Derivation of the Effective Dynamics for the Bose Polaron

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Derivation of the Effective Dynamics for the Bose Polaron

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Titre : Dérivation de la Dynamique Effective du Polaron Bosonique

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Résumé : Nous considérons la dynamique d'un gaz quantique dense et de grande taille composé de N bosons évoluant dans $\mathbb{R}^3$ en présence d'une particule d'impureté dans la limite de champs moyen. Dans l'état initial du système, presque tous les bosons sont dans le condensat de Bose-Einstein, avec quelques particules hors du condensat appelées excitations. Notre objectif est de prouver la persistance de la condensation et la formation d'un polaron bosonique, une quasi-particule composée de l'impureté et des bosons environnants.
Dans les expériences, les particules d'impureté, qui se distinguent du gaz bosonique, sont utilisées pour sonder les propriétés du gaz, telles que la superfluidité [Gru+24] ou sa distribution de densité [SHD10]. Le gaz de Bose est mathématiquement intriguant car la condensation de Bose-Einstein permet une analyse rigoureuse, donnant une compréhension profonde des systèmes quantiques à nombreux corps.
Dans notre résultat principal, nous dérivons une description effective de la théorie quantique des champs pour de tels systèmes à partir de la dynamique microscopique. Cette description est régie par l'hamiltonien de Bogoliubov-Fröhlich dans la limite des grandes densités de gaz ρ et des volumes Λ, modélisant la dynamique de l'impureté et des excitations, qui peuvent former une quasi-particule connue sous le nom de polaron bosonique [HL24]. Les travaux précédents ont traité le problème du champ moyen correspondant sur un volume unitaire avec des conditions aux limites périodiques [MS20; LP22]. La principale nouveauté de cette thèse réside dans la prise en compte d'un volume initial important par rapport à la plage d'interaction des bosons et dans la suppression des conditions aux limites périodiques, ce qui conduit à un condensat non constant. Ces adaptations rapprochent le modèle de la réalité physique.
Pour prouver notre résultat principal, qui est formulé dans la dynamique effective de Bogoliubov, nous contrôlons le nombre d'excitations afin d'assurer la persistance de la condensation. De plus, comme l'impureté doit rester dans le nuage de gaz pour la formation d'une quasi-particule, nous prouvons sa localisation. Pour cette deuxième partie en particulier, un contrôle raffiné de l'état initial du condensat est nécessaire afin d'éviter d'importants transferts d'énergie du condensat vers l'impureté. Le contrôle des quantités susmentionnées est établi par des estimations de propagation et par le lemme de Grönwall. En combinant le contrôle du nombre d'excitations et la localisation de l'impureté, nous validons la dynamique de Bogoliubov-Fröhlich.

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Keywords: mathematical physics, mean-field theory, cold bose gases

Abstract: We consider the dynamics of a dense and large quantum gas consisting of N bosons evolving in $\mathbb{R}^3$ in the presence of an impurity particle in the mean-field scaling. In the initial state of the system almost all bosons are in the Bose-Einstein condensate, with a few particles out of the condensate called excitations. Our goal is to prove the persistence of condensation and the formation of a Bose polaron – a quasi-particle consisting of the impurity and the surrounding bosons.
In experiments, impurity particles, distinguishable from the bosonic gas, are used to probe the gas's properties such as superfluidity [Gru+24] or its density distribution [SHD10]. The Bose gas is mathematically intriguing because Bose-Einstein condensation enables rigorous analysis, providing deep insights into many-body quantum systems.
In our main result, we derive an effective quantum field theory description for such systems from the microscopic dynamics. This description is governed by the Bogoliubov-Fröhlich Hamiltonian in the limit of large gas densities ρ and volumes Λ, modeling the dynamics of both the impurity and the excitations, which can form a quasi-particle known as the Bose polaron [HL24].
Previous works have treated the corresponding mean-field problem on unit volume with periodic boundary conditions [MS20; LP22]. The main novelty of this thesis lies in considering a large initial volume relative to the interaction range of the bosons and the removal of the periodic boundary conditions, leading to a non-constant condensate. These adaptations bring the model closer to physical reality.
To prove our main result which is formulated in the effective Bogoliubov dynamics, we control the number of excitations to ensure the persistence of the condensation. Additionally, since the impurity must remain within the gas cloud for a quasi-particle formation, we prove its localization. For this second part in particular, a refined control of the initial state of the condensate is required in order to avoid large energy transfers from the condensate to the impurity. The control of the above quantities is established by propagation estimates and Grönwall's Lemma.
Combining both the control of the number of excitations and the impurity localization, we validate the Bogoliubov-Fröhlich dynamics.

Zusammenfassung

Wir betrachten die Dynamik eines dichten und großvolumigen Quantengases, das aus N Bosonen besteht, die sich in $\mathbb{R}^3$ in Gegenwart eines Verunreinigungsteilchens in der Mean-Field-Skalierung entwickeln. Im Anfangszustand des Systems befinden sich fast alle Bosonen im Bose-Einstein-Kondensat, mit einigen wenigen Teilchen außerhalb des Kondensats, den sogenannten Anregungen. Unser Ziel ist es, die Persistenz der Kondensation und die Bildung eines Bose-Polarons nachzuweisen – ein Quasiteilchen, das aus der Verunreinigung und den umgebenden Bosonen besteht.

In Experimenten werden Verunreinigungsteilchen, die sich vom bosonischen Gas unterscheiden, verwendet, um Eigenschaften des Gases wie die Suprafluidität [Gru+24] oder seine Dichteverteilung [SHD10] zu untersuchen. Das Bose-Gas ist mathematisch faszinierend, weil die Bose-Einstein-Kondensation eine rigorose Analyse ermöglicht, die tiefe Einblicke in Vielteilchen-Quantensysteme gewährt.

In unserem Hauptergebnis leiten wir eine effektive quantenfeldtheoretische Beschreibung für solche Systeme aus der mikroskopischen Dynamik her. Diese Beschreibung wird durch den Bogoliubov-Fröhlich-Hamiltonian im Grenzfall großer Gasdichten ρ und Volumina Λ bestimmt und modelliert die Dynamik sowohl der Verunreinigung als auch der Anregungen, die ein als Bose-Polaron bekanntes Quasiteilchen bilden können [HL24].
Vorangegangene Arbeiten haben das entsprechende Mean-field-Problem auf den Einheitsvolumen mit periodischen Randbedingungen behandelt [MS20; LP22]. Die wichtigste Neuerung dieser Arbeit besteht in der Berücksichtigung eines großen Anfangsvolumens im Verhältnis zum Wechselwirkungsbereich der Bosonen und in der Aufhebung der periodischen Randbedingungen, was zu einem nicht konstanten Kondensat führt. Diese Anpassungen bringen das Modell näher an die physikalische Realität.
Um unser Hauptergebnis zu beweisen, das in der effektiven Bogoliubov-Dynamik formuliert ist, kontrollieren wir die Anzahl der Anregungen, um die Persistenz der Kondensation zu gewährleisten. Da die Verunreinigung innerhalb der Gaswolke verbleiben muss, um eine Quasiteilchenbildung zu ermöglichen, beweisen wir außerdem ihre Lokalisierung. Insbesondere für diesen zweiten Teil ist eine verfeinerte Kontrolle des Anfangszustands des Kondensats erforderlich, um einen großen Energieübertrag vom Kondensat auf die Verunreinigung zu vermeiden. Die Kontrolle der oben genannten Größen wird durch Zeitentwicklungsabschätzungen und das Grönwallsche Lemma nachgewiesen.
Durch die Kombination der Kontrolle der Anzahl der Anregungen und der Verunreinigungslokalisierung validieren wir die Bogoliubov-Fröhlich-Dynamik.

Contents

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Notation

Our notation is based on [Dec+16; LP22; NNS16; LT24; LNS15]. Let $\mathscr{H}$, $\mathscr{K}$ be $\mathbb{C}$–Hilbert spaces.

1. By C we denote a universal constant, which is independent of our scaling parameters Λ and ρ and whose value may change from one line to another.
2. We denote $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
3. For a Banach space E we denote by $L^p(\mathbb{R}^d, E)$, $p \in [1, \infty]$, the Bochner-Lebesgue spaces and by $L^0(\mathbb{R}^d, E)$ the space of all strongly measurable functions modulo functions vanishing almost everywhere. We set $\| \,.\, \|_p := \| \,.\, \|_{L^p}$. By $L^p_s(\mathbb{R}^{dn}, E)$ we denote the subspace of $L^p(\mathbb{R}^{dn}, E)$ which is symmetric under exchange of the n particle.
4. For a Banach space E, $p \in [1, \infty]$ and $m \in \mathbb{N}_0$ we denote the Sobolev spaces by $W^{m,p}(\mathbb{R}^d, E) := \{f \in L^p(\mathbb{R}^d, E) \,|\, D^\alpha f \in L^p(\mathbb{R}^d, E) \ for \ |\alpha| \leq m\}$ by $H^m = W^{m,2}(\mathbb{R}^d, E)$ and by $H^\infty = \cap_{m \in \mathbb{N}_0} H^m$.
5. Let $f \in L^2(\mathbb{R}^d, \mathscr{H})$ we denote the Fourier transform by
$$\hat{f}(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-ikx} f(x) dx \,, \quad k \in \mathbb{R}^d \,.$$
6. For $f, g \in L^2(\mathbb{R}^3, \mathbb{C})$ we denote $A\,(f \oplus Jg) := a(f) + a^*(g)$, where $a^\#(f)$ are the bosonic creation and annihilation operators.
7. We denote by $\mathscr{H} \bar{\otimes} \mathscr{K} := \mathrm{lin}\{f \otimes g | f \in \mathscr{H}, g \in \mathscr{K}\}$ the algebraic tensor product and by $\mathscr{H} \otimes_s \mathscr{K}$ the symmetric tensor product. For $\phi \in \mathscr{H}^{\otimes k}$ and $\psi \in \mathscr{H}^{\otimes l}$ we set
$$\phi_k \otimes_s \psi_l(y_1, \dots, y_{k+l}) := \frac{1}{\sqrt{k!l!(k+l)!}} \sum_{\sigma \in P_{k+l}} \phi(y_{\sigma(1)}, \dots, y_{\sigma(k)}) \psi(y_{\sigma(k+1)}, \dots, y_{\sigma(k+l)}) \,.$$
8. We denote by $\mathcal{F}(\mathscr{H}) = \bigoplus_{n=0}^\infty \mathscr{H}^{\otimes_s n}$ the symmetric Fock space over $\mathscr{H}$. For $\psi \in \mathcal{F}(\mathscr{H})$ we denote its n–th component by $\psi_n \in \mathscr{H}^{\otimes_s n}$ and define the particle number operator $\mathcal{N}\psi := \sum_{n \geq 0} n\psi_n$ on a suitable subspace of $\mathcal{F}(\mathscr{H})$.

9. For an operator A on $\mathscr{H}$ we set $A_j = I \otimes ... \otimes I \otimes A \otimes ... \otimes I$, where A is acting on the j–th space and set

$$d\Gamma(A)\psi = \sum_{n\geq 1} \sum_{1\leq j\leq n} A_j \psi_n \,.$$
$$D(d\Gamma(A)) = \Big\{\psi \in \bigoplus_{n\geq 0} D(A)^{\otimes_s n} \Big| \sum_{n\geq 1} \big\| \sum_{1\leq j\leq n} A_j \psi_n \big\|^2 < \infty \Big\}\,.$$

10. Let $\mathscr{H}$ be a Hilbert space. We define the symmetrizer $S_n : \mathscr{H}^{\otimes n} \to \mathscr{H}^{\otimes n}, f_1 \otimes \cdots \otimes f_n \mapsto 1/N! \sum_{\sigma \in P_n} f_{\sigma_1} \otimes \cdots \otimes f_{\sigma_n}$.

11. Let $A \geq -\beta$, for $\beta \in \mathbb{R}$, be an operator on a Hilbert space then q_A denotes the closed symmetric quadratic form associated to A and $Q(A)$ its quadratic form domain (see [RS80, Chapter VIII.6]). We denote $q_A(\psi) = \langle \psi, A\psi \rangle$.

12. For $f : \mathbb{R}^3 \to \mathbb{C}$ we denote $f_x(y) := f(x - y)$.

13. For a map $A : x \mapsto A_x$ on $\mathbb{R}^3$ whose values are quadratic forms we define for suitable $\psi \in Q(A) \subset L^2(\mathbb{R}^3, \mathcal{F}(\mathscr{H}))$

$$\langle \psi, A\psi \rangle = \int \langle \psi(x), A_x \psi(x) \rangle dx \,,$$
$$Q(A) = \Big\{\psi \in L^2(\mathbb{R}^3, \mathcal{F}(\mathscr{H}))\Big| \big(x \mapsto \langle \psi(x), A_x \psi(x) \rangle\big) \in L^1(\mathbb{R}^3, \mathbb{C})\Big\}\,.$$

14. We denote $L^2(\mathbb{R}^d, \mathcal{F}(L^2(\mathbb{R}^d, \mathbb{C}))$ or similar spaces often by $L^2(\mathbb{R}^d_x, \mathcal{F}(L^2(\mathbb{R}^d_y, \mathbb{C}))$ to clearly separate the different arguments of the corresponding functions without introducing too much notation.

15. We call a two-parameter family of operators $U(s, t)$, $s, t \in \mathbb{R}$, on a Hilbert space $\mathscr{H}$ which satisfies $U(r, s)U(s, t) = U(r, t)$, $U(t, t) = I$ and $U(s, t)$ is jointly strongly continuous in s and t a unitary propagator.

16. We denote by $\mathcal{L}(\mathscr{H}, \mathscr{K})$ the space of all bounded operators mapping from $\mathscr{H}$ to $\mathscr{K}$.

17. $J : \mathscr{H} \to \mathscr{H}^* : \psi \to \langle \psi, . \rangle$ denotes the canonical anti-unitary map between a Hilbert space and its dual.

18. We denote by $*$ the convolution:

$$(f * g)(x) = \int f(x - y)g(y)dy$$

for $f, g \in L^0(\mathbb{R}^3, \mathbb{C})$ such that $\big(y \mapsto f(x - y)g(y)\big) \in L^1$ for almost all $x \in \mathbb{R}^3$.

Chapter 1

Introduction

1.1 From Microscopic Dynamics to Effective Theories

Effective Theories. Deriving effective dynamics from a full microscopic description is one of the central challenges in statistical mechanics and a major focus of rigorous research in mathematical physics.
Microscopic dynamics describe the motion of all individual particles in a system, incorporating their detailed interactions – such as Newton's laws of motion or the Schrödinger equation. In contrast, an effective theory is designed to describe specific physical phenomena while ignoring certain details of the underlying system. A well-known example is BCS theory, which models superconductivity by focusing on a few key mechanisms rather than the full microscopic picture. This distinction between microscopic and effective theories raises a fundamental question: can an effective theory be systematically derived from a more fundamental one? Since microscopic models encode a greater level of detail, it is generally believed that effective theories should, in principle, emerge rigorously from them. However, establishing such derivations with full mathematical precision remains a challenging and open problem in many cases. The aim of this work is to address this issue by rigorously deriving an effective dynamical description from its underlying microscopic counterpart.

Examples of Effective Theories in Physics. Effective theories appear across various domains of physics. In classical physics, well-known examples include the Vlasov equation, which models plasma dynamics, and the Navier-Stokes equations, which describe the behavior of viscous fluids.

In the quantum many-body context, effective descriptions vary depending on the nature of the particles. Fermionic systems in the mean-field regime are often modelled by the Hartree-Fock equation [Bar+03; Bar+04; Elg+04; FK11], while bosonic systems exhibiting condensation are described by the Hartree equation, which governs the time evolution of the condensate [Hep74; GV79a; GV79b; Spo80]. Additionally, the Bogoliubov theory captures the dynamics of excitations outside the condensate [Bog47].

Our work focuses on a bosonic gas into which we insert an impurity particle. Systems of bosons with impurities are of significant interest in both experimental (see Section 1.3), theoretical and mathematical physics. Of particular relevance to our study is the formation of a quasi-particle composed of the impurity and surrounding bosons, known as the Bose polaron.

Mathematical Literature on Bose-Einstein Condensation. We provide a brief overview of the mathematical literature on the derivation of effective descriptions for Bose gases in condensation. A Bose gas refers to a quantum many-body system consisting of N bosons. Bose-Einstein condensation occurs when a macroscopic fraction of particles occupies the same quantum state, forming what is known as the condensate.
In a dense system with weak interactions – known as the mean field scaling – the condensate is effectively described by the Hartree equation. Early results on the derivation of the Hartree equation date back to [Hep74], followed by Ginibre and Velo [GV79a; GV79b], and Spohn [Spo80]. They have shown that as the number of particles N approaches infinity, the one-particle marginal of the N-body wave function converges to the one single particle solution of the Hartree equation.

Since then these results have been expanded in various directions (see [Lew15] for an comprehensive overview). One significant extension concerns dilute systems, where the dynamics are governed by the Gross-Pitaevskii equation (see [BPS16] for a detailed review).

In addition to the condensate itself, one can also study the fluctuations around it. In an interacting Bose gas, not all particles remain in the condensate — some leave, forming excitations. The dynamics of these excitations were first modelled by Bogoliubov in 1947 in his seminal work [Bog47], where he introduced a quadratic Hamiltonian, called Bogoliubov Hamiltonian, to describe them. Ginibre, Machedon, and Margetis [GMM10; GMM11] provided the first rigorous validation of the Bogoliubov Hamiltonian, demonstrating that the solution of the many-body Schrödinger equation converges in the L^2-norm to that predicted by Bogoliubov theory.
Since then, substantial progress has been made on understanding both the spectrum of the Bogoliubov Hamiltonian [Sei11; GS13; Lew+15; YY09; DN14; Boc+19; BCS21; NT23; BSS22; FS20; FS23; BPS21; Boß+22; Bro25] as well as its dynamics [LNS15; BOS15; NN17; MPP19; PPS20; COS24; BCS17]. There are also result going beyond the Bogoliubov description [PP19; Boß+20; Boß+22; Fal+23].

1.2 The Bose Polaron: A Quasi-Particle in a Bose Gas

The concept of the Bose polaron originates from the study of an electron moving through a dielectric crystal, where the impurity electron interacts with lattice vibrations known as phonons. In 1948, Landau and Pekar [LP48] introduced the idea of a quasi-particle to describe this phenomenon, wherein the electron becomes "dressed" by phonons, effectively modifying its mass. In the strong coupling regime, this interaction can be so pronounced that the electron becomes completely localized within its surrounding phonon cloud – a phenomenon known as self-trapping [LP48].

The mathematical foundation for polaron physics was later established by Fröhlich through the Fröhlich Hamiltonian [Frö54], which describes an impurity interacting with a phonon field. Holstein later extended this model to incorporate lattice effects [Hol59].

In our work, we focus on a mobile impurity particle in a Bose gas exhibiting Bose-Einstein condensation. Unlike the solid-state setting, this system has no lattice and thus no lattice vibrations. Instead, the impurity interacts with both the condensate and its excitations – particles that have left the condensate. In the weak coupling regime of impurity-boson interaction, this model is described in the physics literature by the Fröhlich Hamiltonian [RS13].

For weakly interacting bosons, the model can be further simplified using the effective Bogoliubov dynamics for the excitations, introduced by Bogoliubov in 1947 [Bog47]. This approach significantly simplifies the dynamics of the excitations. The resulting Bogoliubov-Fröhlich Hamiltonian [GD16; Gru+24] applies for weak boson-boson and impurity-boson interaction. Our goal is to rigorously derive this Hamiltonian from the full microscopic time evolution.

This problem has been previously studied in the mean-field setting on a unit volume with periodic boundary conditions. In 2020, Myśliwy and Seiringer solved the corresponding static problem [MS20], while Lampart and Pickl analysed the dynamics [LP22]. More recently, Lampart and Triay extended the static result to the dilute setting (Gross-Pitaevskii limit) [LT24]. The main novelty of this thesis lies in considering a large initial volume relative to the interaction range of the bosons while removing periodic boundary conditions, leading to a non-constant condensate. These modifications make the model more physically realistic and bring new mathematical challenges to the problem.

1.3 Experiments on Bose Gases with Impurities

There is a wide variety of experiments investigating Bose gases with impurity particles. Beyond revealing the formation of a Bose polaron, impurities serve as tracer particles, providing a valuable tool for probing various properties of the Bose gas. Notably, they have been used to demonstrate superfluidity [Gru+24], measure the density distribution of the gas [SHD10], and observe the formation of vortex lattices in rotating Bose gases [YGP79; Abo+01].

Impurities in these experiments can be ions, neutral atoms or even atoms of the same type as those in the Bose-Einstein condensate but in a different hyperfine state [Zip+10; Hu+16; Jør+16]. For example, the impurity particles used are ^{174}Yb, ^{40}K, ^{41}K, ^{6}Li or ^{133}Cs, while the condensate is formed from, for example, neutral ^{87}Rb, ^{23}Na or ^{41}K atoms. The number of bosons in such systems typically ranges from 10^4 to 10^6 atoms. Unlike the setting of this work, most experimental systems are dilute, meaning that the average interatomic distance is large compared to the interaction range of the bosons [SHD10; Zip+10; Gru+24].

When the impurity is an ion, the underlying impurity-boson interaction is well described by the polarization potential [SHD10; Zip+10], of the form

$$V(r) \sim \frac{1}{r^4} .$$

In contrast, boson-boson interactions in a dilute gas of neutral atoms are typically described by the van der Waals potential [Dal11],

$$V_{\mathrm{vdW}}(r) \sim -\frac{1}{r^6} .$$

The characteristic interaction ranges of these two potentials can differ significantly. For instance, in a ^{174}Yb–^{87}Rb gas, the ratio of the impurity-boson interaction length scale r^* to the interatomic interaction scale R_{vdW} is approximately

$$\frac{r^*}{R_{\mathrm{vdW}}} \sim 10^2 ,$$

meaning that the impurity interacts over a much larger spatial scale than the bosons [SHD10].

However, in the dilute regime considered here, the detailed form of these interaction potentials is less important. Instead, the relevant parameter for describing interactions is the corresponding scattering length a.

In dilute gases, impurity-boson and boson-boson interactions can be tuned using external optical or magnetic fields via Feshbach resonances. A widely used theoretical framework for describing these interactions is the Gross-Pitaevskii equation, which, along with the associated scattering lengths a, provides an effective description of the system (see [TVS93; Fed+96; BJ97] for pure bosonic systems and [Wu+12; Jør+16] for systems with impurity).

For a comprehensive literature review on experiments as well as results from theoretical physics on the Bose polaron, we refer to [Gru+24; AD10]. A detailed discussion on the description of cold gases via the van der Waals potential and its manipulation using Feshbach resonances can be found in [Dal11] and the references therein.

1.4 Defining the Model: A Bose Gas in the Presence of an Impurity

We study the dynamics of a quantum gas consisting of N Bosons evolving in $\mathbb{R}^3$ in the presence of an impurity particle, referred to as the tracer particle. In the setting considered, the Bose gas has high initial density $\rho = \frac{N}{\Lambda}$ and occupies a large initial volume $\Lambda \geq 1$, where the scaling condition $\rho^\alpha = \Lambda$, $\alpha > 0$ is imposed. The system is governed by the Hamiltonian

$$H_\rho = -\frac{\Delta_x}{2m} - \sum_{i=1}^{N} \frac{\Delta_{y_i}}{2} + \frac{1}{\rho} \sum_{1 \leq i < j \leq N} V(y_i - y_j) + \frac{1}{\sqrt{\rho}} \sum_{i=1}^{N} W(x - y_i) \tag{1.1}$$

acting on the Hilbert space $L^2(\mathbb{R}^3) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3N})$. Here, x denotes the position of the impurity, y_i the positions of the bosons, m the impurity's mass, and Δ be the Laplace operator. The interactions are weak of mean-field type with

$$V, \widehat{V} \in L^p(\mathbb{R}^3, \mathbb{R}), \quad W \in L^p(\mathbb{R}^3, \mathbb{R}), \quad p \in \{1, \infty\}.$$

Both potentials are even and rapidly decreasing (see Assumption 2.0.3 for a precise definition of the potentials). The scaling factor $\frac{1}{\rho}$ of the potential V is chosen as a mean-field scaling and the $\frac{1}{\sqrt{\rho}}$ scaling for W ensures that the impurity-excitation interaction remains $\mathcal{O}(1)$, where excitations are particles not in the condensate (see Remark 2.0.2 for details).

The interaction range of W and V defines the $\mathcal{O}(1)$ length scale of the system, meaning that the initial volume Λ is large compared to the length scale of the interaction. The connection between the parameter Λ and the initial volume of the system is established through the condensate, which varies on a scale of $\mathcal{O}(\Lambda^{1/3})$ (see Remark 2.1.9 for details).

The Scaling and Comparison to the β-scaling. Our scaling with $\rho^\alpha = \Lambda$, $\alpha > 0$, provides a framework to approach the thermodynamic limit in the dense regime. The usual mean-field scaling is obtained at $\alpha = 0$, while the thermodynamic limit corresponds to $\alpha = \infty$. Note that in the literature the focus lies on the dilute regime. The primary goal of this work is to rigorously derive the Bogoliubov-Fröhlich theory as an effective description of the system's dynamics in the high-density and large-volume limit ($\alpha < 1/3$).
To analyze this limit, we place the system in a cubic box $\Lambda = [-L/2, L/2]^3$ and rescale it to a unit box. Under this transformation, the bosonic part of the Hamiltonian becomes

$$-\sum_{i=1}^{N} \frac{\Delta_{y_i}}{2L^2} + \frac{1}{\rho} \sum_{1 \leq i < j \leq N} V\big(L(y_i - y_j)\big), \quad y_i \in [-1/2, 1/2]^3. \tag{1.2}$$

Using the relation $N/\Lambda = \rho = \Lambda^{1/\alpha} = L^{3/\alpha}$, we obtain $L = N^\beta$ with $\beta = \frac{\alpha}{3(1+\alpha)}$. This leads to the rescaled Hamiltonian

$$-\sum_{i=1}^{N} \frac{\Delta_{y_i}}{2N^{2\beta}} + N^{3\beta-1} \sum_{1\le i<j\le N} V\big(N^\beta(y_i - y_j)\big)\,, \quad y_i \in [-1/2, 1/2]^3\,. \tag{1.3}$$

Thus, our scaling is not directly comparable to the β-scaling in the literature. While the interaction potential scales in the same way, our formulation involves a semi-classically scaled Laplacian. Finally, note that our main result, Theorem 3.1.2, holds for $0 < \alpha < 1/3$, which corresponds to $0 < \beta < 1/12$. Note that the thermodynamic limit is obtained at $\beta = 1/3$ ($\alpha = \infty$).

Mean-Field Description of the Condensate. In the mean-field regime, the dynamics of the condensate are effectively described by the Hartree equation:

$$\mathrm{i}\partial_t \varphi_t = h_t \varphi_t\,, \tag{1.4}$$

$$h_t = h[\varphi_t] = -\frac{\Delta}{2} + V * |\varphi_t|^2 - \mu_t\,, \tag{1.5}$$

where the convolution term, $V * |\varphi_t|^2(y) = \int V(x-y)|\varphi_t|^2(x)dx$, accounts for the mean-field interaction between bosons. The constant $\mu_t \in \mathbb{R}$ can be freely chosen, as it only affects the global phase of φ_t. We set $\mu_t := \frac{1}{2}\big\langle \frac{\varphi_t}{\Lambda^{1/2}}, V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}}\big\rangle$. To motivate the Hartree equation, consider the scenario where the Bose gas is in complete condensation, meaning that the N-body wave function $\psi_{\rho,t}$ is approximately in a product state of a single-particle function φ_t, the condensate:

$$\psi_{\rho,t}(y_1, \ldots, y_N) \approx \prod_{i=1}^{N} \varphi_t(y_i)\,.$$

The time evolution of $\psi_{\rho,t}$ is given by the full Hamiltonian H_ρ and φ_t should follow it to approximate $\psi_{\rho,t}$. By taking into consideration the mean-field setting of a dense system of many interacting particles and that almost all particles are in the condensate we can approximate the boson-boson interaction term in H_ρ as follows:[1]

$$\frac{1}{\rho}\sum_{1\le j\le N} V(y_i - y_j) \sim \frac{1}{\rho}\sum_{1\le j\le N} \int V(y_i - y)\frac{|\varphi_t(y)|^2}{\|\varphi_t\|_{L^2}^2} dy = \int V(y_i - y)|\varphi_t(y)|^2 dy\,.$$

Here we have to replace the boson-boson interaction in H_ρ by the expected interaction of one particle with another one in the condensate. This substitution leads directly to the interaction term in the Hartree equation.

[1] We set $\|\varphi_t\|_2 = \Lambda^{1/2}$ (see Condition 2.1.7).

The impurity-boson interaction contributes a term of order $\mathcal{O}(1/\sqrt{\rho})$ for any fixed particle i,

$$\frac{1}{\sqrt{\rho}} W(x - y_i) \sim \mathcal{O}\left(\frac{1}{\sqrt{\rho}}\right) ,$$

which is negligible in the high density limit. Thus, at a heuristic level, the Hartree equation emerges as a natural effective description of the condensate dynamics. The rigorous mathematical validation of the Hartree equation in the setting of our work was established in [Dec+16; PPS20].

Excitation Dynamics: The Bogoliubov-Fröhlich Hamiltonian. In addition to the condensate, we aim to describe the excitations, i.e., the particles that are not part of the condensate, through an effective theory. This is achieved via the Bogoliubov approximation, leading to the Bogoliubov Hamiltonian [Bog47]. The derivation starts by rewriting the bosonic part of the full Hamiltonian H_ρ in second quantization, using a time-dependent orthonormal basis $\{u_n(t)\}_{n\in\mathbb{N}}$ of $L^2(\mathbb{R}^3, \mathbb{C})$ including the normalized condensate $u_0 := \varphi_t/\Lambda^{1/2}$. This yields the expression

$$H_\rho^{\mathrm{B}} := \frac{1}{2} \sum_{m,n=0}^{\infty} (-\Delta)_{mn}\, a_m^* a_n + \frac{1}{\rho} \sum_{m,n,p,q=0}^{\infty} V_{mnpq} a_m^* a_n^* a_p a_q \,,$$

where we define the interaction coefficients as

$$V_{mnpq} = \langle u_m \otimes u_n, V(x-y) u_p \otimes u_q \rangle.$$

The operators a_m^* and a_m denote creation and annihilation operators of u_m, respectively (see Appendix F for details).
Since we consider a system exhibiting Bose-Einstein condensation, the number of particles in the condensate is approximately N:

$$\mathcal{N}_0 = a_0^* a_0 \sim N \,.$$

In addition, the commutator $[a_0, a_0^*] = 1$ is small compared to the large number of particles N. Hence it makes sense to treat a_0 and a_0^* simply as numbers and to replace $a_0, a_0^* \sim \sqrt{N}$. This technique, known as the c-number substitution, was first introduced by Bogoliubov [Bog47]. After applying this substitution, we neglect all terms that involve more than quadratic powers of a_m^* and a_m for $m \geq 1$, as they contribute only to subleading corrections. This leads to the effective Bogoliubov Hamiltonian, describing the excitation dynamics without an impurity:

$$H^{\mathrm{Bog}}(t) = d\Gamma(h_t) + \sum_{m,n\geq 1} V_{m0n0} a_m^* a_n + \frac{1}{2} \sum_{m,n\geq 1} (V_{mn00} a_m^* a_n^* + \mathrm{h.c.}) \,, \tag{1.6}$$

where h_t is the mean-field Hamiltonian from the Hartree equation (see Chapter 4 for details). The validity of the Bogoliubov dynamics for the excitations without impurity was rigorously established for our setting in [PPS20].

We repeat the c-number substitution also for the interaction of the impurity with the bosons $\frac{1}{\sqrt{\rho}}\sum_{m,n\geq 0}(W_x)_{mn}a_m^* a_n$, where $W_x(y) := W(x-y)$. This way we obtain the effective interaction terms

$$\rho^{1/2} W * |\varphi_t|^2(x) + \sum_{m\geq 1}\big((W_x)_{m0}a_m^* + \text{h.c.}\big)\,, \tag{1.7}$$

which describe how the impurity couples to the excitations (see Section 4.1.1 for details).

This leads us to the Bogoliubov-Fröhlich Hamiltonian, effectively modelling the dynamics of the full system:

$$H^{\text{BF}}(t) = -\frac{\Delta_x}{2m} + d\Gamma(h_t) + \sum_{m,n\geq 1} V_{m0n0}a_m^* a_n + \frac{1}{2}\sum_{m,n\geq 1}(V_{mn00}a_m^* a_n^* + \text{h.c.}) \tag{1.8}$$

$$+ \sum_{m\geq 1}\big((W_x)_{m0}a_m^* + \text{h.c.}\big)\,. \tag{1.9}$$

This model describes an impurity linearly coupled to excitations, while the excitation-excitation interaction is quadratic in the creation and annihilation operators. The operator H^{BF} acts on the space $L^2(\mathbb{R}^3,\mathbb{C})\otimes\mathcal{F}(L^2(\mathbb{R}^3,\mathbb{C})) = L^2(\mathbb{R}^3,\mathcal{F}(L^2(\mathbb{R}^3,\mathbb{C})))$, where we denote the Fock space over L^2 as $\mathcal{F}(L^2(\mathbb{R}^3)) = \bigoplus_{N=0}^{\infty} L^2_{\text{sym}}(\mathbb{R}^{3N},\mathbb{C})$. Rigorously, H^{BF} only generates a dynamic as a quadratic form (see Section 2.3 and Appendix D.4 for details).

In H^{BF}, we omit the mean-field contribution from the condensate-impurity interaction, $\rho^{1/2}W*|\varphi_t|^2(x)$, from (1.7). In fact, under suitable initial conditions, it behaves approximately as a constant $\rho^{1/2}W*|\varphi_t|^2(x) \sim \rho^{1/2}W*1$, and can thus be neglected in the dynamics (see Chapter 5 for details). This omission is crucial: a non-constant mean-field contribution would dominate the impurity's dynamics, masking its interaction with the excitations and causing it to escape the Bose gas on short timescales (see Section 5.4.3).

The main goal of this thesis is to establish the validity of the Bogoliubov-Fröhlich Hamiltonian H^{BF} by proving the norm convergence of the full N-body wave function $\psi_{\rho,t}$, evolving under the microscopic Hamiltonian H_ρ (1.1), to the solution generated by H^{BF}. The precise statement of this result is given in Theorem 3.1.1 and Theorem 3.1.2. This work extends the analysis of Lampart and Pickl [LP22], who treated the same problem on the unit box with periodic boundary condition and a constant condensate. The main novelty of this work is the extension to $\mathbb{R}^3$ with large initial volume and the treatment of a non-constant condensate, thereby increasing the physical relevance of the system.

Ultimately, establishing the validity of H^{BF} brings us one step closer to rigorously proving the existence of the Bose polaron, a quasi-particle arising from the interaction between an impurity and the surrounding Bose gas [HL24].

Method for Validating the Bogoliubov-Fröhlich Model. To establish the validity of the effective dynamics generated by the Bogoliubov-Fröhlich Hamiltonian $H^{\mathrm{BF}}(t)$, we carry out two main steps:

1. **Bogoliubov Approximation:** We apply the so called Bogoliubov approximation to the Hamiltonian H_ρ to obtain an effective description of the excitations (see Chapter 4).
2. **Mean-field Condensate-Impurity Interaction Approximation:** We approximate the large mean-field interaction between the condensate and the impurity by a constant (see Chapter 5).

To justify step 1, we establish a propagation estimate for the excitation number, ensuring that the number of excitations per unit volume remains bounded. This is crucial for the persistence of Bose-Einstein condensation, as an uncontrolled growth of excitations would destroy the condensate.

Step 2 requires a more delicate analysis. To justify replacing the mean-field impurity-condensate interaction with a constant, we must show that the impurity remains localized in a region where variations in condensate density are negligible. Specifically, we show that:

- Over the timescale of $\mathcal{O}(1)$, the impurity remains confined within a region of size $\mathcal{O}(1)$, while the condensate density varies on the larger scale $\mathcal{O}(\Lambda^{1/3})$.
- As a result, the impurity interacts with an effectively uniform condensate, justifying the approximation of their mean-field interaction by a constant.

To establish localization within a region of size $\mathcal{O}(1)$, we prove that despite the total number of excitations scaling as $\mathcal{O}(\Lambda)$ with the volume, the impurity effectively interacts with only a small number of excitations, of $\mathcal{O}(1)$ (see Lemma 4.2.6). This follows heuristically from the local nature of the interaction, which ensures that the impurity only "sees" a limited number of surrounding excitations.

Together, these two key estimates – bounding the excitation number and proving impurity localization – allow us to rigorously justify the Bogoliubov-Fröhlich dynamics as an effective description of the system.

Chapter 2

Definition of the Microscopic and Effective Dynamics

We begin by rigorously defining all the quantities needed to understand our main Theorem. In particular we introduce the microscopic (Definition 2.0.1) and effective dynamics (Definition 2.3.2) of the system, which were briefly discussed in the introduction.

The microscopic dynamics describe the evolution of a weakly and locally interacting system consisting of N bosons and an impurity particle, governed by the Schrödinger equation.

Definition 2.0.1 (Microscopic Dynamics)**.** We set the microscopic Hamiltonian to be

$$H_\rho = -\frac{\Delta_x}{2m} - \sum_{i=1}^{N} \frac{\Delta_{y_i}}{2} + \frac{1}{\rho} \sum_{1\le i<j\le N} V(y_i - y_j) + \frac{1}{\sqrt{\rho}} \sum_{i=1}^{N} W(x - y_i) \tag{2.1}$$

with the potentials V and W from Assumption 2.0.3 and defined on the Hilbert space $L^2(\mathbb{R}^3_x) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3N}_y)$.

Remark 2.0.2.

Microscopic Hamiltonian. We often refer to H_ρ as the microscopic Hamiltonian because it captures all the details of the dynamics.

Well-posedness. The differential equation $\mathrm{i}\partial_t \psi_{\rho,t} = H_\rho \psi_{\rho,t}, \psi_{\rho,t}|_{t=0} = \psi_{\rho,0}$ has a unique global solution $\psi_{\rho,t} = \mathrm{e}^{-itH_\rho}\psi_{\rho,0}$, $\forall \psi_{\rho,0} \in L^2(\mathbb{R}^3, \mathcal{F}(L^2))$, where e^{-itH_ρ} is given by the functional calculus. Note that the self-adjointness of H_ρ can be seen from Kato's Theorem since $V, W \in L^\infty$.

Bosonic Interaction Scaling. The scaling of the V potential is chosen as the mean-field scaling meaning that one fixed Boson sees only ρ other Bosons in a unit volume surrounding it. Leading to $\frac{1}{\rho}\sum_{1\leq i<j\leq N} V(y_i-y_j)\sim N$. Note that the kinetic term in the Hamiltonian is smaller order $N\Lambda^{-2/3}$ due to the large volume of the condensate (see Condition 2.1.7).

Impurity Interaction Scaling. In the following, we will explain the scaling $r=1/2$ for $\frac{1}{\rho^r}\sum_{i=1}^{N} W(x-y_i)$. The scaling is chosen in such a way that the $a^{\#}(Q_t W_x \varphi_t)$ terms in the Bogoliubov-Fröhlich Hamiltonian

$$H^{\mathrm{BF}} := -\frac{\Delta_x}{2m} + a\left(Q_t W_x \varphi_t\right) + a^*\left(Q_t W_x \varphi_t\right) + H^{\mathrm{Bog}}(t)$$

appear with no prefactor. Here φ_t is the condensate defined in Definition 2.1.1 and Q_t the projection onto the space of excitation (see Notation 2.2.1). For a general r, one obtains $\rho^{1/2-r}a^{\#}(Q_t W_x \varphi_t)$. The prefactor $\mathcal{O}(1)$ is important for two things. First we want the interaction of tracer and excitations to be of $\mathcal{O}(1)$, such that we can formally take the infinite volume limit $\Lambda\to\infty$. Second we want the tracer to only change its position in a region of $\mathcal{O}(1)$. This second effect is actually caused by the first one since the tracer can only gain or lose energy of $\mathcal{O}(1)$ from the excitations. In Section 5.4 we give a more detailed description.

The interaction potentials V and W in Definition 2.0.1 are chosen such that they are local and rapidly decreasing as specified in Assumption 2.0.3. The potentials can depend on the volume Λ and the density ρ but we will not track this dependency with our notation. The case of Λ and ρ independent potentials V and W is compatible with all our assumptions.

Assumption 2.0.3 (Assumptions on the Potentials)**.** *We denote the boson-boson interaction potential by $V\in L^1(\mathbb{R}^3,\mathbb{R})$. The potential V is even and for all $k\in\mathbb{N}_0$ there exists a constant $C>0$ such that for all densities $\rho\geq 1$ and all volumes $\Lambda\geq 1$*

$$\|\widehat{V}\|_1 + \left\| |y|^k V \right\|_1 \leq C\,. \tag{2.2}$$

We denote the boson-impurity interaction potential by $W\in L^1(\mathbb{R}^3,\mathbb{R})\cap L^\infty(\mathbb{R}^3,\mathbb{R})$. The potential W is even and $\forall k\in\mathbb{N}_0\ \exists C>0$ such that $\forall\Lambda,\rho\geq 1$

$$\|W\|_{H^2(\mathbb{R}^3,\mathbb{R})} + \left\| |y|^k W \right\|_\infty \leq C\,. \tag{2.3}$$

We say V and W satisfy Assumption 2.0.3$_M$ for $M\in\mathbb{N}_+$ if the assumptions above are satisfied and $\exists C>0$ such that $\forall\Lambda,\rho\geq 1$

$$\|W\|_{H^M(\mathbb{R}^3,\mathbb{R})} + \|W\|_{W^{M,\infty}(\mathbb{R}^3,\mathbb{R})} \leq C\,. \tag{2.4}$$

Remark 2.0.4.

- Assumption 2.0.3$_1$ for $M=1$ is assumed throughout the whole thesis without being mentioned explicitly. Whenever additional regularity of the boson–tracer interaction potential W is required (i.e., $M > 1$), we refer to this explicitly by Assumption $2.0.3_M$.

- For all $M \in \mathbb{N}_0$ Assumption $2.0.3_M$ is, for example, satisfied by Schwartz functions.
- It is sufficient to assume (2.2) and (2.3) for all $k \leq k_0$, where k_0 is a decreasing function of α, with α given by $\rho = \Lambda^\alpha$.
- Note that if $V \in L^1 \cap L^\infty$ then $V \in L^p$, $\forall p \in [1, \infty]$, because of

$$\int |V|^p d\mu \leq \|V\|_\infty^{p-1} \|V\|_1 \, .$$

Especially $\forall p \in [1, \infty]$, $k \in \mathcal{N}_0$ $\exists C > 0$ such that $\forall \Lambda, \rho \geq 1$ we have

$$\big\| \, |y|^k V \, \big\|_p + \|W\|_p \leq C \, .$$

We also have $\big\| \, |y|^k W \, \big\|_1 \leq C$ as $\big\| \, |y|^k W \, \big\|_1 \leq \big\| \, (1 + |y|)^{\tilde{k}} W \, \big\|_\infty \| \, (1 + |y|)^{-m} \|_1$ for $k =: \tilde{k} - m$.

- The conditions $\| \, |y|^k V(y) \|_1 \leq C_k$ and $\| \, |y|^k W(y) \|_1 \leq C_k$ impose a polynomial decay of our potentials in Λ in the following sense:
For all $k \in \mathbb{N}_0$ there exists a constant $C_k > 0$ such that for all $\epsilon > 0$ and $\Lambda \geq 1$

$$C_k \Lambda^{-\epsilon k} \geq \int_{\mathbb{R}^3 \setminus B(0, \Lambda^\epsilon)} |V|(y) dy \, .$$

This follows from

$$C_k \geq \| |y|^k V \|_1 \geq \int_{\mathbb{R}^3 \setminus B(0, \Lambda^\epsilon)} |y|^k |V|(y) dy \geq \Lambda^{\epsilon k} \int_{\mathbb{R}^3 \setminus B(0, \Lambda^\epsilon)} |V|(y) dy \, .$$

In this sense, for large volumes Λ, the potentials V and W describe only local interactions of range $\mathcal{O}(1)$.

2.1 Condensate Definition and Hartree Equation

In Definition 2.1.1, we specify that for all $\Lambda \geq 1$ the initial data for the Hartree equation – and consequently for the condensate evolution – is given by $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$. The precise conditions on φ_0 vary between different theorems and are therefore stated separately in Condition 2.1.7, Condition 2.1.8, and Condition 2.1.11. Intuitively, one can think of the condensate as a rescaled function,

$$\text{“}\varphi_0(x) = \eta(\Lambda^{-1/3} x) \, ,\text{”}$$

which is flat around the origin, where $\eta \in H^\infty$ is independent of Λ and normalized in L^2.

Definition 2.1.1 (Hartree, Time Evolution of the Condensate)**.** For all volumes $\Lambda \geq 1$, let $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$. For each $\Lambda \geq 1$ set φ_t to be the global solution of the Cauchy problem

$$\mathrm{i}\partial_t \varphi_t(y) = h_y[\varphi_t]\varphi_t(y), \qquad h_t := h_y[\varphi_t] := -\frac{1}{2}\Delta + V * |\varphi_t|^2(y) - \mu_t, \tag{2.1}$$

with initial condition $\varphi_t|_{t=0} = \varphi_0$. $\mu_t := \mu(\varphi_t) \in \mathbb{R}$ is an arbitrary but real phase, which satisfies the conditions of Lemma 2.1.3 and is uniformly bound in Λ. We call φ_t the condensate.

Remark 2.1.2.

1. **Well-posedness of the Cauchy Problem.** The well-posedness of the Cauchy problem above is discussed in Lemma 2.1.3.

2. **Definition of μ_t for the Bogoliubov Approximation.** We set

$$\mu_t := \frac{1}{2}\Big\langle \frac{\varphi_t}{\Lambda^{1/2}}, V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}} \Big\rangle$$

which is needed for the Bogoliubov approximation.

Lemma 2.1.3 (Well-Posedness of the Hartree Initial Value Problem)**.** *Let $V \in L^\infty(\mathbb{R}^3, \mathbb{R})$ be a general potential and $\mu : L^2 \to \mathbb{R}, u \mapsto \mu(u)$ with:*

i) (Local Lipschitz) For all $k \in \mathbb{N}_0$ and $\forall B \subset H^k(\mathbb{R}^3, \mathbb{C})$ bounded $\exists L_B$: $\forall u, v \in B$: $|\mu(u) - \mu(v)| \leq L_B \|u - v\|_{H^k}$.

ii) For all $k \in \mathbb{N}_0$ $\exists C \in C(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $C(|t|)$ monotonically increasing such that $\forall u \in H^k$

$$|\mu(u)| \leq C(\|u\|_2)\|u\|_{H^k}.$$

Then

a) (Existence) The initial value problem (2.1) with $\varphi_t|_{t=0} = \varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$, $\mu_t = \mu(\varphi_t)$ has a solution $(t \mapsto \varphi_t) =: \varphi \in C(\mathbb{R}, H^\infty) \cap C^1(\mathbb{R}, H^\infty)$. Esp. $\varphi, \dot{\varphi} \in H^\infty$ and we have conserved charge $\|\varphi_t\|_2 = \|\varphi_0\|_2$.

b) (Uniqueness) Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $I \subset \mathbb{R}$ interval with $0 \in I$ and $\tilde{u} \in C(I, H^{k+1}) \cap C^1(I, H^{k-1})$ solution of the Hartree equation (2.1) in H^{k-1} with initial data φ_0, then $\tilde{u} = \varphi|_I$.

Therefore we have a unique global solution φ of the Hartree equation.

Example 2.1.4. The choice $\mu(u) = \lambda\langle u, V * |u|^2 u\rangle_{L^2}$, $\lambda \in \mathbb{R}$, $u \in L^2$, fulfils all the properties of μ in Lemma 2.1.3. Especially we can choose $\mu(\varphi_t) = \mu_t := \frac{1}{2}\big\langle \frac{\varphi_t}{\Lambda^{1/2}}, V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}}\big\rangle$, which especially satisfies $|\mu_t| \leq C(t)$.

Proof of Lemma 2.1.3. The proof follows standard techniques for ordinary differential equations. We begin by applying the Picard–Lindelöf theorem and the Duhamel formula (precisely Lemma F.0.8) to obtain a local solution. This solution can then be extended to a maximal interval of existence, which, due to the conservation of the L^2-norm, can be shown to be global. see for example [Bag, Section: 5 Prolongation of solutions] or [Ler11, Section: 2.1 Ordinary Differential Equations]. The methods there can easily be generalized to the Hartree equation by using Duhamel. For a detailed discussion of these arguments, see for example [Bag, Section 5: Prolongation of Solutions] or [Ler11, Section 2.1: Ordinary Differential Equations]. The methods presented there extend naturally to the Hartree equation by incorporating the Duhamel formula. ∎

To control the dynamics of φ_t, we define an auxiliary function $\tilde{\varphi}_t$ as in [Dec+16]. It is used to approximate and control the condensate (see Appendix B).

Definition 2.1.5 (Auxiliary Function)**.** For all volumes $\Lambda \geq 1$, let $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$. We call

$$\tilde{\varphi}_t = \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2 - \int_0^t \mu_s ds\right)} \varphi_0 \tag{2.2}$$

the auxiliary function, where μ_t is defined in Definition 2.1.1.

Remark 2.1.6. As it can be seen from Condition 2.1.7, the kinetic term in the time evolution of φ_t from Definition 2.1.1 can be omitted, as it is subleading compared to the interaction term. Specifically,

$$\| -\Delta\varphi_0\|_2 \leq C\Lambda^{1/2-2/3}\,, \quad \text{while} \quad \|V * |\varphi_0|^2\varphi_0\|_2 \leq C\Lambda^{1/2}\,.$$

Due to the appropriate scaling of φ_0, these properties also hold for the time evolved state. This justifies the approximation of φ_t through $\tilde{\varphi}_t$ in Appendix B.

2.1.1 Conditions on the Condensate

The following conditions on the condensate's initial state specify our interpretation of it as a rescaled function (see Condition 2.1.7 and Condition 2.1.8) that is flat around the origin (see Condition 2.1.11).

In particular, Condition 2.1.7 ensures that we can apply known results on the time evolution of the condensate from [Dec+16] (see Appendix B for details).

Condition 2.1.7 (Initial Condition of the Condensate)**.** *For all volumes $\Lambda \geq 1$, let the condensate $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$. We say φ_0 satisfies Condition 2.1.7 if there exists a constant $C > 0$ such that for all $\Lambda \geq 1$*

$$((2\pi)^{3/2}\|\varphi_0\|_\infty \leq)\, \|\,\widehat{\varphi_0}\,\|_1 \leq C\,, \tag{2.3}$$

$$\|\varphi_0\|_2 = \Lambda^{1/2} \tag{2.4}$$

and

$$\|\nabla\varphi_0\|_\infty \leq C\Lambda^{-\frac{1}{3}}\,, \quad \|\nabla\varphi_0\|_2 \leq C\Lambda^{\frac{1}{2}-\frac{1}{3}}\,, \quad \|\Delta\varphi_0\|_2 \leq C\Lambda^{\frac{1}{2}-\frac{2}{3}}\,. \tag{2.5}$$

Next, we specify a condition that tracks the required bounds on the derivatives of the condensate.

Condition 2.1.8 (Initial Condition for higher Derivatives of the Condensate)**.** *For all volumes $\Lambda \geq 1$, let the condensate $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$.*

i) We say φ_0 satisfies Condition 2.1.8i)$_k$ for a given $k \in \mathbb{N}_0$ if for all $\beta \in \mathbb{N}_0^3$ with $0 \leq |\beta| \leq k$, there exists a constant $C > 0$ such that for all $\Lambda \geq 1$

$$\|D^\beta\varphi_0\|_\infty \leq C\Lambda^{-|\beta|/3}\,. \tag{2.6}$$

ii) We say φ_0 satisfies Condition 2.1.8ii)$_k$ for a given $k \in \mathbb{N}_0$ if for all $\beta \in \mathbb{N}_0^3$ with $0 \leq |\beta| \leq k$, there exists a constant $C > 0$ such that for all $\Lambda \geq 1$

$$\|D^\beta\varphi_0\|_2 \leq C\Lambda^{-|\beta|/3+1/2}\,. \tag{2.7}$$

We say φ_0 satisfies Condition 2.1.8$_k$ for a given $k \in \mathbb{N}_0$ if both Condition 2.1.8i)$_k$ and Condition 2.1.8ii)$_k$ are satisfied. And we say φ_0 satisfies Condition 2.1.8 if Condition 2.1.8$_k$ is satisfied for all k.

Remark 2.1.9.

1. **Example of a Suitable Initial Condensate.** A function satisfying both Condition 2.1.8 and Condition 2.1.7 is a rescaled function of the form $\varphi_0(x) = \eta(\Lambda^{-1/3}x)$, where $\eta \in H^\infty$ is independent of Λ and normalized in L^2.

2. **Connection of Λ to the Initial Volume.** The parameter Λ relates to the initial volume of the gas through the scaling of the condensate, which varies on the order of $\mathcal{O}(\Lambda^{1/3})$. However, our results do not require the condensate to be confined within a box of volume Λ, nor do we assume it has compact support. Instead, the condensate may form multiple bumps, each varying on the scale $\mathcal{O}(\Lambda^{1/3})$. The relevant region for our analysis is the bump around the origin, where the tracer particle is initially placed (see Condition 2.3.4).

Remark 2.1.10. If Condition 2.1.8$_k$ is satisfied for some $k \in \mathbb{N}_0$, then it also holds for all $0 \leq \tilde{k} \leq k$.

Condition 2.1.11 (Condensate, Flat Around the Origin)**.** *For all volumes $\Lambda \geq 1$, let the condensate $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$.*

- *We say* φ_0 *satisfies Condition 2.1.11*$_{k,s}$ *for a given* $k \in \mathbb{N}_+$ *and* $1/3 > s > 0$ *if for all* $0 \leq |\beta| \leq k-1$, *there exists a constant* $C > 0$ *such that for all* $\Lambda \geq 1$

$$|D^\beta(\varphi_0(0) - 1)| \leq C\|D^\beta(\varphi_0 - 1)\|_\infty \Lambda^{-(k-|\beta|)(1/3-s)} \tag{2.8}$$

- *We say* φ_0 *satisfies Condition 2.1.11*$_s$ *for a given* $1/3 > s > 0$ *if Condition 2.1.11*$_{k,s}$ *is satisfied for all* k.

Remark 2.1.12.

1. **Monotonicity in** k**.** If Condition 2.1.11$_{k,s}$ is satisfied for some $k \in \mathbb{N}_+$ then it also holds for all $\tilde{k} \leq k$. By Condition 2.1.11$_{-k,s}$, $k \in \mathbb{N}_0$, where $-k$ is negative, we mean that this assumption is empty.

2. **Connection to Lemma A.0.6.** Condition 2.1.11 is chosen in such a way that the condensate satisfies the requirements of Lemma A.0.6.

3. **Generalization to Points Near the Origin.** The key reason for requiring condensate flatness is to control its behaviour around the tracer particle at initial time. Instead of assuming Condition 2.1.11$_{k,s}$ at the origin, we can generalize it by allowing a small displacement of $\mathcal{O}(1)$:
   Let $k \in \mathbb{N}_0, s \geq 0$ and $x_0 \in \mathbb{R}^3, c > 0$ such that $\forall \Lambda, \rho$: $|x_0| \leq c$. Then, for all $0 \leq |\beta| \leq k-1$ $\exists C > 0$ such that $\forall \Lambda \geq 1$ we assume

$$|D^\beta(\varphi_0(x_0) - 1)| \leq C\|D^\beta(\varphi_0 - 1)\|_\infty \Lambda^{-(k-|\beta|)(1/3-s)} . \tag{2.9}$$

   This more flexible assumption is still sufficient for our results.

4. **Flatness Interpretation.** The inequality

$$|D^\beta(\varphi_0(0) - 1)| \leq C\|D^\beta(\varphi_0 - 1)\|_\infty \Lambda^{-(k-|\beta|)(1/3-s)}$$

   implies that for $0 < s < 1/3$, the function $D^\beta(\varphi_0(0) - 1)$ is even flatter around the origin than what one would expect from its supremum (see Condition 2.1.8). In particular, for $\beta = 0$, this gives

$$|\varphi_0(0) - 1| \leq C\|\varphi_0 - 1\|_\infty \Lambda^{-k(1/3-s)}.$$

   If we also assume Condition 2.1.7 ($\|\varphi_0\|_\infty \leq C$) this means that the condensate at the origin is approximately 1.

5. **Flatness of a Rescaled Condensate.** If we consider φ_0 as a rescaled function,

$$\varphi_0(y) = \eta(\Lambda^{-1/3}y),$$

where $\eta \in H^\infty$ is independent of Λ and satisfies $\|\eta\|_2 = 1$, then Condition $2.1.11_{k,s}$ reduces to

$$|D^\beta(\eta(0) - 1)| \leq C\|D^\beta(\eta - 1)\|_\infty \Lambda^{-(k-|\beta|)(1/3-s)}\,, \quad \forall |\beta| \leq k - 1\,.$$

Since η is independent of Λ, we conclude that $D^\beta(\eta(0) - 1) = 0$, $\forall |\beta| \leq k - 1$.

6. Alternative Plateau Condition. Instead of Condition 2.1.11, we could impose a plateau condition for φ_0 as in [Dec+16]:

$$\varphi_0(y) - 1 = 0 \quad \text{for } y \in B_{\frac{1}{2}\Lambda^{1/3}}\,. \tag{2.10}$$

This condition would directly imply Condition $2.1.11_{k,s}$ for all $k \in \mathbb{N}_0$ and $s \geq 0$.
Another alternative is to use flatness around the origin in a stricter sense such that

$$\text{“}\varphi_0(0) = 1\,, D^\beta \varphi_0(0) = 0\,, \quad \forall 1 \leq |\beta| \leq k - 1\,.\text{”} \tag{2.11}$$

One can immediately see that (2.11) would imply Condition $2.1.11_{k,s}$.
However, it turns out that the flatness assumption stated in (2.8) is sufficient to prove all results. Neither a plateau condition nor stricter assumptions are required.

The control of the condensate wave function φ_t is established in Appendix B, where we show that the properties of the initial condition, namely the picture of φ_0 as a rescaled function “$\varphi_0(x) = \eta(\Lambda^{-1/3}x)$” with $\eta \in H^\infty$ independent of Λ and normalized in L^2, persists under time evolution.

2.2 Excitation Representation and Hamiltonian

To effectively model excitations out of the condensate, we introduce the excitation representation. In this framework, a given N-body wave function $\psi \in L^2(\mathbb{R}^3)^{\otimes N}$ is decomposed into a component in the direction of the condensate φ_t and a component orthogonal to it. This representation was first introduced in [Lew+15]. To formalize this approach, we introduce the following notation.

Notation 2.2.1 (Projection onto the Condensate)**.** For all volumes $\Lambda \geq 1$, let φ_t be the condensate as defined in Definition 2.1.1 satisfying $\|\varphi_0\|_2 = \Lambda^{1/2}$. We define the orthogonal projection $P_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ by

$$P_t\psi := \left|\frac{\varphi_t}{\Lambda^{1/2}}\right\rangle \left\langle \frac{\varphi_t}{\Lambda^{1/2}}\right| \psi := \frac{\varphi_t}{\Lambda^{1/2}} \left\langle \frac{\varphi_t}{\Lambda^{1/2}}, \psi \right\rangle$$

and the projection onto excitations is given by $Q_t = 1 - P_t$.

To exploit the special role of the condensate, we work with a time-dependent orthonormal basis incorporating the condensate wave function.

Notation 2.2.2 (Basis of $L^2(\mathbb{R}^3)$)**.** For all volumes $\Lambda \geq 1$, let φ_t be the condensate satisfying $\|\varphi_0\|_2 = \Lambda^{1/2}$. If nothing else is mentioned we use a time-dependent orthonormal basis $\{u_n\}_{n\in\mathbb{N}_0}$ with $u_0(t) := \varphi_t/\Lambda^{1/2}$ for the Hilbert space $L^2(\mathbb{R}^3)$.

Lemma 2.2.3 (Excitation Representation)**.** *For all volumes $\Lambda \geq 1$, let φ_t be the condensate satisfying $\|\varphi_0\|_2 = \Lambda^{1/2}$, and $\psi \in L^2(\mathbb{R}^3)^{\otimes N}$. We can write*

$$\psi = (P_t + Q_t)^{\otimes N}\psi = \sum_{k=0}^{N} \left(\frac{\varphi_t}{\Lambda^{1/2}}\right)^{\otimes N-k} \otimes_s \chi_t^{(k)}$$

with $\chi_t^{(k)} \in \left(\{\varphi_t\}^{\perp}\right)^{\otimes_s k}$. Therefore, we can uniquely represent $\psi \in L^2(\mathbb{R}^3)^{\otimes N}$ by

$$(\chi_t^{(k)})_{k\leq N} \in \bigoplus_{k=0}^{N} \left(\{\varphi_t\}^{\perp}\right)^{\otimes_s k}.$$

To analyze excitations out of the condensate, we define a unitary transformation that maps into the excitation space $\mathscr{F}_{+,t}^{\leq N} \subset \mathscr{F}_{+,t} := \bigoplus_{k=0}^{\infty} \left(\{\varphi_t\}^{\perp}\right)^{\otimes_s k}$.

Definition 2.2.4 (Excitation Map)**.** For all volumes $\Lambda \geq 1$, let φ_t be the condensate satisfying $\|\varphi_0\|_2 = \Lambda^{1/2}$. The operator

$$U_\rho(\varphi_t) : L^2(\mathbb{R}^3)^{\otimes_s N} \to \mathscr{F}_{+,t}^{\leq N} := \bigoplus_{k=0}^{N} \left(\{\varphi_t\}^{\perp}\right)^{\otimes_s k}$$
$$\psi \mapsto \bigoplus_{k=0}^{N} \chi_t^{(k)},$$

where $\psi = (P_t + Q_t)^{\otimes N}\psi = \sum_{k=0}^{N} \left(\frac{\varphi_t}{\Lambda^{1/2}}\right)^{\otimes N-k} \otimes_s \chi_t^{(k)}$, $\chi_t^{(k)} \in \left(\{\varphi_t\}^{\perp}\right)^{\otimes_s k}$, defines an isometric isomorphism. We call $U_\rho(\varphi_t) =: U_{\rho,t}$ the excitation map.

Remark 2.2.5. For convenience, we often denote $I \otimes U_{\rho,t} : L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)^{\otimes_s N} \to L^2 \otimes \bigoplus_{k=0}^{N} \left(\{\varphi_t\}^{\perp}\right)^{\otimes_s k}$, which incorporates the impurity degrees of freedom, simply by $U_{\rho,t}$ as well.

The transformation properties of U_ρ with the creation and annihilation operators can be found in Lemma 4.1.3.

The transformation $U_{\rho,t}$ into the excitation space gives rise to a corresponding transformed Hamiltonian, describing the dynamics of the excitations.

Definition 2.2.6 (Excitation Hamiltonian)**.** For all volumes $\Lambda \geq 1$, let φ_t be the condensate. For given density $\rho \geq 1$ and volume $\Lambda \geq 1$ we define the excitation Hamiltonian

$$H_\rho^{\mathrm{ex}}(t) = U_{\rho,t} H_\rho U_{\rho,t}^* + \mathrm{i}(\partial_t U_{\rho,t}) \cdot U_{\rho,t}^* \,.$$

Remark 2.2.7. The excitation Hamiltonian $H_\rho^{\mathrm{ex}}(t)$ describes the microscopic dynamics in the excitation space, by satisfying

$$\mathrm{i}\partial_t \psi_{\rho,t} = H_\rho \psi_{\rho,t} \quad \Leftrightarrow \quad \mathrm{i}\partial_t U_{\rho,t}\psi_{\rho,t} = H_\rho^{\mathrm{ex}}(t) \cdot U_{\rho,t}\psi_{\rho,t} \,. \tag{2.1}$$

2.3 Bogoliubov-Fröhlich Dynamics

A brief motivation and heuristic derivation of the effective Bogoliubov and Bogoliubov-Fröhlich dynamics can be found in the introduction of this work. We now provide a rigorous definition of these Hamiltonians.

Definition 2.3.1 (Bogoliubov Hamiltonian)**.** For all volumes $\Lambda \geq 1$, let φ_t be the condensate. The Bogoliubov Hamiltonian is an operator on $\mathcal{F}(L^2(\mathbb{R}^3, \mathbb{C}))$ given by

$$H^{\mathrm{Bog}}(t) = d\Gamma(h_t + K_1(t)) + \frac{1}{2} \sum_{m,n \geq 0} \left((K_2(t)J)_{mn} a_m^* a_n^* + \mathrm{h.c.} \right) \,, \tag{2.1}$$

where h_t is defined in Definition 2.1.1 and $\tilde{K}_1(t) : L^2(\mathbb{R}^3, \mathbb{C}) \to L^2$, $K_1(t) := Q_t \tilde{K}_1(t) Q_t$, $\tilde{K}_2(t) : (L^2(\mathbb{R}^3, \mathbb{C}))^* \to L^2$, $K_2(t) := Q_t \tilde{K}_2(t) J Q_t J^*$ with

$$[\tilde{K}_1(t)\psi](x) := \int [\varphi_t(x) V(x-y) \varphi_t^*(y)] \psi(y) dy \,, \tag{2.2}$$

$$[\tilde{K}_2(t)J\psi](x) := \int [\varphi_t(x) \varphi_t(y) V(x-y)] \psi^*(y) dy \,. \tag{2.3}$$

Definition 2.3.2 (Bogoliubov-Fröhlich Hamiltonian)**.** For all volumes $\Lambda \geq 1$, let φ_t be the condensate. The Bogoliubov-Fröhlich Hamiltonian is an operator on $L^2(\mathbb{R}_x^3, \mathcal{F}(L^2(\mathbb{R}^3, \mathbb{C})))$ given by

$$H^{\mathrm{BF}}(t) := -\frac{\Delta_x}{2m} + A\left(Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t\right) + H^{\mathrm{Bog}}(t) \,, \tag{2.4}$$

where $W_x(y) = W(x-y)$.

Remark 2.3.3 (Properties of $H^{\mathrm{Bog}}(t)$ and $H^{\mathrm{BF}}(t)$)**.**

Domain of the Hamiltonians. $H^{\mathrm{Bog}}(t)$ and $H^{\mathrm{BF}}(t)$ are well defined operators with the domain $D(d\Gamma(-\Delta+1))$ and $D(I \otimes d\Gamma(-\Delta+1)) \cap D(-\Delta_x \otimes I)$ respectively (see Lemma C.0.1). Moreover, $H^{\mathrm{Bog}}(t)$ and $H^{\mathrm{BF}}(t)$ are independent of ρ.

Well-posedness of the corresponding Cauchy Problem. The differential equations

$$\mathrm{i}\partial_t \psi_t^{\mathrm{Bog}} = H^{\mathrm{Bog}}(t)\psi_t^{\mathrm{Bog}}\,, \quad \psi_{t=0}^{\mathrm{Bog}} = \psi_0^{\mathrm{Bog}}\,,$$
$$\mathrm{i}\partial_t \psi_t^{\mathrm{BF}} = H^{\mathrm{BF}}(t)\psi_t^{\mathrm{BF}}\,, \quad \psi_{t=0}^{\mathrm{BF}} = \psi_0^{\mathrm{BF}}$$

have in a weak sense unique global solutions, given by

$$\psi_t^{\mathrm{Bog}} = U^{\mathrm{Bog}}(t,0)\psi_0^{\mathrm{Bog}}\,, \quad \forall \psi_0^{\mathrm{Bog}} \in L^2(\mathbb{R}^3, \mathcal{F}(L^2))\,,$$
$$\psi_t^{\mathrm{BF}} = U^{\mathrm{BF}}(t,0)\psi_0^{\mathrm{BF}}\,, \quad \forall \psi_0^{\mathrm{BF}} \in L^2(\mathbb{R}^3, \mathcal{F}(L^2))\,,$$

where $U^{\mathrm{Bog}}(t,0)$ and $U^{\mathrm{BF}}(t,0)$ are the propagators of $H^{\mathrm{Bog}}(t)$ and $H^{\mathrm{BF}}(t)$, respectively. Precise statements and proofs can be found in Corollary D.2.9 and Corollary D.4.3.

Bogoliubov Transformation. The propagator $U^{\mathrm{Bog}}(t,t_0)$ of $H^{\mathrm{Bog}}(t)$ is a time-dependent Bogoliubov transformation (see Corollary D.2.9). Section D.2.1 provides an overview of Bogoliubov transformations.

Invariance of the Excitation Space. The dynamics generated by $H^{\mathrm{Bog}}(t)$ and $H^{\mathrm{BF}}(t)$ leave the excitation space invariant:

$$U^{\mathrm{Bog}}(t,t_0)\big(\mathcal{F}(\{\varphi_{t_0}\}^\perp) \cap Q(\mathcal{N})\big) \subset \mathcal{F}(\{\varphi_t\}^\perp)\,,$$
$$U^{\mathrm{BF}}(t,t_0)\big(\mathcal{F}(\{\varphi_{t_0}\}^\perp) \cap Q(\mathcal{N})\big) \subset \mathcal{F}(\{\varphi_t\}^\perp)\,,$$

which can be proven analogously to [LNS15, Theorem 7].

We now proceed to specify the initial conditions for ψ_0^{BF}. To establish a connection with the solution of the full dynamics, we set in our main theorem (see Chapter 3)

$$\psi_0^{\mathrm{BF}} := U_{\rho,0}\psi_{\rho,0}\,.$$

Consequently, Condition 2.3.4 imposes assumptions on the initial state of the system.

Condition 2.3.4 (Initial Conditions for the Tracer Particle and Excitation Number)**.** *For all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$, let $\psi_0 \in L^2(\mathbb{R}^3, \mathcal{F}(L^2))$.*

i) We say ψ_0 satisfies Condition 2.3.4i)$_{\psi_0,M}$ with power $M \in \mathbb{N}_0$ if there exists constants $C_M, C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$ there exists a unitarily implementable Bogoliubov map $\mathcal{Q}_0 \in \mathcal{L}(\mathscr{H} \oplus \mathscr{H}^*)$ such that we have $\psi_0 \in Q\big((-\Delta_x + x^2 + U_{\mathcal{Q}_0}^*(\mathcal{N}+1)U_{\mathcal{Q}_0})^M\big)$ and

$$q_{\big((-\Delta_x+x^2)\otimes I + I\otimes U_{\mathcal{Q}_0}^*(\mathcal{N}+1)U_{\mathcal{Q}_0}\big)^M}(\psi_0) \leq C_M \tag{2.5}$$

as well as $\|\mathcal{Q}_0\|_{\mathcal{L}(\mathscr{H}\oplus\mathscr{H}^*)} \leq C$.

ii) We say ψ_0 satisfies Condition 2.3.4ii)$_{\psi_0,M}$ with power $M \in \mathbb{N}_0$ if

- *ψ_0 satisfies Condition 2.3.4i)$_{\psi_0,M}$*
- *and there exists a constant $C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$ we have $\|\mathcal{Q}_0\mathcal{Q}_0^* - 1\|_{\mathrm{HS}(L^2 \oplus JL^2)} \leq C\Lambda^{1/2}$.*

Remark 2.3.5.

- A natural choice for ψ_0 is $\psi_0^{\mathrm{BF}} = U_{\rho,0}\psi_{\rho,0} \in \mathcal{F}\big(\{\varphi_0\}^\perp\big)$. In this case, ψ_0 depends on both ρ and Λ due to the excitation map $U_{\rho,0}$ (see Definition 2.2.4) and the dependence of φ_0 on Λ. In this setting the particle number operator $\mathcal{N}$, when acting on ψ_0^{BF}, represents the number of excitations in the excitation space $\mathcal{F}\big(\{\varphi_0\}^\perp\big)$.
- We set tracer momenta, tracer position, and the Bogoliubov transformed number of excitations to be of $\mathcal{O}(1)$ at time $t = 0$. For technical reasons we demand the same a priori bounds on the harmonic oscillator $(-\Delta_x + x^2) \otimes I + I \otimes U_{\mathcal{Q}_0}^*(\mathcal{N}+1)U_{\mathcal{Q}_0}$. Note that $q_{(-\Delta_x)^M} + q_{x^{2M}} \leq Cq_{\big((-\Delta_x+x^2)\otimes I + I \otimes U_{\mathcal{Q}_0}^*(\mathcal{N}+1)U_{\mathcal{Q}_0}\big)^M}$ due to Lemma D.3.1. A detailed description of Condition 2.3.4 can be found in Remark 3.1.3.
- The reason for controlling the transformed excitation number $U_{\mathcal{Q}_0}^*(\mathcal{N}+1)U_{\mathcal{Q}_0}$ instead of $\mathcal{N}+1$ directly is explained in Remark 3.1.3.

