha^{1-\tau}(k)^* e^{\tau B} \psi\| \Big) \\
&\leq C \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}\tau \; \lambda k_{\mathrm{F}} \|\hat{V}\|_2 \Bigg(\Big(\sum_{k \in \Gamma} \sum_{\alpha \in \mathcal{I}_k} \|\mathcal{R}_\alpha^{1-\tau}(k) e^{\tau B} \psi\|^2 \Big)^{1/2} + \Big(\sum_{k \in \Gamma} \sum_{\alpha \in \mathcal{I}_k} \|\mathcal{R}_\alpha^{1-\tau}(k)^* e^{\tau B} \psi\|^2 \Big)^{1/2} \Bigg) \\
&\leq C\lambda k_{\mathrm{F}} M N^{-\frac{2}{3}+\delta} \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}\tau \; (e^{C\|\eta_s\|(1-\tau)} - 1) \|(\mathcal{N}+3) e^{\tau B} \psi\| \\
&\leq C\lambda k_{\mathrm{F}} M N^{-\frac{2}{3}+\delta} \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}\tau \; (e^{C\|\eta_s\|} - e^{C\|\eta_s\|\tau}) \|(\mathcal{N}+5) \psi\| \\
&\leq C (f_C(\lambda k_{\mathrm{F}} t) - 1) \lambda k_{\mathrm{F}} t M N^{-\frac{2}{3}+\delta} \|(\mathcal{N}+5) \psi\|. \qquad (3.5.36)
\end{aligned}
$$

Again, by using $\psi = \phi \otimes \Omega$ we obtain the desired bound.

With the subsequent Lemma 73, we can conclude with a bound on $h_0 = -\beta \Delta_y$ of the form

$$
\begin{aligned}
&\int_0^t \mathrm{d}s \|h_0 W(\eta_s) \phi \otimes \Omega\| \leq C\beta \int_0^t \mathrm{d}s \left\{ (\|\eta_s\| + \|\eta_s\|^4)(e^{C\|\eta_s\|} + 1) \right\} \\
&\leq C\beta t \left\{ \log\left[f_1(\lambda k_{\mathrm{F}} t)\right] + \log\left[f_1(\lambda k_{\mathrm{F}} t)\right]^4 \right\} \left\{ f_C(\lambda k_{\mathrm{F}} t) + 1 \right\} \qquad (3.5.37)
\end{aligned}
$$

where we used (3.5.25) and (3.5.26) in the second inequality. Together with (3.5.35) and (3.5.36) inserted in (3.5.27), we obtain the desired result. ■

Lemma 73. *Under the assumptions of Theorem 56, it holds that for all $t \geq 0$*

$$
\|\Delta_y W(\eta_t) \phi \otimes \Omega\| \leq C(\|\eta_t\| + \|\eta_t\|^4)(e^{C\|\eta_t\|} + 1)
$$

for $C > 0$ independent of k_{F}.

Proof of Lemma 73. We explicitly calculate the action of the Laplacian on the coupled coherent state $W(\eta_s)\psi = e^B\psi$ i.e.

$$-\Delta_y W(\eta_s)\psi = -\big(\Delta_y W(\eta_s)\big)\psi - 2\beta\nabla_y W(\eta_s)\cdot\nabla_y\psi - W(\eta_s)\Delta_y\psi. \tag{3.5.38}$$

In total we expect, that all terms can be bounded by assumption on the initial condition on ϕ. We will first focus on the term $\Delta_y W(\eta_s)$. Recall that it holds

$$W(\eta_s)^*\partial_{y_i}W(\eta_s) = \int_0^1 \mathrm{d}\tau\; e^{-\tau B}\left(\partial_{y_i}B_s\right)e^{\tau B} \tag{3.5.39}$$

due to the same calculation as in the proof of Lemma 69. With

$$\partial_{y_i}B_s = \partial_{y_i}\left\{c^*(\eta_s) - c(\eta_s)\right\} = c^*(\partial_{y_i}\eta_s) - c(\partial_{y_i}\eta_s), \tag{3.5.40}$$

$$\partial_{y_i}\eta_s = \frac{e^{-is\epsilon_\alpha(k)}-1}{\epsilon_\alpha(k)}\lambda\hat{V}(k)n_\alpha(k)ik_ie^{iky} = ik_i\eta_s \tag{3.5.41}$$

it follows analogously to Lemma 69 that

$$\begin{aligned}\partial_{y_i}W(\eta_s) &= \int_0^1 \mathrm{d}\tau\; e^{(1-\tau)B_s}\left(\partial_{y_i}B_s\right)e^{\tau B_s}\\ &= \left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i\mathrm{Im}\langle\eta_s, ik_i\eta_s\rangle\right)W(\eta_s)\\ &\quad + 2i\int_0^1 \mathrm{d}\tau\; W_{1-\tau}(\eta_s)\mathrm{Im}\langle ik_i\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s).\end{aligned} \tag{3.5.42}$$

And repeating the differentiation with

$$\begin{aligned}\partial_{y_i}W_\tau(\eta_s) &= \tau\left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i\tau\mathrm{Im}\langle\eta_s, ik_i\eta_s\rangle\right)W_\tau(\eta_s)\\ &\quad + 2i\int_0^\tau \mathrm{d}\sigma\; W_{1-\sigma}(\eta_s)\mathrm{Im}\langle ik_i\eta_s, \mathcal{R}^{\tau-\sigma}\rangle_\Gamma W_\sigma(\eta_s)\end{aligned} \tag{3.5.43}$$

yields

$$\begin{aligned}&\Delta W(\eta_s) = \sum_{i=1}^3 \partial_{y_i}^2 W(\eta_s)\\ &= \left(c^*(k^2\eta_s) - c(k^2\eta_s) + i\mathrm{Im}\langle\eta_s, k^2\eta_s\rangle\right)W(\eta_s) + 2i\int_0^1 \mathrm{d}\tau\; W_{1-\tau}(\eta_s)\mathrm{Im}\langle k^2\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s)\\ &\quad + \sum_{i=1}^3\left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i\mathrm{Im}\langle\eta_s, ik_i\eta_s\rangle\right)\partial_{y_i}W(\eta_s)\end{aligned}$$

$$+\, 2i \sum_{i=1}^{3} \int_0^1 \mathrm{d}\tau \ \mathrm{Im}\langle ik_i\eta_s, \partial_{y_i} \left\{ W_{1-\tau}(\eta_s)\mathcal{R}_\alpha^{1-\tau}(k)W_\tau(\eta_s)\right\}\rangle_\Gamma \tag{3.5.44}$$

$$= \left(c^*(k^2\eta_s) - c(k^2\eta_s) + i\mathrm{Im}\langle \eta_s, k^2\eta_s\rangle\right) W(\eta_s) + 2i\int_0^1 \mathrm{d}\tau \ \mathrm{Im}\langle k^2\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s)$$
$$+ \sum_{i=1}^{3} \left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i\mathrm{Im}\langle \eta_s, ik_i\eta_s\rangle\right) \times$$
$$\times \left\{ \left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i\mathrm{Im}\langle \eta_s, ik_i\eta_s\rangle\right) W(\eta_s) + 2i\int_0^1 \mathrm{d}\tau \ \mathrm{Im}\langle ik_i\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s)\right\}$$
$$+\, 2i \sum_{i=1}^{3} \int_0^1 \mathrm{d}\tau \ \mathrm{Im}\langle ik_i\eta_s, \partial_{y_i} \left\{ W_{1-\tau}(\eta_s)\mathcal{R}_\alpha^{1-\tau}(k)W_\tau(\eta_s)\right\}\rangle_\Gamma \tag{3.5.45}$$
$$=: I_1 + I_2 + I_3 + I_4$$

with

$$I_1 := \left(c^*(k^2\eta_s) - c(k^2\eta_s) + i\mathrm{Im}\langle \eta_s, k^2\eta_s\rangle\right) W(\eta_s)$$
$$- \sum_{i=1}^{3} \left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i\mathrm{Im}\langle \eta_s, ik_i\eta_s\rangle\right)\left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i\mathrm{Im}\langle \eta_s, ik_i\eta_s\rangle\right) W(\eta_s), \tag{3.5.46}$$

$$I_2 := 2i\int_0^1 \mathrm{d}\tau \ W_{1-\tau}(\eta_s)\mathrm{Im}\langle k^2\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s), \tag{3.5.47}$$

$$I_3 := 2\sum_{i=1}^{3} \left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i\mathrm{Im}\langle \eta_s, ik_i\eta_s\rangle\right) \int_0^1 \mathrm{d}\tau \ iW_{1-\tau}(\eta_s)\mathrm{Im}\langle ik_i\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s), \tag{3.5.48}$$

$$I_4 := 2i \sum_{i=1}^{3} \int_0^1 \mathrm{d}\tau \ \mathrm{Im}\langle ik_i\eta_s, \partial_{y_i} \left\{ W_{1-\tau}(\eta_s)\mathcal{R}_\alpha^{1-\tau}(k)W_\tau(\eta_s)\right\}\rangle_\Gamma. \tag{3.5.49}$$

We show that each term can be bounded here by a constant at most of order 1.

For I_1, we can treat all $c^*(\cdots)$ and $c(\cdots)$ terms with Lemma 70 and Lemma 72. Furthermore, we use that $c_\alpha(k)\mathcal{N} = (\mathcal{N}+2)c_\alpha(k)$ to estimate

$$\|I_1\psi\| \leq C\|\eta_s\| \ \|(\mathcal{N}+1)^{1/2}W(\eta_s)\psi\| + C(\|\eta_s\|^2 + \|\eta_s\|^4)\|W(\eta_s)\psi\| + C\|\eta_s\|^2\|(\mathcal{N}+3)W(\eta_s)\psi\|$$
$$\leq C(\|\eta_s\| + \|\eta_s\|^2)e^{C\|\eta_s\|}\|(\mathcal{N}+5)\psi\| + C(\|\eta_s\|^2 + \|\eta_s\|^4) \tag{3.5.50}$$

where we used Proposition 68.

For I_2, we use a similar approach to (3.5.30) to obtain

$$\begin{aligned}\|I_2\psi\| &\le C\|\eta_s\| MN^{-\frac{2}{3}+\delta}\int_0^1 \mathrm{d}\tau\ (e^{C\|\eta_s\|(1-\tau)}-1)\|(\mathcal{N}+3)W_\tau(\eta_s)\psi\| \\ &\le C\|\eta_s\| MN^{-\frac{2}{3}+\delta}\int_0^1 \mathrm{d}\tau\ (e^{C\|\eta_s\|}e^{(\tilde{C}-C)\|\eta_s\|\tau}-e^{\tilde{C}\|\eta_s\|\tau})\|(\mathcal{N}+5)\psi\| \\ &\le CMN^{-\frac{2}{3}+\delta}e^{C\|\eta_s\|}(e^{C\|\eta_s\|}-1)\|(\mathcal{N}+5)\psi\|. \end{aligned} \tag{3.5.51}$$

For I_3, we first observe that for $n\in\mathbb{N}$ and $\mu\in[0,1]$

$$\begin{aligned}\|\mathcal{N}^n\mathcal{R}^\mu_\beta(k)\psi\| &\le \int_0^\mu \mathrm{d}\tau\ \|\mathcal{N}^n e^{-\tau B}\left(\sum_{l\in\Gamma}\eta_\beta(l)\mathcal{E}_\beta(l,k)\right)e^{\tau B}\psi\| \\ &\le \int_0^\mu \mathrm{d}\tau\ e^{C\|\eta_s\|\tau}\sum_{l\in\Gamma}|\eta_\beta(l)|\ \|\mathcal{E}_\beta(l,k)(\mathcal{N}+3)^n e^{\tau B}\psi\| \\ &\le C\int_0^\mu \mathrm{d}\tau\ e^{C\|\eta_s\|\tau}\sum_{l\in\Gamma}|\eta_\beta(l)|\ MN^{-\frac{2}{3}+\delta}\|\mathcal{N}(\mathcal{N}+3)^n e^{\tau B}\psi\| \\ &\le C\|\eta_s\|^{-1}(e^{C\|\eta_s\|\mu}-1)MN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\beta(l)|\ \|(\mathcal{N}+5)^{n+1}\psi\|. \end{aligned} \tag{3.5.52}$$

We use a similar approach to (3.5.34) and insert the above inequality (3.5.52) to obtain

$$\begin{aligned}\|I_3\psi\| &\le C\|\eta_s\|\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\int_0^1 \mathrm{d}\tau\ \|\left(\|\eta_s\|(\mathcal{N}+1)^{1/2}+C\|\eta_s\|^2\right)W_{1-\tau}(\eta_s)\mathcal{R}^{1-\tau}_\alpha(k)W_\tau(\eta_s)\psi\| \\ &\le C\|\eta_s\|^2\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\int_0^1 \mathrm{d}\tau\ \|\mathcal{N}W_{1-\tau}(\eta_s)\mathcal{R}^{1-\tau}_\alpha(k)W_\tau(\eta_s)\psi\|+C\|\eta_s\|^2\|I_2\psi\| \\ &\le C\|\eta_s\|^2\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\int_0^1 \mathrm{d}\tau\ e^{C\|\eta_s\|(1-\tau)}\|\mathcal{N}\mathcal{R}^{1-\tau}_\alpha(k)W_\tau(\eta_s)\psi\|+C\|\eta_s\|^2\|I_2\psi\| \\ &\le C\|\eta_s\|^2MN^{-\frac{2}{3}+\delta}\int_0^1 \mathrm{d}\tau\ e^{C\|\eta_s\|}(e^{C\|\eta_s\|(1-\tau)}-1)\|(\mathcal{N}+5)^2\psi\|+C\|\eta_s\|^2\|I_2\psi\| \\ &\le C\|\eta_s\|MN^{-\frac{2}{3}+\delta}e^{C\|\eta_s\|}(e^{C\|\eta_s\|}-1)\|(\mathcal{N}+5)^2\psi\| \\ &\quad +CMN^{-\frac{2}{3}+\delta}\|\eta_s\|^2e^{C\|\eta_s\|}(e^{C\|\eta_s\|}-1)\|(\mathcal{N}+5)\psi\| \\ &\le CMN^{-\frac{2}{3}+\delta}(\|\eta_s\|+\|\eta_s\|^2)e^{C\|\eta_s\|}(e^{C\|\eta_s\|}-1)\|(\mathcal{N}+5)^2\psi\|. \end{aligned} \tag{3.5.53}$$