Chapter 3

Main Results

3.1 Validity of the Bose-Polaron Dynamics

We now state our main result, which establishes the validity of the Bogoliubov-Fröhlich dynamics in the limit of large initial volumes $\Lambda = \rho^{\alpha}$ and large initial densities ρ for $0 < \alpha < 1/3$.

We present the result in two versions. The first, in Theorem 3.1.1, provides a simplified statement under intuitive assumptions on the condensate and the initial state of the system. The second, in Theorem 3.1.2, gives a more general formulation that retains the precise initial conditions required for the proof.

Theorem 3.1.1 (Validity of the Bogoliubov-Fröhlich Dynamics, for a Rescaled Condensate)**.** *For given $0 < \alpha < 1/3$ choose $n \in \mathbb{N}_+$ large enough.*[1] *And assume that the potentials V and W are Schwartz functions.*

i) (Condensate conditions) Let the condensate for all volumes $\Lambda \geq 1$ be a rescaled function varying on the scale $\Lambda^{1/3}$, namely $\varphi_0(y) = \eta(\Lambda^{-1/3}y)$, with $\eta \in H^{\infty}(\mathbb{R}^3, \mathbb{C})$ independent of volume Λ and density ρ and normalized in L^2. Assume that the condensate is flat around the origin, namely for all $\beta \in \mathbb{N}_+^3$ with $1 \leq |\beta| \leq 2n - 1$ we have

$$\eta(0) = 1\,, \quad D^{\beta}\eta(0) = 0\,.$$

ii) (Tracer localization and excitation number bound) For all volumes $\Lambda \geq 1$ and densities $\rho \geq 1$ let $\psi_{\rho,0} \in L^2(\mathbb{R}^3, H^1_{\mathrm{sym}}(\mathbb{R}^{3N}, \mathbb{C}))$ and $\psi_0^{\mathrm{BF}} := U_{\rho,0}\psi_{\rho,0} \in L^2(\mathbb{R}^3, Q(d\Gamma(1 - \Delta)))$, where $U_{\rho,t}$ is the excitation map defined in Definition 2.2.4.
Assume that in the initial data ψ_0^{BF} the tracer particle is localized and the number of

[1] We choose $n \in \mathbb{N}_+$ such that $n > 9/4(1/\alpha - 2)$.

excitations small, namely that there exists a constant $C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$ we have $\psi_0^{\mathrm{BF}} \in Q\big((-\Delta_x + x^2 + \mathcal{N} + 1)^{2n}\big)$ and

$$\left\langle \psi_0^{\mathrm{BF}}, \left(-\Delta_x + x^2 + \mathcal{N} + 1\right)^{2n} \psi_0^{\mathrm{BF}} \right\rangle \leq C \,. \tag{3.1}$$

Let $\psi_t^{\mathrm{BF}} \in \mathcal{F}(\{\varphi_t\}^\perp)$ be the solution of the effective Bogoliubov-Fröhlich dynamics with initial data ψ_0^{BF} (see Definition 2.3.2) and $\mu_t \in \mathbb{R}$ as in Remark 2.1.2.
Then for all times $T \geq 0$ there exist a constant $C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda = \rho^\alpha$

$$\sup_{t \in [-T,T]} \|\mathrm{e}^{\mathrm{i} \int_0^t (\rho^{1/2} \int W - \mu_s) ds} I \otimes U_{\rho,t} \mathrm{e}^{-\mathrm{i} t H_\rho} \psi_{\rho,0} - \psi_t^{\mathrm{BF}}\| \leq C \rho^{\frac{3\alpha - 1}{2}} \,. \tag{3.2}$$

Proof of Theorem 3.1.1. Theorem 3.1.1 is a direct consequence of the more general Theorem 3.1.2. ∎

Theorem 3.1.2 (Validity of the Bogoliubov-Fröhlich Dynamics)**.** *For given $0 < \alpha < 1/3$ and $0 < s < 1/3$ choose $n, k \in \mathbb{N}_+$ large enough. We assume the following:*

i) (Interaction Potentials) Assume that the potentials V and W satisfy Assumption 2.0.3$_n$, which ensures the regularity of the boson-impurity interaction potential $W \in W^{n,\infty} \cap H^n$.

ii) (Condensate conditions) We assume that for all volumes $\Lambda \geq 1$ the condensate $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$ varies on the scale $\Lambda^{1/3}$. By this we mean that it satisfies Condition 2.1.7 with additional regularity in the derivatives of φ_0, namely Condition 2.1.8i)$_{k+2n-1}$ and Condition 2.1.8$_m$ for $m = \max\{k+2, 2+2\}$. Furthermore, we require that the condensate is flat around the origin, namely Condition 2.1.11$_{2n,s}$.

iii) (Tracer localization and excitation number bound) For all volumes $\Lambda \geq 1$ and densities $\rho \geq 1$ let $\psi_{\rho,0} \in L^2(\mathbb{R}^3, H^1_{\mathrm{sym}}(\mathbb{R}^{3N}, \mathbb{C}))$ and $\psi_0^{\mathrm{BF}} := U_{\rho,0}\psi_{\rho,0} \in L^2(\mathbb{R}^3, Q(d\Gamma(1-\Delta)))$, where $U_{\rho,t}$ is the excitation map defined in Definition 2.2.4.
Assume that ψ_0^{BF} is a perturbation of a quasi-free state with localized tracer particle, namely that ψ_0^{BF} satisfies Condition 2.3.4ii)$_{2n}$.

Let $\psi_t^{\mathrm{BF}} \in \mathcal{F}(\{\varphi_t\}^\perp)$ be the solution of the effective Bogoliubov-Fröhlich dynamics with initial data ψ_0^{BF} (see Definition 2.3.2) and $\mu_t \in \mathbb{R}$ as in Remark 2.1.2.
Then for all times $T \geq 0$ there exist a constant $C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda = \rho^\alpha$

$$\sup_{t \in [-T,T]} \|\mathrm{e}^{\mathrm{i} \int_0^t (\rho^{1/2} \int W - \mu_s) ds} I \otimes U_{\rho,t} \mathrm{e}^{-\mathrm{i} t H_\rho} \psi_{\rho,0} - \psi_t^{\mathrm{BF}}\| \leq C \rho^{\frac{3\alpha - 1}{2}} \,. \tag{3.3}$$

Remark 3.1.3.

The Microscopic Dynamics. The term $I \otimes U_{\rho,t} \mathrm{e}^{-\mathrm{i}tH_\rho}\psi_{\rho,0}$ represents the excitation part of the solution to the full microscopic dynamics (see Definition 2.0.1).

Convergence and Scaling. After extracting a constant phase, we obtain both convergence and a convergence rate for $0 < \alpha < 1/3$. Note that the mean-field scaling with volume $\Lambda = 1$ corresponds to $\alpha = 0$, while the thermodynamic limit corresponds to $\alpha = \infty$.

Condition on α. We first assume $1 \geq \alpha$, which is necessary for Remark 4.2.2 and simplifies the estimate (3.3). Additionally, we restrict $1/3 > \alpha$, ensuring that the excitation number condition (4.1) in Theorem 4.2.1 follows from Condition 2.3.4ii)$_{2n}$. Note that we need Theorem 4.2.1 in the proof of Theorem 3.1.2.
If we directly assume (4.1) for ψ_0^{BF} then all $1 \geq \alpha$ are admissible. In this case, the validity of Theorem 3.1.2 also holds under the weaker Condition 2.3.4i)$_{2n}$ instead of Condition 2.3.4ii)$_{2n}$.

Lower Bound on n and k. We have explicit control on the lower bounds on n and k such that Theorem 3.1.2 is valid:

$$n \geq \frac{3}{4(1/3 - s)} \left(\frac{1}{\alpha} - 2 - \frac{s}{3} \right), \quad k \geq \frac{(2n - 1/2)(1/3 - s)}{1/3 + s}. \tag{3.4}$$

The lower bound on n is strictly increasing with $0 < s < 1/3$ for fixed $0 < \alpha < 1/3$. Conversely, the lower bound on k is strictly decreasing in $0 < s < 1/3$ with fixed $n \in \mathbb{N}_+$.

Initial State Assumptions. The assumptions are chosen such that at $t = 0$, the tracer is localized at the origin, and the condensate φ_0 is flat in its vicinity. This allows us to extract the tracer-condensate mean-field interaction as approximately a phase from the dynamics: $\rho^{1/2} W * |\varphi_t|^2(x) \sim \rho^{1/2} \int W$ (see Section 5.4 for a detailed argument).

Flatness Condition of the Condensate. The flatness of the condensate is used to control the tracer particle and has two main effects:

Dominant Excitation Interaction. The interaction of the tracer with condensate particles is subleading compared to its interaction with excitations (see Chapter 5 for details).

Tracer Localization. The tracer remains inside the Bose gas over the considered timescale of $\mathcal{O}(1)$ (see Section 5.4.3 for details).

Excitation Number in the Initial State. The initial state should satisfy the following heuristic conditions:

Global Excitation Number. The total number of excitations in the Bose gas is at most $\mathcal{O}(\Lambda)$, growing with the volume. This is required for the validity of the Bogoliubov approximation in Theorem 4.2.1.

Local Excitation Number. The number of excitations effectively interacting with the tracer is $\mathcal{O}(1)$. This corresponds to the number of excitations within a unit volume due to the tracer's $\mathcal{O}(1)$ interaction range (see Remark 2.0.4). Control over this local excitation number is crucial for ensuring tracer localization in Theorem 3.2.1.

In Condition 2.3.4 we control the transformed number of excitations $U^*_{\mathcal{Q}_0}(\mathcal{N}+1)U_{\mathcal{Q}_0}$, where $U_{\mathcal{Q}_0}$ is a Bogoliubov transformation (see Section D.2.1 for a definition).

- If we set $U_{\mathcal{Q}_0} = I$, Condition 2.3.4 on our initial state allows only $\mathcal{O}(1)$ excitations globally and hence also locally.
- By introducing $U_{\mathcal{Q}_0}$, we can meet both the global and local excitation criteria outlined above. The Bogoliubov transformation $U_{\mathcal{Q}_0}$ can change the global excitation number by $\|\mathcal{Q}_0\mathcal{Q}_0^* - 1\|^2_{\mathrm{HS}} + \|\mathcal{Q}_0\|^2_{\mathrm{op}}$ (see Lemma D.2.4). The localization of the tracer, being a local phenomenon, only requires bounds on $\|\mathcal{Q}_0\|_{\mathrm{op}}$ (see Theorem 3.2.1). Thus, we interpret $\|\mathcal{Q}_0\mathcal{Q}_0^* - 1\|^2_{\mathrm{HS}}$ as the global excitation number and $\|\mathcal{Q}_0\|_{\mathrm{op}}$ as the local excitation number in our initial conditions (see Remark 5.4.6 for further discussion).

The proof of Theorem 3.1.2 can be separated into two parts:

Chapter 4: Bogoliubov approximations.

Chapter 5: Control of the tracer-condensate mean-field interaction.

More precisely, we split

$$\|\mathrm{e}^{\mathrm{i}\int_0^t(\rho^{1/2}\int W - \mu_s)ds} I \otimes U_{\rho,t}\psi_{\rho,t} - \psi_t^{\mathrm{BF}}\| \le \|\mathrm{e}^{\mathrm{i}\int_0^t(-\mu_s)ds} I \otimes U_{\rho,t}\psi_{\rho,t} - \psi_t^{\mathrm{BF},\rho}\| \tag{3.5}$$

$$+ \|\mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho} - \psi_t^{\mathrm{BF}}\|\,. \tag{3.6}$$

Here $\psi_t^{\mathrm{BF},\rho}$ is generated by the Bogoliubov-Fröhlich Hamiltonian, which still contains the tracer-condensate mean-field interaction $H^{\mathrm{Bog},\rho}(t)$ with initial state $\psi_0^{\mathrm{BF},\rho} := \psi_0^{\mathrm{BF}}$ (see Proposition 4.1.1).

- The first term (3.5) corresponds to the Bogoliubov approximation, which establishes the effective description of the full dynamics by $H^{\mathrm{Bog},\rho}(t)$ (see Chapter 4).
- In the second term (3.6) we extract the ρ-dependent tracer-condensate mean-field interaction term $\rho^{1/2}W * |\varphi_t|^2$ from the dynamics by approximating it with $\rho^{1/2}\int W$, which is large but constant. For the second part we use our results from Chapter 5, especially the localization of the tracer particle. For more details we refer to Chapter 5 and in particular Section 5.4.

Proof of Theorem 3.1.2. We split $\|e^{i\int_0^t(\rho^{1/2}\int W-\mu_s)ds}U_{\rho,t}\psi_{\rho,t}-\psi_t^{\mathrm{BF}}\|$ into (3.5) and (3.6) and estimate both terms separately. The estimate of (3.5) is given in Theorem 4.2.1 and the estimate of (3.6) is given in Theorem 5.2.1$_\gamma$, where we set $\gamma>0$ such that $\Lambda^{-\gamma}=(\Lambda^3/\rho)^{1/2}$. Note that the conditions of Theorem 4.2.1 are satisfied, since $\forall 1\leq M\leq 4$:

$$\begin{aligned}&\left\langle\psi_0^{\mathrm{BF},\rho}, I\otimes(\mathcal{N}+1)^M\psi_0^{\mathrm{BF},\rho}\right\rangle=\left\langle I\otimes U_{\mathcal{Q}_0}\psi_0^{\mathrm{BF},\rho}, I\otimes(U_{\mathcal{Q}_0}(\mathcal{N}+1)^M U_{\mathcal{Q}_0}^*)I\otimes U_{\mathcal{Q}_0}\psi_0^{\mathrm{BF},\rho}\right\rangle\\ &\leq C_M(1+\|b\|_{\mathrm{HS}}^2+\|c\|_{\mathrm{op}}^2)^M\left\langle I\otimes U_{\mathcal{Q}_0}\psi_0^{\mathrm{BF},\rho}, I\otimes(\mathcal{N}+1)^M I\otimes U_{\mathcal{Q}_0}\psi_0^{\mathrm{BF},\rho}\right\rangle\\ &\leq C_M(1+\|\mathcal{Q}_0\mathcal{Q}_0^*-1\|_{\mathrm{HS}}^2+\|\mathcal{Q}_0\|_{\mathrm{op}}^2)^M\leq C_M\Lambda^M\,,\end{aligned}$$

where we used Lemma D.2.4 together with $\mathcal{Q}_0=:\begin{pmatrix}c & J^*bJ^*\\ b & JcJ^*\end{pmatrix}$, and $\|\mathcal{Q}_0\|_{\mathcal{L}(L^2\oplus JL^2)}\leq C$, $\|\mathcal{Q}_0\mathcal{Q}_0^*-1\|_{\mathrm{HS}(L^2\oplus JL^2)}\leq C\Lambda^{1/2}$ as well as Condition 2.3.4ii)$_{(I\otimes U_{\mathcal{Q}_0}\psi_0^{\mathrm{BF}},2M)}$ for $M=2$. Note that $n\geq\frac{3}{4(1/3-s)}\left(\frac{1}{\alpha}-2-\frac{s}{3}\right)\geq 2=M$ for $\alpha,s\in(0,1/3)$. ■

3.2 Tracer Localization

One of the important ingredients for Theorem 3.1.2 is the localization of the tracer particle, which is an interesting result on its own. For details about its interpretation we refer to Section 5.4.

Theorem 3.2.1 (Tracer Localization in Position and Momenta for the Effective Dynamics)**.** *Let $M\in\mathbb{N}_+$.*

i) (Interaction Potentials) Assume that the potentials V and W satisfy Assumption 2.0.3$_M$, which ensures the regularity of the boson-impurity interaction potential $W\in W^{M,\infty}\cap H^M$.

ii) (Condensate conditions) We assume that for all volumes $\Lambda\geq 1$ the condensate $\varphi_0\in H^\infty(\mathbb{R}^3,\mathbb{C})$ varies on the scale $\Lambda^{1/3}$. By this we mean that it satisfies Condition 2.1.7 and Condition 2.1.8$_{k=2+2}$.

iii) (Tracer localization) For all densities $\rho\geq 1$ and volumes $\Lambda\geq 1$ let $\psi_0^{\mathrm{BF}}\in L^2(\mathbb{R}^3,Q(d\Gamma(1-\Delta)))$. Assume that ψ_0^{BF} is a perturbation of a quasi-free state with localized tracer particle, namely that ψ_0^{BF} satisfies Condition 2.3.4i)$_M$ with power M.

Let ψ_t^{BF} be the solution of the effective Bogoliubov-Fröhlich dynamics with initial data ψ_0^{BF} (see Definition 2.3.2). Then

a) $(t\mapsto\psi_t^{\mathrm{BF}})\in\mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R},Q((-\Delta_x)^M))\cap\mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R},Q(x^{2M}))$.

b) For all times $T\geq 0$ there exists a constant $C>0$ such that for all densities $\rho\geq 1$ and *volumes* $\Lambda \geq 1$

$$\sup_{t\in[-T,T]} \left\{ q_{(-\Delta_x)^M}(\psi_t^{\mathrm{BF}}) + q_{x^{2M}}(\psi_t^{\mathrm{BF}}) \right\} \leq C\,.$$

Proof of Theorem 3.2.1. Set $h_{\mathrm{oc}} := (-\Delta_x + x^2) \otimes I + I \otimes (\mathcal{N}+1)$. Due to Condition 2.3.4i)$_{\psi_0^{\mathrm{BF}},M}$ there exists a unitarily implementable Bogoliubov map $\mathcal{Q}_0 \in \mathcal{L}(\mathscr{H} \oplus \mathscr{H}^*)$ such that we have $\psi_0^{\mathrm{BF}} \in I \otimes U_{\mathcal{Q}_0}^* Q(h_{\mathrm{oc}}^M) \cap Q(d\Gamma(1-\Delta)) \subset Q(h_{\mathrm{oc}}) \cap Q(d\Gamma(1-\Delta))$ and hence ψ_t^{BF} exists (see Corollary D.4.3). From Corollary 5.4.5 we know the claim for $\widetilde{\psi}_t^{\mathrm{BF}}$ with $\widetilde{\psi}_0^{\mathrm{BF}} := I \otimes U_{\mathcal{Q}_0} \psi_0^{\mathrm{BF}}$, where $\widetilde{\psi}_t^{\mathrm{BF}}$ is defined in Corollary 5.4.5 and $\widetilde{\psi}_t^{\mathrm{BF}} = I \otimes U_{\mathcal{Q}_0} U^{\mathrm{Bog}}(t,0)^* \psi_t^{\mathrm{BF}}$ due to Lemma 5.4.3, where $U^{\mathrm{Bog}}(t,0)$ is the propagator of the Bogoliubov Hamiltonian (see Remark 2.3.3). Since $I \otimes U_{\mathcal{Q}_0} U^{\mathrm{Bog}}(t,0)^*$ commutes with $x^{2M} \otimes I$ and $(-\Delta_x)^M \otimes I$ we conclude the claim for ψ_t^{BF}. Note that $q_{h_{\mathrm{oc}}^M}(I \otimes U_{\mathcal{Q}_0} \psi_0^{\mathrm{BF}}) \leq C_M$ due to Condition 2.3.4i)$_{\psi_0^{\mathrm{BF}},M}$. ■

Chapter 4

Bogoliubov Approximation

In Section 4.1, we first determine an effective intermediate Bogoliubov-Fröhlich Hamiltonian for the excitation and tracer dynamics. This is followed by the Bogoliubov approximation in Section 4.2, where we establish its validity rigorously.

4.1 Determination of the Effective Hamiltonian

In this section, we derive the intermediate Bogoliubov-Fröhlich Hamiltonian $H^{\mathrm{BF},\rho}(t)$ (see Proposition 4.1.1), which still includes the tracer-condensate mean-field interaction. This Hamiltonian is obtained from the excitation Hamiltonian $H_\rho^{\mathrm{ex}}(t) = U_{\rho,t} H_\rho U_{\rho,t}^* + \mathrm{i}(\partial_t U_{\rho,t}) \cdot U_{\rho,t}^*$. To achieve this, we isolate all terms in the excitation Hamiltonian that are small when the number of excitations is small compared to the total number of particles N. These terms are collected into an error term R_N. Additionally, we extract the constant $-\mu_t$ from the dynamics. The process of obtaining the effective Hamiltonian is referred to as the Bogoliubov approximation. The analysis in this section is based on results from [LNS15; PPS20; LP22].
The intermediate Bogoliubov-Fröhlich Hamiltonian $H^{\mathrm{BF},\rho}(t)$ still contains the mean-field term of the tracer-condensate interaction $\rho^{1/2} W * |\varphi_t|^2$. This condensate contribution could potentially dominate the tracer-excitation dynamics. To avoid this, we choose the initial data such that it can be effectively approximated by a constant, allowing us to remove it from the dynamics (see Chapter 5).

We start by showing the following Proposition 4.1.1. Together with the R_N estimate in Lemma 4.2.3 and the excitation number estimate in Lemma 4.2.6, this result completes the Bogoliubov approximation and leads to Theorem 4.2.1.

Proposition 4.1.1 (Intermediate Bogoliubov-Fröhlich Hamiltonian)**.** *For all volumes $\Lambda \geq 1$ let φ_t be the condensate. If we choose $\|\varphi_0\|_2 = \Lambda^{1/2}$ and $\mu_t := \frac{1}{2}\left\langle \frac{\varphi_t}{\Lambda^{1/2}}, V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}} \right\rangle$ in*

Definition 2.1.1 then

$$H_\rho^{ex}(t) = H^{\mathrm{BF},\rho}(t) - \mu_t + R_N(t)\,, \tag{4.1}$$

$$H^{\mathrm{BF},\rho}(t) = \rho^{1/2} W * |\varphi_t|^2(x) - \frac{\Delta_x}{2m} + A\left(Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t\right) + H^{Bog}(t)\,, \tag{4.2}$$

where $H^{\mathrm{BF},\rho}(t)$ is the intermediate Bogoliubov-Fröhlich Hamiltonian with tracer-condensate mean-field interaction term, $H^{Bog}(t)$ the Bogoliubov Hamiltonian defined in Definition 2.3.1 and

$$\begin{aligned} R_{1,N} = & -\frac{1}{2} d\Gamma\left(Q_t\left[V * |\varphi_t|^2 + K_1(t) - \mu_t\right] Q_t\right) \frac{\mathcal{N}_+(t)}{N} \\ & - \frac{(\mathcal{N}_+(t)+1)\sqrt{N - \mathcal{N}_+(t)}}{N} a\left(Q_t V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}}\right) \\ & + \frac{1}{2} \sum_{m,n \geq 1} \Lambda V_{mn00} a_m^* a_n^* \left(\frac{\sqrt{(N - \mathcal{N}_+(t))(N - \mathcal{N}_+(t) - 1)}}{N} - 1\right) \\ & + \sum_{m,n,p \geq 1} \Lambda V_{0mnp} \frac{\sqrt{N - \mathcal{N}_+(t)}}{N} a_m^* a_n a_p + \frac{1}{2} \mu_t \frac{\mathcal{N}_+(t)}{N} + \text{h.c.}\,, \end{aligned} \tag{4.3}$$

$$R_{2,N} = \frac{1}{2N} \sum_{m,n,p,q \geq 1} \Lambda V_{mnpq} a_m^* a_n^* a_p a_q\,, \tag{4.4}$$

$$R_{3,N} = a^*(Q_t W_x \varphi_t)\left(\frac{\sqrt{N - \mathcal{N}_+(t)}}{\sqrt{N}} - 1\right) + \text{h.c.}\,, \tag{4.5}$$

$$R_{4,N} = -\frac{1}{\sqrt{\rho}} W * |\varphi_t|^2(x) \mathcal{N}_+(t) + \frac{1}{\sqrt{\rho}} d\Gamma(Q_t W_x Q_t)\,, \tag{4.6}$$

$$R_N = \sum_{i=1}^{4} R_{i,N}\,, \tag{4.7}$$

where $V_{mnpq} = \langle u_m \otimes u_n\,,\, V(x-y) u_p \otimes u_q\rangle$ and $\mathcal{N}_+(t) = \mathcal{N} - a^*\left(\frac{\varphi_t}{\Lambda^{1/2}}\right) a\left(\frac{\varphi_t}{\Lambda^{1/2}}\right)$ is the number operator on the excitation space $\mathcal{F}^{\leq N}\left(\{\varphi_t\}^\perp\right)$.

Proof of Proposition 4.1.1. Equation (4.1) follows directly from the definition of H_ρ^{ex} in Definition 2.2.6 as well as Lemma 4.1.5 and Lemma 4.1.6. ■

Remark 4.1.2.

- The differential equation $\mathrm{i}\partial_t \psi_t^{\mathrm{BF},\rho} = H^{\mathrm{BF},\rho}(t)\psi_t^{\mathrm{BF},\rho}, \psi_{t=0}^{\mathrm{BF},\rho} = \psi_0^{\mathrm{BF},\rho}$ has in a weak sense a unique global solution $\psi_t^{\mathrm{BF},\rho} = U^{\mathrm{BF},\rho}(t,0)\psi_0^{\mathrm{BF},\rho}$, $\forall \psi_0^{\mathrm{BF},\rho} \in L^2(\mathbb{R}^3, \mathcal{F}(L^2))$, where $U^{\mathrm{BF},\rho}(t,0)$ is the propagator of $H^{\mathrm{BF},\rho}(t)$ (see Corollary D.4.3 for a more precise statement).
- The dynamics generated by $H^{\mathrm{BF},\rho}(t)$ leaves the excitation space invariant, meaning that $U^{\mathrm{BF},\rho}(t,t_0)\big(\mathcal{F}(\{\varphi_{t_0}\}^\perp) \cap Q(\mathcal{N})\big) \subset \mathcal{F}(\{\varphi_t\}^\perp)$, which can be proven analogous to [LNS15,

Theorem 7].

- Note that $\mathcal{N}_+(t)$ and $\mathcal{N}$ coincide on the excitation space $\mathcal{F}^{\leq N}\big(\{\varphi_t\}^\perp\big)$.

To prove Proposition 4.1.1 we need to understand $H_\rho^{\mathrm{ex}}(t)$ in detail and therefore also the excitation map $U_{\rho,t}$ defined in Definition 2.2.4. From [Lew+15] we know the following Lemma giving basic properties of $U_{\rho,t}$.

Lemma 4.1.3 (Properties of the Excitation Map). *For all volumes $\Lambda \geq 1$ let φ_t be the condensate.*

a) We have the following representation $U_{\rho,t} = \bigoplus_{k=0}^{N} Q_t^{\otimes k} \frac{a\left(\frac{\varphi_t}{\Lambda^{1/2}}\right)^{N-k}}{\sqrt{(N-k)!}}$.

b) For $f, g \in \{\varphi_t\}^\perp$ the have the following identities on $\mathcal{F}^{\leq N}\big(\{\varphi_t\}^\perp\big)$

$$U_{\rho,t}\, a^*\left(\frac{\varphi_t}{\Lambda^{1/2}}\right) a\left(\frac{\varphi_t}{\Lambda^{1/2}}\right) U_{\rho,t}^* = N - \mathcal{N}_+(t), \tag{4.8}$$

$$U_{\rho,t}\, a^*(f) a\left(\frac{\varphi_t}{\Lambda^{1/2}}\right) U_{\rho,t}^* = a^*(f)\sqrt{N - \mathcal{N}_+(t)}, \tag{4.9}$$

$$U_{\rho,t}\, a^*\left(\frac{\varphi_t}{\Lambda^{1/2}}\right) a(f)\, U_{\rho,t}^* = \sqrt{N - \mathcal{N}_+(t)}\, a(f), \tag{4.10}$$

$$U_{\rho,t}\, a^*(f) a(g)\, U_{\rho,t}^* = a^*(f) a(g)\,, \tag{4.11}$$

where $\mathcal{N}_+(t) = \mathcal{N} - a^*\left(\frac{\varphi_t}{\Lambda^{1/2}}\right) a\left(\frac{\varphi_t}{\Lambda^{1/2}}\right)$ is the number operator on $\mathcal{F}^{\leq N}\big(\{\varphi_t\}^\perp\big)$.

The Hamiltonian of the excitations, $H_\rho^{\mathrm{ex}}(t) = U_{\rho,t} H_\rho U_{\rho,t}^* + \mathrm{i}(\partial_t U_{\rho,t}) \cdot U_{\rho,t}^*$, can be treated by analyzing both terms on the right-hand side separately. For $\mathrm{i}(\partial_t U_{\rho,t}) \cdot U_{\rho,t}^*$ we use [LNS15, Lemma 6], which we restate here for the reader's convenience.

Lemma 4.1.4. *Let $u \in C^1(\mathbb{R}, L^2(\mathbb{R}^3))$ satisfying $\|u(t)\|_2 = \|u(0)\|_2$ for all $t \in \mathbb{R}$. We will denote its derivative in L^2 by $\dot{u}(t)$. Then $U_\rho \in C^1\big(\mathbb{R}, \mathcal{L}(L_s^2(\mathbb{R}^{3N}), \mathcal{F}^{\leq N}(L^2))\big)$ and*

$$\begin{aligned}\mathrm{i}(\partial_t U_{\rho,t}) = \Big(& a^*(u(t)) a(Q(t)\mathrm{i}\dot{u}(t)) - \sqrt{N - \mathcal{N}}\, a(Q(t)\mathrm{i}\dot{u}(t)) \\ & - a^*(Q(t)\mathrm{i}\dot{u}(t))\sqrt{N-\mathcal{N}} - \langle \mathrm{i}\dot{u}(t), u(t)\rangle (N - \mathcal{N})\Big) U_{\rho,t}\,,\end{aligned} \tag{4.12}$$

where $U_{\rho,t}$ is defined in Definition 2.2.4.

Next, we give another result from [LNS15].

Lemma 4.1.5 (Excitation Space Representation of the purely Bosonic Hamiltonian). *Set*

$$H_\rho^{\mathrm{B}} = -\sum_{i=1}^{N} \frac{\Delta_{y_i}}{2} + \frac{1}{\rho} \sum_{1 \leq i < j \leq N} V(y_i - y_j)\,,$$

the microscopic Hamiltonian without the tracer particle. For all volumes $\Lambda \geq 1$ *let* φ_t *be the condensate. If we choose* $\|\varphi_0\|_2 = \Lambda^{1/2}$ *and* $\mu_t := \frac{1}{2}\left\langle \frac{\varphi_t}{\Lambda^{1/2}}, V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}} \right\rangle$ *in Definition 2.1.1 then*

$$\begin{aligned} H_\rho^{\mathrm{ex,B}}(t) &:= U_{\rho,t} H_\rho^{\mathrm{B}} U_{\rho,t}^* + \mathrm{i}\,(\partial_t U_{\rho,t})\, U_\rho^* \\ &= H^{\mathrm{Bog}}(t) - \mu_t + R_{1,N} + R_{2,N}\,, \end{aligned} \tag{4.13}$$

where $H^{\mathrm{Bog}}(t)$ *is the Bogoliubov Hamiltonian defined in Definition 2.3.1, and* $R_{1,N}$ *and* $R_{2,N}$ *from Proposition 4.1.1.*

Proof of Lemma 4.1.5. The proof of Lemma 4.1.5 is analogous to [LNS15, Appendix B] if one substitutes in [LNS15, Appendix B] W with $\Lambda \cdot V$ and u_t with $\frac{\varphi_t}{\Lambda^{1/2}}$. Note that $\|\frac{\varphi_t}{\Lambda^{1/2}}\|_2 = 1$ is normalized. ■

4.1.1 Adding the Tracer Particle

We follow the method displayed in [LP22, Lemma 3.2] to transform the tracer particle contributions to the microscopic Hamiltonian H_ρ into the excitation space using Lemma 4.1.3.

Lemma 4.1.6. *For all volumes* $\Lambda \geq 1$ *let* φ_t *be the condensate. Then*

$$\begin{aligned} -\frac{\Delta_x}{2m} + \frac{1}{\rho^{1/2}} U_{\rho,t} \sum_{n=1}^N (W_x)_n U_{\rho,t}^* = \rho^{1/2} W * |\varphi_t|^2(x) - \frac{\Delta_x}{2m} + A\left(Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t\right) \\ + R_{3,N} + R_{4,N}\,, \end{aligned}$$

where $R_{3,N}$ *and* $R_{4,N}$ *are defined in Proposition 4.1.1.*

Proof of Lemma 4.1.6. For $t \in \mathbb{R}$ let $\{u_{k,t}\}_{k\in\mathbb{N}_0} =: \{u_k\}_{k\in\mathbb{N}_0}$ be an orthonormal basis of $L^2(\mathbb{R}^3, \mathbb{C})$ with $u_{0,t} = \frac{\varphi_t}{\Lambda^{1/2}}$. Our analysis is carried out for a fixed $x \in \mathbb{R}^3$. Note that for any $\phi, \psi \in L^2(\mathbb{R}^3_x, \mathcal{F}^{\leq N})$), we have

$$\begin{aligned} &\left\langle \phi, \left(-\frac{\Delta_x}{2m} + \frac{1}{\rho^{1/2}} U_{\rho,t} \sum_{n=1}^N (W_x)_n U_{\rho,t}^* \right) \psi \right\rangle \\ &= \int dx \left\langle \phi(x), \left(-\frac{\Delta_x}{2m} + \frac{1}{\rho^{1/2}} U_{\rho,t} \sum_{n=1}^N (W_x)_n U_{\rho,t}^* \right) \psi(x) \right\rangle \end{aligned}$$

such that we can work with a fixed x.
Since $W \in L^\infty$, we have that $\sum_{n=1}^N (W_x)_n \in \mathscr{L}(L^2(\mathbb{R}^3)^{\otimes_s N})$ and

$$\sum_{n=1}^N (W_x)_n = \sum_{j,k=0}^\infty \langle u_j, W_x u_k \rangle a_j^* a_k$$

on $L^2(\mathbb{R}^3)^{\otimes_s N}$, where the right-hand side converges strongly on the definition space $L^2(\mathbb{R}^3)^{\bar{\otimes}_s N}$ and is extended to the full space by its closure. We use the decomposition $I = P_t + Q_t$ (see Notation 2.2.1) to achieve

$$\begin{aligned}\sum_{n=1}^{N}(W_x)_n &= \sum_{j,k=0}^{\infty} \langle u_j, (P_t + Q_t)W_x(P_t + Q_t)u_k\rangle a_j^* a_k \\ &= \sum_{j,k=1}^{\infty} (W_x)_{jk} a_j^* a_k + \sum_{j=1}^{\infty}(W_x)_{0j} a_0^* a_j \\ &\quad + \sum_{j=1}^{\infty}(W_x)_{j0} a_j^* a_0 + (W_x)_{00} a_0^* a_0 \,,\end{aligned}$$

on $L^2(\mathbb{R}^3)^{\otimes_s N}$, where all series converge strongly in the same sense as $\sum_{j,k=0}^{\infty} \langle u_j, W_x u_k\rangle a_j^* a_k$. Since $U_{\rho,t}$ is continuous we conclude with Lemma 4.1.3 that

$$\begin{aligned}U_{\rho,t}\sum_{n=1}^{N}(W_x)_n U_{\rho,t}^* &= \sum_{j,k=1}^{\infty} (W_x)_{jk} a_j^* a_k + \sum_{j=1}^{\infty}(W_x)_{0j}\sqrt{N-\mathcal{N}}a_j \\ &\quad + \sum_{j=1}^{\infty}(W_x)_{j0} a_j^* \sqrt{N-\mathcal{N}} + (W_x)_{00}(N-\mathcal{N})\,,\end{aligned} \tag{4.14}$$

on the full space $\mathcal{F}_s^{\leq N}\big(\{\varphi_t\}^\perp\big)$, where all operator converge strongly on the definition space $U_\rho L^2(\mathbb{R}^3)^{\bar{\otimes}_s N}$ and are extended by their closure. Now we note that for $f_n \to f$ in L^2, we have $\|a^*(f_n - f)\phi\| \leq \|f_n - f\|_2 \|(\mathcal{N}+1)^{1/2}\phi\| \to 0$ and hence

$$\|a^*(f_n - f)\phi\| \to 0\,, \tag{4.15}$$

for all $\phi \in \mathcal{F}_s^{\leq N}\big(\{\varphi_t\}^\perp\big)$. We conclude, since $f \mapsto a^*(f)$ is linear, that for $\psi \in \mathcal{F}_s^{\leq N}\big(\{\varphi_t\}^\perp\big)$

$$\begin{aligned}\sum_{j=1}^{\infty}(W_x)_{j0} a_j^*\sqrt{N-\mathcal{N}}\psi &= \overset{\mathcal{F}_s^{\leq N}}{\lim_{n\to\infty}} \sum_{j=1}^{n} \langle u_j, W_x u_0\rangle a^*(u_j)\sqrt{N-\mathcal{N}}\psi \\ &= \overset{\mathcal{F}_s^{\leq N}}{\lim_{n\to\infty}} a^*\left(\sum_{j=1}^{n}\langle u_j, W_x u_0\rangle u_j\right)\sqrt{N-\mathcal{N}}\psi \\ = a^*\left(\sum_{j=1}^{\infty}\langle u_j, W_x u_0\rangle u_j\right)\sqrt{N-\mathcal{N}}\psi &= a^*\left(Q_t W_x \frac{\varphi_t}{\Lambda^{1/2}}\right)\sqrt{N-\mathcal{N}}\psi\,.\end{aligned} \tag{4.16}$$

Analogous, since $f \mapsto a(f)$ is antilinear,

$$\sum_{j=1}^{\infty}(W_x)_{j0} a_j \psi = a\left(Q_t W_x \frac{\varphi_t}{\Lambda^{1/2}}\right)\psi\,. \tag{4.17}$$

By noting

$$\sum_{j,k=1}^{\infty}(W_x)_{jk}a_j^*a_k = d\Gamma^{\leq N}\left(Q_tW_xQ_t\right) \tag{4.18}$$

on $\mathcal{F}_s^{\leq N}\left(\{\varphi_t\}^\perp\right)$, we get with (4.14), (4.16), (4.17) and (4.18) that

$$\begin{aligned} U_\rho\sum_{n=1}^{N}(W_x)_nU_\rho^* &= W*\frac{|\varphi_t|^2}{\Lambda}(x)(N-\mathcal{N}) + d\Gamma^{\leq N}\left(Q_tW_xQ_t\right) \\ &+\sqrt{N-\mathcal{N}}a\left(Q_tW_x\frac{\varphi_t}{\Lambda^{1/2}}\right) + a^*\left(Q_tW_x\frac{\varphi_t}{\Lambda^{1/2}}\right)\sqrt{N-\mathcal{N}}\,, \end{aligned}$$

on $\mathcal{F}_s^{\leq N}\left(\{\varphi_t\}^\perp\right)$, which proves the claim. ■

4.2 Bogoliubov Approximation

We want to estimate $\|\mathrm{e}^{-\mathrm{i}\int_0^t\mu_s ds}\psi_t^{\mathrm{ex}} - \psi_t^{\mathrm{BF},\rho}\|$, where $\psi_t^{\mathrm{ex}} = U_{\rho,t}\psi_{\rho,t}$ and $\psi_t^{\mathrm{BF},\rho}$ is the solution of the formal equation $\mathrm{i}\partial_t\psi_t^{\mathrm{BF},\rho} = H^{\mathrm{BF},\rho}(t)\psi_t^{\mathrm{BF},\rho}$ (see Corollary D.4.3 and (4.2) for the definition of $H^{\mathrm{BF},\rho}(t)$.).

We use the methods of [PPS20], which we transfer to the second quantization, to show the validity of the intermediate Bogoliubov-Fröhlich dynamics in Theorem 4.2.1.

Theorem 4.2.1 (Bogoliubov Approximation). *Assume that for all volumes $\Lambda \geq 1$ the condensate $\varphi_0 \in H^\infty(\mathbb{R}^3,\mathbb{C})$ varies on the scale $\Lambda^{1/3}$, namely that it satisfies Condition 2.1.7. For all volumes $\Lambda \geq 1$ and densities $\rho \geq 1$ let $\psi_{\rho,0} \in L^2(\mathbb{R}^3, H^1_{\mathrm{sym}}(\mathbb{R}^{3N},\mathbb{C}))$ and $\psi_0^{\mathrm{BF}} := U_{\rho,0}\psi_{\rho,0} \in H^1(\mathbb{R}^3,\mathcal{F}(L^2)) \cap L^2(\mathbb{R}^3, Q(d\Gamma(1-\Delta))) = Q(-\Delta_x\otimes I)\cap Q(I\otimes Q(d\Gamma(1-\Delta))$, where $U_{\rho,t}$ is the excitation map defined in Definition 2.2.4.*
If $\psi_0^{\mathrm{BF},\rho} \in Q(\mathcal{N}^4)$ and there exists a constant $C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$ and $1 \leq n \leq 4$

$$\left\langle\psi_0^{\mathrm{BF},\rho}, I\otimes(\mathcal{N}+1)^n\psi_0^{\mathrm{BF},\rho}\right\rangle \leq C\Lambda^n\,, \tag{4.1}$$

then for all times $T \geq 0$ there exists a constant $C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$

$$\sup_{t\in[-T,T]}\|\mathrm{e}^{-\mathrm{i}\int_0^t\mu_s ds}U_{\rho,t}\psi_{\rho,t} - \psi_t^{\mathrm{BF},\rho}\| \leq C\left(\frac{\Lambda^3}{\rho}\right)^{1/2}\left(1+\left(\frac{\Lambda}{\rho}\right)^{1/2}\right)\,. \tag{4.2}$$

Remark 4.2.2. If we choose $\Lambda = \rho^\alpha$ with $\alpha \leq 1$ then the right-hand side of (4.2) can be simplified. In this case we have that $\Lambda/\rho \leq 1$ and thus

$$\sup_{t\in[-T,T]} \|\mathrm{e}^{-\mathrm{i}\int_0^t \mu_s ds} U_{\rho,t}\psi_{\rho,t} - \psi_t^{\mathrm{BF},\rho}\| \leq C\left(\frac{\Lambda^3}{\rho}\right)^{1/2} ,$$

which is convergent for $\rho \to \infty$ if $\alpha < 1/3$.

Proof of Theorem 4.2.1. To shorten our notation, we write $\mathrm{e}^{\mathrm{i}\int_0^t \mu_s ds}\psi_t^{\mathrm{BF},\rho} =: \Phi_t$ and $U_{\rho,t}\psi_t =: \psi_t^{\mathrm{ex}}$. For the proof we use a Grönwall estimate. We start by calculating the time derivative by adding ± 0

$$\begin{aligned}
\frac{d}{dt}\|\psi_t^{\mathrm{ex}} - \Phi_t\|^2 &= 2\mathrm{Re}\big\langle \psi_t^{\mathrm{ex}} - \Phi_t, -\mathrm{i}H_\rho^{\mathrm{ex}}\psi_t^{\mathrm{ex}} - (-\mathrm{i})(H^{\mathrm{BF},\rho} - \mu_t)\Phi_t\big\rangle \\
&= 2\mathrm{Re}\big\langle \psi_t^{\mathrm{ex}} - \Phi_t, -\mathrm{i}(H_\rho^{\mathrm{ex}} - H^{\mathrm{BF},\rho} + \mu_t)\psi_t^{\mathrm{ex}}\big\rangle \\
&\quad + 2\mathrm{Re}\big\langle \psi_t^{\mathrm{ex}} - \Phi_t, -\mathrm{i}(H^{\mathrm{BF},\rho} - \mu_t)(\psi_t^{\mathrm{ex}} - \Phi_t)\big\rangle \\
&= 2\mathrm{Im}\big\langle \psi_{\mathrm{t}}^{\mathrm{ex}} - \Phi_{\mathrm{t}}, (\mathrm{H}_\rho^{\mathrm{ex}} - \mathrm{H}^{\mathrm{BF},\rho} + \mu_{\mathrm{t}})\psi_{\mathrm{t}}^{\mathrm{ex}}\big\rangle = -2\mathrm{Im}\langle \Phi_{\mathrm{t}}, \mathrm{R}_{\mathrm{N}}\psi_{\mathrm{t}}^{\mathrm{ex}}\rangle\,,
\end{aligned}$$

where in step three we have used the real part and $H^{\mathrm{BF},\rho}$ symmetric and in step four the imaginary part.

We now use the estimate on the remainder term in Lemma 4.2.3 to bound the right-hand side by the particle number operator acting on the effective dynamics $\psi_t^{\mathrm{BF},\rho}$

$$\begin{aligned}
&\frac{d}{dt}\|\psi_t^{\mathrm{ex}} - \Phi_t\|^2 \\
&\leq 4\rho^{-1/2}(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2 + 1)(\|W\|_1 + \|W\|_2 + \|W\|_\infty)\|\psi_t^{\mathrm{ex}} - \Phi_t\|\|(\mathcal{N}+1)\Phi_t\| \\
&+ C\rho^{-\frac{1}{2}}\left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2\right)(\|V\|_1 + \|V\|_2)\|\psi_t^{\mathrm{ex}} - \Phi_t\|\|(\mathcal{N}+1)^{\frac{3}{2}}\Phi_t\| \\
&+ C\rho^{-1}\left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2 + 1\right)(\|V\|_1 + \|V\|_2 + \|V\|_\infty)\|\psi_t^{\mathrm{ex}} - \Phi_t\|\|(\mathcal{N}+1)^2\Phi_t\|\,.
\end{aligned}$$

It follows with our initial condition (4.1), the excitation number estimates Lemma 4.2.6, and Corollary B.1.2 with $(a+b)^n = \sum_{i=0}^n \binom{n}{k} a^k b^{n-k}$ that $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$ and $-T \leq t \leq T$

$$\begin{aligned}
\frac{d}{dt}\|\psi_t^{\mathrm{ex}} - \Phi_t\|^2 \leq \|\psi_t^{\mathrm{ex}} - \Phi_t\|\Big\{&C\rho^{-1/2}\big\langle \Phi_0, (\Lambda + \mathcal{N} + 1)^2\Phi_0\big\rangle^{1/2} \\
&+ C\rho^{-1/2}\big\langle \Phi_0, (\Lambda + \mathcal{N} + 1)^3\Phi_0\big\rangle^{1/2} \\
&+ C\rho^{-1}\big\langle \Phi_0, (\Lambda + \mathcal{N} + 1)^4\Phi_0\big\rangle^{1/2}\Big\} \\
\leq \|\psi_t^{\mathrm{ex}} - \Phi_t\|&\Big\{C\rho^{-1/2}\Lambda + C\rho^{-1/2}\Lambda^{3/2} + C\rho^{-1}\Lambda^2\Big\} \\
\leq \|\psi_t^{\mathrm{ex}} - \Phi_t\|&C\left(\left(\frac{\Lambda^2}{\rho}\right)^{1/2} + \left(\frac{\Lambda^3}{\rho}\right)^{1/2} + \frac{\Lambda^2}{\rho}\right)
\end{aligned}$$

$$\leq \|\psi_t^{\mathrm{ex}} - \Phi_t\| C \left(\left(\frac{\Lambda^3}{\rho} \right)^{1/2} + \frac{\Lambda^2}{\rho} \right),$$

where in the last step we used that $\rho^{-1/2}\Lambda \leq \rho^{-\frac{1}{2}}\Lambda^{3/2}$, for $\Lambda \geq 1$. We conclude with Grönwall

$$\|\psi_t^{\mathrm{ex}} - \Phi_t\| \leq C \left(\frac{\Lambda^3}{\rho} \right)^{1/2} \left(1 + \left(\frac{\Lambda}{\rho} \right)^{1/2} \right),$$

which proves the claim. ■

Lemma 4.2.3 (Remainder Term Estimate, part of the proof of Theorem 4.2.1)**.** *For all volumes $\Lambda \geq 1$ let φ_t be the condensate satisfying Condition 2.1.7 then*

$$\begin{aligned}
&-2\mathrm{Im}\langle \Phi_t, R_N \psi_t^{\mathrm{ex}} \rangle \\
&\leq 4\rho^{-1/2}(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2 + 1)(\|W\|_1 + \|W\|_2 + \|W\|_\infty)\|\psi_t^{\mathrm{ex}} - \Phi_t\| \|(\mathcal{N}+1)\psi_t^{\mathrm{BF},\rho}\| \\
&+ C\rho^{-\frac{1}{2}} \left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2 \right) (\|V\|_1 + \|V\|_2) \|\psi_t^{\mathrm{ex}} - \Phi_t\| \|(\mathcal{N}+1)^{\frac{3}{2}} \psi_t^{\mathrm{BF},\rho}\| \\
&+ C\rho^{-1} \left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2 + 1 \right) (\|V\|_1 + \|V\|_2 + \|V\|_\infty) \|\psi_t^{\mathrm{ex}} - \Phi_t\| \|(\mathcal{N}+1)^2 \psi_t^{\mathrm{BF},\rho}\| . \qquad (4.3)
\end{aligned}$$

The proof of Lemma 4.2.3 can be found in the Appendix E.1.1.

In the following Lemma we state the key estimate for Lemma 4.2.3 and thus for Theorem 4.2.1.

Lemma 4.2.4. *For all volumes $\Lambda \geq 1$ let φ_t be the condensate satisfying Condition 2.1.7. Let $\tilde{\Phi}, \Phi \in L^2(\mathbb{R}^3, D(\mathcal{N})) \subset L^2(\mathbb{R}^3, \mathcal{F}_s(L^2))$. We estimate*

$$\begin{aligned}
\mathrm{Im}\left\langle \tilde{\Phi}, \sum_{j,k\geq 1} \Lambda V_{jk00} a_j^* a_k^* \Phi \right\rangle &\leq \|\varphi_t\|_\infty^2 \|V\|_1 \|\mathcal{N}^{1/2}\tilde{\Phi}\| \|\mathcal{N}^{1/2}\Phi\| \\
&+ \Lambda^{1/2} \|\varphi_t\|_\infty \|V\|_2 \|\mathcal{N}^{1/2}\tilde{\Phi}\| \|\Phi\| . \qquad (4.4)
\end{aligned}$$

Remark 4.2.5.

a) We obtained a similar result to [PPS20, Lemma 3.5, Part γ_N^c]. However, from the proof of Lemma 4.2.4, it remains unclear whether the estimate can be further improved.

b) The second term in (4.4), with the prefactor $\Lambda^{1/2}$, determines the order in Λ in the estimate Lemma 4.2.6.

c) Note that $\sum_{jk\geq 1} \Lambda V_{jk00} a_j^* a_k^* \Phi$ is well defined due to part b) and d) of Lemma C.0.1.

The proof of Lemma 4.2.4 can be found in the Appendix E.1.2.

Lemma 4.2.6 (Excitation Number Estimate, part of the proof of Theorem 4.2.1)**.** *For all volumes $\Lambda \geq 1$ let φ_t be the condensate with initial data $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$ satisfying Condition 2.1.7. Then for all $n \in \mathbb{N}_0$ and times $T \geq 0$ there exists a constant $C > 0$ such that for all* *densities* $\rho \geq 1$ *and volumes* $\Lambda \geq 1$ *we have*

$$\sup_{t\in[-T,T]} \left\langle \psi_t^{\mathrm{BF},\rho}, I \otimes (\mathcal{N}+1)^n \psi_t^{\mathrm{BF},\rho} \right\rangle \leq C \left\langle \psi_0^{\mathrm{BF},\rho}, I \otimes (\Lambda + \mathcal{N}+1)^n \psi_0^{\mathrm{BF},\rho} \right\rangle . \tag{4.5}$$

The proof of Lemma 4.2.6 can be found in Appendix E.1.3.

Chapter 5

Control of the Tracer-Condensate Mean-Field Interaction

We started with the Hamiltonian H_ρ and established the validity of $H^{\mathrm{BF},\rho} = H^{\mathrm{BF}} + \sqrt{\rho} W * |\varphi_t|^2(x)$ for the corresponding excitation dynamics in Chapter 4.
In this section, we show that the tracer-condensate mean-field interaction $\sqrt{\rho} W * |\varphi_t|^2(x)$ is approximately constant; allowing us to remove it from the effective dynamics. As a result, the system is well described by $H^{\mathrm{BF}}(t)$.

We first outline our approach in Section 5.1. The main result – showing that the dynamics of $H^{\mathrm{BF},\rho}$ can be well approximated by those of H^{BF} – is presented in Section 5.2. Finally, in Sections 5.3 and 5.4, we provide a detailed discussion of the two main steps of the proof: the local control of the tracer-condensate interaction and the localization of the tracer.

5.1 Outline of the Tracer-Condensate Control

Overview of the Method. Since the tracer-condensate mean-field interaction term $\sqrt{\rho} W * |\varphi_t|^2(x)$, is of $\mathcal{O}(\sqrt{\rho})$, it could potentially dominate the dynamics of the tracer particle and the energy gain of the tracer due to $\sqrt{\rho} W * |\varphi_t|^2(x)$ could lead to the tracer leaving the Bose gas in $\mathcal{O}(1)$ times (for details see Section 5.4.3). However, our focus is on the interaction between the tracer and the excitations, so we aim to choose a setting where $\sqrt{\rho} W * |\varphi_t|^2(x)$ does not outweigh the other interactions. In fact, for the tracer to be able to interact with excitations it has to stay inside the gas cloud.
To achieve this, we show that $\sqrt{\rho} W * |\varphi_t|^2(x)$ is approximately a constant, although large. For this, we want the condensate to remain flat around the position of the tracer particle,

specifically $|\varphi_t|^2 \sim 1$ if $|\varphi_0|^2 \sim 1$. In this case

$$\sqrt{\rho} W * |\varphi_t|^2(x) \sim \sqrt{\rho} W * 1 \sim \text{constant} . \tag{5.1}$$

We only need to control $|\varphi_t|^2$ within the interaction range of W around the tracer particle's position.
The idea above is realized in two separated steps:

1. **Flatness of the Condensate.** Show that the condensate remains flat around the origin, ensuring that (5.1) holds rigorously in this region.

2. **Tracer Localization.** Show that the tracer is localized around the origin, assuming the condensate remains flat in this region.

Since we do not control the precise position of the tracer, we assume it is initially localized around a fixed point – taken to be the origin – and show that the condensate remains approximately flat around this point. We then show that the tracer remains localized near this position over time.
Step 1 is achieved by controlling the condensate localized around the origin, specifically via the bound of $\|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2$ from Proposition B.3.5, where Θ_Λ is the localization function defined in (5.3) below. This, in turn, leads to a local control of $\sqrt{\rho} W * |\varphi_t|^2$ in the same region, as discussed in Section 5.3. In particular, since $|\tilde{\varphi}_t|^2 = |\varphi_0|^2$, the auxiliary function $\tilde{\varphi}_t$ preserves the initial flatness of the condensate φ_0 in the mean-field interaction term.
In Step 2, we establish the tracer localization. We implement the assumption that the condensate remains flat around the origin by using the effective dynamics generated by H^{BF}, where $\sqrt{\rho} W * |\varphi_t|^2(x)$ has already been extracted, to describe the time evolution. The details of this step can be found in Section 5.4.

Detailed Implementation. Now, we discuss how to implement our approach on a more technical level. Our goal is to prove the convergence

$$\|\mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W} \psi_t^{\mathrm{BF},\rho} - \psi_t^{\mathrm{BF}}\| \to 0 .$$

We are interested in controlling $\|\sqrt{\rho} W * (|\varphi_t|^2 - 1)(x)\psi_t^{\mathrm{BF}}\|$ which can be seen directly from Duhamel's formula. To simplify the notation, we define $W_\Lambda := \sqrt{\rho} W * (|\varphi_t|^2 - 1)$.

Lemma 5.1.1 (Duhamel). *Set $\psi_0^{\mathrm{BF},\rho} = \psi_0^{\mathrm{BF}}$. Then we have*

$$\|\mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W} \psi_t^{\mathrm{BF},\rho} - \psi_t^{\mathrm{BF}}\|^2 = \int_0^t 2\mathrm{Im}\Big\langle \mathrm{e}^{\mathrm{i}s\rho^{1/2}\int W} \psi_s^{\mathrm{BF},\rho}, W_\Lambda(x)\psi_s^{\mathrm{BF}} \Big\rangle ds . \tag{5.2}$$

Proof of Lemma 5.1.1.

$$
\begin{aligned}
&\partial_t \|\mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho} - \psi_t^{\mathrm{BF}}\|^2 \\
&= 2\mathrm{Re}\Big\langle \mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho} - \psi_t^{\mathrm{BF}}, -\mathrm{i}(H^{\mathrm{BF},\rho} - \rho^{1/2}\int W)\mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho} + \mathrm{i}H^{\mathrm{BF}}\psi_t^{\mathrm{BF}}\Big\rangle \\
&= 2\mathrm{Re}\Big\langle \mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho}, \mathrm{i}H^{\mathrm{BF}}\psi_t^{\mathrm{BF}}\Big\rangle + 2\mathrm{Re}\Big\langle -\psi_t^{\mathrm{BF}}, -\mathrm{i}(H^{\mathrm{BF},\rho} - \rho^{1/2}\int W)\mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho}\Big\rangle \\
&= 2\mathrm{Re}\Big\langle \mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho}, -\mathrm{i}\left(H^{\mathrm{BF},\rho} - \rho^{1/2}\int W - H^{\mathrm{BF}}\right)\psi_t^{\mathrm{BF}}\Big\rangle \\
&= 2\mathrm{Re}\Big\langle \mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho}, -\mathrm{i}W_\Lambda(x)\psi_t^{\mathrm{BF}}\Big\rangle .
\end{aligned}
$$

The claim follows by integration. ■

To implement steps 1 and 2 outlined above, we introduce for $s \geq 0$ and $n \in \mathbb{N}_0$ the localization function around the origin

$$
\Theta_\Lambda(x) = \frac{1}{1 + (\Lambda^{-s}x)^{2n}} , \tag{5.3}
$$

which is discussed in detail in the Appendix A. By multiplying and dividing with the localization function, we obtain the bound

$$
\|\sqrt{\rho}W * (|\varphi_t|^2 - 1)(x)\psi_t^{\mathrm{BF}}\| \leq \underbrace{\|\Theta_\Lambda\sqrt{\rho}W * (|\varphi_t|^2 - 1)(x)\|_\infty}_{\text{Step 1}} \underbrace{\|1/\Theta_\Lambda(x)\psi_t^{\mathrm{BF}}\|}_{\text{Step 2}} . \tag{5.4}
$$

Here, the control of the localized mean-field interaction $\|\Theta_\Lambda\sqrt{\rho}W * (|\varphi_t|^2 - 1)(x)\|_\infty$ corresponds to step 1 and the control of $\|1/\Theta_\Lambda(x)\psi_t^{\mathrm{BF}}\|$ corresponds to step 2, where we bound moments of the tracer position operator x acting on the effective dynamics ψ_t^{BF}.

Next, we give a remark on the scaling of the tracer in relation to the condensate scaling, which is important for both steps above.

Remark 5.1.2 (Condensate and tracer scaling)**.** Our image of the condensate is that it varies on the scale $\Lambda^{1/3}$, in line with our initial conditions. This scaling ensures that the condensate undergoes only small changes on an $\mathcal{O}(1)$-scale, remaining approximately constant around the origin. Notably, this $\mathcal{O}(1)$ region is precisely where we later prove the tracer to be localized, giving the scale of the tracer particle (see Theorem 3.2.1).
Note that the above reasoning motivated the introduction of a large volume Λ for the gas in the first place. Contrary, if the initial condensate φ_0 has a $\mathcal{O}(1)$ scaling, e.g. being approximately constant~ 1 within a unit ball and quickly decaying to zero at the edges, then the Laplacian term in the Hartree equation would induce fluctuations at the boundaries, where $-\Delta\varphi_0 \neq 0$. Given an $\mathcal{O}(1)$ condensate scaling, these fluctuations would reach the tracer in $\mathcal{O}(1)$ time, breaking the approximation $\rho^{1/2}W * |\varphi_t|^2(x) \sim \rho^{1/2}W * 1$.

However, in the rescaled case, where the tracer's distance from the condensate edges is $\mathcal{O}(\Lambda^{1/3})$, the large volume Λ ensures that fluctuations take much longer to propagate across this region. The smooth decay at the condensate edges further suppresses these fluctuations. The choice of $s < 1/3$ ensures that Θ_Λ has a smaller scaling than the condensate, serving as a localization function.

5.2 Control of the Tracer-Condensate Mean-Field Interaction

The control of the tracer-condensate mean-field interaction around the origin can be found in Section 5.3 and especially Corollary 5.3.2. The tracer localization, namely the estimate of $\|1/\Theta_\Lambda(x)\psi_t^{\mathrm{BF}}\|$, is stated in one of our main Theorems: Theorem 3.2.1. Its detailed proof can be found in Section 5.4.

With Theorem 3.2.1 and Corollary 5.3.2 at hand we are able to prove the convergence of $\|\mathrm{e}^{it\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho} - \psi_t^{\mathrm{BF}}\|$ as well as obtain a convergence rate.

Theorem 5.2.1 (Control of the Tracer-Condensate Mean-Field Interaction)**.** *Let $\alpha > 0$, $\gamma > 0$, $1/3 > s > 0$ and $n, k \in \mathbb{N}_+$ with*

$$n \geq \frac{1}{(1/3 - s)}\left(\frac{1}{4\alpha} + \gamma - \frac{s}{4}\right), \quad k \geq \frac{(2n-1/2)(1/3-s)}{1/3+s}. \tag{5.1}$$

i) (Interaction Potentials) Assume that the potentials V and W satisfy Assumption 2.0.3$_n$, which ensures the regularity of the boson-impurity interaction potential $W \in W^{n,\infty} \cap H^n$.

ii) (Condensate conditions) We assume that for all volumes $\Lambda \geq 1$ the condensate $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$ varies on the scale $\Lambda^{1/3}$. By this we mean that it satisfies Condition 2.1.7 with additional regularity in the derivatives of φ_0, namely Condition 2.1.8i)$_{k+2n-1}$ and Condition 2.1.8$_m$, $m = \max\{k+2, 2+2\}$. Furthermore, we require that the condensate is flat around the origin, namely Condition 2.1.11$_{2n,s}$.

iii) (Tracer Localization) For all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$ let $\psi_0^{\mathrm{BF},\rho} := \psi_0^{\mathrm{BF}} \in I \otimes Q(d\Gamma(1-\Delta))$. Assume that ψ_0^{BF} is a perturbation of a quasi-free state with localized tracer particle, namely that ψ_0^{BF} satisfies Condition 2.3.4i)$_{2n}$.

Then for all times $T \geq 0$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and densities $\rho = \Lambda^{1/\alpha}$ we have

$$\sup_{t\in[-T,T]} \|\mathrm{e}^{it\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho} - \psi_t^{\mathrm{BF}}\| \leq C\Lambda^{-\gamma}.$$

Proof of Theorem 5.2.1. Let $\gamma, \alpha > 0$, $n, k \in \mathbb{N}_+$, and $\psi_0^{\mathrm{BF},\rho} := \psi_0^{\mathrm{BF}} \in Q(h_{\mathrm{oc}}^{2n}) \cap Q(d\Gamma(1-\Delta)) \subset Q(h_{\mathrm{oc}})$.[1] Applying Duhamel's formula, Lemma 5.1.1, we obtain the estimate

$$\begin{aligned} \|\mathrm{e}^{\mathrm{i}t\rho^{1/2}\int W}\psi_t^{\mathrm{BF},\rho} - \psi_t^{\mathrm{BF}}\|^2 &\leq 2\int_0^t \|\psi_\tau^{\mathrm{BF},\rho}\|\|W_\Lambda(x)\psi_\tau^{\mathrm{BF}}\|d\tau \\ &\leq 2\int_0^t \|\Theta_\Lambda W_\Lambda\|_\infty \|\psi_0^{\mathrm{BF},\rho}\| \left\|\frac{\psi_\tau^{\mathrm{BF}}}{\Theta_\Lambda(x)}\right\| d\tau\,, \end{aligned} \tag{5.2}$$

where we inserted the localization function $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}|x|)^{2n}}$ from Example A.0.3, $0 < s < 1/3$, and used the norm conservation of $\psi_t^{\mathrm{BF},\rho}$.

Now let $T \geq 0$. Using the tracer localization result from Theorem 3.2.1$_M$, for $M = n$, we conclude $\exists C > 0$ such that $\forall \Lambda \geq 1$ and $-T \leq t \leq T$

$$\left\|\frac{\psi_t^{\mathrm{BF}}}{\Theta_\Lambda(x)}\right\| \leq \|\psi_0\|_2 + C\Lambda^{-2ns}\,. \tag{5.3}$$

Next, we apply Corollary 5.3.2$_{n,k}$, choosing $k \in \mathbb{N}_+$ as in (5.1) such that $\Lambda^{\frac{1}{2\alpha}-\left(\frac{1}{6}+\frac{k}{3}+ks\right)} + \Lambda^{\frac{1}{2\alpha}-\frac{s}{2}-2n(1/3-s)} \leq 2\Lambda^{\frac{1}{2\alpha}-\frac{s}{2}-2n(1/3-s)}$. Using Corollary 5.3.2$_{n,k}$ and (5.3), we conclude that $\exists C > 0$ such that $\forall \Lambda \geq 1$ and $-T \leq t \leq T$

$$\begin{aligned} \int_0^t \|\Theta_\Lambda W_\Lambda\|_\infty \left\|\frac{\psi_\tau^{\mathrm{BF}}}{\Theta_\Lambda(x)}\right\| d\tau &\leq C\Lambda^{\frac{1}{2\alpha}-\frac{s}{2}-2n(1/3-s)} \cdot \left(1+\Lambda^{-2ns}\right) \\ &\leq C\Lambda^{\frac{1}{2\alpha}-\frac{s}{2}-2n(1/3-s)} \\ &\leq C\Lambda^{-2\gamma}\,. \end{aligned}$$

In the last step, we used (5.1) for n. The claim now follows from the above estimate and (5.2). Finally, we note that $\|\psi_0^{\mathrm{BF},\rho}\| \leq C$ due to Condition 2.3.4i)$_{\psi_0^{\mathrm{BF}},0}$ and the fact that $\psi_0^{\mathrm{BF},\rho} := \psi_0^{\mathrm{BF}}$. ∎

5.3 Local Control of the Tracer-Condensate Interaction

To control $\|\Theta_\Lambda\sqrt{\rho}W * (|\varphi_t|^2 - 1)\|_\infty$ for the proof of Theorem 5.2.1 we need precise control over the localized condensate especially over $\|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2$. The desired estimates are achieved in Appendix B.3.

We start the estimate by splitting $\|\Theta_\Lambda W_\Lambda\|_\infty$ into two parts

$$\|\Theta_\Lambda W_\Lambda\|_\infty = \|\Theta_\Lambda\sqrt{\rho}W * (|\varphi_t|^2 - 1)\|_\infty$$

[1] Note that $\psi_0^{\mathrm{BF}} \in I \otimes U_{\mathcal{Q}_0}^* Q(h_{\mathrm{oc}}^{2n}) = Q(h_{\mathrm{oc}}^{2n})$ due to Condition 2.3.4i)$_{\psi_0^{\mathrm{BF}},2n}$ and Lemma D.2.4.

$$\leq \|\Theta_\Lambda \sqrt{\rho} W * (|\varphi_0|^2 - 1)\|_\infty + \|\Theta_\Lambda \sqrt{\rho} W * (|\varphi_t|^2 - |\varphi_0|^2)\|_\infty . \tag{5.1}$$

Our analysis now consists of the two steps of estimating both terms above. If we assume Condition 2.1.11$_{k,s}$, for a sufficiently large k, and $0 < s < 1/3$, then the term $\Theta_\Lambda \sqrt{\rho} W * (|\varphi_0|^2 - 1)$ is flat around the origin which makes it easy to be estimated. We give the following Lemma.

Lemma 5.3.1. *Let $\alpha, s > 0$, $n \in \mathbb{N}_0$, and $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}x)^{2n}}$ be the localization function. Assume that for all volumes $\Lambda \geq 1$ the condensate $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$ varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.8i)$_{2n}$. Furthermore, we require that the condensate is flat around the origin, namely Condition 2.1.11$_{2n,s}$. Then there exists a constant $C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda = \rho^\alpha$ we have*

$$\|\Theta_\Lambda \sqrt{\rho} W * (|\varphi_0|^2 - 1)\|_\infty \leq \|W\|_1 C \Lambda^{\frac{1}{2\alpha} - 2n\left(\frac{1}{3} - s\right)} . \tag{5.2}$$

Proof of Lemma 5.3.1. The case $n = 0$ is trivial. Now let $n \in \mathbb{N}_+$. We move Θ_Λ inside the convolution. Therefore we use Lemma A.0.6b), especially $\Lambda \geq 1$. This yields, $\forall m \in \mathbb{N}_+$

$$\begin{aligned} &\left|\Theta_\Lambda W * (|\varphi_0|^2 - 1)\right|(x) \\ &\qquad \leq \sqrt{\rho} C_m \|(\varphi_0^* + 1)\Theta_\Lambda(\varphi_0 - 1)\|_\infty + \sqrt{\rho}\Lambda^{-sm} C_m (\|\varphi_0\|_\infty^2 + 1) \\ &\qquad \leq C_n \sqrt{\rho} \Lambda^{-2n\left(\frac{1}{3} - s\right)} , \end{aligned} \tag{5.3}$$

where we used $\|\varphi_0\|_\infty \leq C$, Lemma B.3.1i) (which requires flatness around the origin of the condensate) and that we can choose m large enough (for fixed $n, s > 0$), such that $\Lambda^{-sm} \leq \Lambda^{-2n(\frac{1}{3} - s)}$. The claim follows from $\Lambda = \rho^\alpha$. ■

Since $|\varphi_t|^2 - |\varphi_0|^2$ is missing a flatness condition around the origin, the second term in (5.1) is much harder to estimate. However, using the identity $|\varphi_0|^2 = |\tilde{\varphi}_t|^2$ and the estimate of $\|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2$ from Appendix B.3 we can complete the argument.

Corollary 5.3.2 (Local Control of the Mean-Field Tracer-Condensate Interaction)**.** *Let $\alpha, s > 0$, $n, k \in \mathbb{N}_+$, and $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}x)^{2n}}$ be the localization function. Assume that for all volumes $\Lambda \geq 1$ the condensate $\varphi_0 \in H^\infty(\mathbb{R}^3, \mathbb{C})$ varies on the scale $\Lambda^{1/3}$. By this we mean that it satisfies Condition 2.1.7 with additional regularity in the derivatives of φ_0, namely Condition 2.1.8i)$_{k+2n-1}$ and Condition 2.1.8ii)$_{k+2}$. Furthermore, we require that the condensate is flat around the origin, namely Condition 2.1.11$_{2n,s}$.*
Then for all times $T \geq 0$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and densities $\rho = \Lambda^{1/\alpha}$ we have

$$\sup_{t \in [-T,T]} \|\Theta_\Lambda W_\Lambda\|_\infty \leq C \left\{ \Lambda^{\frac{1}{2\alpha} - \left(\frac{1}{6} + \frac{k}{3} + ks\right)} + \Lambda^{\frac{1}{2\alpha} - \frac{s}{2} - 2n(1/3 - s)} \right\} , \tag{5.4}$$

where $W_\Lambda = \sqrt{\rho} W * (|\varphi_t|^2 - 1)$.

Remark 5.3.3 (Arbitrary Good Convergence)**.** We obtained an arbitrarily good convergence rate for $\|\Theta_\Lambda W_\Lambda\|_\infty$ in Λ through our bound in (5.4). More precisely, the exponent of all Λ-dependent terms on the right-hand side of (5.4) can be made arbitrarily small, provided that Condition 2.1.8$_k$ and Condition 2.1.11$_{n,s}$ are satisfied for all k, n, and $0 < s < 1/3$.

Proof of Corollary 5.3.2. The proof of Corollary 5.3.2 is separated into several Lemmas controlling the condensate φ_t, which can be found in Appendix B.3. Let $T \geq 0$ and $-T \leq t \leq T$. We use Lemma A.0.6b) to conclude that $\forall m \in \mathbb{N}_+$

$$\begin{aligned}\|\Theta_\Lambda \sqrt{\rho} W * (|\varphi_t|^2 - |\varphi_0|^2)\|_\infty &\leq \sqrt{\rho} C \|\Theta_\Lambda(|\varphi_t|^2 - |\varphi_0|^2)\|_{1\wedge 2\wedge\infty} \\ &\quad + C\sqrt{\rho}\Lambda^{-ms} \|\, |\varphi_t|^2 - |\varphi_0|^2\|_{1\wedge 2\wedge\infty} \\ &\leq \sqrt{\rho} C \|\Theta_\Lambda(|\varphi_t|^2 - |\varphi_0|^2)\|_{1\wedge 2\wedge\infty} + C\Lambda^{\frac{1}{2\alpha} - \left(\frac{1}{6} + ms\right)}\,. \end{aligned} \tag{5.5}$$

Note that $\| \,.\, \|_{1\wedge 2\wedge\infty}$ is defined in Definition A.0.5. Corollary 5.3.2 follows directly from (5.1), (5.5), Lemma 5.3.1 and Proposition B.3.5, for $(\beta = 0, \tilde{k} = k - 1)$ with

$$\begin{aligned}\Lambda^{\frac{1}{2\alpha}} \|\Theta_\Lambda(|\varphi_t|^2 - |\varphi_0|^2)\|_{1\wedge 2\wedge\infty} &\leq \Lambda^{\frac{1}{2\alpha}} \|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2 \|\varphi_t - \tilde{\varphi}_t\|_2 \\ &\quad + \Lambda^{\frac{1}{2\alpha}} \|\varphi_0\|_\infty \|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2 \\ &\leq C\Lambda^{\frac{1}{2\alpha}} \|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2\,, \end{aligned}$$

where we used the identity $|\varphi_t|^2 - |\varphi_0|^2 = |\varphi_t - \tilde{\varphi}_t|^2 + 2\mathrm{Re}\tilde{\varphi}_t^*(\varphi_t - \tilde{\varphi}_t)$ and Proposition B.1.1. Note that since $s > 0$ we can choose m in (5.5) large enough such that $\Lambda^{-\frac{1}{6}}\Lambda^{-ms} \leq \Lambda^{-s/2-2n(1/3-s)}$.

■

5.4 Tracer Localization

This section is focused on proving the tracer localization in both position and momentum space in the dynamics generated by $H^{\mathrm{BF}}(t)$. The tracer localization in the effective dynamics is one of our main results and is stated in Theorem 3.2.1. Our goal is to show that the tracer remains confined within a region of $\mathcal{O}(1)$ over timescales of $\mathcal{O}(1)$. To do this, we first explain the main ideas behind the proof in Section 5.4.1 and then present the result in Section 5.4.2.

5.4.1 Outline of the Localization Argument

To control the tracer position we have to control two contributions to its dynamics, one coming from the condensate φ_t and one from the excitations.
The condensate gives a minor contribution, as the largest term that arises from its interaction with the tracer, $\sqrt{\rho} W * |\varphi_t|^2(x)$, has already been extracted from the dynamics: $H^{\mathrm{ex}}_\rho =$

$H^{\mathrm{BF}} + \sqrt{\rho} W * |\varphi_t|^2(x) - \mu_t + \text{error}$. The remaining condensate contributions in H^{BF} are controlled by Corollary B.1.2 in the Appendix.

The excitation contribution is controlled by showing that the tracer only gains $\mathcal{O}(1)$ energy from its interactions with the excitations. This involves proving two key points:

1. **The Number of Effective Interactions.** The number of excitations which effectively interact with the tracer is $\mathcal{O}(1)$.

2. **The Energy Gain per Excitation.** The tracer gains at most $\mathcal{O}(1)$ energy from each excitation it interacts with.

The first point is established using the Bogoliubov transformation U_t^{Bog} (see Corollary D.2.9). The second point is ensured by the proper scaling of the boson-tracer interaction in the Hamiltonian (see Remark 2.0.2, and the detailed description below).

Number of Effectively Interacting Excitations with the Tracer. We begin with a heuristic explanation supported by two main observations:

I. Range of the Interaction Potential. The range of the interaction potential W of tracer and bosons is $\mathcal{O}(1)$ (see Assumption 2.0.3).

II. Global Number of Excitations. The global number of excitations is at most $\mathcal{O}(\Lambda)$, as shown by the bound in Lemma 4.2.6.

If we think of the excitations to be evenly distributed in the gas, then the tracer would only interact with $\mathcal{O}(1)$ many of the global $\mathcal{O}(\Lambda)$ excitations. This heuristic supports the second key point of the proof: the number of effective interactions.

To make the tracer localization rigorous, we aim to bound $\left\langle \psi_t^{\mathrm{BF}}, |x|^{2M} \psi_t^{\mathrm{BF}} \right\rangle$ by an $\mathcal{O}(1)$ bound. For a simpler understanding, consider $\left\langle \psi_t^{\mathrm{BF}}, x_i \psi_t^{\mathrm{BF}} \right\rangle$, which can be analyzed with a Grönwall argument.
Starting with a general operator A and the solution ψ_t of $\mathrm{i}\partial_t \psi_t = H\psi_t$, we have

$$\partial_t \langle \psi_t, A\psi_t \rangle = 2\mathrm{Im}\langle \psi_t, [A, H]\psi_t \rangle \,. \tag{5.1}$$

Estimating the right-hand side and applying Grönwall's Lemma leads to a bound for $\langle \psi_t, A\psi_t \rangle$. The Hamiltonians relevant to our analysis are the Bogoliubov-Fröhlich Hamiltonian H^{BF} (see Definition 2.3.2) and its transformed counterpart $\widetilde{H}^{\mathrm{BF}}$ (defined below in (5.4)), obtained via the Bogoliubov transformation U_t^{Bog}.
For both Hamiltonians we apply Grönwall's Lemma with the operator x_i. Using the commutator

identity $[\Delta_x, x_i] = (\Delta_x x_i) + 2(\nabla_x x_i)\nabla_x$, the operator x_i changes to ∂_{x_i}. Reapplying (5.1) to ∂_{x_i} we obtain $(\mathcal{N}+1)^{1/2}$ as the new operator to be estimated.[2] The procedure can be summarized as follows:

$$\widetilde{H}^{\rm BF} : x_i \overset{-\Delta_x}{\longrightarrow} \quad \partial_{x_i} \overset{a^{\#}((U_t - J^* V_t) Q_t W_x \varphi_t)}{\longrightarrow} \quad (\mathcal{N}+1)^{1/2} \underset{\text{Corollary 5.4.5}}{\overset{\text{Estimate}}{\longrightarrow}} \quad \mathcal{O}(1)\,, \tag{5.2}$$

$$H^{\rm BF} : x_i \overset{-\Delta_x}{\longrightarrow} \quad \partial_{x_i} \overset{a^{\#}(Q_t W_x \varphi_t)}{\longrightarrow} \quad (\mathcal{N}+1)^{1/2} \underset{\text{Lemma 4.2.6}}{\overset{\text{Estimate}}{\longrightarrow}} \quad \mathcal{O}(\Lambda^{1/2})\,. \tag{5.3}$$

The critical difference arises from the Bogoliubov transformation $U_t^{\rm Bog}$. In $\widetilde{H}^{\rm BF}$ we have extracted with the Bogoliubov transformation the excitations which are effectively non-interacting with the tracer. This allows us to bound $\left\langle \widetilde{\psi}_t^{\rm BF}, (\mathcal{N}+1)^{1/2} \widetilde{\psi}_t^{\rm BF} \right\rangle \leq C$ for $t \in [-T, T]$, which is sufficient for the purpose of tracer localization to an $\mathcal{O}(1)$ region. To see this, keep (5.2) in mind and note that $I \otimes U_t^{\rm Bog}$ commutes with $x \otimes I$.
The bound on the excitation number in the transformed dynamics indicates that indeed the excitations are evenly distributed in the gas as claimed in our heuristic argument (see below for a detailed discussion).

Bogoliubov-Transformation. Section D.2.1 provides an overview of Bogoliubov transformations $U_{\mathcal{V}}$ and the associated Bogoliubov maps $\mathcal{V}$. We use $U_t^{\rm Bog} = U_{\mathcal{V}_t}$ to extract the excitations which are effectively non-interacting with the tracer particle, resulting in a transformed Hamiltonian:

$$\begin{aligned} \widetilde{H}^{\rm BF}(t) &= I \otimes (U_t^{\rm Bog})^* H^{\rm BF}(t) I \otimes U_t^{\rm Bog} + \mathrm{i} \left(I \otimes \partial_t U_t^{\rm Bog} \right)^* I \otimes U_t^{\rm Bog} \\ &= -\frac{\Delta_x}{2m} + A(S\mathcal{V}_t^* S \cdot Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t)\,. \end{aligned} \tag{5.4}$$

In the Grönwall argument (5.1) for the excitation number operator we use that

$$A\left(S\mathcal{V}_t^* S \cdot Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t\right) \leq C \|\mathcal{V}_t\|_{\rm op} (\mathcal{N}+1)^{1/2}\,.$$

And hence due to $\|\mathcal{V}_t\|_{\rm op} \leq C$ (see Lemma D.2.10) and Grönwall:

$$\left\langle \widetilde{\psi}_t^{\rm BF}, (\mathcal{N}+1) \widetilde{\psi}_t^{\rm BF} \right\rangle \leq C \left\langle \widetilde{\psi}_0^{\rm BF}, (\mathcal{N}+1) \widetilde{\psi}_0^{\rm BF} \right\rangle, \quad t \in [-T, T]\,.$$

Although we have extracted a global number of excitations growing with volume $\mathcal{O}(\Lambda)$ with the Bogoliubov transformation $U_t^{\rm Bog}$, the interaction term with the tracer particle has changed only by $\mathcal{O}(1)$ due to $\|\mathcal{V}_t\|_{\rm op} \leq C$. This indicates that the tracer effectively interacts with only $\mathcal{O}(1)$ many excitations. This way $\|\mathcal{V}_t\|_{\rm op}$ controls the local quantity of effective interacting excitations with the tracer.

[2] Explicitly, $[\partial_{x_i}, a^{\#}(Q_t W_x \varphi_t)] = a^{\#}(Q_t \partial_{x_i} W_x \varphi_t)$ and $2\mathrm{Im}\langle \psi_t, a^{\#}(Q_t \partial_{x_i} W_x \varphi_t) \psi_t \rangle \leq \|\psi_t\| \sup_x \|\partial_{x_i} W_x\|_2 \|\varphi_t\|_\infty \|(\mathcal{N}+1)^{1/2} \psi_t\|$.

Energy Gain per Excitation. For the tracer localization, it is important that the energy gain of the tracer per excitation it interacts with is $\mathcal{O}(1)$ as discussed in Remark 2.0.2. Technically, this corresponds to the interaction term $A(Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t)$ in H^{BF} having a prefactor of $\mathcal{O}(1)$. Rescaling this term as $\rho^{1/2-r} A(S \mathcal{V}_t^* S \cdot Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t)$ with $r < 1/2$ results in an energy change of the tracer of $\mathcal{O}(\rho^{1/2-r})$. This scaling no longer ensures the tracer localization $\| |x|^{2n} \psi_t^{\mathrm{BF}} \| \leq C$ for $-T \leq t \leq T$. The proof of Corollary 5.4.5 provides further details on this argument.

It is worth noting that $\mathcal{O}(1) \cdot A(Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t)$ is not needed for the Bogoliubov approximation (see Section 4.2). In the Bogoliubov approximation we estimate also the terms coming from H^{Bog}, which create excitations on the whole volume Λ. By checking the proof of Theorem 4.2.1 step by step we see, that we can choose $\sqrt{\frac{\Lambda}{\rho}} \sum_{i=1}^{N} W(x - y_i)$, without changing the convergence rate in Theorem 4.2.1. This choice leads to the scaling $\rho^{-r} = \rho^{\alpha/2 - 1/2}$ for $\Lambda = \rho^\alpha$ in the original Hamiltonian.

5.4.2 Tracer Localization Theorem

In the following, a rigorous proof of the tracer localization estimate is given. This proof includes higher powers in the x_i operator and is based on a Grönwall argument. Due to (5.2) it is not sufficient to control the x_i operator on its own but rather $x_i, \partial_{x_i}, (\mathcal{N}+1)$ at the same time. This is done with the well studied harmonic oscillator in the x-direction: $h_{\mathrm{oc}} = -\Delta_x + x^2 + \mathcal{N} + 1$ (for details see Lemma D.3.1 and Corollary 5.4.5 below).

The Grönwall estimate is done for a class of quadratic forms, including the quadratic form generated by $\widetilde{H}^{\mathrm{BF}}(t)$ (for details see Definition D.3.2 and Theorem D.3.3). To achieve this, we apply Theorem D.1.1 ([LNS15, Theorem 8]) with comparison operator $A := h_{\mathrm{oc}}$. This theorem provides a rigorous Grönwall argument for dynamics generated by time-dependent quadratic forms.

In addition to the Bogoliubov transformation U_t^{Bog}, we apply another Bogoliubov transformation, $U_{\mathcal{Q}_0}$, with $\|\mathcal{Q}_0\|_{\mathrm{op}} \sim \mathcal{O}(1)$. This way we can still follow the same argument as above but generalize our initial state in Theorem 3.2.1 to $I \otimes U_{\mathcal{Q}_0} \psi_0^{\mathrm{BF}} = \widetilde{\psi}_0^{\mathrm{BF}}$ (see Lemma 5.4.3) (see Remark 5.4.6 for more details).

Definition 5.4.1 (Transformed Bogoliubov-Fröhlich Hamiltonian)**.** For all volumes $\Lambda \geq 1$ let φ_t be the condensate. For all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$ let $\mathcal{Q}_0 \in \mathcal{S}$ satisfy the symplectic conditions (see Definition D.2.1). We define the transformed Bogoliubov-Fröhlich

Hamiltonian as

$$\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t) = -\frac{\Delta_x}{2m} + A(\mathcal{Q}_0 S\mathcal{V}_t^* S \cdot Q_t W_x \varphi_t \oplus JQ_t W_x \varphi_t)\,, \tag{5.5}$$

where $\mathcal{V}_t := \mathcal{V}(t,0)$ is the Bogoliubov map corresponding to $U^{\mathrm{Bog}}(t,0) = U_{\mathcal{V}(t,0)}$ (see Corollary D.2.9) and $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Remark 5.4.2. Observe that $\mathcal{Q}_0 S\mathcal{V}_t^* S \cdot Q_t W_x\varphi_t \oplus JQ_t W_x \varphi_t = (c+J^*b)(U_t - V_t^* J^*)Q_t W_x \varphi_t \oplus J(c+J^*b)(U_t - V_t^* J)Q_t W_x \varphi_t =: g_x \oplus Jg_x$ such that the Hamiltonian in Definition 5.4.1 is symmetric. Note that

$$\mathcal{Q} =: \begin{pmatrix} c & J^*bJ^* \\ b & JcJ^* \end{pmatrix}\,, \quad \mathcal{V}_t =: \begin{pmatrix} U_t & J^*V_tJ^* \\ V_t & JU_tJ^* \end{pmatrix}\,.$$

Lemma 5.4.3 (Transformation Properties of the Bogoliubov-Fröhlich Hamiltonian)**.** *For all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$ let $\mathcal{Q}_0 \in \mathcal{S}$ satisfy the symplectic conditions and be unitarily implementable. Let $U_t^{Bog} := U^{Bog}(t,0)$ be the propagator of $H^{Bog}(t)$ (see Corollary D.2.9). Then*

$$I \otimes U_{\mathcal{Q}_0}\left(I \otimes (U_t^{Bog})^* H^{\mathrm{BF}}(t) I \otimes U_t^{Bog} + \mathrm{i} I \otimes \left(\partial_t U_t^{Bog}\right)^* U_t^{Bog}\right) I \otimes U_{\mathcal{Q}_0}^* = \widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)\,. \tag{5.6}$$

Therefore

$$\mathrm{i}\partial_t \psi_t = H^{\mathrm{BF}}(t)\psi_t \quad \Leftrightarrow \quad \mathrm{i}\partial_t I \otimes U_{\mathcal{Q}_0}(U_t^{Bog})^*\psi_t = \widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t) I \otimes U_{\mathcal{Q}_0}(U_t^{Bog})^*\psi_t\,. \tag{5.7}$$

Remark 5.4.4. The term $\mathrm{i}\left(I \otimes \partial_t U_t^{\mathrm{Bog}}\right)^* I \otimes U_t^{\mathrm{Bog}}$ is used to cancel the Bogoliubov Hamiltonian $H^{\mathrm{Bog}}(t)$ from the dynamics, "diagonalizing" the quadratic part of the Bogoliubov-Fröhlich Hamiltonian. In the case of time-independent quadratic Hamiltonians, where we aim to apply a time-independent diagonalization transformation, achieving diagonalization becomes significantly more challenging (see for example [NNS16; Nam20]).

Corollary 5.4.5 (Tracer Localization for the Transformed Dynamics)**.** *Let $M \in \mathbb{N}_+$ and $h_{\mathrm{oc}} = -\Delta_x + x^2 + \mathcal{N} + 1$ be the harmonic oscillator.*

i) (Interaction Potentials) Assume that the potentials V and W satisfy Assumption 2.0.3$_M$, which ensures the regularity of the boson-impurity interaction potential $W \in W^{M,\infty} \cap H^M$.

ii) (Condensate conditions) Assume that for all volumes $\Lambda \geq 1$ the condensate φ_0, which varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.7 and Condition 2.1.8$_{k=2+2}$.

ii) (Bogoliubov map) For all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$ let $\mathcal{Q}_0 \in \mathcal{S}$ satisfy the symplectic conditions. In addition assume that there exists a constant $C > 0$ such that for all $\rho, \Lambda \geq 1$: $\|\mathcal{Q}_0\|_{\mathcal{L}(L^2 \oplus JL^2)} \leq C$.

Then $\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)$ *is defined as a quadratic form* $(q_{\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)}, Q(h_{\mathrm{oc}}))$ *as in Definition D.3.2 and satisfies all conditions of Theorem D.3.3.*[3] *In particular, this implies, that for all times* $T \geq 0$ *there exists a constant* $C > 0$ *such that for all densities* $\rho \geq 1$*, volumes* $\Lambda \geq 1$*,* $-T \leq t \leq T$ *and* $\widetilde{\psi}^{\mathrm{BF}}_0 \in Q(h^M_{\mathrm{oc}}) \subset Q(q_{\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)})$ *we have*

$$q_{h^M_{\mathrm{oc}}}(\widetilde{\psi}^{\mathrm{BF}}_t) \leq C q_{h^M_{\mathrm{oc}}}(\widetilde{\psi}^{\mathrm{BF}}_0) \,.$$

Here, for a given initial state $\widetilde{\psi}^{\mathrm{BF}}_0 \in Q(h_{\mathrm{oc}})$*, the function* $\widetilde{\psi}^{\mathrm{BF}}_t$ *is the unique solution of the evolution equation*

$$\begin{aligned} \mathrm{i}\partial_t \widetilde{\psi}^{\mathrm{BF}}_t &= q_{\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)}(\widetilde{\psi}^{\mathrm{BF}}_t, \,.\,) \,, \\ \widetilde{\psi}^{\mathrm{BF}}_{t=0} &= \widetilde{\psi}^{\mathrm{BF}}_0 \,, \end{aligned}$$

as described in Theorem D.3.3.

Remark 5.4.6.

Position, Momentum, and Excitation Number Bound. By Lemma D.3.1, we have the bound

$$q_{(-\Delta_x)^M}(\widetilde{\psi}^{\mathrm{BF}}_t) + q_{(\mathcal{N}+1)^M}(\widetilde{\psi}^{\mathrm{BF}}_t) + q_{x^{2M}}(\widetilde{\psi}^{\mathrm{BF}}_t) \leq C_M q_{h^M_{\mathrm{oc}}}(\widetilde{\psi}^{\mathrm{BF}}_t) \,.$$

Extracted Mean-field Contribution. Extracting the term $\rho^{1/2} \int W$ from the dynamics via the unitary transformation $\mathrm{e}^{-it\rho^{1/2}\int W}$ is crucial. Otherwise, one would need to estimate $|\rho^{1/2} \int W| \leq \rho^{1/2} C$, leading to bounds in Theorem D.3.3 that grow with the density ρ.

Simplified Notation. For simplicity, we use the shorthand notation $q_{x^{2M}} := q_{M_{x^{2M}} \otimes I_{\mathcal{F}(L^2)}}$ and $q_{(-\Delta_x)^M} := q_{(-\Delta_x)^M \otimes I_{\mathcal{F}(L^2)}}$.

Distinguishing Global and Local Excitations: The Role of $U_{\mathcal{Q}_0}$. We consider the setting of our main Theorem, where $\mathcal{Q}_0$ is unitarily implementable and the initial state is given by

$$\psi^{\mathrm{BF}}_0 = I \otimes U^*_{\mathcal{Q}_0} \widetilde{\psi}^{\mathrm{BF}}_0 \,.$$

The idea of Corollary 5.4.5 is to first extract the Bogoliubov Hamiltonian from the dynamics using $U^{\mathrm{Bog}}_t = U_{\mathcal{V}_t}$, which allows us to control the tracer localization. In a second step, we apply the time-independent Bogoliubov transformation $U_{\mathcal{Q}_0}$ to generalize the initial conditions, allowing for $\mathcal{O}(\Lambda)$ excitations in the gas rather than restricting to $\mathcal{O}(1)$ global excitations.

To clarify the role of $U_{\mathcal{Q}_0}$, consider the case where it is not applied, i.e., setting $\mathcal{Q}_0 = I$.

[3] In fact, the rescaled $2m \cdot \widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)$ does, as the Laplacian $-\Delta_x$ appears here without prefactor as in Theorem D.3.3.

Since $I = U_0^{\mathrm{Bog}}$, the strong initial conditions of Corollary 5.4.5 would then require that

$$C \geq q_{(\mathcal{N}+1)^M}(\widetilde{\psi}_0^{\mathrm{BF}}) = q_{(\mathcal{N}+1)^M}(\psi_0^{\mathrm{BF}}),$$

implying that the global number of excitations must be $\mathcal{O}(1)$. However, heuristically, only control of the local excitations surrounding the tracer is necessary for its localization. In addition, the $\mathcal{O}(1)$-control of the global number of excitations is lost once the time evolution begins (see Lemma 4.2.6).

To overcome this limitation and allow initial states with a global excitation number scaling with the volume, we introduce the second Bogoliubov transformation $U_{\mathcal{Q}_0}$. This transformation is time-independent and can be applied to the Hamiltonian straightforwardly since the first Bogoliubov transformation U_t^{Bog} has already extracted the contribution of H^{Bog}. The initial state then takes the form $\psi_0^{\mathrm{BF}} = I \otimes U_{\mathcal{Q}_0}^* \widetilde{\psi_0^{\mathrm{BF}}}$ (see Lemma 5.4.3), where only $\widetilde{\psi_0^{\mathrm{BF}}}$ needs to satisfy the condition $C \geq q_{(\mathcal{N}+1)^M}(\widetilde{\psi}_0^{\mathrm{BF}})$, as required by Corollary 5.4.5.

In this case, using Lemma D.2.4, we obtain the estimate

$$\begin{aligned} q_{(\mathcal{N}+1)^M}(\psi_0^{\mathrm{BF}}) &= q_{(\mathcal{N}+1)^M}(I \otimes U_{\mathcal{Q}_0} \widetilde{\psi_0^{\mathrm{BF}}}) \\ &\leq C_M (1 + \|\mathcal{Q}_0 \mathcal{Q}_0^* - 1\|_{\mathrm{HS}}^2 + \|\mathcal{Q}_0\|_{\mathrm{op}}^2)^M q_{(\mathcal{N}+1)^M}(\widetilde{\psi_0^{\mathrm{BF}}}). \end{aligned}$$

This estimate indicates that $U_{\mathcal{Q}_0}$ can increase the global excitation number by a factor scaling with

$$\|\mathcal{Q}_0 \mathcal{Q}_0^* - 1\|_{\mathrm{HS}}^2 + \|\mathcal{Q}_0\|_{\mathrm{op}}^2,$$

which grows with $\|\mathcal{Q}_0 \mathcal{Q}_0^* - 1\|_{\mathrm{HS}}^2$. In contrast, the estimate for the tracer localization, a local quantity, in Corollary 5.4.5 only depends on bounds for $\|\mathcal{Q}_0\|_{\mathrm{op}}$. Understood in this way, $\|\mathcal{Q}_0 \mathcal{Q}_0^* - 1\|_{\mathrm{HS}}^2$ represent the global number of excitations and $\|\mathcal{Q}_0\|_{\mathrm{op}}$ the local number of excitations in the initial conditions.

Then by choosing

- The global number of excitation: $\|\mathcal{Q}_0 \mathcal{Q}_0^* - 1\|_{\mathrm{HS}}^2 \leq C\Lambda$.
- The local number of excitation: $\|\mathcal{Q}_0\|_{\mathrm{op}} \leq C$.

we achieve a setting with $\mathcal{O}(\Lambda)$ global excitations and $\mathcal{O}(1)$ local excitations.

Proof of Corollary 5.4.5. We have to verify the conditions of Theorem D.3.3 for $f := \big(t \mapsto (x \mapsto 2m(U + J^*V)(U_t^* - V_t^* J)Q_t W_x \varphi_t)\big)$, $g = 0$ and $\lambda = 0$. This is done in Lemma 5.4.7 and Lemma 5.4.8. Note that we rescaled the Hamiltonian with twice the relative impurity mass $2m$, in order to to ensure the impurity Laplacian has the correct scaling. After applying Theorem D.3.3 we can remove this additional factor by rescaling the time variable.

Hence Theorem D.3.3 is applicable. From the same Lemmas as above, we further know $\forall T \geq 0$

$\exists C > 0$ such that $\forall \rho, \Lambda \geq 1$ and $-T \leq t \leq T$

$$\|f_t\|_{W^{M,\infty}(\mathbb{R}^3, L^2(\mathbb{R}^3,\mathbb{C}))} + \|\dot{f}_t\|_{L^\infty(\mathbb{R}^3, L^2(\mathbb{R}^3,\mathbb{C}))} \leq C\,. \tag{5.8}$$

Now, let $M \in \mathbb{N}_+, T \in \mathbb{R}_{\geq 0}$, $I_b := [-T, T] \subset \mathbb{R}$ bounded. By applying Theorem D.3.3 and (5.8) we conclude $\exists C > 0$ such that $\forall \rho, \Lambda \geq 1$, $\widetilde{\psi}_0^{\mathrm{BF}} \in Q(A^M)$ and $t \in I_b$

$$q_{A^M}(\widetilde{\psi}_t^{\mathrm{BF}}) \leq C_{I_b}(t) q_{A^M}(\widetilde{\psi}_0^{\mathrm{BF}}) \leq \mathrm{e}^{|t-t_0|C} C q_{A^M}(\widetilde{\psi}_0^{\mathrm{BF}})\,.$$

Here $\widetilde{\psi}_t^{\mathrm{BF}}$ is the unique solution of

$$\begin{aligned} \mathrm{i}\partial_t \widetilde{\psi}_t^{\mathrm{BF}} &= q_{\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)}(\widetilde{\psi}_t^{\mathrm{BF}})\,, \\ \widetilde{\psi}_t^{\mathrm{BF}}|_{t=0} &= \widetilde{\psi}_0^{\mathrm{BF}}\,, \end{aligned}$$

as described in Theorem D.3.3. ■

Corollary 5.4.5 follows directly form the general statement Theorem D.3.3 also including the Hamiltonian $\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)$. The conditions of Theorem D.3.3 for the transformed dynamics generated by $\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)$ are verified in the following Lemma.

Lemma 5.4.7. *Let $M \in \mathbb{N}_+$. We make the following assumptions:*

i) (Interaction Potentials) Assume that the potentials V and W satisfy Assumption 2.0.3$_M$, which ensures the regularity of the boson-impurity interaction potential $W \in W^{M,\infty} \cap H^M$.

i) (Condensate condition) For all volumes $\Lambda \geq 1$ let φ_t be the condensate with initial data φ_0, which varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.7.

iii) (Bogoliubov map) For all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$ let $U_t^{\mathrm{Bog}} = U_{\mathcal{V}_t}$ be the propagator of the Bogoliubov dynamics and $\mathcal{Q}_0 \in \mathcal{S}$ satisfy the symplectic conditions and assume that there exists a constant $C > 0$ such that $\|\mathcal{Q}_0\|_{\mathcal{L}(L^2 \oplus JL^2)} \leq C$. We can write

$$\mathcal{Q}_0 =: \begin{pmatrix} c & J^* b J^* \\ b & J c J^* \end{pmatrix}\,, \quad \mathcal{V}_t =: \begin{pmatrix} U_t & J^* V_t J^* \\ V_t & J U_t J^* \end{pmatrix}\,.$$

Set $f := \big(t \mapsto (x \mapsto (c + J^* b)(U_t^* - V_t^* J) Q_t W_x \varphi_t)\big)$ then

a) For almost all $x \in \mathbb{R}^d$ we have $(t \mapsto f_t(x, .)) \in C^1(\mathbb{R}_t, L^2(\mathbb{R}^3_y, \mathbb{C}))$.

b) For all $t \in \mathbb{R}$ we have that $\partial_t f_t := (x \mapsto \partial_t f_t(x, .)) \in L^\infty(\mathbb{R}^3_x, L^2(\mathbb{R}^d_y, \mathbb{C}))$ and $f_t \in W^{M,\infty}(\mathbb{R}^d_x, L^2(\mathbb{R}^3_y, \mathbb{C}))$.

c) If in addition we assume Condition 2.1.8$_{k=2+2}$ for the condensate then for all times $T \geq 0$ there exists a constant $C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$

$$\sup_{t\in[-T,T]} \Big\{ \|f_t\|_{W^{M,\infty}(\mathbb{R}^3, L^2(\mathbb{R}^3,\mathbb{C}))} + \|\partial_t f_t\|_{L^\infty(\mathbb{R}^3, L^2(\mathbb{R}^3,\mathbb{C}))} \Big\} \leq C\,. \tag{5.9}$$

The proof of Lemma 5.4.7, using Lemma 5.4.8, can be found in the Appendix E.2.1.
We divide the proof of Lemma 5.4.7 into two parts. First we prove that $F := (t \mapsto Q_t W_x \varphi_t)$ is smooth in a suitable sense (see Lemma 5.4.8 below). Then, we verify that the function $t \mapsto (c + J^* b)(U_t^* - V_t^* J) F_t$ satisfies the assumptions of Lemma 5.4.7 in the Appendix E.2.1.

Lemma 5.4.8. *Let $M \in \mathbb{N}_+$. We make the following assumptions:*

i) (Interaction Potentials) Assume that the potentials V and W satisfy Assumption 2.0.3$_M$, which ensures the regularity of the boson-impurity interaction potential $W \in W^{M,\infty} \cap H^M$.

ii) (Condensate condition) For all volumes $\Lambda \geq 1$ let φ_t be the condensate with initial data $\varphi_0 \in H^\infty$, which varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.7.

Set $F := \big(t \mapsto (x \mapsto Q_t W_x \varphi_t)\big)$ then

a) $F \in C^1\big(\mathbb{R}_t, W^{M,\infty}(\mathbb{R}^3_x, L^2(\mathbb{R}^3_y, \mathbb{C}))\big)$ and for $|\beta| \leq M$

$$\begin{aligned} D_x^\beta Q_t W_x \varphi_t &= Q_t (D^\beta W_x) \varphi_t\,, \\ \partial_t \big(x \mapsto Q_t (D^\beta W_x)\varphi_t\big) &= \big(x \mapsto \dot{Q}_t (D^\beta W_x)\varphi_t + Q_t (D^\beta W_x)\dot{\varphi}_t\big)\,. \end{aligned}$$

b) For all $x \in \mathbb{R}^3$, $\beta \in \mathbb{N}_0^3$, $k \in \mathbb{N}_0$ with $k + |\beta| \leq M$ we have $\big(t \mapsto D_x^\beta F_t(x,\,.\,) = Q_t (D^\beta W_x)\varphi_t\big) \in C^1(\mathbb{R}_t, H^k(\mathbb{R}^3, \mathbb{C}))$ and

$$\partial_t Q_t (D^\beta W_x)\varphi_t = \dot{Q}_t (D^\beta W_x)\varphi_t + Q_t (D^\beta W_x)\dot{\varphi}_t\,.$$

c) If in addition we assume Condition 2.1.8$_{k=2+2}$ for the condensate then for all times $T \geq 0$ and $\beta \in \mathbb{N}_0^3$, $k \in \mathbb{N}_0$ with $k + |\beta| \leq M$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and $-T \leq t \leq T$

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^3} \big(\|D_x^\beta F_t(x,\,.\,)\|_{H^k(\mathbb{R}^3)} + \|\partial_t D_x^\beta F_t(x,\,.\,)\|_{H^k(\mathbb{R}^3)} \big) \leq C\,. \tag{5.10}$$

Especially we are able to conclude that $f := F$ satisfies all the conditions of Theorem D.3.3 on f.

Remark 5.4.9.

- Note that for $-T \leq t \leq T$, $|\beta| \leq M$ and $k = 0$ the estimate (5.10) implies:

$$\|F_t\|_{W^{M,\infty}(\mathbb{R}^3, L^2(\mathbb{R}^d, \mathbb{C}))} + \|\partial_t F_t\|_{W^{M,\infty}(\mathbb{R}^3, L^2(\mathbb{R}^d, \mathbb{C}))} \leq C\,.$$

- Lemma 5.4.7 and Lemma 5.4.8 giving the conditions for the tracer localization in Corollary 5.4.5 are the reason why we need control over the derivatives of the boson-tracer interaction potential W in our main Theorem 3.1.2.

The proof of Lemma 5.4.8 can be found in the Appendix E.2.2.

5.4.3 Necessity of a Flat Condensate for the Tracer Confinement

In this section, we provide a heuristic explanation of why the flatness condition on the condensate is essential to ensure that the tracer remains confined within the Bose gas. Assume that the condensate is given by $\varphi_0(y) = \eta(\Lambda^{-1/3}y)$, where η is independent of Λ and ρ, without additional constraints like flatness around the origin. We can then ask whether the tracer remains within the Bose gas cloud over timescales of $\mathcal{O}(1)$.
Without flatness, the mean-field interaction between the tracer and the condensate $\sqrt{\rho}W * |\varphi_t|^2$ imposes the condition

$$\rho \ll \Lambda^{4/3}$$

to ensure that the tracer remains inside the gas. However, this is incompatible with the requirement

$$\Lambda^3 \ll \rho\,,$$

which arises from the Bogoliubov approximation in Theorem 4.2.1.
When φ_0 is not flat, we have to consider the term $\sqrt{\rho}W * |\varphi_t|^2$ in the Hamiltonian: $H^{\mathrm{BF}} + \sqrt{\rho}W * |\varphi_t|^2$. When we estimate the tracer position we encounter as in (5.2) the commutator of ∂_{x_i} and the Hamiltonian, tracking the kinetic energy. Due to $\sqrt{\rho}W * |\varphi_t|^2$ a new term appears in the estimate of the commutator:

$$\|\partial_{x_i}\sqrt{\rho}W * |\varphi_t|^2\|_\infty \|\psi_t\|^2 \leq C\sqrt{\rho}\Lambda^{-1/3}\,. \tag{5.11}$$

Combining this with the estimate of the tracer position without $\sqrt{\rho}W * |\varphi_t|^2$ sketched in (5.10), we see that

$$\langle \psi_t, x_i \psi_t \rangle \leq C(1 + \sqrt{\rho}\Lambda^{-1/3})\,.$$

To ensure that the tracer remains within the gas of volume Λ, we require that $\sqrt{\rho}\Lambda^{-1/3} \ll \Lambda^{1/3}$. Hence $\rho \ll \Lambda^{4/3}$.
However, the Bogoliubov approximation demands $\Lambda^3 \ll \rho$, which is incompatible with $\rho \ll \Lambda^{4/3}$. Therefore, by assuming $\varphi_0(y) = \eta(\Lambda^{-1/3}y)$ without any additional constraints to limit

the kinetic energy gain of the tracer particle from the condensate, the tracer may leave the condensate within $\mathcal{O}(1)$ timescales.

Remark 5.4.10 (Intuitive Perspective)**.** For a less technical explanation, consider the classical force on the tracer due to the potential $\sqrt{\rho}W * |\varphi_t|^2$. Since the force is given by $F = \nabla V$, and considering the characteristic length scale of the condensate, we estimate it to be of order $\sqrt{\rho}\Lambda^{-1/3}$. Over a timescale of $\mathcal{O}(1)$, the resulting position change is also of order $\sqrt{\rho}\Lambda^{-1/3}$. To ensure the tracer remains inside the condensate, we obtain the constraint $\rho \ll \Lambda^{4/3}$.

Example 5.4.11 (The Effect of a Flat Condensate)**.** If we can replace $\sqrt{\rho}W * |\varphi_t|^2$ with the approximation for a flat condensate, $\sqrt{\rho}W * 1$, the tracer remains inside the condensate. In this case, the kinetic energy term simplifies:

$$\|\partial_{x_i}\sqrt{\rho}W * 1\|_\infty \|\psi_t\|^2 \leq 0\,,$$

indicating that there is no contribution to the kinetic energy of the tracer particle from this term.

Appendix A

The Localization Function

We define the localization function, which is used to localize the condensate around the origin.

Definition A.0.1 (The Localization Function)**.** Let $\Theta_1 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ with

a)

$$|\Theta_1| \leq C\,, \tag{A.1}$$

$$|\mathrm{D}^\beta \Theta_1| \leq C_\beta |\Theta_1|\,, \quad \forall \beta \in \mathbb{N}_0^3\,. \tag{A.2}$$

b) There exists an $n \in \mathbb{N}$ with

$$|x| \geq a > 0 \quad \Rightarrow \quad |\Theta_1(x)| \leq C a^{-2n}\,. \tag{A.3}$$

We define a localization function Θ_Λ as

$$\Theta_\Lambda(x) := \Theta_1(\Lambda^{-s} x)\,, \tag{A.4}$$

where $s \in \mathbb{R}$ is a parameter.

Remark A.0.2. In order for Θ_Λ to localize φ_0 we have to choose $s < 1/3$. To understand this, one can imagine $\varphi_0 = \eta(\Lambda^{-1/3} x)$ to be a rescaled function varying on the scale of $\Lambda^{1/3}$. Localizing φ_0 with $\Theta_\Lambda \varphi_0$ requires the localization scale of $\Theta_\Lambda(x) = \Theta_1(\Lambda^{-s} x)$ to be smaller than that of φ_0. This heuristic argument will be made rigorous in our propositions (see for example Remark B.3.6).

Example A.0.3. The localization function $\Theta_1(x) = \frac{1}{1+|x|^{2n}}$ satisfies all the properties of Definition A.0.1.

Remark A.0.4.

- One advantage of choosing $\Theta_1(x) = \frac{1}{1+|x|^{2n}}$ as the localization function is that, for large n it is nearly 1 within $B_0(1)$, forming a plateau. Additionally, it rapidly decreases to 0 outside $B_0(1)$, making it a smooth approximation of the characteristic function of $B_0(1)$. Note that we consider φ_0 to be a smooth function that is flat around the origin and $\varphi_0(0) = 1$ (see Condition 2.1.11). Therefore $\Theta_\Lambda(x) = \Theta_1(\Lambda^{-s}x)$ modulates φ_0 within Λ^{3s} and smoothly cuts it off outside this region. This interpretation becomes even clearer when we view φ_0 as having a plateau inside $B_0(1/2\Lambda^{1/3})$.

- Since $1 + |x|^{2n}$ has a commutator with Δ that is easy to handle, the $\|1/\Theta_\Lambda\psi_t\|_2$ estimate in the Grönwall argument within the proof of Theorem 5.2.1 is simplified by choosing Θ_1 as in Example A.0.3.

To facilitate our estimates, we introduce some norms from [Dec+16], which are particularly useful when applying Young's inequality.

Definition A.0.5. For $1 \leq p_1, \ldots, p_M \leq \infty$, $M \in \mathbb{N}_+$, we define the norm

$$\|f\|_{p_1\wedge\ldots\wedge p_M} := \inf_{f=f_{p_1}+\ldots+f_{p_M}} \Big(\|f_{p_1}\|_{p_1} + \ldots + \|f_{p_M}\|_{p_M}\Big).$$

In order to compress the notation we also use

$$\|f\|_{p_1,\ldots,p_M} := \|f\|_{p_1} + \ldots + \|f\|_{p_M} .$$

All important estimates concerning the localization function can be found in the following Lemma.

Lemma A.0.6 (Localized Estimates)**.** *Let $n \in \mathbb{N}_0$ and $s > 0$. Let $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}x)^{2n}}$ be the localization function.*

a) (Functions flat around the origin) Let $k \in \mathbb{N}_0$ with $k \leq 2n$. For all volumes $\Lambda \geq 1$ let $f := f_\Lambda \in C^k(\mathbb{R}^3, \mathbb{C})$. If we have flatness around the origin of f, namely there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and $0 \leq |\beta| \leq k-1$ we have $|D^\beta f(0)| \leq \|D^\beta f\|_\infty C\Lambda^{-(k-|\beta|)(1/3-s)}$ then

i) There exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$

$$\|\Theta_\Lambda f\|_\infty \leq C\Lambda^{-k(1/3-s)} \left(\sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{|\beta|/3}\right) . \tag{A.5}$$

ii) If in addition $n \geq 1$ and $k \leq 2(n-1)$ then there exists a constant $C > 0$ such that *for all volumes* $\Lambda \geq 1$

$$\|\Theta_\Lambda f\|_2 \leq C\Lambda^{-k(1/3-s)+3/2s} \left(\sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{|\beta|/3} \right) . \tag{A.6}$$

b) *(Convolution) Then for all orders* $m \in \mathbb{N}_+$ *there exists a constant* $C > 0$ *such that for all volumes* $\Lambda \geq 1$*, potentials* $W \in L^0(\mathbb{R}^3, \mathbb{C})$ *and functions* $f \in L^0(\mathbb{R}^3, \mathbb{C})$ *we have*

$$\|\Theta_\Lambda W * f\|_2 \leq C \sum_{0\leq|\beta|\leq m-1} \| \, |y|^{|\beta|} W\|_{1,2} \|\Theta_\Lambda f\|_{1\wedge 2} + C\Lambda^{-sm} \| \, |y|^m W\|_{1,2} \|f\|_{1\wedge 2} \, , \tag{A.7}$$

$$\|\Theta_\Lambda W * f\|_\infty \leq C \sum_{0\leq|\beta|\leq m-1} \| \, |y|^{|\beta|} W\|_{1,2,\infty} \|\Theta_\Lambda f\|_{1\wedge 2\wedge\infty} + C\Lambda^{-sm} \| \, |y|^m W\|_{1,2,\infty} \|f\|_{1\wedge 2\wedge\infty} \, , \tag{A.8}$$

Remark A.0.7.

Grönwall Estimates. Part b) of the Lemma is required to close Grönwall-type estimates of the form

$$\partial_t \|\Theta_\Lambda f_t\| \leq \alpha(t) + \beta(t) \|\Theta_\Lambda f_t\| \, ,$$

where we need Θ_Λ inside the convolution $W * f$. In this context, the terms $C\Lambda^{-sm} \cdot \| \, |y|^m W\|_{1,2} \|f\|_{1\wedge 2}$ and $C\Lambda^{-sm} \| \, |y|^m W\|_{1,2,\infty} \|f\|_{1\wedge 2\wedge\infty}$ act as error terms.

Rescaling of Θ_1 to Θ_Λ. The rescaling of Θ_1 to Θ_Λ is introduced to ensure that the error terms in (A.7) and (A.8) decrease as Λ grows. However, this rescaling comes with the cost of additional growth terms: Λ^{ks} in (A.5) and $\Lambda^{ks+3/2s}$ in (A.6).

Flatness Condition. The condition

$$|D^\beta f(0)| \leq \|D^\beta f\|_\infty C\Lambda^{-(k-|\beta|)(1/3-s)} \, , \quad \forall 0 \leq |\beta| \leq k-1$$

in Lemma A.0.6a) implies that, for $0 < s < 1/3$, the derivatives $D^\beta f$ at the origin are significantly smaller than their supremum norm when Λ is large, meaning that f exhibits a higher degree of flatness near the origin than a bound on its supremum alone would suggest.

Arbitrarily Small Errors. For fixed $s > 0$, the error terms in (A.7) and (A.8) can be made arbitrarily small by increasing n. If we assume $\|D^\beta f\|_\infty \leq C\Lambda^{-\frac{|\beta|}{3}}$ for $0 \leq |\beta| \leq k-1$ (e.g., $f = \varphi_0$ in Condition 2.1.8), then (A.5) and (A.6) can be made arbitrarily small by increasing m for fixed $0 < s < 1/3$. The term $\Lambda^{3/2s}$ in (A.6) comes from the square root of the volume occupied by Θ_Λ.

The proof of Lemma A.0.6 can be found in the Appendix A.1.1.

A.1 Proofs of Appendix A

A.1.1 Proof of Lemma A.0.6

Proof of Lemma A.0.6. Part a):
The case $k = 0$ is trivial. Now let $k \geq 1$. By the Taylor expansion formula of f up to order $k-1$ around 0 we have $\forall x \in \mathbb{R}^3\ \exists \xi := \xi_x \in [0,1]$ such that

$$\begin{aligned} |(\Theta_\Lambda f)(x)| &= \Theta_\Lambda(x) \left| \sum_{0\leq|\beta|\leq k-1} \frac{D^\beta f(0)}{\beta!} x^\beta + \sum_{|\beta|=k} \frac{D^\beta f(\xi x)}{\beta!} x^\beta \right| \\ &\leq \Theta_\Lambda(x) \left(\sum_{|\beta|\leq k-1} \frac{\|D^\beta f\|_\infty C \Lambda^{-(k-|\beta|)(1/3-s)}}{\beta!} \left|x^\beta\right| + \sum_{|\beta|=k} \frac{\|D^\beta f\|_\infty}{\beta!} \left|x^\beta\right| \right) \\ &\leq C \sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{-(k-|\beta|)(1/3-s)} \frac{|x|^{|\beta|}}{1+(\Lambda^{-s}|x|)^{2n}}\,, \end{aligned} \tag{A.1}$$

where in step 2 we used $|D^\beta f(0)| \leq \|D^\beta f\|_\infty \Lambda^{-(k-|\beta|)(1/3-s)}$, $\forall 0 \leq |\beta| \leq k-1$, and in step 3 the definition $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}|x|)^{2n}} \geq 0$.
Now we use that $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}|x|)^{2n}}$ is a smoothed version of the characteristic function of $B_0(\Lambda^s)$. To make this property clear we distinct the cases $x \in B_0(\Lambda^s)$ and $x \notin B_0(\Lambda^s)$.
Case $|x| \geq \Lambda^s$*:* Then $\Lambda^{-s}|x| \geq 1$ and since $|\beta| \leq k \leq 2n$ we have $(\Lambda^{-s}|x|)^{2n-|\beta|} \geq 1$ and thus $1 \geq (\Lambda^{-s}|x|)^{|\beta|-2n}$. We conclude since $|x| \geq \Lambda^s > 0$

$$\frac{|x|^{|\beta|}}{1+(\Lambda^{-s}|x|)^{2n}} \overset{x\neq 0}{\leq} \frac{|x|^{|\beta|}}{0+(\Lambda^{-s}|x|)^{2n}} = \Lambda^{s|\beta|} \cdot (\Lambda^{-s}|x|)^{|\beta|-2n} \leq \Lambda^{s|\beta|}\,.$$

Case $|x| < \Lambda^s$*:* In this case we get the same estimate

$$\frac{|x|^{|\beta|}}{1+(\Lambda^{-s}|x|)^{2n}} \leq \frac{|x|^{|\beta|}}{1+0} < \Lambda^{s|\beta|}\,.$$

So we have $\forall x \in \mathbb{R}^3$

$$\frac{|x|^{|\beta|}}{1+(\Lambda^{-s}|x|)^{2n}} \leq \Lambda^{s|\beta|}\,. \tag{A.2}$$

We conclude from (A.1) and (A.2)

$$|(\Theta_\Lambda f)(x)| \leq C \sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{-(k-|\beta|)(1/3-s)} \Lambda^{s|\beta|}\,. \tag{A.3}$$

From here we conclude (A.5).
Now we want to prove the $\|\Theta_\Lambda f\|_2$ estimate. Let us assume $k \leq 2(n-1)$ and thus $n \geq 1$. We use (A.1) to conclude

$$
\begin{aligned}
\|\Theta_\Lambda f\|_2 &= \left(\int_{\mathbb{R}^3} |\Theta_\Lambda f(x)|^2 dx \right)^{1/2} \\
&\overset{(A.1)}{\leq} C \left(\int_{\mathbb{R}^3} |\sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{-(k-|\beta|)(1/3-s)}|^2 \left(\frac{\Lambda^{s|\beta|}(\Lambda^{-s}|x|)^{|\beta|}}{1+(\Lambda^{-s}|x|)^{2n}} \right)^2 dx \right)^{1/2} \\
&\overset{y=\Lambda^{-s}x}{\leq} C\Lambda^{-k(1/3-s)} \sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{|\beta|/3} \left(\int_{\mathbb{R}^3} \left(\frac{|y|^{|\beta|}}{1+|y|^{2n}} \right)^2 \Lambda^{3s} dy \right)^{1/2} \\
&\leq C\Lambda^{-k(1/3-s)+3/2s} \sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{|\beta|/3} \\
&\quad \cdot \left(\int_{\mathbb{R}^3 \setminus B(0,1)} \left(\frac{|y|^{|\beta|}}{0+|y|^{2n}} \right)^2 dy - \int_{B(0,1)} \left(\frac{|y|^{|\beta|}}{1+1} \right)^2 dy \right)^{1/2} \\
&\leq C\Lambda^{-k(1/3-s)+3/2s} \sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{|\beta|/3} \left(\int_1^\infty 4\pi r^2 r^{2(|\beta|-2n)} dr - \int_0^1 4\pi r^2 r^{2|\beta|} dy \right)^{1/2} \\
&\leq C\Lambda^{-k(1/3-s)+3/2s} \sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{|\beta|/3} 4\pi \left(\left[\frac{r^{3+2(|\beta|-2n)}}{3+2(|\beta|-2n)} \right]_1^\infty - \left[\frac{r^{3+2|\beta|}}{3+2|\beta|} \right]_0^1 \right)^{1/2} \\
&\leq C\Lambda^{-k(1/3-s)+3/2s} \sum_{|\beta|\leq k} \|D^\beta f\|_\infty \Lambda^{|\beta|/3},
\end{aligned}
$$

where the last step we used $3 + 2(|\beta| - 2n) \leq -1$, $\forall 0 \leq |\beta| \leq k$ which is equivalent to $|\beta| \leq k \leq 2(n-1)$.
Part b):
We want to change the argument of Θ_Λ from x to $x - y$ such that we can move it inside the convolution. Note that for $x, y \in \mathbb{R}^3$ it follows by expanding Θ_Λ around $x - y$ that $\exists \theta \in [0,1]$ such that

$$
\begin{aligned}
|\Theta_\Lambda(x)| \leq &\sum_{0\leq|\beta|\leq m-1} |\mathrm{D}^\beta \Theta_1 \left(\Lambda^{-s}(x-y)\right)| \frac{|(\Lambda^{-s}y)^\beta|}{\beta!} \\
&+ \sum_{|\beta|=m} |\mathrm{D}^\beta \Theta_1 \left(\Lambda^{-s}(x-y+\theta y)\right)| \frac{|(\Lambda^{-s}y)^\beta|}{\beta!} \\
\leq &\sum_{0\leq|\beta|\leq m-1} C_m |\Theta_1 \left(\Lambda^{-s}(x-y)\right)| \cdot \Lambda^{-s|\beta|} \frac{|y|^{|\beta|}}{\beta!} \\
&+ \sum_{|\beta|=m} C_m \cdot \Lambda^{-s|\beta|} \frac{|y|^{|\beta|}}{\beta!}, \qquad \text{(A.4)}
\end{aligned}
$$

where in the last inequality we have used Definition A.0.1. Now with (A.4)

$$
\begin{aligned}
|\Theta_\Lambda(x) W * f(x)| &\leq \Big| \int dy W(y) f(x-y) \Theta_\Lambda(x) \Big| \\
&\leq C_m \sum_{0 \leq |\beta| \leq m-1} \int dy |y|^{|\beta|} \cdot |W(y) \Theta_\Lambda \left(x-y\right) f(x-y)| \\
&\quad + C_m \Lambda^{-sm} \int dy |y|^m \cdot |W(y) f(x-y)|
\end{aligned}
$$

with Young's inequality for $p \geq 1$

$$
\begin{aligned}
\|\Theta_\Lambda W * f\|_p &\leq C_m \sum_{0 \leq |\beta| \leq m-1} \| \Big| |y|^{|\beta|} W \Big| * |\Theta_\Lambda f| \, \|_p \\
&\quad + C_m \Lambda^{-sm} \| \big| |y|^m W \big| * |f| \|_p \\
&\leq C_m \sum_{0 \leq |\beta| \leq m-1} \| \, |y|^{|\beta|} W \|_{1,p} \|\Theta_\Lambda f\|_{1 \wedge p} + C_m \Lambda^{-sm} \| \, |y|^m W \|_{1,p} \|f\|_{1 \wedge p} \, .
\end{aligned}
$$

This proves (A.7). The proof of and (A.8) is analogous. ■

Appendix B

Control of the Condensate

B.1 Preliminary Results

The following preliminaries Proposition B.1.1 and Corollary B.1.2 form the foundation of our control of the condensate.

The estimates provided in Proposition B.1.1 and Corollary B.1.2 are already known results from [Dec+16]. For the reader's convenience, we also include their proofs in Appendix B.4.1 and B.4.2. Notably, their proofs rely only on Condition 2.1.7, and unlike in [Dec+16], a plateau condition is not required.

Proposition B.1.1. *For all volumes $\Lambda \geq 1$ let φ_t be the condensate with initial data φ_0, which varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.7. Then for all times $T \geq 0$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and $-T \leq t \leq T$*

$$\|\varphi_t - \tilde{\varphi}_t\|_2 \leq C\Lambda^{-\frac{1}{6}}. \tag{B.1}$$

Corollary B.1.2. *For all volumes $\Lambda \geq 1$ let φ_t be the condensate with initial data φ_0, which varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.7. Then for all times $T \geq 0$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and $-T \leq t \leq T$*

$$\|\varphi_t\|_\infty \leq C. \tag{B.2}$$

Remark B.1.3. In the proof of Proposition B.1.1 it is necessary to fix a finite time interval $t \in [-T, T]$ as described in the Proposition. Since Proposition B.1.1 and Corollary B.1.2 are our fundamental control properties for the condensate, this finite time interval appears in most of our Lemmas and Theorems.

B.2 Condensate Control

In Definition 2.1.5, we defined the auxiliary function $\tilde{\varphi}_t$. In this section, we use $\tilde{\varphi}_t$ to approximate the condensate φ_t. The heuristic motivation for $\tilde{\varphi}_t$ is discussed in Remark 2.1.6, and it proves useful as $|\tilde{\varphi}_t| = |\varphi_0|$ allows us to transfer properties of the initial condensate φ_0 to $\tilde{\varphi}_t$.

Lemma B.2.1 (Estimates of $\tilde{\varphi}_t$)**.** *Let $\beta \in \mathbb{N}_0^3$. For all $\Lambda \geq 1$ let φ_0 be the condensate, which varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.8$_{|\beta|}$. Then there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and times $t \in \mathbb{R}$*

$$\|D^\beta \tilde{\varphi}_t\|_\infty \leq C\Lambda^{-\frac{|\beta|}{3}}\,, \tag{B.1}$$

$$\|D^\beta \tilde{\varphi}_t\|_2 \leq C\Lambda^{-\frac{|\beta|}{3}+\frac{1}{2}}\,. \tag{B.2}$$

Remark B.2.2. Lemma B.2.1 shows that the derivative estimates for φ_0 remain valid for $\tilde{\varphi}_t$.

The proof of Lemma B.2.1 can be found in the Appendix B.4.3.

The following Lemma B.2.3 extends Proposition B.1.1 to derivatives of $\varphi_t - \tilde{\varphi}_t$, enabling us to transfer estimates from $\tilde{\varphi}_t$ to the condensate φ_t. Additionally, Lemma B.2.3 proves the condition (B.9) required in Lemma B.3.7 below.

Lemma B.2.3 (Condition (B.9) in Lemma B.3.7)**.** *Let $\beta \in \mathbb{N}_0^3$. For all volumes $\Lambda \geq 1$ let φ_t be the condensate with initial data φ_0, which varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.7 and Condition 2.1.8$_{|\beta|+2}$.*
Then for all times $T \geq 0$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and $-T \leq t \leq T$

$$\|D^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Lambda^{-1/6-|\beta|/3}\,. \tag{B.3}$$

The proof of Lemma B.2.3 can be found in the Appendix B.4.4.

Combining the estimates of Lemma B.2.1 and Lemma B.2.3 we gain control of the condensate φ_t as stated in the following Lemma.

Corollary B.2.4 (Estimates of the Condensate φ_t)**.** *Let $\beta \in \mathbb{N}_0^3$. For all volumes $\Lambda \geq 1$ let φ_t be the condensate with initial data φ_0, which varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.7 and Condition 2.1.8$_{|\beta|+2}$.*
Then for all times $T \geq 0$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and $-T \leq t \leq T$

$$\|D^\beta \varphi_t\|_2 \leq C\Lambda^{1/2-|\beta|/3}\,, \tag{B.4}$$

$$\|D^\beta \varphi_t\|_{2\wedge\infty} \leq C\Lambda^{-|\beta|/3}\,. \tag{B.5}$$

Remark B.2.5. The norm $\| \,.\, \|_{2\wedge\infty}$ is defined in Definition A.0.5.

Proof of Corollary B.2.4. Corollary B.2.4 follows directly from Lemma B.2.1 and Lemma B.2.3. ■

B.3 Localized Condensate

This section is focused on the question how to modulate φ_0 such that the condition of φ_0 being flat around the origin persists for times $t > 0$. We follow [Dec+16] as a guideline for this analysis.

One essential ingredient we need, is control over the localized φ_0 and $\tilde{\varphi}_t$, which is provided by the following Lemma B.3.1 and Lemma B.3.3.

Lemma B.3.1 (Estimates of the Localized Initial Condensate φ_0)**.** *For all volumes $\Lambda \geq 1$ let φ_0 be the condensate. Let $k, n \in \mathbb{N}_0$, $s > 0$. Set the localization function to be $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}x)^{2n}}$.*

i) We assume that the condensate φ_0 varies on the scale $\Lambda^{1/3}$, namely it satisfies Condition 2.1.8i)$_{k+2n}$, and that it is flat around the origin, namely it satisfies the Condition 2.1.11$_{2n,s}$. Then for all $0 \leq |\beta| \leq k$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$

$$\|\Theta_\Lambda D^\beta(\varphi_0 - 1)\|_\infty \leq C\Lambda^{-2n\left(\frac{1}{3}-s\right)-\frac{|\beta|}{3}}\,. \tag{B.1}$$

ii) Assume that the condensate φ_0 varies on the scale $\Lambda^{1/3}$, namely it satisfies the Condition 2.1.8i)$_{k+2(n-1)}$, and that it is flat around the origin, namely it satisfies Condition 2.1.11$_{2(n-1),s}$. Then for all $0 \leq |\beta| \leq k$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$

$$\|\Theta_\Lambda D^\beta(\varphi_0 - 1)\|_2 \leq C\Lambda^{-2(n-1)\left(\frac{1}{3}-s\right)-\frac{|\beta|}{3}+\frac{3}{2}s}\,. \tag{B.2}$$

Remark B.3.2. The term $\Lambda^{\frac{3}{2}s}$ in (B.2) corresponds to the square root of the volume occupied by the localization functions Θ_Λ. This factor appears due to be integration in the L^2-norm.

Proof of Lemma B.3.1.
Proof of Part i):
The case $n = 0$ is trivial and follows from Condition 2.1.8$_k$. Now let $n \geq 1$. Then with Lemma A.0.6a)$_{\tilde{k}=2n}$ we get $\forall 0 \leq |\beta| \leq k$

$$\|\Theta_\Lambda D^\beta(\varphi_0 - 1)\|_\infty \leq C\Lambda^{-2n(1/3-s)}\left(\sum_{|\gamma|\leq 2n} \|D^\gamma D^\beta \varphi_0\|_\infty \Lambda^{|\gamma|/3}\right)$$

$$\overset{\text{Condition 2.1.8i})_{|\beta|+2n}}{\leq} C\Lambda^{-2n\left(\frac{1}{3}-s\right)-\frac{|\beta|}{3}}\,. \tag{B.3}$$

Proof of Part ii):
Now let $n \geq 1$. Then with Lemma A.0.6a)$_{\tilde{k}=2(n-1)}$ we get $\forall 0 \leq |\beta| \leq k$

$$\|\Theta_\Lambda D^\beta(\varphi_0 - 1)\|_2 \leq C\Lambda^{-2(n-1)(1/3-s)+3/2s}\left(\sum_{|\gamma|\leq 2(n-1)} \|D^\gamma D^\beta \varphi_0\|_\infty \Lambda^{|\gamma|/3}\right)$$
$$\overset{\text{Condition 2.1.8i})_{|\beta|+2(n-1)}}{\leq} C\Lambda^{-2(n-1)\left(\frac{1}{3}-s\right)-\frac{|\beta|}{3}+\frac{3}{2}s}\,. \tag{B.4}$$

■

Lemma B.3.3 (Estimates of the Localized Auxiliary Function $\tilde{\varphi}_t$)**.** *For all volumes $\Lambda \geq 1$ let φ_0 be the condensate. Let $\beta \in \mathbb{N}_0^3$, $n \in \mathbb{N}_0$ and $s > 0$. Set the localization function to be $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}x)^{2n}}$.*

i) We assume that the condensate φ_0 varies on the scale $\Lambda^{1/3}$, namely it satisfies Condition 2.1.8i)$_{|\beta|+2n}$, and that it is flat around the origin, namely it satisfies Condition 2.1.11$_{2n,s}$. If $|\beta| \geq 1$ then there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and times $t \in \mathbb{R}$

$$\|\Theta_\Lambda D^\beta \tilde{\varphi}_t\|_\infty \leq C\Lambda^{-2n(1/3-s)-\frac{|\beta|}{3}}\,. \tag{B.5}$$

ii) Assume $n \geq 1$ and that the condensate φ_0 varies on the scale $\Lambda^{1/3}$, namely it satisfies Condition 2.1.8i)$_{|\beta|+2(n-1)}$ and Condition 2.1.8ii)$_{|\beta|}$, and that it is flat around the origin, namely it satisfies Condition 2.1.11$_{2(n-1),s}$. If $|\beta| \geq 1$ then there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and times $t \in \mathbb{R}$

$$\|\Theta_\Lambda D^\beta \tilde{\varphi}_t\|_2 \leq C\Lambda^{-2(n-1)(1/3-s)-\frac{|\beta|}{3}+\frac{3}{2}s}\,. \tag{B.6}$$

Remark B.3.4. We obtain estimates of the localized quantities $\Theta_\Lambda D^\beta \tilde{\varphi}_t$, which coincide with the estimates of $\Theta_\Lambda D^\beta \varphi_0$ from Lemma B.3.1. In this sense, $\tilde{\varphi}_t$ has the essential properties of φ_0.

The proof of Lemma B.3.3 can be found in the Appendix B.4.3.

We are now able to give the main statement of this section proving that the localized condensate $\Theta_\Lambda \varphi_t$ can be arbitrarily well approximated by $\Theta_\Lambda \tilde{\varphi}_t$.

Proposition B.3.5 (Localized Condensate Approximation)**.** *Let $n \in \mathbb{N}_+$, $k \in \mathbb{N}_0$ and $s > 0$. Set the localization function to be $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}x)^{2n}}$. Assume that for all volumes $\Lambda \geq 1$ the condensate φ_0 varies on the scale $\Lambda^{1/3}$, namely it satisfies Condition 2.1.7, Condition 2.1.8$_{k+3}$*

and Condition 2.1.8i)$_{(k+2)+2(n-1)}$. Furthermore, we require that the condensate is flat around the origin, namely Condition 2.1.11$_{2(n-1),s}$.
Then for all $\beta \in \mathbb{N}_0^3$ with $|\beta| \leq k$, and $T \geq 0$ there exists a constant $C > 0$ such that for all volumes $\Lambda \geq 1$ and $-T \leq t \leq T$

$$\|\Theta_\Lambda D^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s} + \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\}. \tag{B.7}$$

Remark B.3.6.

i) The first term on the right-hand side of (B.7) can be made arbitrary small by increasing k.

ii) For given $0 < s < 1/3$ and k, we can choose n large enough such that the last Term on the right-hand side of (B.7) is smaller than the first one. Consequently, as long as we can choose n arbitrary large and Condition 2.1.8$_k$ and Condition 2.1.11$_{n,s}$ are satisfied for all k, n, the right-hand side of (B.7) can be made as small as desired.

Proof of Proposition B.3.5. The claim follows directly from Lemma B.3.7, due to Lemma B.2.3 is the condition of Lemma B.3.7 for $\tilde{k} = 0$ and each step in Lemma B.3.7 outputs the condition for the next step in $\tilde{k}$. The sequence stops if we reach the limit of our initial condition, Condition 2.1.8i)$_{(k+2)+2(n-1)}$ and Condition 2.1.8ii)$_{k+2}$, after step $\tilde{k} = k$.
Note that condition (B.9) of Lemma B.3.7 is always satisfied for $\tilde{k} \leq k$, due to Lemma B.2.3, as pointed out in Remark B.3.8. This holds in particular because Condition 2.1.8$_{(k+3)}$ is assumed. ■

The Proof of Proposition B.3.5 is supported by the following Lemma. In order to get control over the localized quantity $\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2$ it is first necessary to control the not localized version $\|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2$ (see Lemma B.2.3, Lemma B.3.7 and Remark B.3.8 below).

Lemma B.3.7 (Induction Step for Proposition B.3.5)**.** *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}_+$ and $s > 0$. Assume that for all volumes $\Lambda \geq 1$ the condensate φ_0 satisfies Condition 2.1.7, Condition 2.1.11$_{2(n-1),s}$, Condition 2.1.8i)$_{(k+2)+2(n-1)}$ and Condition 2.1.8ii)$_{(k+2)}$.*
If $\forall \beta, \tilde{\beta} \in \mathbb{N}_0^3$ with $0 \leq |\beta| \leq k-1$ and $0 \leq |\tilde{\beta}| \leq k+1$, and $\forall T \geq 0$ $\exists C > 0$ such that $\forall \Lambda \geq 1$ and $-T \leq t \leq T$

$$\|\Theta_\Lambda D^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Big\{\Lambda^{-\frac{1}{6}-\frac{k}{3}-(k-|\beta|)s} + \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\}, \tag{B.8}$$

$$\|\Theta_\Lambda D^{\tilde{\beta}}(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\|D^{\tilde{\beta}}(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Lambda^{-1/6-|\tilde{\beta}|/3}.[1] \tag{B.9}$$

Then for $\forall \beta \in \mathbb{N}_0^3$ *with* $0 \le |\beta| \le k$, *and* $\forall T \ge 0$ $\exists C > 0$ *such that* $\forall \Lambda \ge 1$ *and* $-T \le t \le T$

$$\|\Theta_\Lambda D^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \le C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s} + \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\}. \tag{B.10}$$

Remark B.3.8.

- If we additionally assume Condition 2.1.8$_{k+3}$, then due to Lemma B.2.3 the condition (B.9) in Lemma B.3.7 is always satisfied.
- To prove the desired estimate in Proposition B.3.5 for k, we assume that it holds for $k-1$, i.e., we assume (B.8). Using an additional "external" condition (B.9), we then establish the estimate for k. Note that (B.8)$_{k+1}$=(B.10)$_k$.
  Viewed in this way, Lemma B.3.7 serves as an induction step in proving Proposition B.3.5. For further details, we refer to the proof of Proposition B.3.5.
- We now give the idea behind Lemma B.3.7 and therefore Proposition B.3.5. Let $T \ge 0$ and $-T \le t \le T$.
  - Our goal was to improve Proposition B.1.1, namely $\|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2 \le C\|(\varphi_t - \tilde{\varphi}_t)\|_2 \le C\Lambda^{-1/6}$, by using Θ_Λ.
  - In the estimation of $\|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2$ using Grönwall's Lemma, a commutator term $[\Delta\,,\,\Theta_\Lambda] = (\Delta\Theta_\Lambda) + 2(\nabla\Theta_\Lambda)\nabla$ appears, for which it is necessary to estimate terms $\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2$, $|\beta| = 1$. This term leads to the first term in (B.10).
  - In first instance these terms can again be estimated by Lemma B.2.3:

    $$\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \le C\|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \le \Lambda^{-1/6-1/3},$$

    which then gives a better result for $\|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2$. This is the step $k = 0$ in Lemma B.3.7.
  - To improve the result for $\|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2$ further, it is necessary to improve the estimate of $\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2$, $|\beta| = 1$. So we use Grönwall for this term, and again in the estimate, terms like $\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2$, but with $|\beta| = 2$, appear, which can be estimated by Lemma B.2.3: $\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \le C\Lambda^{-1/6-2/3}$.
  - This pattern continues and controlling all terms finally gives Lemma B.3.7. For details we refer to the proof.

The proof of Lemma B.3.7 can be found in the Appendix B.4.5.

[1]For $k = 0$ the only condition is $\|\Theta_\Lambda \mathrm{D}^{\tilde{\beta}}(\varphi_t - \tilde{\varphi}_t)\|_2 \le C\Lambda^{-1/6-|\tilde{\beta}|/3}$, $0 \le |\tilde{\beta}| \le 1$.

B.4 Proofs of Appendix B

B.4.1 Proof of Lemma B.1.1

Before we start the proof, we give a useful Lemma from real analysis.

Lemma B.4.1. *Let* $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *a sequence and* $a \in \mathbb{R}$*. It holds*

$$\liminf_{n\to\infty} a_n = a$$

if and only if for all $\epsilon > 0$*:*

i) For all but a finite number of indices $n \in \mathbb{N}$ *holds*

$$a_n > a - \epsilon\,.$$

ii) There are infinite many indices $m \in \mathbb{N}$ *with*

$$a_m < a + \epsilon\,.$$

Proof of Lemma B.4.1. This is a standard result from real analysis. ∎

Proof of Proposition B.1.1. We use a Grönwall argument. Important for the proof is the identity $|\varphi_t|^2 - |\varphi_0|^2 = |\varphi_t - \tilde{\varphi}_t|^2 + 2\mathrm{Re}\tilde{\varphi}_t^*(\varphi_t - \tilde{\varphi}_t)$ which allows us to close the Grönwall argument.

In order to provide the bound (B.1) we estimate the time derivative for a Grönwall argument and use that μ_t is a real-valued function

$$\begin{aligned}\partial_t \left\|\varphi_t - \widetilde{\varphi}_t\right\|_2 &\leq \left\| -\frac{1}{2}\Delta\widetilde{\varphi}_t + V * \left(|\varphi_t|^2 - |\varphi_0|^2\right)\widetilde{\varphi}_t \right\|_2 \\ &\leq \left\|\frac{1}{2}\Delta\widetilde{\varphi}_t\right\|_2 + \|\widetilde{\varphi}_t\|_\infty \left(\|V\|_2 \, \|\varphi_t - \tilde{\varphi}_t\|_2^2 + \|V\|_1 \, \|\tilde{\varphi}_t^*\|_\infty \, \|\varphi_t - \tilde{\varphi}_t\|_2 \right)\,. \qquad \text{(B.1)}\end{aligned}$$

We estimate the terms on the right-hand side of (B.1) separately:
We give an estimate for $\|\Delta\tilde{\varphi}_t\|_2$ similar to [Dec+16]

$$\begin{aligned}\nabla\tilde{\varphi}_t &= \left([-itV * \nabla|\varphi_0|^2]\varphi_0 + \nabla\varphi_0\right)\exp\left(-\mathrm{i}\left(tV * |\varphi_0|^2 - \int_0^t \mu_s ds\right)\right) \\ \Delta\tilde{\varphi}_t &= \Big([-itV * \Delta|\varphi_0|^2]\varphi_0 + [-itV * \nabla|\varphi_0|^2]^2\varphi_0 \\ &\quad + 2[-itV * \nabla|\varphi_0|^2]\nabla\varphi_0 + \Delta\varphi_0\Big)\exp\left(-\mathrm{i}\left(tV * |\varphi_0|^2 - \int_0^t \mu_s ds\right)\right)\end{aligned}$$

and

$$\begin{aligned}\|\Delta\tilde{\varphi}_t\|_2 &\le 2|t|\|V\|_1\|\varphi_0\|_\infty\big(\|\varphi_0\|_\infty\|\Delta\varphi_0\|_2+\|\nabla\varphi_0\|_\infty\|\nabla\varphi_0\|_2\big)\\ &\quad+4t^2\|V\|_1^2\|\varphi_0\|_\infty^2\|\nabla\varphi_0\|_\infty^2\|\varphi_0\|_2\\ &\quad+4|t|\|V\|_1\|\varphi_0\|_\infty\|\nabla\varphi_0\|_\infty\|\nabla\varphi_0\|_2\\ &\quad+\|\Delta\varphi_0\|_2\\ &\le C(|t|+t^2)\Lambda^{-\frac{1}{6}}\,,\end{aligned}$$

where in the last step we used Condition 2.1.7. This together with (2.3), $|\widetilde{\varphi}_t| = |\varphi_0|$ and (B.1) ensure

$$\partial_t\left\|\varphi_t-\widetilde{\varphi}_t\right\|_2 \le C(|t|+t^2)\Lambda^{-\frac{1}{6}} + C\left(\|\varphi_t-\tilde{\varphi}_t\|_2^2+\|\varphi_t-\tilde{\varphi}_t\|_2\right). \tag{B.2}$$

To close the Grönwall argument we have to estimate $\|\varphi_t-\tilde{\varphi}_t\|_2^2$ by $\|\varphi_t-\tilde{\varphi}_t\|_2$, which is done in the following.