For I_4, we first calculate

$$\partial_{y_i}\left\{W_{1-\tau}(\eta_s)\mathcal{R}^{1-\tau}_\alpha(k)W_\tau(\eta_s)\right\}$$

$$
\begin{aligned}
&= \int_\tau^1 \mathrm{d}\sigma\ \partial_{y_i} e^{(1-\sigma)B} \left(\sum_{l\in\Gamma} \eta_\alpha(l)\mathcal{E}_\alpha(l,k) \right) e^{\sigma B} + \int_\tau^1 \mathrm{d}\sigma\ e^{(1-\sigma)B} \left(\sum_{l\in\Gamma} ik_i\eta_\alpha(l)\mathcal{E}_\alpha(l,k) \right) e^{\sigma B} \\
&\quad + \int_\tau^1 \mathrm{d}\sigma\ e^{(1-\sigma)B} \left(\sum_{l\in\Gamma} \eta_\alpha(l)\mathcal{E}_\alpha(l,k) \right) \partial_{y_i} e^{\sigma B} \qquad (3.5.54) \\
&=: I_{4,1} + I_{4,2} + I_{4,3} + I_{4,4} + I_{4,5}
\end{aligned}
$$

with

$$
\begin{aligned}
I_{4,1} &= \int_\tau^1 \mathrm{d}\sigma\ (1-\sigma)\left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i(1-\sigma)\mathrm{Im}\langle \eta_s, ik_i\eta_s\rangle\right) \\
&\qquad\qquad \times e^{(1-\sigma)B} \left(\sum_{l\in\Gamma} \eta_\alpha(l)\mathcal{E}_\alpha(l,k) \right) e^{\sigma B}, \qquad (3.5.55)
\end{aligned}
$$

$$
I_{4,2} = 2i \int_\tau^1 \mathrm{d}\sigma \int_0^{1-\sigma} \mathrm{d}a\ e^{(1-a)B}\mathrm{Im}\langle ik_i\eta_s, \mathcal{R}^{1-\sigma-a}\rangle_\Gamma e^{aB} \left(\sum_{l\in\Gamma} \eta_\alpha(l)\mathcal{E}_\alpha(l,k) \right) e^{\sigma B}, \qquad (3.5.56)
$$

$$
I_{4,3} = \int_\tau^1 \mathrm{d}\sigma\ e^{(1-\sigma)B} \left(\sum_{l\in\Gamma} ik_i\eta_\alpha(l)\mathcal{E}_\alpha(l,k) \right) e^{\sigma B}, \qquad (3.5.57)
$$

$$
I_{4,4} = \int_\tau^1 \mathrm{d}\sigma\ e^{(1-\sigma)B} \left(\sum_{l\in\Gamma} \eta_\alpha(l)\mathcal{E}_\alpha(l,k) \right) \sigma\left(c^*(ik_i\eta_s) - c(ik_i\eta_s) + i\sigma\mathrm{Im}\langle \eta_s, ik_i\eta_s\rangle\right) e^{\sigma B}, \qquad (3.5.58)
$$

$$
I_{4,5} = 2i \int_\tau^1 \mathrm{d}\sigma\ e^{(1-\sigma)B} \left(\sum_{l\in\Gamma} \eta_\alpha(l)\mathcal{E}_\alpha(l,k) \right) \int_0^\sigma \mathrm{d}a\ e^{(1-a)B}\mathrm{Im}\langle ik_i\eta_s, \mathcal{R}^{\sigma-a}\rangle_\Gamma e^{aB}. \qquad (3.5.59)
$$

We approach each term similarly to (3.5.30).

For $I_{4,1}$, it holds

$$
\begin{aligned}
&\|I_{4,1}\psi\| \\
&\leq C\sum_{l\in\Gamma}|\eta_\alpha(l)| \int_\tau^1 \mathrm{d}\sigma\ (1-\sigma)\left(\|\eta_s\|\ \|(\mathcal{N}+1)e^{(1-\sigma)B}\mathcal{E}_\alpha(l,k)e^{\sigma B}\psi\| + C\|\eta_s\|^2\|\mathcal{E}_\alpha(l,k)e^{\sigma B}\psi\|\right) \\
&\leq C\|\eta_s\|\sum_{l\in\Gamma}|\eta_\alpha(l)| \int_\tau^1 \mathrm{d}\sigma\ (1-\sigma)\left(e^{C\|\eta_s\|(1-\sigma)}\|(\mathcal{N}+3)\mathcal{E}_\alpha(l,k)e^{\sigma B}\psi\| + C\|\eta_s\|\ \|\mathcal{E}_\alpha(l,k)e^{\sigma B}\psi\|\right) \\
&\leq C\|\eta_s\|MN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)| \int_\tau^1 \mathrm{d}\sigma\ \left((1-\sigma)e^{C\|\eta_s\|(1-\sigma)} + C\|\eta_s\|\right)\|(\mathcal{N}+3)^2e^{\sigma B}\psi\| \\
&\leq C\|\eta_s\|MN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)| \int_\tau^1 \mathrm{d}\sigma\ \left((1-\sigma)e^{C\|\eta_s\|(1-\sigma)} + C\|\eta_s\|\right)e^{\tilde{C}\|\eta_s\|\sigma}\|(\mathcal{N}+5)^2\psi\|
\end{aligned}
\qquad (3.5.60)
$$

$$\leq CMN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\left((1-\tau)e^{C\|\eta_s\|(1-\tau)}+C\|\eta_s\|e^{C\|\eta_s\|}\right)\|(\mathcal{N}+5)^2\psi\| \tag{3.5.61}$$

where we used Lemma 84, Lemma 70 and Lemma 72 in the first inequality, Lemma 83 in the third inequality and Proposition 68 in the second and forth inequality. Therefore, using $\int_0^1(1-\tau)e^{y(1-\tau)}\mathrm{d}\tau = y^{-2}\left((y-1)e^y+1\right)$ yields

$$\int_0^1 \mathrm{d}\tau\ \|\eta_s\|\sqrt{\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\|I_{4,1}\psi\|^2} \tag{3.5.62}$$

$$\leq CMN^{-\frac{2}{3}+\delta}\left((\|\eta_s\|^3+\|\eta_s\|-1)e^{C\|\eta_s\|}+1\right)\|(\mathcal{N}+5)^2\psi\| \tag{3.5.63}$$

where we used $\sum_{\alpha\in\mathcal{I}_l}\left(\sum_{l\in\Gamma}|\eta_\alpha(l)|\right)^2\leq\sum_{l',l\in\Gamma}\|\eta(l)\|_{l^2}\|\eta(l')\|_{l^2}\leq C\|\eta\|^2$ by Lemma 70 and Lemma 72.

For $I_{4,2}$, it holds

$$\begin{aligned}
&\|I_{4,2}\psi\|\\
&\leq C\|\eta_s\|MN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\int_\tau^1\mathrm{d}\sigma\int_0^{1-\sigma}\mathrm{d}a\ \|(\mathcal{N}+3)e^{aB}\mathcal{E}_\alpha(l,k)e^{\sigma B}\psi\|\\
&\leq C\|\eta_s\|\left(MN^{-\frac{2}{3}+\delta}\right)^2\sum_{l\in\Gamma}|\eta_\alpha(l)|\int_\tau^1\mathrm{d}\sigma\int_0^{1-\sigma}\mathrm{d}a\ e^{C\|\eta_s\|a}\|(\mathcal{N}+5)^2e^{\sigma B}\psi\|\\
&\leq C\left(MN^{-\frac{2}{3}+\delta}\right)^2\sum_{l\in\Gamma}|\eta_\alpha(l)|\int_\tau^1\mathrm{d}\sigma(e^{C\|\eta_s\|(1-\sigma)}-1)e^{\tilde{C}\|\eta_s\|\sigma}\|(\mathcal{N}+8)^2\psi\|\\
&\leq C\|\eta_s\|^{-1}\left(MN^{-\frac{2}{3}+\delta}\right)^2\sum_{l\in\Gamma}|\eta_\alpha(l)|e^{C\|\eta_s\|(1-\tau)}\|(\mathcal{N}+8)^2\psi\|
\end{aligned} \tag{3.5.64}$$

where we used Cauchy-Schwarz with (3.5.30), Lemma 70 and Lemma 72 in the first inequality, Proposition 68 and Lemma 83 in the second inequality. Therefore, it follows

$$\int_0^1\mathrm{d}\tau\ \|\eta_s\|\sqrt{\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\|I_{4,2}\psi\|^2}\leq CM^2N^{-\frac{4}{3}+2\delta}(e^{C\|\eta_s\|}-1)\|(\mathcal{N}+8)^2\psi\|.$$

The term $I_{4,3}$ is estimated similarly to (3.5.30) by

$$\|I_{4,3}\psi\|\leq C\|\eta_s\|^{-1}MN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|(e^{C\|\eta_s\|}-e^{C\|\eta_s\|\tau})\|(\mathcal{N}+3)\psi\| \tag{3.5.65}$$

and therefore

$$\int_0^1\mathrm{d}\tau\ \|\eta_s\|\sqrt{\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\|I_{4,3}\psi\|^2}\leq CMN^{-\frac{2}{3}+\delta}\|(\mathcal{N}+3)\psi\|. \tag{3.5.66}$$

Similarly for $I_{4,4}$, we estimate

$$\begin{aligned}
&\|I_{4,4}\psi\| \\
&\le CMN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\int_\tau^1 \mathrm{d}\sigma\ \sigma\|\mathcal{N}\left(c^*(ik_i\eta_s)-c(ik_i\eta_s)+i\sigma\mathrm{Im}\langle\eta_s,ik_i\eta_s\rangle\right)e^{\sigma B}\psi\| \\
&\le C(\|\eta_s\|+\|\eta_s\|^2)MN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\int_\tau^1 \mathrm{d}\sigma\ \sigma\|(\mathcal{N}+1)^2e^{\sigma B}\psi\| \\
&\le C(\|\eta_s\|^{-1}+1)MN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\ \left(e^{C\|\eta_s\|}+(1-C\|\eta_s\|\tau)e^{C\|\eta_s\|\tau}\right)\|(\mathcal{N}+3)^2\psi\| \\
&\le C(\|\eta_s\|^{-1}+1)MN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\ e^{C\|\eta_s\|}\|(\mathcal{N}+3)^2\psi\| \qquad (3.5.67)
\end{aligned}$$

using $\int_\tau^1 \mathrm{d}\sigma\ \sigma e^{y\sigma}=y^{-2}\left((y-1)e^y-(\tau y-1)e^{y\tau}\right)$ and therefore

$$\begin{aligned}
&\int_0^1 \mathrm{d}\tau\ \|\eta_s\|\sqrt{\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\|I_{4,4}\psi\|^2} \\
&\le CMN^{-\frac{2}{3}+\delta}(\|\eta_s\|+\|\eta_s\|^2)e^{C\|\eta_s\|}\|(\mathcal{N}+1)^2\psi\|. \qquad (3.5.68)
\end{aligned}$$

For $I_{4,5}$, we estimate

$$\begin{aligned}
\|I_{4,5}\psi\| &= \|2i\int_\tau^1 \mathrm{d}\sigma\ e^{(1-\sigma)B}\left(\sum_{l\in\Gamma}\eta_\alpha(l)\mathcal{E}_\alpha(l,k)\right)\int_0^\sigma \mathrm{d}a\ e^{(1-a)B}\mathrm{Im}\langle ik_i\eta_s,\mathcal{R}^{\sigma-a}\rangle_\Gamma e^{aB}\psi\| \\
&\le CMN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\int_\tau^1\mathrm{d}\sigma\int_0^\sigma \mathrm{d}a\ \|\mathcal{N}e^{(1-a)B}\mathrm{Im}\langle ik_i\eta_s,\mathcal{R}^{\sigma-a}\rangle_\Gamma e^{aB}\psi\| \\
&\le C\|\eta_s\|MN^{-\frac{2}{3}+\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\int_\tau^1\mathrm{d}\sigma\int_0^\sigma \mathrm{d}a\ e^{C\|\eta_s\|(1-a)}\left(\sum_{\alpha\in\mathcal{I}_k}\|(\mathcal{N}+3)\mathcal{R}_\alpha^{\sigma-a}(k)e^{aB}\psi\|^2\right)^{1/2} \\
&\le C\|\eta_s\|M^2N^{-\frac{4}{3}+2\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\int_\tau^1\mathrm{d}\sigma\int_0^\sigma \mathrm{d}a\ e^{C\|\eta_s\|(1-a)}(e^{C\|\eta_s\|(\sigma-a)}-1)\|(\mathcal{N}+8)^2e^{aB}\psi\| \\
&\le C\|\eta_s\|M^2N^{-\frac{4}{3}+2\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\int_\tau^1\mathrm{d}\sigma\int_0^\sigma \mathrm{d}a\ e^{C\|\eta_s\|}e^{C'\|\eta_s\|\sigma}e^{-C''\|\eta_s\|a}\|(\mathcal{N}+10)^2\psi\| \\
&\le C\|\eta_s\|^{-1}M^2N^{-\frac{4}{3}+2\delta}\sum_{l\in\Gamma}|\eta_\alpha(l)|\ (e^{C\|\eta_s\|}-e^{C\|\eta_s\|\tau})\|(\mathcal{N}+10)^2\psi\| \qquad (3.5.69)
\end{aligned}$$

where we used (3.5.52) in the third inequality. Therefore, we obtain

$$\int_0^1 \mathrm{d}\tau\ \|\eta_s\|\sqrt{\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\|I_{4,5}\psi\|^2}\le CM^2N^{-\frac{4}{3}+2\delta}\|(\mathcal{N}+10)^2\psi\|. \qquad (3.5.70)$$