We now define the Λ dependent quantity

$$t_\Lambda = \inf\left\{|t|\,\middle|\, t\in\mathbb{R}, \|\varphi_t-\tilde{\varphi}_t\|_2 = 1\right\} \in \mathbb{R}\cup\{\infty\}\,, \tag{B.3}$$

where we choose the convention $\inf\emptyset=\infty$. We will show that

$$\lim_{\Lambda\to\infty} t_\Lambda = \infty\,. \tag{B.4}$$

Proof. *Proof by contra position.* Assume there is a sequence $(\Lambda_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty}\Lambda_n=\infty$, but

$$c := \lim_{n\to\infty} i_n := \liminf_{n\to\infty} t_{\Lambda_n} < \infty\,, \tag{B.5}$$

where we introduced the notation $i_n := \inf\{t_{\Lambda_k}\,|\,k\ge n\}$. From (B.5) we know that (i_n) is bounded and therefore

$$\exists \text{ infinite many } t_{\Lambda_k} < \infty\,.^2 \tag{B.6}$$

Due to (B.6) it is possible to take the subsequence $t_{\Lambda_{n_k}}$, which omits only the $t_{\Lambda_i}=\infty$. Then $t_{\Lambda_{n_k}}\in\mathbb{R}$ and

$$i_n = \inf\{t_{\Lambda_k}\,|\,k\ge n\} = \inf\{t_{\Lambda_{j_k}}\,|\,k\ge n\}\,,$$

as $j_k \ge k$ and we left out just $t_{\Lambda_i}=\infty$, which do not affect the value of

[2] For all $n\in\mathbb{N}$ there is at least one $t_{\Lambda_k}<\infty$, $k\ge n$, since (i_n) is bounded.

$\inf\{t_{\Lambda_k} \,|\, k \geq n\}$, because of (B.6).
Now we have with $a_k := t_{\Lambda_{j_k}} \in \mathbb{R}$, $c = \liminf_{n\to\infty} a_n < \infty$ and Lemma B.4.1 that for $\epsilon > 0$ there exists infinitely many a_n with

$$c - \epsilon < a_n < c + \epsilon$$

and we conclude that there exists a subsequence (a_{n_k}) with $a_{n_k} \to c$. Finally

$$\begin{aligned} \exists \tau > 0 : \; a_{n_k} &< \tau \quad k \in \mathbb{N} \\ \Rightarrow \quad t_{\Lambda_{j_{n_k}}} &< \tau \quad k \in \mathbb{N}\,. \end{aligned}$$

From the Definition (B.3) we know that $\|\varphi_t - \tilde{\varphi}_t\|_2 \leq 1$ for all $t \in [-t_{\Lambda_{j_{n_k}}}, t_{\Lambda_{j_{n_k}}}] \subset [-\tau, \tau]$. By using (B.2) we conclude for such t

$$\partial_t \left\|\varphi_t - \widetilde{\varphi}_t\right\|_2 \leq C(\tau + \tau^2)\Lambda_{j_{n_k}}^{-1/6} + C\left\|\varphi_t - \tilde{\varphi}_t\right\|_2,$$

with C independent of k, which thanks to Grönwall's Lemma implies

$$\left\|\varphi_t - \widetilde{\varphi}_t\right\|_2 \leq C_\tau \Lambda_{j_{n_k}}^{-\frac{1}{6}} \qquad \text{for } t \in [-t_{\Lambda_{j_{n_k}}}, t_{\Lambda_{j_{n_k}}}]. \tag{B.7}$$

From (B.3) we get that

$$\begin{aligned} 1 = \left\|\varphi_{t_{\Lambda_{j_{n_k}}}} - \widetilde{\varphi}_{t_{\Lambda_{j_{n_k}}}}\right\|_2 &\leq C_\tau \Lambda_{j_{n_k}}^{-\frac{1}{6}} \\ \Rightarrow \quad \Lambda_{j_{n_k}}^{\frac{1}{6}} &\leq C_\tau\,, \end{aligned}$$

which is a contradiction, since $\Lambda_{j_{n_k}} \to \infty$ for $k \to \infty$ and C_τ is independent of k.

Thus our assumption is false. This implies that for $\lim_{n\to\infty} \Lambda_n = \infty$ we have $\liminf_{n\to\infty} t_{\Lambda_n} = \infty$. Now let $K > 0$, then $\exists N \in \mathbb{N}$ with $\inf\{t_{\Lambda_k} \,|\, k \geq n\} > K$ for $n \geq N$. Then it follows, that $t_{\Lambda_n} \geq \inf\{t_{\Lambda_k} \,|\, k \geq n\} > K$ for $n \geq N$ and therefore

$$\lim_{n\to\infty} t_{\Lambda_n} = \infty\,,$$

for arbitrary (Λ_n) with $\lim_{n\to\infty} \Lambda_n = \infty$ and therefore

$$\lim_{\Lambda\to\infty} t_\Lambda = \infty\,.$$

□

Equation (B.4) means that for $T \in \mathbb{R}$ there exists $\Lambda_0(T)$ such that for all $\Lambda \geq \Lambda_0(T)$:

$$t_\Lambda > T\,.$$

But for all $t \in (-t_\Lambda, t_\Lambda)$ we have $\|\varphi_t - \tilde{\varphi}_t\|_2 \leq 1$, where we have chosen the open interval because t_Λ could be infinite. The estimate (B.2) together with Grönwall now ensures that

$$\|\varphi_t - \widetilde{\varphi}_t\|_2 \leq \Lambda^{-\frac{1}{6}}\mathrm{sgn}(t)\int_0^t C(|s| + s^2)\mathrm{e}^{|t-s|C} ds\,, \qquad \text{for } t \in (-t_\Lambda, t_\Lambda)$$

and therefore for all $t \in [-T, T]$. By taking the supremum over all times $t \in [-T, T]$ we get

$$\|\varphi_t - \widetilde{\varphi}_t\|_2 \leq \Lambda^{-\frac{1}{6}} C_T\,,$$

which proves the claim for $\Lambda \geq \Lambda_0(T)$. Note that C is dependent of the choice of T.
In the case $\Lambda < \Lambda_0(T)$ we use the triangle inequality to conclude the claim. For $t \in [-T, T]$

$$\|\varphi_t - \widetilde{\varphi}_t\|_2 \leq \|\varphi_t\|_2 + \|\tilde{\varphi}_t\|_2 = 2\Lambda^{1/2} \leq 2\Lambda_0^{1/2}\Lambda^{+1/6}\Lambda^{-1/6} \leq C(\Lambda_0(T))\Lambda^{-1/6}\,.$$

The C obtained here is again T dependent. ■

B.4.2 Proof of Lemma B.1.2

For the proof we need a Lemma from [Dec+16] which uses the notation from Definition A.0.5.

Lemma B.4.2. *Let $V \in L^\infty(\mathbb{R}^3, \mathbb{R})$, even, with $\widehat{V} \in L^p$, $p \in \{1, 2, \infty\}$, be a general potential and $\mu_t \in \mathbb{R}$ a constant. Let ζ_t be solution of the non-linear equation*

$$\mathrm{i}\partial_t\zeta_t(x) = \left(-\frac{1}{2}\Delta + V * |\zeta_t|^2(x) - \mu_t\right)\zeta_t(x),$$

for an initial value $\zeta_t|_{t=0} = \zeta_0$ such that for all times $T \geq 0$ there exists a constant $C > 0$ such that for all $-T \leq t \leq T$

$$\left((2\pi)^{3/2}\|\zeta_0\|_\infty \leq\right)\left\|\widehat{\zeta_0}\right\|_1 \leq C \text{ and } \left((2\pi)^{3/2}\|\zeta_t\|_{2\wedge\infty} \leq\right)\left\|\widehat{\zeta_t}\right\|_{1\wedge 2} \leq C\,. \tag{B.8}$$

Then for all times $T \geq 0$ there exists a constant $C > 0$ such that for all $-T \leq t \leq T$

$$\left((2\pi)^{3/2}\|\zeta_t\|_\infty \leq\right)\left\|\widehat{\zeta_t}\right\|_1 \leq C.$$

Proof. Grönwall's Lemma, the bound on the time derivative

$$\begin{aligned}
\partial_t \left\|\widehat{\zeta}_t\right\|_1 &\leq \int dk \, \frac{\mathrm{Im}\Big\{(\widehat{\zeta}_\mathrm{t})^*(\mathrm{k})\left(\frac{\mathrm{k}^2}{2}\widehat{\zeta}_\mathrm{t}(\mathrm{k}) + \widehat{V * |\zeta_t|^2\zeta_t}(\mathrm{k}) - \mu_\mathrm{t}\widehat{\zeta}_\mathrm{t}(\mathrm{k})\right)\Big\}}{\left|\widehat{\zeta}_t(k)\right|} \\
&\leq \int dk \left|\left(\widehat{V}\cdot\widehat{\zeta_t^*} * \widehat{\zeta}_t\right) * \widehat{\zeta}_t(k)\right| \\
&\leq \left\|\widehat{V}\cdot\widehat{\zeta_t^*} * \widehat{\zeta}_t\right\|_1 \left\|\widehat{\zeta}_t\right\|_1 \\
&\leq \left\|\widehat{V}\right\|_{1,2,\infty} \left\|\widehat{\zeta}_t\right\|_{1\wedge 2}^2 \left\|\widehat{\zeta}_t\right\|_1 \\
&\leq C\left\|\widehat{\zeta}_t\right\|_1,
\end{aligned}$$

and the assumption on the initial condition (B.8) imply the claim. ∎

Proof of Corollary B.1.2. We observe

$$((2\pi)^{3/2}\,\|\varphi_t\|_{2\wedge\infty} \leq)\,\|\,\widehat{\varphi_t}\,\|_{1\wedge 2} \leq \left\|\,\widehat{\widetilde{\varphi}_t}\,\right\|_1 + \|\varphi_t - \widetilde{\varphi}_t\|_2\;, \tag{B.9}$$

where in step 1 we used $\widehat{\varphi_t} = \widehat{\widetilde{\varphi}_t} + (\widehat{\varphi_t} - \widehat{\widetilde{\varphi}_t})$. Similar to the proof of Lemma B.4.2 we get

$$\partial_t \left\|\,\widehat{\widetilde{\varphi}_t}\,\right\|_1 \leq \left\|\widehat{V}\right\|_{1,2,\infty} \|\widehat{\varphi_0}\|_{1\wedge 2}^2 \left\|\,\widehat{\widetilde{\varphi}_t}\,\right\|_1$$

and hence with Grönwall and Condition 2.1.7 that $\left\|\,\widehat{\widetilde{\varphi}_t}\,\right\|_1 \leq C$. Note that $\widehat{V} \in L^p$ for all $1 \leq p \leq \infty$ as $V \in L^1$ and $\widehat{V} \in L^1$. With (B.9) and Proposition B.1.1 we are now able to conclude: $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$ and $-T \leq t \leq T$

$$\|\,\widehat{\varphi_t}\,\|_{1\wedge 2} \leq C + C\Lambda^{-\frac{1}{6}}$$

and the claim (B.2) follows from Lemma B.4.2. ∎

B.4.3 Proof of Lemma B.2.1 and Lemma B.3.3

For the proof we need one additional Lemma.

Lemma B.4.3. *For all $\Lambda \geq 1$ let φ_0 be the condensate. Let $n \in \mathbb{N}_0$, $s > 0$, $\beta \in \mathbb{N}_0^3$. Set $\Theta_\Lambda(x) = \frac{1}{1+(\Lambda^{-s}x)^{2n}}$.*

i) Assume that the condensate φ_0 satisfies Condition 2.1.8i)$_{|\beta|}$. Then $\exists C > 0$ such that $\forall \Lambda \geq 1$

$$\|D^\beta \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2 - \int_0^t \mu_s ds\right)}\|_\infty \leq C\Lambda^{-\frac{|\beta|}{3}}\,. \tag{B.10}$$

ii) Assume that the condensate φ_0 *satisfies Condition 2.1.8i)*$_{|\beta|+2n}$ *and Condition 2.1.11*$_{2n,s}$ *then if* $|\beta|\geq 1$ $\exists C>0$ *such that* $\forall\Lambda\geq 1$

$$\|\Theta_\Lambda D^\beta \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t\mu_s ds\right)}\|_\infty \leq C_n(t)\Lambda^{-2n(1/3-s)-\frac{|\beta|}{3}}\,. \tag{B.11}$$

iii) For $n\geq 1$ *assume that the condensate* φ_0 *satisfies Condition 2.1.11*$_{2(n-1),s}$*, Condition 2.1.8ii)*$_{|\beta|}$*, and Condition 2.1.8i)*$_{|\beta|+2(n-1)}$ *then if* $|\beta|\geq 1$ $\exists C>0$ *such that* $\forall\Lambda\geq 1$

$$\|\Theta_\Lambda D^\beta \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t\mu_s ds\right)}\|_2 \leq C_n(t)\Lambda^{-2(n-1)\left(\frac{1}{3}-s\right)-\frac{|\beta|}{3}+\frac{3}{2}s}\,. \tag{B.12}$$

Proof of Lemma B.4.3. Proof of (B.10)*:* We prove (B.10) by induction in $|\beta|$.

Base Case: $|\beta|=0$**.** For $|\beta|=0$ the assertion is clear.

Induction Step: $|\beta|=n+1$**.** Now let $n\in\mathbb{N}_0$ be fixed. Assume that the assertion holds for all $|\beta|\leq n$.
Let $\beta\in\mathbb{N}_0^3$ with $|\beta|=n+1$. Then there exists a $i\in\{1,2,3\}$ and $\hat\beta\in\mathbb{N}_0^3$ with $|\hat\beta|=n$ such that $D^\beta=D^{\hat\beta}\partial_{x_i}$.
Now if for all $0\leq|\tilde\beta|\leq|\beta|=n+1$: $\|\mathrm{D}^{\tilde\beta}\varphi_0\|_\infty\leq C\Lambda^{-|\tilde\beta|/3}$ we know by the induction hypothesis that

$$\|\mathrm{D}^{\tilde\beta}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t\mu_s ds\right)}\|_\infty\leq C\Lambda^{-\frac{|\tilde\beta|}{3}}$$

for all $0\leq|\tilde\beta|\leq|\hat\beta|$. And we get with the product rule

$$\begin{aligned}
\|\mathrm{D}^\beta\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t\mu_s ds\right)}\|_\infty &= \|\mathrm{D}^{\hat\beta}\left[-itV*\partial_{x_i}|\varphi_0|^2\right]\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t\mu_s ds\right)}\|_\infty\\
&=\|\sum_{\gamma=0}^{\hat\beta}\binom{\hat\beta}{\gamma}\left(\mathrm{D}^{\hat\beta-\gamma}\left[-itV*\partial_{x_i}|\varphi_0|^2\right]\right)D^\gamma\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t\mu_s ds\right)}\|_\infty\\
&\overset{|\gamma|\leq|\hat\beta|}{\leq}\sum_{\gamma=0}^{\hat\beta}C\Lambda^{-|\gamma|/3}\|\mathrm{D}^{\hat\beta-\gamma}\left[-itV*2\mathrm{Re}\varphi_0^*\partial_{x_i}\varphi_0\right]\|_\infty\\
&\leq\sum_{\gamma=0}^{\hat\beta}C\Lambda^{-|\gamma|/3}\sum_{\delta=0}^{\hat\beta-\gamma}\|V*2\mathrm{Re}\left(\mathrm{D}^{\hat\beta-\gamma-\delta}\varphi_0^*\partial_{x_i}\right)\mathrm{D}^\delta\partial_{x_i}\varphi_0\|_\infty\\
&\leq\sum_{\gamma=0}^{\hat\beta}\sum_{\delta=0}^{\hat\beta-\gamma}C\Lambda^{-|\gamma|/3}\|V\|_1\Lambda^{-|\hat\beta-\gamma-\delta|/3}\Lambda^{-(|\delta|+1)/3}\\
&\leq C\Lambda^{-(|\hat\beta|+1)/3}=C\Lambda^{-|\beta|/3}\,.
\end{aligned}$$

Which completes the induction step and therefore the proof of (B.10).

Proof of (B.11)*:*
Let $n \in \mathbb{N}_0$, $s > 0$ and $\beta \in \mathbb{N}_0^3$ with $|\beta| \geq 1$. Since $|\beta| \geq 1$, there exists a multi-index $\tilde{\beta} \in \mathbb{N}_0^3$ with $|\tilde{\beta}| = |\beta| - 1$ such that $\beta = \tilde{\beta} + e_i$, where e_i is the standard unit vector in the i-th coordinate direction. This implies that the derivative operator can be written as $\mathrm{D}^\beta = \mathrm{D}^{\tilde{\beta}} \partial_{x_i}$.
Assume Condition 2.1.8i)$_{|\beta|+2n}$ and Condition 2.1.11$_{2n,s}$. Then we get from Lemma B.3.1i) that $\forall 0 \leq |\gamma| \leq |\beta|$

$$\|\Theta_\Lambda D^\gamma(\varphi_0 - 1)\|_\infty \leq C_{n,s} \Lambda^{-2n\left(\frac{1}{3}-s\right)-\frac{|\gamma|}{3}} . \tag{B.13}$$

Then in a similar way as above for the proof of (B.10) with Lemma A.0.6b), we get with part i) of this Lemma that

$$\begin{aligned} &\|\Theta_\Lambda \mathrm{D}^\beta \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2 - \int_0^t \mu_s ds\right)}\|_\infty \\ &= \|\Theta_\Lambda \mathrm{D}^{\tilde{\beta}} \left[-itV * \partial_{x_i}|\varphi_0|^2\right] \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2 - \int_0^t \mu_s ds\right)}\|_\infty \\ &\leq \sum_{\gamma=0}^{\tilde{\beta}} C\|\Theta_\Lambda V * \mathrm{D}^\gamma \partial_{x_i}|\varphi_0|^2\|_\infty \|D^{\tilde{\beta}-\gamma} \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2 - \int_0^t \mu_s ds\right)}\|_\infty \\ &\leq \sum_{\gamma=0}^{\tilde{\beta}} C\|\Theta_\Lambda V * \mathrm{D}^\gamma 2\mathrm{Re}\varphi_0^* \partial_{x_i}\varphi_0\|_\infty \Lambda^{-\frac{|\tilde{\beta}|-|\gamma|}{3}} . \end{aligned} \tag{B.14}$$

We conclude with Condition 2.1.8i)$_{|\beta|}$ and (B.13) as well as $|\tilde{\beta}| + 1 = |\beta|$ that $\forall m \in \mathbb{N}_+$

$$\begin{aligned} \text{(B.14)} \overset{\text{Lemma A.0.6b)}}{\leq} & C_m(t)\Lambda^{-\frac{|\tilde{\beta}|-|\gamma|}{3}} \sum_{\gamma=0}^{\tilde{\beta}} \sum_{\delta=0}^{\gamma} \Big(\|\Theta_\Lambda D^\delta \partial_{x_i}\varphi_0\|_\infty \|D^{\gamma-\delta}\varphi_0\|_\infty \\ & + \Lambda^{-sm} \|D^\delta \partial_{x_i}\varphi_0\|_\infty \|D^{\gamma-\delta}\varphi_0\|_\infty \Big) \\ & \leq C_m(t)\Lambda^{-\frac{|\tilde{\beta}|-|\delta|}{3}} \sum_{\gamma=0}^{\tilde{\beta}} \sum_{\delta=0}^{\gamma} \Big(\|\Theta_\Lambda D^\delta \partial_{x_i}(\varphi_0 - 1)\|_\infty + \Lambda^{-sm-\frac{|\delta|+1}{3}} \Big) \\ & \leq C_m(t) \Big(\Lambda^{-2n(1/3-s)-\frac{|\beta|}{3}} + \Lambda^{-sm-\frac{|\beta|}{3}} \Big) . \end{aligned}$$

Now we conclude that for fixed n, s and m large enough $0 < \Lambda^{-sm} \leq \Lambda^{-2n(1/3-s)}$, since $s > 0$, and thus it follows the claim (B.11).
Proof of (B.12)*:*
We repeat the argument from above for the L^2 norm. As in (B.14) we get

$$\begin{aligned} &\|\Theta_\Lambda \mathrm{D}^\beta \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2 - \int_0^t \mu_s ds\right)}\|_2 \\ &\leq \sum_{\gamma=0}^{\tilde{\beta}} C\|\Theta_\Lambda V * \mathrm{D}^\gamma 2\mathrm{Re}\varphi_0^* \partial_{x_i}\varphi_0\|_2 \Lambda^{-\frac{|\tilde{\beta}|-|\gamma|}{3}} . \end{aligned} \tag{B.15}$$

Now we use Condition 2.1.8$_{|\beta|}$ and Lemma B.3.1ii)$_{|\beta|}$ as well as $|\tilde{\beta}|+1=|\beta|$ to conclude

$$\text{(B.15)} \overset{\text{Lemma A.0.6b)}}{\leq} C_m(t)\Lambda^{-\frac{|\tilde{\beta}|-|\gamma|}{3}} \sum_{\gamma=0}^{\tilde{\beta}} \sum_{\delta=0}^{\gamma} \Big(\|\Theta_\Lambda D^\delta \partial_{x_i}\varphi_0\|_2 \|D^{\gamma-\delta}\varphi_0\|_\infty + \Lambda^{-sm} \|D^\delta \partial_{x_i}\varphi_0\|_2 \|D^{\gamma-\delta}\varphi_0\|_\infty \Big)$$
$$\leq C_m(t)\Big(\Lambda^{-2(n-1)\left(\frac{1}{3}-s\right)-\frac{|\beta|}{3}+\frac{3}{2}s} + \Lambda^{-sm-\frac{|\beta|}{3}+\frac{1}{2}}\Big).$$

By choosing m large enough we conclude as above $\|\Theta_\Lambda \mathrm{D}^\beta \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\|_2 \leq C_n(t)\cdot \Lambda^{-2(n-1)\left(\frac{1}{3}-s\right)-\frac{|\beta|}{3}+\frac{3}{2}s}$.

■

Now we can start with the proof of Lemma B.2.1.

Proof of Lemma B.2.1. Let $t\in\mathbb{R}$.
Proof of (B.1)*:*
Assume Condition 2.1.8i)$_{|\beta|}$ then is follows with Lemma B.4.3i)

$$\begin{aligned}
\|\mathrm{D}^\beta\tilde{\varphi}_t\|_\infty &= \|\mathrm{D}^\beta \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\varphi_0\|_\infty \\
&= \|\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\left(\mathrm{D}^{\beta-\gamma}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\right) D^\gamma\varphi_0\|_\infty \\
&\leq \sum_{\gamma=0}^{\beta} C\Lambda^{-(|\beta-\gamma|)/3}\Lambda^{-|\gamma|/3} \leq C\Lambda^{-|\beta|/3}.
\end{aligned}$$

Proof of (B.2)*:*
Assume Condition 2.1.8$_{|\beta|}$ then analogous to the above we find with Lemma B.4.3i)

$$\begin{aligned}
\|\mathrm{D}^\beta\tilde{\varphi}_t\|_2 &= \|\mathrm{D}^\beta \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\varphi_0\|_2 \\
&= \|\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\left(\mathrm{D}^{\beta-\gamma}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\right) D^\gamma\varphi_0\|_2 \\
&\leq \sum_{\gamma=0}^{\beta} C\Lambda^{-(|\beta-\gamma|)/3}\|D^\gamma\varphi_0\|_2 \leq C\Lambda^{-(|\beta|)/3+1/2}.
\end{aligned}$$

■

Proof of Lemma B.3.3. Let $t\in\mathbb{R}$, $n\in\mathbb{N}_0$, $s>0$ and $\beta\in\mathbb{N}_0^3$ with $|\beta|\geq 1$.
Proof of (B.5)*:*
Assume Condition 2.1.11$_{2n,s}$ and Condition 2.1.8i)$_{|\beta|+2n}$. Then we use Lemma B.4.3 and

Lemma B.3.1 to get

$$\begin{aligned}
\|\Theta_\Lambda \mathrm{D}^\beta \tilde{\varphi}_t\|_\infty &= \|\Theta_\Lambda \mathrm{D}^\beta \mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\varphi_0\|_\infty \\
&\le \sum_{\gamma=0}^{\beta} C\|\Theta_\Lambda \left(\mathrm{D}^{\beta-\gamma}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\right) D^\gamma\varphi_0\|_\infty \\
&\le \sum_{\gamma=0,|\gamma|\ge 1}^{\beta} C\| \left(\mathrm{D}^{\beta-\gamma}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\right) \Theta_\Lambda D^\gamma\varphi_0\|_\infty \\
&\quad + C\| \left(\Theta_\Lambda \mathrm{D}^{\beta}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\right) \varphi_0\|_\infty . \qquad \text{(B.16)}
\end{aligned}$$

Then we apply $\|\varphi_0\|_\infty \le C$ and Lemma B.3.1, note that $D^\gamma\varphi_0 = D^\gamma(\varphi_0 - 1)$ for $|\gamma| \ge 1$, to conclude that

$$\begin{aligned}
\text{(B.16)} \le &\sum_{\gamma=0,|\gamma|\ge 1}^{\beta} C\| \left(\mathrm{D}^{\beta-\gamma}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\right) \|_\infty \cdot \Lambda^{-2n(1/3-s)-|\gamma|/3} \\
&+ C\| \left(\Theta_\Lambda \mathrm{D}^{\beta}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\right) \|_\infty \cdot C . \qquad \text{(B.17)}
\end{aligned}$$

We finish the estimate with the help of Lemma B.4.3 i) and ii)

$$\begin{aligned}
\text{(B.17)} \le &\sum_{\gamma=0,|\gamma|\ge 1}^{\beta} C\Lambda^{-|\beta-\gamma|/3}\Lambda^{-2n(1/3-s)-|\gamma|/3} + C\Lambda^{-2n(1/3-s)}\Lambda^{-|\beta|/3}\cdot C \\
&\le C\Lambda^{-2n(1/3-s)}\Lambda^{-|\beta|/3} .
\end{aligned}$$

Proof of (B.6)*:*
Assume Condition $2.1.11_{2(n-1),s}$, Condition $2.1.8\mathrm{i})_{|\beta|+2(n-1)}$ and Condition $2.1.8\mathrm{ii})_{|\beta|}$. Analogous to the above we get with Lemma B.4.3 and Lemma B.3.1ii) that

$$\begin{aligned}
\|\Theta_\Lambda \mathrm{D}^\beta \tilde{\varphi}_t\|_2 \le &\sum_{\gamma=0,|\gamma|\ge 1}^{\beta} C\|\mathrm{D}^{\beta-\gamma}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\|_\infty \|\Theta_\Lambda D^\gamma\varphi_0\|_2 \\
&+ C\|\Theta_\Lambda \mathrm{D}^{\beta}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\|_2 \|\varphi_0\|_\infty \\
\le &\sum_{\gamma=0,|\gamma|\ge 1}^{\beta} C\|\mathrm{D}^{\beta-\gamma}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\|_\infty \cdot \Lambda^{-2(n-1)(1/3-s)-|\gamma|/3+3/2s} \\
&+ C\|\Theta_\Lambda \mathrm{D}^{\beta}\mathrm{e}^{-\mathrm{i}\left(tV*|\varphi_0|^2-\int_0^t \mu_s ds\right)}\|_2 \cdot C \\
\le &\, C\Lambda^{-2(n-1)(1/3-s)-|\beta|/3+3/2s} .
\end{aligned}$$

■

B.4.4 Proof of Lemma B.2.3

For the proof we need the following Lemma, which allows us to control the convolution of the condensate with the potential V.

Lemma B.4.4. *Let $\beta \in \mathbb{N}_0^3$. Assume that the condensate φ_0 satisfies Condition 2.1.7 and Condition 2.1.8$_{|\beta|}$.*
If $\forall 0 \leq |\tilde{\beta}| \leq |\beta| - 1$ $\forall T \geq 0$ $\exists C > 0$ such that $\forall \Lambda \geq 1$ and $-T \leq t \leq T$

$$\|D^{\tilde{\beta}}(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Lambda^{-1/6-\tilde{\beta}/3}\,, \tag{B.18}$$

then $\forall T \geq 0$ $\exists C > 0$ such that $\forall \Lambda \geq 1$ and $-T \leq t \leq T$

$$\|\,|V| * D^{\beta}|\varphi_t|^2\,\|_\infty \leq C\Lambda^{-|\beta|/3} + C\|D^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2\,, \tag{B.19}$$

$$\|\,|V| * \left|D^{\beta}(|\varphi_t|^2 - |\tilde{\varphi}_t|^2)\right|\,\|_2 \leq C\Lambda^{-1/6-|\beta|/3} + C\|D^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2\,. \tag{B.20}$$

Proof of Lemma B.4.4. Let $T \geq 0$, $-T \leq t \leq T$ and $\beta \in \mathbb{N}_0^3$.
Proof of (B.19)*:*
We consider the case $|\beta| \geq 1$, the case $|\beta| = 0$ follows analogously. By subtracting and adding $\tilde{\varphi}_t$ we get

$$\begin{aligned}
&\|\,|V| * \mathrm{D}^{\beta}|\varphi_t|^2\,\|_\infty \leq C\Big\|\,|V| * \Big|\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\left(\mathrm{D}^{\beta-\gamma}\varphi_t^*\right)\mathrm{D}^{\gamma}\varphi_t\Big|\,\Big\|_\infty \\
\leq &\sum_{\gamma=0}^{\beta} C\Big\{\|\,|V| * \Big|\left(\mathrm{D}^{\beta-\gamma}(\varphi_t - \tilde{\varphi}_t)^*\right)\mathrm{D}^{\gamma}(\varphi_t - \tilde{\varphi}_t)\Big|\,\|_\infty \\
&+ \|\,|V| * \Big|\left(\mathrm{D}^{\beta-\gamma}(\varphi_t - \tilde{\varphi}_t)^*\right)\mathrm{D}^{\gamma}\tilde{\varphi}_t\Big|\,\|_\infty \\
&+ \|\,|V| * \Big|\left(\mathrm{D}^{\beta-\gamma}\tilde{\varphi}_t^*\right)\mathrm{D}^{\gamma}(\varphi_t - \tilde{\varphi}_t)\Big|\,\|_\infty \\
&+ \|\,|V| * \Big|\left(\mathrm{D}^{\beta-\gamma}\tilde{\varphi}_t^*\right)\mathrm{D}^{\gamma}\tilde{\varphi}_t\Big|\,\|_\infty\Big\}\,.
\end{aligned} \tag{B.21}$$

Now we use our assumption $\|\mathrm{D}^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Lambda^{-1/6-\beta/3}$ and Lemma B.2.1 to conclude $\exists C > 0$ such that $\forall \Lambda \geq 1$

$$\begin{aligned}
\text{(B.21)} \leq &\Big(\sum_{\gamma=0,|\gamma|\leq|\beta|-1}^{\beta}\|V\|_\infty C\Lambda^{-\frac{1}{6}-\frac{|\beta-\gamma|}{3}}\Lambda^{-\frac{1}{6}-\frac{|\gamma|}{3}} + \Lambda^{-\frac{1}{6}}C\|\mathrm{D}^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2\Big) \\
&+ \Big(\sum_{\gamma=0,|\gamma|\leq|\beta|-1}^{\beta}\|V\|_2 C\Lambda^{-\frac{1}{6}-\frac{|\beta-\gamma|}{3}}\Lambda^{-\frac{|\gamma|}{3}} + C\|\mathrm{D}^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2\Big) \\
&+ \|V\|_1 C\Lambda^{-\frac{|\beta|}{3}} \\
\leq &\, C\Lambda^{-\frac{|\beta|}{3}} + C\|\mathrm{D}^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2\,.
\end{aligned}$$

Proof of (B.20)*:*
It holds that

$$\| \, |V| * \left| \mathrm{D}^\beta (|\varphi_t|^2 - |\tilde{\varphi}_t|^2) \right| \, \|_2 \leq \| \, |V| * \left| \mathrm{D}^\beta |\varphi_t - \tilde{\varphi}_t|^2 \right| \, \|_2 + \| \, |V| * \left| \mathrm{D}^\beta 2\mathrm{Re} \tilde{\varphi}_t^* (\varphi_t - \tilde{\varphi}_t) \right| \, \|_2 \, .$$

Both terms have been estimated in the proof of (B.19) above. ■

Proof of Lemma B.2.3. We use a Grönwall estimate for $\|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2^2$ and do a straight forward estimate for the appearing commutator terms with the established control over φ_t and $\tilde{\varphi}_t$ in Lemma B.2.1, Proposition B.1.1, Corollary B.1.2 and Lemma B.4.4.

Let $T \geq 0$ and $-T \leq t \leq T$.
We prove Lemma B.2.3 by induction in $|\beta|$, denoted as n.

Base Case: $|\beta| = 0$. For $|\beta| = 0$, the claim follows directly from Proposition B.1.1, noting that we assume Condition 2.1.7.

Induction Step: $|\beta| = n + 1$. Now let $n \in \mathbb{N}_0$ be fixed. Assume that the Assumptions of Lemma B.2.3 are satisfied for n+1, namely Condition 2.1.7 and Condition $2.1.8_{(n+1)+2}$.
As the induction hypothesis, suppose that Lemma B.2.3 hold for all multi-indices $|\beta| \leq n$. Since Condition 2.1.7 and Condition $2.1.8_{n+3}$ are assumed, we obtain

$$\|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Lambda^{-1/6 - |\beta|/3} \, .$$

Now, consider a multi-index $\beta \in \mathbb{N}_0^3$ with $|\beta| = n + 1$. Then, there exists a multi-index $\tilde{\beta}$ with $|\tilde{\beta}| = n$ and a coordinate direction i such that $\mathrm{D}^\beta = \mathrm{D}^{\tilde{\beta}} \partial_{x_i}$.
We use a Grönwall estimate to get the desired bound:

$$\begin{aligned}
&\partial_t \|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2^2 \\
&= 2\mathrm{Re} \left\langle \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t) \, , \, \mathrm{D}^\beta \left((-\mathrm{i}) \left[-\frac{1}{2}\Delta + V * |\varphi_t|^2 - \mu_t \right] (\varphi_t \pm \tilde{\varphi}_t) + \mathrm{i} \left(V * |\varphi_0|^2 - \mu_t \right) \tilde{\varphi}_t \right) \right\rangle \\
&= 2\mathrm{Re} \left\langle \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t) \, , \, (-\mathrm{i}) \left[\mathrm{D}^\beta \, , \, V * |\varphi_t|^2 - \mu_t \right] (\varphi_t - \tilde{\varphi}_t) \right\rangle && \text{(B.22)} \\
&\quad + 2\mathrm{Re} \left\langle \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t) \, , \, (-\mathrm{i}) \left[-\frac{1}{2}\Delta + V * |\varphi_t|^2 - \mu_t \right] \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t) \right\rangle && \text{(B.23)} \\
&\quad + 2\mathrm{Re} \left\langle \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t) \, , \, (-\mathrm{i}) \mathrm{D}^\beta V * \left(|\varphi_t|^2 - |\varphi_0|^2 \right) \tilde{\varphi}_t \right\rangle && \text{(B.24)} \\
&\quad + 2\mathrm{Re} \left\langle \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t) \, , \, (-\mathrm{i}) \mathrm{D}^\beta \left(-\frac{1}{2}\Delta \right) \tilde{\varphi}_t \right\rangle \, , && \text{(B.25)}
\end{aligned}$$

where we used Definition 2.1.1: $\mu_t \in \mathbb{R}$. The Term (B.23) vanishes, because we only consider its real part and $\mu_t \in \mathbb{R}$. In the following, we estimate the remaining terms (B.22), (B.24) and

(B.25).

To (B.25)*:*

(B.25) can be estimated with Lemma B.2.1 and Condition 2.1.8$_{n+3=|\beta|+2}$:

$$\begin{aligned}(\mathrm{B}.25) &\leq \|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\Lambda^{\frac{1}{2}-\frac{|\beta|+2}{3}} \\ &\leq \|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\Lambda^{-\frac{1}{6}-\frac{|\beta|}{3}}\,. \end{aligned} \tag{B.26}$$

To (B.24)*:*

We now use Lemma B.2.1 and in the last step below the induction hypothesis, which provides that Lemma B.4.4 holds for all $\beta-\gamma$ with $|\beta-\gamma| \leq |\beta|-1$:

$$\begin{aligned}\|\mathrm{D}^\beta V * \left(|\varphi_t|^2 - |\varphi_0|^2\right)\tilde{\varphi}_t\|_2 &\leq \sum_{\gamma=0}^{\beta} C\|V * \left[\mathrm{D}^{\beta-\gamma}\left(|\varphi_t|^2 - |\varphi_0|^2\right)\right]\mathrm{D}^\gamma\tilde{\varphi}_t\|_2 \\ &\leq \sum_{\gamma=0}^{\beta} C\|V * \left[\mathrm{D}^{\beta-\gamma}\left(|\varphi_t|^2 - |\varphi_0|^2\right)\right]\|_2\|\mathrm{D}^\gamma\tilde{\varphi}_t\|_\infty \\ &\leq \sum_{\gamma=0}^{\beta} C\left(\Lambda^{-\frac{1}{6}-\frac{|\beta-\gamma|}{3}} + \|\mathrm{D}^{\beta-\gamma}(\varphi_t - \tilde{\varphi}_t)\|_2\right)\Lambda^{-\frac{|\gamma|}{3}} \\ &\leq C\Lambda^{-\frac{1}{6}-\frac{|\beta|}{3}} + C\|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2\,. \end{aligned}$$

It follows that

$$(\mathrm{B}.24) \leq \|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\left(\Lambda^{-\frac{1}{6}-\frac{|\beta|}{3}} + \|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2\right)\,. \tag{B.27}$$

To (B.22)*:*

Due to the commutator in (B.22) the last term in the product formula below shortens and with the help of Lemma B.4.4 and the induction hypothesis we conclude

$$\begin{aligned}&\|\left[\mathrm{D}^\beta\,,\, V * |\varphi_t|^2 - \mu_t\right](\varphi_t - \tilde{\varphi}_t)\|_2 \\ &= \|\mathrm{D}^\beta V * (|\varphi_t|^2 - \mu_t)(\varphi_t - \tilde{\varphi}_t) - V * (|\varphi_t|^2 - \mu_t)\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \\ &\quad\leq \sum_{\gamma=0}^{\tilde{\beta}} C\|V * \mathrm{D}^{\beta-\gamma}|\varphi_t|^2\|_\infty\|\mathrm{D}^\gamma(\varphi_t - \tilde{\varphi}_t)\|_2 \\ &\quad\leq \sum_{\gamma=0}^{\tilde{\beta}} C\left(\Lambda^{-\frac{|\beta-\gamma|}{3}} + C\|\mathrm{D}^{\beta-\gamma}(\varphi_t - \tilde{\varphi}_t)\|_2\right)\Lambda^{-\frac{1}{6}-\frac{|\gamma|}{3}} \\ &\quad\leq C\Lambda^{-\frac{1}{6}-\frac{|\beta|}{3}} + C\Lambda^{-\frac{1}{6}}\|\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2\,. \end{aligned}$$

It follows that

$$(B.22) \leq \|D^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2 C \left(\Lambda^{-\frac{1}{6}-\frac{|\beta|}{3}} + \Lambda^{-\frac{1}{6}}\|D^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2\right) . \tag{B.28}$$

To summarize

$$\begin{aligned}\partial_t\|D^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2^2 &\leq (B.22) + (B.24) + (B.25)\\ &\leq (B.28) + (B.27) + (B.26)\\ &\leq \|D^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2 C \left(\Lambda^{-\frac{1}{6}-\frac{|\beta|}{3}} + \|D^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2\right) ,\end{aligned}$$

with Grönwall and $\tilde{\varphi}_0 = \varphi_0$ we get the claim $\|D^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Lambda^{-\frac{1}{6}-\frac{|\beta|}{3}}$ for $|\beta| = n+1$, thereby completing the induction step. ∎

B.4.5 Proof of Lemma B.3.7

The proof uses the following technical Lemma B.4.5. In part a) we give estimates on the convolution of the potential V with the condensate φ_t, which are simple conclusions from Lemma B.4.4. In part b) we do an estimate on the same quantity but localized by Θ_Λ.

Lemma B.4.5.

a) *Let $\beta \in \mathbb{N}_0^3$. We assume that the condensate φ_0 satisfies Condition 2.1.7 and Condition 2.1.8$_{|\beta|+2}$.*
Then $\forall T \geq 0$ $\exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\| \, |V| * D^{\beta}|\varphi_t|^2 \, \|_\infty \leq C\Lambda^{-|\beta|/3} , \tag{B.29}$$

$$\| \, |V| * \left| D^{\beta}\left(|\varphi_t|^2 - |\varphi_0|^2\right)\right| \, \|_2 \leq C\Lambda^{-|\beta|/3-1/6} . \tag{B.30}$$

b) *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}_+$ and $s > 0$. Assume that the condensate φ_0 satisfies Condition 2.1.8$_k$. If $\forall \beta, \tilde{\beta} \in \mathbb{N}_0^3$ with $0 \leq |\beta| \leq k-1$, $0 \leq |\tilde{\beta}| \leq k+1$ and $\forall T \geq 0$ $\exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$*

$$\|\Theta_\Lambda D^{\beta}(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\left\{\Lambda^{-\frac{1}{6}-\frac{k}{3}-(k-|\beta|)s} + \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\right\} \tag{B.31}$$

$$\|\Theta_\Lambda D^{\tilde{\beta}}(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\|D^{\tilde{\beta}}(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Lambda^{-1/6-|\tilde{\beta}|/3} .[3] \tag{B.32}$$

Then for $\forall \beta \in \mathbb{N}_0^3$ with $0 \leq |\beta| \leq k$, $\forall T \geq 0$ $\exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\|\Theta_\Lambda V * D^{\beta}\left(|\varphi_t|^2 - |\varphi_0|^2\right)\|_2 \leq C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s}$$

[3] For $k = 0$ the only condition is $\|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Lambda^{-1/6}$.

$$+ \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\}$$
$$+ C\|\Theta_\Lambda D^\beta(\varphi_t - \tilde{\varphi}_t)\|_2\,. \tag{B.33}$$

Proof of Lemma B.4.5. Let $T \geq 0$ and $-T \leq t \leq T$.
Part a):
Follows directly from Lemma B.4.4 and Lemma B.2.3.
Part b):
We use Lemma A.0.6b) and $|\varphi_t|^2 - |\varphi_0|^2 = |\varphi_t - \tilde{\varphi}_t|^2 + 2\mathrm{Re}\tilde{\varphi}_t^*(\varphi_t - \tilde{\varphi}_t)$, $\mathrm{D}^\beta(|\varphi_t|^2 - |\varphi_0|^2) = \sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\big((\mathrm{D}^{\beta-\gamma}(\varphi_t - \tilde{\varphi}_t)^*)\mathrm{D}^\gamma(\varphi_t - \tilde{\varphi}_t) + 2\mathrm{Re}(\mathrm{D}^{\beta-\gamma}\tilde{\varphi}_t^*)\mathrm{D}^\gamma(\varphi_t - \tilde{\varphi}_t)\big)$ to conclude $\forall m \in \mathbb{N}_+$

$$\begin{aligned}
&\|\Theta_\Lambda V * \mathrm{D}^\beta\left(|\varphi_t|^2 - |\varphi_0|^2\right)\|_2 \\
&\leq C\|\Theta_\Lambda \mathrm{D}^\beta\left(|\varphi_t|^2 - |\varphi_0|^2\right)\|_{1\wedge 2} + C\Lambda^{-sm}\|\mathrm{D}^\beta\left(|\varphi_t|^2 - |\varphi_0|^2\right)\|_{1\wedge 2} \\
&\leq C\sum_{\gamma=0}^{\beta}\Big(\|\mathrm{D}^{\beta-\gamma}(\varphi_t - \tilde{\varphi}_t)\|_2\|\Theta_\Lambda \mathrm{D}^\gamma(\varphi_t - \tilde{\varphi}_t)\|_2 + \|\mathrm{D}^{\beta-\gamma}\tilde{\varphi}_t\|_\infty\|\Theta_\Lambda \mathrm{D}^\gamma(\varphi_t - \tilde{\varphi}_t)\|_2 \\
&\quad + \Lambda^{-sm}\|\mathrm{D}^{\beta-\gamma}(\varphi_t - \tilde{\varphi}_t)\|_2\|\mathrm{D}^\gamma(\varphi_t - \tilde{\varphi}_t)\|_2 + \Lambda^{-sm}\|\mathrm{D}^{\beta-\gamma}\tilde{\varphi}_t\|_\infty\|\mathrm{D}^\gamma(\varphi_t - \tilde{\varphi}_t)\|_2\Big)\,.
\end{aligned} \tag{B.34}$$

The case $k = 0$ now follows directly from (B.34), since in this case, we have that $\beta = 0$. It remains to show the case $k \geq 1$. We used the conditions (B.31) and (B.32)

$$\begin{aligned}
(\text{B.34}) \leq &\sum_{\gamma=0,|\gamma|<|\beta|}^{\beta}\Lambda^{-|\beta-\gamma|/3}\cdot C\Big\{\Lambda^{-\frac{1}{6}-\frac{k}{3}-(k-|\gamma|)s} + \Lambda^{-\frac{1}{2}s-\frac{|\gamma|}{3}-2n(1/3-s)}\Big\} \\
&+ C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \\
&+ C\Lambda^{-sm}\Lambda^{-|\beta|/3-2/6} + C\Lambda^{-sm}\Lambda^{-|\beta|/3-1/6}
\end{aligned} \tag{B.35}$$

and now that $0 \leq |\gamma| \leq |\beta| - 1$, $\Lambda \geq 1$ as well as that m can be chosen large enough such that we can absorb both last terms into $\Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}$

$$\begin{aligned}
(\text{B.35}) \leq C&\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k-(|\beta|-1))s} + \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\} \\
&+ C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2\,,
\end{aligned}$$

which completes the proof. ■

We now start with the proof of Lemma B.3.7.

Proof of Lemma B.3.7. We use a Grönwall estimate to control $\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2^2$. Most of the required estimates are straightforward, except for the term coming from the commutator $[\Theta_\Lambda, \Delta]$, namely $\|(\nabla\Theta_\Lambda)\nabla \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2$. This term includes a $|\beta| + 1$ derivative, which is not controlled by our first condition (B.8) if $|\beta| = k$.

To control it, we need an additional "external" condition, namely (B.9). With this condition, we can prove our desired estimate (B.10) for $|\beta| = k$. The claim (B.10) for $0 \leq |\beta| \leq k-1$ then follow in a cascading manner from $(\text{B.10})_{|\beta|+1}$ starting with $|\beta| = k-1$ and working downward. This process can be visualized as follows:

$$(\text{B.10})_{|\beta|=k} \to (\text{B.10})_{|\beta|=k-1} \to \cdots \to (\text{B.10})_{|\beta|=0} . \tag{B.36}$$

The case $|\beta| = 0$ in (B.10) is of particular importance, as it is the only one relevant in the proof of Corollary 5.3.2 and therefore gives the desired bound on $\|\Theta_\Lambda W_\Lambda\|_\infty$. Here we also see why we need the higher-orders derivatives of $\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2$: they improve our estimate for the $|\beta| = 0$ case, as seen in (B.36). The more derivatives we control, the better our final $|\beta| = 0$ estimate becomes.

Finally, it is important to emphasize that the process shown in (B.36) unfolds step by step. We first have to prove the base case for $k = 0$, then use this result to derive $(\text{B.10})_{k=1}$, and so forth. Each step builds upon the previous one, with the result of step $k-1$ serving as a condition for step k (see (B.8)).

Let $T \geq 0$ and $-T \leq t \leq T$. We remark for $\beta \in \mathbb{N}_0^3$

$$\begin{aligned}
&2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\,,\,(-\mathrm{i})[\Theta_\Lambda\,,\,-\frac{1}{2}\Delta]\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\right\rangle \\
&= 2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\,,\,(-\mathrm{i})\left(-\frac{1}{2}\right)[(\nabla\Theta_\Lambda)\nabla + \nabla(\nabla\Theta_\Lambda)]\,\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\right\rangle \\
&= -2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\,,\,(-\mathrm{i})(\nabla\Theta_\Lambda)\nabla\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\right\rangle ,
\end{aligned} \tag{B.37}$$

where the last step follows from

$$\begin{aligned}
&2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\,,\,(-\mathrm{i})\nabla(\nabla\Theta_\Lambda)\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\right\rangle \\
&= 2\mathrm{Re}\left\langle (-\mathrm{i})\nabla\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\,,\,(\nabla\Theta_\Lambda)\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\right\rangle \\
&= 2\mathrm{Re}\left\langle (-\mathrm{i})\Theta_\Lambda\nabla \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\,,\,(\nabla\Theta_\Lambda)\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\right\rangle \\
&= 2\mathrm{Re}\left\langle (-\mathrm{i})(\nabla\Theta_\Lambda)\nabla \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\,,\,\Theta_\Lambda\mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\right\rangle .
\end{aligned}$$

Here in step 2 we used that we consider only the real part and in the last step, that Θ_Λ is a real function.

Now we start a Grönwall argument to prove the Lemma. Note that $\mu_t \in \mathbb{R}$ (see Definition 2.1.1). We get

$$\begin{aligned}
&\partial_t\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2^2 \\
&= 2\mathrm{Re}\Big\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\,,\,\Theta_\Lambda \mathrm{D}^\beta\Big[(-\mathrm{i})\Big(-\frac{1}{2}\Delta + V * |\varphi_t|^2 - \mu_t\Big)(\varphi_t \pm \tilde{\varphi}_t)
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{i}\left(V*|\varphi_0|^2-\mu_t\right)\tilde{\varphi}_t\Big]\Big\rangle \\
=&\,2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\,,\,(-\mathrm{i})\left[\Theta_\Lambda \mathrm{D}^\beta\,,\,-\frac{1}{2}\Delta+V*|\varphi_t|^2-\mu_t\right](\varphi_t-\tilde{\varphi}_t)\right\rangle \\
&+2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\,,\,(-\mathrm{i})\left(-\frac{1}{2}\Delta+V*|\varphi_t|^2-\mu_t\right)\Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\right\rangle \\
&+2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\,,\,(-\mathrm{i})\Theta_\Lambda \mathrm{D}^\beta\left(-\frac{1}{2}\Delta\right)\tilde{\varphi}_t\right\rangle \\
&+2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\,,\,(-\mathrm{i})\Theta_\Lambda \mathrm{D}^\beta V*\left(|\varphi_t|^2-|\varphi_0|^2\right)\tilde{\varphi}_t\right\rangle\,,
\end{aligned}
$$

due to the real part the second term on the right-hand side is zero. Form here it follows with (B.37), that

$$
\begin{aligned}
&\partial_t\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\|_2^2 \\
=&\,-2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\,,\,(-\mathrm{i})(\nabla\Theta_\Lambda)\nabla \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\right\rangle && \text{(B.38)}\\
&+2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\,,\,(-\mathrm{i})\Theta_\Lambda \mathrm{D}^\beta\left(-\frac{1}{2}\Delta\right)\tilde{\varphi}_t\right\rangle && \text{(B.39)}\\
&+2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\,,\,(-\mathrm{i})\Theta_\Lambda \mathrm{D}^\beta V*\left(|\varphi_t|^2-|\varphi_0|^2\right)\tilde{\varphi}_t\right\rangle && \text{(B.40)}\\
&+2\mathrm{Re}\left\langle \Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\,,\,(-\mathrm{i})\Theta_\Lambda\left[\mathrm{D}^\beta\,,\,V*|\varphi_t|^2-\mu_t\right](\varphi_t-\tilde{\varphi}_t)\right\rangle\,. && \text{(B.41)}
\end{aligned}
$$

Let $k\in\mathbb{N}_0$.
Case $k=0$*:*
Note that from $(\text{B.9})_{k=|\beta|=0}$ follows $(\text{B.8})_{k=|\beta|=0}$. So we can just use $(\text{B.8})_{k=|\beta|=0}$ and follow the proof from below in the case $k\geq 1$.
Case $k\geq 1$*:*
Assume the condition of the Lemma for k. Let $\beta\in\mathbb{N}_0^3$ with $0\leq|\beta|\leq k$. We will now estimate $\partial_t\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\|_2^2$ by estimating the individual terms (B.38), (B.39), (B.40) and (B.41).
To (B.41)*:*
Case $\beta=0$*:*
$(\text{B.41})_{\beta=0}$=0.
Case $|\beta|\geq 1$*:*
We use Lemma B.4.5a and the condition (B.8), $|\gamma|\leq|\beta|-1\leq k-1$ to conclude

$$
\begin{aligned}
&\|\Theta_\Lambda\left[\mathrm{D}^\beta\,,\,V*|\varphi_t|^2-\mu_t\right](\varphi_t-\tilde{\varphi}_t)\|_2 \\
=&\,\|\Theta_\Lambda\left(\mathrm{D}^\beta V*|\varphi_t|^2(\varphi_t-\tilde{\varphi}_t)-V*|\varphi_t|^2\mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\right)\|_2 \\
\leq&\sum_{\gamma=0,\,|\gamma|\leq|\beta|-1}^{\beta} C\|V*D^{\beta-\gamma}|\varphi_t|^2\|_\infty\|\Theta_\Lambda \mathrm{D}^\gamma(\varphi_t-\tilde{\varphi}_t)\|_2
\end{aligned}
$$

$$\leq \sum_{\gamma=0,\,|\gamma|\leq|\beta|-1}^{\beta} C\Lambda^{-\frac{|\beta|-|\gamma|}{3}}\Big\{\Lambda^{-\frac{1}{6}-\frac{k}{3}-(k-|\gamma|)s}+\Lambda^{-\frac{1}{2}s-\frac{|\gamma|}{3}-2n(1/3-s)}\Big\}$$
$$\leq \sum_{\gamma=0,\,|\gamma|\leq|\beta|-1}^{\beta} C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k-(|\beta|-1))s}+\Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\},$$

where in the last step we used in the first term $|\gamma|\leq|\beta|-1$ and $\Lambda\geq 1$. This and Cauchy-Schwarz give us the bound for (B.41) for all $0\leq|\beta|\leq k$:

$$\begin{aligned}(\mathrm{B.41})_{\beta}\leq\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\|_2 C\Big\{&\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s}\\&+\Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\}.\end{aligned}\tag{B.42}$$

To (B.40)*:*
We use Lemma B.2.1, $|\beta|\leq k$ and Lemma B.4.5 a) and b), which hold due to conditions (B.8), (B.9), $|\gamma|\leq k\leq k+2$, to conclude

$$\begin{aligned}&\|\Theta_\Lambda \mathrm{D}^\beta V*\left(|\varphi_t|^2-|\varphi_0|^2\right)\tilde{\varphi}_t\|_2\\
\leq&\sum_{\gamma=0}^{\beta}C\|\Theta_\Lambda\left(\mathrm{D}^\gamma V*\left(|\varphi_t|^2-|\varphi_0|^2\right)\right)D^{\beta-\gamma}\tilde{\varphi}_t\|_2\\
\leq&\sum_{\gamma=0}^{\beta}C\Lambda^{-\frac{|\beta-\gamma|}{3}}\|\Theta_\Lambda V*\mathrm{D}^\gamma\left(|\varphi_t|^2-|\varphi_0|^2\right)\|_2\\
\leq&\sum_{\gamma=0}^{\beta}C_n(t)\Lambda^{-\frac{|\beta-\gamma|}{3}}\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\gamma|)s}\\
&+\Lambda^{-\frac{1}{2}s-\frac{|\gamma|}{3}-2n(1/3-s)}+\|\Theta_\Lambda \mathrm{D}^\gamma(\varphi_t-\tilde{\varphi}_t)\|_2\Big\}\end{aligned}\tag{B.43}$$

Then with the condition (B.8) for $|\gamma|\leq|\beta|-1\leq k-1$,

$$\begin{aligned}(\mathrm{B.43})\leq&\sum_{\gamma=0}^{\beta}C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s}+\Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\}\\
&+\sum_{\gamma=0,|\gamma|<|\beta|}^{\beta}\Lambda^{-\frac{|\beta-\gamma|}{3}}C\Big\{\Lambda^{-\frac{1}{6}-\frac{k}{3}-(k-|\gamma|)s}+\Lambda^{-\frac{1}{2}s-\frac{|\gamma|}{3}-2n(1/3-s)}\Big\}\\
&+C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t-\tilde{\varphi}_t)\|_2\end{aligned}\tag{B.44}$$

and for the third term we use that $|\gamma|\geq 0$, $\Lambda\geq 1$, $-|\beta-\gamma|/3\leq-1/3$ and $|\gamma|s\leq(|\beta|-1)s$ to conclude

$$(\mathrm{B.44})\leq\sum_{\gamma=0}^{\beta}C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s}+\Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\}$$

$$+ C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \,.$$

Therefore for all $0 \leq |\beta| \leq k$

$$\begin{aligned}(\text{B.40})_\beta \leq \|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\Big\{&\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s} \\ &+ C\Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\} + C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2^2 \,.\end{aligned} \tag{B.45}$$

To (B.39)*:*
It follows directly from Lemma B.3.3ii) that $\forall 0 \leq |\beta| \leq k$

$$\begin{aligned}(\text{B.39}) &\leq \|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\Lambda^{-\frac{|\beta|+2}{3}+\frac{3}{2}s}\Lambda^{-2(n-1)(1/3-s)} \\ &= \|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}}\Lambda^{-2n(1/3-s)} \,.^4\end{aligned} \tag{B.46}$$

To (B.38)*:*
The desired estimates for $(\text{B.38})_\beta$ build on each other. Therefore we start with the estimation of $(\text{B.38})_{|\beta|=k}$ here we use (B.9) to get an estimate:

$$\begin{aligned}(\text{B.38})_{|\beta|=k} &\leq C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \|(\nabla\Theta_\Lambda)\nabla \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \\ &\leq C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \Lambda^{-s}\|\Theta_\Lambda\nabla \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \\ &\leq C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \Lambda^{-s}\Lambda^{-\frac{1}{6}-\frac{k+1}{3}}\end{aligned} \tag{B.47}$$

where in step 1 we used the definition of Θ_Λ and in the second condition (B.9) and $|\beta|+1 = k+1$. For $|\beta| = k$ our gathered information leads to

$$\begin{aligned}\partial_t\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2^2 \leq\ & ((\text{B.38}) + (\text{B.39}) + (\text{B.40}) + (\text{B.41}))_{|\beta|=k} \\ \leq\ & ((\text{B.47}) + (\text{B.46}) + (\text{B.45}) + (\text{B.42}))_{|\beta|=k} \\ \leq\ & \|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s} \\ & + \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\} \\ & + C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2^2 \,,\end{aligned}$$

with Grönwall and $\tilde{\varphi}_0 = \varphi_0$ we conclude for $|\beta| = k$

$$\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-s} + \Lambda^{-\frac{1}{2}s-\frac{k}{3}-2n(1/3-s)}\Big\} \,. \tag{B.48}$$

We have proven (B.10) for $|\beta| = k$. With this we have improved our estimate for $\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2$ from (B.9) to (B.10). This knowledge is now used to improve the cases $0 \leq \beta \leq k-1$.

[4] This estimate is the reason why we need Condition $2.1.11_{2(n-1),s}$, Condition $2.1.8\text{i})_{(k+2)+2(n-1)}$ and Condition $2.1.8\text{ii})_{k+2}$.

As we will show below all estimates $(\text{B.8})_{|\beta|}$ are improved one after another by the one which is one order higher in $|\beta|$, as mentioned at the beginning of the proof.

Now let $1 \leq m \leq k$ and (B.10) be true for all $m \leq |\beta| \leq k$. We will show that (B.10) also holds for $|\beta| = m-1$, which proves the Lemma.
It remains to estimate $(\text{B.38})_{|\beta|=m-1}$:
With the help of (B.10) which holds for $|\beta|+1 = m$

$$\begin{aligned}(\text{B.38})_{|\beta|=m-1} &\leq C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \Lambda^{-s} \|\Theta_\Lambda \nabla \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \\ &\leq \|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\Lambda^{-s}\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-(|\beta|+1))s} \\ &\quad + \Lambda^{-\frac{1}{2}s-\frac{|\beta|+1}{3}-2n(1/3-s)}\Big\} \\ &\leq \|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s} \\ &\quad + \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\}, \end{aligned} \tag{B.49}$$

where in the second term of the last step we used that $s \geq 0$, $\Lambda \geq 1$. This leads us to

$$\begin{aligned}\partial_t \|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2^2 &\leq ((\text{B.38}) + (\text{B.39}) + (\text{B.40}) + (\text{B.41}))_{|\beta|=m-1} \\ &\leq ((\text{B.49}) + (\text{B.46}) + (\text{B.45}) + (\text{B.42}))_{|\beta|=m-1} \\ &\leq \|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s} \\ &\quad + \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\} \\ &\quad + C\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2^2, \end{aligned}$$

from here we conclude with Grönwall for $|\beta| = m-1$

$$\|\Theta_\Lambda \mathrm{D}^\beta(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Big\{\Lambda^{-\frac{1}{6}-\frac{k+1}{3}-(k+1-|\beta|)s} + \Lambda^{-\frac{1}{2}s-\frac{|\beta|}{3}-2n(1/3-s)}\Big\},$$

as mentioned above, this proves the Lemma. ■

Appendix C

Properties of the Bogoliubov Hamiltonian

In the following Lemma C.0.1 we analyze the properties of K_1, K_2 and h_t, which determine the Bogoliubov Hamiltonian. These properties are essential for the proof of existence of dynamics, estimates involving H^{Bog} or H^{BF} or for proving that the propagator of the Bogoliubov dynamics is a Bogoliubov transformation.

The definition of K_1 and K_2 can be found in Definition 2.3.1 and the definition of h_t in Definition 2.1.1.

Lemma C.0.1 (Properties of K_1, K_2, h_t)**.** *For all volumes $\Lambda \geq 1$ let φ_t be the condensate with initial data φ_0, which varies on the scale $\Lambda^{1/3}$, meaning it satisfies Condition 2.1.7.*

a) (Theorem D.2.8 conditions) Then $K_1 \in C^1(\mathbb{R}, HS(L^2))$, $K_2 \in C^1(\mathbb{R}, HS((L^2)^*, L^2))$, $K_1(t)$ self-adjoint and $K_2(t)^* = JK_2(t)J$. In addition $\forall T \geq 0$ $\exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$ and $i \in \{1, 2\}$ we have

$$\|K_i(t)\| \leq C\Lambda^{1/2}\,, \quad \|K_i(t)\|_{\mathrm{op}} \leq C \tag{C.1}$$

and if we assume in addition Condition 2.1.8$_{k=2+2}$ for the condensate φ_0 then $\forall T \geq 0$ $\exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$ and $i \in \{1, 2\}$ we have

$$\|\partial_t K_i(t)\|_{\mathrm{HS}} \leq C\Lambda^{1/2}\,, \quad \|\partial_t K_i(t)\|_{\mathrm{op}} \leq C\,. \tag{C.2}$$

b) Then we have the identities $(\tilde{K}_2(t)J)_{mn} = \langle u_m \otimes u_n\,,\, V(x-y)\varphi_t \otimes \varphi_t\rangle$ and $(\tilde{K}_1(t))_{mn} = \langle u_m \otimes \varphi_t\,,\, V(x-y)\varphi_t \otimes u_n\rangle$.

c) Then h_t is self-adjoint and $D(h_t) = H^2(\mathbb{R}^3)$ and therefore $h_t + K_1(t)$ self-adjoint and

$D(h_t + K_1(t)) = H^2(\mathbb{R}^3)$.

d) Then $\mathcal{K} := \sum_{m,n\geq 0}((K_2(t)J)_{mn}a_m^* a_n^* + h.c.)$ *is strongly convergent on* $D(\mathcal{N})$ *and in addition* $(\mathcal{K}, D(\mathcal{N}))$ *essentially self-adjoint on every core of* $\mathcal{N}$ *with*

$$\|\mathcal{K}\psi\| \leq 2\|K_2(t)\|_{\mathrm{HS}}\|(\mathcal{N}+2)\psi\|\,, \quad \forall x \in D(\mathcal{N})\,. \tag{C.3}$$

e) (Lemma D.2.6 conditions) Set $h(t) = h_1 + h_2(t)$, $h_1 := -\Delta/2$, $h_2(t) := V * |\varphi_t|^2 - \mu_t + K_1(t)$ *and*

$$A_0 := \begin{pmatrix} h_1 & 0 \\ 0 & -Jh_1J^* \end{pmatrix}, \tag{C.4}$$

$$A_1(t) := \begin{pmatrix} h_2(t) & -K_2(t) \\ K_2(t)^* & -Jh_2(t)J^* \end{pmatrix}. \tag{C.5}$$

Then $A_1 := (t \mapsto A_1(t)) \in C(\mathbb{R}, \mathcal{L}(D(A_0)) \cap C(\mathbb{R}, \mathcal{L}(L^2 \oplus JL^2))$*. If we assume in addition Condition 2.1.8*$_{k=2+2}$ *for the condensate* φ_0 *then* $\forall T \geq 0$ $\exists C > 0$ *such that* $\forall \Lambda \geq 1$ *and* $-T \leq t \leq T$

$$\|A_1(t)\|_{\mathcal{L}(L^2 \oplus JL^2)} + \|A_1(t)\|_{\mathcal{L}(D(A_0))} \leq C\,. \tag{C.6}$$

The proof of Lemma C.0.1 can be found in the Appendix C.1.1.

C.1 Proofs of Appendix C

C.1.1 Proof of Lemma C.0.1

Proof of Lemma C.0.1. Let $T \geq 0$ and $-T \leq t \leq T$ then $\exists C > 0$ such that $\forall \Lambda \geq 1$: $\|\varphi_t\|_\infty \leq C$ due to Corollary B.1.2.

Proof of a): We start by proving $\tilde{K}_2(t)^* = J\tilde{K}_2(t)J$, $\tilde{K}_1(t)$ self-adjoint, $\tilde{K}_i(t) \in \mathrm{HS}$, along with the necessary bounds $\tilde{K}_i(t)$. Next, we analyze their derivatives. Finally, we use $Q_t \in \mathcal{L}(L^2)$ to transfer all properties from $\tilde{K}_i(t)$ to $K_i(t)$, thereby completing the proof of part (a).

We start by proving $J\tilde{K}_2(t)J = \tilde{K}_2(t)^*$ and $\tilde{K}_1(t)$ self-adjoint.

Proof. Let $\psi \in L^2, J\phi \in (L^2)^*$. We show $\left\langle \tilde{K}_2(t)J\phi\,,\,\psi \right\rangle = \left\langle J\phi\,,\,J\tilde{K}_2J\psi \right\rangle$ and therefore $J\tilde{K}_2J \subset \tilde{K}_2^*$ and hence $J\tilde{K}_2J = \tilde{K}_2^*$, since $D(J\tilde{K}_2J) = L^2$. A similar argument does the job for $\tilde{K}_1(t)$ self-adjoint. So we use part (b) of the Lemma and $V(x-y) = V(y-x)$ to get

$$\left\langle J\phi\,,\,J\tilde{K}_2J\psi \right\rangle = \left\langle \tilde{K}_2J\psi\,,\,\phi \right\rangle = \langle \phi \otimes \psi\,,\,V(x-y)\varphi_t \otimes \varphi_t \rangle^*$$

$$= \langle \psi \otimes \phi \,,\, V(x-y)\varphi_t \otimes \varphi_t \rangle^* = \left\langle \tilde{K}_2 J\phi \,,\, \psi \right\rangle ,$$

$$\left\langle \phi \,,\, \tilde{K}_1(t)\psi \right\rangle = \int dx \phi^*(x) \int dy \varphi_t(x) V(x-y) \varphi_t^*(y)\psi(y)$$

$$= \int dx \int dy \Big(\varphi_t^*(x) V(y-x) \varphi_t(y) \phi(x) \Big)^* \psi(y)$$

$$= \left\langle \tilde{K}_1 \phi \,,\, \psi \right\rangle ,$$

which proves $J\tilde{K}_2(t)J = \tilde{K}_2(t)^*$ and $\tilde{K}_1(t)$ self-adjoint. □

Next we give a formula to calculate the Hilbert-Schmidt norm of $J\tilde{K}_2J$:

$$\sum_{l=0}^{\infty} \|\tilde{K}_2(t)Ju_l\|^2 = \sum_{l=0}^{\infty} \int \left| \int \tilde{K}_2(x,y) u_l^*(y) dy \right|^2 dx$$

$$= \int \sum_{l=0}^{\infty} \left| \left\langle u_l \,,\, \tilde{K}_2(x,\,.\,) \right\rangle \right|^2 dx = \int \|\tilde{K}_2(x,\,.\,)\|_2^2 dx$$

$$= \int \int |\tilde{K}_2(x,y)|^2 dy dx = \|\tilde{K}_2(\,.\,,\,.\,)\|_2^2 \,, \tag{C.1}$$

where $\tilde{K}_2(\,.\,,\,.\,)$ is the integral kernel of $\tilde{K}_2$.
We show $\tilde{K}_2(t) \in \mathrm{HS}((L^2)^*, L^2)$, $\tilde{K}_1(t) \in \mathrm{HS}(L^2)$ and $\|\tilde{K}_i(t)\|_{\mathrm{HS}} \leq C\Lambda^{1/2}$.

Proof. $\forall T \geq 0 \; \exists C > 0$ such that $\forall \Lambda \geq 1$ and $-T \leq t \leq T$

$$\int \int |\tilde{K}_i(t)(x,y)|^2 dy dx = \int \int |\tilde{K}_2(t)(x,y)|^2 dy dx$$

$$= \int \int |\varphi_t|^2(x) |\varphi_t|^2(y) |V(x-y)|^2 dy dx$$

$$\leq \|\varphi_t\|_2^2 \|V\|_2^2 \|\varphi_t\|_\infty^2 = C\|V\|_2^2 \Lambda < \infty \,,$$

where we used Condition 2.1.7 and Corollary B.1.2. With (C.1) we have $K_2(t)$ everywhere defined and for $t \in \mathbb{R}^3$ exists a $T \geq 0$ such that $t \in [-T, T]$ and $\sum_{l=0}^{\infty} \|J\tilde{K}_2(t)Ju_l\|^2 < \infty$ we conclude that $\tilde{K}_2(t) \in \mathrm{HS}((L^2)^*, L^2)$. An analogous argument holds for $\tilde{K}_1(t)$. □

Now we show $\|\tilde{K}_i\|_{\mathrm{op}} \leq C$.