Taking the five bounds together we finally obtain

$$\begin{aligned}
\|I_4\psi\| &= \|2i\sum_{i=1}^{3}\int_0^1 \mathrm{d}\tau\ \mathrm{Im}\langle ik_i\eta_s, \partial_{y_i}\left\{W_{1-\tau}(\eta_s)\mathcal{R}_\alpha^{1-\tau}(k)W_\tau(\eta_s)\right\}\rangle_\Gamma\psi\| \\
&\le C\int_0^1 \mathrm{d}\tau \sum_{n=1}^{5}\|\eta_s\|\Big(\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\|I_{4,n}\psi\|^2\Big)^{1/2} \\
&\le CMN^{-\frac{2}{3}+\delta}\left((\|\eta_s\|^3+\|\eta_s\|-1)e^{C\|\eta_s\|}+1\right)\|(\mathcal{N}+8)^2\psi\|. \qquad (3.5.71)
\end{aligned}$$

Combining (3.5.50), (3.5.51), (3.5.53) and (3.5.71) with $\psi\equiv\phi\otimes\Omega$ yields the desired final result of

$$\|\big(\Delta_y W(\eta_s)\big)\psi\| \le C(\|\eta_s\|+\|\eta_s\|^4)(e^{C\|\eta_s\|}+1). \qquad (3.5.72)$$

Similarly to (3.5.50) and (3.5.51), we further estimate the second term of (3.5.38)

$$\begin{aligned}
&\|\nabla_y W(\eta_s)\cdot\nabla_y\psi\| \\
&= \|\sum_{i=1}^{3}\Big(\left(c^*(ik_i\eta_s)-c(ik_i\eta_s)+i\mathrm{Im}\langle\eta_s, ik_i\eta_s\rangle\right)W(\eta_s) \\
&\qquad + 2i\int_0^1 \mathrm{d}\tau\ \mathrm{Im}\langle ik_i\eta_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s)\Big)\partial_{y_i}\psi\| \\
&\le C\left((\|\eta_s\|+\|\eta_s\|^2)e^{C\|\eta_s\|}+(\|\eta_s\|^2+\|\eta_s\|^4)\right)\sum_{i=1}^{3}\|\partial_{y_i}\phi\| \\
&\qquad + CMN^{-\frac{2}{3}+\delta}e^{C\|\eta_s\|}(e^{C\|\eta_s\|}-1)\sum_{i=1}^{3}\|\partial_{y_i}\phi\| \qquad (3.5.73) \\
&\le C(\|\eta_s\|+\|\eta_s\|^4)(e^{C\|\eta_s\|}+1)\sum_{i=1}^{3}\|\partial_{y_i}\phi\|. \qquad (3.5.74)
\end{aligned}$$

Therefore, in total for all $t\in\mathbb{R}$

$$\begin{aligned}
&\|\Delta_y W(\eta_t)\phi\otimes\Omega\| \qquad (3.5.75) \\
&\le C\left\{(\|\eta_s\|+\|\eta_s\|^4)(e^{C\|\eta_s\|}+1)+\sum_{i=1}^{3}\left((\|\eta_s\|+\|\eta_s\|^4)(e^{C\|\eta_s\|}+1)\|\partial_{y_i}\phi\|+\|\partial_{y_i}^2\phi\|\right)\right\}. \qquad (3.5.76)
\end{aligned}$$

■

Proofs of properties of η

Proof of Lemma 70. Recall that by definition it holds

$$\|\eta_s\|^2 = \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} |\big(\eta_s\big)_\alpha(k)|^2 = \lambda^2\sum_{k\in\Gamma}|\hat{V}(k)|^2 \underbrace{\sum_{\alpha\in\mathcal{I}_k}\left|n_\alpha(k)\frac{\sin\big(\epsilon_\alpha(k)s/2\big)}{\epsilon_\alpha(k)/2}\right|^2}_{=:S_k} \tag{3.5.77}$$

with $\epsilon_\alpha(k) = 2k_F|k\cdot\omega_\alpha|$. We will first approximate S_k and then give a upper bound. Firstly, we approximate the α-sum S_k by an integral by identifying

$$n_\alpha(k)^2 = k_{\mathrm{F}}^2|k|\sigma(p_\alpha)u_\alpha(k)^2\left(1+\mathcal{O}(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta})\right) \tag{3.5.78}$$

with $\cos\theta_\alpha := |\hat{k}\cdot\hat{\omega}_\alpha| \equiv u_\alpha(k)^2$ analogously to Lemma 81. Thus, we calculate the half-sphere integral as approximation for S_k

$$\begin{aligned} S_k &= \frac{1}{|k|}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi/2}\mathrm{d}\theta\frac{\sin(k_{\mathrm{F}}|k|s\cos\theta)^2}{\cos\theta}\sin\theta + \mathcal{E} \\ &= \frac{2\pi}{|k|}\int_0^1\mathrm{d}u\frac{\sin(k_{\mathrm{F}}|k|su)^2}{u} + \mathcal{E} \\ &= \frac{\pi}{|k|}\left\{\log(2k_{\mathrm{F}}|k|s) - \mathrm{Ci}(2k_{\mathrm{F}}|k|s) + \gamma\right\} + \mathcal{E} \end{aligned} \tag{3.5.79}$$

with total error $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$ consisting three terms: the error $\mathcal{E}_1$ from (3.5.78), the error $\mathcal{E}_2$ from approximating the discrete variables θ_α by continuous θ given by

$$\begin{aligned} \mathcal{E}_2 &= \left|\int_{p_\alpha}\mathrm{d}\sigma\ \frac{\sin(k_{\mathrm{F}}|k|s\cos\theta)^2}{\cos\theta} - \sigma(p_\alpha)\frac{\sin(k_{\mathrm{F}}|k|s\cos\theta_\alpha)^2}{\cos\theta_\alpha}\right| \\ &\le \sup_{\hat{\omega}(\theta,\varphi)\in p_\alpha}\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\sin(k_{\mathrm{F}}|k|s\cos\theta)^2}{\cos\theta}\right|\sup_{(\theta,\varphi)\in p_\alpha}|\theta-\theta_\alpha|\,\sigma(p_\alpha) \\ &\le C(k_{\mathrm{F}}|k|s)^2M^{-\frac{3}{2}} \end{aligned} \tag{3.5.80}$$

and the error $\mathcal{E}_3$ from the patch construction which is given by

$$\mathcal{E}_3 = \sup_{\hat{\omega}(\theta,\varphi)\in p_\alpha}\left|\frac{\sin(k_{\mathrm{F}}|k|s\cos\theta)^2}{\cos\theta}\right|\left|\sigma\Big(\bigcup_{\alpha\in\mathcal{I}_k}p_\alpha\Big) - \sigma(S)\right| \le Ck_{\mathrm{F}}|k|s\left(N^{-\delta} + M^{\frac{1}{2}}N^{-\frac{1}{3}}\right). \tag{3.5.81}$$

Thus, with the triangle inequality

$$\left|S_k - \frac{\pi}{|k|}\left\{\log(2k_{\mathrm{F}}|k|s) - \mathrm{Ci}(2k_{\mathrm{F}}|k|s) + \gamma\right\}\right|$$

$$\leq \frac{C}{|k|}\left\{M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}+(k_{\mathrm{F}}|k|s)^2M^{-\frac{3}{2}}+k_{\mathrm{F}}|k|s\left(N^{-\delta}+M^{\frac{1}{2}}N^{-\frac{1}{3}}\right)\right\}. \tag{3.5.82}$$

We will now give an estimate for S_k by approximating for any $m\in\mathbb{R}$

$$\frac{\sin(mx)^2}{x}=\frac{1}{2}\frac{1-\cos(2mx)}{x}\leq\begin{cases}m^2x & \text{if } 2mx\in[0,\pi/2],\\ \frac{1}{x} & \text{if } 2mx>\pi/2\end{cases}$$

where we used $1-x^2/2\leq\cos(x)$ for all $x\in[0,\pi/2]$. Therefore,

$$\begin{aligned}
&\frac{2\pi}{|k|}\int_0^1 \mathrm{d}u\frac{\sin(k_{\mathrm{F}}|k|su)^2}{u}\\
&\leq\frac{2\pi}{|k|}\int_0^1\mathrm{d}u\ \left\{(k_{\mathrm{F}}|k|s)^2u\ \chi(2k_{\mathrm{F}}|k|su\leq\pi/2)+\frac{1}{u}\chi(2k_{\mathrm{F}}|k|su>\pi/2)\right\}\\
&\leq\frac{2\pi}{|k|}\int_0^1\mathrm{d}u\ \left\{(k_{\mathrm{F}}|k|s)^2u\ \chi\Big(u\leq\frac{\pi}{4k_{\mathrm{F}}|k|s}\Big)+\frac{1}{u}\ \chi\Big(u>\frac{\pi}{4k_{\mathrm{F}}|k|s}\Big)\right\}\\
&=\frac{2\pi}{|k|}\left\{\frac{1}{2}(k_{\mathrm{F}}|k|s)^2\Big(\frac{\pi}{4k_{\mathrm{F}}|k|s}\Big)^2-\log\Big(\frac{\pi}{4k_{\mathrm{F}}|k|s}\Big)\right\}\ \chi\Big(1>\frac{\pi}{4k_{\mathrm{F}}|k|s}\Big)\\
&\quad+\frac{2\pi}{|k|}\frac{1}{2}(k_{\mathrm{F}}|k|s)^2\ \chi\Big(1\leq\frac{\pi}{4k_{\mathrm{F}}|k|s}\Big)\\
&=\frac{2\pi}{|k|}\left\{\frac{\pi^2}{32}-\log(\pi)+\log(4k_{\mathrm{F}}|k|s)\right\}\ \chi\Big(k_{\mathrm{F}}|k|s>\frac{\pi}{4}\Big)\\
&\quad+\frac{2\pi}{|k|}\frac{1}{2}(k_{\mathrm{F}}|k|s)^2\ \chi\Big(k_{\mathrm{F}}|k|s\leq\frac{\pi}{4}\Big)\\
&\leq\frac{2\pi}{|k|}\min\left\{\frac{1}{2}(k_{\mathrm{F}}|k|s)^2,\log(4k_{\mathrm{F}}|k|s+1)\right\}.
\end{aligned}\tag{3.5.83}$$

Note that $\log(4k_{\mathrm{F}}|k|s+1)\leq\log(4k_{\mathrm{F}}s+1)+\log|k|$ using $\frac{1}{2}x^2\leq\log(4x+1)$ for all $x\leq 2$ and $|k|\geq 1$ for all $k\in\mathbb{Z}^3_*$. Thus, it holds

$$\begin{aligned}
\|\eta_s\|^2&\leq 2\pi\lambda^2\sum_{k\in\mathbb{Z}^3_*}|\hat{V}(k)|^2\min\left\{\frac{(k_{\mathrm{F}}s)^2}{2}|k|,\frac{\log(4k_{\mathrm{F}}|k|s+1)}{|k|}\right\}\\
&\leq 2\pi\lambda^2\min\left\{\|\hat{V}(\cdot)^{1/2}\|_2^2(k_{\mathrm{F}}s)^2/2,\|\hat{V}\|_2^2\left(\log(4k_{\mathrm{F}}s+1)+1\right)\right\}.
\end{aligned}\tag{3.5.84}$$

Thus, by using $\sqrt{x+y}\leq\sqrt{x}+\sqrt{y}$ and $\sqrt{\log(x+1)}\leq\log(x+2)$ we obtain the desired

$$\|\eta_s\|\leq\lambda\min\left\{bk_{\mathrm{F}}s,\log(4k_{\mathrm{F}}s+2)^a+a\right\}, \tag{3.5.85}$$

$$e^{c_0\|\eta_s\|}\leq\min\left\{e^{bc_0\lambda k_{\mathrm{F}}s},e^{ac_0\lambda}(4k_{\mathrm{F}}s+2)^{ac_0\lambda}\right\} \tag{3.5.86}$$

with $a=\sqrt{2\pi}\|\hat{V}\|_2, b=\sqrt{\pi}\|\hat{V}(\cdot)^{1/2}\|_2$. ■

Proof of Lemma 72. For the inequalities, we observe that for $n \in \mathbb{N}_0$ it holds

$$\sum_{k\in\Gamma} \| |k|^n \eta_s(k) \|_{l^2} \equiv \sum_{k\in\Gamma} |k|^n \Big(\sum_{\alpha\in\mathcal{I}_k} |(\eta_s)_\alpha(k)|^2 \Big)^{1/2} = \lambda \sum_{k\in\Gamma} |k|^n |\hat{V}(k)| \, (S_k)^{1/2} \tag{3.5.87}$$

with S_k from (3.5.77) Since by assumption $\|(\cdot)^n \hat{V}\|_1$ is bounded for each $n \in \mathbb{N}$, the first statement follows. The second statement simply follows from the same calculation and recalling that $\langle \eta_s, |k|^n \eta_s \rangle \equiv \sum_{k\in\Gamma} \sum_{\alpha\in\mathcal{I}_k} |k|^n |\eta_\alpha(k)|^2$. The third statement follows from Lemma 84 and

$$\|c^*(|k|^n \eta_s)\psi\| \le \sum_{k\in\Gamma} \| \sum_{\alpha\in\mathcal{I}_k} |k|^n \eta_\alpha(k) c^*_\alpha(k)\psi \| \le \sum_{k\in\Gamma} \| |k|^n \eta_s(k) \|_{l^2} \| (\mathcal{N}+1)^{1/2}\psi \|. \tag{3.5.88}$$