Proof. We estimate

$$\|\tilde{K}_2(t)J\psi\|_2 = \|\varphi_t V * (\varphi_t \psi^*)\|_2 \leq \|\varphi_t\|_\infty \|V\|_2 \|\varphi_t\|_\infty \|\psi\|_2$$

and conclude with Corollary B.1.2 that $\|\tilde{K}_2(t)\|_{\mathrm{op}} \leq \|\varphi_t\|_\infty^2 \|V\|_2 \leq C$ for $t \in [-T, T]$. With the same argument and $\tilde{K}_1(t)\psi = \varphi_t \cdot V * (\varphi_t^* \psi)$ we estimate

$$\|\tilde{K}_1(t)\|_{\text{op}} \le C. \qquad \square$$

We now continue with showing continuity and differentiability in time and also prove the bounds on the derivatives. With a similar calculation to (C.1) we are able to show $\tilde{K}_i \in C^1(\mathbb{R}, \text{HS})$ if the kernel $\tilde{K}_i(\,.\,,\,.\,) := (t \mapsto ((y_1, y_2) \mapsto \tilde{K}_i(t)(y_1, y_2))) \in C^1(\mathbb{R}, L^2)$. In this case $[\partial_t \tilde{K}_i(\,.\,)](y_1) = \int \partial_t \tilde{K}_i(t)(y_1, y_2)(\,.\,)(y_2)dy_2$ and $\|\partial_t \tilde{K}_i(t)\|_{\text{HS}} = \left\|\partial_t \tilde{K}_i(t)(\,.\,,\,.\,)\right\|_{L^2}$.[1]

Proof. We start by proving $\tilde{K}_i(\,.\,,\,.\,) \in C^1(\mathbb{R}, L^2)$:

$$\begin{aligned}
&\Big\| \frac{\tilde{K}_1(t+h)(\,.\,,\,.\,) - \tilde{K}_1(t)(\,.\,,\,.\,)}{h} \\
&\quad - \big((x,y) \mapsto \partial_t \varphi_t(x) V(x-y) \varphi_t(y) + \varphi_t(x) V(x-y) \partial_t \varphi_t(y)\big) \Big\|_2 \\
=& \Big(\int \Big| \frac{\varphi_{t+h}(x) V(x-y) \varphi_{t+h}(y) - \varphi_t(x) V(x-y) \varphi_t(y)}{h} \\
&\quad - \partial_t \varphi_t(x) V(x-y) \varphi_t(y) - \varphi_t(x) V(x-y) \partial_t \varphi_t(y) \Big| dxdy \Big)^{1/2} \\
\le& \left\| \frac{\varphi_{t+h} - \varphi_t}{h} - \partial_t \varphi_t \right\|_2 \|V\|_\infty \|\varphi_{t+h}\|_2 + \|\varphi_t\|_2 \|V\|_\infty \left\| \frac{\varphi_{t+h} - \varphi_t}{h} - \partial_t \varphi_t \right\|_2 \to 0\,.
\end{aligned}$$

Here we used in the last step that $\varphi \in C^1(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}))$. Analogous one can show $(t \mapsto ((x,y) \mapsto \varphi_t(x) V(x-y) \varphi_t(y))), (t \mapsto ((x,y) \mapsto \partial_t \varphi_t(x) V(x-y) \varphi_t(y) + \varphi_t(x) V(x-y) \partial_t \varphi_t(y))) \in C(\mathbb{R}, L^2(\mathbb{R}^{3+3}, \mathbb{C}))$. Note that $\varphi \in C^1(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}))$. The proof for $\tilde{K}_2$ is done analogously, since $\tilde{K}_2(t)(x,y) = \varphi_t(x) V(x-y) \varphi_t^*(y)$. $\square$

In the next step we calculate the bounds on $\|\partial_t \tilde{K}_i(t)\|_{\text{HS}}$ and $\|\partial_t \tilde{K}_i(t)\|_{\text{op}}$: $\forall T \ge 0\ \exists C > 0$ such that $\forall \Lambda \ge 1$ and $-T \le t \le T$

$$\|\partial_t \tilde{K}_i(t)\|_{\text{op}} \le C\,, \quad \|\partial_t \tilde{K}_i(t)\|_{\text{HS}} \le C \Lambda^{1/2}\,.$$

Proof. We use V even and the Hartree equation to conclude

$$\begin{aligned}
&\left\|\partial_t \tilde{K}_1(t)\right\|_{\text{HS}} = \left\|\partial_t \tilde{K}_1(t)(\,.\,,\,.\,)\right\|_{L^2} \\
=\;& \|\partial_t \varphi_t(x) V(x-y) \varphi_t(y) + \varphi_t(x) V(x-y) \partial_t \varphi_t(y)\|_2 \\
\le\;& 2 \left(\int \left|\partial_t \varphi_t(x) \varphi_t(y) V(x-y)\right|^2 dxdy \right)^{1/2} \\
=\;& 2 \left\|(-\Delta/2 \varphi_t + V * |\varphi_t|^2 \varphi_t - \mu_t \varphi_t) V * \varphi_t\right\|_2 \\
\le\;& \Big(\|\Delta \varphi_t\|_2 + 2\|V\|_1 \|\,|\varphi_t|^2\|_\infty \|\varphi_t\|_2 + |\mu_t| \cdot \|\varphi_t\|_2 \Big) \|V\|_1 \|\varphi_t\|_\infty
\end{aligned}$$

[1] The proof is done by checking the definition of pointwise differentiability with (C.1). Note that $(\tilde{K}_i(t+h)(\,.\,,\,.\,) - \tilde{K}_i(t)(\,.\,,\,.\,))/h - \partial_t \tilde{K}_i(t)(\,.\,,\,.\,)$ is again a linear operator that you can insert into (C.1).

$$\leq C\Lambda^{1/2} \,,$$

where in the last step we used Corollary B.1.2, $|\mu_t| \leq C$ and Corollary B.2.4$_{|\beta|=2}$ and thus we have to consider times $t \in [-T, T]$.
Analogous is the proof for $\partial_t \tilde{K}_2(t)(\,.\,,\,.\,)$, since $\partial_t \tilde{K}_2(t)(x,y) = \partial_t \varphi_t(x) V(x-y)\varphi_t^*(y) + \varphi_t(x)V(x-y)\partial_t\varphi_t^*(y)$.
We proceed with the $\|\partial_t \tilde{K}_i(t)\|_{\mathrm{op}}$ estimate: $\forall \psi, \phi \in L^2$

$$\begin{aligned}\left\langle \phi, \partial_t \tilde{K}_1(t)\psi \right\rangle = &\int \big(\phi^*(x)\partial_t\varphi_t(x)V(x-y)\varphi_t^*(y)\psi(y) \\ &\quad + \phi^*(x)\varphi_t(x)V(x-y)\partial_t\varphi_t^*(y)\psi(y)\big)dxdy \\ =&\langle \phi, \partial_t\varphi_t V * (\varphi_t^*\psi)\rangle + \langle \psi, \partial_t\varphi_t V * (\varphi_t^*\phi)\rangle^* \,, \end{aligned} \tag{C.2}$$

where we used that V is even. Both terms in (C.2) are estimated in the same way using the Hartree equation:

$$\begin{aligned} &|\langle \phi, \partial_t\varphi_t V * (\varphi_t^*\psi)\rangle| \\ \leq\, &\|\phi\|_2\big\{\|\Delta_x/2\varphi_t \cdot V * (\varphi_t^*\psi)\|_2 + \|V * |\varphi_t|^2 \cdot \varphi_t \cdot V * (\varphi_t^*\psi)\|_2 \\ &\quad + \|\mu_t\varphi_t V * (\varphi_t^*\psi)\|_2\big\} \\ \leq\, &\|\phi\|_2\big\{\|\Delta_x/2\varphi_t\|_2\|V * (\varphi_t^*\psi)\|_\infty \\ &\quad + \big(\|V * |\varphi_t|^2 \cdot \varphi_t\|_\infty + |\mu_t|\|\varphi_t\|_\infty\big)\|V * (\varphi_t^*\psi)\|_2\big\} \\ \leq\, &\|\phi\|_2\big\{\|\Delta_x/2\varphi_t\|_2\|V\|_2\|\varphi_t\|_\infty\|\psi\|_2 \\ &\quad + \big(\|V\|_1\||\varphi_t|^2\|_\infty\|\varphi_t\|_\infty + |\mu_t|\|\varphi_t\|_\infty\big)\|V\|_1\|\varphi_t\|_\infty\|\psi\|_2\big\} \\ \leq\, &\|\phi\|_2\|\psi\|_2\big\{C\Lambda^{1/2-2/3} + C\big\} \\ \leq\, &\|\phi\|_2\|\psi\|_2 C \,, \end{aligned} \tag{C.3}$$

where in step 4, we used the bound $|\mu_t| \leq C$ as well as Corollary B.1.2 and Corollary B.2.4, which apply under the assumption of Condition 2.1.7 and Condition 2.1.8$_{k=2+2}$. We conclude with (C.2) and (C.3) that $\|\partial_t\tilde{K}_1(t)\|_{\mathrm{op}} \leq C$. Analogous one shows $\|\partial_t\tilde{K}_2(t)\|_{\mathrm{op}} \leq C$. Note that $\partial_t\tilde{K}_2(t)J\psi = \partial_t\varphi_t \cdot V * (\varphi_t\psi^*) + \varphi_t \cdot V * (\partial_t\varphi_t\psi^*)$. □

We have proven all properties for $\tilde{K}_i$. Now the want to prove that they are also valid for K_i:

$$\begin{aligned} K_i &\in C^1(\mathbb{R}, \mathrm{HS}) \,, \\ \partial_t K_i(t) &= \dot{Q}_t\tilde{K}_i(t)Q_t + Q_t\dot{\tilde{K}}_i(t)Q_t + Q_t\tilde{K}_i(t)\dot{Q}_t \end{aligned}$$

and $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$ and $-T \leq t \leq T$

$$\|K_i(t)\|_{\text{op}} + \|\partial_t K_i(t)\|_{\text{op}} \leq C\,, \quad \|K_i(t)\|_{\text{HS}} + \|\partial_t K_i(t)\|_{\text{HS}} \leq C\Lambda^{1/2}\,.$$

Proof. $K_i \in C^1(\mathbb{R}, \text{HS})$ and $\partial_t K_i(t) = \dot{Q}_t \tilde{K}_i(t) Q_t + Q_t \dot{\tilde{K}}_i(t) Q_t + Q_t \tilde{K}_i(t) \dot{Q}_t$ follow from

$$(t \mapsto Q_t) \in C^1(\mathbb{R}, \text{HS}(L^2))\,, \quad \partial_t Q_t = -|\dot{u}_t\rangle\langle u_t| - |u_t\rangle\langle \dot{u}_t|$$

which is true due to $\varphi \in C^1(\mathbb{R}, L^2)$ and Lemma F.0.2. The bounds now follow from $\|Q_t\|_{\text{op}} \leq 1$ and $\|\partial_t Q_t\|_{\text{op}} \leq C$:

$$\begin{aligned}\|\partial_t Q_t \psi\|_2 &\leq \|\dot{u}_t\|_2 \|u_t\|_2 \|\psi\|_2 + \|u_t\|_2 \|\dot{u}_t\|_2 \|\psi\|_2 \\ &\leq \frac{2}{\Lambda^{1/2}} \left(\|\Delta_x/2 \varphi_t\|_2 + \|V * |\varphi_t|^2 \varphi_t\|_2 + |\mu_t| \|\varphi_t\|_2 \right) \|\psi\|_2 \\ &\leq C \|\psi\|_2\end{aligned}$$

similar to the above. The bounds on $K_i(t)$ are trivial. □

Proof of b):
Part (b) follows directly if we use the definition of $\tilde{K}_1(t)$ and $\tilde{K}_2(t)$. For $\tilde{K}_2(t)$ we have

$$\begin{aligned}\left\langle \phi\,, \tilde{K}_2(t) J\psi \right\rangle &= \int dx \phi^*(x) \int [\varphi_t(x)\varphi_t(y) V(x-y)](J\psi)(y) dy \\ &= \langle \phi \otimes \psi\,, V(x-y) \varphi_t \otimes \varphi_t \rangle\,.\end{aligned}$$

Proof of c):
By the Kato–Rellich theorem, the operator $-\frac{\Delta}{2} - \mu_t$ is self-adjoint on $H^2(\mathbb{R}^3)$. Similarly, $h_t = -\frac{\Delta}{2} - \mu_t + V * |\varphi_t|^2$ self-adjoint due to Kato-Rellich, since $V * |\varphi_t|^2$ is a bounded multiplication operator, $\|V * |\varphi_t|^2\|_\infty \leq \|V\|_\infty \|\varphi_t\|_2^2 < \infty$. In the same way $h_t + K_1(t)$ self-adjoint, since $K_1(t)$ bounded (see part (a)).
Proof of d):
We show $\sum_{mn} (K_2 J)_{mn} a_m^* a_n^*$ and $\sum_{mn} \overline{(K_2 J)_{mn}} a_m a_n$ strongly convergent on $D(\mathcal{N})$. To prove this we show $\sum_{mn} |(K_2 J)_{mn}|^2 < \infty$, which, by [LP22, Lemma A.1], implies the strong convergence and

$$\| \sum_{mn \geq 0} (K_2(t) J)_{mn} a_m^* a_n^* \psi \| \leq \|K_2(t)\|_{\text{HS}} \|(\mathcal{N} + 2)\psi\|\,, \qquad \forall \psi \in D(\mathcal{N})\,, \tag{C.4}$$

$$\| \sum_{mn \geq 0} \overline{(K_2(t) J)_{mn}} a_m a_n \psi \| \leq \|K_2(t)\|_{\text{HS}} \|\mathcal{N} \psi\|\,, \qquad \forall \psi \in D(\mathcal{N})\,, \tag{C.5}$$

$$\|\mathcal{K}\psi\| \leq 2\|K_2(t)\|_{\mathrm{HS}}\|(\mathcal{N}+2)\psi\|\,, \qquad \forall\psi \in D(\mathcal{N})\,. \tag{C.6}$$

So we have to show $\sum_{mn} |(K_2J)_{mn}|^2 < \infty$:

$$\begin{aligned}\sum_{mn} |(K_2J)_{mn}|^2 &= \sum_{mn} \langle u_m\,,\, K_2Ju_n\rangle\langle K_2Ju_n\,,\, u_m\rangle \\ &= \sum_{n} \langle K_2Ju_n\,,\, K_2Ju_n\rangle = \|K_2J\|^2_{\mathrm{HS}} = \|K_2\|^2_{\mathrm{HS}} < \infty\,,\end{aligned}$$

because due to part (a) $K_2 \in \mathrm{HS}$. We now prove that $(\mathcal{K}, D)$ is essentially self-adjoint for every core D of $D(\mathcal{N})$, using the commutator Theorem [RS75, Theorem X.37]. As a comparison operator we choose $(\mathcal{N}+2, D(\mathcal{N})) \geq I$ self-adjoint. It is clear that $\mathcal{K}$ is symmetric.
We show:
$\exists c \geq 0$: $\forall x \in D(\mathcal{N}^2)$

$$\|\langle \mathcal{K}x, (\mathcal{N}+2)x\rangle - \langle(\mathcal{N}+2)x, \mathcal{K}x\rangle| \leq c\|\mathcal{N}^{1/2}x\|^2\,. \tag{C.7}$$

Proof:
Let $x \in D(\mathcal{N}^2)$

$$\begin{aligned}&|\langle \mathcal{K}x, (\mathcal{N}+2)x\rangle - \langle(\mathcal{N}+2)x, \mathcal{K}x\rangle| \\ =\ &|\langle \mathcal{K}x, \mathcal{N}x\rangle - \langle \mathcal{N}x, \mathcal{K}x\rangle| \\ =\ &\Big| \sum_{mnl\geq 0} \langle ((K_2(t)J)_{mn}a^*_m a^*_n + \text{h.c.})\, x\,,\, a^*_l a_l x\rangle \\ &- \langle a^*_l a_l x\,,\, ((K_2(t)J)_{mn}a^*_m a^*_n + \text{h.c.})\, x\rangle\Big|\,.\end{aligned} \tag{C.8}$$

We reorganize the terms in (C.8) using the CCR

$$\begin{aligned}&\sum_{mnl\geq 0} \langle (K_2(t)J)_{mn}a^*_m a^*_n x\,,\, a^*_l a_l x\rangle \\ =\ &\sum_{mnl\geq 0} \overline{(K_2(t)J)_{mn}}\langle x\,,\, a_n a_m a^*_l a_l x\rangle \\ =\ &\sum_{mnl\geq 0} \overline{(K_2(t)J)_{mn}}\Big\{\delta_{ml}\langle x\,,\, a_n a_l x\rangle + \langle x\,,\, a_n a^*_l a_m a_l x\rangle\Big\} \\ =\ &\sum_{mnl\geq 0} \overline{(K_2(t)J)_{mn}}\Big\{\delta_{ml}\langle x\,,\, a_n a_l x\rangle + \delta_{nl}\langle x\,,\, a_m a_l x\rangle + \langle a^*_l a_l x\,,\, a_n a_m x\rangle\Big\} \\ =\ &\sum_{mnl\geq 0} \overline{(K_2(t)J)_{mn}}\Big\{\delta_{ml}\langle x\,,\, a_n a_m x\rangle + \delta_{nl}\langle x\,,\, a_m a_n x\rangle + \langle a^*_l a_l x\,,\, a_n a_m x\rangle\Big\} \\ =\ &\sum_{mnl\geq 0} \overline{(K_2(t)J)_{mn}}\langle a^*_l a_l x\,,\, a_n a_m x\rangle + 2\sum_{mn\geq 0} \overline{(K_2(t)J)_{mn}}\langle x\,,\, a_n a_m x\rangle\end{aligned} \tag{C.9}$$

and similar for the hermitian conjugate term

$$\begin{aligned}
&\sum_{mnl\geq 0} \Big\langle \overline{(K_2(t)J)_{mn}} a_m a_n x \, , \, a_l^* a_l x \Big\rangle \\
&= \sum_{mnl\geq 0} (K_2(t)J)_{mn} \langle x \, , \, a_n^* a_m^* a_l^* a_l x \rangle \\
&= \sum_{mnl\geq 0} (K_2(t)J)_{mn} \langle x \, , \, a_l^* a_n^* a_m^* a_l x \rangle \\
&= \sum_{mnl\geq 0} (K_2(t)J)_{mn} \Big\{ (-\delta_{lm}) \langle x \, , \, a_l^* a_n^* x \rangle + \langle x \, , \, a_l^* a_n^* a_l a_m^* x \rangle \Big\} \\
&= \sum_{mnl\geq 0} (K_2(t)J)_{mn} \Big\{ (-\delta_{lm}) \langle x \, , \, a_l^* a_n^* x \rangle + (-\delta_{ln}) \langle x \, , \, a_l^* a_m^* x \rangle + \langle x \, , \, a_l^* a_l a_n^* a_m^* x \rangle \Big\} \\
&= \sum_{mnl\geq 0} (K_2(t)J)_{mn} \Big\{ (-\delta_{lm}) \langle x \, , \, a_m^* a_n^* x \rangle + (-\delta_{ln}) \langle x \, , \, a_n^* a_m^* x \rangle + \langle a_l^* a_l x \, , \, a_n^* a_m^* x \rangle \Big\} \\
&= \sum_{mnl\geq 0} (K_2(t)J)_{mn} \langle a_l^* a_l x \, , \, a_n^* a_m^* x \rangle - 2 \sum_{mn\geq 0} (K_2(t)J)_{mn} \langle x \, , \, a_n^* a_m^* x \rangle \, . \qquad \text{(C.10)}
\end{aligned}$$

We conclude with (C.9) and (C.10)

$$\begin{aligned}
&|\langle \mathcal{K}x, (\mathcal{N}+2)x \rangle - \langle (\mathcal{N}+2)x, \mathcal{K}x \rangle| = \text{(C.8)} \\
&= \Big| \Big(\sum_{mnl\geq 0} \overline{(K_2(t)J)_{mn}} \langle a_l^* a_l x \, , \, a_n a_m x \rangle + 2 \sum_{mn\geq 0} \overline{(K_2(t)J)_{mn}} \langle x \, , \, a_n a_m x \rangle \Big) \\
&\quad + \Big(\sum_{mnl\geq 0} (K_2(t)J)_{mn} \langle a_l^* a_l x \, , \, a_n^* a_m^* x \rangle - 2 \sum_{mn\geq 0} (K_2(t)J)_{mn} \langle x \, , \, a_n^* a_m^* x \rangle \Big) \\
&\quad - \sum_{mnl\geq 0} \langle a_l^* a_l x \, , \, ((K_2(t)J)_{mn} a_m^* a_n^* + \text{h.c.}) \, x \rangle \Big| \\
&= 2 \Big| \sum_{mn\geq 0} \overline{(K_2(t)J)_{mn}} \langle x \, , \, a_n a_m x \rangle - \sum_{mn\geq 0} (K_2(t)J)_{mn} \langle x \, , \, a_n^* a_m^* x \rangle \Big| \, . \qquad \text{(C.11)}
\end{aligned}$$

Now we have to equilibrate both sides of the scalar product in powers of $\mathcal{N}$, hence we at $\mathcal{N}^{1/2}\mathcal{N}^{-1/2}$ in the second argument

$$\begin{aligned}
\text{(C.11)} &= 2 \Big| \sum_{mn\geq 0} \overline{(K_2(t)J)_{mn}} \Big\langle x \, , \, a_n a_m \mathcal{N}^{1/2} \mathcal{N}^{-1/2} x \Big\rangle \\
&\quad - \sum_{mn\geq 0} (K_2(t)J)_{mn} \Big\langle x \, , \, a_n^* a_m^* (\mathcal{N}+2)^{1/2} (\mathcal{N}+2)^{-1/2} x \Big\rangle \Big| \\
&= 2 \Big| \sum_{mn\geq 0} \overline{(K_2(t)J)_{mn}} \Big\langle (\mathcal{N}+2)^{1/2} x \, , \, a_n a_m \mathcal{N}^{-1/2} x \Big\rangle \\
&\quad - \sum_{mn\geq 0} (K_2(t)J)_{mn} \Big\langle \mathcal{N}^{1/2} x \, , \, a_n^* a_m^* (\mathcal{N}+2)^{-1/2} x \Big\rangle \Big| \, . \qquad \text{(C.12)}
\end{aligned}$$

We use (C.4) and (C.5) to get

$$\begin{aligned}(\mathrm{C}.12) &\leq 2\|\mathcal{N}+2)^{1/2}x\|\|\mathcal{N}^{1/2}\mathcal{N}^{-1/2}x\|\|K_2\|_{\mathrm{HS}} \\ &\quad + 2\|\mathcal{N}^{1/2}x\|\|(\mathcal{N}+2)^{1/2}(\mathcal{N}+2)^{-1/2}x\|\|K_2\|_{\mathrm{HS}} \\ &\leq 4\|K_2\|_{\mathrm{HS}}\|(\mathcal{N}+2)^{1/2}x\|\,. \end{aligned} \tag{C.13}$$

We conclude

$$\begin{aligned}|\langle\mathcal{K}x,(\mathcal{N}+2)x\rangle - \langle(\mathcal{N}+2)x,\mathcal{K}x\rangle| &= (\mathrm{C}.11) = (\mathrm{C}.12) \leq (\mathrm{C}.13) \\ &\leq 4\|K_2\|_{\mathrm{HS}}\|(\mathcal{N}+2)^{1/2}x\|\,.\end{aligned}$$

Now with the commutator Theorem [RS75, Theorem X.37] and $D(\mathcal{N}^2)$ core of $\mathcal{N}+2$ we conclude $(\mathcal{K}, D(\mathcal{N}^2))$ essentially self-adjoint and $\overline{\mathcal{K}|_{D(\mathcal{N}^2)}}$ essentially self-adjoint on each core of $\mathcal{N}$. But since $\mathcal{K}|_{D(\mathcal{N}^2)} \subset (\mathcal{K}, D(\mathcal{N}))$ and $(\mathcal{K}, D(\mathcal{N}))$ symmetric and hence closable we have $\overline{\mathcal{K}|_{D(\mathcal{N}^2)}} \subset \overline{(\mathcal{K}, D(\mathcal{N}))}$. Now since $(\mathcal{K}, D(\mathcal{N}))$ symmetric extension of the self-adjoint operator $\overline{\mathcal{K}|_{D(\mathcal{N}^2)}}$ we know $\overline{\mathcal{K}|_{D(\mathcal{N}^2)}} = \overline{(\mathcal{K}, D(\mathcal{N}))}$. Therefore $\overline{(\mathcal{K}, D(\mathcal{N}))}$ essentially self-adjoint on each core of $D(\mathcal{N})$ and hence $(\mathcal{K}, D(\mathcal{N}))$ essentially self-adjoint on each core of $D(\mathcal{N})$.

Proof of e):
To prove (e), we prove that the entries of the 2×2 matrix A_1 are continuous in time: $(A_1)_{ij} \in C(\mathbb{R}, \mathcal{L}(H^k))$ and $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$: $\|(A_1)_{ij}(t)\|_{\mathcal{L}(H^k)} \leq C$, $\forall 0 \leq k \leq 2$. Note that $D(A_0) = H^2 \oplus JH^2$.
We split the proof into two parts. First we prove the continuity and then in the second part the bounds in the operator norm.
Continuity:
Because we want the continuity for all volumes Λ and all times we treat them separately from the operator norm bounds which require additional regularity. To prove the continuity we just need the assumption $\varphi \in C^1(\mathbb{R}, H^2(\mathbb{R}^3,\mathbb{C}))$ and $\|\varphi_t\|_2$ constant. Let $\Lambda \geq 1$, $t \in \mathbb{R}$, $0 \leq k \leq 2$ and $\psi \in H^k$.
<u>$t \mapsto M_{V*|\varphi_t|^2}$:</u>
We use the identity $|\varphi_t|^2 - |\varphi_s|^2 = |\varphi_t - \varphi_s|^2 + 2\mathrm{Re}\varphi_s^*(\varphi_t - \varphi_s)$, that $\|\varphi_s^*\|_2 = \Lambda^{1/2}$ independent of s and that $(t \mapsto \varphi_t) \in C^1(\mathbb{R}, H^2) \subset C(\mathbb{R}, H^k)$ to conclude

$$\begin{aligned}\|D^\alpha V * \left(|\varphi_t|^2 - |\varphi_s|^2\right)\psi\|_2 &\leq \sum_{\gamma=0}^{\alpha} C\|V * D^\gamma\left(|\varphi_t|^2 - |\varphi_s|^2\right)\|_\infty \|D^{\alpha-\gamma}\psi\|_2 \\ &\leq C\sum_{\gamma=0}^{\alpha}\left\|D^\gamma\left(|\varphi_t|^2 - |\varphi_s|^2\right)\right\|_1 \|\psi\|_{H^k} \\ &\leq C\left(\|\varphi_t - \varphi_s\|_{H^k}^2 + 2\|\varphi_s^*\|_{H^k}\|\varphi_t - \varphi_s\|_{H^k}\right)\|\psi\|_{H^k}\end{aligned}$$

$$\overset{s\to t}{\to} 0\,. \tag{C.14}$$

<u>$t \mapsto M_{\mu_t}$:</u>
Continuity of $M_{\mu_t} \in \mathcal{L}(H^k)$ is a direct consequence of $\sum_{0\leq|\alpha|\leq k}\|D^\alpha(\mu_t-\mu_s)\psi\|_2^2 = |\mu_t - \mu_s|^2\|\psi\|_{H^k}^2$ and the continuity of $(t\mapsto \mu_t = 1/2\langle \frac{\varphi_t}{\Lambda^{1/2}}, V*|\varphi|^2\frac{\varphi_t}{\Lambda^{1/2}}\rangle) \in C(\mathbb{R},\mathbb{C})$, which holds true since $(t\mapsto\varphi_t)\in C(\mathbb{R},L^2)$ and $\|\varphi_t\|_2 = \Lambda^{1/2}$:

$$\begin{aligned}
&\|V*|\varphi_t|^2\varphi_t - V*|\varphi_s|^2\varphi_s\|_2\\
\leq\;& \|V*(|\varphi_t|^2-|\varphi_s|^2)\|_\infty\|\varphi_t\|_2 + \|V*|\varphi_s|^2\|_\infty\|\varphi_t-\varphi_s\|_2\\
\leq\;& \big\{\|V\|_\infty\|\varphi_t-\varphi_s\|_2^2 + \|V\|_\infty 2\|\varphi_s^*\|_2\|\varphi_t-\varphi_s\|_2)\big\}\|\varphi_t\|_2 + \|V\|_\infty\|\varphi_s\|_2^2\|\varphi_t-\varphi_s\|_2\\
\overset{t\to s}{\to}\;& 0\,.
\end{aligned}$$

<u>$t\mapsto K_i(t)$:</u>
Note that $K_1(t) = Q_t\tilde{K}_1(t)Q_t, K_2 = Q_t\tilde{K}_2(t)JQ_tJ^*$, $Q_t = 1-|u_t\rangle\langle u_t|$, $u_t = \varphi_t/\Lambda^{1/2}$. Due to Lemma F.0.2 and $(t\mapsto\varphi_t)\in C^1(\mathbb{R},H^2)$ we have

$$(t\mapsto Q_t)\in C^1(\mathbb{R},\mathrm{HS}(H^k))\subset C(\mathbb{R},\mathcal{L}(H^k))\,. \tag{C.15}$$

Now we prove $(t\mapsto\tilde{K}_1(t))\in C(\mathbb{R},\mathcal{L}(H^k)), (t\mapsto\tilde{K}_2(t))\in C(\mathbb{R},\mathcal{L}(JH^k,H^k))$. From this it is easy to conclude $(t\mapsto K_1(t))\in\mathcal{L}(H^k), (t\mapsto K_2(t))\in\mathcal{L}(JH^k,H^k)$:

Proof. Assume $(t\mapsto\tilde{K}_1(t))\in C(\mathbb{R},\mathcal{L}(H^k)), (t\mapsto\tilde{K}_2(t))\in C(\mathbb{R},\mathcal{L}(JH^k,H^k))$.

$$\begin{aligned}
&\|Q_{t+h}\tilde{K}_1(t+h)Q_{t+h} - Q_t\tilde{K}_1(t)Q_t\|_{\mathcal{L}(H^k)}\\
\overset{\pm 0}{\leq}\;&\|Q_{t+h}-Q_t\|\|\tilde{K}_1(t)\|\|Q_{t+h}\| + \|Q_t(\tilde{K}_1(t+h)Q_{t+h}-\tilde{K}_1(t)Q_t)\|\\
\leq\;&\|Q_{t+h}-Q_t\|\|\tilde{K}_1(t)\|\|Q_{t+h}\| + \|Q_t\|\|\tilde{K}_1(t+h)-\tilde{K}_1(t)\|\|Q_{t+h}\|\\
&+\|Q_t\|\|\tilde{K}_1(t)\|\|Q_{t+h}-Q_t\|\\
\leq\;& C_\Lambda(t)\Big\{\|Q_{t+h}-Q_t\|_{\mathcal{L}(H^k)} + \|\tilde{K}_1(t+h)-\tilde{K}_1(t)\|_{\mathcal{L}(H^k)}\Big\}\overset{h\to 0}{\to} 0\,,
\end{aligned}$$

where in step 3 we used that continuous function map compact sets into compact sets and that we can choose $h\in\bar{B}(t,1)$ and in the last step the assumption $(t\mapsto\tilde{K}_1(t)), (t\mapsto Q_t)\in C(\mathbb{R},\mathcal{L}(H^k))$. In the same way $\|Q_t\tilde{K}_1(t)Q_t\|\leq C$. $(t\mapsto K_2(t))\in C(\mathbb{R},\mathcal{L}(JH^k,H^k))$ and $\|K_2(t)\|_{\mathcal{L}(JH^k,H^k)}\leq C$ are proven analogously. □

Back to be proof of $(t\mapsto\tilde{K}_1(t))\in C(\mathbb{R},\mathcal{L}(H^k)), (t\mapsto\tilde{K}_2(t))\in C(\mathbb{R},\mathcal{L}(JH^k,H^k))$.
By the definition of $\tilde{K}_1$: $\tilde{K}_1(t)\psi(x) = \int\varphi_t(x)V(x-y)\varphi_t^*(y)\psi(y)dy = \varphi_t(x)V*(\varphi_t^*\psi)(x)$, $|\alpha|\leq k$

and

$$\begin{aligned}
&\|D^\alpha\left(\varphi_t\cdot V*(\varphi_t^*\psi)-\varphi_s\cdot V*(\varphi_s^*\psi)\right)\|_2\\
&\leq\sum_{\gamma=0}^{\alpha}C\|(D^\gamma\varphi_t)\cdot V*D^{\alpha-\gamma}(\varphi_t^*\psi)-(D^\gamma\varphi_s)\cdot V*D^{\alpha-\gamma}(\varphi_s^*\psi)\|_2\\
&\leq\sum_{\gamma=0}^{\alpha}C\Big(\|D^\gamma\varphi_t-D^\gamma\varphi_s\|_2\|V*D^{\alpha-\gamma}(\varphi_t^*\psi)\|_2\\
&\quad+\|D^\gamma\varphi_s\|_2\|V*(D^{\alpha-\gamma}(\varphi_t^*-\varphi_s^*)\psi)\|_2\Big)\\
&\leq\|\varphi_t-\varphi_s\|_{H^k}C\|\varphi_t\|_{H^k}\|\psi\|_{H^k}+\|\varphi_s\|_{H^k}C\|\varphi_t-\varphi_s\|_{H^k}\|\psi\|_{H^k}\\
&\overset{t\to s}{\to}0
\end{aligned}$$

hence

$$\begin{aligned}
\|\tilde{K}_1(t)-\tilde{K}_1(s)\|_{\mathcal{L}(H^k)}&\leq C\|\varphi_t\|_{H^k}\|\varphi_t-\varphi_s\|_{H^k}+C\|\varphi_s\|_{H^k}\|\varphi_t-\varphi_s\|_{H^k}\\
&\overset{t\to s}{\to}0\,.
\end{aligned}$$

We conclude $(t\mapsto\tilde{K}_1(t))\in C(\mathbb{R},\mathcal{L}(H^k))$.
Due to $\tilde{K}_2(t)J\psi(x)=\int\varphi_t(x)\varphi_t(y)V(x-y)\psi^*(y)dy=\varphi_t(x)V*(\varphi_t\psi^*)(x)$ one can show $(t\mapsto K_2(t))\in C(\mathbb{R},\mathcal{L}(JH^k,H^k))$ analogously to the K_1-case above.
All the considered cases above allow us to conclude: $A_1\in C(\mathbb{R},\mathcal{L}(H^k\oplus JH^k))$.

Operator norm bounds:
Let $0\leq k\leq 2$, $T\geq 0$, $-T\leq t\leq T$, $\psi\in H^k$ and assume Condition 2.1.8$_{k=2+2}$ to use Corollary B.2.4 for derivatives up to order two.
$t\mapsto M_{V*|\varphi|^2}$:
We use Lemma B.4.5a) to get for $t\in[-T,T]$

$$\begin{aligned}
\sum_{0\leq|\alpha|\leq k}\|D^\alpha V*|\varphi_t|^2\psi\|_2^2&\leq\sum_{0\leq|\alpha|\leq k}\sum_{\gamma=0}^{\alpha}C\|V*D^\gamma|\varphi_t|^2\|_\infty^2\|D^{\alpha-\gamma}\psi\|_2^2\\
&\leq C\|\psi\|_{H^k}^2 \qquad\text{(C.16)}
\end{aligned}$$

Because $\forall t\in\mathbb{R}\ \exists T\geq 0$ such that $t\in[-T,T]$ we conclude $M_{V*|\varphi_t|^2}\in\mathcal{L}(H^k)$, $\forall t\in\mathbb{R}$.
$t\mapsto M_{\mu_t}$:
Since μ_t constant we have $D^\alpha\mu_t\psi=\mu_t D^\alpha\psi$ and $\sum_{0\leq|\alpha|\leq k}\|D^\alpha\mu_t\psi\|_2^2\leq|\mu_t|\cdot\|\psi\|_{H^k}$ and by the definition of μ_t and Corollary B.1.2 for $t\in[-T,T]$

$$|\mu_t|=\frac{1}{2}\left|\left\langle\frac{\varphi_t}{\Lambda^{1/2}},V*|\varphi_t|^2\frac{\varphi_t}{\Lambda^{1/2}}\right\rangle\right|\leq\frac{1}{2}\|V*|\varphi_t|^2\|_\infty\leq C\,.$$

We conclude

$$\sum_{0\leq|\alpha|\leq k} \|D^\alpha \mu_t \psi\|_2^2 \leq C\|\psi\|_{H^k} . \tag{C.17}$$

Hence $M_{\mu_t} \in \mathcal{L}(H^k)$, $\forall t \in \mathbb{R}$.

<u>$t \mapsto K_i(t)$:</u>

Again note that $K_1(t) = Q_t\tilde{K}_1(t)Q_t, K_2 = Q_t\tilde{K}_2(t)JQ_tJ^*$, $Q_t = 1 - |u_t\rangle\langle u_t|$, $u_t = \varphi_t/\Lambda^{1/2}$. We show

$$\|Q_t\psi\|_{H^k}^2 \leq C\|\psi\|_{H^k}^2 . \tag{C.18}$$

Proof. We estimate

$$\begin{aligned}\sum_{0\leq|\alpha|\leq k} \|D^\alpha Q_t\psi\|_2^2 &\leq \sum_{0\leq|\alpha|\leq k} \|D^\alpha\psi\|_2^2 + \frac{1}{\Lambda^2}\|D^\alpha\varphi_t\|_2^2|\langle\varphi_t,\psi\rangle|^2 \\ &= \|\psi\|_{H^k}^2 + \|\varphi_t\|_{H^k}^2\frac{\Lambda}{\Lambda^2}\|\psi\|_2^2 \\ &\leq \|\psi\|_{H^k}^2\left(1 + \frac{\|\varphi_t\|_{H^k}^2}{\Lambda}\right)\end{aligned}$$

and with Corollary B.2.4 we get (C.18). □

Now we prove $\|\tilde{K}_1(t)\|_{\mathcal{L}(H^k)} \leq C$, $\|\tilde{K}_2(t)\|_{\mathcal{L}(JH^k,H^k)} \leq C$. From this and (C.18) it is easy to conclude $\|K_1(t)\|_{\mathcal{L}(H^k)} \leq C$, $\|K_2(t)\|_{\mathcal{L}(JH^k,H^k)} \leq C$.

With definition $\tilde{K}_1(t)\psi(x) = \int \varphi_t(x)V(x-y)\varphi_t^*(y)\psi(y)dy = \varphi_t(x)V*(\varphi_t^*\psi)(x)$, Definition A.0.5 and Corollary B.2.4$_{|\beta|=k\leq 2}$ for $t \in [-T,T]$ we conclude for $\forall|\alpha| \leq k$

$$\begin{aligned}&\|D^\alpha\tilde{K}_1(t)\psi\|_2 \\ &= \|D^\alpha\varphi_t \cdot V*(\varphi_t^*\psi)\|_2 \\ &\leq \sum_{\gamma=0}^{\alpha} C\|D^{\alpha-\gamma}\varphi_t \cdot V*D^\gamma(\varphi_t^*\psi)\|_2 \\ &\leq \sum_{\gamma=0}^{\alpha} C\|D^{\alpha-\gamma}\varphi_t\|_{2\wedge\infty}\left(\|V*D^\gamma(\varphi_t^*\psi)\|_2 + \|V*D^\gamma(\varphi_t^*\psi)\|_\infty\right) \\ &\leq \sum_{\gamma=0}^{\alpha}\sum_{\beta=0}^{\gamma} C\|D^{\alpha-\gamma}\varphi_t\|_{2\wedge\infty}\left(\|V\|_{1,2}\|D^\gamma\varphi_t\|_{2\wedge\infty}\|\psi\|_{H^k} + \|V\|_{2,\infty}\|D^\gamma\varphi_t\|_{2\wedge\infty}\|\psi\|_{H^k}\right) \\ &\leq C\|\psi\|_{H^k} .\end{aligned}$$

Hence $\|\tilde{K}_1(t)\|_{\mathcal{L}(H^k)} \leq C$.

Due to $\tilde{K}_2(t)J\psi(x) = \int \varphi_t(x)\varphi_t(y)V(x-y)\psi^*(y)dy = \varphi_t(x)V*(\varphi_t\psi^*)(x)$ one can show

$\|K_2(t)\|_{\mathcal{L}(JH^k,H^k)} \leq C$ for $t \in [-T,T]$ analogously to the argument for K_1 above.
By collecting the results above we conclude $\|A_1(t)\|_{\mathcal{L}(H^k \oplus JH^k)} \leq C$ for $t \in [-T,T]$. ■

Appendix D

Existence of Dynamics and Regularity

We prove the existence of all considered Hamiltonians in the quadratic form sense using Grönwall-type arguments.

D.1 Existence of the Dynamics Generated by Quadratic Forms

The result presented in this section is a variant of [LNS15, Theorem 8], which establishes readily verifiable conditions for the existence of dynamics associated with time-dependent quadratic forms. Its proof is fundamentally based on a Grönwall argument. For the reader's convenience, we restate this theorem here in the precise form needed for our purposes and with full details.

Theorem D.1.1 (Dynamics Generated by Time-dependent Quadratic Forms)**.** *Let $\mathscr{H}$ be a Hilbert space, $A \geq 1$ self-adjoint operator, $I \subset \mathbb{R}$ an interval, $t_0 \in I$ and $\{q_{H(t)}\}_{t \in I}$ a family of symmetric quadratic forms on $\mathscr{H}$. Assume there exists $B \geq 0$ self-adjoint operator such that for all bounded intervals $I_b \subset I$ we have*

a) That $q_{H(t)}$ is comparable to A and B on I_b, meaning

i) B commutes with A in the sense of self-adjoint operators.[1]

ii) $B \leq A$.

[1]See [RS80, Chapter VIII.5] for a precise definition of commuting self-adjoint operators.

iii) $\exists C_1, C_2 > 0$ *such that* $\forall t \in I_b$*:*

$$C_1 q_A \geq q_{H(t)} \geq C_1^{-1} q_A - C_2 q_B \,.$$

Especially $Q(q_{H(t)}) = Q(A)$ *and* $|q_{H(t)}| \geq C_{12} q_A$*,* $C_{12} := \max\{C_1, C_2\}$*.*

iv) $\exists C_4 > 0$ *such that* $\forall u \in Q(A)$*:* $(t \mapsto q_{H(t)}(u)) \in C^1(I_b, \mathbb{R})$ *and*

$$\left| \frac{d}{dt} q_{H(t)}(u) \right| \leq C_4 q_A(u) \,.$$

b) The operator A *bounds the commutator of* B *with* $q_{H(t)}$*, meaning* $\exists C_3 > 0$*:*

$$\mp 2 \mathrm{Im}\, q_{H(t)}(u, Bu) \leq C_3 q_A(u) \,, \quad \forall u \in D(A^{3/2}) \,. \tag{D.1}$$

Then for all $u_0 \in Q(A)$ $\exists! u \in L^2_{loc}(I; Q(A)) \cap C(\bar{I}, \mathscr{H})$ *with* $j \circ u \in L^2_{loc}(I; Q(A)^*)$ *weakly differentiable with weak derivative* $\dot{u} \in L^2_{loc}(I; Q(A)^*)$ *such that*

$$\begin{aligned} \mathrm{i}\dot{u}(t) &= q_{H(t)}(\,.\,, u(t)) \,, \\ u(t_0) &= u_0 \,, \end{aligned}$$

where $j : \mathscr{H} \to Q(A)^*, u \mapsto \langle\,.\,, u\rangle_{\mathscr{H}}$*. In addition we have*

i) $\|u(t)\|_{\mathscr{H}} = \|u(s)\|_{\mathscr{H}}$*,* $\forall t, s \in \bar{I}$*.*

ii) $u(t) \in Q(A)$*,* $\forall t \in \mathbb{R}$*, and* $\forall I_b \subset I$ *bounded intervals with* $t_0 \in I_b$ *we have* $\forall t \in I_b$

$$q_A(u(t)) \leq 2C_{12}^2(I_b) \mathrm{e}^{C_{1234}(I_b)|t-t_0|} q_A(u_0) \,,$$

where $C_{12} := \max\{C_1, C_2\}$*,* $C_{1234} := 2C_1C_2C_3 + C_1C_4$*. Note that the* C_i *are coming from a),b) and depend in general on* I_b*.*

iii) $u \in L^\infty_{loc}(I, Q(A))$*,* $\dot{u} \in L^\infty_{loc}(I, Q(A)^*)$*.*

Remark D.1.2.

- It is sufficient to check the condition a)iv) on a $D \subset D(A^{3/2})$ dense, which directly implies condition a)iv).
- If B and A commute and $0 \leq B \leq A$ it is easy to conclude $B^s \leq A^s, D(A^s) \subset D(B^s)$, $\forall s \in \mathbb{R}_{\geq 0}$, and $B^r D(A^s) \subset D(A^{s-r})$, $\forall 0 \leq r \leq s$.
- We need that $\mathscr{H}$ is separable only to show $(t \mapsto q_{H(t)}(\,.\,, u_0)) \in L^0(\mathbb{R}, Q(A)^*)$ for all $u_0 \in Q(A)$. If we assume the strong-measurability as above, Corollary D.1.3 is valid for general Hilbert spaces $\mathscr{H}$ without the separability constraint.

The proof of Theorem D.1.1 can be found in the Appendix D.5.1.

We get as a direct Corollary of Theorem D.1.1 the existence of an unitary propagator.

Corollary D.1.3 (Unitary Propagator of Time-dependent Quadratic Forms)**.** *Let $\mathscr{H}$ be a Hilbert space, $A \geq 1$ self-adjoint operator and $\{q_{H(t)}\}_{t\in\mathbb{R}}$ a family of symmetric quadratic forms on $\mathscr{H}$. Assume the conditions of Theorem D.1.1. For $(t_0, u_0) \in \mathbb{R} \times Q(A)$ let $u \in C(\mathbb{R}, \mathscr{H}) \cap L^2_{loc}(\mathbb{R}, Q(A))$, $\dot{u} \in L^2_{loc}(\mathbb{R}, Q(A)^*)$ be the unique solution of $\mathrm{i}\dot{u}(t) = q_{H(t)}(\,.\,, u(t))$, $u(t_0) = u_0$ given by Theorem D.1.1.*
Then $\forall t, t_0 \in \mathbb{R}$:

$$\begin{aligned} \tilde{U}(t, t_0) : Q(A) &\to Q(A) \\ u_0 &\mapsto u(t) \end{aligned}$$

is linear and continuous extendable and we denote its unique extension by $U(t, t_0)$. $U(t, s)$ is a unitary propagator and $\forall u_0 \in Q(A)$: $jU(t, s)u_0$ is weakly differentiable in $Q(A)^*$ with respect to t and s with weak derivatives

$$\begin{aligned} \frac{d}{dt} jU(t, s)u_0 &= -\mathrm{i}q_{H(t)}(\,.\,, U(t, s)u_0)\,, \\ \frac{d}{ds} jU(t, s)u_0 &= \mathrm{i}q_{H(s)}(U(s, t)\,.\,, u_0)\,, \end{aligned}$$

where $j : \mathscr{H} \to Q(A)^*, u \mapsto \langle\,.\,, u\rangle_{\mathscr{H}}$. Note that $U(t, t_0)u_0$ is the unique solution of $\mathrm{i}\dot{u}(t) = q_{H(t)}(\,.\,, u(t))$, $u(t_0) = u_0$.

Remark D.1.4. We need the separability of $\mathscr{H}$ only to show $(t \mapsto q_{H(t)}(\,.\,, u_0))$, $(t \mapsto q_{H(t)}(U(t, s)\,.\,, u_0)) \in L^0(\mathbb{R}, Q(A)^*)$ for all $u_0 \in Q(A)$. If we assume the strong–measurability as above, Corollary D.1.3 is valid for general Hilbert spaces $\mathscr{H}$ without the separability constraint.

The proof of Corollary D.1.3 can be found in the Appendix D.5.1.

D.2 Existence of the Bogoliubov Dynamics

Next we apply Theorem D.1.1 to Hamiltonians appearing in our framework such as the Bogoliubov Hamiltonian $H^{\mathrm{Bog}}(t)$. However, to rigorously analyze the Bogoliubov dynamics, it is essential to examine the Bogoliubov transformation and the symplectic structure of $\mathscr{H} \oplus \mathscr{H}^*$.

D.2.1 Bogoliubov Transformation and Symplectic Structure

Let $\mathscr{H}$ be a Hilbert space, $J : \mathscr{H} \to \mathscr{H}^*, \psi \mapsto \langle\, . \,, \psi\rangle$ antiunitary, $\mathcal{J} = \begin{pmatrix} 0 & J^* \\ J & 0 \end{pmatrix} : \mathscr{H} \oplus \mathscr{H}^* \to \mathscr{H} \oplus \mathscr{H}^*$, $\mathcal{J}^* = \mathcal{J} = \mathcal{J}^{-1}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. To define the Bogoliubov transformation it is helpful to work with generalized creation and annihilation operators acting on $\mathscr{H} \otimes \mathscr{H}^*$:

$$A(f \oplus Jg) = a(f) + a^*(g)\,, \quad A^*(f \oplus Jg) = a^*(f) + a(g)\,, \quad \forall f, g \in \mathscr{H} \tag{D.1}$$

satisfying $\forall F, G \in \mathscr{H} \oplus \mathscr{H}^*$

$$A^*(F) = A(\mathcal{J}F)\,, \quad [A(F), A^*(G)] = \langle F, SG\rangle\,, \quad [A(F), A(G)] = 0\,.$$

Definition D.2.1 (Bogoliubov Map and Bogoliubov Transformation)**.** Let $\mathscr{H}$ be a Hilbert space.

i) We call a $\mathcal{V} \in \mathcal{L}(\mathscr{H} \oplus \mathscr{H}^*)$ a Bogoliubov map if it fulfils the symplectic conditions, namely $\mathcal{V} \in \mathcal{S}$ with

$$\mathcal{S} = \left\{ \mathcal{V} \in \mathcal{L}(\mathscr{H} \oplus \mathscr{H}^*) \,\middle|\, \mathcal{J}\mathcal{V}\mathcal{J}\,,\ \mathcal{V}^* S \mathcal{V} = S = \mathcal{V} S \mathcal{V}^* \right\}.$$

ii) We call a $\mathcal{V} \in \mathcal{S}$ unitarily implementable if $\exists U_\mathcal{V}$ unitary on $\mathcal{F}(\mathscr{H})$ such that

$$U_\mathcal{V} A(F) U_\mathcal{V}^* = A(\mathcal{V}F)\,, \quad \forall F \in \mathscr{H} \oplus \mathscr{H}^*\,.$$

If such a $U_\mathcal{V}$ exists we call it a Bogoliubov transformation.

Remark D.2.2. Note that $U_\mathcal{V}$ is uniquely determined by $\mathcal{V}$ up to a phase, which can be fixed through $U_\mathcal{V}\Omega$. We denote the equivalence class of all $U_\mathcal{V}$ associated to $\mathcal{V}$ by $[U_\mathcal{V}]$ (for details see e.g. [Nam20]).

Lemma D.2.3. *Let $\mathscr{H}$ be a Hilbert space.*

i) Let $\mathcal{V} \in \mathcal{L}(\mathscr{H} \oplus \mathscr{H})$. Then $\mathcal{V} \in \mathcal{S}$ iff $\exists U \in \mathcal{L}(\mathscr{H}), V \in \mathcal{L}(\mathscr{H}, \mathscr{H}^*)$ such that $\mathcal{V} = \begin{pmatrix} U & J^*VJ^* \\ V & JUJ^* \end{pmatrix}$ where U, V satisfy

$$U^*U = 1 + V^*V\,, \quad UU^* = 1 + J^*VV^*J\,, \quad V^*JU = U^*J^*V\,.$$

ii) If $\mathscr{H}$ is separable then $\mathcal{V} \in \mathcal{S}$ is unitarily implementable iff the Shale-Stinespring condition is satisfied, namely $\mathcal{V}^*\mathcal{V} - 1 \in \mathrm{HS}(\mathscr{H})$.

Proof of Lemma D.2.3. A proof of Lemma D.2.3 can be found in [Nam20]. ■

Lemma D.2.4 (Invariance of the Particle Number under Bogoliubov Transformation)**.** *Let $\mathscr{H}$ be a separable Hilbert space, $b \in \mathbb{R}_{\geq 0}$ and $U_{\mathcal{V}}$ Bogoliubov transformation on $\mathcal{F}(\mathscr{H})$. Then*

$$q_{U_{\mathcal{V}}(\mathcal{N}+1)^b U_{\mathcal{V}}^*} \leq C_{\mathcal{V}}^b (b+1)^b q_{(\mathcal{N}+1)^b} ,$$

and $U_{\mathcal{V}} Q(\mathcal{N}^b) = Q(\mathcal{N}^b) = U_{\mathcal{V}}^* Q(\mathcal{N}^b)$ and

$$C_{\mathcal{V}} := 1 + 3\|V\|_{\mathrm{HS}} + \|U\|_{\mathrm{op}} .$$

Proof of Lemma D.2.4. A proof of Lemma D.2.4 can be found in [Boß+22, Lemma 4.4]. ■

Definition D.2.5 (Generalized One-body Density Matrix)**.** Let $\mathscr{H}$ be a Hilbert space. For $\psi \in Q(\mathcal{N})$ we define the generalized one-body density matrix as

$$\Gamma_\psi = \begin{pmatrix} \gamma_\psi & \alpha_\psi^* \\ \alpha_\psi & J(\gamma_\psi + 1)J^* \end{pmatrix} \in \mathcal{L}(\mathscr{H} \oplus \mathscr{H}^*) ,$$

where density matrix γ_ψ and the pairing matrix α_ψ are defined below

$$\langle g, \gamma_\psi f \rangle = \langle a(f)\psi, a(g)\psi \rangle , \quad \langle Jg, \alpha_\psi f \rangle = \langle a(f)\psi, a^*(g)\psi \rangle , \quad \forall f, g \in \mathscr{H} .$$

D.2.2 Symplectic Dynamics

This section is based on the results from [Boß+22, Lemma 4.8], [AKS13, Lemma 3.10] and [Bac+22].

Our goal is to show existence of the propagator $U^{\mathrm{Bog}}(t, t_0)$ of $H^{\mathrm{Bog}}(t)$ and then prove that $U^{\mathrm{Bog}}(t, t_0)$ is a Bogoliubov transformation. Due to the time dependence of $U^{\mathrm{Bog}}(t, t_0)$ also the corresponding Bogoliubov transformation $U_{\mathcal{V}(t,t_0)}$ has to be time dependent. The $\mathcal{V}(t, t_0)$ generating $U_{\mathcal{V}(t,t_0)}$ is given by a differential equation of the following form

$$\partial_t \mathcal{V}(t, t_0) = -\mathrm{i}\mathcal{A}(t)\mathcal{V}(t, t_0)$$

for a generating operator $\mathcal{A}(t)$ [Boß+22, Lemma 4.8]. In the following Lemma we study this differential equation. Then we proceed in Theorem D.2.8 to show the existence of $U^{\mathrm{Bog}}(t, t_0)$ and identify $U^{\mathrm{Bog}}(t, t_0) = U_{\mathcal{V}(t,t_0)}$ (for a more precise statement see Corollary D.2.9 below).

Lemma D.2.6 (Symplectic Dynamics)**.** *Let $\mathscr{H}$ be a Hilbert space and h_1 self-adjoint in $\mathscr{H}$, $h_2 \in C(\mathbb{R}, \mathcal{L}(\mathscr{H})) \cap C(\mathbb{R}, \mathcal{L}(D(h)))$, $h_2(t)$ self-adjoint in $\mathscr{H}$ and $k_2 \in C(\mathbb{R}, \mathcal{L}(\mathscr{H}, \mathscr{H}^*)) \cap$*

$C(\mathbb{R},\mathcal{L}(D(h_1),JD(h_1)))$, $k_2(t)^* = J^*k_2(t)J^*$. Set $\mathcal{A}(t) := A_0 + A_1(t)$ and

$$A_0 := \begin{pmatrix} h_1 & 0 \\ 0 & -Jh_1J^* \end{pmatrix} \qquad A_1(t) := \begin{pmatrix} h_2(t) & -J^*k_2(t)J^* \\ k_2(t) & -Jh_2(t)J^* \end{pmatrix}.$$

Then A_0 self-adjoint in $\mathscr{H}\oplus\mathscr{H}^*$ and $A_1 \in C(\mathbb{R},\mathcal{L}(D(A_0)) \cap C(\mathbb{R},\mathcal{L}(\mathscr{H}\oplus\mathscr{H}^*))$. Especially $S\mathcal{A}(t)^*S = \mathcal{A}(t)$.
And $\exists!\mathcal{V}:\mathbb{R}^2\to\mathcal{L}(\mathscr{H}\oplus\mathscr{H}^*),(t,s)\mapsto\mathcal{V}(t,s)$ with $\forall u_0\in D(A_0), t_0, t\in\mathbb{R}$: $(s\mapsto\mathcal{V}(s,t_0)u_0)\in C(\mathbb{R},D(A_0))\cap C^1(\mathbb{R},\mathscr{H}\oplus\mathscr{H}^*)$ and

$$\begin{aligned} \partial_t\mathcal{V}(t,t_0)u_0 &= -\mathrm{i}\mathcal{A}(t)\mathcal{V}(t,t_0)u_0\,, \\ \mathcal{V}(t_0,t_0) &= I\,. \end{aligned}$$

In addition we have the properties:

i) (Symplectic Propagator)

a) $\mathcal{V}(t,t) = I$.

b) $\mathcal{V}(t,s)\mathcal{V}(s,t_0) = \mathcal{V}(t,t_0)$.

c) $\mathcal{V}(t,s)$ strongly continuous in t and s.

In addition $\mathcal{V}(t,s)D(A_0) = D(A_0)$ and $\mathcal{V}(t,s)$ fulfils the symplectic conditions, namely $\mathcal{V}(t,s)\in\mathcal{S}$ (see Definition D.2.1), and $\forall x\in\mathscr{H}\oplus\mathscr{H}^*$

$$\langle\mathcal{V}(t,s)x, S\mathcal{V}(t,s)x\rangle = \langle x, Sx\rangle_{\mathscr{H}\oplus\mathscr{H}^*}\,.$$

ii) (Derivative)
$\forall u_0\in D(A_0)$, $t\in\mathbb{R}$: $(s\mapsto\mathcal{V}(t,s)u_0)\in C(\mathbb{R},D(A_0))\cap C^1(\mathbb{R},\mathscr{H}\oplus\mathscr{H}^*)$ and

$$\partial_s\mathcal{V}(t,s)u_0 = \mathcal{V}(t,s)\mathrm{i}\mathcal{A}(s)u_0\,.$$

iii) (Estimates)

a) $\|\mathcal{V}(t,s)\|_{\mathcal{L}(\mathscr{H}\oplus\mathscr{H}^*)} \le \mathrm{e}^{\mathrm{sgn}(t-s)\int_s^t \|k_2(\tau)\|_{\mathcal{L}(\mathscr{H},\mathscr{H}^*)}d\tau}$.

b) $\mathcal{V}(t,s)\in\mathcal{L}(D(A_0))$ and

$$\|\mathcal{V}(t,s)\|_{\mathcal{L}(D(A_0))} \le 1 + \mathrm{sgn}(t-s)\int_s^t \|A_1(\tau)\|_{\mathcal{L}(D(A_0))}\mathrm{e}^{\mathrm{sgn}(t-s)\int_\tau^t \|A_1(\lambda)\|_{\mathcal{L}(D(A_0))}d\lambda}d\tau\,.$$

vi) (Implementability)
Assume $\mathscr{H}$ separable[2] and $k_2\in C(\mathbb{R},\mathrm{HS}(\mathscr{H},\mathscr{H}^*))$ then $\mathcal{V}(t,s)^*\mathcal{V}(t,s)-1\in\mathrm{HS}(\mathscr{H}\oplus\mathscr{H}^*)$ and thus $\mathcal{V}(t,s)$ is unitarily implementable.

[2] Such that Hilbert-Schmidt operators are well-defined.

Proof of Lemma D.2.6. A simple proof of Lemma D.2.6 can be found in [Boß+22, Lemma 4.8]. ∎

D.2.3 Bogoliubov Dynamics

We apply Theorem D.1.1 to the Bogoliubov Hamiltonian H^{Bog} (see Definition 2.3.1). But first we state two more general statement about quadratic operators from [NN17, Proposition 7] and [Boß+22, Lemma 4.8], including also H^{Bog} as it can be seen in Corollary D.2.9 below.

Definition D.2.7 (Time-dependent Quadratic Operator)**.** Let $\mathscr{H}$ be a separable Hilbert space, $h_1 \geq 0$ a self-adjoint operator, $h_2(t) \in \mathcal{L}(\mathscr{H})$ self-adjoint, $k_2(t) \in \mathrm{HS}(\mathscr{H}^*, \mathscr{H})$ and $k_2^*(t) = Jk_2(t)J$, for all times $t \in \mathbb{R}$.

i) (Definition quadratic Hamiltonian)
We define $(H^{\mathrm{qua}}(h_1, h_2(t), k_2(t)), D(d\Gamma(h_1+1))$ as an operator on $\mathscr{H}$ with

$$H^{\mathrm{qua}}(t) := H^{\mathrm{qua}}(h_1, h_2(t), k_2(t)) := d\Gamma(h_1 + h_2(t)) + \frac{1}{2}\sum_{mn} \left((k_2(t)J)_{mn} a_m^* a_n^* + \mathrm{h.c.}\right) .$$

We call $H^{\mathrm{qua}}(t)$ a quadratic operator.

ii) (Definition quadratic form)
We define the symmetric quadratic form associated with $H(t)$ as

$$\begin{aligned} q_{H^{\mathrm{qua}}(t)} := q_{H^{\mathrm{qua}}(h_1,h_2(t),k_2(t))} &: Q(d\Gamma(h_1+1)) \to \mathbb{R} \\ \psi \mapsto \langle \psi, d\Gamma(h_1)\psi\rangle_{Q\times Q^*} + \langle \psi, d\Gamma(h_2(t))\psi\rangle_{Q\times Q^*} &+ \mathrm{ReTr}(k_2(t)\alpha_\psi) , \end{aligned}$$

where α_ψ is the pairing operator (see Definition D.2.5).

H^{qua} is well defined due to Lemma C.0.1d).

Theorem D.2.8 (Dynamics of Time-dependent Quadratic Operators)**.** *Let $\mathscr{H}$ be a separable Hilbert space, $h_1 \geq 0$ a self-adjoint operator, $h_2 \in C^1(\mathbb{R}, \mathcal{L}(\mathscr{H}))$, $h_2(t)$ self-adjoint, $k_2 \in C^1(\mathbb{R}, \mathrm{HS}(\mathscr{H}^*, \mathscr{H}))$ and $k_2^*(t) = Jk_2(t)J$, $\forall t \in \mathbb{R}$. Let $q_{H^{\mathrm{qua}}(t)}$ be the quadratic from of a quadratic operator corresponding to Definition D.2.7.*

*i) ($q_{H^{\mathrm{qua}}(t)}$ generates a dynamic)
Then the symmetric quadratic forms $\{q_{H^{\mathrm{qua}}(t)}\}_{t\in\mathbb{R}}$ satisfy the conditions of Theorem D.1.1 for the comparison operators $A := d\Gamma(h_1+1)+1$ and $B := \mathcal{N}+1$.*

ii) (Bogoliubov transformation) Set

$$A_0 = \begin{pmatrix} h_1 & 0 \\ 0 & -h_1^T \end{pmatrix} , \quad A_1(t) = \begin{pmatrix} h_2(t) & -k_2(t) \\ Jk_2(t)J & -h_2(t)^T \end{pmatrix} .$$

If in addition we assume $h_2 \in C(\mathbb{R}, \mathcal{L}(D(h_1))), k_2 \in C(\mathbb{R}, \mathcal{L}(JD(h_1), D(h_1)))$ *such that* $A_1 \in C(\mathbb{R}, \mathcal{L}(D(A_0))) \cap C(\mathbb{R}, \mathcal{L}(\mathscr{H} \oplus J\mathscr{H}))$ *then we can define* $\mathcal{V}(t,s)$ *as in Lemma D.2.6 and get* $\forall t_0, t \in \mathbb{R}\ \exists! U_{\mathcal{V}(t,t_0)} \in [U_{\mathcal{V}(t,t_0)}]$ *with*

$$U^{\text{qua}}(t,t_0) = U_{\mathcal{V}(t,t_0)}\,,$$

where $U^{\text{qua}}(t,t_0)$ *is the unitary propagator of our dynamics as defined in Corollary D.1.3.*

Proof of Theorem D.2.8. For a proof of Theorem D.2.8i) see for example [NN17, Proposition 7]. A proof of Theorem D.2.8ii) can be found in [Boß+22, Lemma 4.8] and also in a slightly different setting in [Bac+22, Proposition 3.9] and [AKS13, Theorem 2.2].

In addition we provide a detailed overview of the rigorous proof of Theorem D.2.8ii) in Appendix D.5.2, where we discuss the regularity requirements necessary for each step, as well as the underlying idea of the proof. ■

The Bogoliubov Hamiltonian $H^{\text{Bog}}(t)$ is a quadratic operator and with help of Lemma C.0.1 we can show that Theorem D.2.8 applies for $H^{\text{Bog}}(t)$.

Corollary D.2.9 (Existence of the Bogoliubov Dynamics)**.** *For all volumes* $\Lambda \geq 1$ *let* φ_t *be the condensate with initial data* φ_0 *satisfying Condition 2.1.7. Then Theorem D.2.8 i) and ii) can be applied to* $H^{Bog}(t)$ *for* $\mathscr{H} := L^2(\mathbb{R}^3, \mathbb{C}), h_1 = -\Delta/2, h_2(t) = V * |\varphi_t|^2 - \mu_t + K_1, k_2 = K_2$*. Especially the unitary propagator* $U^{Bog}(t,t_0)$ *of the dynamics is a Bogoliubov transformation.*

Proof of Corollary D.2.9. That all conditions of Theorem D.2.8 are satisfied follows directly from Lemma C.0.1. ■

Lemma D.2.10 ($\mathcal{V}(t,0)$ Estimate)**.** *Assume that for all volumes* $\Lambda \geq 1$ *the condensate* φ_0 *satisfies Condition 2.1.7 and Condition* $2.1.8_{k=2+2}$*. Let* $\mathcal{V}(t,s) \in \mathcal{S}$ *with* $U_{\mathcal{V}(t,s)} = U^{\text{Bog}}(t,s)$ *due to Corollary D.2.9 then* $\forall T \geq 0\ \exists C > 0$ *such that* $\forall \Lambda \geq 1$ *and* $-T \leq t \leq T$

$$\|U(t,0)\|_{op} + \|V(t,0)\|_{op} \leq \|\mathcal{V}(t,0)\|_{op} \leq C\,, \tag{D.2}$$

where

$$\mathcal{V}(t,s) = \begin{pmatrix} U(t,s) & J^*V(t,s)J^* \\ V(t,s) & JU(t,s)J^* \end{pmatrix}.$$

Proof of Lemma D.2.10. Lemma D.2.10 is a direct consequence of $\|K_2(t)\|_{\text{op}} \leq C$, for $-T \leq t \leq T$ (see Lemma C.0.1, Lemma D.2.6 and Corollary D.2.9). ■

D.3 Existence of the Tracer Dynamics and Tracer Localization

In this section, we prove the existence of the dynamics generated by the transformed Bogoliubov-Fröhlich Hamiltonian (Definition 5.4.1):

$$\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t) = -\frac{\Delta_x}{2m} + A(\mathcal{Q}_0 S\mathcal{V}_t^* S \cdot Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t)\,, \tag{D.1}$$

where the excitation field dynamics have been extracted from the Hamiltonian, but the tracer and the field remain interacting. We refer to the dynamics generated by Hamiltonians of the form of (D.1) as the tracer dynamics.
By proving the existence of the dynamics of the tracer Hamiltonian, we also show the localization of the tracer. This is achieved by applying the rigorous Grönwall estimate Theorem D.1.1 to the Hamiltonian (D.1) and using a comparison operator with tracks the tracer position x, the harmonic oscillator.

We begin by giving some fundamental properties of the harmonic oscillator used to prove our existence Theorem below.

Lemma D.3.1 (Harmonic Oscillator)**.** *Let $d \in \mathbb{N}_+$ and $\mathscr{H}$ be a Hilbert space. Set*

$$h_{\mathrm{oc}} := (-\Delta_x + x^2) \otimes I + I \otimes (\mathcal{N} + 1)$$

$h_{\mathrm{oc}} \geq 1$ *self-adjoint on $L^2(\mathbb{R}^d, \mathbb{C}) \otimes \mathcal{F}(\mathscr{H})$.*

i) Let $M \in \mathbb{N}_0$ then $h_{\mathrm{oc}}^M \geq 1$ self-adjoint and for all $\tilde{D} \subset \mathscr{H}$ dense we have that $D := \mathscr{S}(\mathbb{R}^d, \mathbb{C}) \bar{\otimes} \bigcup_{L\geq 0} \bigoplus_{N=0}^{L} \tilde{D}^{\bar{\otimes} N}$ core of h_{oc}^M and $h_{\mathrm{oc}} D \subset D$.

ii) (Bounds) For $M \in \mathbb{N}_0$ $\exists C_{d,M} > 0$:

$$q_{(-\Delta_x)^M \otimes I} + q_{I \otimes (\mathcal{N}+1)^M} + q_{x^{2M} \otimes I} \leq C_{d,M} q_{h_{\mathrm{oc}}^M}\,.$$

Proof. The proof of i) uses [RS80, Appendix of V.3] where it is proven that the hermite functions form an orthonormal basis of $L^2(\mathbb{R})$ lying in $\mathscr{S}(\mathbb{R}^d, \mathbb{C})$. This and that the hermite functions are eigenfunctions of the harmonic oscillator is then used to prove the essential self-adjointness of h_{oc}^M on each section of the Fock space, which is then just extended. ii) is a consequence of the simple commutator $[\partial_{x_i}, x_i] = 1$. ■

Lemma D.3.1d) allows us to control the position operator of the tracer through the harmonic oscillator A up to an arbitrary large power.

The Grönwall estimate for the harmonic oscillator is an application of Theorem D.1.1 and is summarized in the following Theorem for which we first define the tracer Hamiltonian to be a abstract version of the transformed Bogoliubov-Fröhlich Hamiltonian (D.1).

Definition D.3.2 (Tracer Hamiltonian)**.** Let $d \in \mathbb{N}_+$, $f_t \in L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d, \mathbb{C}))$, $g_t \in L^\infty(\mathbb{R}^d_x, \mathbb{R})$ and $\lambda_t \in \mathbb{R}$ for all $t \in \mathbb{R}$. Set

$$\begin{aligned} &q_{H^{\mathrm{T}}(t)}(f_t, g_t, \lambda_t) := q_{H^{\mathrm{T}}(t)} : Q(\mathcal{N} - \Delta_x) \times Q(\mathcal{N} - \Delta_x) \to \mathbb{C} \\ &(\psi, \varphi) \mapsto q_{-\Delta_x}(\psi, \varphi) + \langle \psi, A(f_{t,x} \oplus J f_{t,x})\varphi\rangle + \lambda_t \langle \psi, \varphi\rangle + \langle \psi, M_{g_{t,x}} \otimes I_{\mathcal{F}}\varphi\rangle \,. \end{aligned}$$

Note that $q_{H^{\mathrm{T}}(t)}$ is a symmetric quadratic form on $L^2(\mathbb{R}^d_x, \mathcal{F}(L^2))$.

Theorem D.3.3 (Grönwall Estimate for the Harmonic Oscillator in the Tracer Dynamics)**.** *Let $M, d \in \mathbb{N}_+$ and $h_{\mathrm{oc}} = (-\Delta_x + x^2) \otimes I + I \otimes (\mathcal{N}+1)$ be the harmonic oscillator in x direction on $L^2(\mathbb{R}^d_x, \mathbb{C}) \otimes \mathcal{F}(L^2(\mathbb{R}^d_y, \mathbb{C})) = L^2(\mathbb{R}^d_x, \mathcal{F}(L^2))$.*

i) Let $f \in l^\infty_{loc}(\mathbb{R}_t, W^{M,\infty}(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C})))$ such that for a.e. $x \in \mathbb{R}^d$: $(t \mapsto f_t(x, .)) \in C^1(\mathbb{R}_t, L^2(\mathbb{R}^d_y, \mathbb{C}))$ and $\forall t$: $\partial_t f_t := (x \mapsto \partial_t f_t(x, .)) \in L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C}))$ and $\dot{f} := (t \mapsto \partial_t f_t) \in l^\infty_{loc}(\mathbb{R}_t, L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C})))$.

ii) Let $g \in l^\infty_{loc}(\mathbb{R}_t, W^{M,\infty}(\mathbb{R}^d_x, \mathbb{R}))$ with $g \in C^1(\mathbb{R}_t, L^\infty(\mathbb{R}^d_x, \mathbb{R}))$.