■

3.6 Proof of Corollary 61

Proof of Corollary 61. We observe that by the inverse triangle inequality it holds

$$\begin{aligned}
&\|R^* e^{-i\mathbb{H}t} R\psi - e^{-i\widetilde{\mathbb{H}^{\text{eff}}}t}\psi\| \\
&\ge \|e^{iP(t)} W(\eta_t)\psi - e^{-i\widetilde{\mathbb{H}^{\text{eff}}}t}\psi\| && (3.6.1a)\\
&\quad - \|e^{-i\mathbb{H}^{\text{eff}}t}\psi - e^{iP(t)}W(\eta_t)\psi\| - \|R^* e^{-i\mathbb{H}t} R\psi - e^{-i\mathbb{H}^{\text{eff}}t}\psi\|. && (3.6.1b)
\end{aligned}$$

The second line (3.6.1b) is to be bounded from below by bounds of order $o(1)$ from Theorem 51 and Theorem 56. Therefore, it is sufficient to show that the first line (3.6.1a) is large.

The first line (3.6.1a) can be explicitly estimated by the same approach as in the proof of Theorem 56. That is use Duhamel's formula, commute all $c_\alpha(k)$-operators with $W(\eta_s)$ and collect the error terms via Lemma 65:

$$\begin{aligned}
&\Big(e^{i\widetilde{\mathbb{H}^{\text{eff}}}t} e^{iP(t)} W(\eta_t) - 1 \Big)\psi \\
&= i \int_0^t \mathrm{d}s \; e^{i\widetilde{\mathbb{H}^{\text{eff}}}s} e^{iP(s)} \Big(\big(\widetilde{\mathbb{H}^{\text{eff}}} - E_N^{\text{pw}} + 2\mathrm{Im}(\dot{\nu}_s) - \mathrm{Im}\langle \eta_s, \dot{\eta}_s \rangle \big) W(\eta_s)\psi - i\partial_s W(\eta_s)\psi \Big) && (3.6.2)\\
&= i \int_0^t \mathrm{d}s \; e^{i\widetilde{\mathbb{H}^{\text{eff}}}s} e^{iP(s)} \Big(\big(h_0 - c^*(h_y) - c(h_y) \big) W(\eta_s)\psi + \text{Error} \Big) && (3.6.3)\\
&= i \int_0^t \mathrm{d}s \; e^{i\widetilde{\mathbb{H}^{\text{eff}}}s} e^{iP(s)} \Big(- W(\eta_s) \left(c^*(h_y) + \langle \eta_s, h_y \rangle + \langle h_y, \eta_s \rangle \right) \psi + h_0 W(\eta_s)\psi + \widetilde{\text{Error}} \Big) && (3.6.4)
\end{aligned}$$

with $\int_0^t \mathrm{d}s$ Error $=$ Error$_1$ + Error$_2$ from (3.5.27) and where we also commuted $c^*(h_y)$ and $c(h_y)$ with $W(\eta_s)$ such that the new error term is of the form

$$\begin{aligned}
\widetilde{\text{Error}} := &\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k}\left(\epsilon_\alpha(k)c^*_\alpha(k) - e^{is\epsilon_\alpha(k)}\right)\overline{(h_y)_\alpha(k)}W(\eta_s)\mathcal{R}^1_\alpha(k)\psi \\
&- \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} W(\eta_s)\mathcal{R}^1_\alpha(k)^*(h_y)_\alpha(k)\psi \\
&- 2i\int_0^1 \mathrm{d}\tau\ W_{1-\tau}(\eta_s)\mathrm{Im}\langle\dot{\eta}_s, \mathcal{R}^{1-\tau}\rangle_\Gamma W_\tau(\eta_s)\psi.
\end{aligned} \tag{3.6.5}$$

Note that $\widetilde{\text{Error}}$ is s-dependent even though we did not include the dependence explicitly in the notation.

We employ the Cauchy-Schwarz inequality, insert our previous finding with the triangle inequality and use that $e^{-i\widetilde{\mathbb{H}^{\mathrm{eff}}}s}$ is unitary to estimate

$$\begin{aligned}
&\|e^{iP(t)}W(\eta_t)\psi - e^{-i\widetilde{\mathbb{H}^{\mathrm{eff}}}t}\psi\| \\
&= \|\left(e^{i\widetilde{\mathbb{H}^{\mathrm{eff}}}t}e^{iP(t)}W(\eta_t) - 1\right)\psi\| \geq \left|\langle\psi, \left(1 - e^{i\widetilde{\mathbb{H}^{\mathrm{eff}}}t}e^{iP(t)}W(\eta_t)\right)\psi\rangle\right| \\
&= \left|\int_0^t \mathrm{d}s\langle e^{-iP(s)}e^{-i\widetilde{\mathbb{H}^{\mathrm{eff}}}s}\psi, \left(\widetilde{\mathbb{H}^{\mathrm{eff}}} - E_N^{\mathrm{pw}} + 2\mathrm{Im}(\dot{\nu}_s) - \mathrm{Im}\langle\eta_s,\dot{\eta}_s\rangle\right)W(\eta_s)\psi - i\partial_s W(\eta_s)\psi\rangle\right| \\
&\geq \left|\int_0^t \mathrm{d}s\langle\psi, e^{i\widetilde{\mathbb{H}^{\mathrm{eff}}}s}e^{iP(s)}W(\eta_s)\left(c^*(h_y) + 2\mathrm{Re}\langle h_y,\eta_s\rangle\right)\psi\rangle\right| \\
&\qquad - \int_0^t \mathrm{d}s\left|\langle e^{-iP(s)}e^{-i\widetilde{\mathbb{H}^{\mathrm{eff}}}s}\psi, \widetilde{\text{Error}} + h_0 W(\eta_s)\psi\rangle\right| \qquad (3.6.6) \\
&\geq \left|\int_0^t \mathrm{d}s\langle\psi, \left(1 + e^{i\widetilde{\mathbb{H}^{\mathrm{eff}}}s}e^{iP(s)}W(\eta_s) - 1\right)\left(c^*(h_y) + 2\mathrm{Re}\langle h_y,\eta_s\rangle\right)\psi\rangle\right| \\
&\qquad - \int_0^t \mathrm{d}s\left\{\|\widetilde{\text{Error}}\| + \|h_0 W(\eta_s)\psi\|\right\} \qquad (3.6.7) \\
&\geq \left|\int_0^t \mathrm{d}s\left\{\langle\psi, c^*(h_y)\psi\rangle + 2\mathrm{Re}\langle h_y,\eta_s\rangle\right\}\right| \\
&\qquad - \left|\int_0^t \mathrm{d}s\langle\left(e^{i\widetilde{\mathbb{H}^{\mathrm{eff}}}s}e^{iP(s)}W(\eta_s) - 1\right)^*\psi, \left(c^*(h_y) + 2\mathrm{Re}\langle h_y,\eta_s\rangle\right)\psi\rangle\right| \\
&\qquad - \int_0^t \mathrm{d}s\left\{\|\widetilde{\text{Error}}\| + \|h_0 W(\eta_s)\psi\|\right\} \qquad (3.6.8) \\
&\geq 2\left|\int_0^t \mathrm{d}s\ \mathrm{Re}\langle h_y,\eta_s\rangle\right| - \left(\|c^*(h_y)\psi\| + 2\sup_{s\in[0,t]}|\mathrm{Re}\langle h_y,\eta_s\rangle|\right)\int_0^t \mathrm{d}s\|\left(e^{i\widetilde{\mathbb{H}^{\mathrm{eff}}}s}e^{iP(s)}W(\eta_s) - 1\right)\psi\| \\
&\qquad - t\sup_{s\in[0,t]}\left\{\|\widetilde{\text{Error}}\| + \|h_0 W(\eta_s)\psi\|\right\} \qquad (3.6.9)
\end{aligned}$$

where we used $\langle\psi, c^*(h_y)\psi\rangle = 0$ since $c(h_y)\phi \otimes \Omega = 0$.

The three parts of the above inequality (3.6.9) can be bounded in the following:

Firstly, the error term can be bounded with (3.5.37) and analogously to (3.5.35) and (3.5.36):

$$\begin{aligned}
&t \sup_{s\in[0,t]} \left\{ \|\widetilde{\mathrm{Error}}\| + \|h_0 W(\eta_s)\psi\| \right\} \\
&\leq C\lambda k_{\mathrm{F}} t M N^{-\frac{2}{3}+\delta}(f_C(\lambda k_{\mathrm{F}} t) - 1) + C\beta \left(\log\left[f_1(\lambda k_{\mathrm{F}} t)\right] + \log\left[f_1(\lambda k_{\mathrm{F}} t)\right]^4\right)\left(f_C(\lambda k_{\mathrm{F}} t) + 1\right) \\
&\leq C Q_V(k_{\mathrm{F}} t) \max\{\lambda k_{\mathrm{F}} t M N^{-\frac{2}{3}+\delta}, \beta t\} \\
&\leq C \max\{M N^{-\frac{2}{3}+\delta}, \beta k_{\mathrm{F}}^{-1}\lambda^{-1}\} =: d
\end{aligned} \tag{3.6.10}$$

where we used $0 \leq t \lesssim \mathcal{O}(k_{\mathrm{F}}^{-1}\lambda^{-1})$ and defined the error variable d.

Secondly, the prefactor of the integral term is bounded by Lemma 84

$$\|c^*(h_y)\psi\| + 2 \sup_{s\in[0,t]} |\mathrm{Re}\langle h_y, \eta_s\rangle|$$

$$\leq \|h_y\|\|(\mathcal{N}+1)^{1/2}\psi\| + 2 \sup_{s\in[0,t]} \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \frac{1-\cos(s\epsilon_\alpha(k))}{\epsilon_\alpha(k)} |(h_y)_\alpha(k)|^2 \tag{3.6.11}$$

$$\leq \|h_y\| + \frac{2}{k_{\mathrm{F}}}\|h_y\|^2 \leq C\lambda k_{\mathrm{F}}\left(\|\hat{V}\| + \lambda\|\hat{V}\|^2\right) =: \theta \tag{3.6.12}$$

where we introduced the variable θ.

Thirdly, we want to show that the remaining term $|\int_0^t \mathrm{d}s\, \mathrm{Re}\langle h_y, \eta_s\rangle|$ is large:

$$\begin{aligned}
\left|\int_0^t \mathrm{d}s\, \mathrm{Re}\langle h_y, \eta_s\rangle\right| &= \left|\int_0^t \mathrm{d}s\ \mathrm{Re}\sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \frac{e^{-is\epsilon_\alpha(k)} - 1}{\epsilon_\alpha(k)} |(h_y)_\alpha(k)|^2\right| \\
&= \left|\int_0^t \mathrm{d}s\ \mathrm{Re}\sum_{k\in\Gamma} \lambda^2\hat{V}(k)^2 \sum_{\alpha\in\mathcal{I}_k} \frac{e^{-is\epsilon_\alpha(k)} - 1}{\epsilon_\alpha(k)} n_\alpha(k)^2\right| \\
&= \frac{\pi\lambda^2 k_{\mathrm{F}}^2}{k_{\mathrm{F}}}\left|\int_0^t \mathrm{d}s\ \mathrm{Re}\sum_{k\in\Gamma}\hat{V}(k)^2 \frac{e^{-i2k_{\mathrm{F}}|k|s} - 1}{2k_{\mathrm{F}}|k|s}\right| \left\{1 + \mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta} + N^{-\delta}\right)\right\}
\end{aligned} \tag{3.6.13}$$

where we used that the sum over α is approximated similarly to (82) by integrating over the half sphere

$$\sum_{\alpha\in\mathcal{I}_k} \frac{e^{-is\epsilon_\alpha(k)} - 1}{\epsilon_\alpha(k)} n_\alpha(k)^2 = 2\pi\int_0^{\pi/2} \mathrm{d}\theta\ \frac{e^{-is2k_{\mathrm{F}}|k|\cos\theta} - 1}{2k_{\mathrm{F}}|k|\cos\theta} \cos\theta\sin\theta \left\{1 + \mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta} + N^{-\delta}\right)\right\}$$

$$= i\frac{2\pi k_{\mathrm{F}}^2}{2k_{\mathrm{F}}}\frac{e^{-i2k_{\mathrm{F}}|k|s}-1}{2k_{\mathrm{F}}|k|s}\left\{1+\mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}+N^{-\delta}\right)\right\}. \tag{3.6.14}$$

Therefore, we obtain

$$\left|\int_0^t \mathrm{d}s \ \mathrm{Re}\sum_{k\in\Gamma}\hat{V}(k)^2\frac{e^{-i2k_{\mathrm{F}}|k|s}-1}{2k_{\mathrm{F}}|k|s}\right| = \left|\sum_{k\in\Gamma}\hat{V}(k)^2\int_0^t \mathrm{d}s\frac{1-\cos(2k_{\mathrm{F}}|k|s)}{2k_{\mathrm{F}}|k|s}\right|. \tag{3.6.15}$$

It is well-known that for all $x>0$ it holds $\cos(x)<1-4x^2/\pi^2$ and thus $1-\cos(x)\geq 4x^2/\pi^2$ for all $x\in(0,\pi/2)$. Also, it holds for all $x>\pi/2$ that

$$\int_{\frac{\pi}{2}}^x \frac{1-\cos y}{y}\mathrm{d}y = \ln(x)-\ln(\frac{\pi}{2})+\mathrm{Ci}(\frac{\pi}{2})-\mathrm{Ci}(x)\geq \ln(x)-\ln(\frac{\pi}{2})$$

since $\mathrm{Ci}(\pi/2)>0$ and $\mathrm{Ci}(\pi/2)\geq\mathrm{Ci}(x)$ for all $x\geq\pi/2$. Thus we obtain

$$\int_0^x \frac{1-\cos y}{y}\mathrm{d}y \geq \chi(x>\frac{\pi}{2})\left\{\int_0^{\frac{\pi}{2}}\frac{4y}{\pi^2}\mathrm{d}y+\int_{\frac{\pi}{2}}^x\frac{1-\cos y}{y}\mathrm{d}y\right\}+\chi(x\leq\frac{\pi}{2})\int_0^x\frac{4y}{\pi^2}\mathrm{d}y \tag{3.6.16}$$

$$= \chi(x>\frac{\pi}{2})\left\{\frac{1}{2}+\ln(x)-\ln(\frac{\pi}{2})\right\}+\chi(x\leq\frac{\pi}{2})\frac{2x^2}{\pi^2} \tag{3.6.17}$$

which is a differentiable lower bound.

Therefore, we find

$$\begin{aligned}
&2\left|\int_0^t \mathrm{d}s \ \mathrm{Re}\langle h_y,\eta_s\rangle\right| \\
&\geq \frac{2\pi\lambda^2k_{\mathrm{F}}^2}{k_{\mathrm{F}}}\sum_{k\in\Gamma}\frac{\hat{V}(k)^2}{2k_{\mathrm{F}}|k|}\Bigg(\chi(2k_{\mathrm{F}}|k|t>\frac{\pi}{2})\left\{\frac{1}{2}+\log(2k_{\mathrm{F}}|k|t)-\ln(\frac{\pi}{2})\right\} \\
&\qquad +\chi(2k_{\mathrm{F}}|k|t\leq\frac{\pi}{2})\frac{2(2k_{\mathrm{F}}|k|t)^2}{\pi^2}\Bigg)\left\{1+\mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}+N^{-\delta}\right)\right\} \\
&=: b_t+d
\end{aligned} \tag{3.6.18}$$

with $d\in\max\{MN^{-\frac{2}{3}+\delta},\beta k_{\mathrm{F}}^{-1}\lambda^{-1}\}$ defined in (3.6.10) and

$$\dot{b}_t = \frac{\pi\lambda^2k_{\mathrm{F}}^2}{k_{\mathrm{F}}}\sum_{k\in\Gamma}\hat{V}(k)^2\left(\chi(2k_{\mathrm{F}}|k|t>\frac{\pi}{2})\frac{1}{2k_{\mathrm{F}}|k|t}+\chi(2k_{\mathrm{F}}|k|t\leq\frac{\pi}{2})\frac{8k_{\mathrm{F}}|k|t}{\pi^2}\right). \tag{3.6.19}$$

In total, we can re-write the inequality (3.6.9) for $g_t := \|\big(e^{i\widetilde{\mathbb{H}^{\mathrm{eff}}}t}e^{iP(t)}W(\eta_t)-1\big)\psi\|$ as the following integral inequality

$$g_t\geq b_t-\theta\int_0^t \mathrm{d}s g_s \tag{3.6.20}$$

with b_t and θ defined in (3.6.18) and (3.6.12), respectively.

Claim. *We can bound g_t from below for all $t \geq 0$ by a differentiable map h_t which obeys the initial value problem*

$$h_t = b_t - \theta \int_0^t \mathrm{d}s h_s \qquad \textit{with } h_0 = -d < 0 = g_0. \tag{3.6.21}$$

Proof of claim. Assume for the proof by contradiction that there exists a $t_0 > 0 : h_{t_0} > g_{t_0}$ and set $I \subseteq \mathbb{R}_{>0}$ as the largest open interval satisfying $t_0 \in I$ and $h_t > g_t$ for all $t \in I$. Note that since h_t is differentiable such an open interval exists. It holds $t_1 := \inf I > 0$ since $h_0 < g_0$ and for all $t \in I$

$$\dot{g}_t \geq \dot{b}_t - \theta g_t > \dot{b}_t - \theta h_t = \dot{h}_t$$

and therefore

$$\int_{t_1}^{t_0} \dot{g}_s \mathrm{d}s > \int_{t_1}^{t_0} \dot{h}_s \mathrm{d}s \implies h_{t_1} - g_{t_1} > h_{t_0} - g_{t_0} > 0.$$

Thus, we obtain the desired contradiction to t_1 being the infimum of the largest set satisfying $h_t > g_t$. ■

The solution of the initial value problem (3.6.21) is uniquely given by

$$h_t = e^{-\theta t} \int_0^t \dot{b}_s e^{\theta s} \mathrm{d}s - d. \tag{3.6.22}$$

Consequently it holds by inserting (3.6.19)

$$\begin{aligned} h_t &= e^{-\theta t} \int_0^t \dot{b}_s e^{\theta s} \mathrm{d}s \\ &= \frac{\pi \lambda^2 k_{\mathrm{F}}^2}{k_{\mathrm{F}}} \sum_{k \in \Gamma} \hat{V}(k)^2 e^{-\theta t} \int_0^t \Bigg(\chi(2k_{\mathrm{F}}|k|s > \frac{\pi}{2}) e^{\theta s} \frac{1}{2k_{\mathrm{F}}|k|s} \\ &\qquad + \chi(2k_{\mathrm{F}}|k|s \leq \frac{\pi}{2}) e^{\theta s} \frac{8k_{\mathrm{F}}|k|s}{\pi^2} \Bigg) \mathrm{d}s - d \qquad (3.6.23) \\ &\geq \pi \lambda^2 k_{\mathrm{F}} \sum_{k \in \Gamma} \hat{V}(k)^2 \Bigg(\chi(|k|t \leq \frac{\pi}{4k_{\mathrm{F}}}) \frac{8k_{\mathrm{F}}|k|}{\theta^2 \pi^2} \big(e^{-\theta t} + \theta t - 1\big) + \\ &\qquad + \chi(|k|t > \frac{\pi}{4k_{\mathrm{F}}}) \frac{8k_{\mathrm{F}}|k|}{\theta^2 \pi^2} \big(e^{-\theta \pi/(4k_{\mathrm{F}}|k|)} + \frac{\theta \pi}{4k_{\mathrm{F}}|k|} - 1\big) \Bigg) - d \qquad (3.6.24) \\ &\geq \frac{\lambda^2 k_{\mathrm{F}}^2}{\theta^2 \pi} \sum_{k \in \Gamma} \hat{V}(k)^2 |k| \min \Bigg(f(\theta t), f\big(\frac{\pi \theta}{4k_{\mathrm{F}}|k|}\big) \Bigg) - d \qquad (3.6.25) \end{aligned}$$

with $f(t) := (e^{-t} + t - 1)$ defining a non-negative monotonically increasing function. Recall that $d \in \max\{MN^{-\frac{2}{3}+\delta}, \beta k_{\mathrm{F}}^{-1} \lambda^{-1}\}$ from (3.6.10) for all $0 \leq t \lesssim k_{\mathrm{F}}^{-1} \lambda^{-1}$ and $\theta \leq C \lambda k_{\mathrm{F}}$ with a constant $C > 0$ depending only on V from (3.6.12). Thus, we obtain the desired result that (3.5.34) has a lower bound of order 1. ■

3.7 Comparison to the Landau-Pekar equations

The derived effective equation for η_t is connected to the Landau-Pekar equations, which are an effective equation for the Fröhlich polaron model in the strong coupling regime. In the following we present a brief discussion of this topic on the torus Λ.

The *Fröhlich polaron model* is given by Hamiltonian of the form

$$H = -\Delta_y + \alpha^{-2} \sum_{k \in \mathbb{Z}^3} b_k^* b_k + \alpha^{-1} \sum_{k \in \mathbb{Z}^3} \left(G_y(k) b_k^* + \overline{G_y(k)} b_k \right) \tag{3.7.1}$$

where $G_y(k) := \frac{1}{|k|} e^{-iky}$, $b_k, b_{k'}^*$ satisfy the bosonic CCR and α is the coupling parameter which is imagined to be large in the relevant regime. This models a system where a fast impurity particle (usually associated to an electron) interacts with a slow bosonic field with constant dispersion relation. The separation of scales comes from the fact that the kinetic term of the bosonic field is suppressed by α^{-2} whereas the kinetic term of the electron is not.

The *Landau-Pekar equations* are then given by the coupled equations

$$\begin{cases} i\partial_t \phi_t & = (-\Delta_y + V_\eta)\phi_t \\ i\alpha^2 \partial_t \eta_t & = \eta_t + \sigma_\phi \end{cases} \tag{3.7.2}$$

where $V_\eta : \Lambda \to \mathbb{C}$ with

$$V_\eta(y) := \sum_{k \in \mathbb{Z}^3} \left(G_y(-k)\eta(k) + G_y(k)\overline{\eta(k)} \right) \qquad \text{for all } y \in \Lambda \tag{3.7.3}$$

corresponds to an effective potential for the impurity particle and $\sigma_\phi : \mathbb{Z}^3 \to \mathbb{C}$ with

$$\sigma_\phi(k) := \int_\Lambda \mathrm{d}y \; G_y(k) |\phi_t|^2(y) \qquad \text{for all } k \in \mathbb{Z}^3 \tag{3.7.4}$$

to a source term. Recent, in [LMR+21] a norm approximation, similar to Theorem 56, has been shown for the Fröhlich polaron model in the strong coupling regime: On times of order α^2 it holds

$$\| e^{-iHt} \left(\phi_{\eta_0} \otimes W(\alpha^2 \eta_0) \Upsilon_0 \right) - e^{i \int_0^t \mathrm{d}u \omega(u)} \phi_t \otimes W(\alpha^2 \eta_t) \Upsilon_t \| \leq C\alpha^{-1} \tag{3.7.5}$$

where ϕ_{η_0} is the ground state of $h_{\eta_0} = (-\Delta_y + V_{\eta_0})$ and Υ_t describes quantum fluctuations incorporating the correlations within the bosonic field. The dynamics of the fluctuations Υ_t

is explicitly determined by the Landau-Pekar equations. Since in the above case $W(\alpha^2\eta_0)\Upsilon_0$ is allowed to be the vacuum, the result is of a similar form as Theorem 56 describing a formation of a coherent state which is coupled to the impurity via (3.7.2).

However, in our case the impurity particle corresponds to the slow degree of freedom whereas the almost-bosonic excitation field takes the role of the fast degree of freedom. Thus, we obtain a slightly modified version of the above effective description:

$$\begin{cases} i\partial_t \phi_t &= \tilde{V}\phi_t, \\ i\partial_t \eta_t &= \epsilon\eta_t + \tilde{\sigma} \end{cases} \tag{3.7.6}$$

with

$$\tilde{\sigma} = h_y, \tag{3.7.7}$$

$$\begin{aligned} \tilde{V} &= -\langle \epsilon\eta_t, \eta_t\rangle_\Gamma \\ &= \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \frac{2\cos(\epsilon_\alpha(k)t) - 2}{\epsilon_\alpha(k)} |h_y(k)|^2 \\ &= \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \Big((h_y)_\alpha(-k)\eta_\alpha(k) + (h_y)_\alpha(k)\overline{\eta_\alpha(k)} \Big) \end{aligned} \tag{3.7.8}$$

where $(h_y)_\alpha(k) = \lambda\hat{V}(k)e^{iky}n_\alpha(k)$. The differences are: Firstly, on our time scale the kinetic term of the impurity particle is negligible by Lemma 73 in comparison to the contribution from the almost-bosonic field. Secondly, the almost-bosonic field has a linear dispersion relation ϵ instead of a constant one as for the Fröhlich Hamiltonian (3.7.1). Thus, we cannot exploit the existence of a spectral gap as it is done in the proofs in [LMR+21]. Thirdly, the bosonic field, as the faster degree of freedom, is able to resolve the contribution of the impurity particle. Thus, the source term of the bosonic function η depends on the impurity particle position y instead of averaging over the density $|\phi_t|^2$ associated with the impurity particle. In particular, η decouples from the slow degree of freedom ϕ and becomes easily solvable at the cost of making η a function of the position y of the impurity particle. As a consequence of the y-dependence of η the effective potential becomes a constant, i.e. $V_\eta(y) = \text{const}$ for all $y \in \Lambda$.