Set $q_{H^{\mathrm{T}}(t)} = q_{H^{\mathrm{T}}(t)}(f_t, g_t, \lambda_t)$ as in Definition D.3.2.
Then $\{(q_{H^{\mathrm{T}}(t)})|_{Q(h^M_{\mathrm{oc}})}\}_{t\in\mathbb{R}}$ satisfies the conditions of Theorem D.1.1 with comparison operator $A := B := h^M_{\mathrm{oc}}$. Especially we get $\forall (u_0, t_0) \in Q(h^M_{\mathrm{oc}}) \times \mathbb{R}$, $\forall I_b \subset \mathbb{R}$ bounded intervals, $t_0 \in I_b$

$$q_{h^M_{\mathrm{oc}}}(u(t)) \le C_{I_b}(t) q_{h^M_{\mathrm{oc}}}(u_0) \,, \quad \forall t \in I_b \,,$$

where $x(t)$ unique solution of

$$\begin{aligned} \mathrm{i}\partial_t u(t) &= q_{H^{\mathrm{T}}(t)}(u(t)) \,, \\ u(t_0) &= u_0 \end{aligned}$$

as described in Theorem D.1.1 and

$$\begin{aligned} C_{I_b}(t) = \exp\{|t - t_0|\} \cdot 2 \sup_{t \in I_b} \Bigg[&(2\|f_t\|_\infty + \|g_t\|_\infty + |\lambda_t| + 1)^2 \\ &\cdot \exp\Big\{ 2\,(2\|f_t\|_\infty + \|g_t\|_\infty + |\lambda_t| + 1) \Big[\Big(2\|\dot{f}_t\|_\infty + \|\dot{g}_t\|_\infty + |\dot{\lambda}_t|\Big) \\ &+ (2\|f_t\|_\infty + \|g_t\|_\infty + |\lambda_t| + 1) \\ &\cdot C_{d,M}\left(1 + \|f_t\|_{W^{M,\infty}(\mathbb{R}^d, L^2(\mathbb{R}^d, \mathbb{C}))} + \|g_t\|_{W^{M,\infty}(\mathbb{R}^d, \mathbb{R})}\right) \Big] \Big\} \Bigg] \,. \end{aligned}$$

Remark D.3.4.

i) The idea was to make a Grönwall estimate of $q_{h^M_{\mathrm{oc}}}(u(t))$, where $u(t)$ is generated by $q_{H^{\mathrm{T}}(t)}$. Later, we use the bound $Ch^M_{\mathrm{oc}} \geq x^{2M}$ (see Lemma D.3.1) to derive an estimate on the tracer position x^{2M}, namely the first d coordinates.
To perform the Grönwall estimate we use Theorem D.1.1, which reduces the problem to a commutator estimate on a suitable domain D, where the commutator can be explicitly computed. As a by-product, we obtain the existence of the dynamics. Since our primary goal is to estimate h^M_{oc}, we have not investigated alternative domains for $q_{H^{\mathrm{T}}(t)}$ beyond $Q(h^M_{\mathrm{oc}})$ in detail. However, the argument extends naturally to the domain $Q(\mathcal{N} - \Delta_x)$ without additional difficulties (see Remark D.4.4).

ii) The conditions on f are chosen such that we can differentiate $a(f_{t,x})\psi$ and that all terms in $C_{I_b}(t)$ in our final estimate remain bounded. These conditions arise due to the structure of f
$$f := (t \mapsto (x \mapsto (U_t - J^*V_t)Q_tW_x\varphi_t) \in l^\infty_{\mathrm{loc}}(\mathbb{R}_t, W^{M,\infty}(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C}))) ,$$
where U_t and V_t come from $U_t^{\mathrm{Bog}} = U_{\mathcal{V}_t}$ (see Corollary D.2.9). Since $t \mapsto U_t - J^*V_t$ and $t \mapsto \mathcal{V}_t$ are only strongly continuous and differentiable as operators on L^2 we cannot evaluate them at specific points in the "y"-variable and we get differentiability only for fixed x. Nonetheless, the resulting time derivative behaves well, as stated in Theorem D.3.3.

iii) The application of Theorem D.3.3 to $\widetilde{H}^{\mathrm{BF}}_{\mathcal{Q}_0}(t)$ can be found in Corollary 5.4.5, leading to our result on the tracer localization in Theorem 3.2.1.

Proof of Theorem D.3.3. Theorem D.3.3 is a direct Corollary of Theorem D.1.1. Hence your task is to prove that the conditions of Theorem D.1.1 are satisfied for $q_{H^F(t)}$ and comparison operators $A = B = h^M_{\mathrm{oc}}$.
We need to satisfy two condition. First, we show that $H^F(t)$ and h^M_{oc} are comparable, this is done in Lemma D.3.5, and second, we estimate commutator of $H^F(t)$ with h^M_{oc}, this is done below.
Let $d, M \in \mathbb{N}_+$, $I_b \subset \mathbb{R}$ bounded set and $t \in I_b$. Set $D := \mathscr{S}(\mathbb{R}^d, \mathbb{C}) \bar{\otimes} \bigcup_{L\geq 0} \bigoplus_{N=0}^{L} \tilde{D}^{\bar{\otimes}_s N}$, $\tilde{D} \subset L^2(\mathbb{R}^d, \mathbb{C})$ dense. Note that $D \subset A^{3/2}$ dense, hence it is sufficient to validate the commutator conditions of Theorem D.1.1 on D. We prove $\forall \psi \in D$
$$\mp 2\mathrm{Im}\, q_{H^{\mathrm{T}}(t)}(\psi, B\psi) \leq C_{d,M} \left(1 + \|f_t\|_{W^{M,\infty}(\mathbb{R}^d, L^2(\mathbb{R}^d,\mathbb{C}))} + \|g_t\|_{W^{M,\infty}(\mathbb{R}^d,\mathbb{R})}\right) q_A(\psi) .$$
But this can be immediately be seen from Lemma D.3.6 and
$$\begin{aligned}\mp\mathrm{Im} q_{H^{\mathrm{T}}(t)}(\psi, h^M_{\mathrm{oc}}\psi) = \mp \mathrm{Im}\Big\{ &\left\langle \psi, (-\Delta_x + M_{g_{t,x}})h^M_{\mathrm{oc}}\psi\right\rangle - \left\langle h^M_{\mathrm{oc}}\psi, (-\Delta_x + M_{g_t})\psi\right\rangle \\ &+ \left\langle A(f_{t,x} \oplus Jf_{t,x})\psi, h^M_{\mathrm{oc}}\psi\right\rangle - \left\langle h^M_{\mathrm{oc}}\psi, A(f_{t,x} \oplus Jf_{t,x})\psi\right\rangle \Big\} .\end{aligned}$$
■

The following Lemma D.3.5 and Lemma D.3.6 support the proof of Theorem D.3.3. In Lemma D.3.5 we estimate the tracer Hamiltonian through the harmonic oscillator h_{oc} and Lemma D.3.6 is used to estimate the commutator of the harmonic oscillator h_{oc} with the Hamiltonian $H^{\mathrm{T}}(t)$ on a suitable domain.

Lemma D.3.5 (The Tracer Hamiltonian Estimated Through the Harmonic Oscillator)**.** *Let $d, M \in \mathbb{N}_+$ and $I_b \subset \mathbb{R}$ a bounded interval.*

i) Let $f \in l^\infty_{\mathrm{loc}}(\mathbb{R}, L^\infty(\mathbb{R}^d, L^2(\mathbb{R}^d, \mathbb{C})))$, $g \in l^\infty_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R})$, $\lambda \in l^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ then for all $\psi \in Q(h^M_{\mathrm{oc}})$ and $t \in I_b$

$$(C_1 + 1)\, q_{h^M_{\mathrm{oc}}}(\psi) \geq q_{H^{\mathrm{T}}(t)}(\psi) \geq -C_1 q_{h^M_{\mathrm{oc}}}(\psi) \tag{D.2}$$

for $C_1 = \sup_{t \in I_b}(2\|f_t\|_\infty + \|g_t\|_\infty + |\lambda_t|)$.

ii) Let $f \in l^\infty_{loc}(\mathbb{R}_t, L^\infty(\mathbb{R}^d, L^2(\mathbb{R}^d)))$ such that for almost all $x \in \mathbb{R}^d$: $(t \mapsto f_t(x, \,.\,)) \in C^1(\mathbb{R}_t, L^2(\mathbb{R}^d_y, \mathbb{C}))$ and $\forall t$: $\partial_t f_t := (x \mapsto \partial_t f_t(x, \,.\,)) \in L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C}))$ and $\dot{f} := (t \mapsto \partial_t f_t) \in l^\infty_{loc}(\mathbb{R}_t, L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C})))$. Then for all $\psi \in Q(h^M_{\mathrm{oc}})$ $(t \mapsto q_{H^{\mathrm{T}}(t)}(\psi)) \in C^1(\mathbb{R}, \mathbb{R})$ and for all $t \in I_b$

$$\left| \frac{d}{dt} q_{H^{\mathrm{T}}(t)}(\psi) \right| \leq \sup_{t \in I_b} \Big(2\|\dot{f}_t\|_\infty + \|\dot{g}_t\|_\infty + |\dot{\lambda}_t| \Big)\, q_{h^M_{\mathrm{oc}}}(\psi)\,. \tag{D.3}$$

Proof. To the first part. (D.2) follows directly from the estimates $0 \leq -\Delta_x \leq h_{\mathrm{oc}}$, $\pm A(f_{t,x} \oplus Jf_{t,x}) \leq 2\|f_t\|_\infty (\mathcal{N}+1)^{1/2} \leq 2\|f_t\|_\infty h_{\mathrm{oc}}$.
To the second part. The regularity of $(t \mapsto q_{H^{\mathrm{T}}(t)}(\psi))$ follows directly from the regularity of f, g, λ. The bound in (D.3) can immediately be seen by calculating the derivative $\frac{d}{dt} q_{H^{\mathrm{T}}(t)}(\psi)$ explicitly. ■

Lemma D.3.6 (Commutator of h_{oc} with the Hamiltonian $H^{\mathrm{T}}(t)$)**.** *Let $d, M \in \mathbb{N}_+$. Then there exists a constant $C_{d,M} > 0$ such that for all $\psi \in \bigcup_{L \geq 0} \bigoplus_{n=0}^{L} \mathscr{S}(\mathbb{R}^d, \mathbb{C}) \bar{\otimes} L^2(\mathbb{R}^d, \mathbb{C})^{\bar{\otimes}_s n} =: D$, $g \in W^{M,\infty}(\mathbb{R}^d, \mathbb{R})$ and $f \in W^{M,\infty}(\mathbb{R}^d, L^2(\mathbb{R}^d, \mathbb{C}))$ we have*

$$\mp \mathrm{Im}\Big\{ \big\langle g_x \psi_n, h^M_{\mathrm{oc}} \psi_n \big\rangle - \big\langle h^M_{\mathrm{oc}} \psi_n, g_x \psi_n \big\rangle \Big\} \leq C_{d,M} \|g\|_{W^{M,\infty}(\mathbb{R}^d,\mathbb{R})} \big\langle \psi, h^M_{\mathrm{oc}} \psi \big\rangle\,. \tag{D.4}$$

$$\mp \mathrm{Im} \big\langle \psi, [-\Delta_x, h^M_{\mathrm{oc}}] \psi \big\rangle \leq C_{d,M} \big\langle \psi, h^M_{\mathrm{oc}} \psi \big\rangle\,. \tag{D.5}$$

$$\begin{aligned} \mp \mathrm{Im}\Big\{ \big\langle A(f_x \oplus Jf_x)\psi, h^M_{\mathrm{oc}} \psi \big\rangle - \big\langle h^M_{\mathrm{oc}} \psi, A(f_{t,x} \oplus Jf_{t,x})\psi \big\rangle \Big\} \\ \leq C_{d,M} \|f\|_{W^{M,\infty}(\mathbb{R}^d, L^2(\mathbb{R}^d,\mathbb{C}))} \big\langle \psi, h^M_{\mathrm{oc}} \psi \big\rangle\,. \end{aligned} \tag{D.6}$$

Proof of Lemma D.3.6. We start with the proof of (D.6). To ensure better control over regularity, we first assume $f \in C^\infty_b(\mathbb{R}^d, L^2) \subset W^{M,\infty}$, and extend the result to general $f \in W^{M,\infty}$ by a density argument. Let $\psi \in D$ then $a^\#(f_x)\psi \in D$, since $f \in C^\infty_b(\mathbb{R}^d, L^2)$.

We consider the case of $M =: 2m$ even. This allows us to symmetrically split $h_{\text{oc}}^{2m} =: h^{2m}$ between the arguments of the scalar product in $\langle A(f_x \oplus Jf_x)\psi, h^{2m}\psi\rangle - \langle h^{2m}\psi, A(f_{t,x} \oplus Jf_{t,x})\psi\rangle$. To handle this, we commute h with $a^{\#}(f_x)$ one operator after another. We define the iterated commutator by the adjoint map recursively:

$$\mathrm{ad}_h^{(0)}(a^{\#}(f_x)) := a^{\#}(f_x)\,, \tag{D.7}$$

$$\mathrm{ad}_h^{(k)}(a^{\#}(f_x))\psi := \left[\mathrm{ad}_h^{(k-1)}(a^{\#}(f_x)), h\right], \quad \forall k \in \mathbb{N}_+\,. \tag{D.8}$$

We start by commuting the h's to the left argument in the first term of the commutator in (D.6):

$$\begin{aligned}&\langle \psi, a^{\#}(f_x)h^{2m}\psi\rangle = \langle h\psi, a^{\#}(f_x)h^{2m-1}\psi\rangle + \Big\langle \psi, \mathrm{ad}_h^{(1)}(a^{\#}(f_x))h^{2m-1}\psi\Big\rangle \\ &= \langle h^2\psi, a^{\#}(f_x)h^{2m-2}\psi\rangle + 2\Big\langle h\psi, \mathrm{ad}_h^{(1)}(a^{\#}(f_x))h^{2m-2}\psi\Big\rangle + \Big\langle \psi, \mathrm{ad}_h^{(2)}(a^{\#}(f_x))h^{2m-2}\psi\Big\rangle\,.\end{aligned}$$

By proceeding as above until only h^m remains in the right argument, we obtain

$$\langle \psi, a^{\#}(f_x)h^{2m}\psi\rangle = \sum_{k=0}^{m}\binom{m}{k}\Big\langle h^k\psi, \mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))h^m\psi\Big\rangle\,. \tag{D.9}$$

Similar for the second term of the commutator in (D.6):

$$\langle h^{2m}\psi, a^{\#}(f_x)\psi\rangle = \sum_{k=0}^{m}\binom{m}{k}\Big\langle h^m\psi, (-1)^{m-k}\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))h^k\psi\Big\rangle\,. \tag{D.10}$$

Subtracting (D.10) from (D.9), we observe that the $k = m$ terms cancel, as they have identical numbers of h operators on both sides. This gives:

$$\begin{aligned}&\pm\,\mathrm{Im}\big\{\langle (a^{\#}(f_x))^*\psi, h^{2m}\psi\rangle - \langle h^{2m}\psi, a^{\#}(f_x)\psi\rangle\big\} \\ &= \pm\mathrm{Im}\sum_{k=0}^{m-1}\binom{m}{k}\Big\{\Big\langle h^k\psi, \mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))h^m\psi\Big\rangle - \Big\langle h^m\psi, (-1)^{m-k}\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))h^k\psi\Big\rangle\Big\}\,.\end{aligned} \tag{D.11}$$

We will show below that the cancellation of the $k = m$ term is sufficient to show the desired estimate in (D.6). To proceed, we need a structural representation for the nested commutators $\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))$ terms (for $m - k \geq 1$):

$$\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))\psi = \sum_{\substack{|\beta|+|\gamma|+|\delta|\leq 2(m-k)\\ |\gamma|+|\delta|\leq 2(m-k)-1}} C_{\beta,\gamma,\delta}^{(m-k)}\, a^{\#}(D_x^{\beta}f_x)x^{\gamma}D_x^{\delta}\psi\,. \tag{D.12}$$

We emphasize that this representation does not hold for $m - k = 0$, as $\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x)) =$

$a^\#(f_x)$. Thus, it is essential that the sum in (D.11) only runs over $k \le m-1$, so that (D.12) can be applied throughout.

Moreover, in the representation (D.12) the monomials in x and D_x appear only up to total order $2(m-k)-1$, which is strictly less than the order of monomials appearing in h^{m-k}, namely $2(m-k)$. This drop in order is what ultimately allows us to close the estimate. The validity of (D.12) can be verified by induction and is easily confirmed for $m-k=1$. Indeed, by the definition of the harmonic oscillator h, $[a^\#(f_x), h]\psi = (a^\#(\Delta_x f_x) + 2a^\#(\nabla_x f_x)\nabla_x - \mathrm{sgn}(\#)a^\#(f_x))\psi$, where $\mathrm{sgn}(\#) = 1$ for $\# = *$ and $\mathrm{sgn}(\#) = -1$ for $\# = \emptyset$ (for a detailed argument see Lemma D.3.7).

To estimate the right-hand side of (D.11), we now combine the representation (D.12) with the following inequality for monomials $x^\gamma D_x^\delta$:

$$(-iD_x)^\delta x^{2\gamma}(-iD_x)^\delta \le C h^{|\delta|+|\gamma|}\,, \tag{D.13}$$

which follows from the definition of the harmonic oscillator $h = -\Delta_x + x^2 + \mathcal{N} + 1$ (see again Lemma D.3.7 for details). Combining (D.12) and (D.13), and using that $\mathcal{N}+1$ is commuting with x^γ, D_x^δ and h, we get we following estimate

$$\begin{aligned}
&\|\mathrm{ad}_h^{(m-k)}(a^\#(f_x))h^k\psi\| \\
&\le \sum_{\substack{|\beta|+|\gamma|+|\delta|\le 2(m-k)\\ |\gamma|+|\delta|\le 2(m-k)-1}} C_{m,k}\big(\operatorname*{ess\,sup}_x \|D_x^\beta f_x\|_{L^2}\big)\|(\mathcal{N}+1)^{1/2}x^\gamma D_x^\delta h^k\psi\| \\
&\le \sum_{\substack{|\beta|+|\gamma|+|\delta|\le 2(m-k)\\ |\gamma|+|\delta|\le 2(m-k)-1}} C_{m,k}\|f\|_{W^{|\beta|,\infty}(\mathbb{R}^d,L^2)}C\|h^{|\delta|+|\gamma|}(\mathcal{N}+1)^{1/2}h^k\psi\| \\
&\le C_{m,k}\|f\|_{W^{2(m-k),\infty}(\mathbb{R}^d,L^2)}\|h^m\psi\|\,.
\end{aligned} \tag{D.14}$$

We conclude from (D.11) and (D.14)

$$\pm\mathrm{Im}\big\{\big\langle (a^\#(f_x))^*\psi, h^{2m}\psi\big\rangle - \big\langle h^{2m}\psi, a^\#(f_x)\psi\big\rangle\big\} \le C_m\|f\|_{W^{2m,\infty}(\mathbb{R}^d,L^2)}\|h^m\psi\|^2\,.$$

This proves the bound (D.6) in the case of even $M = 2m$ for all $f \in C_b^\infty \subset W^{2m,\infty}$. The result then extends to all $f \in W^{2m,\infty}$ by density.

Now we consider the case $M = 2m+1$ uneven, which is technically more delicate since it is not directly possible to split the powers of h evenly between both arguments of the scalar product in (D.6) as we did in the even case. Attempting to distribute the h^{2m+1} in a straight forward way, we run into commutators of $[a^\#(f_x), h^{1/2}]$, which we can not calculate. To circumvent this, we rewrite the extra factor of h, we have compared to the M even case, in from of operators that are bounded by $h^{1/2}$. Namely

$$h = (-\nabla_x + x)(\nabla_x + x) + (\mathcal{N}+1)^{1/2}(\mathcal{N}+1)^{1/2}\,. \tag{D.15}$$

We can now proceed analogously to the even case. We find with the known splitting of the h-terms as in (D.10) that

$$\langle \psi, a^{\#}(f_x)h^{2m+1}\psi\rangle = \sum_{k=0}^{m}\binom{m}{k}\Big\langle h^k\psi, \mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))h\cdot h^m\psi\Big\rangle\,. \tag{D.16}$$

Using the decomposition (D.15) for the singled-out h we get

$$\begin{aligned}\langle \psi, a^{\#}(f_x)h^{2m+1}\psi\rangle = \sum_{k=0}^{m}\binom{m}{k}\sum_{B\in\{(\nabla_x+x),(\mathcal{N}+1)^{1/2}\}}&\Big\{\Big\langle Bh^k\psi, \mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))Bh^m\psi\Big\rangle\\ &+\Big\langle h^k\psi, \big[\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x)), B^*\big]Bh^m\psi\Big\rangle\Big\}\,.\end{aligned} \tag{D.17}$$

And again with (D.10) we get a similar form for the other commutator term appearing in (D.6):

$$\begin{aligned}\langle h^{2m+1}\psi, a^{\#}(f_x)\psi\rangle &= \sum_{k=0}^{m}\sum_{B\in\{(\nabla_x+x),(\mathcal{N}+1)^{1/2}\}}(-1)^{m-k}\binom{m}{k}\\ &\cdot\Big\{\Big\langle Bh^m\psi, \mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))Bh^k\psi\Big\rangle + \Big\langle Bh^m\psi, \big[\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x)), B^*\big]h^k\psi\Big\rangle\Big\}\,.\end{aligned} \tag{D.18}$$

We want to point out that we proceeded exactly as in the case of M even only that we commute once with B instead of h. Subtracting (D.18) from (D.17) also cancels the $k = M$ term but only in the first term in (D.17) and (D.18):

$$\pm\,\mathrm{Im}\big\{\langle (a^{\#}(f_x))^*\psi, h^{2m+1}\psi\rangle - \langle h^{2m+1}\psi, a^{\#}(f_x)\psi\rangle\big\} = \sum_{B\in\{(\nabla_x+x),(\mathcal{N}+1)^{1/2}\}}\Bigg(\sum_{k=0}^{m-1}\binom{m}{k}$$

$$\cdot\Big\{\pm\,\mathrm{Im}\Big\langle Bh^k\psi, \mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))Bh^m\psi\Big\rangle \mp (-1)^{m-k}\mathrm{Im}\Big\langle Bh^m\psi, \mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))Bh^k\psi\Big\rangle \tag{D.19}$$

$$\pm\,\mathrm{Im}\Big\langle h^k\psi, \big[\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x)), B^*\big]Bh^m\psi\Big\rangle \mp (-1)^{m-k}\mathrm{Im}\Big\langle Bh^m\psi, \big[\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x)), B^*\big]h^k\psi\Big\rangle\Big\} \tag{D.20}$$

$$\pm\,\mathrm{Im}\big\langle h^m\psi, \big[a^{\#}(f_x), B^*\big]Bh^m\psi\big\rangle \mp \mathrm{Im}\big\langle Bh^m\psi, \big[a^{\#}(f_x), B^*\big]h^m\psi\big\rangle\Bigg)\,. \tag{D.21}$$

We now estimate the pair of terms (D.19), (D.20) and (D.21).
We start with (D.19), which is estimated by our bound on the adjoint mapping $\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))$

in (D.14) and

$$\|h^{k/2}x_l\psi\|^2 + \|h^{k/2}\partial_{x_l}\psi\|^2 \leq C_k\|h^{(k+1)/2}\psi\|^2\,, \quad \forall k \in \mathbb{N}_0 \tag{D.22}$$

which is shown in (D.42) as part of the proof of Lemma D.3.7. Let $B \in \{(\nabla_x + x), (\mathcal{N}+1)^{1/2}\}$. We get with $(\mathcal{N}+1)^{1/2} \leq h^{1/2}$ and $k \leq m-1$ that

$$\|\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x)^*)Bh^k\psi\| \leq C_{m,k}\|f\|_{W^{2(m-k),\infty}(\mathbb{R}^d,L^2)}\|h^{m+1/2}\psi\| \tag{D.23}$$

Thus

$$\text{(D.19)} \leq C_m\|f\|_{W^{2m,\infty}(\mathbb{R}^d,L^2)}\|h^{m+1/2}\psi\|^2\,. \tag{D.24}$$

Next we estimate (D.20). We use again the representation of the adjoint map (D.12) and $\big[x^\gamma D_x^\delta, -\partial_{x_i} + x_i\big] = C_1 x^{\tilde{\gamma}} D_x^{\tilde{\delta}} + C_2 x^{\overline{\gamma}} D^{\overline{\delta}}$ with $|\tilde{\gamma}| + |\tilde{\delta}| \leq |\gamma| + |\delta|$ and $|\overline{\gamma}| + |\overline{\delta}| \leq |\gamma| + |\delta|$ to get

$$\begin{aligned}
&\|\big[\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x)), -\partial_{x_i} + x_i\big]h^k\psi\| \\
&\leq C_{m,k} \sum_{\substack{|\beta|+|\gamma|+|\delta|\leq 2(m-k)\\ |\gamma|+|\delta|\leq 2(m-k)-1}} \Big\{\|\big[a^{\#}(D_x^\beta f_x), -\partial_{x_i} + x_i\big]x^\gamma D_x^\delta h^k\psi\| \\
&\qquad + \|a^{\#}(D_x^\beta f_x)\big[x^\gamma D_x^\delta, -\partial_{x_i} + x_i\big]h^k\psi\|\Big\} \\
&\leq C_{m,k} \sum_{\substack{|\beta|+|\gamma|+|\delta|\leq 2(m-k)\\ |\gamma|+|\delta|\leq 2(m-k)-1}} \Big\{\|a^{\#}(\partial_{x_i} D_x^\beta f_x)x^\gamma D_x^\delta h^k\psi\| + \|a^{\#}(D_x^\beta f_x)x^\gamma D_x^\delta h^k\psi\|\Big\} \\
&\leq C_m\|f\|_{W^{2m+1,\infty}}\|h^m\psi\|\,,
\end{aligned} \tag{D.25}$$

where in the last step we proceeded as in (D.14). Thus we have estimated (D.20) in the case of $B = \nabla_x + x$. The estimate in the case $B = (\mathcal{N}+1)^{1/2}$ follows in a similar fashion, as with the representation of the adjoint mapping in (D.12), we get

$$\big[\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x)), (\mathcal{N}+1)^{1/2}\big]\psi = \mathrm{ad}_h^{(m-k)}(a^{\#}(f_x))\big\{(\mathcal{N}+1)^{1/2} - (\mathcal{N}+1+\mathrm{sgn}(\#))^{1/2}\big\}\psi\,. \tag{D.26}$$

But for $b = 1 + \mathrm{sgn}(\#) \geq 0$

$$\pm\left((\mathcal{N}+1)^{1/2} - (\mathcal{N}+b)^{1/2}\right) = \frac{\pm\big((\mathcal{N}+1) - (\mathcal{N}+b)\big)}{(\mathcal{N}+1)^{1/2} + (\mathcal{N}+b)^{1/2}} \leq 1\,. \tag{D.27}$$

Thus

$$\|\big[\mathrm{ad}_h^{(m-k)}(a^{\#}(f_x)), (\mathcal{N}+1)^{1/2}\big]h^k\psi\| \leq C_m\|f\|_{W^{2m,\infty}}\|h^m\psi\|\,. \tag{D.28}$$

The estimates (D.25) and (D.28) allow us to conclude the desired estimate of (D.20):

$$(D.20) \leq C_m \|f\|_{W^{2m+1,\infty}} \|h^{m+1/2}\psi\|^2 \,. \tag{D.29}$$

We final estimate of (D.21) works similar to the (D.20) estimate but instead of the representation of the adjoint map (D.12) we use just $a^{\#}(f_x)$. The result follows analogously:

$$(D.21) \leq C_m \|f\|_{W^{1,\infty}} \|h^{m+1/2}\psi\|^2 \tag{D.30}$$

Combining estimates (D.24), (D.28) and (D.30), we obtain

$$\pm \mathrm{Im}\big\{ \big\langle a^{\#}(f_x)\psi, h^{2m+1}\psi \big\rangle - \big\langle h^{2m+1}\psi, a^{\#}(f_x)\psi \big\rangle \big\} \leq C_m \|f\|_{W^{2m+1,\infty}(\mathbb{R}^d, L^2)} \|h^{m+1/2}\psi\|^2 \,.$$

This completes the proof of (D.6) for uneven $M = 2m+1$ and for all $f \in C_b^\infty \subset W^{2m+1,\infty}$. The result extends to general $f \in W^{2m+1,\infty}$ by a standard density argument.

With the use of the following identities,

$$\mathrm{ad}_h^{(k)}(g)\psi = \sum_{\substack{|\beta|+|\gamma|+|\delta| \leq 2k \\ |\gamma|+|\delta| \leq 2k-1}} C_{\gamma,\delta}^{(k)} (D_x^\beta g_x) x^\gamma D_x^\delta \psi \,, \quad \forall k \in \mathbb{N}_+ \,, \tag{D.31}$$

$$\mathrm{ad}_h^{(k)}(-\Delta_x)\psi = \sum_{|\gamma|+|\delta| \leq 2} C_{\gamma,\delta}^{(k)} x^\gamma D_x^\delta \psi \,, \quad \forall k \in \mathbb{N}_0 \,, \tag{D.32}$$

which are used instead of the representation of the adjoint map in (D.12) the proofs of the estimates (D.4) and (D.5) follow in a similar fashion as the proof of the (D.6) estimate. Note that g and $-\Delta_x$ commute with $\mathcal{N}$, which simplifies the argument from above even further. This completes the proof of Lemma D.3.6. ■

The following lemma is used in the proof of Lemma D.3.6.

Lemma D.3.7. *Let* $d \in \mathbb{N}_+$.

i) For all $k \in \mathbb{N}_+$ *exists a family of constants* $\{C_{\alpha,\beta,\gamma}^{(k)}\}_{\alpha,\beta,\gamma \in \mathbb{N}_0^d} \subset \mathbb{R}$ *such that for all* $\psi \in \bigcup_{L \geq 0} \bigoplus_{n=0}^{L} \mathscr{S}(\mathbb{R}^d, \mathbb{C}) \bar{\otimes} L^2(\mathbb{R}^d, \mathbb{C})^{\bar{\otimes}_s n} =: D$ *and* $f \in C_b^\infty(\mathbb{R}^d, L^2(\mathbb{R}^d, \mathbb{C}))$ *we have*

$$\mathrm{ad}_{h_{\mathrm{oc}}}^{(k)}(a^{\#}(f_x))\psi = \sum_{\substack{|\beta|+|\gamma|+|\delta| \leq 2k \\ |\gamma|+|\delta| \leq 2k-1}} C_{\beta,\gamma,\delta}^{(k)} a^{\#}(D_x^\beta f_x) x^\gamma D_x^\delta \psi \,. \tag{D.33}$$

ii) For all $\gamma, \delta \in \mathbb{N}_0^d$ *exists a constant* $C > 0$ *such that*

$$(-iD_x)^\delta x^{2\gamma} (-iD_x)^\delta \leq C h^{|\delta|+|\gamma|} \tag{D.34}$$

on $\bigcup_{L\geq 0}\bigoplus_{n=0}^{L}\mathscr{S}(\mathbb{R}^d,\mathbb{C})\bar{\otimes}L^2(\mathbb{R}^d,\mathbb{C})^{\bar{\otimes}_s n}$.

Proof of Lemma D.3.7. We start with the proof of part (i), which is proven by induction argument on $k\in\mathbb{N}_+$. The base case follows by using the explicit form of the harmonic oscillator $h := h_{\mathrm{oc}}$:

$$[a(f_x),h]\psi = (a^{\#}(\Delta_x f_x) + 2a^{\#}(\nabla_x f_x)\nabla_x - \mathrm{sgn}(\#)a^{\#}(f_x))\psi\,, \tag{D.35}$$

where $\mathrm{sgn}(\#)=1$ for $\#=*$ and $\mathrm{sgn}(\#)=-1$ for $\#=\emptyset$. This matches the claimed structure.

Now let us assume the assertion holds for k. Then we prove it to hold for $k+1$. We use the definition of the adjoint map in (D.7) and (D.8), and the induction hypotheses to conclude

$$\begin{aligned}\mathrm{ad}_h^{(k+1)}(a^{\#}(f_x))\psi &= \big[\mathrm{ad}_h^{(k)}(a^{\#}(f_x)),h\big]\psi = \sum_{\substack{|\beta|+|\gamma|+|\delta|\leq 2k\\ |\gamma|+|\delta|\leq 2k-1}} C^{(k)}_{\beta,\gamma,\delta}\big[a^{\#}(D_x^\beta f_x)x^\gamma D_x^\delta,h\big]\psi\\ &= \sum_{\substack{|\beta|+|\gamma|+|\delta|\leq 2k\\ |\gamma|+|\delta|\leq 2k-1}} C^{(k)}_{\beta,\gamma,\delta}\Big\{\big[a^{\#}(D_x^\beta f_x),h\big]x^\gamma D_x^\delta\psi + a^{\#}(D_x^\beta f_x)\big[x^\gamma D_x^\delta,h\big]\psi\Big\}\,. \end{aligned}\tag{D.36}$$

The calculation of the first term in (D.36) is straight forward and analogous to (D.35):

$$\big[a^{\#}(D_x^\beta f_x),h\big]x^\gamma D_x^\delta\psi = (a^{\#}(\Delta_x D_x^\beta f_x) + 2a^{\#}(\nabla_x D_x^\beta f_x)\nabla_x - \mathrm{sgn}(\#)a^{\#}(D_x^\beta f_x))x^\gamma D_x^\delta\psi\,. \tag{D.37}$$

For the second term in (D.36) we remark that the commutators

$$[x^\gamma,-\Delta_x] = \sum_{l=1}^{d} x^{\gamma-\gamma_l e_l}\big(\gamma_l(\gamma_l-1)x_l^{\gamma_l-2} + 2\gamma_l x_l^{\gamma-1}\partial_{x_l}\big)\,, \tag{D.38}$$

$$[D_x^\delta,x^2] = \sum_{l=1}^{d}\big(2\delta_l x_l\partial_{x_l}^{\delta_l-1} + 2(\delta_l-1)!\partial_{x_l}^{\delta_l-2}\big)D_x^{\delta-\delta_l e_l} \tag{D.39}$$

are again polynomials in x and D_x of the same order $|\gamma|$ and $|\delta|$, respectively. Hence, all resulting terms from (D.36) are in the desired form. This allows us to conclude that, by choosing the right coefficients $C^{(k+1)}_{\beta,\gamma,\delta}$,

$$\mathrm{ad}_h^{(k+1)}(a^{\#}(f_x))\psi = \quad = \sum_{\substack{|\beta|+|\gamma|+|\delta|\leq 2(k+1)\\ |\gamma|+|\delta|\leq 2(k+1)-1}} C^{(k)}_{\beta,\gamma,\delta}a^{\#}(D_x^\beta f_x)x^\gamma D_x^\delta\psi\,, \tag{D.40}$$

which proves part (i) of the lemma.

We proceed with the proof of part (ii), which is done by induction on the two variables $(|\gamma|,|\delta|)$ in three steps. First we prove the base case $(0,0)$, then $(k,0)$ and finally that if $(\tilde{k},\tilde{n})$ is true for all $\tilde{k}\leq k,\tilde{n}\leq n$ then also $(k,n+1)$.

The case $(0,0)$ is trivial and also $(k,0) = (|\gamma|,0)$ follows directly form Lemma D.3.1 as $h_{\text{oc}}^{|\gamma|} \geq Cx^{2|\gamma|} \geq Cx^{2\gamma}$.

We proceed with the induction step. Set $|\gamma| =: k$ and $|\delta| =: n$. Then by the induction hypotheses

$$(-iD_x)^{\delta+e_l} x^{2\gamma} (-iD_x)^{\delta+e_l} \leq -C\partial_{x_l} h^{|\delta|+|\gamma|} \partial_{x_l} \,. \tag{D.41}$$

From here we conclude the claim from the following estimate

$$\|h_{\text{oc}}^{\frac{m}{2}} \partial_{x_l} \psi\|^2 + \|h_{\text{oc}}^{\frac{m}{2}} x_l \psi\|^2 \leq C_m \|h_{\text{oc}}^{\frac{m+1}{2}} \psi\|^2 \tag{D.42}$$

for all $m \in \mathbb{N}_0$ and $\psi \in \bigcup_{L\geq 0} \bigoplus_{n=0}^{L} \mathscr{S}(\mathbb{R}^d,\mathbb{C}) \bar{\otimes} L^2(\mathbb{R}^d,\mathbb{C})^{\bar{\otimes}_s n} =: D$, where $C_m > 0$ only depends on m.

Proof. Proof by induction on $m \in \mathbb{N}_0$.

The base case $m = 0$ is trivial. We also treat the case $m = 1$ separately, since the induction step from m to $m+1$ requires $m \geq 1$ as a prerequisite. We set $F_1 := x_l$, $F_2 := \partial_{x_l}$ and $h := h_{\text{oc}}$. Then

$$\sum_{j=1}^{2} \|h_{\text{oc}}^{\frac{m}{2}} F_j \psi\|^2 = \sum_{j=1}^{2} \langle \psi, F_j^* h F_j \psi \rangle \,. \tag{D.43}$$

To estimate this in terms of $\langle \psi, h^2 \psi \rangle$ we commute the operators F_j and half a power of the h. This is done by splitting h in the right way

$$h = (-\nabla_x)(\nabla + x) + (\mathcal{N}+1)^{1/2}(\mathcal{N}+1)^{1/2} \,. \tag{D.44}$$

By the base case $m = 0$ both $B := (\nabla + x)$ and $(\mathcal{N}+1)^{1/2}$ are bounded by $h^{1/2}$ and serve as substitutes for $h^{1/2}$. We get from (D.44) that

$$\begin{aligned}
\sum_{j=1}^{2} \|h_{\text{oc}}^{\frac{m}{2}} F_j \psi\|^2 &= \sum_j \Big\{ \langle \psi, F_j^* (\mathcal{N}+1) F_j \psi \rangle + \langle \psi, F_j^* B^* B F_j \psi \rangle \Big\} \\
&= \sum_j \Big\{ \Big\langle \psi, (\mathcal{N}+1)^{1/2} F_j^* F_j (\mathcal{N}+1)^{1/2} \psi \Big\rangle + \langle \psi, B^* F_j^* F_j B \psi \rangle \\
&+ 2\mathrm{Re}\langle [B,F_j]\psi, BF_j\psi \rangle + \langle [F_j,B]\psi, [B,F_j]\psi \rangle \Big\} \,.
\end{aligned} \tag{D.45}$$

Omitting the last term which is negative and using $[\partial_{x_j} + x_j, x_l] = \delta_{j,l}$ and $[\partial_{x_j} + x_j, \partial_{x_l}] = -\delta_{j,l}$ we get

$$\sum_{j=1}^{2} \|h_{\text{oc}}^{\frac{m}{2}} F_j \psi\|^2 \leq \Big\langle \psi, (\mathcal{N}+1)^{1/2} (-\partial_{x_l}^2 + x_l^2) (\mathcal{N}+1)^{1/2} \psi \Big\rangle$$

$$
\begin{aligned}
&+ \left\langle \psi, B^*(-\partial_{x_l}^2 + x_l^2)B\psi \right\rangle + 2\mathrm{Re}\langle (-1)(-\partial_{x_l} + x_l)\psi, \partial_{x_l}\psi\rangle \\
&+ 2\mathrm{Re}\langle (-\partial_{x_l} + x_l)\psi, \partial_{x_l}\psi\rangle \\
\leq &\left\langle \psi, (\mathcal{N}+1)^{1/2} h (\mathcal{N}+1)^{1/2}\psi \right\rangle + \left\langle \psi, B^*(-\partial_{x_l}^2 + x_l^2)B\psi \right\rangle \\
&+ 2\mathrm{Re}\langle (-\partial_{x_l} + x_l)\psi, (-\partial_{x_l} + x_l)\psi\rangle \, . \qquad \text{(D.46)}
\end{aligned}
$$

And by the base case $m = 0$ we get $2\|(-\partial_{x_l} + x_l)\psi\|^2 \leq 4(\|\partial_{x_l}\psi\|^2 + \|x_l\psi\|^2) \leq 4\|h^{1/2}\psi\|^2$. Hence

$$
\sum_{j=1}^{2} \|h_{\mathrm{oc}}^{\frac{m}{2}} F_j \psi\|^2 \leq \left\langle \psi, h^2\psi \right\rangle + \langle \psi, B^* h B\psi\rangle + 4\|h^{1/2}\psi\|^2 \, . \qquad \text{(D.47)}
$$

It remains to prove that $\langle \psi, B^* h B\psi\rangle$ is bounded by $\left\langle \psi, h^2\psi \right\rangle$. To prove this we start from $\left\langle \psi, h^2\psi \right\rangle$ and derive a lower bound. With (D.44) we get

$$
\begin{aligned}
\left\langle \psi, h^2\psi \right\rangle &= \langle \psi, h(\mathcal{N}+1)\psi\rangle + \langle \psi, hB^*B\psi\rangle \\
&\geq \langle \psi, B^* h B\psi\rangle + \langle \psi, [hB^*]B\psi\rangle \\
&\geq \langle \psi, B^* h B\psi\rangle + \langle \psi, 2B^*B\psi\rangle \\
&\geq \langle \psi, B^* h B\psi\rangle \, . \qquad \text{(D.48)}
\end{aligned}
$$

From (D.47) and (D.48) we conclude the claim for $m = 1$, namely

$$
\sum_{j=1}^{2} \|h_{\mathrm{oc}}^{\frac{m}{2}} F_j \psi\|^2 \leq 6\|h\psi\|^2 \, . \qquad \text{(D.49)}
$$

Now we are ready to begin with the induction step. Let $m \in \mathbb{N}_+$ be fixed and assume that the assertion is true for all $\tilde{m} \leq m$. We set as before $F_1 := x_l$ and $F_2 := \partial_{x_l}$. We begin the estimate by using the induction hypotheses for $m-1$

$$
C_{m-1}\|h^{\frac{(m+1)+1}{2}}\psi\|^2 \geq \sum_{j=1}^{2} \|h^{\frac{(m-1)}{2}} F_j h\psi\|^2 = \sum_{j} \left\langle \psi, hF_j^* h^{m-1} F_j h\psi \right\rangle . \qquad \text{(D.50)}
$$

It is important to apply the induction hypotheses for $m-1$, rather than for m directly, in order to retain a full power of h, rather than $h^{1/2}$, appearing before F_j in (D.50). This is necessary because commutators involving the complete operator h and F_j can be explicitly computed, whereas commutators with fractional powers $h^{1/2}$ are more subtle and not directly accessible.

We now proceed by commuting both F_j and F_j^* with the single h to receive the terms on the left-hand side of (D.42):

$$
\begin{aligned}
\text{(D.50)} &= \sum_j \Big\{ \langle \psi, hF_j^* h^m F_j \psi \rangle + \langle \psi, hF_j^* h^{m-1}[F_j, h]\psi \rangle \Big\} \\
&= \sum_j \Big\{ \langle \psi, F_j^* h^{m+1} F_j \psi \rangle + \langle \psi, [h, F_j^*] h^m F_j \psi \rangle \\
&\quad + \langle \psi, F_j^* h^m [F_j, h]\psi \rangle + \langle \psi, [h, F_j^*] h^{m-1}[F_j, h]\psi \rangle \Big\} . \qquad \text{(D.51)}
\end{aligned}
$$

The last term in (D.51) is positive and can be neglected. For the terms with the single commutator we use $2\mathrm{Re}\langle x, y\rangle = \|x+y\|^2 - \|x\|^2 - \|y\|^2 \geq -\|x\|^2 - \|y\|^2$ to obtain

$$
\text{(D.51)} = \sum_j \Big\{ \langle \psi, F_j^* h^{m+1} F_j \psi \rangle - \|h^{m/2} F_j \psi\|^2 - \|h^{m/2}[F_j, h]\psi\|^2 \Big\} . \qquad \text{(D.52)}
$$

Next, we compute the commutators explicitly: $[F_1, h] = 2F_2$ and $[F_2, h] = 2F_1$, leading to

$$
\begin{aligned}
\text{(D.52)} &= \sum_j \Big\{ \langle \psi, F_j^* h^{m+1} F_j \psi \rangle - 3\|h^{m/2} F_j \psi\|^2 \Big\} \\
&\geq \sum_j \Big\{ \langle \psi, F_j^* h^{m+1} F_j \psi \rangle - 3C_k \|h^{(m+1)/2}\psi\|^2 \Big\} \\
&\geq \sum_j \Big\{ \langle \psi, F_j^* h^{m+1} F_j \psi \rangle - 3C_k \|h^{((m+1)+1)/2}\psi\|^2 \Big\} , \qquad \text{(D.53)}
\end{aligned}
$$

where we used the induction hypothesis for m and $h \geq 1$. Hence

$$
(C_{m-1} + 3C_m)\|h^{\frac{(m+1)+1}{2}}\psi\|^2 \geq \sum_j \langle \psi, F_j^* h^{m+1} F_j \psi \rangle ,
$$

which proves (D.42). □

■

D.4 Existence of the Bogoliubov-Fröhlich Dynamics

Definition D.4.1 (The Polaron Hamiltonian)**.** Let $\mathscr{H}$ be a separable Hilbert space, $h_1 \geq 0$ self-adjoint, $h_2(t) \in \mathcal{L}(\mathscr{H})$ self-adjoint, $k_2(t) \in \mathrm{HS}(\mathscr{H}^*, \mathscr{H}))$ and $k_2^*(t) = Jk_2(t)J$, $\forall t \in \mathbb{R}$. Let $d \in \mathbb{N}_+$, $f_t \in L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d, \mathbb{C})))$, $g_t \in L^\infty(\mathbb{R}^d_x, \mathbb{R})$ for all $t \in \mathbb{R}$. Set the polaron quadratic form as

$$
\begin{aligned}
&q_{H^{\mathrm{pol}}(t)} := q_{H^{\mathrm{pol}}(t)}(h_1, h_2(t), k_2(t), f_t, g_t) : Q(d\Gamma(h_1+1)-\Delta_x) \times Q(d\Gamma(h_1+1)-\Delta_x) \to \mathbb{C} \\
&(\psi, \varphi) \mapsto q_{H^{\mathrm{qua}}(t)}(\psi, \varphi) + q_{H^{\mathrm{F}}(t)}(\psi, \varphi) \,,
\end{aligned}
$$

where $\forall (\psi, \varphi) \in Q(d\Gamma(h_1+1)-\Delta_x) \times Q(d\Gamma(h_1+1)-\Delta_x)$ and

$$
\begin{aligned}
q_{H^{\mathrm{qua}}(t)}(\psi) &= q_{d\Gamma(h_1)}(\psi) + \tilde{q}_{d\Gamma(h_2(t))}(\psi) + \mathrm{ReTr}(k_2(t)\alpha_\psi) \,, \\
q_{H^{\mathrm{T}}(t)}(\psi, \varphi) &= \langle \psi, M_{g_{t,x}} \otimes I_{\mathcal{F}} \cdot \varphi \rangle + q_{-\Delta_x}(\psi, \varphi) + \langle \psi, A(f_{t,x} \oplus J f_{t,x})\varphi \rangle \,,
\end{aligned}
$$

from Definition D.2.7 and Definition D.3.2. Note that $q_{H^{\mathrm{Pol}}}$ is a symmetric quadratic form in $L^2(\mathbb{R}^d_x, \mathcal{F}(L^2))$.

Theorem D.4.2 (Dynamics of Time-dependent Quadratic Operators with Tracer Particle)**.** *Let $d \in \mathbb{N}_+$.*

i) (Theorem D.2.8 conditions) Let $h_1 \geq 0$ self-adjoint operator on $\mathscr{H} := L^2(\mathbb{R}^d, \mathbb{C})$, $h_2 \in C^1(\mathbb{R}, \mathcal{L}(\mathscr{H}))$ such that $h_2(t)$ self-adjoint, $k_2 \in C^1(\mathbb{R}, \mathrm{HS}(\mathscr{H}^*, \mathscr{H}))$ such that $k_2(t)^* = J k_2(t) J$, $\forall t \in \mathbb{R}$. Set $h(t) = h_1 + h_2(t)$.

ii) (Theorem D.3.3 conditions for $M = 1$) Let $f \in l^\infty_{loc}(\mathbb{R}_t, W^{1,\infty}(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C})))$ such that for a.e. $x \in \mathbb{R}^d$: $(t \mapsto f_t(x, .)) \in C^1(\mathbb{R}_t, L^2(\mathbb{R}^d_y, \mathbb{C}))$ and $\forall t$: $\partial_t f_t := (x \mapsto \partial_t f_t(x, .)) \in L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C}))$ and $\dot{f} := (t \mapsto \partial_t f_t) \in l^\infty_{loc}(\mathbb{R}_t, L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C})))$. Let $g \in l^\infty_{loc}(\mathbb{R}_t, W^{1,\infty}(\mathbb{R}^d_x, \mathbb{R}))$ with $g \in C^1(\mathbb{R}_t, L^\infty(\mathbb{R}^d_x, \mathbb{R}))$.

Set $q_{H^{\mathrm{pol}}(t)} := q_{H^{\mathrm{pol}}(t)}(h_1, h_2(t), k_2(t), f_t, g_t)$. Then $\{q_{H^{\mathrm{pol}}(t)}\}_{t \in \mathbb{R}}$ satisfies the conditions of Theorem D.1.1 with comparison operator $A := d\Gamma(h_1) - \Delta_x + x^2 + \mathcal{N} + 1$ and $B := -\Delta_x + x^2 + \mathcal{N} + 1$.

Proof of Theorem D.4.2. Theorem D.4.2 is a Corollary of Theorem D.1.1. Due to $q_{H^{\mathrm{pol}}(t)} = q_{H^{\mathrm{qua}}(t)} + q_{H^{\mathrm{T}}(t)}$, one can obtain all the estimates needed for the Theorem D.1.1 conditions by adding the estimates done to prove Theorem D.2.8 and Theorem D.3.3. The comparison operators $A := d\Gamma(h_1) - \Delta_x + x^2 + \mathcal{N} + 1$ and $B := -\Delta_x + x^2 + \mathcal{N} + 1$ for Theorem D.1.1 are chosen in a way that they control the comparison operators chosen in Theorem D.2.8 and Theorem D.3.3. ■

Corollary D.4.3 (Existence of the Bogoliubov-Fröhlich Dynamics)**.** *For volumes $\Lambda \geq 1$ let φ_0 be the condensate satisfying Condition 2.1.7.*

Then Theorem D.4.2 can be applied to

i) $H^{\mathrm{BF}}(t)$ (see Definition 2.3.2).

ii) $H^{\mathrm{BF},\rho}(t)$ *(see* (4.2)*).*

Both are defined as quadratic forms $(q_{H^{\mathrm{BF}}(t)}, Q(-\Delta_x+x^2+d\Gamma(1-\Delta)))$ *and* $(q_{H^{\mathrm{BF},\rho}(t)}, Q(-\Delta_x+x^2+d\Gamma(1-\Delta)))$ *as in Definition D.4.1.*

Remark D.4.4. In Corollary D.4.3, we establish the existence of the dynamics on the quadratic form domain $Q(-\Delta_x + x^2 + d\Gamma(1-\Delta))$. However, the natural space of definition for the associated quadratic forms – and thus for the existence of the dynamics – is $Q(-\Delta_x+d\Gamma(1-\Delta))$. Extending the result to this case is straightforward. Reviewing the proofs of Theorem D.3.3 and Theorem D.4.2, we see that they remain valid when the x^2 term in the comparison operators A and B is omitted. This proves the existence of the dynamics for $H^{\mathrm{BF}}(t)$ and $H^{\mathrm{BF},\rho}(t)$ on $Q(-\Delta_x+d\Gamma(1-\Delta))$. Note that the reason why we need to globally assume $W \in W^{1,\infty} \cap H^2$ in Assumption $2.0.3_1$ is to prove the existence of the Bogoliubov-Fröhlich dynamics shown above.

Proof of Corollary D.4.3. The Corollary is proven by applying Theorem D.4.2 to the quadratic forms $q_{H^{\mathrm{BF}}(t)}$ and $q_{H^{\mathrm{BF},\rho}(t)}$, after rescaling them by $2m$ – twice the impurity mass – to ensure the impurity Laplacian appears with the correct scaling. After applying the theorem, this additional factor can be absorbed into the time variable. To use Theorem D.4.2 we have to verify its Theorem D.2.8 and Theorem D.3.3 condition.
For $q_{H^{\mathrm{BF}}(t)}$ the Theorem D.3.3 condition is proven in Lemma 5.4.8, for $M = 1$, and the Theorem D.2.8 condition is verified in Lemma C.0.1.
For the quadratic form $q_{H^{\mathrm{BF},\rho}(t)}$ the Theorem D.3.3 condition is proven in Lemma 5.4.8, for $M = 1$, and Lemma D.4.5, and the Theorem D.2.8 condition is verified in Lemma C.0.1. ■

Lemma D.4.5 (Conditions on g in Theorem D.3.3 for $H^{\mathrm{BF},\rho}$)**.** *For volumes* $\Lambda \geq 1$ *let* φ_t *be the condensate with initial data* $\varphi_0 \in H^\infty(\mathbb{R}^3,\mathbb{C})$ *satisfying Condition 2.1.7.*
Let $g = (t \mapsto \rho^{1/2} W * |\varphi_t|^2)$ *and* $k \in \mathbb{N}_0$*, then* $g \in C^1(\mathbb{R}_t, W^{k,\infty}(\mathbb{R}^d,\mathbb{R})) \subset l^\infty_{loc}(\mathbb{R}_t, W^{k,\infty}(\mathbb{R}^d,\mathbb{R}))$ *and* $g_t \in C_b^\infty(\mathbb{R}^d,\mathbb{R})$ *and* $D^\alpha \rho^{1/2} W * |\varphi_t|^2 = \rho^{1/2} W * D^\alpha |\varphi_t|^2$, $\forall t \in \mathbb{R}, \alpha \in \mathbb{N}_0^3$.
If in addition we assume Condition $2.1.8_{k=2+2}$ *for the condensate then we have* $\forall T \geq 0\ \exists C > 0$ *such that* $\forall \Lambda \geq 1$ *and* $-T \leq t \leq T$

$$\|g_t\|_{W^{k,\infty}(\mathbb{R}^d,\mathbb{R})} + \|\dot{g}_t\|_{W^{k,\infty}(\mathbb{R}^d,\mathbb{R})} \leq C\rho^{1/2}\,. \tag{D.1}$$

Proof of Lemma D.4.5. The regularity in spatial argument $\rho^{1/2} W * |\varphi_t|^2 \in C_b^\infty(\mathbb{R}^3,\mathbb{R})$ and $D^\alpha \rho^{1/2} W * |\varphi_t|^2 = \rho^{1/2} W * D^\alpha |\varphi_t|^2$, $\forall t \in \mathbb{R}, \alpha \in \mathbb{N}_0^3$ is clear from $\varphi_t \in H^\infty(\mathbb{R}^3,\mathbb{C})$ and $W \in L^\infty$. Form this it is easy to conclude

$$(t \mapsto \rho^{1/2} W * |\varphi_t|^2) \in C^1(\mathbb{R}_t, W^{k,\infty}(\mathbb{R}^d,\mathbb{R})) \subset l^\infty_{\mathrm{loc}}(\mathbb{R}_t, W^{k,\infty}(\mathbb{R}^d,\mathbb{R}))$$

and $k \in \mathbb{N}_0$, $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\|g_t\|_{W^{k,\infty}(\mathbb{R}^d,\mathbb{R})} + \|\dot{g}_t\|_{W^{k,\infty}(\mathbb{R}^d,\mathbb{R})} \leq C\rho^{1/2}\,.$$

Proof. Let $0 \le |\alpha| \le k$. We start by proving the regularity

$$\begin{aligned} &\left\| \rho^{1/2} W * D^\alpha \left(\frac{|\varphi_{t+h}|^2 - |\varphi_t|^2}{h} - \partial_t |\varphi_t|^2 \right) \right\|_\infty \\ \le\, &\rho^{1/2} \|W\|_\infty \left\| D^\alpha \left(\frac{|\varphi_{t+h}|^2 - |\varphi_t|^2}{h} - \partial_t |\varphi_t|^2 \right) \right\|_1 \to 0 \end{aligned}$$

there we have use $\varphi \in C^1(\mathbb{R}, H^{|\alpha|})$ and

$$\begin{aligned} &\left\| D^\alpha \left(\frac{|\varphi_{t+h}|^2 - |\varphi_t|^2}{h} - \partial_t |\varphi_t|^2 \right) \right\|_{L^1} \\ = &\left\| D^\alpha \left(\frac{\varphi_{t+h}^* \varphi_{t+h} - \varphi_t^* \varphi_t}{h} - \varphi_t^* \partial_t \varphi_t - \partial_t \varphi_t^* \cdot \varphi_t \right) \right\|_{L^1} \to 0\,. \end{aligned}$$

This proves differentiability of $(t \mapsto \rho^{1/2} W * D^\alpha |\varphi_t|^2)$. Its continuity and continuity of the derivative is proven in a similar way.
Now the face the bounds:

$$\left\| \rho^{1/2} W * D^\alpha |\varphi_t|^2 \right\|_\infty \le \rho^{1/2} \|W\|_{1,\infty} \|D^\alpha \varphi_t\|_{2\wedge\infty}^2 \le C\rho^{1/2}$$

where in step 2 we used Corollary B.2.4. Next we use the Hartree equation, $\mathrm{Re}\varphi_t^*(-\mathrm{i})V * |\varphi_t|^2 \varphi_t = 0$ and Corollary B.1.2 and Corollary B.2.4 to conclude $\forall T \ge 0\ \exists C > 0$ such that $\forall \Lambda \ge 1$, $-T \le t \le T$

$$\begin{aligned} &\left\| D^\alpha \rho^{1/2} W * 2\mathrm{Re}\varphi_t^* \partial_t \varphi_t \right\|_\infty \\ = &\left\| D^\alpha \rho^{1/2} W * 2\mathrm{Re}\varphi_t^* (-\mathrm{i})(-\Delta/2) \varphi_t \right\|_\infty \\ \le\, &\rho^{1/2} \|W\|_{1,\infty} C \sum_{\beta=0}^{\alpha} \left\| D^{\alpha-\beta} \varphi_t^* \right\|_{2\wedge\infty} \|D^\alpha \Delta \varphi_t\|_{2\wedge\infty} \\ \le\, &\rho^{1/2} C \Lambda^{-(|\alpha|+2)/3}\,. \end{aligned}$$

□

■

D.5 Proofs of Appendix D

D.5.1 Proofs of Theorem D.1.1 and Corollary D.1.3

First we prove Theorem D.1.1 following the proof of [LNS15, Theorem 8] and then Corollary D.1.3 using standard methods.

Proof of Theorem D.1.1

Proof of Theorem D.1.1. Let $(t_0, u_0) \in I \times Q(A)$. We establish the existence and uniqueness of the formal Cauchy problem

$$\begin{aligned} \mathrm{i}\partial_t u(t) &= H(t)u(t)\,, \\ u(t_0) &= u_0 \end{aligned}$$

in the weak sense, as described in Theorem D.1.1. The proof of existence is divided into three main steps:

Regularized Dynamics: We prove the existence of solutions $u_n(t)$ to the differential equation with regularized Hamiltonian $H_n(t)$, for $n \in \mathbb{N}_0$.

Candidates for the Full Dynamics: Using a Grönwall argument, we derive energy bounds for $\langle u_n(t), Au_n(t)\rangle$ and $\langle \dot{u}_n(t), A^{-1}\dot{u}_n(t)\rangle$. These bounds lead to candidates $u(t) \in Q(A)$ and $\dot{u}(t) \in Q(A)^*$ as a potential solution to the differential equation.

Validation of the Full Dynamics: We show that u and $\dot{u}$ fulfil $\mathrm{i}\dot{u} = q_{H(t)}(\,.\,, u(t))$, and that $\dot{u}$ is indeed the derivative of u.

The uniqueness of the solution u follows from the norm conservation: $\|u(t)\|_{\mathscr{H}} = \|u(t_0)\|_{\mathscr{H}}$. The steps above are carried out on bounded time intervals $I_b \subset I$. Finally, the proof is extended to potentially unbounded intervals I.

Existence of a solution. Let $n \in \mathbb{N}_0$, $I_b \subset I$ bounded interval with $t_0 \in I_b$. We set $P_n := \mathbb{1}_{B_n}(A)$ to be the spectral projection of A on the ball B_n around the origin of radius $n \in \mathbb{N}_0$. From [RS80, Theorem VIII.15] we know that there exists a unique self-adjoint Hamiltonian $H_n(t) \in \mathcal{L}(\mathscr{H})$ such that $\forall u \in \mathscr{H}, t \in I_b$

$$\langle u, H_n(t)u\rangle = q_{H(t)}(P_n u, P_n u)\,. \tag{D.1}$$

$H_n(t)$ generates a dynamic as described by the following Lemma.

Lemma D.5.1 (Regularized dynamics)**.** *Let $n \in \mathbb{N}_0$. Then* *i) (Dynamics of $H_n(t)$)* $\exists! u_n \in C^1(I_b, \mathscr{H})$ *with* $u_n(t_0) = P_n u_0$ *and*

$$\mathrm{i}\dot{u}_n(t) = H_n(t) u_n(t) \quad \forall t \in I_b \,.$$

In addition $\|u_n(t)\|_{\mathscr{H}} = \|P_n u_0\|_{\mathscr{H}}$ *and* $u_n(t) = P_n u_n(t) \in Q(A^k)$ *for all* $k \in \mathbb{N}, t \in I$.

ii) (Estimates on $H_n(t)$) $\|H_n(t)\|_{\mathcal{L}(\mathscr{H})} \leq C_{12} n$, $-C_{12} n \leq H_n(t) \leq C_{12} n$, $H_n(t) = P_n H_n(t) = H_n(t) P_n$, *where* $C_{12} := \max\{C_1, C_2\}$ *and* $C_1 A \geq H_n(t)$, $\|A^{-1/2} H_n(t) A^{-1/2}\|_{\mathrm{op}} \leq C_{12}$.

iii) (Derivative of $H_n(t)$) For all $u \in \mathscr{H}$ *we have* $(t \mapsto \langle u, H_n(t) u \rangle) \in C^1(I_b, \mathbb{R})$ *and*

$$\partial_t \langle u, H_n(t) u \rangle) = \partial_t q_{H(t)}(P_n u) \leq C_4 q_A(P_n u) \,.$$

Remark D.5.2. For the regularized dynamics we set the initial data as $u_n(t_0) = P_n u_0$ instead of u_0, which allows us to conclude $P_n u_n(t) = u_n(t)$ and hence simplifies the estimates below.

Proof. For the first part i) we use the same methods as to prove the well-posedness of the Hartree equation in Lemma 2.1.3. In fact the argument here is less complicated because $H_n(t)$ is a bounded operator.
The estimates in the second and third part ii) and iii) are readily verified by using $C_1 q_A \geq q_{H(t)} \geq C_1^{-1} q_A - C_2 q_B$, $0 \leq B \leq A$ and the differentiability of $t \mapsto q_{H(t)}(u)$ for all $u \in Q(A)$. ■

We arrived at the second step in the proof, where we do a Grönwall estimate for the regularized dynamics u_n.

Lemma D.5.3. *Under the conditions of Theorem D.1.1 we have that for all* $t \in I_b, n \in \mathbb{N}_0$

$$\langle u_n(t), A u_n(t) \rangle \leq 2 C_{12}^2 e^{C_{1234}|t - t_0|} q_A(u_0) \,, \tag{D.2}$$

$$\left\langle \dot{u}_n(t), A^{-1} \dot{u}_n(t) \right\rangle \leq 2 C_{12}^4 e^{C_{1234}|t - t_0|} q_A(u_0) \tag{D.3}$$

with $C_{1234} := 2 C_1 C_2 C_3 + C_1 C_4$, $C_{12} := \max\{C_1, C_2\}$ *and* u_n *as defined in Lemma D.5.1. In addition the sequence* (u_n) *is equicontinuous in* t*, meaning it converges uniform in* n*:*

$$\|u_n(t) - u_n(s)\|_{\mathscr{H}} \leq |t - s|^{1/2} \left(4 C_{12}^3 e^{C_{1234} \cdot \sup I_b} q_A(u_0)\right)^{1/2} \,. \tag{D.4}$$

Proof. We begin with the proof of (D.2). Because we can not control (D.2) directly we introduce an auxiliary operator controlling A

$$\mathcal{L}(\mathscr{H}) \ni A_n(t) := H_n(t) + C_2 P_n B P_n \geq C_1^{-1} P_n A P_n \,.$$

For $A_n(t)$ the estimate (D.2) can be achieve

$$\langle u_n(t), A_n(t) u_n(t) \rangle \leq \mathrm{e}^{C_{1234}|t - t_0|} \langle u_0, A_n(0) u_0 \rangle \,. \tag{D.5}$$

Because $\langle u_n(t), Au_n(t)\rangle = \langle u_n(t), P_nAP_nu_n(t)\rangle \leq C_1\langle u_n(t), A_n(t)u_n(t)\rangle$ and $\langle u_0, A_n(0)u_0\rangle = \langle u_0, H_n(0)u_0\rangle + C_2\langle u_0, P_nBP_nu_0\rangle \leq (C_1+C_2)q_A(P_nu_0) \leq (C_1+C_2)q_A(u_0)$ we can conclude (D.2).

To prove (D.5) we use the definition of $A_n(t)$ and $P_nu_n(t) = u_n(t)$

$$\begin{aligned}\frac{d}{dt}\langle u_n(t), A_n(t)u_n(t)\rangle &= \frac{d}{dt}q_{H(t)}(u_n(t)) + C_2\langle u_n(t), Bu_n(t)\rangle \\ &= q_{\dot{H}(t)}(u_n(t)) \qquad \text{(D.6)} \\ &\quad + \langle u_n(t), (-\mathrm{i})[A_n(t), H_n(t)]u_n(t)\rangle \qquad \text{(D.7)}\end{aligned}$$

where we denote by $q_{\dot{H}(t)}(y) \in C(I_b, \mathbb{R})$ we derivative of $q_{H(t)}(y) \in C^1$. The first term is estimated directly by condition a)iv) of Theorem D.1.1

$$\text{(D.6)} \leq C_4q_A(u_n(t)) \leq C_4C_1q_{A_n(t)}(u_n(t))\,. \tag{D.8}$$

For the second term (D.3) we compute the commutator and use the identity $\langle\, .\, , H_n(t)\, .\rangle = q_{H(t)}(P_n\, .\, , P_n\, .)$ as well as that B commutes with A and hence with $P_n = \mathbb{1}_{B_n}(A)$

$$\begin{aligned}\text{(D.7)} &= (-\mathrm{i})C_2\langle u_n(t), [B, H_n(t)]u_n(t)\rangle \\ &= 2C_2\mathrm{Im}q_{H(t)}(P_nBu_n(t), P_nu_n(t)) \\ &= 2C_2\mathrm{Im}q_{H(t)}(Bu_n(t), u_n(t))\,. \qquad \text{(D.9)}\end{aligned}$$

Note that $Bu_n(t), u_n(t) \in Q(A)$. With the commutator condition $\mp 2\mathrm{Im}\, q_{H(t)}(u, Bu) \leq C_3q_A(u)$ for $u \in D(A^{3/2})$ as well as $u_n(t) = P_nu_n(t)$ we can prove

$$\text{(D.9)} \leq 2C_2C_3q_A(u_n(t)) \leq 2C_1C_2C_3q_{A_n(t)}(u_n(t))\,. \tag{D.10}$$

Combining (D.8) and (D.10) we conclude

$$\frac{d}{dt}\langle u_n(t), A_n(t)u_n(t)\rangle \leq C_1\big(C_4 + 2C_2C_3)q_{A_n(t)}(u_n(t))\,.$$

From here (D.5) follows using Grönwall.
Facing (D.3), we use $\|A^{-1/2}H_n(t)A^{-1/2}\|_{\mathrm{op}} \leq C_{12}$ from Lemma D.5.1 and (D.2) to conclude

$$\begin{aligned}\big\langle \dot{u}_n(t), A^{-1}\dot{u}_n(t)\big\rangle &= \big\langle u_n(t), H_n(t)A^{-1}H_n(t)u_n(t)\big\rangle \\ &\leq \|A^{1/2}u_n(t)\|^2_{\mathscr{H}}\|\|A^{-1/2}H_n(t)A^{-1/2}\|^2_{\mathrm{op}} \\ &\leq C_{12}^2\langle u_n(t), Au_n(t)\rangle \\ &\leq C_{12}^2 2C_{12}^2 e^{C_{1234}|t-t_0|}q_A(u_0)\,.\end{aligned}$$

Now we prove the equicontinuity of $(u_n) \subset C^1(I_b, \mathscr{H})$. We use their differentiability, the fundamental Theorem of calculus and use the estimates (D.2) and (D.3) to conclude

$$\begin{aligned}\|u_n(t)-u_n(s)\|^2_{\mathscr{H}} &= \int_s^t 2\mathrm{Re}\langle u_n(\tau),\dot u_n(\tau)\rangle_{\mathscr{H}} d\tau \\ &\le 2\int_s^t \|u_n(\tau)\|_{Q(A)}\|\dot u(\tau)\|_{Q(A)^*} d\tau \\ &\le |t-s| 4C_{12}^3 e^{C_{1234}\cdot \sup I_b} q_A(u_0)\,.\end{aligned}$$

■

The bounds (D.2) and (D.3) are enough to provide us with a candidate for the solution and its derivative of the not regularized dynamics through the Banach-Alaoglu Theorem.

Lemma D.5.4. *Let u_n be as defined in Lemma D.5.1. Under the conditions of Theorem D.1.1 there exists a subsequence n_l, $u \in L^\infty(I_b, Q(A))$ and $\dot u \in L^\infty(I_b, Q(A)^*)$ such that*

$$\int \langle u_{n_l} - u, y\rangle_{Q(A)\times Q(A)^*} dt \to 0\,, \quad \forall y \in L^1(I_b, Q(A)^*)\,, \tag{D.11}$$

$$\int \langle \dot u_{n_l} - \dot u, y\rangle_{Q(A)^*\times Q(A)} dt \to 0\,, \quad \forall y \in L^1(I_b, Q(A))\,, \tag{D.12}$$

$$\operatorname*{ess\,sup}_{t\in I_b} \|u(t)\|^2_{Q(A)} \le 2C_{12}^2 \mathrm{e}^{C_{1234}\sup I_b} q_A(u_0)\,, \tag{D.13}$$

$$\operatorname*{ess\,sup}_{t\in I_b} \|\dot u(t)\|^2_{Q(A)^*} \le 2C_{12}^4 \mathrm{e}^{C_{1234}\sup I_b} q_A(u_0)\,. \tag{D.14}$$

Proof. We use the isometric isomorphism $\phi_{\mathscr{H}} : L^\infty(\Omega, \mathscr{H}) \to (L^1)^*(\Omega, \mathscr{H}^*), f \mapsto (g \mapsto \int \langle f, g\rangle_{\mathscr{H}\times\mathscr{H}^*} dt)$, $\Omega \subset \mathbb{R}^d$ [Kre15, Theorem 2.22] and that $L^1(\Omega, \mathscr{H}^*)$ is separable if $\mathscr{H}$ is separable to conclude the claim from Lemma D.5.3, Banach-Alaoglu: Lemma F.0.4 and $\|\,.\,\|_{Q(A)^*} = \|A^{-1/2} j\,.\,\|_{\mathscr{H}}$ with the embedding $j : \mathscr{H} \to Q(A)^*, \psi \mapsto \langle\,.\,,\psi\rangle_{\mathscr{H}}$. ■

Now we connect our candidates u, $\dot u$ from Lemma D.5.4 to the differential equation.

Lemma D.5.5. *If we assume the conditions of Theorem D.1.1 we have for $u, \dot u$ as defined in Lemma D.5.4 that*

i) $\dot u$ *is the weak derivative of* $j \circ u$ *with the embedding* $j : \mathscr{H} \to Q(A)^*, \psi \mapsto \langle\,.\,,\psi\rangle_{\mathscr{H}}$.

ii) $\mathrm{i}\dot u(t) = q_{H(t)}(\,.\,, u(t))$.

iii) $u \in C(\overline{I_b}, \mathscr{H})$ *and* $u(t_0) = u_0$.

iv) For all $t \in I_b$ *we have*

$$q_A(u(t)) \le 2C_{12}^2 e^{C_{1234}|t-t_0|} q_A(u_0)\,.$$

Proof. Due to [Hun14, Proposition 6.36] it is enough to show the weak differentiability of $\langle \omega, j \circ u \rangle_{Q(A)\times Q(A)^*}$ and show $\partial_t \langle \omega, j \circ u \rangle = \langle \omega, \dot{u} \rangle$ for all $\omega \in Q(A)$ to prove part i). Let $\omega \in Q(A)$ and $\varphi \in C_c^\infty(I_b, \mathbb{C})$ then

$$\int \dot{\varphi}(s) \langle \omega, j \circ u(s) \rangle_{Q(A)\times Q(A)^*} ds = \int \langle \dot{\varphi}^*(s)\omega, u(s) \rangle_{\mathscr{H}} ds \tag{D.15}$$

due to Lemma D.5.4 and $\dot{\varphi}^*(\,.\,)\omega \in L^1$ this can be expressed in the following limit

$$\begin{aligned} \text{(D.15)} &= \lim_{n\to\infty} \int \langle \dot{\varphi}^*(s)\omega, u_n(s) \rangle_{\mathscr{H}} ds \\ &= \lim_{n\to\infty} \int \langle \omega, \dot{\varphi}(s) u_n(s) \rangle_{\mathscr{H}} ds \,. \end{aligned} \tag{D.16}$$

Using [Hun14, Proposition 6.36], $u_n \in C^1(I_b, \mathscr{H})$ (see Lemma D.5.1) and the convergence from Lemma D.5.4 we conclude

$$\begin{aligned} \text{(D.16)} &= \lim_{n\to\infty} \int \langle \omega, \varphi(s) \dot{u}_n(s) \rangle_{\mathscr{H}} ds \\ &= \lim_{n\to\infty} \int \langle \varphi(s)^*\omega, j \circ \dot{u}_n(s) \rangle_{Q(A)\times Q(A)^*} ds \\ &= \int \langle \varphi(s)^*\omega, \dot{u}(s) \rangle_{Q(A)\times Q(A)^*} ds \\ &= \int \varphi(s) \langle \omega, \dot{u}(s) \rangle_{Q(A)\times Q(A)^*} ds \,, \end{aligned}$$

which concludes the proof of part i).
We proceed with the proof of ii). We want to show that $\mathrm{i}\dot{u}(t) = q_{H(t)}(\,.\,, u(t))$. For this, we use the uniqueness of the weak*-convergence in $(L^1)^*(I_b, Q(A))$ of $\dot{u}_n \rightharpoonup u$ from Lemma D.5.4 and prove $(t \mapsto q_{H(t)}(\,.\,, u(t))) \in L^\infty(I_b, Q(A)^*)) = (L^1)^*(I_b, Q(A))$ as well as

$$\int_{I_b} \big\langle \varphi(t), j(\mathrm{i}\dot{u}_n(t)) - q_{H(t)}(\,.\,, u(t)) \big\rangle dt \to 0 \,, \quad \forall \varphi \in L^1(I_b, Q(A)) \,. \tag{D.17}$$

We start with the proof of $(t \mapsto q_{H(t)}(\,.\,, u(t))) \in L^\infty(I_b, Q(A)^*)$.

Proof. First we show

$$((t \mapsto q_{H(t)}(\,.\,, y)) \in L^0(I_b, Q(A)^*) \,, \quad \forall y \in Q(A) \,. \tag{D.18}$$

then it is an easy task to conclude $(t \mapsto q_{H(t)}(\,.\,, u(t))) \in L^\infty(I_b, Q(A)^*)$ with Petti's measurability Theorem [Are+11, Theorem 1.1.1] which states that a function f mapping into a Banach space E is strongly measurable iff $g(f)$ is measurable for all $g \in E^*$ and ran f is separable.
Since we assumed that $\mathscr{H}$ is separable we conclude that $Q(A)$ is separable with

Lemma F.0.7. And because $Q(A)$ as a Hilbert space is reflexive also $Q(A)^*$ is separable. By Petti's measurability Theorem and $Q(A)^{**} = Q(A)$ it remains to prove $(t \mapsto q_{H(t)}(z,y) \in L^0(I_b, \mathbb{C})$ for all $y, z \in Q(A)$, which is a simple consequence of the continuity of $t \mapsto q_{H(t)}(y)$ and the polarization identity.
We processed with the proof of $(t \mapsto q_{H(t)}(\,.\,, u(t))) \in L^0(I_b, Q(A)^*)$. Again $(t \mapsto q_{H(t)}(y, u(t))) \in L^0(I_b, \mathbb{C})$ for all $y \in Q(A)$ due to Lemma D.5.6. Now by Petti's measurability Theorem it remains to show $\operatorname{ran}(t \mapsto q_{H(t)}(\,.\,, u(t)))$ is separable. Since $u \in L^0(I_b, Q(A))$ we know $\operatorname{ran} u$ separable. Let $D \subset \operatorname{ran} u$ dense in $Q(A)$–norm and countable. Since for all $y \in \operatorname{ran} u$ we have $(t \mapsto q_{H(t)}(\,.\,, y)) \in L^0(I_b, Q(A)^*)$ there exists $D_y \subset \operatorname{ran}(t \mapsto q_{H(t)}(\,.\,, y)$ dense in $Q(A)^*$–norm and countable. It remains to show

$$\bigcup_{y \in \operatorname{ran} u} \operatorname{ran}(t \mapsto q_{H(t)}(\,.\,, y)) \subset \overline{\bigcup_{y \in D} D_y}^{Q(A)^*} .$$

This follows from the densities of the sets from above and $(y \mapsto q_{H(t)}(\,.\,, y)) \in \mathcal{L}(Q(A), Q(A)^*)$:

$$\begin{aligned} q_{H(t)}(\varphi, \psi) &= q_{H(t)}(A^{-1/2}A^{1/2}\varphi, A^{-1/2}A^{1/2}\psi) = q_{S(t)}(A^{1/2}\varphi, A^{1/2}\psi) \\ &\leq \|S(t)\|_{\mathrm{op}} \|A^{1/2}\varphi\| \|A^{1/2}\psi\| , \end{aligned}$$

where we used Lemma D.5.6i). The boundedness of $t \mapsto q_{H(t)}(\,.\,, u(t))$ follows from the last estimate and Lemma D.5.6i). □

Now we prove (D.17). So let $\varphi \in L^1(I_b, Q(A))$ then with Lemma D.5.1

$$\begin{aligned} &\left| \int_{I_b} \langle \varphi(t), j(\mathrm{i}\dot{u}_n(t)) - q_{H(t)}(\,.\,, u(t)) \rangle dt \right| \\ &= \left| \int_{I_b} \langle \varphi(t), H_n(t) u_n(t) \rangle - q_{H(t)}(\varphi(t), u(t)) dt \right| \\ &= \left| \int_{I_b} q_{H(t)}(P_n\varphi(t), P_n u_n(t)) - q_{H(t)}(\varphi(t), u(t)) dt \right| . \end{aligned} \tag{D.19}$$

Now we use the definition of the operator $S(t)$ from Lemma D.5.6 to conclude

$$\begin{aligned} (\text{D.19}) &= \left| \int_{I_b} \left\langle A^{1/2} P_n \varphi(t), S(t) A^{1/2} u_n(t) \right\rangle - \left\langle A^{1/2}\varphi(t), S A^{1/2} u(t) \right\rangle dt \right| \\ &= \left| \int_{I_b} \left\langle A^{1/2}\varphi(t), P_n S A^{1/2} u_n(t) - S(t) A^{1/2} u(t) \right\rangle dt \right| \\ &\overset{\pm 0}{\leq} \left| \int_{I_b} \left\langle A^{1/2}\varphi(t), S(t) A^{1/2} (u_n(t) - u(t)) \right\rangle dt \right| \end{aligned} \tag{D.20}$$

$$+\left|\int_{I_b}\Big\langle A^{1/2}\varphi(t),(P_n-1)S(t)A^{1/2}u_n(t)\Big\rangle dt\right|. \tag{D.21}$$

(D.21) is converging to zero as shown below with the help of the bounds from Lemma D.5.4, Lemma D.5.6 and dominated convergence

$$\begin{aligned}(\text{D.21}) &\leq \int_{I_b}\|(P_n-1)A^{1/2}\varphi(t)\|_{\mathscr{H}}\|S(t)A^{1/2}u_n(t)\|_{\mathscr{H}}dt\\ &\leq \int\|(P_n-1)A^{1/2}\varphi(t)\|_{\mathscr{H}}dt\cdot\sup_{t\in I_b}\|S(t)\|_{\text{op}}\|A^{1/2}u_n(t)\|_{\mathscr{H}}\to 0\,.\end{aligned} \tag{D.22}$$

The convergence of (D.20) to zero is shown with the convergence of u_n to u specified in Lemma D.5.4. For is we show that $(t\mapsto\langle A^{1/2}\varphi(t),S(t)A^{1/2}\,.\,\rangle)\in L^1(I_b,Q(A)^*)$ then (D.20) $\to 0$ follows from Lemma D.5.4. But $(t\mapsto\langle A^{1/2}\varphi(t),S(t)A^{1/2}\,.\,\rangle)\in L^1(I_b,Q(A)^*)$ is already proven in the proof of (D.17). Hence we conclude that both (D.20) and (D.21) converge to zero and we conclude (D.17), which completes the proof of part ii).
We continue with the proof of iii). That $u\in C(\overline{I_b},\mathscr{H})$ follows from i), ii) and Lemma D.5.8. $u(t_0)=u_0$ is a direct consequence of $u_n(t_0)=P_nu_0\to u_0$ in $\mathscr{H}$ and Lemma D.5.7 which can be applied due to $u\in C(\overline{I_b},\mathscr{H})$, Lemma D.5.3 and Lemma D.5.4. This concludes the proof of part iii).
The last part iv) of the Lemma follows from Lemma D.5.3 and Lemma D.5.7. Let $t\in I_b$ be fixed. We have $u_{n_l}(t)\rightharpoonup u(t)$ weak in $\mathscr{H}$ and $\langle u_n(t),Au_n(t)\rangle\leq 2C_{12}^2e^{C_{1234}|t-t_0|}q_A(u_0)$ hence we conclude from Lemma F.0.5 that $u(t)\in Q(A)$ and $q_A(u(t))\leq 2C_{12}^2e^{C_{1234}|t-t_0|}q_A(u_0)$. This completes the proof of the Lemma. ∎

The following two Lemma D.5.6 and Lemma D.5.7 support the proof of Lemma D.5.5.

Lemma D.5.6. *Under the conditions of Theorem D.1.1 we have*

i) $\exists!\ S(t)\in\mathcal{L}(\mathscr{H})$ *self-adjoint such that* $\langle\,.\,,S\,.\,\rangle_{\mathscr{H}}=q_{H(t)}(A^{-1/2}\,.\,,A^{-1/2}\,.\,)$. *In addition* $\|S(t)\|_{\mathcal{L}(\mathscr{H})}\leq C_{12}$ *for all* $t\in I_b$.

ii) Let $p,q,r\in[1,\infty]\cap\{0\}$ *with* $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, $\phi\in L^p(I_b,Q(A))$ *and* $\psi\in L^q(I_b,Q(A))$ *then* $(t\mapsto q_{H(t)}(\psi(t),\phi(t)))\in L^r(I_b,\mathbb{C})$.

Proof. Part i) follows from [RS80, Theorem VIII.15] and $|q_{H(t)}(A^{-1/2}\psi,A^{-1/2}\psi)|\leq C_{12}\|\psi\|_{\mathscr{H}}^2$ for all $\psi\in\mathscr{H}$.
In part ii) the strong measurability of $(t\mapsto q_{H(t)}(\psi(t),\phi(t)))$ follows directly from $I_b\to I_b\times Q(A)^2, t\mapsto(t,\psi(t),\phi(t))$ strongly measurable and $f:I_b\times Q(A)^2\to\mathbb{C},(t,\psi,\phi)\mapsto q_{H(t)}(\psi,\phi)$ continuous. Where the continuity of f is readily proven with part i) of the Lemma. Also $(t\mapsto q_{H(t)}(\psi(t),\phi(t)))\in L^r(I_b,\mathbb{C})$ is a simple consequence of part i). ∎

Lemma D.5.7. *Let $\mathscr{H}$ be a Hilbert space, $I \subset \mathbb{R}$ an interval, $A \geq 1$ self-adjoint operator on $\mathscr{H}$, $u \in L^\infty_{\text{loc}}(I, Q(A)) \cap C(\overline{I}, \mathscr{H})$ and $\{u_n\} \subset C(I, \mathscr{H})$ equicontinuous and $u_n(t) \in Q(A)$ with*

$$\int_I \langle u_n(t) - u(t), \varphi(t) \rangle_{Q(A) \times Q(A)^*} dt \to 0 \tag{D.23}$$

as $n \to \infty$ for all $\varphi \in L^1(I, Q(A)^*)$. Then for all $t \in I$ and $\tilde{\varphi} \in \mathscr{H}$ we have

$$\langle u_n(t) - u(t), \tilde{\varphi} \rangle_{\mathscr{H}} \to 0 \tag{D.24}$$

as $n \to \infty$.

Proof. The case $\tilde{\varphi} = 0$ is trivial. So let $t \in I$, $\tilde{\varphi} \in \mathscr{H} \setminus \{0\}$ and $\epsilon > 0$. Since $u \in C(\overline{I}, \mathscr{H})$ and $\{u_n\}$ equicontinuous $\exists \delta > 0$ such that $\forall n$, $s \in B(t, \delta) \cap I$:

$$\|u(s) - u(t)\|_{\mathscr{H}} < \frac{\epsilon}{3\|\tilde{\varphi}\|_{\mathscr{H}}}, \tag{D.25}$$

$$\|u_n(s) - u_n(t)\|_{\mathscr{H}} < \frac{\epsilon}{3\|\tilde{\varphi}\|_{\mathscr{H}}}. \tag{D.26}$$

Set $\phi(s) := \frac{\mathbb{1}_{B(t,\delta)}}{2\delta}(s) \cdot j\tilde{\varphi}$ then $\phi \in L^1(I, Q(A)^*)$ and from (D.23) we conclude $\exists N$ dependent on δ and $\tilde{\varphi}$ such that $\forall n \geq N$

$$\int_I \langle u_n(s) - u(s), \phi(s) \rangle_{Q(A) \times Q(A)^*} ds < \frac{\epsilon}{3}. \tag{D.27}$$

Now we combine (D.25) to (D.27) to prove the claim

$$\begin{aligned}
\left| \langle u_n(t) - u(t), \tilde{\varphi} \rangle_{Q(A) \times Q(A)^*} \right| &= \left| \int_I \frac{\mathbb{1}_{B(t,\delta)}(s)}{2\delta} \langle u_n(t) - u(t), \tilde{\varphi} \rangle ds \right| \\
&\overset{\pm 0}{\leq} \left| \int_I \frac{\mathbb{1}_{B(t,\delta)}(s)}{2\delta} \langle u_n(s) - u(s), \tilde{\varphi} \rangle ds \right| \\
&\quad + \left| \int_I \frac{\mathbb{1}_{B(t,\delta)}(s)}{2\delta} \langle u_n(t) - u_n(s), \tilde{\varphi} \rangle ds \right| \\
&\quad + \left| \int_I \frac{\mathbb{1}_{B(t,\delta)}(s)}{2\delta} \langle u(s) - u(t), \tilde{\varphi} \rangle ds \right| \\
&< \epsilon.
\end{aligned}$$

■

Uniqueness. We first prove the $\mathscr{H}$–norm conservation and then use it to prove the uniqueness.

Lemma D.5.8 ($\mathscr{H}$–norm conservation)**.** *Let $I \subset \mathbb{R}$ an interval. For every $u \in L^2_{\text{loc}}(I, Q(A))$*

with $j \circ u$ weak differentiable in $Q(A)^*$, $\dot{u} \in L^2_{\mathrm{loc}}(I, Q(A)^*)$ and $\mathrm{i}\dot{u}(t) = q_{H(t)}(\,.\,, u(t))$ we have $u \in C(\overline{I}, \mathscr{H})$ and

$$\|u(t)\|_{\mathscr{H}} = \|u(s)\|_{\mathscr{H}}\,, \quad \forall t, s \in I_b\,.$$

Proof. From [Hun14, Theorem 6.35] we know $u \in C(\overline{I_b}, \mathscr{H})$ and $t \mapsto \|u(t)\|^2_{\mathscr{H}}$ weakly differentiable with for almost all t

$$\frac{d}{dt}\|u(t)\|^2_{\mathscr{H}} = 2\mathrm{Re}\langle \dot{u}(t), u(t)\rangle = 2\mathrm{Re}(-\mathrm{i})q_{H(t)}(u(t), u(t)) = 0$$

where we used that $q_{H(t)}$ is symmetric. Hence the weak derivative of $\|u(t)\|^2_{\mathscr{H}}$ is zero and therefore $\|u(t)\|_{\mathscr{H}}$ constant almost everywhere [Hun14, Proposition 6.34]. The claim follows from the continuity of u. ■

Lemma D.5.9 (Uniqueness)**.** *Let $i \in \{1, 2\}$ and $I_i \subset \mathbb{R}$ intervals. If $u_i \in L^2_{\mathrm{loc}}(I_i, Q(A)) \cap C(\overline{I_i}, \mathscr{H})$ with $j \circ u_i$ weak differentiable in $Q(A)^*$, $\dot{u}_i \in L^2_{\mathrm{loc}}(I, Q(A)^*)$, $\mathrm{i}\dot{u}_i(t) = q_{H(t)}(\,.\,, u_i(t))$ and there exists $s \in I_1 \cap I_2$ such that $u_1(s) = u_2(s)$ then*

$$u_1|_{I_1 \cap I_2} = u_2|_{I_1 \cap I_2}\,.$$

Proof. Due to the linearity of the problem also $(u_1 - u_2)|_{I_1 \cap I_2}$ solves the differential equation on $I_1 \cap I_2$ in the sense of Lemma D.5.8. Thus its $\mathscr{H}$–norm is conserved and

$$\|u_1(t) - u_2(t)\| = \|u_1(s) - u_2(s)\| = 0\,.$$

■

Global existence. So far we have established Theorem D.1.1 for bounded intervals. The next step is to extend this result to the arbitrary interval I. To achieve this, we once again rely on standard methods commonly used for ordinary differential equations. These methods mirror those employed for the Hartree equation, as shown in Lemma 2.1.3. For a comprehensive guideline, one can refer to [Bag, Section: 5 Prolongation of solutions] or [Ler11, Section: 2.1 Ordinary Differential Equations].

From the first part of the proof, we already know that, in our setting, $\mathrm{i}\dot{u}(t) = q_{H(t)}(\,.\,, u(t))$ admits unique solutions on bounded intervals. Building on this, we now construct a global solution. The uniqueness of this global solution is guaranteed by Lemma D.5.9. Set $I_n := I \cap [t_0 - n, t_0 + n]$ then $I = \bigcup_{n \in \mathbb{N}_+} I_n$. Let $u^{(n)}$ to be the solution of $\mathrm{i}\dot{u}^{(n)}(t) = q_{H(t)}(\,.\,, u^{(n)}(t))$ on I_n with $u^{(n)}(t_0) = u_0$. Set

$$u(t) := u^{(n)}(t) \text{ for } t \in \bar{I}_n\,, \tag{D.28}$$

$$\dot{u}(t) := \dot{u}^{(n)}(t) \text{ for } t \in I_n\,. \tag{D.29}$$

u and $\dot{u}$ are well-defined maps on I because of Lemma D.5.9. Note that from the definition of weak differentiability it is easy to see that $j \circ u^{(n)}|_{I_n \cap I_k}$ weakly differentiable with weak derivative $\dot{u}^{(n)}|_{I_n \cap I_k}$.
By using (D.28) and (D.29) it is easy to see that $u \in C(\bar{I}, \mathscr{H}) \cap L^\infty_{\text{loc}}(I, Q(A))$ and $\dot{u} \in L^\infty_{\text{loc}}(I, Q(A)^*)$. That $j \circ u$ is weakly differentiable and with weak derivative $\dot{u}$ follows directly from the definition of weak differentiability. Now we show that u is a solution of the Cauchy problem $\mathrm{i}\dot{u}(t) = q_{H(t)}(\,.\,, u(t))$ on I with $u(t_0) = u_0$.

Proof. First we note that $u(t_0) = u^{(n)}(t_0) = u_0$. Let $\phi \in C^\infty_c(I, \mathbb{C})$ then there exists a I_n with $\operatorname{supp}\phi \cap \operatorname{supp}\dot{\phi} \subset I_n$. We calculate

$$\begin{aligned} \int_I \phi(t) q_{H(t)}(\,.\,, u(t)) dt &= \int_{I_n} \phi(t) q_{H(t)}(\,.\,, u^{(n)}(t)) dt \\ &= \int_{I_n} \phi(t) \mathrm{i}\dot{u}^{(n)}(t)) dt = \int_I \phi(t) \mathrm{i}\dot{u}(t) dt \,. \end{aligned} \tag{D.30}$$

Because (D.30) holds for arbitrary $\phi \in C^\infty_c(I, \mathbb{C})$ we can conclude that $\mathrm{i}\dot{u}(t) = q_{H(t)}(\,.\,, u(t))$ [Hun14, Corollary 6.33]. □

Now $\|u(t)\|_{\mathscr{H}} = \|u(s)\|_{\mathscr{H}}$ is a direct consequence of Lemma D.5.9. For $I_b \subset I$ bounded interval with $t_0 \in I_b$ we have for all $t \in I_b$: $q_A(u(t)) \leq 2C_{12}(I_b)\mathrm{e}^{C_{1234}(I_b)|t-t_0} q_A(u_0)$, following directly from the respective properties of $u^{(n)}$. Note that $q_A(u^{I_b}(t)) \leq 2C_{12}(I_b)\mathrm{e}^{C_{1234}(I_b)|t-t_0|} q_A(u_0)$ for the solution u^{I_b} of $\mathrm{i}\dot{u}^{I_b}(t) = q_{H(t)}(\,.\,, u^{I_b}(t))$ on I_b and $\exists I_n \supset I_b$ and due to Lemma D.5.9 $u^{I_b}(t) = u^{(n)}(t) = u(t)$ for all $t \in I_b$. This concludes the proof of Theorem D.1.1. ■

Proof of Corollary D.1.3

We prove the existence of a unitary propagator $U(t,s)$ with $u(t) = U(t,s)u_0$, where is the solution of the formal Cauchy problem $\mathrm{i}\dot{u}(t) = H(t)u(t)$, $u(t_0) = u_0$ stated rigorous in Theorem D.1.1.

Proof of Corollary D.1.3. Let $U(t,s)$ be as defined in Corollary D.1.3 then all its properties except of its continuity and differentiability with respect to s are immediate consequences of the properties of $u(t)$.
The continuity with respect to s follows form $\|U(t,s)y - U(t,t_0)y\|_{\mathscr{H}} = \|y - U(s,t_0)y\|_{\mathscr{H}} \to 0$ for all $y \in \mathscr{H}$ and the continuity with respect to the first argument.

The differentiability of $U(t,s)$ with respect of s follows from its differentiability with respect to t and its properties as a unitary propagator. It is clear that $(s \mapsto jU(t,s)y) \in L^1_{\text{loc}}(\mathbb{R}, Q(A)^*)$ for all $y \in \mathscr{H}$, since $(s \mapsto U(t,s)y) \in C(\mathbb{R}, \mathscr{H})$ and $j \in \mathcal{L}(\mathscr{H}, Q(A)^*)$. And for $u_0, y \in Q(A)$ and $\phi \in C^\infty_c(\mathbb{R}, \mathbb{C})$ we have

$$\left(\int \dot{\phi}(s)\langle\,.\,,U(t,s)u_0\rangle_{\mathscr{H}}ds\right)(y) = \int \dot{\phi}(s)\langle y,U(t,s)u_0\rangle_{\mathscr{H}}ds$$
$$= \left(\int \dot{\phi}^*(s)\langle u_0,U(s,t)y\rangle_{\mathscr{H}}ds\right)^*$$
$$= \left(\left(\int \dot{\phi}^*(s)\langle\,.\,,U(s,t)y\rangle_{\mathscr{H}}ds\right)(u_0)\right)^* . \tag{D.31}$$

Using the weak differentiability of $s \mapsto \langle\,.\,,U(s,t)y\rangle_{\mathscr{H}}$ we conclude

$$(\text{D.31}) = -\left(\left(\int \phi^*(s)(-\mathrm{i})q_{H(s)}(\,.\,,U(s,t)y)ds\right)(u_0)\right)^*$$
$$= -\int \phi(s)\mathrm{i}q_{H(s)}(U(s,t)y,u_0)ds$$
$$\overset{(*)}{=} -\left(\int \phi(s)\mathrm{i}q_{H(s)}(U(s,t)\,.\,,u_0)ds\right)(y)\,,$$

where $(*)$ is proven below. Hence $s \mapsto \langle\,.\,,U(t,s)u_0\rangle_{\mathscr{H}}$ is weakly differentiable and its weak derivative is $\mathrm{i}q_{H(s)}(U(s,t)\,.\,,u_0)$, which proves Corollary D.1.3. Equality $(*)$ is a direct consequence of $(t \mapsto q_{H(s)}(U(s,t)\,.\,,u_0) \in L^1_{\text{loc}}(\mathbb{R},Q(A)^*)$, which follows from Lemma D.5.6i) and Petti's measurability Theorem that can be applied because $\mathscr{H}$ is separable and Lemma D.5.6ii) (for a detailed discussion see (D.17)). ■

D.5.2 Proof of Theorem D.2.8 Part ii)

We prove the second statement in Theorem D.2.8, namely that the propagator $U^{\text{qua}}(t,t_0)$ of a quadratic Hamiltonian is a Bogoliubov transformation $U_{\mathcal{V}(t,t_0)}$. Since $U_{\mathcal{V}(t,t_0)}$ is completely determined by $\mathcal{V}(t,t_0)$ up to a phase, our main two task are:

1. Identify the appropriate candidate for the Bogoliubov map $\mathcal{V}(t,t_0)$ implementing $U^{\text{qua}}(t,t_0)$.

2. Determine the correct phase, i.e., finding $U_{\mathcal{V}(t,t_0)} \in [U_{\mathcal{V}(t,t_0)}]$ such that[3]

$$U_{\mathcal{V}(t,t_0)} = U^{\text{qua}}(t,t_0)\,.$$

For the first part, we use the fact that for a given Bogoliubov transformation U, the associated Bogoliubov map $\mathcal{V}$ can be determined by

$$U^*A(F)U = A(S\mathcal{V}^*SF)\,.$$

[3]By $[U_{\mathcal{V}(t,t_0)}]$ we denote the equivalence class of all Bogoliubov transformations that are equal to $U_{\mathcal{V}(t,t_0)}$ up to a phase.

Thus, to find the $\mathcal{V}(t,t_0)$ corresponding to $U^{\mathrm{qua}}(t,t_0)$, we compute $U^{\mathrm{qua}}(t,t_0)^* A(F) U^{\mathrm{qua}}(t,t_0)$. However, since we do not explicitly know $U^{\mathrm{qua}}(t,t_0)$, but only its generator $-\mathrm{i}H^{\mathrm{qua}}(t)$, we instead analyze its time derivative, leading to the formal identity

$$\begin{aligned}\frac{d}{dt}U^{\mathrm{qua}}(t,t_0)^* A(F) U^{\mathrm{qua}}(t,t_0) &= U^{\mathrm{qua}}(t,t_0)^* \mathrm{i}[H^{\mathrm{qua}}(t), A(F)] U^{\mathrm{qua}}(t,t_0) \\ &= U^{\mathrm{qua}}(t,t_0)^* (-\mathrm{i}) A(\mathcal{A}(t)F) U^{\mathrm{qua}}(t,t_0)\,,\end{aligned}$$

where we used the following formal equality, which can be shown by an easy computation

$$[H^{\mathrm{qua}}(t), A(F)] = -A(\mathcal{A}(t)F)\,.$$

By inserting as a assumption $U^* A(F) U = A(S\mathcal{V}^* S F)$ for $U^{\mathrm{qua}}(t,t_0)$ and use the antilinearity of $F \mapsto A(F)$, we obtain a candidate for the differential equation governing $\mathcal{V}(t,t_0)$:

$$\begin{aligned}&\frac{d}{dt}S\mathcal{V}(t,t_0)^* S = \mathrm{i} S\mathcal{V}(t,t_0)^* S\mathcal{A}(t)\,,\\ &S\mathcal{V}(t_0,t_0)^* S = I\,.\end{aligned}$$

Equivalently, assuming $\mathcal{V}(t,t_0)$ is a symplectic propagator (see Lemma D.2.6), we can write

$$\frac{d}{dt}\mathcal{V}(t,t_0) = -i\mathcal{A}(t)\mathcal{V}(t,t_0)\,, \tag{D.32}$$

$$\mathcal{V}(t_0,t_0) = I\,. \tag{D.33}$$

This is precisely the differential equation we rigorously analyzed in Lemma D.2.6.

Now our tasks is to connect the Bogoliubov map from (D.32) to the correct Bogoliubov transform by fixing a phase. To determine the phase, it suffices to consider the action of both unitaries on a fixed vector x_0. We want to show that there exists a $U_{\mathcal{V}(t,t_0)} \in [U_{\mathcal{V}(t,t_0)}]$ such that

$$U^{\mathrm{qua}}(t,t_0)x_0 = U_{\mathcal{V}(t,t_0)}x_0\,.$$

For this purpose, the generalized one-body density matrix $\Gamma_{U^{\mathrm{qua}}(t,t_0)x_0}$ proves to be a particularly useful tool, for three main reasons:

I. Γ_ψ is Independent of the Phase of ψ. For $\psi, \phi \in \mathscr{H}$ that differ only by a phase, we have $\Gamma_\psi = \Gamma_\phi$. We also denote $\Gamma_{[\psi]} := \Gamma_\psi$, where $[\psi]$ represents the equivalence class of all $\phi \in \mathscr{H}$ that are equal to ψ up to a phase.

II. One-body Density Matrix (OBD) Determines the State. If $\Gamma_\psi = \Gamma_\phi$, then ψ and ϕ must be equal up to a phase [Nam20, Theorem 5.1].

III. Transformation Properties under the Bogoliubov Transformation. The generalized one-body density matrix satisfies the identity $\Gamma_{U_{\mathcal{V}}^* x_0} = \mathcal{V}^* \Gamma_{x_0} \mathcal{V}$ (see [NNS16]).

The equality of $U^{\text{qua}}(t,t_0)$ and $U_{\mathcal{V}(t,t_0)}$ is now proven as follows:

i) Establishing the OBD identity. We show that $\Gamma_{U^{\text{qua}}(t,t_0)x_0} = \mathcal{V}(t_0,t)^* \Gamma_{x_0} \mathcal{V}(t_0,t)$. Using property III above, this implies

$$\Gamma_{U^{\text{qua}}(t,t_0)x_0} = \Gamma_{U_{\mathcal{V}(t,t_0)}x_0} \,.$$

By property II, this proves that for all x_0, there exists a choice of phase for $U_{\mathcal{V}(t,t_0)} \in [U_{\mathcal{V}(t,t_0)}]$ such that $U^{\text{qua}}(t,t_0)x_0 = U_{\mathcal{V}(t,t_0)}x_0$.

ii) Phase Independence of x_0. We show that the phase determined in (i) is actually independent of x_0. To do this, we first choose a fixed reference vector $x_0 = \Omega$ and thereby select a specific phase. We then extend the equality on Ω to the entire Fock space by using the commutator relations of $U_{\mathcal{V}(t,t_0)}$ with the creation and annihilation operators.

Proof of $\Gamma_{U^{\text{qua}}(t,t_0)x_0} = \mathcal{V}(t_0,t)^* \Gamma_{x_0} \mathcal{V}(t_0,t)$

We establish the equality of $U^{\text{qua}}(t,t_0)$ and $U_{\mathcal{V}(t,t_0)}$ on the level of the generalized one-body density matrices by proving

$$\Gamma_{U^{\text{qua}}(t,t_0)x_0} = \mathcal{V}(t_0,t)^* \Gamma_{x_0} \mathcal{V}(t_0,t) \,.$$

The proof proceeds in six steps:

1. Duhamel's Formula for $x(t) := U^{\text{qua}}(t,t_0)x_0$. We prove that for all $y, x_0 \in Q(A)$ and $t, t_0 \in \mathbb{R}$ we have

$$\left\langle y, \mathrm{e}^{\mathrm{i}h_1(t-t_0)} x(t) \right\rangle = \langle y, x(t_0) \rangle - \mathrm{i} \int_{t_0}^{t} q_{H^{\text{qua}}_{h_1=0}(s)}(\mathrm{e}^{-\mathrm{i}h_1(s-t_0)} y, x(s)) ds \,, \tag{D.34}$$

where $q_{H^{\text{qua}}_{h_1=0}}(s)$ denotes the quadratic form $q_{H^{\text{qua}}}(s)$ with $h_1 = 0$.

2. Commutator Relation: $[A(F), H^{\text{qua}}(t)] = A(\mathcal{A}(t)F)$. This commutator identity is needed for step 3. More precisely, we prove that

$$\langle \psi, A(\mathcal{A}(t)F)\psi \rangle = q_{H^{\text{qua}}(t)}(A(\mathcal{J}F)\psi, \psi) - q_{H^{\text{qua}}(t)}(\psi, A(F)\psi) \,, \tag{D.35}$$

for all $\psi \in D(d\Gamma^{1/2}(h_1+1)\mathcal{N}^{1/2})$ and $F \in D(\mathcal{A}(t)) = D(h_1) \oplus JD(h_1)$.

3. Duhamel's Formula for $\Gamma_{x(t)}$. We prove a Duhamel-type equation for $\Gamma_{x(t)}$, namely

$$\left\langle F, \mathrm{e}^{iA_0(t-t_0)} \Gamma_{x(t)} \mathrm{e}^{-iA_0(t-t_0)} G \right\rangle$$

$$= \langle F, \Gamma_{x_0} G\rangle - \mathrm{i}\Big\langle F, \int_{t_0}^{t} \mathrm{e}^{iA_0(s-t_0)}\big(A_1(s)\Gamma_{x(s)} - \Gamma_{x(s)}A_1(s)\big)\mathrm{e}^{-\mathrm{i}A_0(s-t_0)}G ds\Big\rangle , \quad \text{(D.36)}$$

for all $G, F \in \mathscr{H} \oplus \mathscr{H}^*$, $x_0 \in Q(A)$ and $t, t_0 \in \mathbb{R}$. The proof of step 3 consists of two parts:

i) First, we prove (D.36) for sufficiently regular initial data $x_0 \in D(B^{3/2}) \cap Q(A)$ by direct computation.

ii) Then, we extend the result to all $x_0 \in Q(A)$ using an approximation argument and the bound $\|B^k x(t)\| \le \mathrm{e}^{C(t)}\|B^k x_0\|$, for $x_0 \in D(B^k) \cap Q(A)$.

4. Uniqueness of the Solution of the Integral Equation (D.36)**.** Let $t_0 \in \mathbb{R}$ and $\Gamma \in \mathcal{L}(\mathscr{H} \oplus \mathscr{H}^*)$. If it exists, there is only a unique $(t \mapsto \Gamma_t) : \mathbb{R} \to \mathcal{L}(\mathscr{H} \oplus \mathscr{H}^*)$ such that for all $F, G \in \mathscr{H} \oplus \mathscr{H}^*$

$$(t \mapsto \langle F, \Gamma_t G\rangle) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}) ,$$
$$(t \mapsto \|\Gamma_t)\|_{\mathrm{op}}) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C})$$

and for all $t \in \mathbb{R}$

$$\Big\langle F, \mathrm{e}^{iA_0(t-t_0)}\Gamma_t \mathrm{e}^{-iA_0(t-t_0)}G\Big\rangle$$
$$= \langle F, \Gamma G\rangle - \mathrm{i}\Big\langle F, \int_{t_0}^{t} \mathrm{e}^{iA_0(s-t_0)}\big(A_1(s)\Gamma_s - \Gamma_s A_1(s)\big)\mathrm{e}^{-\mathrm{i}A_0(s-t_0)}G ds\Big\rangle . \quad \text{(D.37)}$$

5. $\mathcal{V}(t_0,t)^*\Gamma_{x_0}\mathcal{V}(t_0,t)$ **is a Solution of the Integral Equation** (D.36)**.** We verify that $\Gamma_t = \mathcal{V}(t_0,t)^*\Gamma_{x_0}\mathcal{V}(t_0,t)$ is a solution of the integral equation (D.36).

6. Conclusion: Equality of $\Gamma_{x(t)}$ **and** $\mathcal{V}(t_0,t)^*\Gamma_{x_0}\mathcal{V}(t_0,t)$**.** By the uniqueness result in step 4, we conclude that

$$\Gamma_{x(t)} = \mathcal{V}(t_0,t)^*\Gamma_{x_0}\mathcal{V}(t_0,t) .$$

Independent Phase of the Bogoliubov Transformation

We have already proven that

$$\Gamma_{U^{\mathrm{qua}}(t,t_0)x_0} = \mathcal{V}(t_0,t)^*\Gamma_{x_0}\mathcal{V}(t_0,t) = \Gamma_{U_{\mathcal{V}(t,t_0)}x_0} , \quad \text{(D.38)}$$

which leads to the following conclusion: for all x_0 there exists a phase of $U_{\mathcal{V}(t,t_0)} \in [U_{\mathcal{V}(t,t_0)}]$ such that $U^{\mathrm{qua}}(t,t_0)x_0 = U_{\mathcal{V}(t,t_0)}x_0$. Our goal now is to show that this phase of $U_{\mathcal{V}(t,t_0)}$ is actually independent of x_0. The proof proceeds in four steps:

1. Equality on the Vacuum. The vacuum state Ω provides a natural starting point. The previously established result (D.38) ensures that there is a phase of $U_{\mathcal{V}(t,t_0)} \in [U_{\mathcal{V}(t,t_0)}]$

such that

$$U^{\mathrm{qua}}(t,t_0)\Omega = U_{\mathcal{V}(t,t_0)}\Omega\,. \tag{D.39}$$

2. **Uniqueness of Solutions for the Regularized H^{qua} Time Evolution.** To extend the equality beyond the vacuum state, we establish the uniqueness of solutions to the integral equation (Duhamel) defining $x(t)$, namely (D.34). Since showing uniqueness directly for (D.34) is difficult, we instead prove it for the corresponding differential equation (D.40). We have the following uniqueness statement:
   Let $(t_0, y_0) \in \mathbb{R} \times Q(B)$. If it exists, there is a unique $y \in L^2_{\mathrm{loc}}(\mathbb{R}, Q(B)) \cap C(\mathbb{R}, \mathcal{F}(\mathscr{H}))$ with $j \circ y \in L^2_{\mathrm{loc}}(\mathbb{R}, Q(B)^*)$ weakly differentiable and $\dot{y} \in L^2_{\mathrm{loc}}(\mathbb{R}, Q(B)^*)$ such that for almost all $t \in \mathbb{R}$

$$\dot{y}(t) = -\mathrm{i} q_{H^{\mathrm{qua}}_{h_1=0}(t)}\big(\mathrm{e}^{-\mathrm{i}h_1(t-t_0)}\,.\,, \mathrm{e}^{-\mathrm{i}h_1(t-t_0)}y(t)\big)\,, \tag{D.40}$$

$$y(t_0) = y_0\,, \tag{D.41}$$

   where $q_{H^{\mathrm{qua}}_{h_1=0}}(t)$ denotes the quadratic form $q_{H^{\mathrm{qua}}}(t)$ with $h_1 = 0$.

3. **Equality on a Dense Subset of the Fock Space.** Having established equality on the vacuum, we now extend it to a dense subset of $\mathcal{F}(\mathscr{H})$. A natural choice is the set

$$\bigcup_{n\in\mathbb{N}_+} \mathrm{lin}\big\{A(F_n)...A(F_1)\Omega \,\big|\, F_j \in \mathscr{H} \oplus \mathscr{H}^*\big\} \tag{D.42}$$

   since the commutation properties of $U_{\mathcal{V}(t,t_0)}$ with $A(F)$ as a Bogoliubov transformation allow us to extend (D.39) to the set (D.42). We prove that both $U^{\mathrm{qua}}(t,t_0)$ and $U_{\mathcal{V}(t,t_0)}$ satisfy the same differential equation (D.40) on this subset, leading to their equality on it.

Formal Proof Outline. We first provide a formal proof overview before addressing necessary adaptations for rigour.

i) We compute that

$$\begin{aligned}
\partial_t U_{\mathcal{V}(t,t_0)} A(F)\Omega &= \partial_t A(\mathcal{V}(t,t_0)F) U_{\mathcal{V}(t,t_0)}\Omega \\
&= \partial_t A(\mathcal{V}(t,t_0)F) U^{\mathrm{qua}}(t,t_0)\Omega \\
&= A(-\mathrm{i}\mathcal{A}(t)\mathcal{V}(t,t_0)F) U^{\mathrm{qua}}(t,t_0)\Omega \\
&\quad + A(\mathcal{V}(t,t_0)F)(-\mathrm{i})H^{\mathrm{qua}}(t) U^{\mathrm{qua}}(t,t_0)\Omega \\
&= -\mathrm{i}\big[H^{\mathrm{qua}}(t) A(\mathcal{V}(t,t_0)F)\big] U^{\mathrm{qua}}(t,t_0)\Omega \\
&\quad + A(\mathcal{V}(t-t_0)F)(-\mathrm{i})H^{\mathrm{qua}}(t) U^{\mathrm{qua}}(t,t_0)\Omega
\end{aligned}$$

$$= -\mathrm{i}H^{\mathrm{qua}}(t)A(\mathcal{V}(t,t_0)F)U^{\mathrm{qua}}(t,t_0)\Omega$$
$$= -\mathrm{i}H^{\mathrm{qua}}(t)U_{\mathcal{V}(t,t_0)}A(F)\Omega\,.$$

ii) Thus, $U_{\mathcal{V}(t,t_0)}A(F)\Omega$ satisfies the differential equation

$$\dot{x}(t) = -\mathrm{i}H^{\mathrm{qua}}(t)x(t)\,, \tag{D.43}$$
$$x(t_0) = A(F)\Omega\,. \tag{D.44}$$

By uniqueness of its solutions we have

$$U^{\mathrm{qua}}(t,t_0)A(F)\Omega = U_{\mathcal{V}(t,t_0)}A(F)\Omega\,.$$

iii) We conclude $U^{\mathrm{qua}}(t,t_0) = U_{\mathcal{V}(t,t_0)}$ on the dense set (D.42) via induction on $n \in \mathbb{N}_+$ of the formula

$$U^{\mathrm{qua}}(t,t_0)A(F_n)...A(F_1)\Omega = U_{\mathcal{V}(t,t_0)}A(F_n)...A(F_1)\Omega\,.$$

Making the Proof Rigorous. To ensure that we avoid regularity issues, we adapt the formal argument as follows:

i) Following the steps of the formal proof, we first show that $U_{\mathcal{V}(t,t_0)}A(F)\Omega$ satisfies the same integral equation (Duhamel) as $U^{\mathrm{qua}}(t,t_0)A(F)\Omega$, namely (D.34).

ii) $e^{-th_1(t-t_0)}U_{\mathcal{V}(t,t_0)}A(F)\Omega$ satisfies the differential equation corresponding to (D.34), namely (D.40).

iii) By the uniqueness of solutions of (D.40), it follows that

$$e^{-\mathrm{i}h_1(t-t_0)}U_{\mathcal{V}(t,t_0)}A(F)\Omega = e^{-\mathrm{i}h_1(t-t_0)}U^{\mathrm{qua}}(t,t_0)A(F)\Omega$$

and hence

$$U_{\mathcal{V}(t,t_0)}A(F)\Omega = U^{\mathrm{qua}}(t,t_0)A(F)\Omega\,.$$

iv) Finally, the equality of $U_{\mathcal{V}(t,t_0)}$ and $U^{\mathrm{qua}}(t,t_0)$ extends to the entire dense subset (D.42) in the same manner as in the formal argument.

4. Equality on the Entire Fock Space. Since the subset considered in step 3 is dense in the Fock space, and both operators are continuous, the equality extends to all of $\mathcal{F}(\mathscr{H})$.

Appendix E

Supplementary Proofs

E.1 Proofs of Chapter 4

E.1.1 Proof of Lemma 4.2.3

Proof of Lemma 4.2.3. Let $\mathbb{1}^{\leq M} \in \mathscr{L}(L^2 \otimes \mathcal{F})$ be the projection onto $L^2 \otimes \mathcal{F}^{\leq M}$. Since $\psi_t^{\text{ex}} \in L^2 \otimes \mathcal{F}^{\leq N}$ we can rewrite $-2\text{Im}\langle \Phi_t, R_N \psi_t^{\text{ex}} \rangle = -2\text{Im}\left\langle \Phi_t, \mathbb{1}^{\leq M} R_N \mathbb{1}^{\leq N} \psi_t^{\text{ex}} \right\rangle = 2\text{Im}\left\langle \psi_t^{\text{ex}}, \mathbb{1}^{\leq N} R_N \cdot \mathbb{1}^{\leq M} \Phi_t \right\rangle$ where M has to be chosen for every individual term in R_N such that its action on Φ_t is well defined. Note that R_N contains terms like $\sqrt{N - \mathcal{N}}$. For simplicity, we do not write the projections $1^{\leq M}$ explicitly. In addition we write $\psi_t^{\text{ex}} - \Phi_t =: \tilde{\Phi}_t$.
The definition of the remainder terms $R_{i,N}$ can be found in Proposition 4.1.1. We estimate

$$2\text{Im}\langle \psi_t^{\text{ex}}, R_N \Phi_t \rangle = 2\text{Im}\langle \psi_t^{\text{ex}} - \Phi_t, R_N \Phi_t \rangle = 2\text{Im}\left\langle \tilde{\Phi}_t, R_N \Phi_t \right\rangle .$$

The estimate is done for each $R_{i,N}$ individually.
To $R_{4,N}$:

$$\begin{aligned} &2\text{Im}\left\langle \tilde{\Phi}_t, -\frac{1}{\sqrt{\rho}} \mathcal{N} W * \frac{|\varphi_t|^2}{\Lambda}(x) \Phi_t \right\rangle \\ &\leq \|\tilde{\Phi}_t\| \frac{2}{\sqrt{\rho}\Lambda} \|\varphi_t\|_\infty^2 \|W\|_1 \|\mathcal{N}\Phi_t\| = \frac{2}{\sqrt{N\Lambda}} \|\varphi_t\|_\infty^2 \|W\|_1 \|\tilde{\Phi}_t\| \|\mathcal{N}\Phi_t\| . \end{aligned} \tag{E.1}$$

and

$$\begin{aligned} &\frac{2}{\sqrt{\rho}} \text{Im}\left\langle \tilde{\Phi}_t, d\Gamma(Q_t W_x Q_t) \Phi_t \right\rangle = \frac{2}{\sqrt{\rho}} \text{Im} \int dx \left\langle \tilde{\Phi}_t(x), d\Gamma(Q_t W_x Q_t) \Phi_t(x) \right\rangle \\ &= \frac{2}{\sqrt{\rho}} \int dy dx \text{Im} \left\langle \sum_{m \geq 1} u_m(y) a_m \tilde{\Phi}_t(x), W_x(y) \sum_{n \geq 1} u_n(y) a_n \Phi_t(x) \right\rangle \end{aligned}$$

$$\leq \frac{2}{\sqrt{\rho}}\|W\|_\infty \left(\int dy \| \sum_{m\geq 1} u_m(y) a_m \tilde{\Phi}_t\|^2\right)^{1/2} \left(\int dy \| \sum_{m\geq 1} u_m(y) a_m \Phi_t\|^2\right)^{1/2}, \tag{E.2}$$

where in step 1 we made the integral over the tracer coordinate x explicit and in step 2 we used $d\Gamma(Q_t W_x Q_t) = \sum_{mn\geq 1}(W_x)_{mn} a_m^* a_n$, for fixed x. We estimate both terms in (E.2) in the same way

$$\int dy \| \sum_{m\geq 1} u_m(y) a_m \Phi_t\|^2 = \int dy \sum_{mk\geq 1} u_k^*(y) u_m(y) \langle \Phi_t, a_k^* a_m \Phi_t\rangle$$
$$= \langle \Phi_t, \mathcal{N}\Phi_t\rangle = \|\mathcal{N}^{1/2}\Phi_t\|^2, \tag{E.3}$$

where in step 2 we used $\langle u_k, u_n\rangle = \delta_{k,n}$. We conclude from (E.2) and (E.3)

$$\frac{2}{\sqrt{\rho}}\mathrm{Im}\left\langle \tilde{\Phi}_t, d\Gamma(Q_t W_x Q_t)\Phi_t\right\rangle = \frac{2}{\sqrt{\rho}}\|W\|_\infty \|\mathcal{N}^{1/2}\tilde{\Phi}_t\|\|\mathcal{N}^{1/2}\Phi_t\|. \tag{E.4}$$

Now if the repeat the same argument but insert $(\mathcal{N}+1)^{-1/2}(\mathcal{N}+1)^{1/2}$ in front of Φ_t in the first line of (E.2) and pull $(\mathcal{N}+1)^{-1/2}$ through before $\tilde{\Phi}_t$, we conclude

$$\frac{2}{\sqrt{\rho}}\mathrm{Im}\left\langle \tilde{\Phi}_t, d\Gamma(Q_t W_x Q_t)\Phi_t\right\rangle = \frac{2}{\sqrt{\rho}}\|W\|_\infty \|\tilde{\Phi}_t\|\|(\mathcal{N}+1)\Phi_t\|. \tag{E.5}$$

We get our $R_{4,N}$ estimate from (E.1) and (E.5)

$$2\mathrm{Im}\left\langle \tilde{\Phi}_t, R_{4,N}\Phi_t\right\rangle = 2\sqrt{\frac{\Lambda}{N}}\left(\|\varphi_t\|_\infty^2 + 1\right)(\|W\|_1 + \|W\|_\infty)\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)\Phi_t\| \tag{E.6}$$

To $R_{3,N}$:
We insert $\mathcal{N}^{-1/2}\mathcal{N}^{1/2}$ to obtain

$$2\mathrm{Im}\left\langle \tilde{\Phi}_t, \left(\frac{\sqrt{N-\mathcal{N}}}{\sqrt{N}} - 1\right) a(Q_t W_x \varphi_t)\Phi_t\right\rangle$$
$$= 2\mathrm{Im}\left\langle \tilde{\Phi}_t, \left(\frac{\sqrt{N-\mathcal{N}}}{\sqrt{N}} - 1\right) a(Q_t W_x \varphi_t)\mathcal{N}^{-1/2}\mathcal{N}^{1/2}\Phi_t\right\rangle$$
$$\leq 2\|\left(\frac{\sqrt{N-\mathcal{N}}}{\sqrt{N}} - 1\right)(\mathcal{N}+1)^{-1/2}\tilde{\Phi}_t\|\|a(Q_t W_x \varphi_t)\mathcal{N}^{1/2}\Phi_t\|$$
$$\leq 2\|W\|_2\|\varphi_t\|_\infty N^{-1/2}\|\tilde{\Phi}_t\|\|\mathcal{N}\Phi_t\|, \tag{E.7}$$

where in the last step we have used $\|a(Q_t W_x \varphi_t)\Phi_t\| \leq \operatorname{ess\,sup}_x \|Q_t W_x \varphi_t\|_2 \|\mathcal{N}\psi_t\|$ and

$$\|(\sqrt{N-\mathcal{N}} - \sqrt{N})\psi\|^2 = \left\langle \psi, \left(N - \mathcal{N} - 2\sqrt{N}\sqrt{N-\mathcal{N}} + N\right)\psi\right\rangle$$

$$\leq \left\langle \psi, \left(N - \mathcal{N} - 2\sqrt{N-\mathcal{N}}\sqrt{N-\mathcal{N}} + N \right) \psi \right\rangle \leq \langle \psi, \mathcal{N}\psi \rangle .$$

For the h.c. term we find similar to the above

$$\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, a^*(Q_t W_x \varphi_t) \left(\frac{\sqrt{N-\mathcal{N}}}{\sqrt{N}} - 1 \right) (\mathcal{N}+1)^{-1/2} (\mathcal{N}+1)^{1/2} \Phi_t \right\rangle \\
&\leq 2\| a(Q_t W_x \varphi_t) \mathcal{N}^{-1/2} \tilde{\Phi}_t \| \| \left(\frac{\sqrt{N-\mathcal{N}}}{\sqrt{N}} - 1 \right) (\mathcal{N}+1)^{1/2} \Phi_t \| \\
&\leq 2 \|W\|_2 \|\varphi_t\|_\infty \|\tilde{\Phi}_t\| N^{-1/2} \|(\mathcal{N}_+ + 1)\Phi_t\| , \qquad \text{(E.8)}
\end{aligned}$$

hence

$$\left\langle \tilde{\Phi}_t, R_{3,N} \Phi_t \right\rangle \leq 4N^{-1/2} \|\varphi_t\|_\infty \|W\|_2 \|\tilde{\Phi}_t\| \|(\mathcal{N}_+ + 1)\Phi_t\| . \qquad \text{(E.9)}$$

To $R_{2,N}$:

$$\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, \frac{1}{2N} \sum_{mnpq \geq 1} \Lambda V_{mnpq} a_m^* a_n^* a_p a_q \Phi_t \right\rangle \\
&= \frac{\Lambda}{N} \mathrm{Im} \sum_{mnpq \geq 1} \int dy_1 dy_2 \left\langle V(y_1 - y_2) u_m(y_1) u_n(y_2) \tilde{\Phi}_t, u_p(y_1) u_q(y_2) a_m^* a_n^* a_p a_q \Phi_t \right\rangle \\
&\leq \frac{\Lambda}{N} \left(\int dy_1 dy_2 \| \sum_{mn \geq 1} V(y_1 - y_2) u_m(y_1) u_n(y_2) a_m a_n \tilde{\Phi}_t \|^2 \right)^{1/2} \\
&\quad \cdot \left(\int dy_1 dy_2 \| \sum_{pq \geq 1} u_p(y_1) u_q(y_2) a_p a_q \Phi_t \|^2 \right)^{1/2} \\
&\leq \frac{\Lambda}{N} \|V\|_\infty \|\mathcal{N} \tilde{\Phi}_t\| \|\mathcal{N} \Phi_t\| , \qquad \text{(E.10)}
\end{aligned}$$

where in the last step we have used

$$\begin{aligned}
&\int dy_1 dy_2 \| \sum_{pq \geq 1} u_p(y_1) u_q(y_2) a_p a_q \Phi_t \|^2 \\
&= \int dy_1 dy_2 \sum_{mnpq \geq 1} u_p(y_1) u_q(y_2) u_m^*(y_1) u_n^*(y_2) \langle \Phi_t, a_m^* a_n^* a_p a_q \Phi_t \rangle \\
&= \sum_{pq \geq 1} \left\langle \Phi_t, a_p^* a_q^* a_p a_q \Phi_t \right\rangle = \sum_{pq \geq 1} \left\langle a_q \Phi_t, a_p^* a_p a_q \Phi_t \right\rangle \\
&= \sum_{p \geq 1} \langle a_q \Phi_t, \mathcal{N} a_q \Phi_t \rangle = \langle \Phi_t, (\mathcal{N} - 1) \mathcal{N} \Phi_t \rangle . \qquad \text{(E.11)}
\end{aligned}$$

By inserting $\mathcal{N}^{-1}\mathcal{N}$ before Φ_t in the first line of (E.10) we conclude

$$\begin{aligned} &2\mathrm{Im}\left\langle \tilde{\Phi}_t, R_{2,N}\Phi_t \right\rangle \\ &= 2\mathrm{Im}\left\langle \tilde{\Phi}_t, \frac{1}{2N} \sum_{mnpq\geq 1} \Lambda V_{mnpq} a_m^* a_n^* a_p a_q \mathcal{N}^{-1}\mathcal{N}\Phi_t \right\rangle \\ &\leq \frac{\Lambda}{N} \|V\|_\infty \|\tilde{\Phi}_t\| \|\mathcal{N}^2\Phi_t\| , \end{aligned} \tag{E.12}$$

To $R_{1,N}$:

$R_{1,N}$ consists of several terms. To keep the proof more clear we estimate them separately in Lemma E.1.1, (E.14). The remainder $R_{1,N}$ contains several terms, which we estimate separately in Lemma E.1.1, see also (E.14).

R_N estimate:

We collect the different $R_{i,N}$ estimates done above, i.e.(E.6), (E.9), (E.12), (E.14):

$$\begin{aligned} &2\mathrm{Im}\left\langle \tilde{\Phi}_t, R_N\Phi_t \right\rangle = \sum_{i=1}^{4} 2\mathrm{Im}\left\langle \tilde{\Phi}_t, R_{i,N}\Phi_t \right\rangle \\ &\leq 2\left(\frac{\Lambda}{N}\right)^{1/2} \left(\|\varphi_t\|_\infty^2 + 1\right) (\|W\|_1 + \|W\|_\infty) \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)\Phi_t\| && \text{(E.6)} \\ &+ 4\frac{1}{\sqrt{N}} \|\varphi_t\|_\infty \|W\|_2 \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)\Phi_t\| && \text{(E.9)} \\ &+ \frac{\Lambda}{N} \|V\|_\infty \|\tilde{\Phi}_t\| \|\mathcal{N}^2\Phi_t\| && \text{(E.12)} \\ &+ C\left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2\right) (\|V\|_1 + \|V\|_2) \|\tilde{\Phi}_t\| \\ &\cdot \left(\frac{\Lambda}{N}\right)^{1/2} \left(\|(\mathcal{N}+1)^{3/2}\Phi_t\| + \frac{1}{N^{1/2}} \|(\mathcal{N}+1)^2\Phi_t\| \right) . && \text{(E.14)} \end{aligned}$$

Now using $\Lambda \geq 1$ we simplify

$$\begin{aligned} &2\mathrm{Im}\left\langle \tilde{\Phi}_t, R_N\Phi_t \right\rangle \\ &\leq 4\left(\frac{\Lambda}{N}\right)^{1/2} \left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2 + 1\right) (\|W\|_1 + \|W\|_2 + \|W\|_\infty) \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)\Phi_t\| \\ &+ C\left(\frac{\Lambda}{N}\right)^{1/2} \left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2\right) (\|V\|_1 + \|V\|_2) \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)^{3/2}\Phi_t\| \\ &+ C\frac{\Lambda}{N} \left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2 + 1\right) (\|V\|_1 + \|V\|_2 + \|V\|_\infty) \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)^2\Phi_t\| , \end{aligned} \tag{E.13}$$

which proves the claim. ■

We now give the $R_{1,N}$ estimate used in the proof above. Note that we use the same projection $\mathbb{1}^{\leq M}$ as in the proof of Lemma 4.2.3.

Lemma E.1.1. *For $\Lambda \geq 1$ let φ_t be the condensate satisfying $\|\varphi_t\|_2 = \Lambda^{1/2}$. Set $\psi_t^{\mathrm{ex}} - \Phi_t =: \tilde{\Phi}_t$. We estimate $R_{1,N}$*

$$\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, R_{1,N}\Phi_t \right\rangle \\
&\leq C\left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2\right)\left(\|V\|_1 + \|V\|_2\right)\|\tilde{\Phi}_t\| \\
&\cdot \left(\frac{\Lambda}{N}\right)^{1/2}\left(\|(\mathcal{N}+1)^{3/2}\Phi_t\| + \frac{1}{N^{1/2}}\|(\mathcal{N}+1)^2\Phi_t\|\right). \qquad \text{(E.14)}
\end{aligned}$$

Proof. $R_{1,N}$ consists of several terms

$$\begin{aligned}
R_{1,N} = &-\frac{1}{2}d\Gamma\left(Q_t\left[V * |\varphi_t|^2 + K_1(t) - \mu_t\right]Q_t\right)\frac{\mathcal{N}}{N} \\
&- \frac{(\mathcal{N}+1)\sqrt{N-\mathcal{N}}}{N}a\left(Q_t V * |\varphi_t|^2\frac{\varphi_t}{\Lambda^{1/2}}\right) \\
&+ \frac{1}{2}\sum_{m,n\geq 1}\Lambda V_{mn00}a_m^* a_n^*\left(\frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N} - 1\right) \\
&+ \sum_{mnp\geq 1}\Lambda V_{0mnp}\frac{\sqrt{N-\mathcal{N}}}{N}a_m^* a_n a_p + \frac{1}{2}\mu_t\frac{\mathcal{N}}{N} + \mathrm{h.c},
\end{aligned}$$

see (4.3). In the following, we will estimate each of these terms individually.
<u>To the μ_t terms in $R_{1,N}$:</u>

$$\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, \left[\frac{1}{2}\mu_t\frac{\mathcal{N}}{N} + \frac{1}{2}\mu_t\frac{\mathcal{N}^2}{N} + \mathrm{h.c.}\right]\Phi_t \right\rangle \\
&= 2\mathrm{Im}\left\langle \tilde{\Phi}_t, \left(\mu_t\frac{\mathcal{N}}{N} + \mu_t\frac{\mathcal{N}^2}{N}\right)\Phi_t \right\rangle \\
&\leq \frac{2}{N}\|\varphi_t\|_\infty^2\|V\|_1\|\tilde{\Phi}_t\|\|\mathcal{N}^2\Phi_t\|, \qquad \text{(E.15)}
\end{aligned}$$

where we have used $\mathbb{R} \ni \mu_t = \frac{1}{2}\left\langle \frac{\varphi_t}{\Lambda^{1/2}}, V * |\varphi_t|^2\frac{\varphi_t}{\Lambda^{1/2}}\right\rangle \leq \frac{1}{2}\|V * |\varphi_t|^2\|_\infty\|\frac{\varphi_t}{\Lambda^{1/2}}\|_2^2 \leq \frac{1}{2}\|\varphi_t\|_\infty^2\|V\|_1$.
<u>To the $a(.), a^*(.)$ Terms in $R_{1,N}$:</u>

$$\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, -a^*\left(Q_t V * |\varphi_t|^2\frac{\varphi_t}{\Lambda^{1/2}}\right)\frac{(\mathcal{N}+1)\sqrt{N-\mathcal{N}}}{N}\Phi_t \right\rangle \\
&\leq 2\|\tilde{\Phi}_t\| \cdot \|Q_t V * |\varphi_t|^2\frac{\varphi_t}{\Lambda^{1/2}}\|_2 \cdot \|(\mathcal{N}+1)^{1/2}\frac{(\mathcal{N}+1)\sqrt{N-\mathcal{N}}}{N}\Phi_t\| \\
&\overset{\sqrt{N-\mathcal{N}}/\sqrt{N}\leq 1}{\leq} 2\|\tilde{\Phi}_t\| \cdot \|V * |\varphi_t|^2\|_\infty\frac{\|\varphi_t\|_2}{\Lambda^{1/2}} \cdot \|\frac{(\mathcal{N}+1)^{3/2}}{\sqrt{N}}\Phi_t\| \\
&\leq \frac{2}{\sqrt{N}}\|V\|_1\|\varphi_t\|_\infty^2\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^{3/2}\Phi_t\|, \qquad \text{(E.16)}
\end{aligned}$$

Now the estimate the hermitian conjugated term:

$$
\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, -\frac{(\mathcal{N}+1)\sqrt{N-\mathcal{N}}}{N} a\left(Q_t V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}}\right)\Phi_t \right\rangle \\
= &-2\mathrm{Im}\left\langle \tilde{\Phi}_t, \frac{\sqrt{N-\mathcal{N}}}{N} a\left(Q_t V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}}\right)\mathcal{N}\Phi_t \right\rangle \\
&\leq 2\|\frac{\sqrt{N-\mathcal{N}}}{N}\tilde{\Phi}_t\| \cdot \|a\left(Q_t V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}}\right)\mathcal{N}\Phi_t\| \\
&\overset{\sqrt{N-\mathcal{N}}/\sqrt{N}\leq 1}{\leq} 2\|\frac{1}{\sqrt{N}}\tilde{\Phi}_t\| \cdot \|V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}}\|_2 \|\mathcal{N}^{1/2}\mathcal{N}\Phi_t\| \\
&\leq \frac{2}{\sqrt{N}}\|V\|_1 \|\varphi_t\|_\infty^2 \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)^{3/2}\Phi_t\|\,,
\end{aligned} \tag{E.17}
$$

Form (E.16) and (E.17) we conclude

$$
\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, \left[-\frac{(\mathcal{N}+1)\sqrt{N-\mathcal{N}}}{N} a\left(Q_t V * |\varphi_t|^2 \frac{\varphi_t}{\Lambda^{1/2}}\right) + \text{h.c.}\right]\Phi_t \right\rangle \\
&\leq \frac{4}{\sqrt{N}}\|V\|_1 \|\varphi_t\|_\infty^2 \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)^{3/2}\Phi_t\|
\end{aligned} \tag{E.18}
$$

<u>To the $a_j^* a_k$ Terms in $R_{1,N}$:</u>

Since $(V * |\varphi_t|^2)_{mn} = \langle u_m, V * |\varphi_t|^2 u_n \rangle = \langle u_m \otimes \varphi_t, V(y_1 - y_2) u_n \otimes \varphi_t \rangle = \Lambda \cdot V_{m0n0}$

$$
2\mathrm{Im}\left\langle \tilde{\Phi}_t, -\frac{1}{2} d\Gamma(Q_t V * |\varphi_t|^2 Q_t)\frac{\mathcal{N}}{N}\Phi_t \right\rangle = 2\mathrm{Im}\left\langle \tilde{\Phi}_t, -\frac{1}{2}\sum_{mn\geq 1} \Lambda V_{m0n0} a_m^* a_n \frac{\mathcal{N}}{N}\Phi_t \right\rangle
$$

and

$$
\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, -\frac{1}{2}\sum_{mn\geq 1} \Lambda V_{m0n0} a_m^* a_n \frac{\mathcal{N}}{N}\Phi_t \right\rangle \\
= &-\frac{\Lambda}{N}\mathrm{Im}\sum_{mn\geq 1}\int dy_1 dy_2 \left\langle V(y_1-y_2) u_m(y_1) u_0(y_2) a_m \tilde{\Phi}_t, u_n(y_1) u_0(y_2) a_n \mathcal{N}\Phi_t \right\rangle \\
&\leq \frac{\Lambda}{N}\int dy_1 dy_2 \|u_0\|_\infty^2 \left| \left\langle \sqrt{|V|(y_1-y_2)}\sum_{m\geq 1} u_m(y_1) a_m \tilde{\Phi}_t, \sqrt{|V|(y_1-y_2)}\sum_{n\geq 1} u_n(y_1) a_n \mathcal{N}\Phi_t \right\rangle \right| \\
&\leq \frac{\|\varphi_t\|_\infty^2}{N}\left(\int dy_1 dy_2 \|\sqrt{|V|(y_1-y_2)}\sum_{m\geq 1} u_m(y_1) a_m \tilde{\Phi}_t\|^2\right)^{1/2} \\
&\qquad \cdot \left(\int dy_1 dy_2 \|\sqrt{|V|(y_1-y_2)}\sum_{n\geq 1} u_n(y_1) a_n \mathcal{N}\Phi_t\|^2\right)^{1/2}
\end{aligned}
$$

$$\leq \frac{\|\varphi_t\|_\infty^2}{N}\|V\|_1\|\mathcal{N}^{1/2}\tilde{\Phi}_t\|\|\mathcal{N}^{3/2}\Phi_t\|\,, \tag{E.19}$$

where in the last step we have used

$$\begin{aligned}
&\int dy_1dy_2\|\sqrt{|V|(y_1-y_2)}\sum_{n\geq 1}u_n(y_1)a_n\mathcal{N}\Phi_t\|^2\\
&=\int dy_1dy_2|V|(y_1-y_2)\sum_{mn\geq 1}u_n(y_1)u_m^*(y_1)\big\langle\Phi_t,a_m^*a_n\mathcal{N}^2\Phi_t\big\rangle\\
&=\|V\|_1\sum_{n\geq 1}\big\langle\Phi_t,a_n^*a_n\mathcal{N}^2\Phi_t\big\rangle=\|V\|_1\big\langle\Phi_t,\mathcal{N}\mathcal{N}^2\Phi_t\big\rangle\,.
\end{aligned} \tag{E.20}$$

By inserting $(\mathcal{N}+1)^{-1/2}(\mathcal{N}+1)^{1/2}$ before Φ_t in the first line of (E.20) we conclude

$$2\mathrm{Im}\left\langle\tilde{\Phi}_t,-\frac{1}{2}\sum_{mn\geq 1}\Lambda V_{m0n0}a_m^*a_n\frac{\mathcal{N}}{N}\Phi_t\right\rangle\leq\frac{\|\varphi_t\|_\infty^2}{N}\|V\|_1\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^2\Phi_t\|\,. \tag{E.21}$$

The $K_1(t)$ term gives with Lemma C.0.1b)

$$\begin{aligned}
&2\mathrm{Im}\left\langle\tilde{\Phi}_t,-\frac{1}{2}\frac{\mathcal{N}}{N}d\Gamma(Q_tK_1(t)Q_t)\Phi_t\right\rangle=2\mathrm{Im}\left\langle\tilde{\Phi}_t,-\frac{1}{2}\frac{\mathcal{N}}{N}\sum_{mn\geq 1}(K_1(t))_{mn}a_m^*a_n\Phi_t\right\rangle\\
&=2\mathrm{Im}\left\langle\tilde{\Phi}_t,-\frac{1}{2}\frac{\mathcal{N}}{N}\sum_{mn\geq 1}\Lambda V_{0mn0}a_m^*a_n\Phi_t\right\rangle\\
&=2\mathrm{Im}\left\langle\tilde{\Phi}_t,-\frac{1}{2}\sum_{mn\geq 1}\Lambda V_{0mn0}a_m^*a_n\frac{\mathcal{N}}{N}\Phi_t\right\rangle\\
&\leq\frac{\|\varphi_t\|_\infty^2}{N}\int dy_1dy_2\left|\left\langle\sum_{m\geq 1}\sqrt{|V|(y_1-y_2)}u_m(y_1)a_m\tilde{\Phi}_t,\sum_{n\geq 1}\sqrt{|V|(y_1-y_2)}u_n(y_2)a_n\mathcal{N}\Phi_t\right\rangle\right|\\
&\overset{\text{as above (E.19)}}{\leq}\frac{\|\varphi_t\|_\infty^2}{N}\|V\|_1\|\mathcal{N}^{1/2}\tilde{\Phi}_t\|\|\mathcal{N}^{3/2}\Phi_t\|\,.
\end{aligned} \tag{E.22}$$

By inserting $(\mathcal{N}+1)^{-1/2}(\mathcal{N}+1)^{1/2}$ before Φ_t in the first line of (E.22) we conclude

$$2\mathrm{Im}\left\langle\tilde{\Phi}_t,-\frac{1}{2}\sum_{mn\geq 1}\Lambda V_{0mn0}a_m^*a_n\frac{\mathcal{N}}{N}\Phi_t\right\rangle\leq\frac{\|\varphi_t\|_\infty^2}{N}\|V\|_1\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^2\Phi_t\|\,. \tag{E.23}$$

Putting both estimates (E.21) and (E.23) together we conclude

$$2\mathrm{Im}\left\langle\tilde{\Phi}_t,\left[-\frac{1}{2}\sum_{mn\geq 1}\Lambda V_{m0n0}a_m^*a_n\frac{\mathcal{N}}{N}-\frac{1}{2}\sum_{mn\geq 1}\Lambda V_{0mn0}a_m^*a_n\frac{\mathcal{N}}{N}+h.c.\right]\Phi_t\right\rangle$$

$$= 2\mathrm{Im}\left\langle \tilde{\Phi}_t, \left[-\sum_{mn\geq 1} \Lambda V_{m0n0} a_m^* a_n \frac{\mathcal{N}}{N} - \sum_{mn\geq 1} \Lambda V_{0mn0} a_m^* a_n \frac{\mathcal{N}}{N} \right] \Phi_t \right\rangle$$
$$\leq 4 \frac{\|\varphi_t\|_\infty^2}{N} \|V\|_1 \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)^2 \Phi_t\| \,, \tag{E.24}$$

where we used that the h.c. terms coincide with the original ones due to $V(y_1-y_2) = V(y_2-y_1)$.
To the $a_m^* a_n^*, a_m a_n$ terms in $R_{1,N}$:
For our estimates Lemma 4.2.4 is very important. By using Lemma 4.2.4 we get

$$2\mathrm{Im}\left\langle \tilde{\Phi}_t, \frac{1}{2} \sum_{mn\geq 1} \Lambda V_{mn00} a_m^* a_n^* \left(\frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N} - 1 \right) \Phi_t \right\rangle$$
$$\overset{\text{Lemma 4.2.4}}{\leq} \|\varphi_t\|_\infty^2 \|V\|_1 \|\mathcal{N}^{1/2} \tilde{\Phi}_t\| \left\| \mathcal{N}^{1/2} \left(\frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N} - 1 \right) \Phi_t \right\|$$
$$+ \Lambda^{1/2} \|\varphi_t\|_\infty \|V\|_2 \|\mathcal{N}^{1/2} \tilde{\Phi}_t\| \left\| \left(\frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N} - 1 \right) \Phi_t \right\|$$
$$\leq \frac{1}{N} \|\varphi_t\|_\infty^2 \|V\|_1 \|\mathcal{N}^{1/2} \tilde{\Phi}_t\| \|\mathcal{N}^{3/2} \Phi_t\|$$
$$+ \frac{\Lambda^{1/2}}{N} \|\varphi_t\|_\infty \|V\|_2 \|\mathcal{N}^{1/2} \tilde{\Phi}_t\| \|\mathcal{N} \Phi_t\| \,, \tag{E.25}$$

where in the third line we have used $\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)} - N \leq (N-\mathcal{N}) - N = -\mathcal{N}$. By inserting $(\mathcal{N}+2)^{-1/2}(\mathcal{N}+2)^{1/2}$ before Φ_t in the first line of (E.25) and $(\mathcal{N}+2)^{1/2} \leq (2\mathcal{N}+2)^{1/2} \leq \sqrt{2}(\mathcal{N}+1)^{1/2}$ we conclude

$$2\mathrm{Im}\left\langle \tilde{\Phi}_t, \frac{1}{2} \sum_{mn\geq 1} \Lambda V_{mn00} a_m^* a_n^* \left(\frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N} - 1 \right) \Phi_t \right\rangle$$
$$\leq \frac{1}{N} C \|\varphi_t\|_\infty^2 \|V\|_1 \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)^2 \Phi_t\|$$
$$+ \frac{\Lambda^{1/2}}{N} C \|\varphi_t\|_\infty \|V\|_2 \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)^{3/2} \Phi_t\| \,. \tag{E.26}$$

The hermitian term can be estimated with the same argument as above with $\tilde{\Phi}_t$ and Φ_t interchanged

$$2\mathrm{Im}\left\langle \tilde{\Phi}_t, \left(\frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N} - 1 \right) \frac{1}{2} \sum_{mn\geq 1} \Lambda V_{00mn} a_m a_n \Phi_t \right\rangle$$
$$\leq \frac{1}{N} \|\varphi_t\|_\infty^2 \|V\|_1 \|\mathcal{N}^{1/2} \Phi_t\| \|\mathcal{N}^{3/2} \tilde{\Phi}_t\|$$
$$+ \frac{\Lambda^{1/2}}{N} \|\varphi_t\|_\infty \|V\|_2 \|\mathcal{N}^{1/2} \Phi_t\| \|\mathcal{N} \tilde{\Phi}_t\| \,, \tag{E.27}$$

By inserting $(\mathcal{N}+1)^{-3/2}(\mathcal{N}+1)^{3/2}$ before Φ_t in the first line of (E.27) and $\|(\mathcal{N}+3)^{-3/2}\tilde{\Phi}_t\| \leq$

$\|\tilde{\Phi}_t\|$ we conclude

$$\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, \left(\frac{\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}}{N} - 1\right)\frac{1}{2}\sum_{mn\geq 1}\Lambda V_{00mn}a_m a_n \Phi_t\right\rangle \\
&\leq \frac{1}{N}\|\varphi_t\|_\infty^2 \|V\|_1 \|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^2\Phi_t\| \\
&+ \frac{\Lambda^{1/2}}{N}\|\varphi_t\|_\infty \|V\|_2 \|(\mathcal{N}+3)^{-1/2}\tilde{\Phi}_t\|\|(\mathcal{N}+1)^2\Phi_t\| \\
&\leq C\frac{\Lambda^{1/2}}{N}(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2)(\|V\|_1 + \|V\|_2)\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^2\Phi_t\|\,. \qquad \text{(E.28)}
\end{aligned}$$

Note that (E.28) has a worse prefactor compared to (E.27), namely $\Lambda^{1/2}/N$ instead of $1/N$. But this is not important, since $R_{2,N}$ already causes a worse prefactor Λ/N for a $\|(\mathcal{N}+1)^2\Phi_t\|$ term, due to (E.12).