3.8 Derivation of the Loschmidt echo

The Loschmidt echo $L_{\psi_0}(t)$ is defined as overlap between the non-interacting dynamics and the interacting dynamics, i.e.

$$L_{\psi_0}(t) := \left|\langle e^{-ih_0t}\psi_0, e^{-iH_Nt}\psi_0\rangle\right| \tag{3.8.1}$$

with H_N from (3.1.2) and $h_0 = -\beta\Delta_y$. We give a rigorous approximation of the Loschmidt echo $L_{\psi_0}(t)$ in the high-density regime for the initial state $\psi_0 = \phi \otimes R\Omega \in L(\Lambda, \mathrm{d}y) \otimes \mathcal{H}_N^-$, i.e. we assume initially a product state between the impurity wave function ϕ and the filled Fermi ball $R\Omega$. The assumptions on the initial state are natural since the typical setting in ultracold atom experiments is that the impurity is initially in a non-interacting state with the Fermi gas and then at $t = 0$ an interaction is suddenly switched on [CJL+15, CJL+16]. The Loschmidt echo is a quantity of interest in these experiments, as it is not only accessible but also characterizes the strength of the impurity's perturbation on the Fermi sea. A Loschmidt echo close to 1 indicates that the impurity remains well-defined as an isolated particle. In contrast, for polaron formation, one expects a parabolic decay according to short-time perturbation theory [KSN+12]. Over longer times, the echo exhibits a characteristic power-law decay. Ultimately, the Loschmidt echo decays to zero, signifying that the initial state has become completely orthogonal to the interacting system. This behavior was recently measured in experiments with ^{6}Li atoms [CJL+16].

We will show that the expected behavior of $L_{\psi_0}(t)$ is found for the case of an impurity particle interacting with a dense Fermi gas. Our technique uses the recently established rigorous effective description of the interacting dynamics of the impurity particle. In this effective picture the impurity particle creates and annihilates almost-bosonic collective particle-hole excitations such that the dynamics can be approximated as an explicit coupled coherent state. Since this effective state can be seen as a superposition over different numbers of almost-bosonic excitations, we can identify the Loschmidt echo as the overlap between the non-interacting dynamics with the zero excitation sector which is exactly given by the normalization constant of the coupled coherent state.

3.8.1 Effective Loschmidt echo and discussion

Definition 74. We define the *effective Loschmidt echo* for $t \geq 0$ as

$$L^{\mathrm{eff}}(t) := \prod_{k \in \mathbb{Z}^3 \setminus \{0\}} \exp\left(-\frac{\pi}{4} \lambda^2 \hat{V}(k)^2 \frac{1}{|k|} \left\{ \log(2k_{\mathrm{F}}|k|t) - \mathrm{Ci}(2k_{\mathrm{F}}|k|t) + \gamma \right\} \right)$$

where γ is the Euler-Mascheroni constant and $\mathrm{Ci} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denotes the integral cosine.

Remark 75. Due to the identity

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos(y)}{y} \mathrm{d}y = \gamma + \log x + \int_0^x \frac{\cos(y) - 1}{y} \mathrm{d}y \tag{3.8.2}$$

for all $x \in \mathbb{R}_{\geq 0}$, the map $x \mapsto h(x) := x^{-1} (\log x - \mathrm{Ci}(x) + \gamma)$ is well-defined and bounded for all $x \geq 0$. Thus, for all $t \geq 0$

$$L^{\mathrm{eff}}(t) = \prod_{k \in \mathbb{Z}^3} e^{-\frac{\pi}{2} k_{\mathrm{F}} t \lambda^2 \hat{V}(k)^2 h(2k_{\mathrm{F}}|k|t)} \tag{3.8.3}$$

is bounded and for $t \in \mathcal{O}(\lambda^{-1} k_{\mathrm{F}}^{-1})$ the effective Loschmidt echo is of order 1. Also note that $\{\log(x) - \mathrm{Ci}(x) + \gamma\}/x \xrightarrow{x\to 0} 0$, thus we can exclude $k \neq 0$.

Theorem 76. *Let $\psi_0 = \phi \otimes \Omega_0$ with $\Omega_0 = R\Omega$ denoting the filled Fermi ball and ϕ satisfying $\sum_{i=1}^{3} \left(\|\partial_{y_i}\phi\| + \|\partial_{y_i}^2 \phi\| \right) \leq c$ for a $c > 0$. There exists a non-negative map $t \mapsto C(t)$ such that for all $t \geq 0$*

$$\left| L_{\psi_0}(t) - L^{\mathrm{eff}}(t) \right| \leq C(k_F t) \max\{k_F^{-\frac{2}{15}}(1+t), (\beta t)^2\}$$

where $P(t) \in \mathbb{R}$ and η_t are given by (3.3.14) and (3.3.15), respectively.

Remark 77. In the proof, we obtain a slightly stronger statement than in Theorem 76, as we can also incorporate the phase of $\langle e^{-ih_0 t}\psi_0, e^{-iH_N t}\psi_0\rangle$ into our approximation. However, the phase $P(t)$, as defined in (3.3.14), has its dominant contribution in the energy $E_N^{\mathrm{pw}} = \sum_{k\in B_{\mathrm{F}}} |k|^2$, i.e. $P(t) = E_N^{\mathrm{pw}} t + \mathcal{O}\left(k_{\mathrm{F}}^2 t^2\right)$. Thus the phase is well approximated by a linear function with slope $E_N^{\mathrm{pw}} \in \mathcal{O}(k_{\mathrm{F}}^5)$.

Remark 78 (universal properties of L^{eff}). We emphasize that $t \mapsto L^{\mathrm{eff}}(t)$ qualitatively satisfies the properties the experimental measurement of $L(t)$ obtained in [CJL+16] (the Loschmidt echo is referred to as contrast in this work). For small $t \approx 0$, L^{eff} exhibits a parabolic transient as the Taylor series is of the form

$$L^{\mathrm{eff}}(t) = 1 - \pi (k_{\mathrm{F}} t)^2 \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \lambda^2 \hat{V}(k)^2 |k| + \mathcal{O}(t^4). \tag{3.8.4}$$

The oscillatory behavior is inherited from the Ci-function (see (3.8.2)). The large time behavior is given by a power-law decay: since $-\mathrm{Ci}(x) + \gamma \leq 1$ for all $x \geq 2$, it holds for all $t \geq k_{\mathrm{F}}^{-1}$

$$L^{\mathrm{eff}}(t k_{\mathrm{F}}^{-1}) \geq (t)^{-\alpha} \exp\left(- \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\pi}{4} \frac{\lambda^2 \hat{V}(k)^2}{|k|} (\log(2|k|) + 1) \right) \tag{3.8.5}$$

with

$$\alpha := \lambda^2 \frac{\pi}{4} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\hat{V}(k)^2}{|k|} \tag{3.8.6}$$

Furthermore, since $\mathrm{Ci}(x) \xrightarrow{x\to 0} 0$, it holds for $t \gg k_{\mathrm{F}}^{-1}$

$$L^{\mathrm{eff}}(t k_{\mathrm{F}}^{-1}) = C(t)^{-\alpha} + o\left(t^{-\alpha}\right) \tag{3.8.7}$$

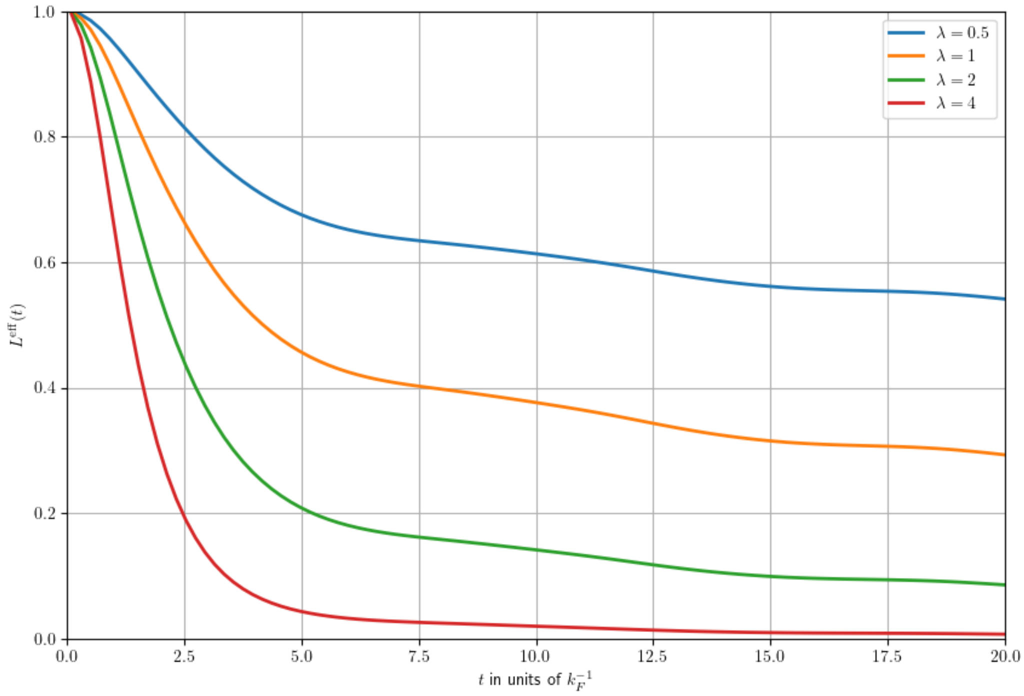

Figure 3.8.1: Exemplary plot of $L^{\mathrm{eff}}(t)$ for a truncated function $\hat{V}(k) = \frac{\chi(|k|<r_0)}{1+k^2}$ for $r_0 > 0$.

with the explicit constant

$$C = \exp\left(- \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\pi}{4} \frac{\lambda^2 \hat{V}(k)^2}{|k|} \left(\log(2|k|) + \gamma\right) \right). \tag{3.8.8}$$

The fact that $L^{\mathrm{eff}}(t)$ tends to zero as $t \to \infty$ indicates that the free evolution becomes orthogonal to its time-evolved interacting state, reflecting Anderson's orthogonality catastrophe [NDD69, KSN+12].

3.8.2 Proof of Theorem 76

Lemma 79. *Under the assumptions of Theorem 56, there exists a non-negative map $t \mapsto C(t)$ such that*

$$\left| e^{-\|\eta_t\|^2/2} - L^{\mathrm{eff}}(t) \right| \leq C(k_F t) k_F^{-\frac{2}{15}}.$$

Proof. Due to Lemma 70 it holds for $\varepsilon_N(t) := g(k_{\mathrm{F}}|k|t)k_{\mathrm{F}}^{-\frac{2}{15}}$ with a non-negative map g

$$
\begin{aligned}
e^{-\|\eta_t\|^2/2} - L^{\mathrm{eff}}(t) &= e^{-\frac{\|\eta_t\|^2}{2}} - e^{-\frac{\|\eta_t\|^2}{2}(1+\varepsilon_N(t))} \\
&= 2e^{-\frac{\|\eta_t\|^2}{2}} e^{-\varepsilon_N(t)\frac{\|\eta_t\|^2}{4}} \sinh\left(\varepsilon_N(t)\frac{\|\eta_t\|^2}{4}\right) \\
&\leq \exp\left(-\frac{\|\eta_t\|^2}{2} - \varepsilon_N(t)\frac{\|\eta_t\|^2}{4} + \varepsilon_N(t)^2\frac{\|\eta_t\|^4}{16}\right)\varepsilon_N(t)\frac{\|\eta_t\|^2}{2} \qquad (3.8.9)
\end{aligned}
$$

where we used the well-known bound $\sinh(x) \leq e^{x^2/6}x$ in the last line. Thus for all $t \in \mathcal{O}(\lambda^{-1}k_{\mathrm{F}}^{-1})$ we can make use of the bound Lemma 70 to obtain the desired bound. ∎

Lemma 80. *Under the assumptions of Theorem 56, there exists a non-negative map $t \mapsto C(t)$ such that*

$$
|\langle \phi \otimes \Omega, (W(\eta_t) - e^{-\|\eta_t\|^2/2})\phi \otimes \Omega\rangle| \leq C(k_F t)k_F^{-\frac{2}{15}}.
$$

for a $b > 0$.

Proof. We recall that $h_0 = -\beta\Delta_y$ and that the almost bosonic operators satisfy $c(\dot{\eta}_s)\phi\otimes\Omega = 0$ and the approximate shift property

$$
c(\dot{\eta}_s)W(\eta_s) = W(\eta_s)c(\dot{\eta}_s) + \langle \eta_s, \dot{\eta}_s\rangle_\Gamma + \mathcal{E} \qquad (3.8.10)
$$

with the error from the almost-bosonic Weyl operator

$$
\mathcal{E} = \langle \dot{\eta}_s, \mathcal{R}\rangle_\Gamma := \sum_{k\in\Gamma}\sum_{\alpha\in\mathcal{I}_k} \overline{(\dot{\eta}_s)_\alpha(k)} \int_0^1 \mathrm{d}\tau\ W(-\tau\eta_s)\left(\sum_{l\in\Gamma}\eta_\alpha(l)\mathcal{E}_\alpha(l,k)\right)W(\tau\eta_s). \qquad (3.8.11)
$$

We obtain for the derivative

$$
\begin{aligned}
&\left|\partial_s\langle \phi\otimes\Omega, \left(W(\eta_s) - e^{-\|\eta_s\|^2/2}\right)\phi\otimes\Omega\rangle\right| \\
&\leq \left|\langle\phi\otimes\Omega, \left((-\langle\dot{\eta}_s,\eta\rangle_\Gamma + i\mathrm{Im}\langle\dot{\eta}_s,\eta_s\rangle_\Gamma)W(\eta_s) + \mathcal{E} + \tilde{\mathcal{E}} - \mathrm{Re}\langle\dot{\eta}_s,\eta_s\rangle_\Gamma e^{-\|\eta_t\|^2/2}\right)\phi\otimes\Omega\rangle\right| \\
&\leq |\mathrm{Re}\langle\dot{\eta}_s,\eta_s\rangle_\Gamma|\left|\langle e^{-ih_0 s}\phi\otimes\Omega, \left(W(\eta_s) - e^{-\|\eta_t\|^2/2}\right)\phi\otimes\Omega\rangle\right| \\
&\quad + \|\left(\mathcal{E}+\tilde{\mathcal{E}}\right)\phi\otimes\Omega\| \qquad (3.8.12)
\end{aligned}
$$

with $\tilde{\mathcal{E}} = 2i\int_0^1 \mathrm{d}\tau\ W((1-\tau)\eta_t)\mathrm{Im}\langle\dot{\eta}_t, \mathcal{R}^{1-\tau}\rangle_\Gamma W(\tau\eta_t)$ as the error from the derivative.

Now in the error analysis we show that $\|\left(\mathcal{E}+\tilde{\mathcal{E}}\right)\phi\otimes\Omega\|$ is small and that $|\mathrm{Re}\langle\dot{\eta}_s,\eta_s\rangle_\Gamma| \leq C$ for $s \in \mathcal{O}(k_{\mathrm{F}}^{-1})$. We estimate using Lemma 70

$$
\|\mathcal{E}\phi\otimes\Omega\| \leq CMN^{-\frac{2}{3}+\delta}\|\dot{\eta}_s\|\left(e^{C\|\eta_s\|}-1\right)\|(\mathcal{N}+3)\phi\otimes\Omega\|
$$

$$\leq CMN^{-\frac{2}{3}+\delta}\|\dot{\eta}_s\|\left(f_C(k_{\mathrm{F}}s)-1\right)\|(\mathcal{N}+3)\phi\otimes\Omega\| \tag{3.8.13}$$

and since $\tilde{\mathcal{E}} = \mathrm{Error}_2$ from (3.5.30) satisfies

$$\begin{aligned}\|\tilde{\mathcal{E}}\phi\otimes\Omega\| &\leq Ck_{\mathrm{F}}sMN^{-\frac{2}{3}+\delta}\int_0^s \mathrm{d}\sigma\int_0^1 \mathrm{d}\tau\ \left(e^{C\|\eta_\sigma\|}-e^{C\|\eta_\sigma\|\tau}\right)\|(\mathcal{N}+5)\phi\otimes\Omega\| \\ &\leq C\left(f_C(k_{\mathrm{F}}s)-1\right)k_{\mathrm{F}}sMN^{-\frac{2}{3}+\delta}\|(\mathcal{N}+5)\phi\otimes\Omega\| \end{aligned}\tag{3.8.14}$$

where we estimated the exponential $e^{C\|\eta_s\|}$ by a monotonically increasing function. Moreover, one can estimate

$$\begin{aligned}\|\dot{\eta}_s\| &= 2\Bigg(\sum_{k\in\Gamma}|\hat{V}(k)|^2\sum_{\alpha\in\mathcal{I}_k}n_\alpha(k)^2|\sin\left(\epsilon_\alpha(k)s/2\right)|^2\Bigg)^{1/2} \\ &\leq Ck_{\mathrm{F}}\Bigg(\sum_{k\in\Gamma}|\hat{V}(k)|^2\ |k|\sum_{\alpha\in\mathcal{I}_k}\sigma(p_\alpha)u_\alpha(k)^2|\sin\left(\epsilon_\alpha(k)s/2\right)|^2\Bigg)^{1/2} \\ &\leq Ck_{\mathrm{F}}\Bigg(\sum_{k\in\Gamma}|\hat{V}(k)|^2\ |k|\int_0^1\mathrm{d}u\ \sin(k_{\mathrm{F}}|k|su)^2u\Bigg)^{1/2} \\ &\leq Ck_{\mathrm{F}}\|(\cdot)^{1/2}\hat{V}\|_2. \end{aligned}\tag{3.8.15}$$

In total it follows for the derivative

$$\begin{aligned}&\left|\partial_s\langle\phi\otimes\Omega,\left(W(\eta_s)-e^{-\|\eta_s\|^2/2}\right)\phi\otimes\Omega\rangle\right| \\ &\leq C\beta(s)\left(\left|\langle\phi\otimes\Omega,\left(W(\eta_s)-e^{-\|\eta_s\|^2/2}\right)\phi\otimes\Omega\rangle\right|+CMN^{-\frac{2}{3}+\delta}\right)\end{aligned}\tag{3.8.16}$$

where

$$\beta(s)=\max\left\{\log\left(f_C(k_{\mathrm{F}}s)\right)k_{\mathrm{F}},\left(f_C(k_{\mathrm{F}}s)-1\right)k_{\mathrm{F}}s\right\} \tag{3.8.17}$$

is a monotonically increasing function which implies by the Grönwall inequality

$$\begin{aligned}\left|\langle\phi\otimes\Omega,\left(W(\eta_t)-e^{-\|\eta_t\|^2/2}\right)\phi\otimes\Omega\rangle\right| &\leq CMN^{-\frac{2}{3}+\delta}\left(e^{C\int_0^t\mathrm{d}s\ \beta(s)}-1\right)\|\phi\otimes\Omega\| \\ &\leq CMN^{-\frac{2}{3}+\delta}\left(e^{Ct\beta(t)}-1\right). \end{aligned}\tag{3.8.18}$$

■

Proof of Theorem 76. Since $\phi\otimes\Omega$ is normed it holds

$$\begin{aligned}&\langle e^{-ih_0t}\phi\otimes R\Omega, e^{-iH_Nt}\phi\otimes R\Omega\rangle \\ &= \langle(e^{-ih_0t}-1)\phi\otimes\Omega,\left(R^*e^{-iH_Nt}R-e^{iP(t)}W(\eta_t)\right)\phi\otimes\Omega\rangle\end{aligned}$$

$$
\begin{aligned}
&+ \langle \phi \otimes \Omega, \left(R^* e^{-iH_N t} R - e^{iP(t)} W(\eta_t)\right) \phi \otimes \Omega \rangle \\
&+ \langle \phi \otimes \Omega, \left(e^{iP(t)} W(\eta_t) - e^{iP(t)} e^{-\|\eta_t\|^2/2}\right) \phi \otimes \Omega \rangle \\
&+ e^{iP(t)} e^{-\|\eta_t\|^2/2} - e^{iP(t)} L^{\mathrm{eff}}(t) \\
&+ e^{iP(t)} L^{\mathrm{eff}}(t)
\end{aligned} \tag{3.8.19}
$$

Thus, by the triangle inequality it holds

$$
\begin{aligned}
&\left| \langle e^{-ih_0 t} \phi \otimes R\Omega, e^{-iH_N t} \phi \otimes R\Omega \rangle - e^{iP(t)} e^{-\|\eta_t\|^2/2} \right| \\
&\leq \|(e^{-ih_0 t} - 1)\phi \otimes \Omega\| \; \| \left(R^* e^{-iH_N t} R - e^{iP(t)} W(\eta_t)\right) \phi \otimes \Omega \| \\
&\quad + \left| \langle \phi \otimes \Omega, \left(W(\eta_t) - e^{-\|\eta_t\|^2/2}\right) \phi \otimes \Omega \rangle \right| + |e^{-\|\eta_t\|^2/2} - L^{\mathrm{eff}}(t)|.
\end{aligned} \tag{3.8.20}
$$

Since $\sum_{i=1}^3 \|\partial_{y_i}^2 \phi\| \leq c$ for a constant $c > 0$ by assumption, it follows by Duhamel's formula

$$
\|(e^{-ih_0 t} - 1)\phi \otimes \Omega\| \leq \beta \int_0^t \mathrm{d}s \|(\Delta_y \phi) \otimes \Omega\| \leq c\beta t \tag{3.8.21}
$$

and we obtain with Theorem 56, Lemma 80, Lemma 79 and the reverse triangle inequality the desired result. ■

A.1 Estimates within the bosonization framework

In this section we collect all relevant estimates of the bosonization framework which was first developed in [BNP+19, BNP+21a, BNP+21b, BPSS23] in the semiclassical regime. In addition to the references we present brief proof sketches where we think they are helpful for the interested reader.

Lemma 81 (Approximation of $n_\alpha(k)$, [BNP+21b, Lemma 5.1])**.** *For $N^{2\delta} \ll M \ll N^{\frac{2}{3}-2\delta}$ and $k \in \Gamma, \alpha \in \mathcal{I}_k$ it holds*

$$n_\alpha(k)^2 = \frac{4\pi k_F^2}{M}|k \cdot \hat{\omega}_\alpha|\big(1 + o(1)\big).$$

Note that $|k \cdot \hat{\omega}_\alpha| > N^{-\delta}$ by construction of $\mathcal{I}_k$.

Lemma 82 (Approximation of $\sum_{\alpha \in \mathcal{I}_k} n_\alpha(k)^2$)**.** *For $N^{2\delta} \ll M \ll N^{\frac{2}{3}-2\delta}$ and $k \in \Gamma$ it holds*

$$\sum_{\alpha \in \mathcal{I}_k} n_\alpha(k)^2 = k_F^2|k|\pi \left\{1 + \mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta} + N^{-\delta}\right)\right\}.$$

Proof. It holds from [BNP+19, Proposition 3.1]

$$n_\alpha(k)^2 = k_{\mathrm{F}}^2|k|\sigma(p_\alpha)u_\alpha(k)^2\left(1 + \mathcal{O}(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta})\right)$$

with $\cos\theta_\alpha := |\hat{k} \cdot \hat{\omega}_\alpha| \equiv u_\alpha(k)^2$. Choose φ_α for the azimuth angle of ω_α. We estimate the α-sum by an appropriate surface integral over the patch p_α

$$\begin{aligned}\left|\int_{p_\alpha} \mathrm{d}\sigma\ \cos\theta - \sigma(p_\alpha)\cos\theta_\alpha\right| &\leq \sup_{\hat{\omega}(\theta,\varphi)\in p_\alpha}\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta\right| \sup_{(\theta,\varphi)\in p_\alpha}|\theta - \theta_\alpha|\,\sigma(p_\alpha)\\ &\leq CM^{-\frac{3}{2}}\end{aligned}$$

where we used $|\theta - \theta_\alpha| \leq CM^{-\frac{1}{2}}$ and $\sigma(p_\alpha) \leq CM^{-1}$ by the patch construction. Note that the integral over $\tilde{S} := \bigcup_{\alpha\in\mathcal{I}_k} p_\alpha$ which excludes a collar of width $N^{-\delta}$ can be approximated by an integral over the half-sphere S

$$\left|\int_{\tilde{S}} \mathrm{d}\sigma\ \cos\theta - \int_S \mathrm{d}\sigma\ \cos\theta\right| < C\left(N^{-\delta} + M^{\frac{1}{2}}N^{-\frac{1}{3}}\right)$$

which can be calculated explicitly

$$\int_S \mathrm{d}\sigma\ \cos\theta = 2\pi\int_0^{\pi/2} \mathrm{d}\theta\ \cos\theta\sin\theta = \pi.$$

Thus in total we obtain with the triangle inequality and $|k| < C$

$$\begin{aligned}\left|\sum_{\alpha\in\mathcal{I}_k} n_\alpha(k)^2 - k_{\mathrm{F}}^2|k|\pi\right| &= \left|\sum_{\alpha\in\mathcal{I}_k} n_\alpha(k)^2 - k_{\mathrm{F}}^2|k|\int_S \mathrm{d}\sigma\ \cos\theta\right| \\ &\le Ck_{\mathrm{F}}^2|k|\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta} + M^{-\frac{1}{2}} + N^{-\delta} + M^{\frac{1}{2}}N^{-\frac{1}{3}}\right) \\ &\le Ck_{\mathrm{F}}^2|k|\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta} + N^{-\delta}\right).\end{aligned}$$

■

Lemma 83 (Estimates on the CCR error term, [BNP+21b, Lemma 5.2])**.** *For $k', k \in \Gamma$ and $\alpha \in \mathcal{I}_k, \beta \in \mathcal{I}_{k'}$ the error term $\mathcal{E}_\alpha(k,k')$ as defined in (3.2.22) satisfies $\mathcal{E}_\alpha(k,k') = \mathcal{E}_\alpha(k',k)^*$, commutes with $\mathcal{N}$ and for all $\gamma \in \mathcal{I}_k \cap \mathcal{I}_{k'}$ it holds for all $\zeta \in \mathcal{F}$*

$$\begin{aligned}|\mathcal{E}_\gamma(k,k')|^2 \le \sum_{\gamma\in\mathcal{I}_k\cap\mathcal{I}_{k'}} |\mathcal{E}_\gamma(k,k')|^2 &\le C\left(MN^{-\frac{2}{3}+\delta}\mathcal{N}\right)^2, \\ \sum_{\gamma\in\mathcal{I}_k\cap\mathcal{I}_{k'}} \|\mathcal{E}_\gamma(k,k')\zeta\| &\le CM^{\frac{3}{2}}N^{-\frac{2}{3}+\delta}\|\mathcal{N}\zeta\|.\end{aligned}$$

Lemma 84 (Pair operator bounds, [BNP+21b, Lemma 5.3])**.** *It holds for all $k \in \Gamma$ and $\psi \in \mathcal{F}, f \in l^2(\mathcal{I}_k)$:*

(i). $\sum_{\alpha\in\mathcal{I}_k} \|c_\alpha(k)\psi\|^2 \le \|\mathcal{N}^{\frac{1}{2}}\psi\|^2$,

(ii). $\sum_{\alpha\in\mathcal{I}_k} \|c_\alpha(k)\psi\| \le M^{\frac{1}{2}}\|\mathcal{N}^{\frac{1}{2}}\psi\|$,

(iii). $\sum_{\alpha\in\mathcal{I}_k} \|c^*_\alpha(k)\psi\| \le M^{\frac{1}{2}}\|(\mathcal{N}+M)^{\frac{1}{2}}\psi\|$,

(iv). $\sum_{\alpha\in\mathcal{I}_k} \|c^*_\alpha(k)\psi\|^2 \le \|(\mathcal{N}+M)^{\frac{1}{2}}\psi\|^2$,

(v). $\|\sum_{\alpha\in\mathcal{I}_k} f_\alpha c^*_\alpha(k)\psi\| \le \|f\|_{l^2}\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\|$,

(vi). $\sum_{\alpha\in\mathcal{I}_k} c^*_\alpha(k)c_\alpha(k) \le \mathcal{N}$.