To the $a_m^* a_n a_p, a_m^* a_n^* a_p$ terms in $R_{1,N}$:

$$\begin{aligned}
&2\mathrm{Im}\left\langle \tilde{\Phi}_t, \sum_{mnp\geq 1}\Lambda V_{0mnp}\frac{\sqrt{N-\mathcal{N}}}{N}a_m^* a_n a_p \Phi_t\right\rangle \\
&= 2\Lambda \mathrm{Im}\int dy_1 dy_2 \sum_{mnp\geq 1} u_0^*(y_1)\left\langle V(y_1-y_2)u_m(y_2)a_m\frac{\sqrt{N-\mathcal{N}}}{N}\tilde{\Phi}_t, u_n(y_1)u_p(y_2)a_n a_p \Phi_t\right\rangle \\
&\leq 2\Lambda^{1/2}\|\varphi_t\|_\infty \\
&\cdot\left(\int dy_1 dy_2 \|\sum_{m\geq 1}V(y_1-y_2)u_m(y_2)a_m\frac{\sqrt{N-\mathcal{N}}}{N}\tilde{\Phi}_t\|^2\right)^{1/2} \qquad \text{(E.29)} \\
&\cdot\left(\int dy_1 dy_2 \|\sum_{np\geq 1}u_n(y_1)u_p(y_2)a_n a_p \Phi_t\|^2\right)^{1/2} \qquad \text{(E.30)}
\end{aligned}$$

where in the second equation we have used the definition $u_0 = \varphi_t/\Lambda^{1/2}$. Now we estimate both terms (E.29) and (E.30) separately.

$$\begin{aligned}
\text{(E.29)}^2 &= \int dy_1 dy_2 \|\sum_{m\geq 1}V(y_1-y_2)u_m(y_2)a_m\frac{\sqrt{N-\mathcal{N}}}{N}\tilde{\Phi}_t\|^2 \\
&= \int dy_1 dy_2 \sum_{mn\geq 1}V^2(y_1-y_2)u_n^*(y_2)u_m(y_2)\left\langle \tilde{\Phi}_t, a_n^* a_m \frac{N-\mathcal{N}}{N^2}\tilde{\Phi}_t\right\rangle \\
&= \|V\|_2^2 \sum_{m\geq 1}\left\langle \tilde{\Phi}_t, a_m^* a_m \frac{N-\mathcal{N}}{N^2}\tilde{\Phi}_t\right\rangle \\
&\leq \|V\|_2^2\left\langle \tilde{\Phi}_t, \frac{\mathcal{N}}{N}\tilde{\Phi}_t\right\rangle \qquad \text{(E.31)}
\end{aligned}$$

where in the third equation we have used that u_n are perpendicular in L^2 and in the last equality that $(N-\mathcal{N})/N \leq 1$. Now we estimate the second term from above

$$\begin{aligned}
&(\text{E.30})^2 = \int dy_1 dy_2 \| \sum_{np\geq 1} u_n(y_1)u_p(y_2)a_n a_p \Phi_t\|^2 \\
&= \int dy_1 dy_2 \sum_{mpnq\geq 1} u_n(y_1)u_p(y_2)u_m^*(y_1)u_q^*(y_2)\big\langle \Phi_t, a_m^* a_q^* a_n a_p \Phi_t\big\rangle \\
&= \sum_{np\geq 1} \big\langle \Phi_t, a_n^* a_p^* a_n a_p \Phi_t\big\rangle \\
&\overset{[\text{LP22}],(43)}{\leq} \big\langle \Phi_t, \mathcal{N}^2 \Phi_t\big\rangle \,.
\end{aligned} \tag{E.32}$$

We conclude

$$\begin{aligned}
&2\text{Im}\Big\langle \tilde{\Phi}_t, \sum_{mnp\geq 1} \Lambda V_{0mnp} \frac{\sqrt{N-\mathcal{N}}}{N} a_m^* a_n a_p \Phi_t \Big\rangle \\
&\leq 2\Lambda^{1/2}\|\varphi_t\|_\infty (\text{E.29})\cdot(\text{E.30}) \leq 2\Lambda^{1/2}\|\varphi_t\|_\infty (\text{E.31})^{1/2}\cdot(\text{E.32})^{1/2} \\
&\leq 2\Lambda^{1/2}\|\varphi_t\|_\infty \frac{1}{N^{1/2}}\|V\|_2 \|\mathcal{N}^{1/2}\tilde{\Phi}_t\| \|\mathcal{N}\Phi_t\| \,.
\end{aligned} \tag{E.33}$$

Now by inserting $(\mathcal{N}+1)^{-1/2}(\mathcal{N}+1)^{1/2}$ before Φ_t at the start of our estimate we conclude

$$\begin{aligned}
&2\text{Im}\Big\langle \tilde{\Phi}_t, \sum_{mnp\geq 1} \Lambda V_{0mnp} \frac{\sqrt{N-\mathcal{N}}}{N} a_m^* a_n a_p \Phi_t \Big\rangle \\
&\leq 2\frac{\Lambda^{1/2}}{N^{1/2}}\|\varphi_t\|_\infty \|V\|_2 \|\tilde{\Phi}_t\| \|(\mathcal{N}+1)^{3/2}\Phi_t\| \,.
\end{aligned} \tag{E.34}$$

Analogously, one can estimate the hermitian term:

$$\begin{aligned}
&2\text{Im}\Big\langle \tilde{\Phi}_t, \sum_{mnp\geq 1} \Lambda V_{mnp0} a_m^* a_n^* a_p \frac{\sqrt{N-\mathcal{N}}}{N} \Phi_t \Big\rangle \\
&= -2\text{Im}\Big\langle \Phi_t, \sum_{mnp\geq 1} \Lambda V_{0mnp} \frac{\sqrt{N-\mathcal{N}}}{N} a_m^* a_n a_p \tilde{\Phi}_t \Big\rangle \\
&\overset{\text{as in (E.33)}}{\leq} 2\frac{\Lambda^{1/2}}{N^{1/2}}\|\varphi_t\|_\infty \|V\|_2 \|\mathcal{N}^{1/2}\Phi_t\| \|\mathcal{N}\tilde{\Phi}_t\| \,.
\end{aligned} \tag{E.35}$$

By inserting $(\mathcal{N}+1)^{-1}(\mathcal{N}+1)$ before Φ_t in the first line of (E.35) we conclude

$$\begin{aligned}
&2\text{Im}\Big\langle \tilde{\Phi}_t, \sum_{mnp\geq 1} \Lambda V_{mnp0} a_m^* a_n^* a_p \frac{\sqrt{N-\mathcal{N}}}{N} \Phi_t \Big\rangle \\
&\leq 2\frac{\Lambda^{1/2}}{N^{1/2}}\|\varphi_t\|_\infty \|V\|_2 \|(\mathcal{N}+1)^{3/2}\Phi_t\| \|\tilde{\Phi}_t\| \,.
\end{aligned} \tag{E.36}$$

Having estimated all terms in $R_{1,N}$, we now gather the results.
$R_{1,N}$ estimate conclusion:
We gather our estimates of all the terms in $R_{1,N}$, i.e. (E.15), (E.18),(E.24), (E.26),(E.28), (E.34), (E.36)

$$
\begin{aligned}
&2\mathrm{Im}\Big\langle \tilde{\Phi}_t, R_{1,N}\Phi_t \Big\rangle \\
&\leq \frac{2}{N}\|\varphi_t\|_\infty^2\|V\|_1\|\tilde{\Phi}_t\|\|\mathcal{N}^2\Phi_t\| && \text{(E.15)}\\
&+ \frac{4}{\sqrt{N}}\|V\|_1\|\varphi_t\|_\infty^2\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^{3/2}\Phi_t\| && \text{(E.18)}\\
&+ 4\frac{\|\varphi_t\|_\infty^2}{N}\|V\|_1\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^2\Phi_t\| && \text{(E.24)}\\
&+ \frac{1}{N}C\|\varphi_t\|_\infty^2\|V\|_1\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^2\Phi_t\| && \text{(E.26)}\\
&+ \frac{\Lambda^{1/2}}{N}C\|\varphi_t\|_\infty\|V\|_2\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^{3/2}\Phi_t\| && \text{(E.26)}\\
&+ C\frac{\Lambda^{1/2}}{N}(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2)(\|V\|_1+\|V\|_2)\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^2\Phi_t\| && \text{(E.28)}\\
&+ 2\frac{\Lambda^{1/2}}{N^{1/2}}\|\varphi_t\|_\infty\|V\|_2\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^{3/2}\Phi_t\| && \text{(E.34)}\\
&+ 2\frac{\Lambda^{1/2}}{N^{1/2}}\|\varphi_t\|_\infty\|V\|_2\|(\mathcal{N}+1)^{3/2}\Phi_t\|\|\tilde{\Phi}_t\|\,. && \text{(E.36)}
\end{aligned}
$$

Putting together terms with the same order in $\mathcal{N}+1$ and considering $1/N \leq \Lambda^{1/2}/N \leq \Lambda^{1/2}/N^{1/2}$, since $\Lambda, N \geq 1$, leads to

$$
\begin{aligned}
&2\mathrm{Im}\Big\langle \tilde{\Phi}_t, R_{1,N}\Phi_t \Big\rangle \\
&\leq C\left(\frac{\Lambda}{N}\right)^{1/2}\left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2\right)\left(\|V\|_1+\|V\|_2\right)\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^{3/2}\Phi_t\| \\
&+ C\frac{\Lambda^{1/2}}{N}\left(\|\varphi_t\|_\infty + \|\varphi_t\|_\infty^2\right)\left(\|V\|_1+\|V\|_2\right)\|\tilde{\Phi}_t\|\|(\mathcal{N}+1)^2\Phi_t\|\,, \qquad \text{(E.37)}
\end{aligned}
$$

which concludes our estimate for $R_{1,N}$. ■

E.1.2 Proof of Lemma 4.2.4

Proof of Lemma 4.2.4. First, we separate $a_j^* a_k^*$ and place them each on one side of the scalar product. By using Cauchy-Schwartz on the scalar product we get a term $\| \sum_{k\geq 1}\int dy_2 V(y_1 - y_2)u_0(y_2)u_k^*(y_2)a_k^*\Phi\|$ with a_k^*. We want to estimate this term through $\mathcal{N}$ but therefore we need to replace a_k^* with a_k. The replacement is done by using the scalar product structure of the norm and the CCR $[a_n, a_k^*] = a_k^* a_n + \delta_{n,k}$. The additional term received through the $\delta_{n,k}$ has a prefactor $\Lambda^{1/2}$ and determines the order in Λ.

For the proof we are following the ideas of [PPS20, Lemma 3.5, Part γ_N^c]. We compute by using the definition of ΛV_{jk00} and Cauchy-Schwarz

$$
\begin{aligned}
&\mathrm{Im}\left\langle \tilde{\Phi}, \sum_{jk\geq 1} \Lambda V_{jk00} a_j^* a_k^* \Phi \right\rangle \\
&= \Lambda \mathrm{Im} \sum_{jk\geq 1} \left\langle \tilde{\Phi}, \int dy_1 dy_2 V(y_1 - y_2) u_0(y_1) u_0(y_2) u_j^*(y_1) u_k^*(y_2) a_j^* a_k^* \Phi \right\rangle \\
&\leq \Lambda \int dy_1 \| \sum_{j\geq 1} u_j(y_1) a_j \tilde{\Phi} \| \cdot |u_0(y_1)| \qquad \text{(E.38)} \\
&\qquad \cdot \| \sum_{k\geq 1} \int dy_2 V(y_1 - y_2) u_0(y_2) u_k^*(y_2) a_k^* \Phi \| \,. \qquad \text{(E.39)}
\end{aligned}
$$

(E.38) and (E.39) have both one u_0 term. We will try to approximate both by $\|u_0\|_\infty \leq C\Lambda^{-1/2}$ to cancel the Λ in (E.38), which is not possible for all the terms appearing below.
We now estimate (E.39). In the estimate in (E.46) we can estimate the a_k operator through $\mathcal{N}^{1/2}$. For the a_k^* operator in (E.39) this is not directly possible but instead we have to first replace a_k^* by a_k. By rewriting $\| \,.\, \| = \langle \,.\, , \,.\, \rangle^{1/2}$ we find

$$
\begin{aligned}
&\text{(E.39)} = \Big\{ \sum_{kl\geq 1} \int dy_2 V(y_1 - y_2) u_0(y_2) u_k^*(y_2) \int dy_2' V(y_1 - y_2') u_0^*(y_2') u_l(y_2') \langle \Phi, a_l a_k^* \Phi \rangle \Big\}^{1/2} \\
&\leq \Big\{ \sum_{kl\geq 1} \int dy_2 V(y_1 - y_2) u_0(y_2) u_k^*(y_2) \int dy_2' V(y_1 - y_2') u_0^*(y_2') u_l(y_2') \langle \Phi, (a_k^* a_l + \delta_{l,k}) \Phi \rangle \Big\}^{1/2} \\
&\leq \| \sum_{k\geq 1} \int dy_2 V(y_1 - y_2) u_0^*(y_2) u_k(y_2) a_k \Phi \| \qquad \text{(E.40)} \\
&+ \Big\{ \sum_{k\geq 1} \langle V_{y_1} u_0, u_k \rangle \langle u_k, V_{y_1} u_0 \rangle \|\Phi\|^2 \Big\}^{1/2} , \qquad \text{(E.41)}
\end{aligned}
$$

where in the second step we have used $[a_l, a_k^*] = \delta_{l,k}$, in the third the notation $V_{y_1}(y) = V(y_1 - y)$ and $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$ for $a, b \geq 0$. The change in (E.39) from a_k^* to a_k is thus at the expense of the additional term (E.41), coming from the commutator $[a_l, a_k^*]$. We will see below that (E.41) will give us the largest order in Λ. We see

$$
\begin{aligned}
\text{(E.41)}^2 &= \sum_{k\geq 0} |\langle Q_t V_{y_1} u_0, u_k \rangle|^2 \|\Phi\|^2 = \|Q_t V_{y_1} u_0\|_2^2 \|\Phi\|^2 \\
&\leq \|u_0\|_\infty^2 \|V\|_2^2 \|\Phi\|^2 \,. \qquad \text{(E.42)}
\end{aligned}
$$

For the estimate (E.42) it is important that the integral over y_2 is inside the norm in (E.39). With (E.38), (E.40) and (E.42) we conclude

$$\begin{aligned}
&\mathrm{Im}\left\langle \tilde{\Phi}, \sum_{jk\geq 1} \Lambda V_{jk00} a_j^* a_k^* \Phi \right\rangle \\
&\leq \text{(E.38)} \cdot \text{(E.39)} \\
&\leq \text{(E.38)} \cdot (\text{(E.40)} + \text{(E.42)}) \\
&\leq \Lambda \int dy_1 \| \sum_{j\geq 1} u_j(y_1) a_j \tilde{\Phi} \| \cdot |u_0(y_1)| \cdot \| \sum_{k\geq 1} \int dy_2 V(y_1 - y_2) u_0^*(y_2) u_k(y_2) a_k \Phi \| && \text{(E.43)} \\
&+ \Lambda \int dy_1 \| \sum_{j\geq 1} u_j(y_1) a_j \tilde{\Phi} \| \cdot |u_0(y_1)| \cdot \|u_0\|_\infty \|V\|_2 \|\Phi\| \, . && \text{(E.44)}
\end{aligned}$$

In (E.43) we can estimate both u_0 in the $\|.\|_\infty$-Norm, since we can regularise the integrals with $|V|^{1/2}(y_1 - y_2)$. We move the y_2 integral out of the norm to obtain

$$\begin{aligned}
&\text{(E.43)} \\
&\leq \Lambda \int dy_1 dy_2 |V|(y_1 - y_2) |u_0(y_2)| \cdot |u_0(y_1)| \cdot \| \sum_{j\geq 1} u_j(y_1) a_j \tilde{\Phi} \| \cdot \| \sum_{k\geq 1} u_k(y_2) a_k \Phi \| \\
&\leq \|\varphi_t\|_\infty^2 \int dy_1 dy_2 \| |V|^{1/2}(y_1 - y_2) \sum_{j\geq 1} u_j(y_1) a_j \tilde{\Phi} \| \cdot \| |V|^{1/2}(y_1 - y_2) \sum_{k\geq 1} u_k(y_2) a_k \Phi \| \\
&\leq \|\varphi_t\|_\infty^2 \left(\int dy_1 dy_2 \| |V|^{1/2}(y_1 - y_2) \sum_{j\geq 1} u_j(y_1) a_j \tilde{\Phi} \|^2 \right)^{1/2} \\
&\qquad \left(\int dy_1 dy_2 \| |V|^{1/2}(y_1 - y_2) \sum_{k\geq 1} u_k(y_2) a_k \Phi \| \right)^{1/2} \\
&\leq \|\varphi_t\|_\infty^2 \|V\|_1 \left(\int dy_1 \| \sum_{j\geq 1} u_j(y_1) a_j \tilde{\Phi} \|^2 \right)^{1/2} \left(\int dy_2 \| \sum_{k\geq 1} u_k(y_2) a_k \Phi \|^2 \right)^{1/2} \\
&\leq \|\varphi_t\|_\infty^2 \|V\|_1 \|\mathcal{N}^{1/2} \tilde{\Phi}\| \|\mathcal{N}^{1/2} \Phi\| \, , && \text{(E.45)}
\end{aligned}$$

where in the first step we have used the trivial inequality for integrals in a Banach space and in the last step we have used

$$\begin{aligned}
\int dy_2 \| \sum_{k\geq 1} u_k(y_2) a_k \Phi \|^2 &\leq \int dy_2 \sum_{jl\geq 1} u_k^*(y_2) u_l(y_2) \langle \Phi, a_l^* a_k \Phi \rangle \\
&\leq \left\langle \Phi, \sum_{k\geq 1} a_k^* a_k \Phi \right\rangle \leq \|\mathcal{N}^{1/2} \Phi\|^2 \, . && \text{(E.46)}
\end{aligned}$$

In (E.44) it is important, that we do not estimate $|u_0(y_1)|$ with $\|u_0\|_\infty$ as we need it for the integral over y_1 to be finite, as shown below

$$
\begin{aligned}
&(\text{E.44}) \leq \Lambda \left(\int dy_1 |u_0(y_1)|^2 \right)^{1/2} \left(\int dy_1 \| \sum_{j\geq 1} u_j(y_1) a_j \tilde{\Phi} \|^2 \right)^{1/2} \|u_0\|_\infty \|V\|_2 \|\Phi\| \\
&\leq \Lambda^{1/2} \|\varphi_t\|_\infty \|V\|_2 \|\Phi\| \|\mathcal{N}^{1/2} \tilde{\Phi}\| \,,
\end{aligned} \tag{E.47}
$$

where in the second step we have used (E.46). The problem is that $\|V\|_2$ in (E.42) no longer depends on y_1, since its y_1 dependence is cancelled by the L^2 norm. Therefore, we need to use the $u_0(y_1)$ for the Cauchy-Schwarz estimate above.
We finally conclude

$$
\begin{aligned}
\operatorname{Im}\left\langle \tilde{\Phi}, \sum_{jk\geq 1} \Lambda V_{jk00} a_j^* a_k^* \Phi \right\rangle &\leq (\text{E.43}) + (\text{E.44}) \leq (\text{E.45}) + (\text{E.47}) \\
&\leq \|\varphi_t\|_\infty^2 \|V\|_1 \|\mathcal{N}^{1/2}\tilde{\Phi}\| \|\mathcal{N}^{1/2}\Phi\| \\
&\quad + \Lambda^{1/2} \|\varphi_t\|_\infty \|V\|_2 \|\mathcal{N}^{1/2}\tilde{\Phi}\| \|\Phi\| \,,
\end{aligned}
$$

which proves the claim and therefore we find a similar result to [PPS20, Lemma 3.5, Part γ_N^c]. ■

E.1.3 Proof of Lemma 4.2.6

For the proof we need a technical Lemma to estimate terms like $(\mathcal{N}+2)^n - (\mathcal{N}+1)^n$ with the help of the binomial formula.

Lemma E.1.2. *Let $a \geq 1$, $b \geq 0$ then $(a+b)^m - a^m \leq (1+b)^m a^{m-1}$, $\forall m \in \mathbb{N}_0$.*

We now begin with the proof of Lemma 4.2.6.

Proof of Lemma 4.2.6. We have everything we need to estimate the particle number. Especially Lemma 4.2.4 plays an important role in this proof. The $\Lambda^{1/2}$ prefactor in the second term in (4.4) determines the order in Λ in our estimate.

We use a Grönwall estimate. We write for short $\psi_t^{\mathrm{BF}} =: \psi_t$. Let $n \in \mathbb{N}_0$. We consider the time derivative of the quantity we want to estimate

$$
\begin{aligned}
\mathbb{R} \ni \partial_t \langle \psi_t \,, (\mathcal{N}+1)^n \psi_t \rangle &= \operatorname{Re}\langle \psi_t \,, -\mathrm{i} \left[(\mathcal{N}+1)^n, H^{\mathrm{BF}} \right] \psi_t \rangle \\
&= \operatorname{Im}\left\langle \psi_t \,, \left[(\mathcal{N}+1)^n, \frac{1}{2} \sum_{mn} \Lambda \left(V_{mn00} a_m^* a_n^* + \text{h.c.} \right) \right] \psi_t \right\rangle
\end{aligned} \tag{E.48}
$$

$$+\operatorname{Im}\langle\psi_t\,,\,[(\mathcal{N}+1)^n, a^*(Q_tW_x\varphi_t)+a(Q_tW_x\varphi_t)]\,\psi_t\rangle\,. \tag{E.49}$$

We now estimate both terms (E.48) and (E.49) separately.
To (E.49)*:*
The estimate follows directly from Lemma E.1.2 and the commutator of $\mathcal{N}$ with a, a^*

$$\begin{aligned}
&(\text{E.49}) = \operatorname{Im}\langle\psi_t\,,\,[(\mathcal{N}+1)^n, a^*(Q_tW_x\varphi_t)+a(Q_tW_x\varphi_t)]\,\psi_t\rangle\\
&= 2\operatorname{Im}\langle\psi_t\,,\,a^*(Q_tW_x\varphi_t)\{(\mathcal{N}+2)^n-(\mathcal{N}+1)^n\}\psi_t\rangle\\
&= 2\operatorname{Im}\Big\langle\{(\mathcal{N}+1)^n-\mathcal{N}^n\}^{1/2}\psi_t\,,\,a^*(Q_tW_x\varphi_t)\{(\mathcal{N}+2)^n-(\mathcal{N}+1)^n\}^{1/2}\psi_t\Big\rangle\\
&\le C\|(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|\|Q_tW_x\varphi_t\|_2 2\|(\mathcal{N}+1)^{1/2}(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|\\
&\le C\|\varphi_t\|_\infty\|W\|_2\|(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|\|(\mathcal{N}+1)^{\frac{n}{2}}\psi_t\|\\
&\le C\|W\|_2\|(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|^2+C\|W\|_2\|\varphi_t\|_\infty^2\|(\mathcal{N}+1)^{\frac{n}{2}}\psi_t\|^2\,,
\end{aligned} \tag{E.50}$$

where in step 4 we used Lemma E.1.2 and in the last step $ab \le 1/2(a^2+b^2)$, $a, b \ge 0$.
To (E.48)*:*
We use $\operatorname{Im}\langle\psi,[\mathcal{N},A^*]\psi\rangle = \operatorname{Im}\langle\psi,(-[\mathcal{N},A])^*\psi\rangle = \operatorname{Im}\langle\psi,[\mathcal{N},A]\psi\rangle$, Lemma 4.2.4 as well as Lemma E.1.2 to conclude

$$\begin{aligned}
&(\text{E.48}) = \operatorname{Im}\left\langle\psi_t\,,\,\left[(\mathcal{N}+1)^n,\frac{1}{2}\sum_{mn}(\Lambda V_{mn00}a_m^*a_n^*+\text{h.c.})\right]\psi_t\right\rangle\\
&= \operatorname{Im}\left\langle\psi_t\,,\,\left[(\mathcal{N}+1)^n,\sum_{mn}\Lambda V_{mn00}a_m^*a_n^*\right]\psi_t\right\rangle\\
&= \operatorname{Im}\left\langle\psi_t\,,\,\sum_{mn}\Lambda V_{mn00}a_m^*a_n^*\{(\mathcal{N}+3)^n-(\mathcal{N}+1)^n\}\psi_t\right\rangle\\
&= \operatorname{Im}\left\langle\{(\mathcal{N}+1)^n-(\mathcal{N}-1)^n\}^{1/2}\psi_t\,,\,\sum_{mn}\Lambda V_{mn00}a_m^*a_n^*\{(\mathcal{N}+3)^n-(\mathcal{N}+1)^n\}^{1/2}\psi_t\right\rangle\\
&\le C\|\varphi_t\|_\infty^2\|V\|_1\|\mathcal{N}^{1/2}(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|\|\mathcal{N}^{1/2}(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|\\
&\quad+C\Lambda^{1/2}\|\varphi_t\|_\infty\|V\|_2\|\mathcal{N}^{1/2}(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|\|(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|\\
&\le C(\|V\|_1+\|V\|_2)\|\varphi_t\|_\infty^2\|(\mathcal{N}+1)^{\frac{n}{2}}\psi_t\|^2+C\Lambda\|V\|_2\|(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|^2\,.
\end{aligned} \tag{E.51}$$

Finally we conclude

$$\begin{aligned}
&\partial_t\langle\psi_t\,,\,(\mathcal{N}+1)^n\psi_t\rangle \le (\text{E.48})+(\text{E.49}) \le (\text{E.50})+(\text{E.51})\\
&\le C\|W\|_2\|(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|^2+C\|W\|_2\|\varphi_t\|_\infty^2\|(\mathcal{N}+1)^{\frac{n}{2}}\psi_t\|^2\\
&\quad+C(\|V\|_1+\|V\|_2)\|\varphi_t\|_\infty^2\|(\mathcal{N}+1)^{\frac{n}{2}}\psi_t\|^2+C\Lambda\|V\|_2\|(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|^2\\
&\le C\|\varphi_t\|_\infty^2(\|V\|_1+\|V\|_2+\|W\|_2)\|(\mathcal{N}+1)^{\frac{n}{2}}\psi_t\|^2
\end{aligned}$$

$$+ C(\Lambda\|V\|_2 + \|W\|_2)\|(\mathcal{N}+1)^{\frac{n-1}{2}}\psi_t\|^2 . \tag{E.52}$$

With (E.52) we now prove with induction that $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\langle \psi_t\,,\, (\mathcal{N}+1)^n \psi_t\rangle \leq C\big\langle \psi_0\,,\, \big(\Lambda + (\mathcal{N}+1)\big)^n \psi_0\big\rangle\,, \quad \forall n \in \mathbb{N}_0\,. \tag{E.53}$$

Proof.

Base Case: $n = 0$**.** (E.53) is trivial in the case $n = 0$.

Induction step. Now let $n \in \mathbb{N}_0$ and assume as induction hypothesis that (E.53) be true for n. We show that (E.53) is true for n+1.
We know from (E.52) and Corollary B.1.2 that $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\begin{aligned}
\partial_t\big\langle \psi_t\,,\, (\mathcal{N}+1)^{n+1}\psi_t\big\rangle &\leq C\|(\mathcal{N}+1)^{\frac{n+1}{2}}\psi_t\|^2 \\
&\quad + \Lambda C\|(\mathcal{N}+1)^{\frac{n}{2}}\psi_t\|^2 \\
&\leq C\|(\mathcal{N}+1)^{\frac{n+1}{2}}\psi_t\|^2 + C\Lambda\big\langle \psi_0\,,\, \big(\Lambda + (\mathcal{N}+1)\big)^n\psi_0\big\rangle\,,
\end{aligned}$$

where in step 2 we used the induction hypothesis for the second term. With Grönwall we conclude $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\begin{aligned}
\big\langle \psi_t\,,\, (\mathcal{N}+1)^{n+1}\psi_t\big\rangle &\leq C\big\langle \psi_0, (\mathcal{N}+1)^{n+1}\psi_0\big\rangle \\
&\quad + C\Lambda\big\langle \psi_0\,,\, \big(\Lambda + (\mathcal{N}+1)\big)^n\psi_0\big\rangle \\
&= C\Big\langle \psi_0\,,\, \big((\mathcal{N}+1) + \Lambda\big)^{n+1}\psi_0\Big\rangle\,,
\end{aligned} \tag{E.54}$$

where we used $\Lambda \leq \Lambda + (\mathcal{N}+1)$ and $(\mathcal{N}+1) \leq \Lambda + (\mathcal{N}+1)$. The estimate (E.54) proves (E.53) and therefore the claim. □

■

E.2 Proofs of Chapter 5

E.2.1 Proof of Lemma 5.4.7

Proof of Lemma 5.4.7. We extend the regularity of $(x, y) \mapsto Q_t W_x \varphi_t(y)$, proven in Lemma 5.4.8 to $x \mapsto (c + J^*b)(U_t^* - V_t^* J)Q_t W_x \varphi_t$. Important for this proof is $\|c\|_{\mathcal{L}(L^2)} + \|b\|_{\mathcal{L}(L^2, JL^2)} \leq C$ and Lemma D.2.10.

We prove $\forall \beta \in \mathbb{N}_0^3$, $|\beta| \leq M$

$$(x \mapsto (c + J^* b)(U_t^* - V_t^* J) Q_t W_x \varphi_t) \in W^{M,\infty}(\mathbb{R}^3, L^2(\mathbb{R}^3, \mathbb{C}))\,, \tag{E.1}$$

$$\begin{aligned} &(D_x^\beta)_{L^2(\mathbb{R}^3,\mathbb{C})}(c + J^* b)(U_t^* - V_t^* J) Q_t W_x \varphi_t \\ &= (c + J^* b)(U_t^* - V_t^* J)(D_x^\beta)_{L^2(\mathbb{R}^3,\mathbb{C})}(Q_t W_x \varphi_t) \end{aligned} \tag{E.2}$$

and $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\|(x \mapsto (c + J^* b)(U_t^* - V_t^* J) Q_t W_x \varphi_t)\|_{W^{M,\infty}(\mathbb{R}^3, L^2(\mathbb{R}^3,\mathbb{C}))} \leq C\,. \tag{E.3}$$

With Lemma 5.4.8 we get from (E.2) that

$$\begin{aligned} (D_x^\beta)_{L^2(\mathbb{R}^3)} f_t(x) &= (c + J^* b)(U_t^* - V_t^* J) Q_t (D_x^\beta)_{L^2(\mathbb{R}^3)}(W_x \varphi_t) \\ &= (c + J^* b)(U_t^* - V_t^* J) Q_t ((D_x^\beta)_{L^2(\mathbb{R}^3)} W_x) \varphi_t\,. \end{aligned} \tag{E.4}$$

Proof. We start by proving the regularity

$$\begin{aligned} &\Bigg\|(c + J^* b)(U_t^* - V_t^* J) \\ &\cdot \left(\frac{Q_t(D_x^\beta W_{x+he_i})\varphi_t - Q_t(D_x^\beta W_x)\varphi_t}{h} - \partial_{x_i} Q_t(D_x^\beta W_x)\varphi_t \right) \Bigg\|_{L^2} \\ \leq\ &\left(\|c\|_{\mathcal{L}(L^2)} + \|b\|_{\mathcal{L}(L^2, JL^2)} + \|U_t\|_{\mathcal{L}(L^2)} + \|V_t\|_{\mathcal{L}(L^2, JL^2)} \right) \\ &\cdot \left\| \frac{Q_t(D_x^\beta W_{x+he_i})\varphi_t - Q_t(D_x^\beta W_x)\varphi_t}{h} - \partial_{x_i} Q_t(D_x^\beta W_x)\varphi_t \right\|_{L^2} \\ &\to 0 \end{aligned}$$

where we used that $U_t \in \mathcal{L}(L^2), V_t \in \mathcal{L}(L^2, JL^2)$ and differentiability provided by Lemma 5.4.8. Note that $\partial_{x_i} Q_t(D_x^\beta W_x)\varphi_t = Q_t(\partial_{x_i} D_x^\beta W_x)\varphi_t$. The continuity of $(D_x^\beta)_{L^2(\mathbb{R}^3,\mathbb{C})}(U_t - J^* V_t) Q_t W_x \varphi_t$ is proven analogous to the above. The bound (E.3) is also obtained in a similar way: $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\begin{aligned} \operatorname*{ess\,sup}_x \|(c + J^* b)(U_t^* - V_t^* J) Q_t(D_x^\beta W_x)\varphi_t\|_2 &\leq C \operatorname*{ess\,sup}_x \|Q_t(D_x^\beta W_x)\varphi_t\|_2 \\ &\leq C \end{aligned}$$

where we used the assumption $\|c\| + \|b\| \leq C$ and $\|U_t\| + \|V_t\| \leq C$ from Lemma D.2.10 and the fact that we have $t \in [-T, T]$ and then Lemma 5.4.8b): $\operatorname{ess\,sup}_x \|Q_t(D_x^\beta W_x)\varphi_t\|_2 \leq C$ for $t \in [-T, T]$. $\square$

In order to easily show the properties in the time variable we want to work with $\mathcal{V}_t$ to use Lemma D.2.6 directly. Therefore we introduce $G_t(x) := Q_t W_x \varphi_t \oplus J Q_t W_x \varphi_t$ and note $\mathcal{Q}_0 S \mathcal{V}_t^* S G_t(x) = (c + J^* b)(U_t^* - V_t^* J) Q_t W_x \varphi_t \oplus J(c + J^* b)(U_t^* - V_t^* J) Q_t W_x \varphi_t = f_t(x, .) \oplus J f_t(x, .)$. And show $\forall x \in \mathbb{R}^3$

$$(t \mapsto \mathcal{Q}_0 S \mathcal{V}_t^* S G_t(x) = f_t(x, .) \oplus J f_t(x, .)) \in C^1(\mathbb{R}, L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)) , \tag{E.5}$$

$$\partial_t \mathcal{Q}_0 S \mathcal{V}_t^* S G_t(x) = \mathcal{Q}_0 S \dot{\mathcal{V}}_t^* S G_t(x) + \mathcal{Q}_0 S \mathcal{V}_t^* S \partial_t G_t(x) . \tag{E.6}$$

Hence $\forall x \in \mathbb{R}^3$

$$(t \mapsto (c + J^* b)(U_t^* - V_t^* J) Q_t W_x \varphi_t = f_t(x, .)) \in C^1(\mathbb{R}, L^2(\mathbb{R}^3)) . \tag{E.7}$$

This only works for fixed x since $\mathcal{V}_t$ is only strongly differentiable.

Proof. We prove that the conditions for the product rule for operators, namely Lemma F.0.1, are satisfied. We choose as a comparison operator $B := A_0 = \begin{pmatrix} -\frac{\Delta}{2} & 0 \\ 0 & \left(\frac{\Delta}{2}\right)^T \end{pmatrix}$, $D(A_0) = H^2(\mathbb{R}^3) \oplus J H^2(\mathbb{R}^3)$.

- $\mathcal{Q}_0 S \mathcal{V}_t^* S$ is relative A_0 bounded.
- $\forall \phi \in D(A_0)$: $(t \mapsto \mathcal{Q}_0 S \mathcal{V}_t^* S \phi) \in C^1(\mathbb{R}, L^2 \oplus J L^2)$ (see Lemma D.2.6) and $\mathcal{Q}_0 \in \mathcal{L}(L^2 \oplus J L^2)$.
- For $x \in \mathbb{R}^3$: $(t \mapsto G_t(x)) \in C^1(\mathbb{R}, H^2 \oplus J H^2) = C^1(\mathbb{R}, D(A_0))$ (as in Lemma 5.4.8, note that $W \in H^2$).

Hence (E.7) follows from Lemma F.0.1. □

Now we prove the bound on the derivative: $\dot{f} \in l^\infty_{\text{loc}}(\mathbb{R}_t, L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C})))$ and $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\operatorname*{ess\,sup}_x \|\partial_t f_t(x\, .)\|_2 \leq \|\partial_t \mathcal{Q}_0 S \mathcal{V}_t^* S G_t\|_{L^\infty(\mathbb{R}^3, L^2(\mathbb{R}^3, \mathbb{C}))} \leq C . \tag{E.8}$$

Proof. We start with

$$\|\partial_t \mathcal{Q}_0 S \mathcal{V}_t^* S G_t(x)\|_{L^2 \oplus L^2} \leq \|\mathcal{Q}_0 S \mathcal{V}_t^* S \partial_t G_t(x)\|_{L^2 \oplus L^2} \tag{E.9}$$

$$+ \|\mathcal{Q}_0 S \mathcal{V}_t^* S \mathrm{i} \mathcal{A}(t) G_t(x)\|_{L^2 \oplus L^2} . \tag{E.10}$$

We estimate (E.9) and (E.10) separately. First note that due to Lemma D.2.6 and Lemma C.0.1

$$\|\mathcal{V}(t,s)\|_{\mathcal{L}(L^2 \oplus J L^2)} \leq \mathrm{e}^{\operatorname{sgn}(t-s) \int_s^t \|K_2(\tau)\|_{\mathcal{L}(L^2, (L^2)^*)} d\tau} , \tag{E.11}$$

and for $t \in [-T, T]$

$$\|K_2(t)\|_{\mathcal{L}(L^2,(L^2)^*)} + \|A_1(t)\|_{\mathcal{L}(L^2 \oplus L^2)} \leq C\,. \tag{E.12}$$

With (E.11) and $(t \mapsto \|K_2(t)\|_{\mathcal{L}(L^2,(L^2)^*)} + \|A_1(t)\|_{\mathcal{L}(L^2 \oplus L^2)}) \in C(\mathbb{R}, \mathbb{R})$ (see Lemma C.0.1) we conclude similar to below that $\dot{f} \in l^\infty_{\text{loc}}(\mathbb{R}_t, L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d_y, \mathbb{C})))$. In addition with the estimate (E.12) we are able to explicitly bound (E.10): $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\begin{aligned}
\|\mathcal{Q}_0 S \mathcal{V}_t^* S \mathrm{i} \mathcal{A}(t) G_t(x)\|_{L^2 \oplus L^2} &\leq \|\mathcal{Q}_0 S \mathcal{V}_t\|_{\mathcal{L}(L^2 \oplus JL^2)} \|(A_0 + A_1(t)) G_t(x)\|_{L^2 \oplus L^2} \\
&\leq \|\mathcal{Q}_0 S \mathcal{V}_t\|_{\mathcal{L}(L^2 \oplus JL^2)} \{\|A_0 G_t(x)\| \qquad \text{(E.13)} \\
&\quad + \|A_1(t)\|_{\mathcal{L}(L^2 \oplus L^2)} \|G_t(x)\|\} \\
&\leq C \|A_0 G_t(x)\| + C \|G_t(x)\| \\
&\leq C(\|\Delta Q_t W_x \varphi_t\| + \|Q_t W_x \varphi_t\|) \leq C \qquad \text{(E.14)}
\end{aligned}$$

where in step 1 we used $\mathcal{A}(t) = A_0 + A_1(t)$, in step 3 (E.11), (E.12), in step 4 the definition of A_0 and $G_t(x)$ and in step 5 Lemma 5.4.8b). Note that (E.14) is the reason why always assume $W \in H^2$ in Assumption 2.0.3.
Now we proceed with the estimate of (E.9)

$$\begin{aligned}
\|\mathcal{Q}_0 S \mathcal{V}_t^* S \partial_t G_t(x)\|_{L^2 \oplus L^2} &\leq \|\mathcal{Q}_0 S \mathcal{V}_t\|_{\mathcal{L}(L^2 \oplus L^2)} \|\partial_t G_t(x)\|_{L^2 \oplus L^2} \\
&\leq C \|\partial_t G_t(x)\| \\
&\leq C \|\partial_t Q_t W_x \varphi_t\|_2 \\
&\leq C
\end{aligned}$$

where in step 1 we used (E.11), in step 3 the definition of $G_t(x)$ and in the last step Lemma 5.4.8, $t \in [-T, T]$. □

■

E.2.2 Proof of Lemma 5.4.8

Proof of Lemma 5.4.8. We use the regularity and bounds on φ_t and V, W to conclude through standard arguments all regularity and bounds on $Q_t W_x \varphi_t$.

For the regularity arguments the only properties of φ we need are $\varphi \in C^1(\mathbb{R}, H^\infty)$ and that φ solves the Hartree equation. The additional assumptions on φ are needed to get the bounds in Lemma 5.4.8.

We start by noting that for $k \in \mathbb{N}_0$

$$(t \mapsto Q_t) \in C^1(\mathbb{R}, \mathrm{HS}(H^k(\mathbb{R}^3)))\,, \tag{E.15}$$

$$u_t := \varphi_t/\Lambda^{1/2}\,, \quad \partial_t Q_t = -\,|\dot{u}_t\rangle\langle u_t| - |u_t\rangle\langle \dot{u}_t|\,, \tag{E.16}$$

$$\big(x \mapsto W_x = (y \mapsto W(x-y))\big) \in W^{M,\infty}(\mathbb{R}^3, L^\infty(\mathbb{R}^3,\mathbb{R}))\,, \tag{E.17}$$

where (E.15) and (E.16) follow from Lemma F.0.2 and $\varphi_t \in C^1(\mathbb{R}, H^\infty(\mathbb{R}^3,\mathbb{C}))$, and (E.17) from $W \in W^{M,\infty}(\mathbb{R}^3,\mathbb{R})$.
Due to (E.17), $\varphi_t \in L^2(\mathbb{R}^3,\mathbb{C})$ and $Q_t \in \mathcal{L}(L^2)$ we conclude $Q_t W_x \varphi_t \in W^{M,\infty}(\mathbb{R}^3_x, L^2(\mathbb{R}^3,\mathbb{C}))$ and for $|\beta| \leq 2$

$$D^\beta_x Q_t W_x \varphi_t = Q_t (D^\beta_x W_x)\varphi_t\,. \tag{E.18}$$

Next we prove the regularity in the time argument: $\forall x \in \mathbb{R}^3, \beta \in \mathbb{N}_0^3$, $\forall k \in \mathbb{N}_0,\ k + |\beta| \leq M$

$$(t \mapsto Q_t(D^\beta W_x)\varphi_t) \in C^1(\mathbb{R}, H^k(\mathbb{R}^3))\,, \tag{E.19}$$

$$\partial_t^{H^k} Q_t(D^\beta W_x)\varphi_t = \dot{Q}_t(D^\beta W_x)\varphi_t + Q_t(D^\beta W_x)\dot{\varphi}_t\,, \tag{E.20}$$

$$\big(t \mapsto (x \mapsto Q_t W_x \varphi_t)\big) \in C^1(\mathbb{R}, W^{M,\infty}(\mathbb{R}^3, L^2(\mathbb{R}^3,\mathbb{C}))\,, \tag{E.21}$$

$$\partial_t^{L^\infty(\mathbb{R}^3,L^2)}(x \mapsto Q_t(D^\beta W_x)\varphi_t) = \Big(x \mapsto \dot{Q}_t(D^\beta W_x)\varphi_t + Q_t(D^\beta W_x)\dot{\varphi}_t\Big)\,. \tag{E.22}$$

and $\forall T \geq 0\ \exists C > 0$ such that $\forall \Lambda \geq 1$, $-T \leq t \leq T$

$$\sup_x \{\|Q_t(D^\beta W_x)\varphi_t\|_{H^k} + \|\partial_t Q_t(D^\beta W_x)\varphi_t\|_{H^k}\} \leq C\,. \tag{E.23}$$

Proof. We start by proving the regularity. Let $x \in \mathbb{R}^3$ fixed then

$$\begin{aligned}
&\left\|\frac{Q_{t+h}(D^\beta W_x)\varphi_{t+h} - Q_t(D^\beta W_x)\varphi_t}{h} - \dot{Q}_t(D^\beta W_x)\varphi_t - Q_t(D^\beta W_x)\dot{\varphi}_t\right\|_{H^k}\\
\leq &\left\|\frac{Q_{t+h} - Q_t}{h}(D^\beta W_x)\varphi_t - \dot{Q}_t(D^\beta W_x)\varphi_t\right\|_{H^k}\\
&+ \left\|Q_{t+h}(D^\beta W_x)\frac{\varphi_{t+h} - \varphi_t}{h} - Q_{t+h}(D^\beta W_x)\dot{\varphi}_t\right\|_{H^k}\\
&+ \left\|(Q_{t+h} - Q_t)(D^\beta W_x)\dot{\varphi}_t\right\|_{H^k}\\
\leq &\left\|\frac{Q_{t+h} - Q_t}{h} - \dot{Q}_t\right\|_{\mathcal{L}(H^k)} \sum_{|\gamma|\leq k} C\|D^\gamma D^\beta W\|_\infty \|\varphi_t\|_{H^k}\\
&+ C\|Q_{t+h}\|_{\mathcal{L}(H^k)} \sum_{|\gamma|\leq k} C\|D^\gamma D^\beta W\|_\infty \left\|\frac{\varphi_{t+h} - \varphi_t}{h} - \dot{\varphi}_t\right\|_{H^k}\\
&+ \|(Q_{t+h} - Q_t)\|_{\mathcal{L}(H^k)} \sum_{|\gamma|\leq k} C\|D^\gamma D^\beta W\|_\infty \|\dot{\varphi}_t\|_{H^k}
\end{aligned}$$

$$\to 0\,. \tag{E.24}$$

all three terms go to zero because $(t \mapsto Q_t) \in C^1(\mathbb{R}, \mathrm{HS}(H^k))$, especially $\|Q(t+h)\|_{H^k} \le R_t > 0$ for $h \in \bar{B}(t,1)$ and $\varphi \in C^1(\mathbb{R}, H^\infty)$. Note that $(t \mapsto Q_t) \in C^1(\mathbb{R}, \mathrm{HS}(H^k))$ follows from Lemma F.0.2. Now we can easily follow the differentiability of $(t \mapsto Q_t(D^\beta W_x)\varphi_t)$ in H^∞ with the help of Lemma F.0.3. Continuity of $Q_t(D^\beta W_x)\varphi_t$ and $\partial_t Q_t(D^\beta W_x)\varphi_t$ is proven similar to (E.24) since $(t \mapsto Q_t) \in C^1(\mathbb{R}, \mathrm{HS}(H^k))$, $\varphi \in C^1(\mathbb{R}, H^\infty)$. The constraint $k + |\beta| \le M$ is needed because W is only M-times differentiable.
(E.24) also holds true if we take $\operatorname{ess\,sup}_x$. And with (E.18) we conclude $(t \mapsto (x \mapsto Q_t W_x \varphi_t)) \in C^1(\mathbb{R}, W^{M,\infty}(\mathbb{R}^3, L^2(\mathbb{R}^3, \mathbb{C}))$ – the continuity is proven analogous to the differentiability, and $\partial_t^{L^\infty(\mathbb{R}^3, L^2)}(x \mapsto Q_t(D^\beta W_x)\varphi_t) = \big(x \mapsto \dot{Q}_t(D^\beta W_x)\varphi_t + Q_t(D^\beta W_x)\dot{\varphi}_t\big)$.

Now we derive the bounds and therefore assume Condition 2.1.8$_{k=2+2}$. Because of (E.16): $\partial_t Q_t = -|\dot{u}_t\rangle\langle u_t| - |u_t\rangle\langle \dot{u}_t|$ we get

$$\begin{aligned}
&\operatorname*{ess\,sup}_x \|\dot{Q}_t(D^\beta W_x)\varphi_t + Q_t(D^\beta W_x)\dot{\varphi}_t\|_2 \\
=\;& \operatorname*{ess\,sup}_x \{\| - \dot{u}_t\big\langle u_t, (D^\beta W_x)\varphi_t\big\rangle - u_t\big\langle \dot{u}_t, (D^\beta W_x)\varphi_t\big\rangle + Q_t(D^\beta W_x)\dot{\varphi}_t\|_2\} \\
\le\;& \operatorname*{ess\,sup}_x \frac{1}{\Lambda}\Big(\|\dot{\varphi}_t\|_2 |\big\langle \varphi_t, (D^\beta W_x)\varphi_t\big\rangle| + \|\varphi_t\|_2 |\big\langle \dot{\varphi}_t, (D^\beta W_x)\varphi_t\big\rangle|\Big) \\
&+ \|(D^\beta W_x)\dot{\varphi}_t\|_2 \\
\le\;& \frac{1}{\Lambda}\Big(\|\dot{\varphi}_t\|_2 \|\varphi_t\|_2 \|D^\beta W_x\|_2 \|\varphi_t\|_\infty + \|\varphi_t\|_2 \|\dot{\varphi}_t\|_2 \|D^\beta W_x\|_2 \|\varphi_t\|_\infty\Big) && \text{(E.25)} \\
&+ \|D^\beta W_x\|_{2,\infty} \|\dot{\varphi}_t\|_{2\wedge\infty} && \text{(E.26)}
\end{aligned}$$

Now (E.25) can be estimate with Corollary B.1.2, $\|\varphi_t\|_2 = \Lambda^{1/2}$ and

$$\begin{aligned}
\|\dot{\varphi}_t\|_2 &\le \| - \Delta/2\varphi_t\|_2 + \|V * |\varphi_t|^2 \varphi_t\|_2 + |\mu_t| \cdot \|\varphi_t\|_2 \\
&\le C\Lambda^{1/2-2/3} + \|V\|_1 \|\varphi_t\|_\infty^2 \|\varphi_t\|_2 + C\Lambda^{1/2} \le C\Lambda^{1/2}\,,
\end{aligned}$$

where we used the Hartree equation and then Corollary B.2.4 and Corollary B.1.2 for $t \in [-T, T]$. We estimate (E.26) with

$$\begin{aligned}
\|\dot{\varphi}_t\|_{2\wedge\infty} &\le \| - \Delta/2\varphi_t\|_{2\wedge\infty} + \|V * |\varphi_t|^2 \varphi_t\|_\infty + |\mu_t| \cdot \|\varphi_t\|_\infty \\
&\le C\Lambda^{-2/3} + \|V\|_1 \|\varphi_t\|_\infty^2 \|\varphi_t\|_\infty + C \le C\,,
\end{aligned}$$

where we used Corollary B.2.4. Hence

$$\operatorname*{ess\,sup}_x \|\dot{Q}_t W_x \varphi_t + Q_t W_x \dot{\varphi}_t\|_2 \le \text{(E.25)} + \text{(E.26)} \le C\,.$$

The estimate $\operatorname{ess\,sup}_x \|Q_t W_x \varphi_t\|_2 \leq C$ is analogous to the one above. We conclude (E.23). □

Appendix F

Supplementary Collection of Standard Results

This Appendix F presents useful tools from functional analysis that are required for various technical details throughout this thesis, listed in no particular order.

We start with some results on differentiability.

Lemma F.0.1 (Product Rule for Operators)**.** *Let $d, k \in \mathbb{N}_+$, $\mathscr{H}$ be a Hilbert space, B an operator in $\mathscr{H}$, $\{A_x\}_{x\in\mathbb{R}^d}$ a family of operators in $\mathscr{H}$, $D(B) \subset D(A_x)$, and $\{\psi_x\}_{x\in\mathbb{R}^d} \subset D(B)$ with*

i) $\forall \psi \in D(B)$: $(x \mapsto A_x\psi) \in C^k(\mathbb{R}^d, \mathscr{H})$, *and* $\forall x \in \mathbb{R}^d, |\alpha| \leq k-1$: $\partial^\alpha A_x$ *relative B bounded uniformly in $x \in \mathbb{R}^d$,*[1] *where*

$$\begin{aligned} \partial^\alpha A_x : D(B) \to& \mathscr{H} \\ \psi \mapsto& \partial^\alpha (A_x \psi) \end{aligned}$$

linear operator on $\mathscr{H}$.

ii) $(x \mapsto \psi_x) \in C^k(\mathbb{R}^d, D(B))$.

Then $(x \mapsto A_x\psi_x) \in C^k(\mathbb{R}^d, \mathscr{H})$ *and*

$$\partial^\alpha A_x \psi_x = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \partial^\beta A_x \partial^{\alpha-\beta} \psi_x \,.$$

[1] $\exists a, b > 0$: $\forall \psi \in D(B)$: $\|\partial^\alpha A_x \psi\| \leq a\|B\psi\| + b\|\psi\|$.

Proof of Lemma F.0.1. The proof of Lemma F.0.1 is straightforward and follows analogously to the proof of the product rule for complex-valued functions. ■

Lemma F.0.2. *Let $\mathscr{H}$ be a separable Hilbert space and $\psi, \varphi \in C^1(\mathbb{R}, \mathscr{H})$ then $(t \mapsto |\psi_t\rangle\langle\varphi_t|) \in C^1(\mathbb{R}, \mathrm{HS}(\mathscr{H}))$ and*

$$\partial_t |\psi_t\rangle\langle\varphi_t| = |\dot{\psi}_t\rangle\langle\varphi_t| + |\psi_t\rangle\langle\dot{\varphi}_t| \,.$$

Especially by $\mathrm{HS}(\mathscr{H}) \simeq \mathscr{H} \otimes \mathscr{H}$ we have $(t \mapsto \psi_t \otimes \varphi_t) \in C^1(\mathbb{R}, \mathscr{H} \otimes \mathscr{H})$ and

$$\partial_t \psi_t \otimes \varphi_t = \dot{\psi}_t \otimes \varphi_t + \psi_t \otimes \dot{\varphi}_t \,.$$

Proof of Lemma F.0.2. It is easy to prove the differentiability on $\mathscr{H} \otimes \mathscr{H}$ with standard arguments. Then one uses the isometric isomorphism $U : \mathscr{H} \otimes \mathscr{H} \to \mathrm{HS}(\mathscr{H}), f \otimes g \mapsto |f\rangle\langle g|$ to conclude the claim. ■

Lemma F.0.3 (H^∞ is a complete Fréchet space)**.** *Let $H^\infty(\mathbb{R}^d) := \bigcap_{k\in\mathbb{N}_0} H^k(\mathbb{R}^d, \mathbb{C})$. Then*

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|f-g\|_{H^k}}{1 + \|f-g\|_{H^k}} < \infty \,, \quad f, g \in H^\infty$$

is a metric on H^∞ and (H^∞, d) is a complete metric space.
Equivalent are for $f_n, f \in H^\infty$

i) $\|f_n - f\|_{H^k} \to 0$, $\forall k \in \mathbb{N}_0$.

ii) $d(f_n, f) \to 0$.

Proof of Lemma F.0.3. For a proof see for example [Por19, Proposition 3.8]. ■

Lemma F.0.4 (Banach-Alaoglu)**.** *Let X be a Banach space and Z be a separable Banach space with $X \simeq Z'$ isometric isomorph and let $\Phi : X \to Z'$ be a isometric isomorphism. Then for any bounded sequence $(x_n) \subset X$, $\|x_n\|_X \leq C$, we have a weak*–convergent subsequence*

$$\Phi(x_{n_k}) \overset{\sigma(Z',Z)}{\longrightarrow} \Phi(x) \in Z' \,,$$
$$\|x\|_X \leq \liminf_{k\to\infty} \|x_{n_k}\| < \infty \,.$$

Proof of Lemma F.0.4. The proof of Lemma F.0.4 can for example be found in [Wer11]. ■

As a direct corollary of Banach-Alaoglu we get the following Lemma characterizing the form domain of positive operators.

Lemma F.0.5 (Positive Quadratic Form: Characterization of the Form Domain)**.** *Let $\mathscr{H}$ be a Hilbert space and $q \geq 0$ closed quadratic form. Then*

$$Q(q) = \{\psi \in \mathscr{H} \mid \exists C > 0, (\psi_n) \subset Q(q) : q(\psi_n) \leq C,\ \forall n,\ \psi_n \overset{\sigma(\mathscr{H},\mathscr{H}')}{\longrightarrow} \psi\}\,.$$

Especially for $\psi \in Q(A)$ and ψ_n as above there exists a subsequence n_k with

$$q(\psi) + \|\psi\|_{\mathscr{H}} \leq \liminf_{k\to\infty}\{q(\psi_{n_k}) + \|\psi_{n_k}\|_{\mathscr{H}}\}\,,$$
$$\langle\theta, \psi_{n_k} - \psi\rangle_q \to 0\,, \quad \forall\theta \in Q(q)\,.$$

Remark F.0.6. The Lemma above can be formally stated as

$$\psi \in Q(q) \quad \Leftrightarrow \quad \text{“}q(\psi) < \infty\text{”}$$

if $q \geq 0$.
By $\psi_n \overset{\sigma(\mathscr{H},\mathscr{H}')}{\longrightarrow} \psi$ we mean that ψ_n is weakly convergent.

Lemma F.0.7. *Let $\mathscr{H}$ be a separable Hilbert space and $q \geq 0$ closable quadratic form defined in [RS80]. Then $(Q(q), q(\,.\,) + \|\,.\,\|_{\mathscr{H}})$ is a separable Hilbert space.*

Proof of Lemma F.0.7. Since $q \geq 0$ closable quadratic form there exists a $h \geq 0$ self-adjoint with $q(\psi) = \langle\psi, (h+1)^{1/2}\psi\rangle$ [RS80]. Thus $(h+1)^{1/2} : \overline{Q(q)} \to \mathscr{H}$ is an isometric isomorphism, which proves the claim. ■

Lemma F.0.8 (Duhamel: Equivalence of Differential and Integral Equation)**.** *Let $\mathscr{H}$ be a Hilbert space, $U(t)$, $t \in \mathbb{R}$, unitary group with generator $-iA$ on $\mathscr{H}$. Let $(t_0, u_0) \in \mathbb{R} \times D(A)$, $I \subset \mathbb{R}$ interval, $t_0 \in I$, and $f \in C(I \times D(A), D(A))$.*[2]
Equivalent are:

i) $\exists! u \in C(I, D(A))$ *with*

$$u(t) = U(t - t_0)u_0 + \int_{t_0}^{t} U(t-s)f(s, u(s))ds \quad \forall t \in I\,.$$

ii) $\exists! u \in C(I, D(A)) \cap C^1(I, \mathscr{H})$

$$\dot{u}(t) = -iAu(t) + f(t, u(t)) \quad \forall t \in I\,,$$
$$u(t_0) = u_0\,.$$

In addition if a map $u : I \to D(A)$ fulfils one case from above then it fulfils also the other.

[2] We use the graph norm on $D(A)$.

Remark F.0.9. Lemma F.0.8 still holds if I choose: $\mathscr{H}$ Banach space, $U(t)$, $t \in \mathbb{R}_{\geq 0}$, C_0-semi group with generator $-iA$ and $(t_0, u_0) \in \mathbb{R} \times D(A)$, $I \subset \mathbb{R}$ interval, $t_0 \in I$, and $f \in C(I \times D(A), D(A))$.

Proof of Lemma F.0.8. Duhamel's formula is as well known result. For a proof see for example [Kat95, Section 9.1.5]. ■

F.1 Creation and Annihilation Operators

For simplicity we define the creation and annihilation operators only on Fock spaces over the Hilbert space $L^2(\mathbb{R}^d, \mathbb{C})$ (for a general definition see for example [Nam20; ATT]).

Definition F.1.1. Let $d \in \mathbb{N}_+$, $f, g \in L^2(\mathbb{R}^d, \mathbb{C})$ and $\psi_n \in L^2(\mathbb{R}^d, \mathbb{C})^{\otimes_s n}$. We define

$$(a(f)\psi_n)(y_1, \dots, y_{n-1}) = \sqrt{n} \int \psi_n(y_1, \dots, y_{n-1}, y) f^*(y) dy \,,$$
$$(a^*(f)\psi_n)(y_1, \dots, y_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \psi_n(y_1, \dots, \hat{y}_i, \dots, y_{n+1}) f(y_i) \,,$$

where $\hat{y}_i$ means that the variable y_i is missing.

Lemma F.1.2. *For $d \in \mathbb{N}_+$ and $f \in L^2(\mathbb{R}^d, \mathbb{C})$ the creation and annihilation operators $a^*(f_x)$ and $a(f_x)$ are well and densely defined operators on $\bigoplus_{n=0}^{\infty} (L^2)^{\otimes_s n}$ with domains*

$$D(a(f)) = \Big\{ \psi \in \bigoplus_{n=0}^{\infty} (L^2)^{\otimes_s n} \,\Big|\, \sum_{n \geq 1} \|a(f)\psi_n\|^2_{(L^2)^{\otimes_s (n-1)}} < \infty \Big\} \,,$$
$$D(a^*(f_x)) = \Big\{ \psi \in \bigoplus_{n=0}^{\infty} (L^2)^{\otimes_s n} \,\Big|\, \sum_{n \geq 0} \|a^*(f_x)\psi_n\|^2_{(L^2)^{\otimes_s (n+1)}} < \infty \Big\} \,.$$

Lemma F.1.3. *Let $d \in \mathbb{N}_+$ and $f \in L^2(\mathbb{R}^d, \mathbb{C})$. Then*

i) $a^*(f)$ is the adjoint of $a(f)$.

ii) $Q(\mathcal{N}) \subset D(a^*(f)) \cap D(a(f))$ and

$$\|a^*(f)\psi\| \leq \|f\| \|(\mathcal{N}+1)^{1/2}\psi\| \,,$$
$$\|a(f)\psi\| \leq \|f\| \|\mathcal{N}^{1/2}\psi\| \,,$$

for all $\psi \in Q(N)$.

iii) For $\alpha \geq 0$ we have $a^{\#}(f) D(\mathcal{N}^{\alpha+1/2}) \subset \mathcal{N}^{\alpha}$ and on the domain $D(\mathcal{N}^{\alpha+1/2})$

$$a(f)\mathcal{N}^{\alpha} = (\mathcal{N}+1)^{\alpha} a(f) \,,$$

$$a^*(f)\mathcal{N}^\alpha = (\mathcal{N}-1)^\alpha a^*(f)\,.$$

iv) We have the CCR on the domain $D(\mathcal{N})$

$$a(f)a^*(g) - a^*(g)a(f) = \langle f, g\rangle\,,$$

for $g \in L^2(\mathbb{R}^d, \mathbb{C})$.

A useful notation are the generalize creation and annihilation operators:

$$A(f \oplus Jg) = a(f) + a^*(g)\,, \quad A^*(f \oplus Jg) = a^*(f) + a(g)\,, \quad \forall f, g \in L^2(\mathbb{R}^d, \mathbb{C})\,, \tag{F.1}$$

as discussed in Appendix D.2.1, where $J : \mathscr{H} \to \mathscr{H}^*, \psi \mapsto \langle\,.\,, \psi\rangle$ antiunitary.
For example, they allow us to generalize the commutation of $\mathcal{N}$ with $a^\#(f)$.

Lemma F.1.4. *Let* $\mathscr{H}$ *be a Hilbert space,* h *a self-adjoint operator on* $\mathscr{H}$*. Then* $\forall F \in D(h) \oplus JD(h)$ *we have* $D(d\Gamma(h)\mathcal{N}^{1/2}) \subset D(d\Gamma(h)A(F))$ *and*

$$A(F)d\Gamma(h)\psi - d\Gamma(h)A(F)\psi = A\left(\begin{pmatrix} h & 0 \\ 0 & h^T \end{pmatrix} F\right)\psi\,,$$

for $\psi \in D(d\Gamma(h)\mathcal{N}^{1/2})$.

Coordinate Dependent Creation and Annihilation Operators

Due to the presence of an additional tracer particle, we frequently see terms of the form $a(Q_t W_x \varphi_t)$ and $a^*(Q_t W_x \varphi_t)$, where the argument of the creation and annihilation operators depends on the coordinate x of the tracer particle. In this section, we provide a more detailed discussion of such operators, expanding on the brief introduction given in the Notation overview.

Definition F.1.5 (Coordinate Dependent Creation and Annihilation Operators)**.** Let $d \in \mathbb{N}_+$ and $f \in L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d, \mathbb{C}))$. Define the linear operators $a(f_x)$ and $a^*(f_x)$ on $\bigoplus_{n=0}^\infty L^2(\mathbb{R}^d, \mathbb{C}) \otimes L^2(\mathbb{R}^d, \mathbb{C})^{\otimes_s n}$ by

$$\begin{aligned} a(f_x)\psi_n(\tilde{x}, y_1, \ldots, y_{n-1}) &:= n^{1/2} \int \psi_n(\tilde{x}, y_1, \ldots, y_{n-1}, y) f^*_{\tilde{x}}(y) dy\,, \\ &\qquad \forall n \in \mathbb{N}_+\,,\ \forall \psi_n \in L^2 \otimes (L^2)^{\otimes_s n}\,, \\ a(f_x)\psi_0 &:= 0\,, \quad \forall \psi_0 \in L^2 \end{aligned}$$

and

$$a^*(f_x)\psi_n(\tilde{x}, y_1, \ldots, y_{n+1}) := \frac{1}{(n+1)^{1/2}} \sum_{i=1}^{n+1} \psi_n(\tilde{x}, y_1, \ldots, \hat{y}_i, \ldots, y_{n+1}) f_{\tilde{x}}(y_i)\,,$$
$$\forall n \in \mathbb{N}\,,\ \forall \psi_n \in L^2 \otimes (L^2)^{\otimes_s n}\,.^3$$

$a(f_x)L^2 \otimes (L^2)^{\otimes_s n} \subset L^2 \otimes (L^2)^{\otimes_s n-1}$ and $a^*(f_x)L^2 \otimes (L^2)^{\otimes_s n} \subset L^2 \otimes (L^2)^{\otimes_s +1}$. The corresponding definition spaces are

$$D(a(f_x)) = \Big\{\psi \in \bigoplus_{n=0}^{\infty} L^2 \otimes (L^2)^{\otimes_s n} \,\Big|\, \sum_{n \geq 1} \|a(f_x)\psi_n\|^2_{L^2 \otimes (L^2)^{\otimes_s (n-1)}} < \infty\Big\}\,,$$
$$D(a^*(f_x)) = \Big\{\psi \in \bigoplus_{n=0}^{\infty} L^2 \otimes (L^2)^{\otimes_s n} \,\Big|\, \sum_{n \geq 0} \|a^*(f_x)\psi_n\|^2_{L^2 \otimes (L^2)^{\otimes_s (n+1)}} < \infty\Big\}\,.$$

In addition we define $A(F_x) = A(f_x \oplus Jg_x) := a(f_x) + a^*(g_x)$ where $F_x := f_x \oplus Jg_x$ with $f, g \in L^\infty(\mathbb{R}^d_x, L^2(\mathbb{R}^d, \mathbb{C}))$.

Lemma F.1.6. *For $d \in \mathbb{N}_+$ and $f \in L^\infty(\mathbb{R}^d, L^2(\mathbb{R}^d, \mathbb{C}))$ the creation and annihilation operators $a^*(f_x)$ and $a(f_x)$ are well and densely defined operators on $\bigoplus_{n=0}^\infty L^2 \otimes (L^2)^{\otimes_s n}$.*

The following Lemma connects them to the standard creation and annihilation operators.

Lemma F.1.7. *Let $\phi \in \bigoplus_{n=0}^\infty L^2 \otimes (L^2)^{\otimes_s n}$. Then we have that $\big(\mathbb{R}^d \to L^2(\mathbb{R}^{dn}), x \mapsto \phi_x^{(n)}\big) \in L^2(\mathbb{R}^d, L^2(\mathbb{R}^{dn}))$, where $\phi_x^{(n)}(y_1, \ldots, y_n) := \phi^{(n)}(x, y_1, \ldots, y_n)$.[4] Denote $\phi_x := \bigoplus_{n=0}^\infty \phi_x^{(n)}$. For clarification we denote the "standard" creation and annihilation operators as $a_s^{\#}(h)$, $h \in L^2(\mathbb{R}^d, \mathbb{C})$ with an index 's'.*
Then for $\psi \in \bigoplus_{n=0}^\infty L^2 \otimes (L^2)^{\otimes_s n}$ and $\# \in \{\emptyset, *\}$ the following are equivalent:

i) $\psi \in D(a^{\#}(f_x))$.

ii) $\psi_x \in D(a_s^{\#}(f_x))$ *for almost all x and* $\int dx \|a_s^{\#}(f_x)\psi_x\|^2 < \infty$.

In this case

$$\langle \phi, a^{\#}(f_x)\psi\rangle = \int dx \langle \phi_x, a_s^{\#}(f_x)\psi_x\rangle\,.$$

This lemma allows us to transfer many properties of the standard creation and annihilation operators to their coordinate-dependent counterparts.

[3]To be more precise: $\langle \phi_{n+1}, a^*(f_x)\psi_n\rangle_{L^2 \otimes (L^2)^{\otimes_s n+1}} := \int d\tilde{x} dy_1 \ldots dy_{n+1} \phi^*_{n+1}(y_1, \ldots y_{n+1}) \cdot \frac{1}{(n+1)^{1/2}} \sum_{i=1}^{n+1} \psi_n(\tilde{x}, y_1, \ldots, \hat{y}_i, \ldots, y_{n+1}) f_{\tilde{x}}(y_i)$ and a similar definition for $a(f_x)$. The definition inside the scalar product is favourable because elements of L^2 are only defined almost everywhere.

[4]Note: Here we used the isometric isomorphism $T : L^2(\mathbb{R}^{m+n}, E) \to L^2(\mathbb{R}^m, L^2(\mathbb{R}^n; E), g \mapsto g(x, .)$ from [AE08, Theorem 6.22].

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