Lemma 85 (Error of linearized kinetic energy, [BNP+21b, Lemma 8.2])**.** *It holds for $k \in \Gamma$, $\alpha \in \mathcal{I}_k$ and all $\psi \in \mathcal{F}$*

$$[\mathbb{H}_0, c^*_\alpha(k)] = 2k_F|k\cdot\hat{\omega}_\alpha|c^*_\alpha(k) + \mathfrak{E}^{\mathrm{lin}}_\alpha(k)^*$$

with

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}_k} \|\mathfrak{E}^{\mathrm{lin}}_\alpha(k)\psi\|^2 &\le C\left(N^{\frac{1}{3}}M^{-\frac{1}{2}}\right)^2\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\|^2 \\ \sum_{\alpha\in\mathcal{I}_k} \|\mathfrak{E}^{\mathrm{lin}}_\alpha(k)\psi\| &\le CN^{\frac{1}{3}}\|\mathcal{N}^{\frac{1}{2}}\psi\|.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}[\mathbb{H}_0, c^*_\alpha(k)] &= \frac{1}{n_\alpha(k)} \sum_{p\in B_{\mathrm{F}}^c\cap B_\alpha, p-k\in B_{\mathrm{F}}\cap B_\alpha} \sum_{l\in\mathbb{Z}^3} \left[e(l)a^*_l a_l, a^*_p a^*_{p-k}\right] \\ &= \frac{1}{n_\alpha(k)} \sum_{p\in B_{\mathrm{F}}^c\cap B_\alpha, p-k\in B_{\mathrm{F}}\cap B_\alpha} \left(e(p)+e(p-k)\right) a^*_p a^*_{p-k} \\ &= 2k_{\mathrm{F}}|k\cdot\hat\omega_\alpha| c^*_\alpha(k) + \mathfrak{E}^{\mathrm{lin}}_\alpha(k)^*\end{aligned}$$

One makes the identification $\mathfrak{E}^{\mathrm{lin}}_\alpha(k) \equiv c^g_\alpha(k)$ representing a weighted operator (see [BNP+21b, eq. (5.11)]) with

$$\begin{aligned} g(p,k) &= e(p)+e(p-k) - 2k_{\mathrm{F}}|k\cdot\hat\omega_\alpha| = |p|^2 - |p-k|^2 - 2k_{\mathrm{F}}|k\cdot\hat\omega_\alpha| \\ &= \left(2k\cdot(p-k_{\mathrm{F}}\hat\omega_\alpha) - |k|^2\right)\end{aligned}$$

where we used $e(p)$ as defined in (3.2.8). The bound follows from [BNP+21b, Lemma 5.4] which depends on $\|g\|_{l^\infty}$. This can be estimated by using $\operatorname{diam}(B_\alpha) \le CN^{\frac{1}{3}}M^{-\frac{1}{2}}$ such that $|g(p,k)| \le CN^{\frac{1}{3}}M^{-\frac{1}{2}}$. ∎

Lemma 86 (Error of bosonized kinetic energy, [BNP+21b, eq. (8.6)])**.** *It holds for $k\in\Gamma$, $\alpha\in\mathcal{I}_k$ and all $\psi\in\mathcal{F}$*

$$[\mathbb{D}_{\mathrm{B}}, c^*_\alpha(k)] = 2k_F|k\cdot\hat\omega_\alpha| c^*_\alpha(k) + \mathfrak{E}^{\mathrm{B}}_\alpha(k)^*$$

with

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}_k} \|\mathfrak{E}^B_\alpha(k)\psi\|^2 &\le C\left(k_F M N^{-\frac{2}{3}+\delta}\right)^2 \|(\mathcal{N}+1)^{\frac{3}{2}}\psi\|^2, \\ \sum_{\alpha\in\mathcal{I}_k} \|\mathfrak{E}^B_\alpha(k)\psi\| &\le C k_F M^{\frac{3}{2}} N^{-\frac{2}{3}+\delta} \|(\mathcal{N}+1)^{\frac{3}{2}}\psi\|.\end{aligned}$$

Proof. It holds $\mathfrak{E}^{\mathrm{B}}_\alpha(k) = 2k_{\mathrm{F}}\sum_l |l\cdot\hat\omega_\alpha| \mathcal{E}^*_\alpha(l,k) c_\alpha(l)\chi(\alpha\in\mathcal{I}_l)$ and therefore

$$\begin{aligned}&\sum_{\alpha\in\mathcal{I}_k} \|\mathfrak{E}^{\mathrm{B}}_\alpha(k)\psi\|^2 \\ &\le \sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l} \left(\sum_{l\in\Gamma} \|2k_{\mathrm{F}}|l\cdot\hat\omega_\alpha|\mathcal{E}^*_\alpha(l,k)c_\alpha(l)\psi\|\right)^2 \le Ck_{\mathrm{F}} \sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\left(\sum_{l\in\Gamma}\|\mathcal{E}^*_\alpha(l,k)c_\alpha(l)\psi\|\right)^2 \\ &\le Ck_{\mathrm{F}} \sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\left(\sum_{l\in\Gamma} CMN^{-\frac{2}{3}+\delta}\|\mathcal{N}c_\alpha(l)\psi\|\right)^2 \\ &\le Ck_{\mathrm{F}}\left(CMN^{-\frac{2}{3}+\delta}\right)^2 \sum_{\alpha\in\mathcal{I}_k\cap\mathcal{I}_l}\sum_{l\in\Gamma}\sum_{l'\in\Gamma} \|c_\alpha(l)(\mathcal{N}-2)\psi\|\;\|c_\alpha(l')(\mathcal{N}-2)\psi\|\end{aligned}$$

$$\leq Ck_{\mathrm{F}} \left(CMN^{-\frac{2}{3}+\delta} \right)^2 \|(\mathcal{N}+1)^{\frac{3}{2}}\psi\|^2$$

where we used in the second line $l \in \Gamma$ bounded, Lemma 83 in the third line, $\mathcal{N} c_\alpha(l) = c_\alpha(l)(\mathcal{N}-2)$ in the fourth line and Cauchy-Schwarz and Lemma 84 in the last line.

The second statement simply follows from Cauchy-Schwarz. ■

Lemma 87 (Approximation of patch decomposed operators). *It holds for all $k \in \Gamma$*

$$\|\big(b(k) - \sum_{\alpha \in \mathcal{I}_k} n_\alpha(k) c_\alpha(k) + h.c.\big)\psi\| \leq C(N^{\frac{1}{3}-\frac{\delta}{2}} + N^{\frac{1}{6}} M^{\frac{1}{4}})\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\|.$$

Lemma 88 (Estimate of non-bosonizable terms, [BNP+21a, eq. (4.6)]). *It holds for $\mathcal{E}$ as defined in (3.2.9b) the following estimate for all $\psi \in \mathcal{F}$*

$$\|\mathcal{E}\psi\| \leq C\lambda \|\hat{V}\|_1 \|\mathcal{N}\psi\|.$$

Chapter 4

Conclusion

In this thesis, we have made significant contributions to the understanding of strongly interacting fermionic systems and polaron dynamics. Our research has focused on extending the derivation of the Hartree-Fock equations to a novel regime and exploring the dynamics of impurity particles within a dense Fermi gas.

Summary of key findings and implications Firstly, we derived the Hartree-Fock equations for strongly interacting fermions, achieving a rigorous approximation at the level of the reduced one-body density. This derivation is particularly notable for its exploration of a new regime characterized by a "quantum" density profile, where classical phase space density approximations are not expected to hold. Unlike previous works, our setting is, through re-scaling, equivalent to an unscaled system that does not depend on weak coupling constants on volumes of order 1. This makes it highly relevant for both fundamental physics and practical applications. Our result is valid on an optimal time scale, within which the mean-field description is expected to be valid. Furthermore, our approach can incorporate singular and long-range potentials, enhancing its applicability to a broader range of physical systems.

Secondly, we have provided a derivation of the polaron dynamics for an impurity particle interacting with a dense Fermi gas. After deriving a Fröhlich-type Hamiltonian, we went beyond the typical Hamiltonian-level approximations by developing an effective time-dependent state that captures the leading order effect. This work reveals a significant difference from free time evolution on the time scale of interest, contrasting with previous studies that assumed weak couplings. Therefore, we partially refuted a conjecture raised in Subsection 1.3.1.1 that the free-decoupling effect in 3D, due to the Pauli exclusion principle, remains valid for stronger interactions and longer time scales. Additionally, we calculated the response function, which exhibits characteristic features of the polaron quasi-particle, which align with recent experimental and theoretical findings.

Contributions to future research Our work suggests several potential directions for future research, particularly in the application of advanced mathematical techniques to quantum many-body systems.

Firstly, the gauge phase method from Chapter 2, which we employed to extract leading-order effects, can be further extended to investigate short-time regimes. This method is particularly relevant for fermionic systems, where a significant portion of the particles possess high momenta due to the Pauli exclusion principle. Consequently, the time scales of interest for effective theories are relatively short. This can be utilized by the gauge phase method, as it introduces explicit time dependencies into the transformed system, potentially minimizing problematic terms. This methodology may offer insights into new dynamics in strongly interacting systems, potentially enhancing our understanding of formation processes and non-equilibrium phenomena.

Secondly, the dynamical incorporation of almost-bosonic excitations in fermionic systems presents a powerful approach, albeit currently restricted to the torus geometry. The central tool here is to decompose interactions into an almost-bosonic component and a remainder. The almost-bosonic component corresponds to the creation and annihilation of particle-hole pairs, while the transport of excitations or holes constitutes the remainder terms. For future research, one could explore the interplay between internal interactions and impurity-induced interactions. Additionally, it would be valuable to combine this approach with the perturbative resolvent method to demonstrate that, under weak coupling conditions, the impurity particle exhibits free evolution. Recent works into this direction involve the study of Bose-Fermi-mixtures [CMMP25].

Thirdly, our approach of deriving a simpler intermediate Hamiltonian and subsequently identifying an effective state, which captures the leading-order effect, offers potential for exploring the formation of new states in a dynamical sense. By explicitly tracking the real-time evolution of quantum states, we gain insight into how interactions gradually give rise to collective phenomena, rather than assuming their existence from a purely spectral perspective. This perspective could be particularly relevant for superconductivity, where the ground state of an attractive fermionic system is believed to exhibit Cooper pairs. The dynamical counterpart would involve investigating whether the time-evolved state effectively exhibits superconductive

features, such as long-range off-diagonal order.

Limitations and open questions While our research has made significant advances, several limitations and open questions remain.

In the context of the Hartree-Fock equations, future work should aim to include more singular interactions, particularly the Coulomb case, to enhance the model's applicability to real-world systems. One possible approach is to find a better estimate of the bad kinetic energy by propagating a more controllable quantity and varying the auxiliary dynamics to allow for a better norm approximation. In addition, relaxing the L^∞-conditions on the effective dynamics or reducing these time-dependent conditions to initial state conditions would be preferable. This requires propagation-in-time estimates, possibly via Grönwall estimates. Another possibility is to avoid these conditions by finding different estimates for the mean-field quantities, as can be easily done for bounded interactions in confined systems.

For polaron dynamics, future investigations could focus on incorporating more singular and decaying interactions V between the impurity particle and the Fermi gas, as well as weak internal interactions $g_N W$ within the Fermi gas. Singular interactions V are likely feasible as long as the singularity is integrable. One possible approach is to decompose the Fourier transform $\hat{V}$ into singular and decaying parts, where the decaying part contributes minimally and the singular part is treated similarly to a compact support scenario. For internal interactions $g_N W$, we expect no significant effect for sufficiently small coupling g_N (see discussion in Subsection 1.3.1). Otherwise, these interactions might be included in the almost-bosonic description through a Bogoliubov transformation, as demonstrated in [BNP+21a]. This would effectively alter the dispersion relation of the collective excitations. Additionally, considering systems with multiple impurity particles and deriving mediated interactions between polarons would offer insights into the collective behaviors of these quasi-particles [BHF+23]. Furthermore, exploring longer time scales is of particular interest, as physicists expect decaying processes, at least for the repulsive Fermi polaron, due to its finite lifetime [KSN+12]. The same questions can also be addressed for the low density case, where recent progress in bosonization methods has been achieved in static settings [Gia23, GHNS24, Gia24].

Addressing these remaining challenges in future research would further extend and deepen our understanding of quantum many-body systems and their potential applications.

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